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Author(s)	Saito, Mutsumi
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**CRITICAL MODULES
OF THE RING OF DIFFERENTIAL OPERATORS
OF AN AFFINE SEMIGROUP ALGEBRA**

MUTSUMI SAITO

ABSTRACT. Let D be the ring of differential operators of an affine semigroup algebra. Regarding the Krull dimension of finitely generated \mathbb{Z}^d -graded D -modules, we characterize critical \mathbb{Z}^d -graded D -modules. Moreover we explicitly describe cyclic ones.

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1. INTRODUCTION

Critical modules were introduced by Hart (1971) and Goldie (1972). They are the modules whose quotient by any nonzero submodule has strictly smaller Krull dimension (in the sense of (Gabriel, 1962) and (Rentschler, Gabriel, 1967)). The critical modules of Krull dimension 0 are precisely the simple modules. The notions of Krull dimension and critical modules enable us to use Artinian type method to Noetherian rings (see e.g. (Goodearl, Warfield, 1989; Lenagan, 2000; McConnell, Robson, 1987)).

Let F be an algebraically closed field of characteristic 0. Let $A := \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ be a finite subset of \mathbb{Z}^d . Let R_A denote the semigroup algebra over F of the monoid generated by A , and $D(R_A)$ the ring of differential operators of R_A ; both of them have natural \mathbb{Z}^d -gradings. Musson, Van den Bergh (1998) classified the \mathbb{Z}^d -graded simple $D(R_A)$ -modules. (See also (Saito, Traves, 2001).) In this paper, considering the Krull dimension of finitely generated \mathbb{Z}^d -graded $D(R_A)$ -modules, we characterize critical \mathbb{Z}^d -graded $D(R_A)$ -modules. Moreover we explicitly describe cyclic critical \mathbb{Z}^d -graded $D(R_A)$ -modules.

To consider the Krull dimension, it is convenient to assume the Noetherian property. (Saito, Takahashi, 2008) proved that $D(R_A)$ is always right Noetherian. It also gave some conditions for the left Noetherian property; for example, $D(R_A)$ is left Noetherian if R_A satisfies Serre's (S_2) condition. Throughout this paper, **we assume that $D(R_A)$ is left Noetherian** when we consider left $D(R_A)$ -modules.

This paper is organized as follows. In Section 2, we recall some fundamental facts about the rings of differential operators of affine semigroup algebras.

Let $\mathbf{a}, \mathbf{b} \in F^d$. In (Saito, Traves, 2001), a preorder $\mathbf{a} \preceq \mathbf{b}$ was defined; the equivalence relation induced from \preceq was used for the classifications of \mathbb{Z}^d -graded simple modules and A -hypergeometric systems (Musson, Van den Bergh, 1998; Saito, 2001; Saito, Traves, 2001; Saito, 2007). This preorder is for maximal ideals of the polynomial ring $F[s_1, \dots, s_d]$. In Section 3, we generalize this for prime ideals, suitable for considering critical modules.

In Section 4, we introduce $\tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$), the category of \mathbb{Z}^d -graded finitely generated left (resp. right) $D(R_A)$ -modules. For an object $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} M_{\mathbf{a}} \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$), we

define $\delta(M)$ by

$$\delta(M) := \max_{\mathbf{a}} \dim_{F[s]} M_{\mathbf{a}}.$$

In Section 5, we define $\deg(M)$ for $M \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$) so that, for a descending chain

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq \cdots,$$

there exist at most $\deg(M)$ i 's such that $\delta(M_i/M_{i+1}) = \delta(M)$.

In Section 6, we prove that δ gives the Krull dimension for the category $\tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$).

In Section 7, we characterize critical \mathbb{Z}^d -graded $D(R_A)$ -modules (Theorem 7.4), and explicitly describe cyclic ones (Theorem 7.7).

2. RINGS OF DIFFERENTIAL OPERATORS OF AFFINE SEMIGROUP ALGEBRAS

In this section, we briefly recall some fundamental facts about the rings of differential operators of affine semigroup algebras. Let F be an algebraically closed field of characteristic 0. Let $A := \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ be a finite set of column vectors in \mathbb{Z}^d , and $\mathbb{N}A$ the monoid generated by A . Throughout this paper, we assume that \mathbb{Z}^d is generated by A as an abelian group for simplicity.

The ring $F[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \langle \partial_1, \dots, \partial_d \rangle$ of differential operators with Laurent polynomial coefficients is the ring of differential operators on the algebraic torus $(F^\times)^d$, where $[\partial_i, t_j] = \delta_{ij}$, $[\partial_i, t_j^{-1}] = -\delta_{ij} t_j^{-2}$, and the other pairs of generators commute. Here $[,]$ denotes the commutator, and δ_{ij} is 1 if $i = j$ and 0 otherwise.

The semigroup algebra $R_A := F[\mathbb{N}A] = \bigoplus_{\mathbf{a} \in \mathbb{N}A} F t^{\mathbf{a}}$ is the ring of regular functions on the affine toric variety defined by A , where $t^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\mathbf{a} = {}^t(a_1, a_2, \dots, a_d) \in \mathbb{Z}^d$. Its ring of differential operators $D(R_A)$ can be realized as a subring of the ring

$$F[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \langle \partial_1, \dots, \partial_d \rangle$$

of differential operators on the big torus as follows:

$$D(R_A) = \{P \in F[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \langle \partial_1, \dots, \partial_d \rangle : P(R_A) \subset R_A\}.$$

Put $s_j := t_j \partial_j$ for $j = 1, 2, \dots, d$. Then it is easy to see that $s_j \in D(R_A)$ for all j . We introduce a \mathbb{Z}^d -grading on the ring $D(R_A)$ as follows: For $\mathbf{a} = {}^t(a_1, a_2, \dots, a_d) \in \mathbb{Z}^d$, set

$$D(R_A)_{\mathbf{a}} := \{P \in D(R_A) : [s_j, P] = a_j P \text{ for } j = 1, 2, \dots, d\}.$$

Then $D(R_A) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} D(R_A)_{\mathbf{a}}$.

By regarding $D(R_A)_{\mathbf{a}}$ as a subset of $F[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \langle \partial_1, \dots, \partial_d \rangle$, we see that there exists an ideal I of $F[s] := F[s_1, \dots, s_d]$ such that $D(R_A)_{\mathbf{a}} = t^{\mathbf{a}} I$. To describe this ideal I explicitly, we define a subset $\Omega(\mathbf{a})$ of the semigroup $\mathbb{N}A$ by

$$\Omega(\mathbf{a}) = \{\mathbf{b} \in \mathbb{N}A : \mathbf{b} + \mathbf{a} \notin \mathbb{N}A\} = \mathbb{N}A \setminus (-\mathbf{a} + \mathbb{N}A).$$

Then each $D(R_A)_{\mathbf{a}}$ is described as follows.

Theorem 2.1 (Theorem 2.3 in (Musson, 1987)).

$$D(R_A)_{\mathbf{a}} = t^{\mathbf{a}} \mathbb{I}(\Omega(\mathbf{a})) \quad \text{for all } \mathbf{a} \in \mathbb{Z}^d,$$

where

$$\mathbb{I}(\Omega(\mathbf{a})) := \{f(s) \in F[s] := F[s_1, \dots, s_d] : f \text{ vanishes on } \Omega(\mathbf{a})\}.$$

In particular, we have for each $\mathbf{a} \in \mathbb{N}A$

$$D(R_A)_{\mathbf{a}} = t^{\mathbf{a}}F[s] = F[s]t^{\mathbf{a}},$$

since $\Omega(\mathbf{a}) = \emptyset$ in this case.

3. EQUIVALENCE RELATIONS

In this section, for a prime ideal \mathfrak{p} of $F[s]$, we define a preorder $\preceq_{[\mathfrak{p}]}$ and an equivalence relation $\sim_{[\mathfrak{p}]}$, which are generalizations of those considered in (Musson, Van den Bergh, 1998; Saito, 2001; Saito, Traves, 2001; Saito, 2007) for maximal ideals (Proposition 3.3 and Remark 3.4). This equivalence relation plays a central role in this paper.

Notation. For an ideal I of $F[s]$ and $\mathbf{c} \in \mathbb{Z}^d$, we define a new ideal $I + \mathbf{c}$ by

$$I + \mathbf{c} := \{f(s - \mathbf{c}) : f(s) \in I\}.$$

We clearly have the following.

Proposition 3.1. (1) $\mathbb{I}(V) + \mathbf{c} = \mathbb{I}(V + \mathbf{c})$.

(2) $\mathfrak{m}_{\mathbf{a}} + (\mathbf{b} - \mathbf{a}) = \mathfrak{m}_{\mathbf{b}}$, where $\mathfrak{m}_{\mathbf{a}}$ is the maximal ideal $\langle s_1 - a_1, \dots, s_d - a_d \rangle$ for $\mathbf{a} = {}^t(a_1, \dots, a_d) \in F^d$.

(3) If \mathfrak{p} is prime, then so is $\mathfrak{p} + \mathbf{c}$.

Let \mathfrak{p} be a prime ideal of $F[s]$. We shall introduce an equivalence relation in

$$[\mathfrak{p}] := \{\mathfrak{p} + \mathbf{c} : \mathbf{c} \in \mathbb{Z}^d\}.$$

Notation. We write $\mathfrak{p} \preceq_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{c}$ (or simply $\mathfrak{p} \preceq \mathfrak{p} + \mathbf{c}$) if $\mathbb{I}(\Omega(\mathbf{c})) \not\subseteq \mathfrak{p}$.

Recall that a binary relation is called a *preorder* if it is reflexive and transitive.

Proposition 3.2. *The relation $\preceq_{[\mathfrak{p}]}$ is a preorder in the set $[\mathfrak{p}]$.*

Proof. First, since $\mathbb{I}(\Omega(\mathbf{0})) = (1) \not\subseteq \mathfrak{p}$, we have $\mathfrak{p} \preceq_{[\mathfrak{p}]} \mathfrak{p}$.

Second, let $\mathfrak{p} \preceq_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{a}$ and $\mathfrak{p} + \mathbf{a} \preceq_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{a} + \mathbf{b}$. Then we have $\mathbb{I}(\Omega(\mathbf{a})) \not\subseteq \mathfrak{p}$ and $\mathbb{I}(\Omega(\mathbf{b})) \not\subseteq \mathfrak{p} + \mathbf{a}$. The latter is equivalent to $\mathbb{I}(\Omega(\mathbf{b}) - \mathbf{a}) \not\subseteq \mathfrak{p}$. Since \mathfrak{p} is prime, we have

$$\mathbb{I}(\Omega(\mathbf{a}))\mathbb{I}(\Omega(\mathbf{b}) - \mathbf{a}) \not\subseteq \mathfrak{p}.$$

From the inclusion $D(R_A)_{\mathbf{b}}D(R_A)_{\mathbf{a}} \subseteq D(R_A)_{\mathbf{a}+\mathbf{b}}$, we obtain

$$\mathbb{I}(\Omega(\mathbf{a}))\mathbb{I}(\Omega(\mathbf{b}) - \mathbf{a}) \subseteq \mathbb{I}(\Omega(\mathbf{a} + \mathbf{b})).$$

Hence we have

$$\mathbb{I}(\Omega(\mathbf{a} + \mathbf{b})) \not\subseteq \mathfrak{p},$$

or equivalently

$$\mathfrak{p} \preceq_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{a} + \mathbf{b}.$$

□

Let $\mathbf{a}, \mathbf{b} \in F^d$. In (Saito, Traves, 2001, Definition 4.1.1) a preorder $\mathbf{a} \preceq \mathbf{b}$ was defined by $E_{\tau}(\mathbf{a}) \subseteq E_{\tau}(\mathbf{b})$ for all faces τ of the cone $\mathbb{R}_{\geq 0}A$ generated by A , where

$$(3.1) \quad E_{\tau}(\mathbf{a}) = \{\boldsymbol{\lambda} \in F(A \cap \tau) / \mathbb{Z}(A \cap \tau) : \mathbf{a} - \boldsymbol{\lambda} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)\}.$$

Here $\mathbb{Z}(A \cap \tau)$ and $F(A \cap \tau)$ are the abelian group and the vector space generated by $A \cap \tau$, respectively. Our preorder is its generalization in the following sense.

Proposition 3.3. $\mathfrak{m}_{\mathbf{a}} \preceq \mathfrak{m}_{\mathbf{b}}$ if and only if $\mathbf{a} \preceq \mathbf{b}$.

Proof.

$$\begin{aligned}
\mathfrak{m}_a \preceq \mathfrak{m}_b &\Leftrightarrow \mathfrak{m}_a \preceq \mathfrak{m}_a + (\mathbf{b} - \mathbf{a}) \\
&\Leftrightarrow \mathbb{I}(\Omega(\mathbf{b} - \mathbf{a})) \not\subseteq \mathfrak{m}_a \\
&\Leftrightarrow \mathbf{a} \notin \text{ZC}(\Omega(\mathbf{b} - \mathbf{a})) \\
&\Leftrightarrow \mathbf{a} \preceq \mathbf{b}.
\end{aligned}$$

Here ZC stands for taking Zariski closure in F^d , and the last equivalence follows from (Saito, Traves, 2001, Lemma 4.1.4). \square

Notation. The preorder $\preceq_{[\mathfrak{p}]}$ induces an equivalence relation in $[\mathfrak{p}]$; we write $\mathfrak{p} \sim_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{c}$ (or simply $\mathfrak{p} \sim \mathfrak{p} + \mathbf{c}$) if $\mathfrak{p} \preceq_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{c}$ and $\mathfrak{p} + \mathbf{c} \preceq_{[\mathfrak{p}]} \mathfrak{p}$.

Remark 3.4. In (Musson, Van den Bergh, 1998, Lemma 3.1.9 (3)), a similar preorder $\mathbf{a} \xRightarrow{\mathbf{c}} \mathbf{b}$ was defined by $\mathbb{I}(\Omega(\mathbf{b} - \mathbf{a}) - (\mathbf{a} - \mathbf{c}))\mathbb{I}(\Omega(\mathbf{a} - \mathbf{c})) \not\subseteq \mathbb{I}(\Omega(\mathbf{b} - \mathbf{c}))\mathfrak{m}_{\mathbf{c}}$. The preorders $\mathbf{a} \preceq_{\mathbf{c}} \mathbf{b}$ and $\mathbf{a} \xRightarrow{\mathbf{c}} \mathbf{b}$ induce the same equivalence relation (Musson, Van den Bergh, 1998, Lemma 3.1.9 (5); Saito, 2007, Theorem 4.10).

Proposition 3.5. *Let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals. Then, for $\mathbf{c} \in \mathbb{Z}^d$, $\mathfrak{q} \preceq_{[\mathfrak{q}]} \mathfrak{q} + \mathbf{c}$ implies $\mathfrak{p} \preceq_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{c}$. Hence $\mathfrak{q} \sim_{[\mathfrak{q}]} \mathfrak{q} + \mathbf{c}$ implies $\mathfrak{p} \sim_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{c}$.*

Proof. This follows from the fact that $\mathbb{I}(\Omega(\mathbf{c})) \not\subseteq \mathfrak{q}$ implies $\mathbb{I}(\Omega(\mathbf{c})) \not\subseteq \mathfrak{p}$. \square

Corollary 3.6. *Let \mathfrak{p} be a prime ideal of $F[s]$. Then $[\mathfrak{p}]$ has only a finite number of equivalence classes.*

Proof. Let \mathfrak{m}_a be a maximal ideal containing \mathfrak{p} . Let $\mathfrak{m}_b \in [\mathfrak{m}_a]$, or equivalently let $\mathbf{b} \in \mathbf{a} + \mathbb{Z}^d$. Then, for a face τ and an element $\boldsymbol{\lambda} \in F(A \cap \tau)$,

$$\mathbf{a} - \boldsymbol{\lambda} \in \mathbb{Z}^d \Leftrightarrow \mathbf{b} - \boldsymbol{\lambda} \in \mathbb{Z}^d.$$

If such $\boldsymbol{\lambda}$ does not exist, then $E_\tau(\mathbf{a}) = E_\tau(\mathbf{b}) = \emptyset$ (see (3.1) for the definition of $E_\tau(\mathbf{a})$). If such $\boldsymbol{\lambda}$ exists, then $E_\tau(\mathbf{a})$ and $E_\tau(\mathbf{b})$ are contained in the finite set $(\boldsymbol{\lambda} + \mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d)/\mathbb{Z}(A \cap \tau)$ (see the proof of (Saito, 2001, Proposition 2.3 (1))). Therefore $[\mathfrak{m}_a]$ has only a finite number of equivalence classes by Proposition 3.3. Hence by Proposition 3.5 we conclude that $[\mathfrak{p}]$ also has only a finite number of equivalence classes. \square

4. CATEGORIES $\tilde{\mathcal{O}}$

We say that a left (resp. right) $D(R_A)$ -module M is \mathbb{Z}^d -graded if M has a decomposition $M = \bigoplus_{\mathbf{b} \in \mathbb{Z}^d} M_{\mathbf{b}}$ satisfying $D(R_A)_{\mathbf{a}} M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}}$ (resp. $M_{\mathbf{b}} D(R_A)_{\mathbf{a}} \subseteq M_{\mathbf{a}+\mathbf{b}}$) for all $\mathbf{a} \in \mathbb{Z}^d$ and all $\mathbf{b} \in \mathbb{Z}^d$.

Notation. Let $\tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$) denote the category of \mathbb{Z}^d -graded finitely generated left (resp. right) $D(R_A)$ -modules $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} M_{\mathbf{a}}$.

Lemma 4.1. *Let $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} M_{\mathbf{a}} \in \tilde{\mathcal{O}}$ (resp. $M \in {}^R\tilde{\mathcal{O}}$). Then each $M_{\mathbf{a}}$ is a finitely generated $F[s]$ -module.*

Proof. Suppose that M is generated by v_1, \dots, v_m with $v_j \in M_{\mathbf{b}_j}$ ($j = 1, \dots, m$). Then the assertion is clear from the fact that each $D(R_A)_{\mathbf{a}-\mathbf{b}_j}$ is finitely generated as a $F[s]$ -module. \square

Notation. For $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} M_{\mathbf{a}} \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$), set

$$\delta(M) := \max_{\mathbf{a}} \dim_{F[s]} M_{\mathbf{a}}.$$

If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of \mathbb{Z}^d -graded $D(R_A)$ -modules, then $0 \rightarrow L_{\mathbf{a}} \rightarrow M_{\mathbf{a}} \rightarrow N_{\mathbf{a}} \rightarrow 0$ is exact for each \mathbf{a} . Hence we have the following.

Lemma 4.2. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence in $\tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$). Then*

$$\delta(M) = \max\{\delta(L), \delta(N)\}.$$

Lemma 4.3. *Let $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} M_{\mathbf{a}} \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$). Suppose that M is generated by $M_{\mathbf{b}_1}, \dots, M_{\mathbf{b}_m}$. Then*

$$\delta(M) = \max_{1 \leq i \leq m} \dim_{F[s]} M_{\mathbf{b}_i}.$$

Proof. Let $M \in \tilde{\mathcal{O}}$. It suffices to prove that $\dim M_{\mathbf{b}} \geq \dim D(R_A)_{\mathbf{a}} M_{\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$. It follows from $D(R_A)_{\mathbf{a}} = t^{\mathbf{a}} \mathbb{I}(\Omega(\mathbf{a})) \subseteq t^{\mathbf{a}} F[s]$ that

$$(4.1) \quad (\text{Ann}_{F[s]}(M_{\mathbf{b}}) + \mathbf{a}) \subseteq \text{Ann}_{F[s]}(D(R_A)_{\mathbf{a}} M_{\mathbf{b}}).$$

$$\text{(resp. } (\text{Ann}_{F[s]}(M_{\mathbf{b}}) - \mathbf{a}) \subseteq \text{Ann}_{F[s]}(M_{\mathbf{b}} D(R_A)_{\mathbf{a}}) \text{ for a right module.)}$$

(Here $\text{Ann}_{F[s]}(M_{\mathbf{b}})$ denotes the annihilator of the $F[s]$ -module $M_{\mathbf{b}}$.) Hence $\mathfrak{p} + \mathbf{a} \in \text{Supp}(D(R_A)_{\mathbf{a}} M_{\mathbf{b}})$ (resp. $\mathfrak{p} - \mathbf{a} \in \text{Supp}(M_{\mathbf{b}} D(R_A)_{\mathbf{a}})$ for a right module) implies $\mathfrak{p} \in \text{Supp}(M_{\mathbf{b}})$. We have thus proved the assertion. \square

Notation. Let \mathfrak{p} be a prime ideal of $F[s]$. Let $\tilde{\mathcal{O}}_{\mathfrak{p}}$ (resp. ${}^R\tilde{\mathcal{O}}_{\mathfrak{p}}$) denote the subcategory of $\tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$) consisting of $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} M_{\mathbf{a}}$ such that, for all $\mathbf{a} \in \mathbb{Z}^d$, all minimal primes of the finitely generated $F[s]$ -module $M_{\mathbf{a}}$ contain $\mathfrak{p} + \mathbf{a}$ (resp. $\mathfrak{p} - \mathbf{a}$).

Remark 4.4. Let $\mathbf{a} \in \mathbb{Z}^d$. Then by definition $M \in \tilde{\mathcal{O}}_{\mathfrak{p}}$ if and only if $M[\mathbf{a}] \in \tilde{\mathcal{O}}_{\mathfrak{p} + \mathbf{a}}$ (resp. $M \in {}^R\tilde{\mathcal{O}}_{\mathfrak{p}}$ if and only if $M[\mathbf{a}] \in {}^R\tilde{\mathcal{O}}_{\mathfrak{p} - \mathbf{a}}$), where $M[\mathbf{a}]$ is defined by $M[\mathbf{a}]_{\mathbf{b}} = M_{\mathbf{a} + \mathbf{b}}$ for all $\mathbf{b} \in \mathbb{Z}^d$.

Remark 4.5. Let $\beta \in F^d$. Then the category $\tilde{\mathcal{O}}_{\mathfrak{m}_{\beta}}$ consists of a finitely generated \mathbb{Z}^d -graded $D(R_A)$ -module $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} M_{\mathbf{a}}$ such that, for all \mathbf{a} , $(\mathfrak{m}_{\beta + \mathbf{a}})^n M_{\mathbf{a}} = 0$ ($n \gg 0$). An object of $\tilde{\mathcal{O}}_{\mathfrak{m}_{\beta}}$ is an object of the category $\mathcal{O}_{\beta + \mathbb{Z}^d}^{(\infty)}$ in (Musson, Van den Bergh, 1998, p. 12).

Notation. Let $M \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$), and $\delta \in \mathbb{N}$. Set

$$\text{Supp}_{\delta}(M) := \{\mathfrak{p} \in \text{Spec } F[s] : \dim F[s]/\mathfrak{p} = \delta, \mathfrak{p} + \mathbf{a} \in \text{Supp}(M_{\mathbf{a}}) \text{ for some } \mathbf{a}\}.$$

$$\text{(resp. } {}^R\text{Supp}_{\delta}(M) := \{\mathfrak{p} \in \text{Spec } F[s] : \dim F[s]/\mathfrak{p} = \delta, \mathfrak{p} - \mathbf{a} \in \text{Supp}(M_{\mathbf{a}}) \text{ for some } \mathbf{a}\}.)$$

Proposition 4.6. *Let $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} M_{\mathbf{a}} \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$). Then the set $\text{Supp}_{\delta(M)}(M)$ (resp. ${}^R\text{Supp}_{\delta(M)}(M)$) is finite.*

Proof. Let $M \in \tilde{\mathcal{O}}$. Suppose that M is generated by $M_{\mathbf{b}_1}, \dots, M_{\mathbf{b}_j}$. Let $\{\mathfrak{p}_{kl} : l\}$ be the set of minimal primes of the $F[s]$ -module $M_{\mathbf{b}_k}$. Suppose that $\mathfrak{q} \in \text{Supp}_{F[s]}(M_{\mathbf{a}})$. Then there exists k such that $\mathfrak{q} \in \text{Supp}_{F[s]}(D(R_A)_{\mathbf{a}-\mathbf{b}_k}M_{\mathbf{b}_k})$. By the inclusion (4.1), we have $\mathfrak{q} + \mathbf{b}_k - \mathbf{a} \in \text{Supp}(M_{\mathbf{b}_k})$. Hence there exists l such that $\mathfrak{q} + \mathbf{b}_k - \mathbf{a} \supseteq \mathfrak{p}_{kl}$, or $\mathfrak{q} \supseteq (\mathfrak{p}_{kl} - \mathbf{b}_k) + \mathbf{a}$. Hence we have

$$\text{Supp}_{\delta(M)}(M) = \{\mathfrak{p}_{kl} - \mathbf{b}_k : \dim F[s]/\mathfrak{p}_{kl} = \delta(M)\}.$$

(For $M \in {}^R\tilde{\mathcal{O}}$, ${}^R\text{Supp}_{\delta(M)}(M) = \{\mathfrak{p}_{kl} + \mathbf{b}_k : \dim F[s]/\mathfrak{p}_{kl} = \delta(M)\}$, respectively.) □

5. DEGREE OF MODULES

Let $\tau : R \rightarrow R'$ be an isomorphism of commutative rings. Let M be an R -module, N an R' -module. A \mathbb{Z} -module homomorphism $f : M \rightarrow N$ is called a τ -homomorphism if $f(am) = \tau(a)f(m)$ for $a \in R$ and $m \in M$.

Let $\tau : R \rightarrow R$ be an automorphism, and $f : M \rightarrow N$ a τ -homomorphism. Let \mathfrak{p} be a prime ideal of R . Then $\tau(\mathfrak{p})$ is also a prime ideal of R , and

$$\tau_{\mathfrak{p}} : R_{\mathfrak{p}} \ni \frac{a}{s} \mapsto \frac{\tau(a)}{\tau(s)} \in R_{\tau(\mathfrak{p})}$$

is an isomorphism of rings. Furthermore f induces $\tau_{\mathfrak{p}}$ -homomorphism

$$(5.1) \quad f_{\mathfrak{p}} : M_{\mathfrak{p}} \ni \frac{m}{s} \mapsto \frac{f(m)}{\tau(s)} \in N_{\tau(\mathfrak{p})}.$$

Clearly, if L is an R -submodule of M , then $f_{\mathfrak{p}}(L_{\mathfrak{p}})$ is an $R_{\tau(\mathfrak{p})}$ -submodule of $N_{\tau(\mathfrak{p})}$.

Example 5.1. Let $\mathbf{a} \in \mathbb{Z}^d$, and let $\tau_{\mathbf{a}}$ be an automorphism of $F[s]$ defined by $f(s) \mapsto f(s + \mathbf{a})$. Let $Q = t^{\mathbf{a}}q(s) \in D(R_A)_{\mathbf{a}}$, and let M be a \mathbb{Z}^d -graded left $D(R_A)$ -module. Then $Q \times : M_{\mathbf{b}} \ni m \mapsto Qm \in M_{\mathbf{a}+\mathbf{b}}$ is a $\tau_{-\mathbf{a}}$ -homomorphism; for $g(s) \in F[s]$ and $m \in M_{\mathbf{b}}$

$$Q(g(s)m) = t^{\mathbf{a}}q(s)g(s)m = g(s - \mathbf{a})t^{\mathbf{a}}q(s)m = g(s - \mathbf{a})Q(m).$$

For a \mathbb{Z}^d -graded right $D(R_A)$ -module M , $\times Q : M_{\mathbf{b}} \ni m \mapsto mQ \in M_{\mathbf{a}+\mathbf{b}}$ is a $\tau_{\mathbf{a}}$ -homomorphism.

Let R be a commutative Noetherian ring, and N a finitely generated R -module. If \mathfrak{p} is a minimal prime of N , then $N_{\mathfrak{p}}$ is an Artinian $R_{\mathfrak{p}}$ -module, and $N_{\mathfrak{p}}$ has a composition series (see e.g. Eisenbud, 1995, Corollary 2.18); we denote its length by $\text{length}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$.

Lemma 5.2. Let M be a \mathbb{Z}^d -graded left (resp. right) $D(R_A)$ -module. Let \mathfrak{p} be a prime ideal of $F[s]$ with $\dim F[s]/\mathfrak{p} = \delta(M)$. Suppose that $\mathfrak{p} + \mathbf{a} \sim \mathfrak{p} + \mathbf{b}$ (resp. $\mathfrak{p} - \mathbf{a} \sim \mathfrak{p} - \mathbf{b}$). Then

$$\begin{aligned} \text{length}_{F[s]_{\mathfrak{p}+\mathbf{a}}}(M_{\mathbf{a}})_{\mathfrak{p}+\mathbf{a}} &= \text{length}_{F[s]_{\mathfrak{p}+\mathbf{b}}}(M_{\mathbf{b}})_{\mathfrak{p}+\mathbf{b}}, \\ \left(\text{resp. } \text{length}_{F[s]_{\mathfrak{p}-\mathbf{a}}}(M_{\mathbf{a}})_{\mathfrak{p}-\mathbf{a}} &= \text{length}_{F[s]_{\mathfrak{p}-\mathbf{b}}}(M_{\mathbf{b}})_{\mathfrak{p}-\mathbf{b}} \right). \end{aligned}$$

Proof. From now on, we prove assertions only in the left module case. The proofs in the right module case are similar.

Since $\mathfrak{p} + \mathbf{a} \sim \mathfrak{p} + \mathbf{b}$, we have

$$\mathbb{I}(\Omega(\mathbf{b} - \mathbf{a})) \not\subseteq \mathfrak{p} + \mathbf{a}, \quad \mathbb{I}(\Omega(\mathbf{a} - \mathbf{b})) \not\subseteq \mathfrak{p} + \mathbf{b}.$$

Take

$$q_{\mathbf{a}}(s) \in \mathbb{I}(\Omega(\mathbf{b} - \mathbf{a})) \setminus (\mathfrak{p} + \mathbf{a}), \quad q_{\mathbf{b}}(s) \in \mathbb{I}(\Omega(\mathbf{a} - \mathbf{b})) \setminus (\mathfrak{p} + \mathbf{b}),$$

and put

$$Q_{\mathbf{a}} := t^{\mathbf{b}-\mathbf{a}}q_{\mathbf{a}}(s), \quad Q_{\mathbf{b}} := t^{\mathbf{a}-\mathbf{b}}q_{\mathbf{b}}(s).$$

Then

$$Q_{\mathbf{a}} : M_{\mathbf{a}} \rightarrow M_{\mathbf{b}}, \quad Q_{\mathbf{b}} : M_{\mathbf{b}} \rightarrow M_{\mathbf{a}}$$

are a $\tau_{\mathbf{a}-\mathbf{b}}$ -homomorphism and a $\tau_{\mathbf{b}-\mathbf{a}}$ -homomorphism, respectively. They induce

$$(5.2) \quad (Q_{\mathbf{a}})_{\mathfrak{p}+\mathbf{a}} : (M_{\mathbf{a}})_{\mathfrak{p}+\mathbf{a}} \rightarrow (M_{\mathbf{b}})_{\mathfrak{p}+\mathbf{b}}, \quad (Q_{\mathbf{b}})_{\mathfrak{p}+\mathbf{b}} : (M_{\mathbf{b}})_{\mathfrak{p}+\mathbf{b}} \rightarrow (M_{\mathbf{a}})_{\mathfrak{p}+\mathbf{a}}.$$

Furthermore we have

$$Q_{\mathbf{b}}Q_{\mathbf{a}} = t^{\mathbf{a}-\mathbf{b}}q_{\mathbf{b}}(s)t^{\mathbf{b}-\mathbf{a}}q_{\mathbf{a}}(s) = q_{\mathbf{b}}(s + \mathbf{b} - \mathbf{a})q_{\mathbf{a}}(s).$$

Since $q_{\mathbf{b}}(s) \notin \mathfrak{p} + \mathbf{b}$ means $q_{\mathbf{b}}(s + \mathbf{b} - \mathbf{a}) \notin \mathfrak{p} + \mathbf{a}$, we have

$$Q_{\mathbf{b}}Q_{\mathbf{a}} = q_{\mathbf{b}}(s + \mathbf{b} - \mathbf{a})q_{\mathbf{a}}(s) \notin \mathfrak{p} + \mathbf{a}.$$

Similarly

$$Q_{\mathbf{a}}Q_{\mathbf{b}} = q_{\mathbf{a}}(s + \mathbf{a} - \mathbf{b})q_{\mathbf{b}}(s) \notin \mathfrak{p} + \mathbf{b}.$$

Hence from (5.2) we obtain

$$\text{length}_{F[s]_{\mathfrak{p}+\mathbf{a}}}(M_{\mathbf{a}})_{\mathfrak{p}+\mathbf{a}} = \text{length}_{F[s]_{\mathfrak{p}+\mathbf{b}}}(M_{\mathbf{b}})_{\mathfrak{p}+\mathbf{b}}.$$

□

Let $M \in \tilde{\mathcal{O}}$. We know that the set $\text{Supp}_{\delta(M)}(M)$ is finite by Proposition 4.6. Let $\mathfrak{p} \in \text{Supp}_{\delta(M)}(M)$. Then by Lemma 5.2 if $\mathfrak{p} + \mathbf{a} \sim_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{b}$, then $\text{length}_{F[s]_{\mathfrak{p}+\mathbf{a}}}M_{\mathbf{a}} = \text{length}_{F[s]_{\mathfrak{p}+\mathbf{b}}}M_{\mathbf{b}}$.

Notation. Let $M \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$). Set

$$\begin{aligned} \deg M &:= \sum_{\mathfrak{p} \in \text{Supp}_{\delta(M)}(M)} \sum_{\mathfrak{p}+\mathbf{a} \in [\mathfrak{p}]/\sim_{[\mathfrak{p}]}} \text{length}_{F[s]_{\mathfrak{p}+\mathbf{a}}}M_{\mathbf{a}}. \\ (\text{resp. } \deg M &:= \sum_{\mathfrak{p} \in {}^R\text{Supp}_{\delta(M)}(M)} \sum_{\mathfrak{p}-\mathbf{a} \in [\mathfrak{p}]/\sim_{[\mathfrak{p}]}} \text{length}_{F[s]_{\mathfrak{p}-\mathbf{a}}}M_{\mathbf{a}}.) \end{aligned}$$

The following lemma is clear from the definitions.

Lemma 5.3. *Let $M \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$).*

- (1) *If $M \neq 0$, then $\deg M \neq 0$.*
- (2) *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence in $\tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$). Then*
 - (a) *If $\delta(M) = \delta(N)$, then $\deg M \geq \deg N$.*
 - (b) *If $\delta(M) = \delta(N)$ and $\deg M = \deg N$, then $\delta(L) < \delta(M)$.*
 - (c) *If $\delta(L) = \delta(M)$, then $\deg L \leq \deg M$.*
 - (d) *If $\delta(L) = \delta(M)$ and $\deg L = \deg M$, then $\delta(N) < \delta(M)$.*
- (3) *Let*

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq \cdots$$

be a descending chain in $\tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$). Then there exist at most $\deg M$ i 's such that $\delta(M_i/M_{i+1}) = \delta(M)$.

6. KRULL DIMENSION

In this section we consider the Krull dimension for the category $\tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$) in the sense of (Gabriel, 1962) and (Rentschler, Gabriel, 1967), which we simply call it Krull dimension.

Notation. Let $M \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$). Then we denote by $\text{Kdim } M$ the Krull dimension for the lattice of \mathbb{Z}^d -graded $D(R_A)$ -submodules.

Theorem 6.1. *Let $M \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$). Then*

$$\text{Kdim } M = \delta(M).$$

Proof. Let $M \in \tilde{\mathcal{O}}$.

First $\delta(M) < 0$ if and only if $M = 0$ if and only if $\text{Kdim } M < 0$.

Next let $M \neq 0$, and let $\delta := \delta(M)$. We prove $\text{Kdim } M = \delta$ by induction on δ .

By Lemma 4.2 and (McConnel-Robson, 1987, Lemma 6.2.4) we may assume that M is cyclic. Furthermore, by shifting its grading if necessary, we may assume that $M = D(R_A)M_0$. We know $\delta = \dim_{F[s]} M_0$ by Lemma 4.3.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_l$ be the minimal primes of M_0 with $\dim F[s]/\mathfrak{p}_j = \delta$ for $j \leq k$, and $< \delta$ for $j > k$. Then $\text{Supp}_\delta(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. By Lemma 4.2 and (McConnel-Robson, 1987, Lemma 6.2.4) again, we may assume that $k = 1$ and $\deg M = 1$.

Let

$$M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_m \supseteq \dots$$

be a descending chain of \mathbb{Z}^d -graded $D(R_A)$ -modules. Suppose that $\delta(M_i/M_{i+1}) = \delta$. Then clearly $\delta(M/M_{i+1}) = \delta$. Since $\deg M/M_{i+1} = 1 = \deg M$, by Lemma 5.3 $\delta(M_{i+1}) < \delta$. Hence $\delta(M_j/M_{j+1}) < \delta$ for all $j > i$. By the induction hypothesis, $\text{Kdim}(M_j/M_{j+1}) < \delta$ for all $j > i$. Therefore we have proved that $\text{Kdim } M \leq \delta$.

Finally we prove the inequality $\text{Kdim } M \geq \delta$. Let $N_1 \subsetneq N_2$ be proper $F[s]$ -submodules of M_0 . Then $D(R_A)N_1 \subsetneq D(R_A)N_2$ are proper \mathbb{Z}^d -graded $D(R_A)$ -submodules of M , since $(D(R_A)N_i)_0 = N_i$ ($i = 1, 2$). Hence

$$\text{Kdim } M \geq \dim_{F[s]} M_0 = \delta.$$

□

The following is clear from Theorem 6.1, and also from the non-graded version (Musson, Van den Bergh, 1998, Theorem 8.1.2).

Corollary 6.2. *As $D(R_A) \in \tilde{\mathcal{O}}$ (resp. ${}^R\tilde{\mathcal{O}}$),*

$$\text{Kdim } D(R_A) = d.$$

7. CRITICAL MODULES

Let R be a left (resp. right) Noetherian ring. An R -module M of Krull dimension δ is said to be δ -critical if $\text{Kdim}(M/N) < \delta$ for all nonzero R -submodules N of M . The 0-critical modules are precisely the simple modules. Hence critical modules play a fundamental role in the theory of Noetherian modules. (See for example (Goodearl, Warfield, 1989; Lenagan, 2000; McConnel, Robson 1987).) In this section, we characterize critical \mathbb{Z}^d -graded $D(R_A)$ -modules (Theorem 7.4), and explicitly describe cyclic ones (Theorem 7.7).

The following lemma is clear from the definitions.

Lemma 7.1. *Let R be a commutative Noetherian ring, and M a finitely generated R -module. Then M is critical if and only if there exists a prime ideal \mathfrak{p} of R such that $\text{Ass}(M) = \{\mathfrak{p}\}$, and $\text{length}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 1$, where $\text{Ass}(M)$ denotes the set of associated primes of M . If this is the case, then $\text{Ann}_R M = \mathfrak{p}$.*

Lemma 7.2. *Suppose that M is a δ -critical \mathbb{Z}^d -graded left (resp. right) $D(R_A)$ -module. Then for each $\mathbf{a} \in \mathbb{Z}^d$ the $F[s]$ -module $M_{\mathbf{a}}$ is δ -critical if it is not zero.*

Proof. First suppose that $\dim M_{\mathbf{a}} = \delta$. Let N be a nonzero $F[s]$ -submodule of $M_{\mathbf{a}}$ with $\dim M_{\mathbf{a}}/N = \delta'$. Then by Theorem 6.1

$$\text{Kdim}(M/D(R_A)N) \geq \dim_{F[s]}(M/D(R_A)N)_{\mathbf{a}} = \dim_{F[s]}(M_{\mathbf{a}}/N) = \delta'.$$

Since M is δ -critical, $\delta' < \delta$. Hence $M_{\mathbf{a}}$ is δ -critical.

Next suppose that $0 \neq \dim_{F[s]} M_{\mathbf{a}} < \delta$. Then there exists \mathbf{b} such that $\text{Kdim} M = \dim_{F[s]} M_{\mathbf{b}} = \delta$. Consider the submodule $D(R_A)M_{\mathbf{a}}$ of M . Then

$$\text{Kdim} M/D(R_A)M_{\mathbf{a}} \geq \dim_{F[s]}(M/D(R_A)M_{\mathbf{a}})_{\mathbf{b}} = \dim_{F[s]}(M_{\mathbf{b}}/D(R_A)_{\mathbf{b}-\mathbf{a}}M_{\mathbf{a}}).$$

We know by the proof of Lemma 4.3

$$\dim_{F[s]} D(R_A)_{\mathbf{b}-\mathbf{a}}M_{\mathbf{a}} \leq \dim_{F[s]} M_{\mathbf{a}} < \delta.$$

Hence

$$\text{Kdim} M/D(R_A)M_{\mathbf{a}} \geq \dim_{F[s]} M_{\mathbf{b}} = \delta,$$

which contradicts the fact that M is critical. Hence there exists no nonzero $M_{\mathbf{a}}$ with $\dim_{F[s]} M_{\mathbf{a}} < \delta$. \square

Let M be a critical \mathbb{Z}^d -graded left (resp. right) $D(R_A)$ -module with $M_{\mathbf{a}} \neq 0$. Then by Lemmas 7.1 and 7.2 there exists a prime ideal \mathfrak{p} of $F[s]$ such that $\text{Ass}(M_{\mathbf{a}}) = \{\mathfrak{p}\}$. By considering the shift $M[\mathbf{a}]$, we may assume that $M_{\mathbf{0}} \neq 0$.

Proposition 7.3. *Let \mathfrak{p} be a prime ideal of $F[s]$ with $\dim F[s]/\mathfrak{p} = \delta$. Suppose that M is a δ -critical \mathbb{Z}^d -graded left (resp. right) $D(R_A)$ -module with $\text{Ass}(M_{\mathbf{0}}) = \{\mathfrak{p}\}$. Then*

- (1) $M \in \tilde{\mathcal{O}}_{\mathfrak{p}}$ (resp. $M \in {}^R\tilde{\mathcal{O}}_{\mathfrak{p}}$).
- (2) $M_{\mathbf{a}} = 0$ if $\mathfrak{p} \not\sim \mathfrak{p} + \mathbf{a}$ (resp. $\mathfrak{p} \not\sim \mathfrak{p} - \mathbf{a}$).
- (3) $\text{Ass}(M_{\mathbf{a}}) = \{\mathfrak{p} + \mathbf{a}\}$ if $\mathfrak{p} \sim \mathfrak{p} + \mathbf{a}$. (resp. $\text{Ass}(M_{\mathbf{a}}) = \{\mathfrak{p} - \mathbf{a}\}$ if $\mathfrak{p} \sim \mathfrak{p} - \mathbf{a}$.)

Proof. Suppose that $D(R_A)_{-\mathbf{a}}M_{\mathbf{a}} = 0$. Then

$$\dim_{F[s]}(M/D(R_A)M_{\mathbf{a}})_{\mathbf{0}} = \dim_{F[s]}(M_{\mathbf{0}}/D(R_A)_{-\mathbf{a}}M_{\mathbf{a}}) = \dim_{F[s]}(M_{\mathbf{0}}) = \delta,$$

which means $\text{Kdim}(M/D(R_A)M_{\mathbf{a}}) = \delta$. Since M is δ -critical, $D(R_A)M_{\mathbf{a}} = 0$, and hence $M_{\mathbf{a}} = 0$.

Let $M_{\mathbf{a}} \neq 0$, and $\mathfrak{q} \in \text{Ass}(M_{\mathbf{a}})$. Then by Lemmas 7.1 and 7.2 we have $\text{Ass}(M_{\mathbf{a}}) = \{\mathfrak{q}\}$ and $\mathfrak{q} = \text{Ann}_{F[s]} M_{\mathbf{a}}$. By (4.1)

$$\text{Ann}_{F[s]}(D(R_A)_{-\mathbf{a}}M_{\mathbf{a}}) \supseteq \text{Ann}_{F[s]} M_{\mathbf{a}} - \mathbf{a} = \mathfrak{q} - \mathbf{a}.$$

By the first paragraph of the proof, $D(R_A)_{-\mathbf{a}}M_{\mathbf{a}} \neq 0$. Since $\text{Ann}_{F[s]}(x) = \mathfrak{p}$ for each nonzero element $x \in D(R_A)_{-\mathbf{a}}M_{\mathbf{a}} \subseteq M_{\mathbf{0}}$, we have

$$\text{Ann}_{F[s]}(D(R_A)_{-\mathbf{a}}M_{\mathbf{a}}) = \mathfrak{p}.$$

Hence $\mathfrak{q} \subseteq \mathfrak{p} + \mathbf{a}$. Then $\mathfrak{q} = \mathfrak{p} + \mathbf{a}$, since $\dim F[s]/\mathfrak{p} = \text{Kdim} M$, and thus $\text{Ass}(M_{\mathbf{a}}) = \{\mathfrak{p} + \mathbf{a}\}$.

Suppose that $\mathfrak{p} \sim \mathfrak{p} + \mathbf{a}$. Then by Lemma 5.2 $M_{\mathbf{a}} \neq 0$. Hence $\text{Ass}(M_{\mathbf{a}}) = \{\mathfrak{p} + \mathbf{a}\}$ by the previous paragraph.

Finally suppose that $\mathfrak{p} \not\sim \mathfrak{p} + \mathbf{a}$. Then $\mathbb{I}(\Omega(\mathbf{a})) \subseteq \mathfrak{p}$ or $\mathbb{I}(\Omega(-\mathbf{a})) \subseteq \mathfrak{p} + \mathbf{a}$. Hence $\mathbb{I}(\Omega(-\mathbf{a}) - \mathbf{a})\mathbb{I}(\Omega(\mathbf{a})) \subseteq \mathfrak{p}$. Consider the submodule $D(R_A)(D(R_A)_{\mathbf{a}}M_0)$ of M . We have

$$\begin{aligned} (D(R_A)(D(R_A)_{\mathbf{a}}M_0))_0 &= D(R_A)_{-\mathbf{a}}D(R_A)_{\mathbf{a}}M_0 \\ &= t^{-\mathbf{a}}\mathbb{I}(\Omega(-\mathbf{a}))t^{\mathbf{a}}\mathbb{I}(\Omega(\mathbf{a}))M_0 \\ &= \mathbb{I}(\Omega(-\mathbf{a}) - \mathbf{a})\mathbb{I}(\Omega(\mathbf{a}))M_0 \\ &\subseteq \mathfrak{p}M_0 = 0. \end{aligned}$$

Since $\text{Kdim}(M/D(R_A)(D(R_A)_{\mathbf{a}}M_0)) \geq \dim(M/D(R_A)(D(R_A)_{\mathbf{a}}M_0))_0 = \dim M_0 = \delta = \text{Kdim} M$, and since M is critical, $D(R_A)(D(R_A)_{\mathbf{a}}M_0) = 0$. Hence $D(R_A)_{\mathbf{a}}M_0 = 0$. If $M_{\mathbf{a}} \neq 0$, then $\text{Ass}(M_{\mathbf{a}}) = \{\mathfrak{p} + \mathbf{a}\}$, and

$$\text{Kdim} M/D(R_A)M_0 \geq \dim(M/D(R_A)M_0)_{\mathbf{a}} = \dim M_{\mathbf{a}}/D(R_A)_{\mathbf{a}}M_0 = \dim M_{\mathbf{a}} = \delta.$$

This contradicts the fact that M is critical. Therefore $M_{\mathbf{a}} = 0$ in this case. \square

Theorem 7.4. *Let M be a finitely generated left (resp. right) \mathbb{Z}^d -graded $D(R_A)$ -module with $M_0 \neq 0$. Then M is critical if and only if there exists a prime ideal \mathfrak{p} of $F[s]$ such that*

- (1) $M \in \tilde{\mathcal{O}}_{\mathfrak{p}}$ (resp. $M \in {}^R\tilde{\mathcal{O}}_{\mathfrak{p}}$).
- (2) $M_{\mathbf{a}} = 0$ if $\mathfrak{p} \not\sim \mathfrak{p} + \mathbf{a}$ (resp. $\mathfrak{p} \not\sim \mathfrak{p} - \mathbf{a}$).
- (3) $M_{\mathbf{a}}$ is a critical $F[s]$ -module with $\text{Ass}(M_{\mathbf{a}}) = \{\mathfrak{p} + \mathbf{a}\}$ if $\mathfrak{p} \sim \mathfrak{p} + \mathbf{a}$ (resp. $\text{Ass}(M_{\mathbf{a}}) = \{\mathfrak{p} - \mathbf{a}\}$ if $\mathfrak{p} \sim \mathfrak{p} - \mathbf{a}$).

Proof. Proposition 7.3 is the only-if part.

Assume the conditions (1) through (3). Suppose that $\mathfrak{p} \sim \mathfrak{p} + \mathbf{a}$. Let N be a nonzero $F[s]$ -submodule of $M_{\mathbf{a}}$. Consider the quotient $M/D(R_A)N$.

First let $\mathfrak{p} \not\sim \mathfrak{p} + \mathbf{b}$. Then $(M/D(R_A)N)_{\mathbf{b}} = 0$, since $M_{\mathbf{b}} = 0$.

Next let $\mathfrak{p} \sim \mathfrak{p} + \mathbf{b}$. Then $\mathfrak{p} + \mathbf{a} \sim \mathfrak{p} + \mathbf{b}$. By Lemma 5.2

$$\begin{aligned} \text{length}_{F[s]_{\mathfrak{p}+\mathbf{b}}}((D(R_A)N)_{\mathbf{b}})_{\mathfrak{p}+\mathbf{b}} &= \text{length}_{F[s]_{\mathfrak{p}+\mathbf{a}}}((D(R_A)N)_{\mathbf{a}})_{\mathfrak{p}+\mathbf{a}} = \text{length}_{F[s]_{\mathfrak{p}+\mathbf{a}}}N_{\mathfrak{p}+\mathbf{a}} \\ &= 1 \\ &= \text{length}_{F[s]_{\mathfrak{p}+\mathbf{a}}}(M_{\mathbf{a}})_{\mathfrak{p}+\mathbf{a}} = \text{length}_{F[s]_{\mathfrak{p}+\mathbf{b}}}(M_{\mathbf{b}})_{\mathfrak{p}+\mathbf{b}}. \end{aligned}$$

Hence

$$\dim_{F[s]}(M/D(R_A)N)_{\mathbf{b}} < \dim_{F[s]} M_{\mathbf{b}} = \text{Kdim} M.$$

Therefore $\text{Kdim} M/D(R_A)N < \text{Kdim} M$. \square

Example 7.5. Let \mathfrak{p} be a prime ideal of $F[s]$, and let $\delta = \dim F[s]/\mathfrak{p}$. Let ${}^L L(\mathfrak{p}) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} {}^L L(\mathfrak{p})_{\mathbf{a}}$ (resp. ${}^R L(\mathfrak{p}) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} {}^R L(\mathfrak{p})_{\mathbf{a}}$) be the cyclic \mathbb{Z}^d -graded left (resp. right) $D(R_A)$ -module ${}^L L(\mathfrak{p}) = D(R_A)/{}^L I(\mathfrak{p})$ (resp. ${}^R L(\mathfrak{p}) = D(R_A)/{}^R I(\mathfrak{p})$) with

$${}^L I(\mathfrak{p})_{\mathbf{a}} = \begin{cases} t^{\mathbf{a}}(\mathbb{I}(\Omega(\mathbf{a})) \cap \mathfrak{p}) & (\mathfrak{p} \sim_{[\mathfrak{p}]} \mathfrak{p} + \mathbf{a}) \\ D(R_A)_{\mathbf{a}} & (\text{otherwise}). \end{cases}$$

$$\left(\text{resp. } {}^R I(\mathfrak{p})_{\mathbf{a}} = \begin{cases} (\mathbb{I}(\Omega(\mathbf{a}) + \mathbf{a}) \cap \mathfrak{p})t^{\mathbf{a}} & (\mathfrak{p} \sim_{[\mathfrak{p}]} \mathfrak{p} - \mathbf{a}) \\ D(R_A)_{\mathbf{a}} & (\text{otherwise}). \end{cases} \right)$$

Then ${}^L L(\mathfrak{p})$ (resp. ${}^R L(\mathfrak{p})$) is δ -critical by Theorem 7.4.

If $\mathfrak{p} = \mathfrak{m}_\beta$, then

$${}^L L(\mathfrak{m}_\beta)_\mathfrak{a} = \begin{cases} t^\mathfrak{a} \mathbb{I}(\Omega(\mathfrak{a})) / t^\mathfrak{a} (\mathbb{I}(\Omega(\mathfrak{a})) \cap \mathfrak{m}_\beta) & (\beta \sim \beta + \mathfrak{a}) \\ 0 & (\text{otherwise}) \end{cases} = {}^L L(\beta)_{\beta + \mathfrak{a}}.$$

$$\left(\text{resp. } {}^R L(\mathfrak{m}_\beta)_\mathfrak{a} = \begin{cases} \mathbb{I}(\Omega(\mathfrak{a}) + \mathfrak{a}) t^\mathfrak{a} / (\mathbb{I}(\Omega(\mathfrak{a}) + \mathfrak{a}) \cap \mathfrak{m}_\beta) t^\mathfrak{a} & (\beta \sim \beta - \mathfrak{a}) \\ 0 & (\text{otherwise}) \end{cases} = {}^R L(\beta)_{-\beta + \mathfrak{a}} \right)$$

Thus ${}^L L(\mathfrak{m}_\beta) = {}^L L(\beta)[\beta]$ (resp. ${}^R L(\mathfrak{m}_\beta) = {}^R L(\beta)[- \beta]$). (See (Saito, 2007) for ${}^L L(\beta)$ and ${}^R L(\beta)$.)

Example 7.6. Let $n = d = 2$, and let $A = \{ {}^t(1, 0), {}^t(0, 1) \}$. Then $D = D(R_A)$ is the second Weyl algebra $F\langle t_1, t_2, \partial_1, \partial_2 \rangle$. Let \mathfrak{p} be the prime ideal $\langle s_1 \rangle$ of $F[s]$. We know

$$\mathbb{I}(\Omega(\mathfrak{c})) = \left\langle \prod_{c_i < 0} \prod_{m=0}^{-c_i-1} (s_i - m) \right\rangle.$$

Hence

$$\mathbb{I}(\Omega(\mathfrak{c})) \subseteq \mathfrak{p} \Leftrightarrow c_1 < 0,$$

or equivalently

$$\mathfrak{p} \preceq \mathfrak{p} + \mathfrak{c} \Leftrightarrow c_1 \geq 0.$$

Similarly we see that we always have $\mathfrak{p} + \mathfrak{c} \preceq \mathfrak{p}$. Therefore

$$\mathfrak{p} \sim \mathfrak{p} + \mathfrak{c} \Leftrightarrow c_1 \geq 0.$$

Set

$${}^L L(\mathfrak{p})_\mathfrak{c} = \begin{cases} t^\mathfrak{c} \mathbb{I}(\Omega(\mathfrak{c})) / t^\mathfrak{c} (\mathbb{I}(\Omega(\mathfrak{c})) \cap \mathfrak{p}) & (c_1 \geq 0) \\ 0 & (\text{otherwise}). \end{cases}$$

Then ${}^L L(\mathfrak{p}) = D / (Ds_1 + \sum_{c_1 < 0} Dc) = D / D\partial_1$ is 1-critical.

Theorem 7.7. *Let M be a δ -critical \mathbb{Z}^d -graded left (resp. right) $D(R_A)$ -module generated by one element $v \in M_0$ with $\text{Ann}_{F[s]}(v) = \mathfrak{p}$. Then M is isomorphic to ${}^L L(\mathfrak{p})$ (resp. ${}^R L(\mathfrak{p})$).*

Proof. Let $K = \text{Ann}_{D(R_A)}(v)$. Hence $M \simeq D(R_A)/K$.

First we prove that $K \subseteq {}^L I(\mathfrak{p})$. Let $\mathfrak{p} + \mathfrak{a} \not\sim \mathfrak{p}$. Then $K_\mathfrak{a} = {}^L I(\mathfrak{p})_\mathfrak{a} = D(R_A)_\mathfrak{a}$. Let $\mathfrak{p} + \mathfrak{a} \sim \mathfrak{p}$. Let $P = t^\mathfrak{a} p(s) \in D(R_A)_\mathfrak{a}$ annihilate v . Suppose that $p \notin \mathfrak{p}$. Then, since $\mathfrak{p} + \mathfrak{a} \sim \mathfrak{p}$, by the proof of Lemma 5.2 there exists $Q \in D(R_A)_{-\mathfrak{a}}$ such that $v = QPv$ as elements in $(M_0)_\mathfrak{p}$. This is a contradiction; v is not zero, while P annihilate v . Hence $p \in \mathfrak{p}$. Thus we have proved the inclusion $K \subseteq {}^L I(\mathfrak{p})$.

Next suppose that $K \subsetneq {}^L I(\mathfrak{p})$. Let $x \in {}^L I(\mathfrak{p})_\mathfrak{a} \setminus K_\mathfrak{a}$, and \bar{x} its image in $D(R_A)/K \simeq M$. Since M is δ -critical,

$$\delta = \text{Kdim } {}^L L(\mathfrak{p}) \leq \text{Kdim } M / D(R_A)\bar{x} < \text{Kdim } M = \delta.$$

Hence we conclude that $K = {}^L I(\mathfrak{p})$. □

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DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
HOKKAIDO UNIVERSITY
SAPPORO, 060-0810 JAPAN
E-mail address: `saito@math.sci.hokudai.ac.jp`