Large-time asymptotics of the gyration radius for long-range statistical-mechanical models*

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Abstract

The aim of this short article is to convey the basic idea of the original paper [3], without going into too much detail, about how to derive sharp asymptotics of the gyration radius for random walk, self-avoiding walk and oriented percolation above the model-dependent upper critical dimension.

1 Introduction

Let D be the $\mathbb{Z}^d$-symmetric 1-step distribution for random walk (RW) and define the RW 2-point function as

$$\varphi_t^{\mathrm{RW}}(x) = \sum_{\substack{\omega:o\to x\\ |\omega|=t}} \prod_{s=1}^{t} D(\omega_s - \omega_{s-1}) \qquad (x \in \mathbb{Z}^d,\ t \in \mathbb{Z}_+). \tag{1.1}$$

We also consider self-avoiding walk (SAW) and oriented percolation (OP) that are both generated by D. The SAW 2-point function is defined as

$$\varphi_t^{\mathrm{SAW}}(x) = \sum_{\substack{\omega:o\to x\\ |\omega|=t}} \prod_{s=1}^{t} D(\omega_s - \omega_{s-1}) \prod_{0\le i<j\le t} (1-\delta_{\omega_i,\omega_j}), \tag{1.2}$$

where the indicator $\prod_{0\le i<j\le t}(1-\delta_{\omega_i,\omega_j})$, which is absent in (1.1), is 1 if and only if ω does not intersect to itself, hence accounting for the self-avoidance constraint. Oriented percolation is a model for random media in space-time $\mathbb{Z}^d \times \mathbb{Z}_+$. A bond is an ordered pair of vertices in $\mathbb{Z}^d \times \mathbb{Z}_+$, and each bond $((u,t),(v,t+1))$ is either occupied or vacant with probability $pD(v-u)$ and $1-pD(v-u)$, respectively, independently of the other

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bonds. The parameter p equals the expected number of occupied bonds per vertex, and it is known that there is a phase transition at $p = p_c$. We say that (x, s) is connected to (y, t) if either $(x, s) = (y, t)$ or there is a time-increasing sequence of occupied bonds from (x, s) to (y, t). The OP 2-point function $\varphi_t^{\mathrm{OP}}(x)$ is then defined as the probability that the origin $(o, 0)$ is connected to (x, t).

The models are said to be finite-range if D is supported on a finite set of $\mathbb{Z}^d$. The main property of a finite-range D is the existence of the variance $\sigma^2 \equiv \sum_{x\in\mathbb{Z}^d} |x|^2 D(x)$, and because of this, investigation of finite-range models is relatively easier. The situation is basically the same for D that decays faster than any polynomials, such as an exponentially decaying D. However, if $D(x) \approx |x|^{-d-\alpha}$ for large $|x|$, then the existence of the variance depends on $\alpha > 0$ and therefore we cannot always expect that the same results for finite-range models also hold for this long-range models with index α. For example, take the gyration radius of order $r \in (0, \alpha)$, which is defined as

$$\xi_t^{(r)} = \left(\frac{\sum_{x\in\mathbb{Z}^d} |x|^r \varphi_t(x)}{\sum_{x\in\mathbb{Z}^d} \varphi_t(x)} \right)^{1/r}. \tag{1.3}$$

The gyration radius represents a typical end-to-end distance of a linear structure of length t or a typical spatial size of a cluster at time t. It may be natural to guess, at least for random walk, that $\xi_t^{(r)} = O(\sqrt{t})$ if $\alpha > 2$ and $\xi_t^{(r)} = O(t^{1/\alpha})$ if $\alpha < 2$, for every real $r \in (0, \alpha)$. As we state shortly, we have proved affirmative results [3] for random walk in any dimension and for self-avoiding walk and critical/subcritical oriented percolation above the common upper-critical dimension $d_c \equiv 2(\alpha \wedge 2)$.

More precisely, we assume the following properties of D. Given an $L \in [1, \infty)$, we suppose that $D(x) \propto |x/L|^{-d-\alpha}$ for large $|x|$ such that its Fourier transform $\hat{D}(k) \equiv \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} D(x)$ exhibits the $k \to 0$ asymptotics

$$1 - \hat{D}(k) = v_\alpha |k|^{\alpha\wedge 2} \times \begin{cases} 1 + O((L|k|)^\epsilon) & (\alpha \neq 2), \\ \log \frac{1}{L|k|} + O(1) & (\alpha = 2), \end{cases} \tag{1.4}$$

for some $v_\alpha = O(L^{\alpha\wedge 2})$ and $\epsilon > 0$. If $\alpha > 2$ (or D is finite-range), then $v_\alpha = \frac{1}{2d}\sigma^2$. An example that satisfies the above properties is the long-range Kac potential

$$D(x) = \frac{h(y/L)}{\sum_{y\in\mathbb{Z}^d} h(y/L)} \qquad (x \in \mathbb{Z}^d), \tag{1.5}$$

defined by the rotation-invariant function

$$h(x) = \frac{1 + O\big((|x| \vee 1)^{-\rho}\big)}{(|x| \vee 1)^{d+\alpha}} \qquad (x \in \mathbb{R}^d), \tag{1.6}$$

for some $\rho > \epsilon$ (cf., [3]). Under this assumption, we have proved the following sharp asymptotics of a variant of the gyration radius:

Theorem 1.1 ([3]). *For random walk in any dimension with any L, and for self-avoiding walk and critical/subcritical oriented percolation for $d > d_c$ with $L \gg 1$, there is a model-dependent constant $C_\alpha = 1 + O(L^{-d})$ ($C_\alpha \equiv 1$ for random walk) such that, for every $r \in (0, \alpha)$,*

$$\frac{\sum_{x\in\mathbb{Z}^d} |x_1|^r \varphi_t(x)}{\sum_{x\in\mathbb{Z}^d} \varphi_t(x)} \underset{t\uparrow\infty}{\sim} \frac{2\sin\frac{r\pi}{\alpha\vee 2}}{(\alpha\wedge 2)\sin\frac{r\pi}{\alpha}} \frac{\Gamma(r+1)}{\Gamma(\frac{r}{\alpha\wedge 2}+1)} \times \begin{cases} (C_\alpha v_\alpha t)^{\frac{r}{\alpha\wedge 2}} & (\alpha \neq 2), \\ (C_2 v_2 t \log\sqrt{t})^{r/2} & (\alpha = 2), \end{cases} \tag{1.7}$$

where x_1 is the first coordinate of $x \in \mathbb{Z}^d$.

We should emphasize that, except for the actual value of C_α, the expression (1.7) is universal. The result also holds for finite-range models, for which α is considered to be infinity. As far as we notice, even for random walk, the sharp asymptotic expression (1.7) for all real $r \in (0, \alpha)$ is new.

Using $|x_1|^r \leq |x|^r \leq d^{r/2} \sum_{j=1}^d |x_j|^r$ and the $\mathbb{Z}^d$-symmetry of the models, we can conclude the following:

Corollary 1.2 ([3]). *Under the same condition as in Theorem 1.1,*

$$\xi_t^{(r)} = \begin{cases} O(t^{\frac{1}{\alpha\wedge 2}}) & (\alpha \neq 2), \\ O(\sqrt{t\log t}) & (\alpha = 2). \end{cases} \tag{1.8}$$

for every $r \in (0, \alpha)$.

In his recent work [4], Heydenreich proved (1.8) for self-avoiding walk, but only for small $r < \alpha \wedge 2$, with no attempt to identify the proportional constant. Our results are somewhat stronger, because we have derived the exact expression for the proportional constant in (1.7) (also clarifying its model-dependence) and proved (1.8) for all $r < \alpha$.

2 Sketch proof for random walk

In this section, we restrict our attention to random walk, which is obviously simpler than the other two models, and explain the framework of the proof of Theorem 1.1.

First we consider the generating function (= the Fourier-Laplace transform) of the 2-point function. Recall that $\varphi_t^{\mathrm{RW}}(x)$ satisfies the convolution equation

$$\varphi_t^{\mathrm{RW}}(x) = \delta_{t,0}\delta_{x,o} + (D * \varphi_{t-1}^{\mathrm{RW}})(x) \equiv \delta_{t,0}\delta_{x,o} + \sum_{y\in\mathbb{Z}^d} D(y)\, \varphi_{t-1}^{\mathrm{RW}}(x-y), \tag{2.1}$$

where we regard $(D * \varphi_{t-1}^{\mathrm{RW}})(x)$ for $t \leq 0$ as zero. Taking the Fourier-Laplace transform of both sides, we obtain that, for $k \in [-\pi, \pi]^d$ and $m \in [0, m_c^{\mathrm{RW}})$,

$$\hat{\varphi}_m^{\mathrm{RW}}(k) \equiv \sum_{t\in\mathbb{Z}_+} m^t \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} \varphi_t^{\mathrm{RW}}(x) = 1 + m\hat{D}(k)\, \hat{\varphi}_m^{\mathrm{RW}}(k), \tag{2.2}$$

where $m_c^{\rm RW} \equiv 1$ is the radius of convergence for the sequence $\big\{\sum_{x\in\mathbb{Z}^d}\varphi_t^{\rm RW}(x)\big\}_{t\in\mathbb{Z}_+}$. To see this in a different way, take $k=0$ in (2.2) so that

$$\hat\varphi_m^{\rm RW}(0) = 1 + m\hat\varphi_m^{\rm RW}(0) = \frac{1}{1-m}. \tag{2.3}$$

The expansion of the right-hand side is $\sum_{t\in\mathbb{Z}_+} m^t$ and the coefficient of m^t is exactly 1 $(\equiv \sum_{x\in\mathbb{Z}^d}\varphi_t^{\rm RW}(x))$ for every $t\in\mathbb{Z}_+$.

Next we differentiate $\hat\varphi_m^{\rm RW}(k)$ with respect to k_1 (= the first coordinate of k) to yield the generating function of the sequence $\big\{\sum_{x\in\mathbb{Z}^d}|x_1|^r\varphi_t^{\rm RW}(x)\big\}_{t\in\mathbb{Z}_+}$. For example, if $r=2j$ with $j\in\mathbb{N}$ (hence $\alpha>2$), then

$$\nabla_1^{2j}\hat\varphi_m^{\rm RW}(0) \equiv \frac{\partial^{2j}}{\partial k_1^{2j}}\hat\varphi_m^{\rm RW}(k)\bigg|_{k=0} = (-1)^j\sum_{t\in\mathbb{Z}_+} m^t \sum_{x\in\mathbb{Z}^d} x_1^{2j}\varphi_t^{\rm RW}(x). \tag{2.4}$$

On the other hand, by differentiating (2.2) and using the $\mathbb{Z}^d$-symmetry of the model,

$$\begin{aligned}\nabla_1^{2j}\hat\varphi_m^{\rm RW}(0) &= m\nabla_1^{2j}\hat\varphi_m^{\rm RW}(0) + m\sum_{l=1}^{j}\binom{2j}{2l}\nabla_1^{2l}\hat D(0)\,\nabla_1^{2(j-l)}\hat\varphi_m^{\rm RW}(0)\\ &= \frac{m}{1-m}\sum_{l=1}^{j}\binom{2j}{2l}\nabla_1^{2l}\hat D(0)\,\nabla_1^{2(j-l)}\hat\varphi_m^{\rm RW}(0).\end{aligned} \tag{2.5}$$

Solving this recursion by induction under the initial condition (2.3), we obtain (see [3] for more details)

$$\begin{aligned}\nabla_1^{2j}\hat\varphi_m^{\rm RW}(0) &= \binom{2j}{2}\frac{m\nabla_1^2\hat D(0)}{1-m}\,\nabla_1^{2(j-1)}\hat\varphi_m^{\rm RW}(0) + O\big((1-m)^{-j}\big)\\ &= \binom{2j}{2}\binom{2(j-1)}{2}\bigg(\frac{m\nabla_1^2\hat D(0)}{1-m}\bigg)^2\nabla_1^{2(j-2)}\hat\varphi_m^{\rm RW}(0) + O\big((1-m)^{-j}\big)\\ &\;\;\vdots\\ &= \prod_{l=1}^{j}\binom{2l}{2}\bigg(\frac{m\nabla_1^2\hat D(0)}{1-m}\bigg)^j\hat\varphi_m^{\rm RW}(0) + O\big((1-m)^{-j}\big)\\ &= \frac{(2j)!}{2^j}\frac{\big(m\nabla_1^2\hat D(0)\big)^j}{(1-m)^{j+1}} + O\big((1-m)^{-j}\big).\end{aligned} \tag{2.6}$$

Comparing this with (2.4) and using $v_\alpha \equiv \frac{1}{2d}\sigma^2 = \frac{-1}{2}\nabla_1^2\hat D(0)$ for $\alpha>2$, we arrive at

$$\sum_{t\in\mathbb{Z}_+} m^t \sum_{x\in\mathbb{Z}^d} x_1^{2j}\varphi_t^{\rm RW}(x) = (2j)!\,\frac{(mv_\alpha)^j}{(1-m)^{j+1}} + O\big((1-m)^{-j}\big). \tag{2.7}$$

However, by the general binomial expansion,

$$\frac{m^j}{(1-m)^{j+1}} = m^j \sum_{l=0}^{\infty} \binom{-j-1}{l} (-m)^l = m^j \sum_{l=0}^{\infty} \binom{j+l}{j} m^l = \sum_{t=j}^{\infty} \binom{t}{j} m^t. \tag{2.8}$$

Therefore,

$$\sum_{x\in\mathbb{Z}^d} x_1^{2j} \varphi_t^{\text{RW}}(x) \underset{t\uparrow\infty}{\sim} (2j)! \binom{t}{j} v_\alpha^j \sim \frac{\Gamma(2j+1)}{\Gamma(j+1)} (v_\alpha t)^j. \tag{2.9}$$

This completes the proof of (1.7) for $r = 2j < \alpha$.

In order to consider the other values of $r < \alpha$, we use the following integral representation for $|x_1|^q$ with $q \in (0, 2)$ (cf., [3]):

$$|x_1|^q = \frac{1}{K_q} \int_0^\infty \frac{1-\cos(ux_1)}{u^{1+q}} \,\mathrm{d}u, \tag{2.10}$$

where

$$K_q = \int_0^\infty \frac{1-\cos u}{u^{1+q}} \,\mathrm{d}u = \frac{\pi}{2\sin\frac{q\pi}{2}} \frac{1}{\Gamma(q+1)}. \tag{2.11}$$

Let $r = 2j + q$ with $j \in \mathbb{Z}_+$ and $q \in (0, 2)$. Then, by (2.10), the generating function for the fractional moment $\big\{\sum_{x\in\mathbb{Z}^d} |x_1|^{2j+q} \varphi_t^{\text{RW}}(x)\big\}_{t\in\mathbb{Z}_+}$ can be written as

$$\begin{aligned}\sum_{t\in\mathbb{Z}_+} m^t \sum_{x\in\mathbb{Z}^d} |x_1|^{2j+q} \varphi_t^{\text{RW}}(x) &= \frac{1}{K_q} \int_0^\infty \frac{\mathrm{d}u}{u^{1+q}} \sum_{t\in\mathbb{Z}_+} m^t \sum_{x\in\mathbb{Z}^d} \big(1-\cos(ux_1)\big)\, x_1^{2j} \varphi_t^{\text{RW}}(x) \\ &= \frac{(-1)^j}{K_q} \int_0^\infty \frac{\mathrm{d}u}{u^{1+q}} \Big(\nabla_1^{2j} \hat{\varphi}_m^{\text{RW}}(0) - \nabla_1^{2j} \hat{\varphi}_m^{\text{RW}}(\vec{u})\Big),\end{aligned} \tag{2.12}$$

where $\vec{u} = (u, 0, \ldots, 0) \in \mathbb{R}^d$. Therefore, similarly to the above case of $r = 2j$, it suffices to investigate the "derivative"

$$\bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{\varphi}_m^{\text{RW}}(0) \equiv \nabla_1^{2j} \hat{\varphi}_m^{\text{RW}}(0) - \nabla_1^{2j} \hat{\varphi}_m^{\text{RW}}(\vec{u}). \tag{2.13}$$

However, by "differentiating" both sides of (2.2) and using the $\mathbb{Z}^d$-symmetry, we obtain

$$\begin{aligned}\bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{\varphi}_m^{\text{RW}}(0) &= m \bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{\varphi}_m^{\text{RW}}(0) + m \sum_{l=1}^{j} \binom{2j}{2l} \nabla_1^{2l} \hat{D}(0)\, \bar{\Delta}_{\vec{u}} \nabla_1^{2(j-l)} \hat{\varphi}_m^{\text{RW}}(0) \\ &\quad + m \sum_{n=0}^{2j} \binom{2j}{n} \nabla_1^{2j-n} \hat{\varphi}_m^{\text{RW}}(\vec{u})\, \bar{\Delta}_{\vec{u}} \nabla_1^{n} \hat{D}(0) \\ &= \frac{m}{1-m} \bigg(\sum_{l=1}^{j} \binom{2j}{2l} \nabla_1^{2l} \hat{D}(0)\, \bar{\Delta}_{\vec{u}} \nabla_1^{2(j-l)} \hat{\varphi}_m^{\text{RW}}(0) \\ &\qquad + \sum_{n=0}^{2j} \binom{2j}{n} \nabla_1^{2j-n} \hat{\varphi}_m^{\text{RW}}(\vec{u})\, \bar{\Delta}_{\vec{u}} \nabla_1^{n} \hat{D}(0) \bigg),\end{aligned} \tag{2.14}$$

where we regard the sum over $l \in \{1, \dots, j\}$ in the last expression as zero when $j = 0$. Substituting this back to (2.12), performing the integration with respect to $u \in (0, \infty)$ and then reorganizing the resulting terms (see [3] for more details), we will end up with

$$\sum_{t \in \mathbb{Z}_+} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t^{\mathrm{RW}}(x) = \frac{2 \sin \frac{r\pi}{\alpha \vee 2}}{(\alpha \wedge 2) \sin \frac{r\pi}{\alpha}} \, \Gamma(r+1) \, \frac{(m v_\alpha)^{\frac{r}{\alpha \wedge 2}}}{(1-m)^{1+\frac{r}{\alpha \wedge 2}}} \times \begin{cases} 1 + O((1-m)^\epsilon) & (\alpha \neq 2), \\ \big(\log \frac{1}{\sqrt{1-m}}\big)^{r/2} + O(1) & (\alpha = 2), \end{cases} \tag{2.15}$$

for some $\epsilon > 0$. The proof of (1.7) is completed by expanding the right-hand side of the above expression in powers of m and comparing the coefficient of m^t in both sides, for large t.

3 The model-dependence

The key to the proof for self-avoiding walk and oriented percolation is the following lace expansion (see, e.g., [1, 5]):

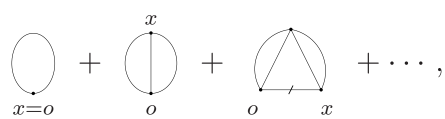
$$\varphi_t(x) = I_t(x) + \sum_{s=1}^{t} (J_s * \varphi_{t-s})(x), \tag{3.1}$$

where

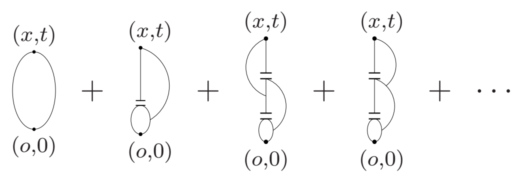
$$I_t(x) = \begin{cases} \delta_{x,o}\delta_{t,0} & (\mathrm{SAW}), \\ \pi_t^{\mathrm{OP}}(x) & (\mathrm{OP}), \end{cases} \qquad J_t(x) = \begin{cases} D(x)\delta_{t,1} + \pi_t^{\mathrm{SAW}}(x) & (\mathrm{SAW}), \\ p(D * \pi_{t-1}^{\mathrm{OP}})(x) & (\mathrm{OP}). \end{cases} \tag{3.2}$$

Recall (2.1) for random walk, so that $I_t^{\mathrm{RW}}(x) = \delta_{x,o}\delta_{t,0}$ and $J_t^{\mathrm{RW}}(x) = D(x)\delta_{t,1}$. The model-dependent $\pi_t(x)$ in (3.2), which accounts for difference from random walk, is an alternating sum of the lace-expansion coefficients and obey the following diagrammatic bounds (cf., [1, 5]):

$$|\pi_t^{\mathrm{SAW}}(x)| \le \tag{3.3}$$



$$|\pi_t^{\mathrm{OP}}(x)| \le \tag{3.4}$$



where each line corresponds to a 2-point function. For self-avoiding walk, the first diagram represents self-avoiding loop of length $t \geq 2$, i.e., $(D * \varphi_{t-1}^{\mathrm{SAW}})(x)$, and the second diagram represents the product of three 2-point functions, $\varphi^{\rm SAW}_{s_1}(x)\,\varphi^{\rm SAW}_{s_2}(x)\,\varphi^{\rm SAW}_{s_3}(x)$, summed over all possible combinations of $s_1, s_2, s_3 \geq 1$ satisfying $s_1 + s_2 + s_3 = t$, and so on. For oriented percolation, the first diagram represents $\varphi^{\rm OP}_t(x)^2$, where the upward direction is the time-increasing direction, and the second diagram represents the product of five 2-point functions concatenated in a depicted way, where unlabeled vertices are summed over $\mathbb{Z}^d \times \mathbb{Z}_+$, and so on. For more details, we refer to [1, 5].

Because of the similarity between (2.1) and (3.1), it is natural to expect that the strategy in §2 for random walk may also work for self-avoiding walk and oriented percolation. To see if it really works, we first take the Fourier-Laplace transform of (3.1). For $k \in [-\pi,\pi]^d$ and $m \in [0, m_{\rm c})$,

$$\hat\varphi_m(k) = \hat I_m(k) + \hat J_m(k)\,\hat\varphi_m(k), \tag{3.5}$$

where $m_{\rm c} \geq 1$ is the model-dependent radius of convergence for $\big\{\sum_{x\in\mathbb{Z}^d}\varphi_t(x)\big\}_{t\in\mathbb{Z}_+}$ for self-avoiding walk and critcal/subcritical oriented percolation ($m^{\rm OP}_{\rm c}$ is a non-increasing function of $p \leq p_{\rm c}$ and $m^{\rm OP}_{\rm c} = 1$ at $p = p_{\rm c}$ [1]). Due to the diagrammatic bounds (3.3)–(3.4), it has been proved [1, 2, 4] that, for $d > d_{\rm c}$ and $L \gg 1$, there are $\epsilon, \delta > 0$ such that

$$\sum_{t\in\mathbb{Z}^d} t^{1+\epsilon} m^t \sum_{x\in\mathbb{Z}^d} |\pi_t(x)|, \qquad\qquad \sum_{t\in\mathbb{Z}^d} m^t \sum_{x\in\mathbb{Z}^d} |x_1|^{\alpha\wedge 2+\delta}|\pi_t(x)|, \tag{3.6}$$

both converge, even at $m = m_{\rm c}$. This implies that $\hat J_{m_{\rm c}}(0) = 1$ and, as $m \uparrow m_{\rm c}$,

$$\begin{aligned}\hat\varphi_m(0) = \frac{\hat I_m(0)}{1-\hat J_m(0)} = \frac{\hat I_m(0)}{\hat J_{m_{\rm c}}(0)-\hat J_m(0)} &\sim \frac{\hat I_{m_{\rm c}}(0)}{m_{\rm c}\partial_m \hat J_{m_{\rm c}}(0)\left(1-\frac{m}{m_{\rm c}}\right)} \\ &= \frac{\hat I_{m_{\rm c}}(0)}{m_{\rm c}\partial_m \hat J_{m_{\rm c}}(0)} \sum_{t\in\mathbb{Z}_+}\left(\frac{m}{m_{\rm c}}\right)^t.\end{aligned} \tag{3.7}$$

On the other hand, for $r = 2j < \alpha$ with $j \in \mathbb{N}$,

$$\begin{aligned}\nabla_1^{2j}\hat\varphi_m(0) &= \nabla_1^{2j}\hat I_m(0) + \sum_{l=0}^{j}\binom{2j}{2l}\nabla_1^{2l}\hat J_m(0)\,\nabla_1^{2(j-l)}\hat\varphi_m(0) \\ &= \frac{1}{1-\hat J_m(0)}\bigg(\nabla_1^{2j}\hat I_m(0) + \sum_{l=1}^{j}\binom{2j}{2l}\nabla_1^{2l}\hat J_m(0)\,\nabla_1^{2(j-l)}\hat\varphi_m(0)\bigg).\end{aligned} \tag{3.8}$$

Suppose that the leading contribution is due to the $l = 1$ term (this is far from trivial and needs to be proved, as in [3]). Then, by induction and using (3.7),

$$\begin{aligned}
\nabla_1^{2j}\hat{\varphi}_m(0) &\sim \binom{2j}{2}\frac{\nabla_1^2\hat{J}_m(0)}{1-\hat{J}_m(0)}\,\nabla_1^{2(j-1)}\hat{\varphi}_m(0)\\
&\vdots\\
&\sim \frac{(2j)!}{2^j}\bigg(\frac{\nabla_1^2\hat{J}_m(0)}{1-\hat{J}_m(0)}\bigg)^j\hat{\varphi}_m(0)\\
&\sim \frac{(2j)!}{2^j}\Bigg(\frac{\nabla_1^2\hat{J}_{m_c}(0)}{m_c\partial_m\hat{J}_{m_c}(0)\,\big(1-\frac{m}{m_c}\big)}\Bigg)^j\frac{\hat{I}_{m_c}(0)}{m_c\partial_m\hat{J}_{m_c}(0)\,\big(1-\frac{m}{m_c}\big)}. \qquad (3.9)
\end{aligned}$$

However, similarly to (2.8),

$$\Big(1-\frac{m}{m_c}\Big)^{-j-1}=\sum_{t\in\mathbb{Z}_+}\binom{t+j}{j}\Big(\frac{m}{m_c}\Big)^t, \qquad (3.10)$$

hence

$$\nabla_1^{2j}\hat{\varphi}_m(0)\sim(2j)!\,\bigg(\frac{\nabla_1^2\hat{J}_{m_c}(0)}{2m_c\partial_m\hat{J}_{m_c}(0)}\bigg)^j\frac{\hat{I}_{m_c}(0)}{m_c\partial_m\hat{J}_{m_c}(0)}\sum_{t\in\mathbb{Z}_+}\binom{t+j}{j}\Big(\frac{m}{m_c}\Big)^t. \qquad (3.11)$$

Therefore, by (3.7) and (3.11),

$$\begin{aligned}
\frac{\sum_{x\in\mathbb{Z}^d}x_1^{2j}\varphi_t(x)}{\sum_{x\in\mathbb{Z}^d}\varphi_t(x)} &\sim \frac{(2j)!}{j!}\,\bigg(\frac{-\nabla_1^2\hat{J}_{m_c}(0)}{2m_c\partial_m\hat{J}_{m_c}(0)}\,t\bigg)^j\\
&=\frac{\Gamma(2j+1)}{\Gamma(j+1)}\Bigg(\underbrace{\frac{1}{m_c\partial_m\hat{J}_{m_c}(0)}\,\frac{\nabla_1^2\hat{J}_{m_c}(0)}{\nabla_1^2\hat{D}(0)}}_{C_\alpha}\;\underbrace{\frac{-\nabla_1^2\hat{D}(0)}{2}}_{v_\alpha}\;t\Bigg)^j. \qquad (3.12)
\end{aligned}$$

This completes a sketch proof for $r=2j$.

The case for the other values of $r<\alpha$ is more involved, but can be proved by following the same strategy as in §2 for random walk. However, since C_α in (3.12) is ill-defined for $\alpha\le 2$ due to the divergence of $\nabla_1^2\hat{D}(0)$, it is replaced by

$$C_\alpha=\frac{1}{m_c\partial_m\hat{J}_{m_c}(0)}\lim_{k\to0}\frac{\bar{\Delta}_k\hat{J}_{m_c}(0)}{\bar{\Delta}_k\hat{D}(0)}\equiv\frac{1}{m_c\partial_m\hat{J}_{m_c}(0)}\lim_{k\to0}\frac{\hat{J}_{m_c}(0)-\hat{J}_{m_c}(k)}{\hat{D}(0)-\hat{D}(k)}. \qquad (3.13)$$

We refrain from showing further details and refer the readers to the original paper [3].

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