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ASYMPTOTIC BEHAVIOR OF THE GYRATION RADIUS FOR LONG-RANGE SELF-AVOIDING WALK AND LONG-RANGE ORIENTED PERCOLATION

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We consider random walk and self-avoiding walk whose 1-step distribution is given by D, and oriented percolation whose bond-occupation probability is proportional to D. Suppose that $D(x)$ decays as $|x|^{-d-\alpha}$ with $\alpha > 0$. For random walk in any dimension d and for self-avoiding walk and critical/subcritical oriented percolation above the common upper-critical dimension $d_c \equiv 2(\alpha \wedge 2)$, we prove large-$t$ asymptotics of the gyration radius, which is the average end-to-end distance of random walk/self-avoiding walk of length t or the average spatial size of an oriented percolation cluster at time t. This proves the conjecture for long-range self-avoiding walk in [*Ann. Inst. H. Poincaré Probab. Statist.* (2010), to appear] and for long-range oriented percolation in [*Probab. Theory Related Fields* **142** (2008) 151–188] and [*Probab. Theory Related Fields* **145** (2009) 435–458].

1. Introduction.

1.1. *Motivation.* Let $\varphi_t^{RW}(x)$ be the t-step transition probability for random walk on $\mathbb{Z}^d$: $\varphi_0^{RW}(x) = \delta_{o,x}$ and

$$(1.1) \qquad \varphi_t^{RW}(x) = (\varphi_{t-1}^{RW} * D)(x) \equiv \sum_{y \in \mathbb{Z}^d} \varphi_{t-1}^{RW}(y) D(x - y) \qquad [t \in \mathbb{N}].$$

Suppose that the 1-step distribution D is $\mathbb{Z}^d$-symmetric. How does the rth moment $\sum_x |x|^r \varphi_t^{RW}(x)$ grow as $t \to \infty$, where $|\cdot|$ denotes the Euclidean distance? When $r = 2$ and $\sigma^2 \equiv \sum_x |x|^2 D(x) < \infty$, the answer is trivial: $\sum_x |x|^2 \varphi_t^{RW}(x) = \sigma^2 t$ since the variance of the sum of independent random variables is the sum of their variances. It is not so hard to see that $\sum_x |x|^r \varphi_t^{RW}(x) = O(t^{r/2})$ as $t \to \infty$ for other values of $r > 2$, as long as $\sum_x |x|^r D(x) < \infty$. Even so, it may not be that easy to identify the constant $C \in (0, \infty)$ such that $(\sum_x |x|^r \varphi_t^{RW}(x))^{1/r} \sim C\sqrt{t}$. Here, and in the rest of the paper, "$f(z) = O(g(z))$" means that $|f(z)/g(z)|$ is

bounded for all z in some relevant set, while "$f(z) \sim g(z)$" means that $f(z)/g(z)$ tends to 1 in some relevant limit for z.

Let $\alpha > 0$, $L \in [1, \infty)$ and suppose that $D(x) \approx |x/L|^{-d-\alpha}$ for large x such that its Fourier transform $\hat{D}(k) \equiv \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} D(x)$ satisfies

$$(1.2) \qquad 1 - \hat{D}(k) = v_\alpha |k|^{\alpha \wedge 2} \times \begin{cases} 1 + O((L|k|)^\epsilon), & \alpha \neq 2, \\ \log \dfrac{1}{L|k|} + O(1), & \alpha = 2 \end{cases}$$

for some $v_\alpha = O(L^{\alpha \wedge 2})$ and $\epsilon > 0$. If $\alpha > 2$ (or D is finite-range), then $v_\alpha \equiv \sigma^2/(2d)$. As shown in Appendix A.1, the long-range Kac potential

$$(1.3) \qquad D(x) = \frac{h(y/L)}{\sum_{y \in \mathbb{Z}^d} h(y/L)} \qquad [x \in \mathbb{Z}^d],$$

defined in terms of a rotation-invariant function h satisfying

$$h(x) = \frac{1 + O((|x| \vee 1)^{-\rho})}{(|x| \vee 1)^{d+\alpha}} \qquad [x \in \mathbb{R}^d]$$

for some $\rho > \epsilon$, satisfies the above properties. Notice that $\sum_x |x|^r D(x) = \infty$ for $r \geq \alpha$ and, in particular, $\sigma^2 = \infty$ if $\alpha \leq 2$. This is of interest in investigating the asymptotic behavior of $\sum_x |x|^r \varphi_t^{RW}(x)$ for *all* $r \in (0, \alpha)$ and understanding its α-dependence.

In fact, our main interest is in proving sharp asymptotics of the gyration radius of order $r \in (0, \alpha)$, defined as

$$\xi_t^{(r)} = \left(\frac{\sum_{x \in \mathbb{Z}^d} |x|^r \varphi_t(x)}{\sum_{x \in \mathbb{Z}^d} \varphi_t(x)} \right)^{1/r},$$

where $\varphi_t(x) \equiv \varphi_t^{SAW}(x)$ is the two-point function for t-step self-avoiding walk whose 1-step distribution is given by D, or $\varphi_t(x) \equiv \varphi_t^{OP}(x)$ is the two-point function for oriented percolation whose bond-occupation probability for each bond $((u, s), (v, s+1))$ is given by $pD(v-u)$, independently of $s \in \mathbb{Z}_+$, where $p \geq 0$ is the percolation parameter. More precisely,

$$(1.4) \quad \varphi_t(x) = \begin{cases} \varphi_t^{RW}(x) \equiv \displaystyle\sum_{\substack{\omega: o \to x \\ (|\omega|=t)}} \prod_{s=1}^{t} D(\omega_s - \omega_{s-1}), \\[2em] \varphi_t^{SAW}(x) \equiv \displaystyle\sum_{\substack{\omega: o \to x \\ (|\omega|=t)}} \prod_{s=1}^{t} D(\omega_s - \omega_{s-1}) \prod_{0 \leq i < j \leq t} (1 - \delta_{\omega_i, \omega_j}), \\[2em] \varphi_t^{OP}(x) \equiv \mathbb{P}_p((o, 0) \to (x, t)), \end{cases}$$

where $\prod_{0 \leq i < j \leq t} (1 - \delta_{\omega_i, \omega_j})$ is the self-avoiding constraint on ω and $\{(o, 0) \to (x, t)\}$ is the event that either $(x, t) = (o, 0)$ or there is a consecutive sequence

of occupied bonds from $(o, 0)$ to (x, t) in the time-increasing direction. The gyration radius $\xi_t^{(r)}$ represents a typical end-to-end distance of a linear structure of length t or a typical spatial size of a cluster at time t. It has been expected (and would certainly be true for random walk in any dimension) that, above the common upper-critical dimension $d_c = 2(\alpha \wedge 2)$ for self-avoiding walk and oriented percolation, for every $r \in (0, \alpha)$,

$$(1.5) \qquad \xi_t^{(r)} = \begin{cases} O\big(t^{1/(\alpha \wedge 2)}\big), & \alpha \neq 2, \\ O(\sqrt{t \log t}), & \alpha = 2. \end{cases}$$

Heydenreich [5] proved (1.5) for self-avoiding walk, but only for small $r < \alpha \wedge 2$. Nevertheless, this small-r result is enough to prove weak convergence of self-avoiding walk to an α-stable process/Brownian motion, depending on the value of α [5].

As stated below in Theorem 1.2, we prove sharp asymptotics (including the proportionality constant) of $\sum_x |x_1|^r \varphi_t(x) / \sum_x \varphi_t(x)$ as $t \to \infty$, where x_1 is the first coordinate of $x \equiv (x_1, \ldots, x_d)$, and show that (1.5) holds for all $r \in (0, \alpha)$, solving the open problems in [3, 5].

1.2. *Main results.* Let $m_c \geq 1$ be the model-dependent radius of convergence for the sequence $\sum_x \varphi_t(x)$. For random walk, $m_c = 1$ since $\sum_x \varphi_t^{RW}(x)$ is always 1. For self-avoiding walk, $m_c > 1$ due to the self-avoiding constraint in (1.4) and, indeed, $m_c = 1 + O(L^{-d})$ for $d > d_c$ and $L \gg 1$ [6]. For oriented percolation, m_c depends on the percolation parameter p [i.e., $m_c = m_c(p)$] and was denoted by m_p in [2, 3]. It has been proven [2] that $m_c(p) > 1$ for $p < p_c$, and $m_c(p_c) = 1$ for $d > d_c$ and $L \gg 1$, where p_c is the critical point characterized by the divergence of the susceptibility: $\sum_{t=0}^{\infty} \sum_{x \in \mathbb{Z}^d} \varphi_t^{OP}(x) \uparrow \infty$ as $p \uparrow p_c$. It has also been proven [2] that $pm_c = 1 + O(L^{-d})$ for all $p \leq p_c$.

Let C_I and C_{II} be the constants in [2, 3, 5] such that, as $t \to \infty$,

$$(1.6) \qquad \sum_{x \in \mathbb{Z}^d} \varphi_t(x) \sim C_I m_c^{-t}, \qquad \frac{\sum_{x \in \mathbb{Z}^d} e^{ik_t \cdot x} \varphi_t(x)}{\sum_{x \in \mathbb{Z}^d} \varphi_t(x)} \sim e^{-C_{II}|k|^{\alpha \wedge 2}},$$

where

$$(1.7) \qquad k_t = k \times \begin{cases} (v_\alpha t)^{-1/(\alpha \wedge 2)}, & \alpha \neq 2, \\ (v_2 t \log \sqrt{t})^{-1/2}, & \alpha = 2. \end{cases}$$

Because of this scaling, we have $C_I^{RW} = C_{II}^{RW} = 1$ for random walk. For self-avoiding walk and critical/subcritical oriented percolation for $d > 2(\alpha \wedge 2)$ with $L \gg 1$ (depending on the models), it has been proven that the model-dependent constants C_I and C_{II} are both $1 + O(L^{-d})$ [2, 5] and that the $O(L^{-d})$ term in C_{II} exhibits crossover behavior at $\alpha = 2$ [3, 5]. We will provide precise expressions for C_I and C_{II} at the end of Section 1.3.

Our first result is the following asymptotic behavior of the generating function for the sequence $\sum_x |x_1|^r \varphi_t(x)$.

THEOREM 1.1.   *Consider the three aforementioned long-range models. For random walk in any dimension d with any L, and for self-avoiding walk and critical/subcritical oriented percolation for $d > d_c \equiv 2(\alpha \wedge 2)$ with $L \gg 1$ (depending on the models), the following holds for all $r \in (0, \alpha)$: as $m \uparrow m_c$,*

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x) = \frac{2\sin(r\pi/(\alpha \vee 2))}{(\alpha \wedge 2)\sin(r\pi/\alpha)} \Gamma(r+1) \frac{C_{\mathrm{I}}(C_{\mathrm{II}} v_\alpha)^{r/(\alpha \wedge 2)}}{(1 - m/m_c)^{1+r/(\alpha \wedge 2)}}$$

(1.8)

$$\times \begin{cases} 1 + O\left(\left(1 - \dfrac{m}{m_c}\right)^{\epsilon}\right), & \alpha \neq 2, \\[3mm] \left(\log \dfrac{1}{\sqrt{1 - m/m_c}}\right)^{r/2} + O(1), & \alpha = 2 \end{cases}$$

for some $\epsilon > 0$ when $\alpha \neq 2$. The $O(1)$ term for $\alpha = 2$ is independent of m.

It is worth emphasizing that, although $C_{\mathrm{I}}, C_{\mathrm{II}}, m_c$ are model-dependent, the formula (1.8) itself is universal. Expanding (1.8) in powers of m and using (1.6), we obtain the following theorem.

THEOREM 1.2.   *Under the same condition as in Theorem 1.1, as $t \to \infty$,*

$$\frac{\sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x)}{\sum_{x \in \mathbb{Z}^d} \varphi_t(x)} \sim \frac{2\sin(r\pi/(\alpha \vee 2))}{(\alpha \wedge 2)\sin(r\pi/\alpha)} \frac{\Gamma(r+1)}{\Gamma(r/(\alpha \wedge 2) + 1)}$$

(1.9)

$$\times \begin{cases} (C_{\mathrm{II}} v_\alpha t)^{r/(\alpha \wedge 2)}, & \alpha \neq 2, \\[2mm] (C_{\mathrm{II}} v_2 t \log \sqrt{t})^{r/2}, & \alpha = 2. \end{cases}$$

We note that C_{II} is the only model-dependent term in (1.9). As far as we are aware, the sharp asymptotics (1.8) and (1.9) for *all real* $r \in (0, \alpha)$ are new, even for random walk.

Although we focus our attention on the long-range models defined by D that satisfies (1.2), our proof also applies to finite-range models, for which α is considered to be infinity.

Using $|x_1|^r \leq |x|^r \leq d^{r/2} \sum_{j=1}^{d} |x_j|^r$ and the $\mathbb{Z}^d$-symmetry of the models, we are finally able to arrive at the following result.

COROLLARY 1.3.   *Under the same condition as in Theorem 1.1, (1.5) holds for all $r \in (0, \alpha)$. In particular, when $r = 2 < \alpha$,*

(1.10)
$$\xi_t^{(2)} \underset{t \to \infty}{\sim} \sqrt{C_{\mathrm{II}} \sigma^2 t}.$$

As mentioned earlier, (1.5) has been proven [5] for self-avoiding walk, but only for small $r < \alpha \wedge 2$. The sharp asymptotics (1.10) has been proven [7] for self-avoiding walk and critical oriented percolation defined by D that has a finite $(2 +$

ϵ)th moment for some $\epsilon > 0$. Our proof is based on a different method than those used in [5, 7]. It is closer to the method, explained in the next subsection, used in [8] for finite-range self-avoiding walk and in [9] for critical/subcritical finite-range oriented percolation.

We strongly believe that the same method should work for lattice trees. Any two points in a lattice tree are connected by a unique path, so the number of bonds contained in that path can be considered as time and we can apply the current method to obtain the same results (with different values for C_I, C_{II}). As this suggests, time, or something equivalent, is important for the current method to work. For unoriented percolation, for example, it is not so clear what should be interpreted as time. However, if D is biased in average in one direction, say, the positive direction of the first coordinate, then x_1 can be treated as time and, after subtracting the effect of the bias, we may obtain the results even for unoriented percolation.

1.3. *Outline and notation.* In this subsection, we outline the proof of Theorem 1.1 and introduce some notation which is used in the rest of the paper. We also refer interested readers to an extended version of this subsection in [11].

One of the key elements for the proof is to represent the left-hand side of (1.8) in terms of the generating function (i.e., the Fourier–Laplace transform) of the two-point function. We now explain this representation.

Given a function $f_t(x)$, where $(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+$, we formally define

$$\hat{f}(k, m) = \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} f_t(x) e^{ik \cdot x} \qquad [k \in [-\pi, \pi]^d, \; m \geq 0].$$

We note that $\hat{\varphi}(k, m)$ is well defined when $m < m_c$ (recall that $m_c \geq 1$, as explained at the beginning of Section 1.2). Let

$$(1.11) \qquad \nabla_1^n \hat{f}(l, m) = \left. \frac{\partial^n \hat{f}(k, m)}{\partial k_1^n} \right|_{k=l} \qquad [l \in [-\pi, \pi]^d, \; n \in \mathbb{Z}_+].$$

Then, for $r = 2j < \alpha$ ($j \in \mathbb{N}$), we obtain the representation

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} x_1^{2j} f_t(x) = (-1)^j \nabla_1^{2j} \hat{f}(0, m).$$

For $r \in (0, \alpha \wedge 2)$, we generate the factor $|x_1|^r$ by using the constant $K_r \in (0, \infty)$, as follows (see [3]):

$$(1.12) \qquad K_r \equiv \int_0^{\infty} \frac{1 - \cos v}{v^{1+r}} \, dv = |x_1|^{-r} \int_0^{\infty} \frac{1 - \cos(ux_1)}{u^{1+r}} \, du.$$

Suppose, from now on, that f_t is $\mathbb{Z}^d$-symmetric. Then,

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r f_t(x) = \frac{1}{K_r} \int_0^{\infty} \frac{du}{u^{1+r}} \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} \left(1 - \cos(ux_1)\right) f_t(x)$$

$$= \frac{1}{K_r} \int_0^{\infty} \frac{du}{u^{1+r}} (\hat{f}(0, m) - \hat{f}(\vec{u}, m)),$$

where $\vec{u} = (u, 0, \ldots, 0) \in \mathbb{R}^d$. Let

(1.13)
$$\bar{\Delta}_l \hat{f}(k, m) \equiv \hat{f}(k, m) - \frac{\hat{f}(k+l, m) + \hat{f}(k-l, m)}{2}$$
$$= \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} (1 - \cos(l \cdot x)) f_t(x) e^{ik \cdot x}.$$

We note that $\bar{\Delta}_l \hat{f}(k, m)$ is equivalent to $\frac{-1}{2}\Delta_l \hat{f}(k, m)$ in the previous papers (e.g., [2, 3]). In particular,

$$\bar{\Delta}_l \hat{f}(0, m) = \hat{f}(0, m) - \hat{f}(l, m).$$

Therefore, for $r \in (0, \alpha \wedge 2)$,

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r f_t(x) = \frac{1}{K_r} \int_0^{\infty} \frac{\mathrm{d}u}{u^{1+r}} \bar{\Delta}_{\vec{u}} \hat{f}(0, m).$$

For $r = 2j + q < \alpha$ [$j \in \mathbb{N}, q \in (0, 2)$], we combine the above representations as

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{2j+q} f_t(x) = \frac{1}{K_q} \int_0^{\infty} \frac{\mathrm{d}u}{u^{1+q}} \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} (1 - \cos(ux_1)) x_1^{2j} f_t(x)$$
$$= \frac{(-1)^j}{K_q} \int_0^{\infty} \frac{\mathrm{d}u}{u^{1+q}} (\nabla_1^{2j} \hat{f}(0, m) - \nabla_1^{2j} \hat{f}(\vec{u}, m))$$
$$= \frac{(-1)^j}{K_q} \int_0^{\infty} \frac{\mathrm{d}u}{u^{1+q}} \bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{f}(0, m).$$

From now on, as long as no confusion arises, we will simply omit m and abbreviate $\hat{f}(k, m)$ to $\hat{f}(k)$. Then, the above three representations are summarized as

(1.14)
$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r f_t(x)$$
$$= \begin{cases} (-1)^j \nabla_1^{2j} \hat{f}(0) & [r = 2j < \alpha, \ j \in \mathbb{N}], \\ \dfrac{(-1)^j}{K_q} \displaystyle\int_0^{\infty} \dfrac{\mathrm{d}u}{u^{1+q}} \bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{f}(0) \\ \qquad [r = 2j + q < \alpha, j \in \mathbb{Z}_+, q \in (0, 2)]. \end{cases}$$

Also, we will abbreviate $\hat{f}(k, m_{\mathrm{c}})$ to $\hat{f}_{\mathrm{c}}(k)$ whenever it is well defined. Moreover, we will use the notation

$$\partial_m \hat{f}_{\mathrm{c}}(k) = \left. \frac{\partial \hat{f}(k, m)}{\partial m} \right|_{m \uparrow m_{\mathrm{c}}}.$$

Another key element for the proof of the main theorem is the lace expansion (see, e.g., [12], Sections 3 and 13),

$$\varphi_t(x) = I_t(x) + \sum_{s=1}^{t} (J_s * \varphi_{t-s})(x), \tag{1.15}$$

where, for $t \geq 0$,

$$I_t(x) = \begin{cases} \delta_{x,o}\delta_{t,0}, & \text{RW/SAW}, \\ \pi_t^{OP}(x), & \text{OP}, \end{cases} \tag{1.16}$$

and for $t \geq 1$,

$$J_t(x) = \begin{cases} D(x)\delta_{t,1}, & \text{RW}, \\ D(x)\delta_{t,1} + \pi_t^{SAW}(x), & \text{SAW}, \\ (\pi_{t-1}^{OP} * pD)(x), & \text{OP}. \end{cases} \tag{1.17}$$

Recall (1.1) for random walk. For self-avoiding walk and oriented percolation, $\pi_t^{SAW}(x)$ and $\pi_t^{OP}(x)$ are (alternating sums of) the model-dependent lace expansion coefficients (see, e.g., [12] for their precise definitions). By (1.15), we obtain

$$\hat{\varphi}(k) = \hat{I}(k) + \hat{J}(k)\hat{\varphi}(k). \tag{1.18}$$

From this, we can derive identities for the "derivatives" of $\hat{\varphi}$ in (1.14). For example,

$$\begin{aligned}
\bar{\Delta}_{\vec{u}}\hat{\varphi}(0) &\equiv \hat{\varphi}(0) - \hat{\varphi}(\vec{u}) = \hat{I}(0) + \hat{J}(0)\hat{\varphi}(0) - \big(\hat{I}(\vec{u}) + \hat{J}(\vec{u})\hat{\varphi}(\vec{u})\big) \\
&= \bar{\Delta}_{\vec{u}}\hat{I}(0) + \hat{J}(0)\hat{\varphi}(0) - \hat{J}(\vec{u})\hat{\varphi}(\vec{u}) \\
&= \bar{\Delta}_{\vec{u}}\hat{I}(0) + \hat{\varphi}(0)\bar{\Delta}_{\vec{u}}\hat{J}(0) + \hat{J}(\vec{u})\bar{\Delta}_{\vec{u}}\hat{\varphi}(0) \\
&= \frac{1}{1 - \hat{J}(\vec{u})}\big(\bar{\Delta}_{\vec{u}}\hat{I}(0) + \hat{\varphi}(0)\bar{\Delta}_{\vec{u}}\hat{J}(0)\big),
\end{aligned} \tag{1.19}$$

where the last line has been obtained by solving the previous equation for $\bar{\Delta}_{\vec{u}}\hat{\varphi}(0)$. Hence, for $r \in (0, \alpha \wedge 2)$,

$$\begin{aligned}
\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x) &= \frac{\hat{\varphi}(0)}{K_r} \int_0^{\infty} \frac{du}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}}\hat{J}(0)}{1 - \hat{J}(\vec{u})} \\
&\quad + \frac{1}{K_r} \int_0^{\infty} \frac{du}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}}\hat{I}(0)}{1 - \hat{J}(\vec{u})}.
\end{aligned} \tag{1.20}$$

It is known [2, 6] that as long as $d > d_c$ (and $L \gg 1$), it is easier to tame $\hat{I}$ and $\hat{J}$, up to $m = m_c$, than to tame $\hat{\varphi}$. We will thus be able to analyze the integrals on the right-hand side of (1.20) and prove the main theorem.

Before closing this subsection, we provide the following representations for the constants C_I and C_{II} in (1.8) in terms of $\hat{I}_c$ and $\hat{J}_c$:

$$C_I = \frac{\hat{I}_c(0)}{m_c\,\partial_m \hat{J}_c(0)}, \qquad C_{II} = \frac{1}{m_c\,\partial_m \hat{J}_c(0)} \lim_{k \to 0} \frac{\bar{\Delta}_k \hat{J}_c(0)}{\bar{\Delta}_k \hat{D}(0)}. \tag{1.21}$$

In Section 2, we will explain the heuristics for the derivation of these representations.

1.4. *Organization.* In the remainder of the paper, whenever we consider self-avoiding walk and oriented percolation, we assume $d > d_c$ and $L \gg 1$, as well as $p \leq p_c$ for oriented percolation.

The paper is organized as follows. In Section 3, we prove Theorem 1.1 for $r \in (0, \alpha \wedge 2)$ (Section 3.1), for $r = 2j < \alpha$ with $j \in \mathbb{N}$ (Section 3.2) and for $r = 2j + q < \alpha$ with $j \in \mathbb{N}$ and $q \in (0, 2)$ (Section 3.3) separately, assuming Propositions 3.1 and 3.2. We prove those key propositions in Section 4.

We strongly believe that the results for self-avoiding walk and oriented percolation are the most important and interesting parts of this work. However, for those who are more interested in random walk, we make the following suggestion: read up to Section 3 for the proof of Theorem 1.1, where Propositions 3.1 and 3.2 are used. However, Proposition 3.1 and a part [i.e., (3.15)] of Proposition 3.2 are trivial for random walk. The remaining part [i.e., (3.16)] of Proposition 3.2 is the result of Lemma 4.1, which is proved in Section 4.1.

2. Preliminaries. In this section, we review in outline the derivation in [2, 3, 5] of the constants C_{I} and C_{II}. During the course of this, we summarize the already known properties of $\hat{I}$ and $\hat{J}$ and introduce some quantities used in the following sections.

We begin by solving (1.18) for $\hat{\varphi}(k)$, which yields

$$(2.1) \qquad \hat{\varphi}(k) = \frac{\hat{I}(k)}{1 - \hat{J}(k)},$$

where, by (1.16) and (1.17),

$$(2.2) \qquad \hat{I}(k) = \begin{cases} 1, & \text{RW/SAW}, \\ \hat{\pi}^{\mathrm{OP}}(k), & \text{OP}, \end{cases}$$

$$(2.3) \qquad \hat{J}(k) = \begin{cases} m\hat{D}(k), & \text{RW}, \\ m\hat{D}(k) + \hat{\pi}^{\mathrm{SAW}}(k), & \text{SAW}, \\ \hat{\pi}^{\mathrm{OP}}(k)\, pm\hat{D}(k), & \text{OP}. \end{cases}$$

It is known [2, 6] that

$$(2.4) \qquad \hat{\pi}^{\mathrm{SAW}}(k) = O(L^{-d})m^2, \qquad \hat{\pi}^{\mathrm{OP}}(k) - 1 = O(L^{-d})(pm)^2,$$

where the $O(L^{-d})$ terms are uniform in $k \in [-\pi, \pi]^d$ and $m \leq m_c$. Therefore, $\hat{I}(k)$ and $\hat{J}(k)$ are both convergent for all $k \in [-\pi, \pi]^d$ and $m \leq m_c$. However, since $\hat{\varphi}(0)$ diverges as $m \uparrow m_c$, we can characterize m_c by the equation

$$(2.5) \qquad 1 = \hat{J}_c(0) = \begin{cases} m_c, & \text{RW}, \\ m_c + \hat{\pi}_c^{\mathrm{SAW}}(0), & \text{SAW}, \\ \hat{\pi}_c^{\mathrm{OP}}(0)\, pm_c, & \text{OP}. \end{cases}$$

Using this identity, we obtain that, as $m \uparrow m_c$ (see [2, 5] for the precise argument),

$$\hat{\varphi}(k) = \frac{\hat{I}(k)}{\hat{J}_c(0) - \hat{J}_c(k) + m_c((\hat{J}_c(k) - \hat{J}(k))/(m_c - m))(1 - m/m_c)}$$

$$\sim \frac{\hat{I}_c(k)}{\bar{\Delta}_k \hat{J}_c(0) + m_c \, \partial_m \hat{J}_c(k)(1 - m/m_c)}$$

$$= \frac{\hat{I}_c(k)}{\bar{\Delta}_k \hat{J}_c(0) + m_c \, \partial_m \hat{J}_c(k)} \sum_{t=0}^{\infty} \left(\frac{m_c \, \partial_m \hat{J}_c(k)}{\bar{\Delta}_k \hat{J}_c(0) + m_c \, \partial_m \hat{J}_c(k)} \frac{m}{m_c} \right)^t,$$

hence,

$$(2.6) \qquad \sum_{x \in \mathbb{Z}^d} \varphi_t(x) e^{ik \cdot x} \underset{t \to \infty}{\sim} \frac{\hat{I}_c(k)}{\bar{\Delta}_k \hat{J}_c(0) + m_c \, \partial_m \hat{J}_c(k)} m_c^{-t}$$

$$\times \left(1 - \frac{\bar{\Delta}_k \hat{J}_c(0)}{\bar{\Delta}_k \hat{J}_c(0) + m_c \, \partial_m \hat{J}_c(k)} \right)^t.$$

In particular, at $k = 0$,

$$(2.7) \qquad \sum_{x \in \mathbb{Z}^d} \varphi_t(x) \sim \frac{\hat{I}_c(0)}{m_c \, \partial_m \hat{J}_c(0)} m_c^{-t},$$

which yields the representation for C_I in (1.21).

In the above computation, we have used the fact that the quantities such as $m_c \, \partial_m \hat{J}_c(0)$ and $\bar{\Delta}_k \hat{J}_c(0)$ are all convergent uniformly in $k \in [-\pi, \pi]^d$. To see this, we note that, by (2.3),

$$(2.8) \qquad m_c \partial_m \hat{J}_c(k) = \begin{cases} m_c \hat{D}(k), & \text{RW}, \\ m_c \hat{D}(k) + m_c \, \partial_m \hat{\pi}_c^{\text{SAW}}(k), & \text{SAW}, \\ (\hat{\pi}_c^{\text{OP}}(k) + m_c \, \partial_m \hat{\pi}_c^{\text{OP}}(k)) p m_c \hat{D}(k), & \text{OP}, \end{cases}$$

$$(2.9) \qquad \bar{\Delta}_k \hat{J}(0) = \begin{cases} m \bar{\Delta}_k \hat{D}(0), & \text{RW}, \\ m \bar{\Delta}_k \hat{D}(0) + \bar{\Delta}_k \hat{\pi}^{\text{SAW}}(0), & \text{SAW}, \\ (\hat{\pi}^{\text{OP}}(0) \bar{\Delta}_k \hat{D}(0) + \hat{D}(k) \bar{\Delta}_k \hat{\pi}^{\text{OP}}(0)) p m, & \text{OP}. \end{cases}$$

However, it is known that π^{SAW} and π^{OP} both satisfy

$$(2.10) \qquad |m_c \, \partial_m \hat{\pi}_c(k)| \leq \sum_{t=0}^{\infty} t m_c^t \sum_{x \in \mathbb{Z}^d} |\pi_t(x)| \leq O(L^{-d}),$$

$$(2.11) \qquad |\bar{\Delta}_k \hat{\pi}(0)| \leq \sum_{t=0}^{\infty} m_c^t \sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) |\pi_t(x)| \leq O(L^{-d}) \bar{\Delta}_k \hat{D}(0)$$

for all $k \in [-\pi, \pi]^d$ and $m \leq m_c$ for the latter (see [3], Proposition 1, [5], the paragraph below Theorem 1.2 and [6], Proposition 4.1, with an improvement due

to monotone convergence). By these bounds and using (2.4) and (2.5) and the fact that m_c^{SAW} and pm_c^{OP} are both $1 + O(L^{-d})$ (see the beginning of Section 1.2), we conclude that $m_c \, \partial_m \hat{J}_c(0) = 1 + O(L^{-d})$ and $\bar{\Delta}_k \hat{J}_c(0) = O(\bar{\Delta}_k \hat{D}(0))$.

Moreover, it has been proven [2, 3, 5] that there exist $\epsilon = \epsilon(d, \alpha) > 0$ and $\delta = \delta(d, \alpha)$, which is zero if $\alpha = 2$ and > 0 if $\alpha \neq 2$, such that π^{SAW} and π^{OP} both satisfy

$$\sum_{t=0}^{\infty} t^{1+\epsilon} m_c^t \sum_{x \in \mathbb{Z}^d} |\pi_t(x)| < \infty, \qquad \sum_{t=0}^{\infty} m_c^t \sum_{x \in \mathbb{Z}^d} |x|^{\alpha \wedge 2 + \delta} |\pi_t(x)| < \infty.$$

These bounds imply (see [2], equations (6.13) and (6.14), [3], equations (3.3)–(3.4), [5], equations (2.25)–(2.28) and (2.64)–(2.70))

$$(2.12) \qquad \frac{\hat{J}_c(0) - \hat{J}(0)}{1 - m/m_c} = m_c \, \partial_m \hat{J}_c(0) + O\left(\left(1 - \frac{m}{m_c}\right)^{\epsilon}\right),$$

$$(2.13) \qquad \frac{\bar{\Delta}_k \hat{J}(0)}{\bar{\Delta}_k \hat{D}(0)} = M + \begin{cases} O(|k|^{\delta}), & \alpha \neq 2, \\ O\left(1/\log \dfrac{1}{|k|}\right), & \alpha = 2, \end{cases}$$

where the error terms in (2.13), which are zero for random walk, are uniform in $m \leq m_c$ and where $M \equiv M(m)$ is defined as

$$(2.14) \qquad M = \begin{cases} m, & \text{RW}, \\ m + \dfrac{\nabla_1^2 \hat{\pi}^{\text{SAW}}(0)}{-2v_\alpha} \mathbb{1}_{\{\alpha > 2\}}, & \text{SAW}, \\ \left(\hat{\pi}^{\text{OP}}(0) + \dfrac{\nabla_1^2 \hat{\pi}^{\text{OP}}(0)}{-2v_\alpha} \mathbb{1}_{\{\alpha > 2\}}\right) pm, & \text{OP}. \end{cases}$$

The crossover terms, which are proportional to $\mathbb{1}_{\{\alpha > 2\}}$, converge for all $m \leq m_c$ [3, 5]. By (2.6) and (2.7) and (2.13), and using $\lim_{t \to \infty} t \bar{\Delta}_{k_t} \hat{D}(0) = |k|^{\alpha \wedge 2}$, due to the scaling (1.7), we obtain that, as $t \to \infty$,

$$\frac{\sum_{x \in \mathbb{Z}^d} \varphi_t(x) e^{ik_t \cdot x}}{\sum_{x \in \mathbb{Z}^d} \varphi_t(x)}$$

$$\sim \left(1 - \frac{\bar{\Delta}_{k_t} \hat{J}_c(0)}{\bar{\Delta}_{k_t} \hat{J}_c(0) + m_c \, \partial_m \hat{J}_c(k_t)}\right)^t \qquad [\because (2.6)_{k=k_t}/(2.7)]$$

$$\sim \exp\left(-\frac{\bar{\Delta}_{k_t} \hat{J}_c(0)}{m_c \, \partial_m \hat{J}_c(0)} t\right) \qquad [\because 1 - \tau \sim e^{-\tau} \text{ as } \tau \to 0]$$

$$= \exp\left(-\frac{1}{m_c \, \partial_m \hat{J}_c(0)} \frac{\bar{\Delta}_{k_t} \hat{J}_c(0)}{\bar{\Delta}_{k_t} \hat{D}(0)} t \bar{\Delta}_{k_t} \hat{D}(0)\right)$$

$$\sim \exp\left(-\frac{M_c}{m_c \, \partial_m \hat{J}_c(0)} |k|^{\alpha \wedge 2}\right),$$

where $M_c = M(m_c)$. This yields the representation for C_{II} in (1.21).

REMARK. It is natural for some readers to wonder why we do not directly prove (1.9) by using the formula (2.6) for $\sum_x \varphi_t(x) e^{ik \cdot x}$, instead of proving the asymptotics (1.8) of its generating function and expanding it in powers of m. In fact, the first-named author was able to derive an asymptotic expression for $\sum_x |x_1|^r \varphi_t(x)$ using (2.6), but the proportionality constant was in a rather complicated sum form. We then concluded that using (2.6) would not be an ideal method for deriving the simplest possible display of the proportionality constant and started searching for another method. That turns out to be the use of the generating function, as explained in this paper. Later, the first-named author proved that the aforementioned sum form is indeed an expansion of the proportionality constant in (1.9).

3. Proof of the main results.

3.1. *Proof of Theorem* 1.1 *for* $r \in (0, \alpha \wedge 2)$. In this subsection, we prove Theorem 1.1 for $r \in (0, \alpha \wedge 2)$. We will discuss the case for $\alpha \neq 2$ and that for $\alpha = 2$ simultaneously, until we arrive at the point where we require separate approaches.

First, we recall (1.14) and split $\int_0^\infty$ into $\int_0^U$ and $\int_U^\infty$ for a given $U > 0$. Using (1.19) for the former integral [as in (1.20)] and (1.13) for the latter, we obtain

$$
\sum_{t=0}^\infty m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x)
$$

$$
(3.1) \qquad = \frac{\hat{\varphi}(0)}{K_r} \int_0^U \frac{du}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{1 - \hat{J}(\vec{u})} + \frac{1}{K_r} \int_0^U \frac{du}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{I}(0)}{1 - \hat{J}(\vec{u})}
$$

$$
+ \frac{1}{K_r} \int_U^\infty \frac{du}{u^{1+r}} \sum_{t=0}^\infty m^t \sum_{x \in \mathbb{Z}^d} (1 - \cos(u x_1)) \varphi_t(x).
$$

We note that, by (2.2) and (2.11), $\bar{\Delta}_k \hat{I}(0) \equiv 0$ for random walk and self-avoiding walk and $\bar{\Delta}_k \hat{I}(0) = O(\bar{\Delta}_k \hat{D}(0))$ uniformly in $m \leq m_c$ for oriented percolation. Since $\bar{\Delta}_k \hat{J}(0)$ is also $O(\bar{\Delta}_k \hat{D}(0))$ uniformly in $m \leq m_c$ [see (2.9)], the integrals in the first two terms of (3.1) are of the same order and therefore the first term dominates the second term as $m \uparrow m_c$, due to the extra factor $\hat{\varphi}(0)$, which exhibits

$$
(3.2) \qquad \begin{aligned} \hat{\varphi}(0) &= \frac{\hat{I}(0)}{\hat{J}_c(0) - \hat{J}(0)} = \frac{\hat{I}_c(0) + O(1 - m/m_c)}{m_c \, \partial_m \hat{J}_c(0)(1 - m/m_c) + O((1 - m/m_c)^{1+\epsilon})} \\ &= \frac{C_{\text{I}}}{1 - m/m_c} + O\left(\left(1 - \frac{m}{m_c}\right)^{-1+\epsilon}\right), \end{aligned}
$$

where the first equality is due to (2.1) and (2.5), and the second equality is due to (2.10) and (2.12). These estimates are valid independently of r and thus used in

the later sections as well. By the fact that $0 \leq 1 - \cos(ux_1) \leq 2$, the last term in (3.1) obeys

(3.3)
$$
0 \leq \frac{1}{K_r} \int_U^\infty \frac{\mathrm{d}u}{u^{1+r}} \sum_{t=0}^\infty m^t \sum_{x \in \mathbb{Z}^d} (1 - \cos(ux_1)) \varphi_t(x)
$$
$$
\leq \frac{2\hat{\varphi}(0)}{K_r} \int_U^\infty \frac{\mathrm{d}u}{u^{1+r}} = \frac{2\hat{\varphi}(0)}{K_r r} U^{-r}.
$$

We will choose U to be relatively small so as to make the first term in (3.1) dominant.

Next, we investigate the integral part of the first term in (3.1),

(3.4)
$$
\int_0^U \frac{\mathrm{d}u}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{1 - \hat{J}(\vec{u})} = \int_0^U \frac{\mathrm{d}u}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{\hat{J}_{\mathrm{c}}(0) - \hat{J}(0) + \bar{\Delta}_{\vec{u}} \hat{J}(0)},
$$

where we have used (2.5). By (1.2) and (2.13), we have that, for small u,

$$
\bar{\Delta}_{\vec{u}} \hat{J}(0) = \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{\bar{\Delta}_{\vec{u}} \hat{D}(0)} \bar{\Delta}_{\vec{u}} \hat{D}(0) = \begin{cases} M v_\alpha u^{\alpha \wedge 2} + O(u^{\alpha \wedge 2 + \epsilon}), & \alpha \neq 2, \\ M v_2 u^2 \log \dfrac{1}{u} + O(u^2), & \alpha = 2 \end{cases}
$$

for some $\epsilon > 0$, where the error terms are uniform in $m \leq m_{\mathrm{c}}$. Let

(3.5)
$$
\mu = \frac{\hat{J}_{\mathrm{c}}(0) - \hat{J}(0)}{M v_\alpha}.
$$

Then,

(3.6)
$$
\frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{\hat{J}_{\mathrm{c}}(0) - \hat{J}(0) + \bar{\Delta}_{\vec{u}} \hat{J}(0)}
$$
$$
= \begin{cases} \dfrac{u^{\alpha \wedge 2}}{\mu + u^{\alpha \wedge 2}} + \dfrac{O(u^{\alpha \wedge 2 + \epsilon})}{\mu + u^{\alpha \wedge 2}}, & \alpha \neq 2, \\ \dfrac{u^2 \log 1/u}{\mu + u^2 \log 1/u} + \dfrac{O(u^2)}{\mu + u^2 \log 1/u}, & \alpha = 2. \end{cases}
$$

We now investigate the integral (3.4) for $\alpha \neq 2$ and $\alpha = 2$ separately, using (3.6) and the following proposition.

PROPOSITION 3.1.   *Under the same conditions as in Theorem* 1.1,

(3.7)
$$
M = M_{\mathrm{c}} + O\left(\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^\epsilon\right),
$$

(3.8)
$$
\mu = \frac{1 - m/m_{\mathrm{c}}}{C_{\mathrm{II}} v_\alpha} + O\left(\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^{1+\epsilon}\right)
$$

for some $\epsilon > 0$, where $M_{\mathrm{c}} = M(m_{\mathrm{c}})$.

The proof is deferred to Section 4. We note that these estimates are trivial for random walk.

3.1.1. *Proof for* $\alpha \neq 2$. We assume that $\epsilon < r$, without loss of generality. By (3.4) and (3.6) for $\alpha \neq 2$, we have that, for small U,

$$
\int_0^U \frac{\mathrm{d}u}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{1 - \hat{J}(\vec{u})} = \int_0^U \frac{\mathrm{d}u}{u} \left(\frac{u^{\alpha \wedge 2 - r}}{\mu + u^{\alpha \wedge 2}} + \frac{O(u^{\alpha \wedge 2 - r + \epsilon})}{\mu + u^{\alpha \wedge 2}} \right)
$$

$$
= \int_0^\infty \frac{\mathrm{d}u}{u} \frac{u^{\alpha \wedge 2 - r}}{\mu + u^{\alpha \wedge 2}} - \int_U^\infty \frac{\mathrm{d}u}{u} \frac{u^{\alpha \wedge 2 - r}}{\mu + u^{\alpha \wedge 2}}
$$

$$
+ \int_0^U \frac{\mathrm{d}u}{u} \frac{O(u^{\alpha \wedge 2 + \epsilon - r})}{\mu + u^{\alpha \wedge 2}} \left(\mathbb{1}_{\{\mu \geq u^{\alpha \wedge 2}\}} + \mathbb{1}_{\{\mu < u^{\alpha \wedge 2}\}} \right)
$$

$$
= \int_0^\infty \frac{\mathrm{d}u}{u} \frac{u^{\alpha \wedge 2 - r}}{\mu + u^{\alpha \wedge 2}} + O(U^{-r}) + O\left(\mu^{-(r-\epsilon)/(\alpha \wedge 2)} \right).
$$

Let $U = \mu^{(1-\epsilon/r)/(\alpha \wedge 2)}$, which is indeed small as $m \uparrow m_c$, due to Proposition 3.1. By the change of variables $u^{\alpha \wedge 2} = \mu z$, we obtain

$$
\int_0^{\mu^{(1-\epsilon/r)/(\alpha \wedge 2)}} \frac{\mathrm{d}u}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{1 - \hat{J}(\vec{u})}
$$

$$
(3.9) \qquad = \int_0^\infty \frac{\mathrm{d}u}{u} \frac{u^{\alpha \wedge 2 - r}}{\mu + u^{\alpha \wedge 2}} + O\left(\mu^{-(r-\epsilon)/(\alpha \wedge 2)} \right)
$$

$$
= \frac{\mu^{-r/(\alpha \wedge 2)}}{\alpha \wedge 2} \int_0^\infty \frac{\mathrm{d}z}{z} \frac{z^{1 - r/(\alpha \wedge 2)}}{1 + z} + O\left(\mu^{-(r-\epsilon)/(\alpha \wedge 2)} \right).
$$

However, by the standard Cauchy integral formula, for $\beta \in (0, 1)$,

$$
(3.10) \qquad \oint_{\gamma_1} \frac{\mathrm{d}z}{z} \frac{z^{1-\beta}}{1+z} = \oint_{\gamma_2} \frac{\mathrm{d}z}{z} \frac{z^{1-\beta}}{1+z} = 2\pi i (-1)^{-\beta} = 2\pi i e^{-\pi i \beta},
$$

where, as depicted in Figure 1, the contour γ_1 consists of two line segments, an arc of the circle with smaller radius $\delta \in (0, 1)$ and an arc of the circle with larger radius $R \in (1, \infty)$, and the contour γ_2 is the circle centered at -1 with radius smaller than 1. On the other hand, by taking $\delta \to 0$ and $R \to \infty$, we obtain
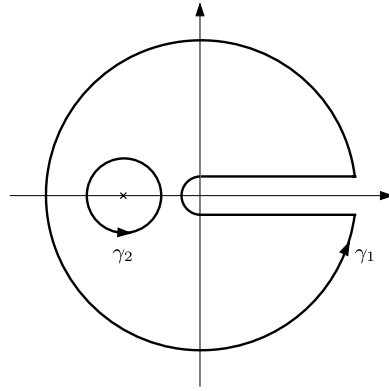
$$
\lim_{\substack{R \to \infty \\ \delta \to 0}} \oint_{\gamma_1} \frac{\mathrm{d}z}{z} \frac{z^{1-\beta}}{1+z} = (1 - e^{-2\pi i \beta}) \int_0^\infty \frac{\mathrm{d}z}{z} \frac{z^{1-\beta}}{1+z}.
$$

Therefore,

$$
\int_0^\infty \frac{\mathrm{d}z}{z} \frac{z^{1-\beta}}{1+z} = \frac{2\pi i e^{-\pi i \beta}}{1 - e^{-2\pi i \beta}} = \frac{\pi}{\sin(\beta \pi)},
$$

which implies that

$$
\int_0^{\mu^{(1-\epsilon/r)/(\alpha \wedge 2)}} \frac{\mathrm{d}u}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{1 - \hat{J}(\vec{u})} = \frac{\pi}{(\alpha \wedge 2) \sin(r\pi/(\alpha \wedge 2))} \mu^{-r/(\alpha \wedge 2)}
$$

$$
+ O\left(\mu^{-(r-\epsilon)/(\alpha \wedge 2)} \right).
$$

FIG. 1.   *The contours γ_1 and γ_2 in the complex plane.*

Finally, by substituting (3.9) back into (3.1) and using (3.2) and (3.8), we conclude that there is an $\epsilon' \in (0, 1)$ such that

$$
\begin{aligned}
\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x) &= \frac{\pi K_r^{-1}}{(\alpha \wedge 2) \sin(r\pi/(\alpha \wedge 2))} \frac{C_{\mathrm{I}}(C_{\mathrm{II}} v_\alpha)^{r/(\alpha \wedge 2)}}{(1 - m/m_{\mathrm{c}})^{1+r/(\alpha \wedge 2)}} \\
&\quad + O\left(\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^{-1-r/(\alpha \wedge 2)+\epsilon'}\right).
\end{aligned}
$$

(3.11)

However, since (see Appendix A.2)

$$
\text{(3.12)} \qquad\qquad \pi K_r^{-1} = 2\Gamma(r+1) \sin\frac{r\pi}{2},
$$

this completes the proof of Theorem 1.1 for $0 < r < \alpha \wedge 2$ with $\alpha \neq 2$.

REMARK.   Although the proportionality constant $(2 \sin\frac{r\pi}{\alpha \vee 2})/((\alpha \wedge 2) \sin\frac{r\pi}{\alpha})$ in (1.8) looks slightly different from the constant $(2 \sin\frac{r\pi}{2})/((\alpha \wedge 2) \sin\frac{r\pi}{\alpha \wedge 2})$ derived from (3.11) and (3.12), they are equal when $0 < r < \alpha \wedge 2$. The reason why we have adopted the former in the main theorem is due to its applicability to larger values of $r < \alpha$, which the latter lacks (e.g., take $r = 3 < \alpha$).

3.1.2. *Proof for $\alpha = 2$.* The proof for $\alpha = 2$ is slightly more involved than the above proof for $\alpha \neq 2$, due to the log corrections in (3.6). By (3.4) and (3.6) for $\alpha = 2$, we have that, for small U,

$$
\begin{aligned}
\int_0^U \frac{\mathrm{d}u}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{1 - \hat{J}(\vec{u})} &= \int_0^U \frac{\mathrm{d}u}{u} \left(\frac{u^{2-r} \log 1/u}{\mu + u^2 \log 1/u} + \frac{O(u^{2-r})}{\mu + u^2 \log 1/u} \right) \\
&= \int_0^U \frac{\mathrm{d}u}{u} \frac{u^{2-r} \log 1/u}{\mu + u^2 \log 1/u} + \frac{O(U^{2-r})}{\mu},
\end{aligned}
$$

where we have obtained the error term by simply ignoring $u^2 \log \frac{1}{u} > 0$ in the denominator. Let $U = \sqrt{\mu}$, which is small as $m \uparrow m_c$, as required, due to Proposition 3.1. By the change of variables $u^2 \log \frac{1}{u} = \mu z$, we obtain

$$
\int_0^{\sqrt{\mu}} \frac{du}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{1 - \hat{J}(\vec{u})} = \int_0^{\sqrt{\mu}} \frac{du}{u} \frac{u^{2-r} \log 1/u}{\mu + u^2 \log 1/u} + O(\mu^{-r/2})
$$

$$
= \frac{\mu^{-r/2}}{2} \int_0^{\log 1/\sqrt{\mu}} \frac{dz}{z} \frac{z^{1-r/2}(\log 1/u(z))^{r/2}}{1 + z} + O(\mu^{-r/2}).
$$

Note that, by taking the logarithm of $u^2 \log 1/u = \mu z$ and using the monotonicity of $(\log \log 1/u)/\log 1/u$ in $0 < u < \sqrt{\mu} \ll 1$, we have

$$
\log \frac{1}{u(z)} = \left(1 + O\left(\frac{\log \log 1/\sqrt{\mu}}{\log 1/\mu}\right)\right) \log \frac{1}{\sqrt{\mu z}}.
$$

Therefore,

$$
\int_0^{\sqrt{\mu}} \frac{du}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}} \hat{J}(0)}{1 - \hat{J}(\vec{u})}
$$

$$
= \frac{\mu^{-r/2}}{2} \left(1 + O\left(\frac{\log \log 1/\sqrt{\mu}}{\log 1/\mu}\right)\right) \int_0^{\log 1/\sqrt{\mu}} \frac{dz}{z} \frac{z^{1-r/2}(\log 1/\sqrt{\mu z})^{r/2}}{1 + z}
$$

$$
+ O(\mu^{-r/2}).
$$

Suppose that $\log \frac{1}{\sqrt{\mu}} \gg 1$. Then, by the Cauchy integral formula (see Figure 1),

$$
\oint_{\gamma_1} \frac{dz}{z} \frac{z^{1-r/2}(\log 1/\sqrt{\mu z})^{r/2}}{1 + z} = \oint_{\gamma_2} \frac{dz}{z} \frac{z^{1-r/2}(\log 1/\sqrt{\mu z})^{r/2}}{1 + z}
$$

$$
= 2\pi i e^{-\pi i r/2} \left(\log \frac{1}{\sqrt{\mu}} - \frac{\pi i}{2}\right)^{r/2}
$$

$$
= 2\pi i e^{-\pi i r/2} \left(\log \frac{1}{\sqrt{\mu}}\right)^{r/2} + O(1),
$$

where, as in (3.10), the contour γ_2 is the circle at -1 with radius smaller than 1, while the contour γ_1 contains an arc of the circle with radius $\delta \in (0, 1)$ and an arc of the circle with radius $R \equiv \log \frac{1}{\sqrt{\mu}}$. On the other hand, by taking $\delta \to 0$, we obtain

$$
\lim_{\delta \to 0} \oint_{\gamma_1} \frac{dz}{z} \frac{z^{1-r/2}(\log 1/\sqrt{\mu z})^{r/2}}{1 + z}
$$

$$
= (1 - e^{-\pi i r}) \int_0^{\log 1/\sqrt{\mu}} \frac{dz}{z} \frac{z^{1-r/2}(\log 1/\sqrt{\mu z})^{r/2}}{1 + z} + O(1),
$$

where the error term is independent of μ. Therefore,

$$\int_0^{\log 1/\sqrt{\mu}} \frac{dz}{z} \frac{z^{1-r/2}(\log 1/\sqrt{\mu z})^{r/2}}{1+z} = \frac{2\pi i e^{-\pi i r/2}}{1 - e^{-\pi i r}}\left(\log \frac{1}{\sqrt{\mu}}\right)^{r/2} + O(1)$$

$$= \frac{\pi}{\sin(r\pi/2)}\left(\log \frac{1}{\sqrt{\mu}}\right)^{r/2} + O(1),$$

which implies that

$$(3.13) \quad \int_0^{\sqrt{\mu}} \frac{du}{u^{1+r}} \frac{\bar{\Delta}_{\vec{u}}\hat{J}(0)}{1 - \hat{J}(\vec{u})} = \frac{\pi}{2\sin(r\pi/2)}\mu^{-r/2}\left(\log \frac{1}{\sqrt{\mu}}\right)^{r/2} + O(\mu^{-r/2}),$$

where we have used

$$O\left(\frac{\log\log 1/\sqrt{\mu}}{\log 1/\mu}\right)\left(\log \frac{1}{\sqrt{\mu}}\right)^{r/2} = o(1) \qquad [\because r < 2].$$

Finally, by substituting (3.13) back into (3.1) and using (3.3) with $U = \sqrt{\mu}$, we obtain

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x) = (\hat{\varphi}(0) + O(1))\frac{\pi K_r^{-1}}{2\sin(r\pi/2)}\mu^{-r/2}\left(\log \frac{1}{\sqrt{\mu}}\right)^{r/2}$$

$$+ \hat{\varphi}(0) O(\mu^{-r/2}).$$

Combining this with (3.2), (3.8) and (3.12) yields (1.8) for $\alpha = 2$. This completes the proof of Theorem 1.1 for $0 < r < \alpha = 2$.

3.2. *Proof of Theorem* 1.1 *for* $r = 2j < \alpha$ [$j \in \mathbb{N}$]. In this subsection, we prove Theorem 1.1 for positive even integers $r = 2j < \alpha$. First, we recall (1.14) for $r = 2j$:

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} x_1^{2j} \varphi_t(x) = (-1)^j \nabla_1^{2j}\hat{\varphi}(0).$$

Differentiating (1.18) and using the $\mathbb{Z}^d$-symmetry of the models [so that $\nabla_1^n \hat{J}(0)$ and $\nabla_1^n \hat{\varphi}(0)$ are both zero when n is odd], we have

$$\nabla_1^{2j}\hat{\varphi}(0) = \nabla_1^{2j}\hat{I}(0) + \hat{J}(0)\nabla_1^{2j}\hat{\varphi}(0) + \sum_{l=1}^{j}\binom{2j}{2l}\nabla_1^{2l}\hat{J}(0)\nabla_1^{2(j-l)}\hat{\varphi}(0).$$

Solving this equation for $\nabla_1^{2j}\hat{\varphi}(0)$ and using (2.1) for $k = 0$, we obtain

$$(3.14) \quad \nabla_1^{2j}\hat{\varphi}(0) = \frac{\hat{\varphi}(0)}{\hat{I}(0)}\left(\nabla_1^{2j}\hat{I}(0) + \sum_{l=1}^{j}\binom{2j}{2l}\nabla_1^{2l}\hat{J}(0)\nabla_1^{2(j-l)}\hat{\varphi}(0)\right).$$

To identify the dominant term of the right-hand side, we use the following proposition.

PROPOSITION 3.2. *Let $\alpha > 2$ and $\langle\alpha\rangle = \max\{j \in \mathbb{N} : j < \alpha\}$ (note that $\langle\alpha\rangle = \alpha - 1$ if $\alpha \geq 3$ is an integer). Under the same conditions as in Theorem* 1.1,

$$(3.15) \quad \left.\begin{aligned} \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{\nu} |I_t(x)| \\ \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{\nu} |J_t(x)| \end{aligned}\right\} \leq \begin{cases} O(1), & 0 \leq \nu \leq 2, \\ O\left(\left(1 - \dfrac{m}{m_{\mathrm{c}}}\right)^{1-\nu/2+\epsilon}\right), & 2 < \nu < \alpha \end{cases}$$

for some $\epsilon > 0$. Moreover,

$$(3.16) \quad |\nabla_1^n \hat{\varphi}(k, me^{i\theta})|$$

$$\leq \begin{cases} O\left(\left(1 - \dfrac{m}{m_{\mathrm{c}}} + |\theta| + |k|^2\right)^{-1-n/2}\right), & n = 0, 1, 2, \\ \dfrac{O((1 - m/m_{\mathrm{c}})^{1-n/2})}{(1 - m/m_{\mathrm{c}} + |\theta| + |k|^2)^2}, & n = 3, \dots, \langle\alpha\rangle, \end{cases}$$

where the $O((1 - \frac{m}{m_{\mathrm{c}}})^{1-n/2})$ term is uniform in $(k, \theta) \in [-\pi, \pi]^{d+1}$.

We will use this proposition again in the next subsection to prove Theorem 1.1 for the remaining case: $r = 2j + q$, where $j \in \mathbb{N}$ and $q \in (0, 2)$. The proof of Proposition 3.2 is deferred to Section 4. Note that (3.15) is trivial for random walk.

Now we resume the proof of Theorem 1.1 for $r = 2j$. Notice that

$$(3.17) \qquad |\nabla_1^{2l} \hat{J}(0)| \leq \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{2l} |J_t(x)|$$

and that a similar bound holds for I. By (3.14)–(3.16), we have the recursion

$$\nabla_1^{2j} \hat{\varphi}(0) = \frac{\hat{\varphi}(0)}{\hat{I}(0)} \left(\nabla_1^{2j} \hat{I}(0) + \binom{2j}{2} \nabla_1^2 \hat{J}(0) \nabla_1^{2(j-1)} \hat{\varphi}(0) \right.$$

$$\left. + \sum_{l=2}^{j} \binom{2j}{2l} \nabla_1^{2l} \hat{J}(0) \nabla_1^{2(j-l)} \hat{\varphi}(0) \right)$$

$$= \binom{2j}{2} \frac{\nabla_1^2 \hat{J}(0)}{\hat{I}(0)} \hat{\varphi}(0) \nabla_1^{2(j-1)} \hat{\varphi}(0) + O\left(\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^{-1-j+\epsilon}\right),$$

where the first term is $O((1 - \frac{m}{m_{\mathrm{c}}})^{-1-j})$, which is dominant as $m \uparrow m_{\mathrm{c}}$. Repeated use of this recursion then yields

$$\nabla_1^{2j} \hat{\varphi}(0) = \binom{2j}{2} \binom{2(j-1)}{2} \left(\frac{\nabla_1^2 \hat{J}(0)}{\hat{I}(0)} \hat{\varphi}(0) \right)^2 \nabla_1^{2(j-2)} \hat{\varphi}(0)$$

$$+ O\left(\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^{-1-j+\epsilon}\right)$$

$$\vdots$$

$$= \prod_{l=2}^{j} \binom{2l}{2} \left(\frac{\nabla_1^2 \hat{J}(0)}{\hat{I}(0)} \hat{\varphi}(0) \right)^{j-1} \nabla_1^2 \hat{\varphi}(0) + O\left(\left(1 - \frac{m}{m_c} \right)^{-1-j+\epsilon} \right)$$

$$= \frac{(2j)!}{2^j} \left(\frac{\nabla_1^2 \hat{J}(0)}{\hat{I}(0)} \right)^j \hat{\varphi}(0)^{j+1} + O\left(\left(1 - \frac{m}{m_c} \right)^{-1-j+\epsilon} \right).$$

However, by comparing (1.11) and (1.13), and using (3.7), we have

$$\nabla_1^2 \hat{J}(0) = -2v_\alpha \lim_{k \to 0} \frac{\bar{\Delta}_k \hat{J}(0)}{\bar{\Delta}_k \hat{D}(0)} = -2v_\alpha M$$

$$= -2v_\alpha M_c + O\left(\left(1 - \frac{m}{m_c} \right)^\epsilon \right).$$

Recall that $\hat{I}(0) = \hat{I}_c(0) + O(1 - \frac{m}{m_c})$ [cf. the numerator in (3.2)]. Therefore,

$$\text{(3.18)} \quad \begin{aligned} \frac{\nabla_1^2 \hat{J}(0)}{\hat{I}(0)} &= -2v_\alpha \frac{M_c}{\hat{I}_c(0)} + O\left(\left(1 - \frac{m}{m_c} \right)^\epsilon \right) \\ &= -2v_\alpha \frac{C_{II}}{C_I} + O\left(\left(1 - \frac{m}{m_c} \right)^\epsilon \right) \qquad [\because (1.21) \text{ and } (2.13)], \end{aligned}$$

hence

$$\begin{aligned} \nabla_1^{2j} \hat{\varphi}(0) &= \frac{(2j)!}{2^j} \left(-2v_\alpha \frac{C_{II}}{C_I} \right)^j \left(\frac{C_I}{1 - m/m_c} \right)^{j+1} + O\left(\left(1 - \frac{m}{m_c} \right)^{-1-j+\epsilon} \right) \\ &= \Gamma(2j+1) \frac{C_I (-C_{II} v_\alpha)^j}{(1 - m/m_c)^{j+1}} + O\left(\left(1 - \frac{m}{m_c} \right)^{-1-j+\epsilon} \right). \end{aligned}$$

This completes the proof of Theorem 1.1 for positive even integers $r = 2j < \alpha$.

3.3. *Proof of Theorem* 1.1 *for* $r = 2j + q < \alpha$ $[j \in \mathbb{N}, q \in (0, 2)]$. In this subsection, we prove Theorem 1.1 for the other values of $r < \alpha$: $r = 2j + q$ with $j \in \mathbb{N}$ and $q \in (0, 2)$. First, we recall (1.14):

$$\text{(3.19)} \quad \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x) = \frac{(-1)^j}{K_q} \int_0^\infty \frac{du}{u^{1+q}} \bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{\varphi}(0),$$

where, by (1.18),

$$\begin{aligned} \bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{\varphi}(0) &= \nabla_1^{2j} \hat{\varphi}(0) - \nabla_1^{2j} \hat{\varphi}(\vec{u}) \\ &= \nabla_1^{2j} \hat{I}(0) - \nabla_1^{2j} \hat{I}(\vec{u}) + \sum_{n=0}^{2j} \binom{2j}{n} (\nabla_1^n \hat{J}(0) \nabla_1^{2j-n} \hat{\varphi}(0) \\ &\qquad\qquad\qquad\qquad\qquad\qquad - \nabla_1^n \hat{J}(\vec{u}) \nabla_1^{2j-n} \hat{\varphi}(\vec{u})) \end{aligned}$$

$$= \bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{I}(0) + \sum_{n=0}^{2j} \binom{2j}{n} \left(\nabla_1^n \hat{J}(0) \, \bar{\Delta}_{\vec{u}} \nabla_1^{2j-n} \hat{\varphi}(0) \right.$$

$$\left. + \nabla_1^{2j-n} \hat{\varphi}(\vec{u}) \bar{\Delta}_{\vec{u}} \nabla_1^n \hat{J}(0) \right).$$

Solving this equation for $\bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{\varphi}(0)$ and using (2.1) for $k = 0$ and $\nabla_1^n \hat{J}(0) = 0$ for odd n, we obtain

$$\bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{\varphi}(0) = \frac{\hat{\varphi}(0)}{\hat{I}(0)} \left(\bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{I}(0) + \sum_{l=1}^{j} \binom{2j}{2l} \nabla_1^{2l} \hat{J}(0) \bar{\Delta}_{\vec{u}} \nabla_1^{2(j-l)} \hat{\varphi}(0) \right.$$

$$\left. + \sum_{n=0}^{2j} \binom{2j}{n} \nabla_1^{2j-n} \hat{\varphi}(\vec{u}) \bar{\Delta}_{\vec{u}} \nabla_1^n \hat{J}(0) \right).$$

Substituting this back into (3.19) yields

(3.20)
$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x)$$
$$= \frac{\hat{\varphi}(0)}{\hat{I}(0)} \left(H^{(1)} + \sum_{l=1}^{j} \binom{2j}{2l} H_{2l}^{(2)} + \sum_{n=0}^{2j} \binom{2j}{n} H_n^{(3)} \right),$$

where

(3.21)
$$H^{(1)} = \frac{(-1)^j}{K_q} \int_0^{\infty} \frac{du}{u^{1+q}} \bar{\Delta}_{\vec{u}} \nabla_1^{2j} \hat{I}(0) \equiv \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r I_t(x),$$

(3.22)
$$H_{2l}^{(2)} = \frac{(-1)^j}{K_q} \int_0^{\infty} \frac{du}{u^{1+q}} \nabla_1^{2l} \hat{J}(0) \bar{\Delta}_{\vec{u}} \nabla_1^{2(j-l)} \hat{\varphi}(0)$$
$$\equiv (-1)^l \nabla_1^{2l} \hat{J}(0) \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{r-2l} \varphi_t(x)$$

and

(3.23)
$$H_n^{(3)} = \frac{(-1)^j}{K_q} \int_0^{\infty} \frac{du}{u^{1+q}} \nabla_1^{2j-n} \hat{\varphi}(\vec{u}) \, \bar{\Delta}_{\vec{u}} \nabla_1^n \hat{J}(0)$$
$$\equiv \sum_{s,t=0}^{\infty} m^{t+s} \sum_{x,y \in \mathbb{Z}^d} x_1^{2j-n} \varphi_t(x) y_1^n J_s(y)$$
$$\times \frac{1}{K_q} \int_0^{\infty} \frac{du}{u^{1+q}} \times \begin{cases} \sin(u x_1) \sin(u y_1), & \text{odd } n, \\ \cos(u x_1)(1 - \cos(u y_1)), & \text{even } n. \end{cases}$$

Next, we isolate error terms from (3.20) using Proposition 3.2. First, by (3.15), we have

$$(3.24) \qquad |H^{(1)}| \le \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r |I_t(x)| \le O\left(\left(1 - \frac{m}{m_c}\right)^{-r/2+1+\epsilon}\right),$$

which gives rise to an error term.

Next, for $H_{2l}^{(2)}$, where $r - 2l = 2j + q - 2l < 2j + 2 - 2l < \alpha$, we first apply Jensen's inequality and then (3.16) to obtain

$$
\begin{aligned}
\sum_{t=0}^{\infty} m^t &\sum_{x \in \mathbb{Z}^d} |x_1|^{r-2l} \varphi_t(x) \\
&\le \left(\frac{1}{\hat{\varphi}(0)} \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{2j+2-2l} \varphi_t(x)\right)^{(r-2l)/(2j+2-2l)} \hat{\varphi}(0) \\
(3.25) \qquad &= \left(\frac{|\nabla_1^{2j+2-2l} \hat{\varphi}(0)|}{\hat{\varphi}(0)}\right)^{(r-2l)/(2j+2-2l)} \hat{\varphi}(0) \\
&\le O\left(\left(1 - \frac{m}{m_c}\right)^{-(2j+2-2l)/2}\right)^{(r-2l)/(2j+2-2l)} O\left(\left(1 - \frac{m}{m_c}\right)^{-1}\right) \\
&= O\left(\left(1 - \frac{m}{m_c}\right)^{-1-(r-2l)/2}\right).
\end{aligned}
$$

Combining this with (3.15) and (3.17) yields

$$(3.26) \qquad |H_{2l}^{(2)}| \le \begin{cases} O\left(\left(1 - \dfrac{m}{m_c}\right)^{-r/2}\right), & l = 1, \\[2mm] O\left(\left(1 - \dfrac{m}{m_c}\right)^{-r/2+\epsilon}\right), & l = 2, 3, \ldots, j. \end{cases}$$

Finally, for $H_n^{(3)}$ with $n \ge 2$ ($H_0^{(3)}$ and $H_1^{(3)}$ will be investigated in detail later), we use

$$
\begin{aligned}
(3.27) \qquad \int_0^{\infty} \frac{du}{u^{1+q}} |\sin(ux_1)\sin(uy_1)| &\le \int_0^{\infty} \frac{du}{u^{1+q}} (|u^2 x_1 y_1| \wedge 1) \\
&= O(|x_1 y_1|^{q/2}),
\end{aligned}
$$

$$
\begin{aligned}
(3.28) \qquad \int_0^{\infty} \frac{du}{u^{1+q}} |\cos(ux_1)(1 - \cos(uy_1))| &\le \int_0^{\infty} \frac{du}{u^{1+q}} \left(\frac{u^2 y_1^2}{2} \wedge 2\right) \\
&= O(|y_1|^q),
\end{aligned}
$$

which are due to the naive bounds $|\sin w| \le |w| \wedge 1$, $|\cos w| \le 1$ and $|1 - \cos w| \le \frac{w^2}{2} \wedge 2$. By (3.27) and (3.28) and using Jensen's inequality for odd n, as in (3.25),

we obtain

$$
|H_n^{(3)}| \le \begin{cases}
\left(\dfrac{|\nabla_1^{2j-n+1} \hat{\varphi}(0)|}{\hat{\varphi}(0)} \right)^{(2j-n+q/2)/(2j-n+1)} \hat{\varphi}(0) \\
\qquad \times \displaystyle\sum_{s=0}^{\infty} m^s \sum_{y \in \mathbb{Z}^d} |y_1|^{n+q/2} |J_s(y)|, & \text{odd } n, \\[2em]
|\nabla_1^{2j-n} \hat{\varphi}(0)| \displaystyle\sum_{s=0}^{\infty} m^s \sum_{y \in \mathbb{Z}^d} |y_1|^{n+q} |J_s(y)|, & \text{even } n.
\end{cases}
$$

Then, by Proposition 3.2 and using $2j + q = r$, we obtain

$$(3.29) \qquad |H_n^{(3)}| \le O\!\left(\left(1 - \frac{m}{m_c}\right)^{-r/2+\epsilon} \right) \qquad [n = 2, 3, \ldots, 2j].$$

Now, by (3.20), (3.24), (3.26) and (3.29), we arrive at

$$
\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x) = \frac{\hat{\varphi}(0)}{\hat{I}(0)} \left(\binom{2j}{2} H_2^{(2)} + H_0^{(3)} + \binom{2j}{1} H_1^{(3)} \right)
$$

$$(3.30) \qquad\qquad\qquad\qquad + O\!\left(\left(1 - \frac{m}{m_c}\right)^{-1-r/2+\epsilon} \right).$$

Finally, we reorganize the main term of (3.30) and complete the proof of Theorem 1.1. First, we note that

$$
\begin{aligned}
\sin(ux_1)\sin(uy_1) &= \frac{\cos(u(x_1 - y_1)) - \cos(u(x_1 + y_1))}{2} \\
&= \frac{1 - \cos(u(x_1 + y_1)) - (1 - \cos(u(x_1 - y_1)))}{2}, \\
\cos(ux_1)(1 - \cos(uy_1)) &= \cos(ux_1) - \frac{\cos(u(x_1 + y_1)) + \cos(u(x_1 - y_1))}{2} \\
&= \frac{(1 - \cos(u(x_1 + y_1))) + (1 - \cos(u(x_1 - y_1)))}{2} \\
&\quad - (1 - \cos(ux_1)).
\end{aligned}
$$

Then, by (1.12), we have the identities

$$
\frac{1}{K_q} \int_0^\infty \frac{du}{u^{1+q}} \sin(ux_1)\sin(uy_1) = \frac{|x_1 + y_1|^q - |x_1 - y_1|^q}{2},
$$

$$
\frac{1}{K_q} \int_0^\infty \frac{du}{u^{1+q}} \cos(ux_1)(1 - \cos(uy_1)) = \frac{|x_1 + y_1|^q + |x_1 - y_1|^q - 2|x_1|^q}{2}.
$$

By these identities and the fact that $r = 2j + q$, we obtain

$$
\binom{2j}{2} H_2^{(2)} + H_0^{(3)} + \binom{2j}{1} H_1^{(3)} = \sum_{s,t=0}^{\infty} m^{t+s} \sum_{x,y \in \mathbb{Z}^d} x_1^{2j-2} \varphi_t(x) J_s(y) \mathcal{H}(x_1, y_1),
$$

where

$$\mathcal{H}(x_1, y_1) = \binom{2j}{2} |x_1|^q y_1^2 + x_1^2 \frac{|x_1 + y_1|^q + |x_1 - y_1|^q - 2|x_1|^q}{2}$$

$$+ \binom{2j}{1} x_1 y_1 \frac{|x_1 + y_1|^q - |x_1 - y_1|^q}{2}.$$

In fact, due to the symmetry $\mathcal{H}(x_1, y_1) = \mathcal{H}(x_1, -y_1) = \mathcal{H}(-x_1, y_1) = \mathcal{H}(-x_1, -y_1)$ for any $x_1, y_1 \in \mathbb{Z}$, the above identity is equivalent to

$$\binom{2j}{2} H_2^{(2)} + H_0^{(3)} + \binom{2j}{1} H_1^{(3)}$$

$$= 4 \sum_{s,t=0}^{\infty} m^{t+s} \sum_{\substack{x,y \in \mathbb{Z}^d \\ (x_1, y_1 > 0)}} x_1^{2j-2} \varphi_t(x) J_s(y) \mathcal{H}(x_1, y_1).$$

Using the Taylor expansion of $|x_1 \pm y_1|^q \equiv x_1^q (1 \pm \frac{y_1}{x_1})^q$ if $x_1 > y_1 > 0$ and the expansion of $|x_1 \pm y_1|^q \equiv y_1^q (1 \pm \frac{x_1}{y_1})^q$ if $y_1 > x_1 > 0$, we have

(3.31)
$$\mathcal{H}(x_1, y_1)$$
$$= \begin{cases} \left(\binom{2j}{2} + \binom{q}{2} + \binom{2j}{1} q \right) x_1^q y_1^2 + O(x_1^{q-1} y_1^3), \\ \qquad x_1 > y_1 > 0, \\ O(y_1^{2+q}), \qquad y_1 \geq x_1 > 0. \end{cases}$$

Notice that

$$\binom{2j}{2} + \binom{q}{2} + \binom{2j}{1} q = j(2j - 1) + \frac{q}{2}(q - 1) + 2jq$$

$$= \left(j + \frac{q}{2} \right)(2j + q) - j - \frac{q}{2} = \frac{r}{2} r - \frac{r}{2} = \binom{r}{2}.$$

We also notice that, as long as $q \in (0, 1]$, we have

$$x_1^{q-1} y_1^3 = \left(\frac{y_1}{x_1} \right)^{1-q} y_1^{2+q} \leq y_1^{2+q} \qquad [x_1 > y_1 > 0].$$

Therefore, by Proposition 3.2, we obtain that, for $q \equiv r - 2j \in (0, 1]$,

$$\binom{2j}{2} H_2^{(2)} + H_0^{(3)} + \binom{2j}{1} H_1^{(3)}$$

$$= 4 \binom{r}{2} \sum_{s,t=0}^{\infty} m^{t+s} \sum_{\substack{x,y \in \mathbb{Z}^d \\ (x_1, y_1 > 0)}} x_1^{2j+q-2} \varphi_t(x) y_1^2 J_s(y)$$

(3.32)
$$+ \sum_{s,t=0}^{\infty} m^{t+s} \sum_{x,y \in \mathbb{Z}^d} x_1^{2j-2} \varphi_t(x) O(|y_1|^{2+q}) J_s(y)$$

$$= \binom{r}{2}(-\nabla_1^2 \hat{J}(0)) \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{r-2} \varphi_t(x)$$

$$+ \underbrace{|\nabla_1^{2j-2} \hat{\varphi}(0)| \sum_{s=0}^{\infty} m^s \sum_{y \in \mathbb{Z}^d} O(|y_1|^{2+q}) J_s(y)}_{O((1-m/m_c)^{-r/2+\epsilon})}.$$

For $q \in (1, 2)$, we have to deal with the contribution from $O(x_1^{q-1} y_1^3)$ in (3.31). However, by Jensen's inequality and Proposition 3.2, we have

$$\sum_{s,t=0}^{\infty} m^{t+s} \sum_{\substack{x,y \in \mathbb{Z}^d \\ (x_1 > y_1 > 0)}} x_1^{2j+q-3} \varphi_t(x) y_1^3 |J_s(y)|$$

$$\leq \left(\frac{|\nabla_1^{2j-1} \hat{\varphi}(0)|}{\hat{\varphi}(0)} \right)^{(2j+q-3)/(2j-1)} \hat{\varphi}(0) \sum_{s=0}^{\infty} m^s \sum_{y \in \mathbb{Z}^d} |y_1|^3 |J_s(y)|$$

$$\leq O\left(\left(1 - \frac{m}{m_c} \right)^{-r/2+\epsilon} \right)$$

and thus (3.32) is valid for any $q \in (0, 2)$.

Now, by substituting (3.32) back into (3.30), we obtain the recursion

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x) = \binom{r}{2} \frac{-\nabla_1^2 \hat{J}(0)}{\hat{I}(0)} \hat{\varphi}(0) \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{r-2} \varphi_t(x)$$

$$+ O\left(\left(1 - \frac{m}{m_c} \right)^{-1-r/2+\epsilon} \right).$$

Repeatedly using this recursion j times and recalling $r - 2j = q$, we obtain

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x) = \prod_{i=0}^{j-1} \binom{r-2i}{2} \left(\frac{-\nabla_1^2 \hat{J}(0)}{\hat{I}(0)} \hat{\varphi}(0) \right)^j$$

$$\times \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{r-2j} \varphi_t(x) + O\left(\left(1 - \frac{m}{m_c} \right)^{-1-r/2+\epsilon} \right)$$

$$= \frac{\Gamma(r+1)}{2^j \Gamma(r-2j+1)} \left(\frac{-\nabla_1^2 \hat{J}(0)}{\hat{I}(0)} \hat{\varphi}(0) \right)^j$$

$$\times \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^q \varphi_t(x) + O\left(\left(1 - \frac{m}{m_c} \right)^{-1-r/2+\epsilon} \right).$$

Notice that, by (3.2) and (3.18),

$$\frac{-\nabla_1^2 \hat{J}(0)}{\hat{I}(0)} \hat{\varphi}(0) = \frac{2C_{\mathrm{II}} v_\alpha}{1 - m/m_{\mathrm{c}}} + O\left(\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^{-1+\epsilon}\right)$$

and that, by (3.11) for $\alpha > 2$ and (3.12),

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^q \varphi_t(x) = \Gamma(q+1) \frac{C_{\mathrm{I}}(C_{\mathrm{II}} v_\alpha)^{q/2}}{(1 - m/m_{\mathrm{c}})^{1+q/2}} + O\left(\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^{-1-q/2+\epsilon}\right).$$

Therefore, we arrive at

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x)$$

$$= \frac{\Gamma(r+1)}{2^j \Gamma(q+1)} \left(\frac{2C_{\mathrm{II}} v_\alpha}{1 - m/m_{\mathrm{c}}}\right)^j \Gamma(q+1) \frac{C_{\mathrm{I}}(C_{\mathrm{II}} v_\alpha)^{q/2}}{(1 - m/m_{\mathrm{c}})^{1+q/2}}$$

$$+ O\left(\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^{-1-r/2+\epsilon}\right)$$

$$= \Gamma(r+1) \frac{C_{\mathrm{I}}(C_{\mathrm{II}} v_\alpha)^{r/2}}{(1 - m/m_{\mathrm{c}})^{1+r/2}} + O\left(\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^{-1-r/2+\epsilon}\right).$$

This completes the proof of Theorem 1.1.

3.4. *Proof of Theorem* 1.2. It is very easy to identify the main term for $\alpha \neq 2$. First, by the binomial expansion of the main term in (1.8),

$$\left(1 - \frac{m}{m_{\mathrm{c}}}\right)^{-1-r/(\alpha \wedge 2)}$$

$$= \sum_{t=0}^{\infty} \frac{(-r/(\alpha \wedge 2) - 1)(-r/(\alpha \wedge 2) - 2) \cdots (-r/(\alpha \wedge 2) - t)}{t!}$$

(3.33) $$\times \left(-\frac{m}{m_{\mathrm{c}}}\right)^t$$

$$= \sum_{t=0}^{\infty} \frac{\Gamma(r/(\alpha \wedge 2) + t + 1)}{t! \Gamma(r/(\alpha \wedge 2) + 1)} \left(\frac{m}{m_{\mathrm{c}}}\right)^t$$

$$= \frac{1}{\Gamma(r/(\alpha \wedge 2) + 1)} \sum_{t=0}^{\infty} \left(\frac{m}{m_{\mathrm{c}}}\right)^t \frac{1}{t!} \int_0^\infty x^{t+r/(\alpha \wedge 2)} e^{-x} \, dx.$$

Then, by the steepest descent method, we obtain that, for every $\beta \in \mathbb{R}$,

$$\int_0^\infty x^{t+\beta} e^{-x} \, dx \sim \sqrt{2\pi(t+\beta)} \left(\frac{t+\beta}{e}\right)^{t+\beta} \qquad \text{as } t \to \infty.$$

Using this for $\beta = 0, \frac{r}{\alpha \wedge 2}$, we conclude that, as $t \to \infty$,

$$\frac{1}{t!} \int_0^\infty x^{t+r/(\alpha \wedge 2)} e^{-x} \, dx \sim \left(\frac{t + r/(\alpha \wedge 2)}{t} \right)^{t+1/2} \left(\frac{t + r/(\alpha \wedge 2)}{e} \right)^{r/(\alpha \wedge 2)}$$

$$\sim t^{r/(\alpha \wedge 2)},$$

which implies that the large-t asymptotic expression for the coefficient of m^t in (3.33) is $m_c^{-t} t^{r/(\alpha \wedge 2)} / \Gamma(\frac{r}{\alpha \wedge 2} + 1)$, hence the expression for the constant in (1.9) for $\alpha \neq 2$.

There are many other ways to derive the above asymptotic expression. One of them is to notice that $x^t e^{-x}/t!$ in (3.33) is the probability density for the sum of independent mean-one exponential random variables. Then, we use Jensen's inequality and apply the law of large numbers if $\frac{r}{\alpha \wedge 2} \leq 1$, or exactly compute integer-power moments for the exponential random variables if $\frac{r}{\alpha \wedge 2} > 1$. We omit the details.

To identify the main term for $\alpha = 2$ in (1.9), as well as to obtain the error estimates for all $\alpha > 0$, we simply use [4], Theorems 3A and 4. For convenience, we summarize a slightly simplified version of these results as follows.

THEOREM 3.3 ([4], Theorems 3A and 4).   (i) *Let*

$$f(z) = (1-z)^{-1-\beta} \left(\log \frac{1}{1-z} \right)^\gamma,$$

where $\beta \notin -\mathbb{N} \equiv \mathbb{Z} \setminus \mathbb{Z}_+$ and $\gamma \notin \mathbb{Z}_+$ are real or complex numbers. Then, the coefficient f_t of $f(z) = \sum_t f_t z^t$ satisfies

$$f_t \sim \frac{t^\beta (\log t)^\gamma}{\Gamma(1+\beta)} \qquad as \; t \to \infty.$$

(ii) *Let $f(z)$ be analytic in $|z| < 1$ and*

$$f(z) = O(|1-z|^{-1-\beta}) \qquad as \; z \to 1$$

for some real number $\beta > 0$. Then, the coefficient f_t of $f(z) = \sum_t f_t z^t$ satisfies

$$f_t = O(t^\beta) \qquad as \; t \to \infty.$$

The main term for $\alpha = 2$ in (1.9) is obtained by setting $\beta = \gamma = \frac{r}{2}$ in Theorem 3.3(i). For the error estimates, we use Theorem 3.3(ii) with $\beta = \frac{r}{2}$ for $\alpha = 2$ and $\beta = \frac{r}{\alpha \wedge 2} - \epsilon > 0$ for $\alpha \neq 2$. This completes the proof of Theorem 1.2.

4. Proof of the key propositions. In this section, we prove Propositions 3.1 and 3.2, these being key propositions used in the previous section to prove the main theorem. In Section 4.1, we first prove Proposition 3.2. Then, in Section 4.2, we use (3.16) in Proposition 3.2 to show Proposition 3.1 for $\alpha > 2$.

4.1. *Proof of Proposition* 3.2.   Below, we prove Proposition 3.2 by using the results already obtained in [2, 3, 5, 6] and alternately applying the following two lemmas.

LEMMA 4.1.   *Let* $\alpha > 2$, $l \in \{1, 2, \ldots, \langle \alpha \rangle\}$ *and suppose that* (3.15) *holds for any* $v \in \{0, 1, \ldots, l \vee 2\}$ *and* (3.16) *holds for any* $n \in \{0, \ldots, l - 1\}$. *Then,* (3.16) *holds for* $n = l$.

LEMMA 4.2.   *Let* $\alpha > 2$ *and suppose that* (3.16) *holds for* $n = 2l$, *where* $l \in \{1, \ldots, \langle \frac{\alpha}{2} \rangle\}$ *(note that* $\alpha - 2 \leq 2\langle \frac{\alpha}{2} \rangle < \alpha$). *Then,* (3.15) *holds for any* $v \in (n, n+2]$ *if* $n + 2 < \alpha$, *or for any* $v \in (n, \alpha)$ *if* $\alpha \leq n + 2$.

We will prove these lemmas after completing the proof of Proposition 3.2. For random walk, (3.15) always holds as mentioned earlier and we therefore only need Lemma 4.1.

We now begin by proving Proposition 3.2. First, we note that (3.15) for $v \in [0, 2]$ and (3.16) for $n = 0$ have been proven in the current setting [2, 3, 5, 6]; the result in [6] for self-avoiding walk is only valid at $\theta = 0$. However, it is not hard to extend the result to nonzero θ by splitting the denominator in (2.1) into $1 - \hat{J}(k, m)$ and $\hat{J}(k, m) - \hat{J}(k, me^{i\theta})$, and estimating the latter as $m^t - (me^{i\theta})^t = m^t(1 - e^{i\theta}) \sum_{s=0}^{t-1} e^{i\theta s}$ [which equals $O(\theta)tm^t$ for $|\theta| \ll 1$]. We omit the details. Then, by Lemma 4.1 with $l = 1$, we obtain (3.16) for $n = 1$. With this conclusion and again using Lemma 4.1, but now with $l = 2$, we obtain (3.16) for $n = 2$. With this conclusion and using Lemma 4.2, we further obtain (3.15) for $v \in (2, 4]$ or $v \in (2, \alpha)$, depending on whether $\alpha > 4$ or $\alpha \leq 4$. We can repeat this, using Lemmas 4.1 and 4.2 alternately, until n reaches $\langle \alpha \rangle$. Let $\tilde{l} = \langle \frac{\alpha}{2} \rangle$. We see that

$$
\left.
\begin{array}{l}
(3.15)_{v \in [0,2]} \\
(3.16)_{n=0}
\end{array}
\right\} \overset{\text{Lemma } 4.1}{\Longrightarrow} (3.16)_{n=1,2} \overset{\text{Lemma } 4.2}{\Longrightarrow} (3.15)_{v \in (2,4]} \overset{\text{Lemma } 4.1}{\Longrightarrow} \cdots
$$

$$
\overset{\text{Lemma } 4.1}{\Longrightarrow} (3.16)_{n=2\tilde{l}-1,2\tilde{l}}
$$

$$
\overset{\text{Lemma } 4.2}{\Longrightarrow} (3.15)_{v \in (2\tilde{l},\alpha)} \left(\overset{\text{Lemma } 4.1}{\underset{\text{if } \alpha > 2\tilde{l}+1}{\Longrightarrow}} (3.16)_{n=2\tilde{l}+1} \right).
$$

This completes the proof of Proposition 3.2.

PROOF OF LEMMA 4.1.   First, by using (3.15) for $v = 2$ and (3.16) for $n = 0$, we prove $|\nabla_1 \hat{I}(k)| \leq O(|\hat{\varphi}(k)|^{-1/2})$; the proof of $|\nabla_1 \hat{J}(k)| \leq O(|\hat{\varphi}(k)|^{-1/2})$ is almost identical and thus we omit it. By the $\mathbb{Z}^d$-symmetry of the models and using

$|\sin(k_1 x_1)| \le |k_1 x_1|$ and (3.15) for $\nu = 2$, we obtain

$$|\nabla_1 \hat{I}(k)| = \left| \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} x_1 \sin(k_1 x_1) I_t(x) e^{i(k_2 x_2 + \cdots + k_d x_d)} \right|$$

$$\le |k_1| \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} x_1^2 |I_t(x)| \le O(|k_1|).$$

However, by (3.16) for $n = 0$, we have $|\hat{\varphi}(k)| \le O(|k|^{-2})$, which implies that $|k_1| \le |k| \le O(|\hat{\varphi}(k)|^{-1/2})$, as required.

We now use this bound to complete the proof of Lemma 4.1. First, by differentiating (1.18) and solving the resulting equation for $\nabla_1^l \hat{\varphi}(k)$, we have that, for $l \in \mathbb{N}$,

$$\nabla_1^l \hat{\varphi}(k) = \nabla_1^l \hat{I}(k) + \sum_{j=0}^{l} \binom{l}{j} \nabla_1^j \hat{J}(k) \nabla_1^{l-j} \hat{\varphi}(k)$$

$$= \frac{1}{1 - \hat{J}(k)} \left(\nabla_1^l \hat{I}(k) + \sum_{j=1}^{l} \binom{l}{j} \nabla_1^j \hat{J}(k) \nabla_1^{l-j} \hat{\varphi}(k) \right).$$

By (2.1), (2.2) and (2.4), we have $|1 - \hat{J}(k)|^{-1} = O(|\hat{\varphi}(k)|)$. By (3.15) for $\nu \ge 2$ or using $|\nabla_1 \hat{I}(k)| \le O(|\hat{\varphi}(k)|^{-1/2})$, we obtain

$$\left| \frac{\nabla_1^l \hat{I}(k)}{1 - \hat{J}(k)} \right| \le O(|\hat{\varphi}(k)|) \times \begin{cases} |\hat{\varphi}(k)|^{-1/2}, & l = 1, \\ \left(1 - \dfrac{m}{m_c}\right)^{1-l/2+\epsilon}, & l = 2, \ldots, \langle \alpha \rangle, \end{cases}$$

which, by (3.16) for $n = 0$, is smaller than the bound in (3.16) for $n = l$, yielding an error term. For $j = 1, 2$, we also use (3.16) for $n \le l - 1$ to obtain

$$\left| \frac{\nabla_1^j \hat{J}(k)}{1 - \hat{J}(k)} \nabla_1^{l-j} \hat{\varphi}(k) \right|$$

$$\le O(|\hat{\varphi}(k)|^{j/2})$$

$$\times \begin{cases} \left(1 - \dfrac{m}{m_c} + |\theta| + |k|^2\right)^{-1-(l-j)/2}, & l = j, j+1, \\ \dfrac{(1 - m/m_c)^{1-(l-j)/2}}{(1 - m/m_c + |\theta| + |k|^2)^2}, & l = j+2, \ldots, \langle \alpha \rangle, \end{cases}$$

which, again by (3.16) for $n = 0$, obeys the required bound in (3.16) for $n = l$. Finally, for $j \ge 3$ (hence for $l \ge 3$),

$$\left| \frac{\nabla_1^j \hat{J}(k)}{1 - \hat{J}(k)} \nabla_1^{l-j} \hat{\varphi}(k) \right|$$

$$\le \frac{O((1 - m/m_c)^{1-j/2+\epsilon})}{1 - m/m_c + |\theta| + |k|^2}$$

$$
\times
\begin{cases}
\left(1 - \dfrac{m}{m_c} + |\theta| + |k|^2\right)^{-1-(l-j)/2} & [l = j, j+1], \\[2ex]
\dfrac{(1 - m/m_c)^{1-(l-j)/2}}{(1 - m/m_c + |\theta| + |k|^2)^2} & [l = j+2, \ldots, \langle\alpha\rangle]
\end{cases}
$$

$$
\leq \frac{O((1 - m/m_c)^{1-l/2+\epsilon})}{(1 - m/m_c + |\theta| + |k|^2)^2},
$$

which is smaller [by the factor $(1 - \frac{m}{m_c})^\epsilon$] than the bound in (3.16), yielding an error term. This completes the proof of Lemma 4.1. □

PROOF OF LEMMA 4.2. First, we recall (1.16) and (1.17). Since $\sum_x |x_1|^\nu D(x) < \infty$ provided that $\nu < \alpha$, (3.15) always holds for random walk. Moreover, for oriented percolation, there is a constant $C_\nu < \infty$ such that

$$
\sum_{t=0}^\infty m^t \sum_{x \in \mathbb{Z}^d} |x_1|^\nu |J_t^{OP}(x)| \leq p \sum_{t=1}^\infty m^t \sum_{x,y \in \mathbb{Z}^d} |y_1 + x_1 - y_1|^\nu |\pi_{t-1}^{OP}(y)| D(x-y)
$$

$$
\leq C_\nu \, pm \sum_{t=1}^\infty m^{t-1} \sum_{y \in \mathbb{Z}^d} (|y_1|^\nu + 1) |\pi_{t-1}^{OP}(y)|,
$$

where we have used the fact that, for any $a_1, \ldots, a_n \in \mathbb{R}$,

$$
(4.1) \qquad \left|\sum_{j=1}^n a_j\right|^\nu \leq \left(n \max_{1 \leq j \leq n} |a_j|\right)^\nu = n^\nu \max_{1 \leq j \leq n} |a_j|^\nu \leq n^\nu \sum_{j=1}^n |a_j|^\nu.
$$

Since $\sum_{s=0}^\infty m^s \sum_{y \in \mathbb{Z}^d} |\pi_s^{OP}(y)| = O(1)$ uniformly in $m \leq m_c$ [2], it suffices to show that, for self-avoiding walk and oriented percolation, (3.16) for $n = 2l$, where $l \in \{1, \ldots, \langle\frac{\alpha}{2}\rangle\}$ implies that

$$
(4.2) \qquad \sum_{t=0}^\infty m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{2l+q} |\pi_t(x)| \leq O\left(\left(1 - \frac{m}{m_c}\right)^{1-(2l+q)/2+\epsilon}\right)
$$

for any $q \in (0, 2]$ if $2l + 2 < \alpha$, or for any $q \in (0, \alpha - 2l)$ if $\alpha \leq 2l + 2$.

As we mentioned earlier, $\pi_t(x)$ is an alternating sum of the lace expansion coefficients. More precisely,

$$
\pi_t(x) = \sum_{N=0}^\infty (-1)^N \pi_t^{(N)}(x),
$$

where $\pi_t^{(N)}(x) \geq 0$ is the model-dependent Nth expansion coefficient (see, e.g., [2, 12] for the precise definitions of the expansion coefficients). Due to the subadditivity argument for self-avoiding walk and by the BK inequality [1] for percolation, it is known that the expansion coefficients satisfy the following diagrammatic

bounds, in which each line corresponds to a 2-point function. For self-avoiding walk,

(4.3)

$$\pi_t^{(0)}(x) \equiv 0, \qquad \pi_t^{(1)}(x) \leq \underset{x=o}{\bigcirc},$$

$$\pi_t^{(2)}(x) \leq \overset{x}{\underset{o}{\bigoplus}}, \qquad \pi_t^{(3)}(x) \leq \underset{o \qquad x}{\triangle\!\!\!\bigcirc},$$

where the bounding diagram for $\pi_t^{(1)}(x)$ is the t-step self-avoiding loop at $x = o$, hence proportional to $\delta_{x,o}$, and the diagram for $\pi_t^{(2)}(x)$ is the product of three 2-point functions $\varphi_s^{\mathrm{SAW}}(x)\varphi_{s'}^{\mathrm{SAW}}(x)\varphi_{s''}^{\mathrm{SAW}}(x)$ summed over all possible combinations of $s, s', s'' \in \mathbb{N}$ satisfying $s + s' + s'' = t$, and so on. The unlabeled vertices in the diagrams for $\pi_t^{(3)}(x)$ and the higher order expansion coefficients are summed over $\mathbb{Z}^d$. For oriented percolation,

(4.4) $\quad \pi_t^{(0)}(x) \leq \overset{(x,t)}{\underset{(o,0)}{\bigcirc}}, \qquad \pi_t^{(1)}(x) \leq \overset{(x,t)}{\underset{(o,0)}{\bigcirc}}, \qquad \pi_t^{(2)}(x) \leq \overset{(x,t)}{\underset{(o,0)}{\bigotimes}} + \overset{(x,t)}{\underset{(o,0)}{\bigotimes}},$

where the bounding diagram for $\pi_t^{(0)}(x)$ is $\varphi_t^{\mathrm{OP}}(x)^2$ and that for $\pi_t^{(1)}(x)$ is the product of five 2-point functions concatenated in the depicted way, and so on. The upward direction of the diagrams is the time-increasing direction and the unlabeled vertices are summed over space–time $\mathbb{Z}^d \times \mathbb{Z}_+$. For more details, we refer to [10].

First, we prove (4.2) for self-avoiding walk. Since $\pi_t^{(0)}(x) \equiv 0$ and $\pi_t^{(1)}(x) \propto \delta_{x,o}$, it suffices to investigate the contributions from $\pi_t^{(N)}(x)$ for $N \geq 2$. For $\pi_t^{(2)}(x)$, since

(4.5) $$\pi_t^{(2)}(x) \leq \sum_{\substack{s,s',s'' \in \mathbb{N} \\ (s+s'+s''=t)}} \varphi_s^{\mathrm{SAW}}(x)\varphi_{s'}^{\mathrm{SAW}}(x)\varphi_{s''}^{\mathrm{SAW}}(x),$$

we obtain

$$\sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{2l+q} \pi_t^{(2)}(x) \leq \left(\sum_{x \in \mathbb{Z}^d} |x_1|^q \sum_{s,s' \in \mathbb{N}} m^{s+s'} \varphi_s^{\mathrm{SAW}}(x)\varphi_{s'}^{\mathrm{SAW}}(x) \right)$$

$$\times \left(\sup_{x \in \mathbb{Z}^d} |x_1|^{2l} \sum_{s'' \in \mathbb{N}} m^{s''} \varphi_{s''}^{\mathrm{SAW}}(x) \right)$$

$$\leq B^{(q)} W^{(2l)},$$

where

$$B^{(\nu)} = \sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} |x_1|^\nu \sum_{t\in\mathbb{N}} m^t \varphi_t^{\mathrm{SAW}}(x) \sum_{s=0}^\infty m^s \varphi_s^{\mathrm{SAW}}(y-x),$$

$$W^{(\nu)} = \sup_{x\in\mathbb{Z}^d} |x_1|^\nu \sum_{t\in\mathbb{N}} m^t \varphi_t^{\mathrm{SAW}}(x).$$

Similarly to the above and the derivation of [5], formula (2.42), by using (4.1) and diagrammatic bounds of the form (4.3), we can show that

(4.6)
$$\sum_{t=0}^\infty m^t \sum_{x\in\mathbb{Z}^d} |x_1|^{2l+q} \pi_t^{(N)}(x)$$
$$\leq N^{2l+q+2} (B^{(0)})^{N-2} B^{(q)} W^{(2l)} \qquad [N \geq 2].$$

It is immediate from the definition (1.4) that $\varphi_t^{\mathrm{SAW}}(x) \leq \delta_{x,o}\delta_{t,0} + (D * \varphi_{t-1}^{\mathrm{SAW}})(x)$. By this, we have

(4.7)
$$B^{(0)} \leq \sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{N}} m^t \varphi_t^{\mathrm{SAW}}(x)\left(\delta_{x,y} + \sum_{s\in\mathbb{N}} m^s (D * \varphi_{s-1}^{\mathrm{SAW}})(y-x)\right)$$
$$\leq W^{(0)} + \sup_{y\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{N}} m^t (D * \varphi_{t-1}^{\mathrm{SAW}})(x) \sum_{s\in\mathbb{N}} m^s (D * \varphi_{s-1}^{\mathrm{SAW}})(y-x)$$
$$\leq W^{(0)} + m^2 \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \hat{D}(k)^2 |\hat{\varphi}^{\mathrm{SAW}}(k,m)|^2$$

and

(4.8)
$$W^{(0)} \leq \sup_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{N}} m^t (D * \varphi_{t-1}^{\mathrm{SAW}})(x)$$
$$\leq m\|D\|_\infty + \sup_{x\in\mathbb{Z}^d} \sum_{t=2}^\infty m^t (D * D * \varphi_{t-2}^{\mathrm{SAW}})(x)$$
$$\leq m\|D\|_\infty + m^2 \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \hat{D}(k)^2 |\hat{\varphi}^{\mathrm{SAW}}(k,m)|.$$

By (3.16) for $n = 0$ and $\|D\|_\infty = O(L^{-d})$, we can show that $B^{(0)} = O(L^{-d})$ uniformly in $m \leq m_c$ if $d > 4$, hence the summability of (4.6) over $N \geq 2$ when $L \gg 1$. Moreover, by (3.16) for $n = 2l$,

(4.9)
$$W^{(2l)} \leq \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} |\nabla_1^{2l} \hat{\varphi}^{\mathrm{SAW}}(k,m)|$$
$$\leq O\left(\left(1 - \frac{m}{m_c}\right)^{1-l}\right) \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{|k|^4}$$
$$\overset{d\geq 4}{=} O\left(\left(1 - \frac{m}{m_c}\right)^{1-l}\right).$$

Therefore,

$$\sum_{N=2}^{\infty} \sum_{t=0}^{\infty} m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{2l+q} \pi_t^{(N)}(x) \le O\left(\left(1 - \frac{m}{m_c}\right)^{1-l}\right) B^{(q)}.$$

To complete the proof of (4.2) for self-avoiding walk, it suffices to show that there is an $\epsilon > 0$ such that $B^{(q)} = O((1 - \frac{m}{m_c})^{-q/2+\epsilon})$. For $q = 2$, we use (3.16) for $n = 2$ and take an arbitrary $\epsilon \in (0, 1 \wedge \frac{d-4}{2})$ to obtain

$$B^{(2)} \le \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} |\hat{\varphi}^{\text{SAW}}(k, m) \nabla_1^2 \hat{\varphi}^{\text{SAW}}(k, m)|$$

$$\le O\left(\left(1 - \frac{m}{m_c}\right)^{-1+\epsilon}\right) \int_{[-\pi,\pi]^d} \frac{d^d k}{|k|^{2(2+\epsilon)}} \le O\left(\left(1 - \frac{m}{m_c}\right)^{-1+\epsilon}\right).$$

For $q \in (0, 2)$, we first note that

$$(4.10) \quad B^{(q)} \le \frac{1}{K_q} \int_0^\infty \frac{du}{u^{1+q}} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} |\hat{\varphi}^{\text{SAW}}(k, m) \bar{\Delta}_{\vec{u}} \hat{\varphi}^{\text{SAW}}(k, m)|.$$

It is known that, by [6], Proposition 2.6, with an improvement due to the same argument as in [3], Proposition 2.1,

$$|\bar{\Delta}_{\vec{u}} \hat{\varphi}^{\text{SAW}}(k, m)|$$

$$\le \sum_{(j,j')=(0,\pm 1),(1,-1)} \frac{O(1 - \hat{D}(\vec{u}))}{1 - m/m_c + 1 - \hat{D}(k + j\vec{u})}$$

$$\times \frac{1}{1 - m/m_c + 1 - \hat{D}(k + j'\vec{u})}$$

holds in the current setting, where the $O(1 - \hat{D}(\vec{u}))$ term is uniform in $k \in [-\pi, \pi]^d$ and $m \le m_c$. Substituting this, and (3.16) for $n = 0$, into (4.10), and using the translation invariance and the $\mathbb{Z}^d$-symmetry of D and the Schwarz inequality (see [3], formulas (4.27)–(4.29)), we end up with

$$B^{(q)} \le \int_0^\infty \frac{du}{u^{1+q}} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{O(1 - \hat{D}(\vec{u}))}{(1 - m/m_c + 1 - \hat{D}(k))^2}$$

$$\times \frac{1}{1 - m/m_c + 1 - \hat{D}(k - \vec{u})}$$

$$\le \int_0^\infty du \frac{1 - \hat{D}(\vec{u})}{u^{1+q}} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{O((1 - m/m_c)^{-q/2+\epsilon})}{(1 - \hat{D}(k))^{2-q/2+\epsilon}(1 - \hat{D}(k - \vec{u}))}$$

for any $\epsilon \in (0, \frac{q}{2})$. However, by following the proof of [3], formula (4.30), we can show that

$$\int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{1}{(1 - \hat{D}(k))^{2-q/2+\epsilon}(1 - \hat{D}(k - \vec{u}))} \le O(u^{(d-6+q-2\epsilon)\wedge 0}),$$

hence

$$
B^{(q)} \leq O\left(\left(1 - \frac{m}{m_c}\right)^{-q/2+\epsilon}\right)\left(\int_0^1 \frac{du}{u} u^{(d-4-2\epsilon)\wedge(2-q)} + \int_1^\infty \frac{du}{u^{1+q}}\right)
$$

$$
(4.11) \qquad = O\left(\left(1 - \frac{m}{m_c}\right)^{-q/2+\epsilon}\right)
$$

if $\epsilon < \frac{d-4}{2}$. This completes the proof of (4.2) for self-avoiding walk.

For oriented percolation, similarly to the proof of [3], Lemma 3, by using (4.1) and diagrammatic bounds of the form (4.4), we can show that, for $N \geq 0$,

$$
\sum_{t=0}^\infty m^t \sum_{x \in \mathbb{Z}^d} |x_1|^{2l+q} \pi_t^{(N)}(x)
$$

$$
(4.12) \qquad \leq (N+1)^{2l+q}\left(T^{(0)}\right)^{N-2}\left(\left(N(1+T^{(0)}) + T^{(0)}\right)T^{(0)}V^{(q)}\right.
$$
$$
\left. + N\left((N-1)(1+T^{(0)}) + 3T^{(0)}\right)T^{(q)}V^{(0)}\right),
$$

where

$$
V^{(\nu)} = \sup_{(x,t)\in\mathbb{Z}^{d+1}} \sum_{(y,s)\in\mathbb{Z}^{d+1}} |y_1|^{2l}(mD * \varphi_s^{\mathrm{OP}})(y)m^s|y_1 - x_1|^\nu
$$
$$
\times (D * \varphi_{s-t}^{\mathrm{OP}})(y - x),
$$

$$
T^{(\nu)} = \sup_{(x,t)\in\mathbb{Z}^{d+1}} \sum_{(y,s),(y',s')\in\mathbb{Z}^{d+1}} (mD * \varphi_s^{\mathrm{OP}})(y)m^s|y_1 - x_1|^\nu
$$
$$
\times (D * \varphi_{s'-t}^{\mathrm{OP}})(y' - x)
$$
$$
\times \left(\varphi_{s-s'}^{\mathrm{OP}}(y - y') + \varphi_{s'-s}^{\mathrm{OP}}(y' - y)\right).
$$

Notice that

$$
(4.13) \qquad T^{(0)} \leq 2m \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \hat{D}(k)^2 \int_{-\pi}^\pi \frac{\mathrm{d}\theta}{2\pi} |\hat{\varphi}^{\mathrm{OP}}(k, me^{i\theta})||\hat{\varphi}^{\mathrm{OP}}(k, e^{i\theta})|^2.
$$

Using (3.16) for $n = 0$ and $\|D\|_\infty = O(L^{-d})$, we can show that $T^{(0)} = O(L^{-d})$ uniformly in $m \leq m_c$ if $d > 4$ and $L \gg 1$, hence the summability of (4.12) over $N \geq 0$. Moreover, by (3.16) for $n = 0, 2l$ and using $|\hat{D}(k)| \leq 1$, we have

$$
V^{(0)} \leq 2^{2l} m \left(\int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{-\pi}^\pi \frac{\mathrm{d}\theta}{2\pi} |\nabla_1^{2l} \hat{\varphi}^{\mathrm{OP}}(k, me^{i\theta})||\hat{\varphi}^{\mathrm{OP}}(k, e^{i\theta})|\right.
$$

$$
(4.14) \qquad + \sum_{x \in \mathbb{Z}^d} |x_1|^{2l} D(x) \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{-\pi}^\pi \frac{\mathrm{d}\theta}{2\pi} |\hat{\varphi}^{\mathrm{OP}}(k, me^{i\theta})|
$$

$$
\left. \times |\hat{\varphi}^{\mathrm{OP}}(k, e^{i\theta})|\right)
$$

$$
\overset{d \geq 4}{=} O\left(\left(1 - \frac{m}{m_c}\right)^{1-l}\right).
$$

To complete the proof of (4.2), it thus suffices to show that there is an $\epsilon > 0$ such that

$$T^{(q)} = O\left(\left(1 - \frac{m}{m_c}\right)^{-q/2+\epsilon}\right), \qquad V^{(q)} = O\left(\left(1 - \frac{m}{m_c}\right)^{1-l-q/2+\epsilon}\right).$$

Here, we only explain the proof of the bound on $V^{(2)}$; the bound on $T^{(2)}$ can be proven quite similarly and the bounds on $T^{(q)}$ and $V^{(q)}$ for $q \in (0, 2)$ can be proven by following a similar line of argument from (4.10) through to (4.11). To prove the bound on $V^{(2)}$, we first note that

$$
\begin{aligned}
V^{(2)} \leq 2^{2l+2} m \bigg(&\int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} |\nabla_1^{2l} \hat{\varphi}^{\mathrm{OP}}(k, me^{i\theta})| |\nabla_1^2 \hat{\varphi}^{\mathrm{OP}}(k, e^{i\theta})| \\
&+ \sigma^2 \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} |\nabla_1^{2l} \hat{\varphi}^{\mathrm{OP}}(k, me^{i\theta})| |\hat{\varphi}^{\mathrm{OP}}(k, e^{i\theta})| \\
&+ \sum_{x \in \mathbb{Z}^d} x_1^{2l} D(x) \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} |\hat{\varphi}^{\mathrm{OP}}(k, me^{i\theta})| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times |\nabla_1^2 \hat{\varphi}^{\mathrm{OP}}(k, e^{i\theta})| \\
&+ \sigma^2 \sum_{x \in \mathbb{Z}^d} x_1^{2l} D(x) \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} |\hat{\varphi}^{\mathrm{OP}}(k, me^{i\theta})| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times |\hat{\varphi}^{\mathrm{OP}}(k, e^{i\theta})| \bigg).
\end{aligned}
$$

(4.15)

It is immediate from (3.16) for $n = 0, 2$ that the last two lines are both $O(1)$ for $d > 4$. Moreover, by (4.14), the second line is $O((1 - \frac{m}{m_c})^{1-l})$ for $d > 4$. For the first line, we use the following bounds due to (3.16) for $n = 2, 2l$: for any $\epsilon \in (0, 1)$,

$$|\nabla_1^{2l} \hat{\varphi}^{\mathrm{OP}}(k, me^{i\theta})| \leq \frac{O((1 - m/m_c)^{-l+\epsilon})}{(|\theta| + |k|^2)^{1+\epsilon}}, \qquad |\nabla_1^2 \hat{\varphi}^{\mathrm{OP}}(k, e^{i\theta})| \leq O(|k|^{-4}),$$

where the $O((1 - \frac{m}{m_c})^{-l+\epsilon})$ term is uniform in $(k, \theta) \in [-\pi, \pi]^{d+1}$ and the $O(|k|^{-4})$ term is uniform in $\theta \in [-\pi, \pi]$. We then obtain that

the first line of (4.15)

$$
\begin{aligned}
&\leq \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{|k|^4} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{O((1 - m/m_c)^{-l+\epsilon})}{(|\theta| + |k|^2)^{1+\epsilon}} \\
&\leq O\left(\left(1 - \frac{m}{m_c}\right)^{-l+\epsilon}\right) \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{|k|^{4+2\epsilon}} = O\left(\left(1 - \frac{m}{m_c}\right)^{-l+\epsilon}\right)
\end{aligned}
$$

if $\epsilon < \frac{d-4}{2}$. This completes the proof of (4.2) for oriented percolation. This completes the proof of Lemma 4.2. $\square$

4.2. *Proof of Proposition* 3.1. First, we note that (3.7) implies (3.8). To see this, we first substitute (2.12) and (3.7) into (3.5) and then use (1.21) [see (2.13)] to obtain

$$\mu = \frac{m_c \, \partial_m \hat{J}_{m_c}(0)(1 - m/m_c) + O((1 - m/m_c)^{1+\epsilon})}{M_c v_\alpha + O((1 - m/m_c)^\epsilon)}$$

$$= \frac{1 - m/m_c}{C_{\mathrm{II}} v_\alpha} + O\left(\left(1 - \frac{m}{m_c}\right)^{1+\epsilon}\right).$$

Therefore, to complete the proof of Proposition 3.1, it suffices to show (3.7).

It is easier to prove (3.7) for $\alpha \leq 2$. In this case, M in (2.14) is reduced to

$$M = \begin{cases} m, & \text{RW/SAW}, \\ \hat{\pi}^{\mathrm{OP}}(0)\, pm, & \text{OP}. \end{cases}$$

Therefore, (3.7) is trivial for random walk and self-avoiding walk. For oriented percolation, we use (2.5) and (2.10) to obtain

$$M_c - M = \hat{\pi}_c^{\mathrm{OP}}(0)\, p(m_c - m) + \left(\hat{\pi}_c^{\mathrm{OP}}(0) - \hat{\pi}^{\mathrm{OP}}(0)\right) pm$$

$$= 1 - \frac{m}{m_c} + O(L^{-d})\left(1 - \frac{m}{m_c}\right),$$

where the $O(L^{-d})$ term is uniform in $m \leq m_c$. This implies (3.7).

It remains to prove (3.7) for $\alpha > 2$. In fact, we only need investigate the crossover terms in (2.14) that are proportional to $\mathbb{1}_{\{\alpha > 2\}}$ and show that

$$(4.16) \qquad |\nabla_1^2 \hat{\pi}_c(0) - \nabla_1^2 \hat{\pi}(0)| \leq O\left(\left(1 - \frac{m}{m_c}\right)^\epsilon\right)$$

since the above proof for $\alpha \leq 2$ directly applies to the noncrossover terms. Notice that, for $\epsilon \in (0, 1)$,

$$0 \leq m_c^t - m^t \leq m_c^t \left(1 - \left(\frac{m}{m_c}\right)^t\right)^{1-\epsilon} \left(\frac{1 - (m/m_c)^t}{1 - m/m_c}\right)^\epsilon \left(1 - \frac{m}{m_c}\right)^\epsilon$$

$$\leq m_c^t t^\epsilon \left(1 - \frac{m}{m_c}\right)^\epsilon$$

so that

$$|\nabla_1^2 \hat{\pi}_c(0) - \nabla_1^2 \hat{\pi}(0)| \leq \sum_{t \in \mathbb{N}} (m_c^t - m^t) \sum_{x \in \mathbb{Z}^d} x_1^2 |\pi_t(x)|$$

$$\leq \left(1 - \frac{m}{m_c}\right)^\epsilon \sum_{t \in \mathbb{N}} t^\epsilon m_c^t \sum_{x \in \mathbb{Z}^d} x_1^2 |\pi_t(x)|.$$

Moreover, since

$$t^\epsilon = \frac{t}{t^{1-\epsilon}} = \frac{t}{\Gamma(1 - \epsilon)} \int_0^\infty \ell^{-\epsilon} e^{-\ell t} \, d\ell,$$

we have

$$(4.17)\qquad \begin{aligned}&|\nabla_1^2\hat{\pi}_c(0)-\nabla_1^2\hat{\pi}(0)|\\ &\qquad\leq\frac{(1-m/m_c)^\epsilon}{\Gamma(1-\epsilon)}\int_0^\infty\frac{d\ell}{\ell^\epsilon}\sum_{t\in\mathbb{N}}t(m_c e^{-\ell})^t\sum_{x\in\mathbb{Z}^d}x_1^2|\pi_t(x)|.\end{aligned}$$

To show (4.16), it thus suffices to prove that the above integral with respect to ℓ is $O(1)$ for sufficiently small ϵ.

First, we consider self-avoiding walk. By the diagrammatic bound on $\pi_t^{(2)}(x)$ in (4.3) [see (4.5)], we readily obtain

$$\begin{aligned}\sum_{t\in\mathbb{N}}tm^t\sum_{x\in\mathbb{Z}^d}x_1^2\pi_t^{(2)}(x)&\leq\sum_{s,s',s''\in\mathbb{N}}(s+s'+s'')m^{s+s'+s''}\\ &\qquad\qquad\times\sum_{x\in\mathbb{Z}^d}x_1^2\varphi_s^{\mathrm{SAW}}(x)\varphi_{s'}^{\mathrm{SAW}}(x)\varphi_{s''}^{\mathrm{SAW}}(x)\\ &\leq3W^{(2)}\sum_{x\in\mathbb{Z}^d}\sum_{s,s'\in\mathbb{N}}sm^{s+s'}\varphi_s^{\mathrm{SAW}}(x)\varphi_{s'}^{\mathrm{SAW}}(x)\\ &\leq3B'W^{(2)},\end{aligned}$$

where

$$(4.18)\qquad B'\equiv B'(m)=\sup_{y\in\mathbb{Z}^d}\sum_{x\in\mathbb{Z}^d}\sum_{t\in\mathbb{N}}tm^t\varphi_t^{\mathrm{SAW}}(x)\sum_{s=0}^\infty m^s\varphi_s^{\mathrm{SAW}}(y-x).$$

Similarly to the above and the derivation of (4.6), we can show that, by (4.3) and (4.1),

$$\sum_{t\in\mathbb{N}}tm^t\sum_{x\in\mathbb{Z}^d}x_1^2\pi_t^{(N)}(x)\leq N^4(B^{(0)})^{N-2}B'W^{(2)}\qquad[N\geq2].$$

Since $B^{(0)}=O(L^{-d})$ and $W^{(2)}=O(1)$ uniformly in $m\leq m_c$ if $d>4$ [see formulas (4.7)–(4.9)], we obtain that, for $L\gg1$,

$$(4.19)\qquad\begin{aligned}&\int_0^\infty\frac{d\ell}{\ell^\epsilon}\sum_{t\in\mathbb{N}}t(m_c e^{-\ell})^t\sum_{x\in\mathbb{Z}^d}x_1^2|\pi_t(x)|\\ &\qquad\leq\underbrace{\sum_{N=2}^\infty O(N^4)O(L^{-d})^{N-2}}_{O(1)}\int_0^\infty\frac{d\ell}{\ell^\epsilon}B'(m_c e^{-\ell}).\end{aligned}$$

We now show that the integral of $B'(m_c e^{-\ell})/\ell^\epsilon$ is uniformly bounded if $\epsilon<\frac{d-4}{2}$. First, we replace $t\varphi_t^{\mathrm{SAW}}(x)$ in (4.18) by the following bound due to sub-additivity:

$$t\varphi_t^{\mathrm{SAW}}(x)=\sum_{s=1}^t\varphi_t^{\mathrm{SAW}}(x)\leq\sum_{s=1}^t(\varphi_{s-1}^{\mathrm{SAW}}*D*\varphi_{t-s}^{\mathrm{SAW}})(x).$$

Then, by using $|\hat{D}(k)| \le 1$ and (3.16) for $n = 0$, we obtain

(4.20)
$$B'(m_{\mathrm{c}}e^{-\ell}) \le m_{\mathrm{c}}e^{-\ell} \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} |\hat{\varphi}^{\mathrm{SAW}}(k, m_{\mathrm{c}}e^{-\ell})|^3$$

$$\le O(1) \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{|k|^4} \frac{e^{-\ell}}{1 - e^{-\ell} + |k|^2},$$

where the $O(1)$ term is independent of ℓ. However, for $\epsilon \in (0, 1)$,

$$\int_0^\infty \frac{\mathrm{d}\ell}{\ell^\epsilon} \frac{e^{-\ell}}{1 - e^{-\ell} + |k|^2} \le \frac{1}{1 - e^{-1}} \left(\int_0^1 \frac{\mathrm{d}\ell}{\ell^\epsilon} \frac{1}{\ell + |k|^2} + \int_1^\infty \frac{\mathrm{d}\ell}{\ell^\epsilon} e^{-\ell} \right)$$

$$\le \frac{1}{1 - e^{-1}} \left(\int_0^{|k|^2} \frac{\mathrm{d}\ell}{\ell^\epsilon} \frac{1}{|k|^2} + \int_{|k|^2}^1 \frac{\mathrm{d}\ell}{\ell^{1+\epsilon}} + 1 \right)$$

$$= O(|k|^{-2\epsilon}).$$

Therefore, if $\epsilon < \frac{d-4}{2}$, then we obtain

(4.21)
$$\int_0^\infty \frac{\mathrm{d}\ell}{\ell^\epsilon} B'(m_{\mathrm{c}}e^{-\ell}) \le O(1) \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{|k|^{4+2\epsilon}} = O(1).$$

Combining (4.17), (4.19) and (4.21), we complete the proof of (4.16) for self-avoiding walk.

For oriented percolation, similarly to the derivation of (4.12), we can show that, for $N \ge 0$,

$$\sum_{t \in \mathbb{N}} t m^t \sum_{x \in \mathbb{Z}^d} x_1^2 \pi_t^{(N)}(x)$$

$$\le (N+1)^2 (T^{(0)})^{N-2} \big((N(1 + T^{(0)}) + T^{(0)}) T^{(0)} V'$$
$$+ N((N-1)(1 + T^{(0)}) + 3T^{(0)}) T' V^{(0)} \big),$$

where

$$V' \equiv V'(m) = \sup_{(x,t) \in \mathbb{Z}^{d+1}} \sum_{(y,s) \in \mathbb{Z}^{d+1}} |y_1|^2 (mD * \varphi_s^{\mathrm{OP}})(y) m^s |s - t + 1|$$
$$\times (D * \varphi_{s-t}^{\mathrm{OP}})(y - x),$$

$$T' \equiv T'(m) = \sup_{(x,t) \in \mathbb{Z}^{d+1}} \sum_{(y,s),(y',s') \in \mathbb{Z}^{d+1}} (mD * \varphi_s^{\mathrm{OP}})(y) m^s |s' - t + 1|$$
$$\times (D * \varphi_{s'-t}^{\mathrm{OP}})(y' - x)$$
$$\times \big(\varphi_{s-s'}^{\mathrm{OP}}(y - y') + \varphi_{s'-s}^{\mathrm{OP}}(y' - y) \big).$$

Since $T^{(0)} = O(L^{-d})$ and $V^{(0)}|_{l=1} = O(1)$ uniformly in $m \leq m_c$ if $d > 4$ and $p \leq p_c$ [see formulas (4.13) and (4.14)], we obtain that, for $L \gg 1$,

(4.22)

$$\int_0^\infty \frac{d\ell}{\ell^\epsilon} \sum_{t \in \mathbb{N}} t (m_c e^{-\ell})^t \sum_{x \in \mathbb{Z}^d} x_1^2 |\pi_t(x)|$$

$$\leq O(1) \int_0^\infty \frac{d\ell}{\ell^\epsilon} (V'(m_c e^{-\ell}) + T'(m_c e^{-\ell})).$$

However, by the Markov property,

$$(t+1)(D * \varphi_t^{\mathrm{OP}})(x) = \sum_{s=0}^t (D * \varphi_t^{\mathrm{OP}})(x) \leq \sum_{s=0}^t (\varphi_s^{\mathrm{OP}} * D * \varphi_{t-s}^{\mathrm{OP}})(x).$$

Applying this bound to the definitions of V' and T' and then using $|\hat{D}(k)| \leq 1$ and (3.16) for $n = 0, 2$, we obtain

$$\left. \begin{array}{c} V'(m_c e^{-\ell}) \\ T'(m_c e^{-\ell}) \end{array} \right\} \leq O(1) \int_{[-\pi,\pi]^d} \frac{d^d k}{|k|^4} \frac{e^{-\ell}}{1 - e^{-\ell} + |k|^2}.$$

Recalling (4.20) and (4.21), we conclude that (4.22) is uniformly bounded. This completes the proof of (4.16) for oriented percolation. We have thus completed the proof of Proposition 3.1.

APPENDIX

A.1. Asymptotics of $1 - \hat{D}(k)$ for small k. In this appendix, we will use the following notation for convenience:

$$\||x\||_\ell = |x| \vee \ell \qquad [\ell > 0].$$

LEMMA A.1. *Let $\alpha, \rho > 0$ and*

$$h(x) = \frac{1 + O(\||x\||_1^{-\rho})}{\||x\||_1^{d+\alpha}} \qquad [x \in \mathbb{R}^d].$$

Suppose that h is a rotation-invariant function. Then, there exist $\epsilon > 0$ and $v_\alpha = O(L^{\alpha \wedge 2})$ such that, for $|k| < 1/L$, the 1-step distribution D in (1.3) satisfies

(A.1) $$1 - \hat{D}(k) = v_\alpha |k|^{\alpha \wedge 2} \times \begin{cases} 1 + O((L|k|)^\epsilon), & \alpha \neq 2, \\ \log \dfrac{1}{L|k|} + O(1), & \alpha = 2. \end{cases}$$

PROOF. The case for $\alpha > 2$ is easy. By the Taylor expansion of $1 - \cos(k \cdot x)$ and using the $\mathbb{Z}^d$-symmetry of D,

$$1 - \hat{D}(k) = \sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) D(x) = \frac{|k|^2}{2d} \sum_{x \in \mathbb{Z}^d} |x|^2 D(x) + O((L|k|)^{2+\epsilon})$$

holds provided that $0 < \epsilon < 2 \wedge (\alpha - 2)$. This proves (A.1) with $v_\alpha \equiv \sigma^2/(2d) = O(L^2)$.

It remains to prove (A.1) for $\alpha \leq 2$. First, we note that, by definition,

$$D(x) = \frac{c_h}{L^d} h(x/L) \qquad [x \in \mathbb{Z}^d],$$

where

$$c_h = \left(\frac{1}{L^d} \sum_{y \in \mathbb{Z}^d / L} h(y) \right)^{-1} = \int_{\mathbb{R}^d} h(y) \, d^d y + O(L^{-1}).$$

Taking the Fourier transform yields

$$1 - \hat{D}(k) = \frac{c_h}{L^d} \sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) h\left(\frac{x}{L}\right)$$

$$= \frac{c_h}{(L|k|)^d} \left(|k|^d \sum_{y \in |k|\mathbb{Z}^d} (1 - \cos(e_k \cdot y)) h\left(\frac{y}{L|k|}\right) \right),$$

where $e_k = k/|k|$. By the Riemann sum approximation for small k and the rotational invariance of h, we obtain

$$1 - \hat{D}(k) = \frac{c_h(1 + O(|k|))}{(L|k|)^d} \int_{|y| \geq |k|} (1 - \cos(e_k \cdot y)) h\left(\frac{y}{L|k|}\right) d^d y$$

$$= \frac{c_h(1 + O(|k|))}{(L|k|)^d} \int_{|y| \geq |k|} (1 - \cos y_1) h\left(\frac{y}{L|k|}\right) d^d y$$

$$= c_h (L|k|)^\alpha (1 + O(|k|))$$

$$\times \int_{|y| \geq |k|} (1 - \cos y_1) \left(\frac{1}{\|\|y\|\|_{L|k|}^{d+\alpha}} + \frac{O((L|k|)^\rho)}{\|\|y\|\|_{L|k|}^{d+\alpha+\rho}} \right) d^d y.$$

This is the starting point of the analysis for $\alpha \leq 2$.

For $\alpha < 2$, we note that

$$\int_{|y| \geq |k|} \frac{1 - \cos y_1}{\|\|y\|\|_{L|k|}^{d+\alpha}} \, d^d y = \int_{|y| \geq L|k|} \frac{1 - \cos y_1}{|y|^{d+\alpha}} \, d^d y$$

$$+ \underbrace{\int_{|k| \leq |y| < L|k|} \frac{1 - \cos y_1}{(L|k|)^{d+\alpha}} \, d^d y}_{O((L|k|)^{2-\alpha})}$$

$$= \int_{\mathbb{R}^d} \frac{1 - \cos y_1}{|y|^{d+\alpha}} \, d^d y - \underbrace{\int_{|y| < L|k|} \frac{1 - \cos y_1}{|y|^{d+\alpha}} \, d^d y}_{O((L|k|)^{2-\alpha})}$$

$$+ O((L|k|)^{2-\alpha}),$$

where we have used $L|k| < 1$ to estimate the error terms. Moreover,

$$\int_{|y| \geq |k|} \frac{1 - \cos y_1}{\||y\||_{L|k|}^{d+\alpha+\rho}} \, d^d y = \underbrace{\int_{|y| \geq 1} \frac{1 - \cos y_1}{|y|^{d+\alpha+\rho}} \, d^d y + \int_{L|k| \leq |y| < 1} \frac{1 - \cos y_1}{|y|^{d+\alpha+\rho}} \, d^d y}_{O(1)}$$

$$+ \underbrace{\int_{|k| \leq |y| < L|k|} \frac{1 - \cos y_1}{(L|k|)^{d+\alpha+\rho}} \, d^d y}_{O((L|k|)^{2-\alpha-\rho})},$$

where

$$\int_{L|k| \leq |y| < 1} \frac{1 - \cos y_1}{|y|^{d+\alpha+\rho}} \, d^d y = \begin{cases} O(1), & \rho < 2 - \alpha, \\ O\left(\log \dfrac{1}{L|k|}\right), & \rho = 2 - \alpha, \\ O((L|k|)^{2-\alpha-\rho}), & \rho > 2 - \alpha. \end{cases}$$

This proves (A.1) with $0 < \epsilon < 1 \wedge (2 - \alpha) \wedge \rho$ and

$$v_\alpha = c_h L^\alpha \int_{\mathbb{R}^d} \frac{1 - \cos y_1}{|y|^{d+\alpha}} \, d^d y.$$

For $\alpha = 2$, we note that

$$\int_{|y| \geq |k|} \frac{1 - \cos y_1}{\||y\||_{L|k|}^{d+2}} \, d^d y = \underbrace{\int_{|y| \geq 1} \frac{1 - \cos y_1}{|y|^{d+2}} \, d^d y + \int_{L|k| \leq |y| < 1} \frac{1 - \cos y_1}{|y|^{d+2}} \, d^d y}_{O(1)}$$

$$+ \underbrace{\int_{|k| \leq |y| < L|k|} \frac{1 - \cos y_1}{(L|k|)^{d+2}} \, d^d y}_{O(1)}.$$

By the Taylor expansion of $1 - \cos y_1$ and using $|y|^2 = \sum_{j=1}^d y_j^2$, we obtain

$$\int_{L|k| \leq |y| < 1} \frac{1 - \cos y_1}{|y|^{d+2}} \, d^d y = \frac{1}{2} \int_{L|k| \leq |y| < 1} \frac{y_1^2}{|y|^{d+2}} \, d^d y + O(1)$$

$$= \frac{1}{2d} \int_{L|k| \leq |y| < 1} \frac{1}{|y|^d} \, d^d y + O(1)$$

$$= \frac{\omega_d}{2d} \log \frac{1}{L|k|} + O(1),$$

where $\omega_d \equiv 2\pi^{d/2} / \Gamma(d/2)$ is the surface area of the unit d-sphere. Moreover,

$$\int_{|y| \geq |k|} \frac{1 - \cos y_1}{\||y\||_{L|k|}^{d+2+\rho}} \, d^d y = \underbrace{\int_{|y| \geq L|k|} \frac{1 - \cos y_1}{|y|^{d+2+\rho}} \, d^d y}_{O((L|k|)^{-\rho})}$$

$$+ \underbrace{\int_{|k| \leq |y| < L|k|} \frac{1 - \cos y_1}{(L|k|)^{d+2+\rho}} \, d^d y}_{O((L|k|)^{-\rho})}.$$

This proves (A.1) with $v_2 = c_h L^2 \omega_d / (2d)$. □

A.2. Identity for the constant K_r.

LEMMA A.2. *For $r \in (0, 2)$,*

(A.2) $$K_r \equiv \int_0^\infty \frac{1 - \cos v}{v^{1+r}} \, dv = \frac{\pi}{2\Gamma(r+1)\sin(r\pi/2)}.$$

PROOF. Below, we prove (A.2) only for $r \in (0, 1]$. Since the definition of K_r and the rightmost expression in (A.2) are both analytic in $r \in \mathbb{C}$ with $0 < \Re(r) < 2$, we can extend (A.2) to $r \in (1, 2)$ using analytic continuation.

First, we rewrite K_r as

(A.3)
$$K_r = \int_0^\infty \frac{du}{u^{1+r}} \int_0^u \sin v \, dv = \frac{1}{r} \int_0^\infty \frac{\sin v}{v^r} \, dv$$
$$= \lim_{\substack{R \to \infty \\ \delta \to 0}} \frac{1}{2ir} \int_\delta^R \frac{e^{iv} - e^{-iv}}{v^r} \, dv.$$

For $a > 0$, we let

$$\gamma_a^\pm = \left\{ z = ae^{\pm i\theta} : \theta \text{ increases from } 0 \text{ to } \frac{\pi}{2} \right\},$$

$$\eta^\pm = \{ z = \pm iv : v \text{ increases from } \delta \text{ to } R \}.$$

Then, by the Cauchy integral formula,

$$\int_\delta^R \frac{e^{iv}}{v^r} \, dv = \int_{\gamma_\delta^+} \frac{e^{iz}}{z^r} \, dz + \int_{\eta^+} \frac{e^{iz}}{z^r} \, dz - \int_{\gamma_R^+} \frac{e^{iz}}{z^r} \, dz$$

$$= i \int_0^{\pi/2} \frac{e^{i\delta e^{i\theta}}}{(\delta e^{i\theta})^{r-1}} \, d\theta + i^{1-r} \int_\delta^R \frac{e^{-v}}{v^r} \, dv - \underbrace{i \int_0^{\pi/2} \frac{e^{iRe^{i\theta}}}{(Re^{i\theta})^{r-1}} \, d\theta}_{O(R^{-r})}.$$

Similarly,

$$\int_\delta^R \frac{e^{-iv}}{v^r} \, dv = \int_{\gamma_\delta^-} \frac{e^{-iz}}{z^r} \, dz + \int_{\eta^-} \frac{e^{-iz}}{z^r} \, dz - \int_{\gamma_R^-} \frac{e^{-iz}}{z^r} \, dz$$

$$= -i \int_0^{\pi/2} \frac{e^{-i\delta e^{-i\theta}}}{(\delta e^{-i\theta})^{r-1}} \, d\theta + (-i)^{1-r} \int_\delta^R \frac{e^{-v}}{v^r} \, dv + O(R^{-r}).$$

Substituting these expressions back into (A.3) yields

(A.4)
$$K_r = \lim_{\substack{R \to \infty \\ \delta \to 0}} \left(\frac{\delta^{1-r}}{2r} \int_0^{\pi/2} \left(\frac{e^{i\delta e^{i\theta}}}{e^{i\theta(r-1)}} + \frac{e^{-i\delta e^{-i\theta}}}{e^{-i\theta(r-1)}} \right) d\theta \right.$$
$$\left. + i^{-r} \frac{1 + (-1)^{-r}}{2r} \int_\delta^R \frac{e^{-v}}{v^r} \, dv \right).$$

If $r = 1$, then the second term is absent due to the cancelation $1 + (-1) = 0$. By dominated convergence, we obtain

$$(A.5) \qquad K_1 = \lim_{\delta \to 0} \frac{1}{2} \int_0^{\pi/2} (e^{i\delta e^{i\theta}} + e^{-i\delta e^{-i\theta}}) \, d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

If $r \in (0, 1)$, on the other hand, the first term in (A.4) is $O(\delta^{1-r})$ and therefore goes to zero as $\delta \to 0$. Since $(-1)^{-r} = (-1)^r = i^{2r}$ and $i^r + i^{-r} = 2\cos\frac{r\pi}{2}$, we obtain

$$K_r = \frac{\cos(r\pi/2)}{r} \int_0^\infty \frac{e^{-v}}{v^r} \, dv = \frac{\cos(r\pi/2)}{r} \Gamma(1 - r).$$

Using the well-known relations $\Gamma(1-r)\Gamma(r) = \pi/\sin(r\pi)$ and $r\Gamma(r) = \Gamma(r+1)$, we finally arrive at

$$K_r = \frac{\cos(r\pi/2)}{r\Gamma(r)} \frac{\pi}{\sin(r\pi)} = \frac{\pi}{2\Gamma(r+1)\sin(r\pi/2)}.$$

This is also valid for $r = 1$, due to (A.5). This completes the proof of Lemma A.2. $\square$

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