Path Integral Representation of the Index of Kähler-Dirac Operators on an Infinite Dimensional Manifold

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#	Author	Title
1.	Y. Okabe,	On the theory of discrete KMO-Langevin equations with reflection positivity (I)
2.	Y. Giga and T. Kambe,	Large time behavior of the vorticity of two-dimensional flow and its application to vortex formation

Path Integral Representation of the Index of Kähler-Dirac Operators on an Infinite Dimensional Manifold

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ABSTRACT

Operators of Kähler-Dirac type are defined in an abstract infinite dimensional Boson-Fermion Fock space and a path integral representation of their index is established. As preliminaries to this end, some trace formulas associated with "Gibbs states" are derived in both an abstract Boson and Fermion Fock space. This is done by introducing Euclidean Bose and Fermi fields at "finite temperature" in each case. In connection with supersymmetric quantum field theories, the result gives a path integral formula of the so-called "Witten index" in a model with cutoffs.

Contents :

I. INTRODUCTION

In this paper, as a first step towards construction of index theorems on infinite dimensional manifolds, we consider Kähler-Dirac type operators in an abstract infinite dimensional Boson-Fermion Fock space and derive a path integral formula of their index.

As for finite dimensional compact manifolds, index theorems have been well established under the name of Atiyah-Singer index theorems. Recently, some alternative, analytical proofs of these index theorems have been given in connection with supersymmetry (e.g.,[1],[6],[19], cf. also [35], [36]), where path integral methods are used as a powerful tool. The idea of our analysis is to extend these path integral methods to the case of infinite dimensional manifolds.

In order to define an infinite dimensional Kähler-Dirac type operator, we first have to construct a framework of differential forms on infinite dimensional manifolds. In the case where the manifold M is a real topological vector space, it may be straightforwardly done on the analogy of the theory of differential forms on R^n, once a measure on M is given. In [2], the case $M = \mathcal{S}'_r$, the space of real tempered distributions, is considered in connection with a supersymmetric quantum field theory. Recently, Shigekawa [27] has given a formulation of differential forms on an abstract Wiener space. Our formulation of differential forms is a slight generalization of that in [27].

As applications of our theory, we have in mind some models in supersymmetric quantum field theories ([35],[36],[2]), but, in the present article, we do not discuss them ; We shall consider them in a separate paper together with the problem of dimensional reduction [4]. We remark only that our result gives a path integral formula of the so-called "Witten index" in a supersymmetric quantum field model with cutoffs.

This paper is organized as follows : Sections II and III are preliminary ones to Section IV which is the main part of this article, although they may have independent mathematical interests. In Section II, we derive some trace formulas in an abstract Boson Fock space, which are essentially abstract generalizations of some results in [17,16]. The point is to represent in terms of path integrals traces associated with "Gibbs states" in the Boson Fock space. This is done by introducing a "sharp time Euclidean Bose field (or Gaussian random process) at finite temperature". In Section III, we introduce Euclidean Fermi fields at finite temperature and, in terms of them, we represent traces associated with "Gibbs states" in an abstract Fermion Fock space. As in the case of usual Euclidean Fermi fields (e.g.,[20],[34],[21]), we have a dufficulty that there does not exist a sharp time Euclidean Fermi field as a well-defined operator on the Euclidean Fermion Fock space. We avoid this difficulty by defining the sharp time field as a sesquilinear form on a dense domain in the Euclidean Fermion Fock space. This is sufficient for our purpose. In Section IV, we first define "free" Kähler-Dirac operators in an abstract Boson-Fermion Fock space, which is regarded as an infinite dimensional exterior algebra , and establish some elemetary properties of them. It is proved that the index of the free Kähler-Dirac operators is equal to 1. Then, we consider a perturbation of the free Kähler-Dirac operators and derive a path integral representation of its index. This is carried out by combining the results in Sections II and III. In Appendix A, we prove an elementary fact on the trace. In Appendix B, we give the definition of an index of a self-adjoint operator acting in a graded Hilbert space and a general principle to compute the index.

II. TRACE FORMULAS IN AN ABSTRACT BOSON FOCK SPACE

We begin with fundamental definitions and elementary facts (e.g., [10],[25],[22],[23],[30],[13]). Let $\mathcal{H}$ be a separable real Hilbert space with inner product $(\ ,\ )_{\mathcal{H}}$ and a strictly positive self-adjoint operator h_B acting in $\mathcal{H}$ be given. Let $\mathcal{H}_{-1}$ be the real Hilbert space obtained by the completion of $\mathcal{H}$ with respect to the inner product

$$(f,g)_{-1} \equiv (h_B^{-1/2} f,\ h_B^{-1/2} g)_{\mathcal{H}}, \qquad f,g \in \mathcal{H}. \tag{2.1}$$

Then we consider the Gaussian mean zero random process $\{ \phi(f) \mid f \in \mathcal{H}_{-1} \}$ indexed by $\mathcal{H}_{-1}$; We denote by Q (resp. $d\mu_0$) the underlying measure space (resp. the Gaussian probability measure) and by $<\ \ >$ the expectation with respect to $d\mu_0$, so that we have

$$< \phi(f)\ \phi(g) > = (f,g)_{-1}, \qquad f,\ g \in \mathcal{H}_{-1}. \tag{2.2}$$

The Boson Fock space over $\mathcal{H}_{-1}$ is given by

$$\mathcal{F}_B(\mathcal{H}_{-1}) = L^2(Q\ ,d\mu_0). \tag{2.3}$$

It is well-known (e.g.,[30],[13]) that $\mathcal{F}_B(\mathcal{H}_{-1})$ is written as the completed infinte direct sum of the closed subspace $\Gamma_n(\mathcal{H}_{-1})$ ($\Gamma_0(\mathcal{H}_{-1}) \equiv C$) generated by vectors of the form $:\phi(f_1)\dots\phi(f_n):$, $f_1,..,f_n \in \mathcal{H}_{-1}$, that is

$$\mathcal{F}_B(\mathcal{H}_{-1}) = \bigoplus_{n=0}^{\infty} \Gamma_n(\mathcal{H}_{-1}), \tag{2.4}$$

where $:\phi(f_1)\ldots\phi(f_n):$ is the Wick product of random variables $\phi(f_1)$, ..., $\phi(f_n)$, defined by the following recursion formula :

$$:\phi(f): = \phi(f),$$

$$:\phi(f_1)\ldots\phi(f_n): = \phi(f_1):\phi(f_2)\ldots\phi(f_n):$$

$$- \sum_{j=2}^{n} < \phi(f_1)\phi(f_j)> :\phi(f_2)..\phi(f_{j-1})\phi(f_{j+1})\ldots\phi(f_n):,$$

$$n=2,3,4,\ldots.. \qquad (2.5)$$

Let $\Gamma_n^{(0)}(H_{-1})$ be the subspace algebraically spanned by vectors of the form $:\phi(f_1)\ldots\phi(f_n):$, $f_j \in H_{-1}$, and

$$\Gamma_0(H_{-1}) = \bigoplus_{n=0}^{\infty} \Gamma_n^{(0)}(H_{-1}) \qquad (2.6)$$

be the incompleted direct sum ($\Gamma_0^{(0)}(H_{-1}) \equiv C$). The subspace $\Gamma_0(H_{-1})$ is dense in $F_B(H_{-1})$.

In general we denote by D(A) the domain of the operator A. It is obvious that h_B has a unique self-adjoint extension $\hat{h}_B$ to H_{-1} such that

$$(g, \hat{h}_B f)_{-1} = (g,f)_H \ , \quad g,f \in D(h_B).$$

For each f in $D(\hat{h}_B^{1/2})$, we define the annihilation operator b(f) on $\Gamma_0(H_{-1})$ by

$$b(f):\phi(f_1)\dots\phi(f_n): = \sum_{j=1}^{n} (\hat{h}_B^{1/2} f, f_j)_{-1} :\phi(f_1)\dots\phi(f_{j-1})\phi(f_{j+1})\dots\phi(f_n): \tag{2.7}$$

and extending it by linearity to $\Gamma_0(H_{-1})$. The creation operator $b^*(f)$ is defined as the adjoint of $b(f)|\Gamma_0(H_{-1})$. Then, $D(b^*(f)) \supset \Gamma_0(H_{-1})$ and we have

$$b^*(f):\phi(f_1)\dots\phi(f_n): = :\phi(\hat{h}_B^{1/2} f)\phi(f_1)\dots\phi(f_n):, \quad f_1,\dots,f_n \in H_{-1}. \tag{2.8}$$

Therefore $b(f)$ is closable and we denote the closure by the same symbol. Both $b(f)$ and $b^*(f)$ leave $\Gamma_0(H_{-1})$ invariant and satisfy the canonical commutation relations on $\Gamma_0(H_{-1})$:

$$\begin{aligned} &[b(f), b^*(g)] = (f,g)_H , \\ &[b(f), b(g)] = 0 = [b^*(f), b^*(g)], \qquad f,g \in D(\hat{h}_B^{1/2}), \end{aligned} \tag{2.9}$$

where [,] denotes the commutator. Further, we have the following standard estimates : For all f in $D(\hat{h}_B^{1/2})$ and Ψ in $D(N_B^{1/2})$

$$\begin{aligned} &||b(f)\Psi|| \le ||\hat{h}_B^{1/2} f||_{-1} ||N_B^{1/2}\Psi||, \\ &||b^*(f)\Psi|| \le ||\hat{h}_B^{1/2} f||_{-1} ||(N_B+1)^{1/2}\Psi||, \end{aligned} \tag{2.10}$$

where N_B is the Boson number operator defined by

$$(N_B\Psi)^{(n)} = n\,\Psi^{(n)}, \quad n=0,1,2,\ldots, \quad \Psi \in \mathcal{F}_B(\mathcal{H}_{-1}). \quad (2.11)$$

We denote by $b^{\#}$ either b or b^*. In terms of $b^{\#}(f)$, $\phi(f)$ is written as

$$\phi(f) = b(\hat{h}_B^{-1/2}f) + b^*(\hat{h}_B^{-1/2}f), \quad f \in \mathcal{H}_{-1} \quad (2.12)$$

on $\Gamma_0(\mathcal{H}_{-1})$. For f in $D(\hat{h}_B)$ we define the canonical conjugate momentum operator $\pi(f)$ by

$$\pi(f) = \frac{1}{2i}[\, b(\hat{h}_B^{1/2}f) - b^*(\hat{h}_B^{1/2}f)]. \quad (2.13)$$

Then, (2.9) implies that for all f and g in $D(\hat{h}_B)$

$$[\phi(f), \pi(g)] = i(f,g)_{\mathcal{H}} \quad , \quad [\phi(f), \phi(g)] = 0 = [\pi(f), \pi(g)] \quad (2.14)$$

on $\Gamma_0(\mathcal{H}_{-1})$. Using estimates (2.10) and Nelson's analytic vector theorem (e.g.,[23]), one can show that $\phi(f)$ and $\pi(f)$ are essentially self-adjoint on $\Gamma_0(\mathcal{H}_{-1})$.

For any contraction linear operator C on $\mathcal{H}_{-1}$, we can define a unique bounded linear operator $\Gamma(C)$ on $\mathcal{F}_B(\mathcal{H}_{-1})$ such that, for all f_j in $\mathcal{H}_{-1}$, $j=1, .., n$, $n=1,2,\ldots,$

$$\Gamma(C){:}\phi(f_1)\ldots\phi(f_n){:} = {:}\phi(Cf_1)\ldots\phi(Cf_n){:} \quad , \ \Gamma(C)1 = 1.$$

For every non-negative self-adjoint operator A acting in $\mathcal{H}_{-1}$, $\{\Gamma(e^{-tA})\}_{t\geq 0}$ is a strongly continuous contraction semigroup on $\mathcal{F}_B(\mathcal{H}_{-1})$. Hence there exists a unique non-negative self-adjoint operator $d\Gamma(A)$ such that for all $t > 0$

$$\Gamma(e^{-tA}) = e^{-td\Gamma(A)}.$$

It is easy to see that $d\Gamma(A)$ acts as

$$d\Gamma(A):\phi(f_1)\ldots\phi(f_n): = \sum_{j=1}^{n} :\phi(f_1)..\phi(f_{j-1})\phi(Af_j)\phi(f_{j+1})\ldots\phi(f_n): \tag{2.15}$$

with $f_1,..,f_n \in D(A)$.

Let

$$H_{0B} = d\Gamma(\hat{h}_B). \tag{2.16}$$

Then we have from (2.10)

$$||b^{\#}(f)\Psi|| \leq c||\hat{h}_B^{1/2}f||_{-1}||(H_{0B}+1)^{1/2}\Psi||, \quad \Psi \in D(H_{0B}^{1/2}), \tag{2.17}$$

with some constant $c > 0$ (This estimate is not best possible, but, for our purpose, it is sufficient). Using (2.15),(2.16) and (2.17), one can prove that

$$[H_{0B}, b(f)] = -b(\hat{h}_B f), \ [H_{0B}, b^*(f)] = b^*(\hat{h}_B f), \qquad f \in D(\hat{h}_B^{3/2}), \tag{2.18}$$

on $D(H_{0B}^{3/2})$, which imply that

$$[H_{0B}, \phi(f)] = -2i\,\pi(f), \qquad [H_{0B}, \pi(f)] = -\frac{1}{2i}\,\phi(\hat{h}_B^2 f), \qquad f \in D(\hat{h}_B^2 f), \tag{2.19}$$

on $D(H_{0B}^{3/2})$. By (2.17) and (2.19), we can prove the following estimate :

$$\|\phi(f)^n\Psi\| \le C_n(f)\ \|(H_{0B}+1)^n\Psi\|, \qquad \Psi \in D(H_{0B}^n),\ n=1,2,3,.., \tag{2.20}$$

where $C_n(f)$ is a polynomial in $\|\hat{h}_B^k f\|_{-1}$, k=0,1,..,n (This estimate is rude).

In what follows we assume the following :

$(A)_B$ For some constant $\gamma > 0$, $h_B^{-\gamma/2}$ is Hilbert-Schmidt on $\mathcal{H}$.

(See Lemma 2.1 below). This assumption implies that, for all $t > 0$, $\exp(-t\hat{h}_B)$ is trace class on $\mathcal{H}_{-1}$ and hence so is $\exp(-tH_{0B})$ on $\mathcal{F}_B(\mathcal{H}_{-1})$. One can easily show that

$$\mathrm{Tr}\ e^{-tH_{0B}} = \frac{1}{\det(1 - e^{-th_B})}, \tag{2.21}$$

where Tr denotes the trace and det the determinant (e.g.,[31,32,24]). Our aim is to derive a Feynman-Kac type formula for quantities such as

$$\mathrm{Tr}\ (G_0 e^{-t_1 H_{0B}} G_1 e^{-t_2 H_{0B}} G_2 \ldots G_{n-1} e^{-t_n H_{0B}}),\quad t_j > 0,$$

with G_j's being suitable measurable functions on Q.

For this purpose, we introduce another Gaussian random process.

<u>LEMMA 2.1</u>. Let $\beta > 0$ be fixed. Assume $(A)_B$. Let $\mathcal{H}_{-\gamma}$ denote the completion of $\mathcal{H}$ with respect to the norm

$$||f||_{-\gamma} = ||h_B^{-\gamma/2} f||_{\mathcal{H}} .$$

Then, there exists a Gaussian mean zero process Φ_t on $[0,\beta]$ with state space $\mathcal{H}_{-\gamma}$ and with continuous sample paths such that, for all f and g in $\mathcal{H}$

$$\int_{\mathcal{H}_{-\gamma}} d\mu_{0,\beta}\, \Phi_t(f)\Phi_s(g) = (f,\ (1-e^{-\beta\hat{h}_B})^{-1}\ (e^{-|t-s|\hat{h}_B} + e^{-(\beta-|t-s|)\hat{h}_B})g\ )_{-1} , \tag{2.22}$$

where $\mu_{0,\beta}$ is the underlying measure on $\mathcal{H}_{-\gamma}$ and $\Phi_t(f) = (\Phi_t\ ,\ f)_{-\gamma}$.

<u>Proof</u>. Quite similar to the proof of Proposition 5.1 in Gross [14]. □

<u>Remark.</u> In quantum field theoretical language, Φ_t may be called the " sharp time Euclidean field at the finite temperature $1/\beta$ ". The " smeared process (or field)" with respect to time t can be constructed as follows : Let $L^2_r(0,\beta)$ be the real Hilbert space of real valued measurable functions in $L^2(0,\beta)$ and put

$$\mathcal{H}_\beta = L^2_r(0,\beta) \otimes \mathcal{H}.$$

Let Δ_P be the <u>periodic</u> Laplacian acting in $L^2(0,\beta)$. Define the norm $||\cdot||_{-1,\beta}$ by

$$||f||^2_{-1,\beta} = 2\ ||(-\Delta_P \otimes I + I \otimes h_B^2\ )^{-1/2} f||^2_{\mathcal{H}_\beta}\ , \quad f \in \mathcal{H}_\beta .$$

We denote by $H_{-1,\beta}$ the completion of H_β with respect to the norm $\|\ \|_{-1,\beta}$. Let f be an H-valued measurable function on $[0,\beta]$ such that

$$\int_0^\beta \|f(t)\|_{-\gamma}^2\, dt < \infty.$$

Then we can define the random variable (the "smeared process")

$$\Phi(f) = \int_0^\beta dt(\Phi_t\ ,\ f(t)\ )_{-\gamma}\ .$$

It is not so difficult to see that

$$\int d\mu_{0,\beta}\ \Phi(f)\Phi(g) = (f,g)_{-1,\beta}\ .$$

Thus we can identify $\{\Phi(f)\}$ with the Gaussian mean zero random process indexed by $H_{-1,\beta}$. As is seen from (2.22), the zero temperature limit $\beta \to \infty$ gives the covariance of the usual sharp time Euclidean field (cf.[14]).

THEOREM 2.2. Assume $(A)_B$. Let $G_0,..,G_n$ be bounded measurable functions on R^m and put

$$G_j^B = G_j(\phi(f_1),..,\ \phi(f_m)),$$

$$G_j(t) = G_j(\ \Phi_t(f_1),...,\ \Phi_t(f_m)),\qquad f_1,..,f_m \in H_{-1}$$

Let $0 < t_1 < t_2 < .... < t_n < \beta$. Then,

$$\frac{\mathrm{Tr}\,(G_0^B\, e^{-t_1 H_{0B}}\, G_1^B\, e^{-(t_2-t_1)H_{0B}}\, G_2^B \ldots\ldots e^{-(t_n-t_{n-1})H_{0B}}\, G_n^B\, e^{-(\beta-t_n)H_{0B}})}{\mathrm{Tr}\; e^{-\beta H_{0B}}}$$

$$= \int d\mu_{0,\beta} G_0(0) G_1(t_1) \ldots G_n(t_n). \tag{2.23}$$

Proof. We give only the outline (cf.[17]). We first note that h_B has a purely discrete spectrum $\{\lambda_n\}_{n=1}^{\infty}$ with $0 < \lambda_1 \leq \lambda_2 \leq \ldots, \lambda_n \uparrow \infty$ $(n \to \infty)$ (counted with algebraic multiplicity), since $\exp(-th_B)$ is trace class on $\mathcal{H}$ and hence compact. Let $\{e_n\}_{n=1}^{\infty}$ be the complete orthonormal system of the corresponding eigenvectors in $\mathcal{H}$:

$$h_B e_n = \lambda_n e_n.$$

Let

$$\phi_n = \phi(e_n), \quad \pi_n = \pi(e_n).$$

Then, we have from (2.14)

$$[\phi_n, \pi_m] = i\delta_{nm}, \quad [\phi_n, \phi_m] = 0 = [\pi_n, \pi_m].$$

It is easy to check that H_{0B} is represented as

$$H_{0B} = \sum_{n=1}^{\infty} T_n$$

on $\Gamma_0(\mathcal{H}_{-1}) \cap D(H_{0B})$, where

$$T_n = \pi_n^2 + \frac{1}{4}\lambda_n^2\phi_n^2 - \frac{1}{2}\lambda_n$$

and the convergence on the right hand side is taken in the strong sense. Let $F_B^{(N)}$ be the closure of the set

$$\{ P(\phi_1,..,\phi_N) \mid P \text{ a polynomial} \}$$

in $F_B(H_{-1})$ and put $F_0^{(N)} = F_B^{(N)} \cap \Gamma_0(H_{-1})$. Then ϕ_n and π_n (n=1,..,N) are essentially self-adjoint on $F_0^{(N)}$ and we have a representation of the Weyl relations :

$$e^{it\phi_n} e^{is\pi_m} = e^{-its\delta_{nm}} e^{is\pi_m} e^{it\phi_n} , \qquad s,t \in R.$$

Further, the constant function 1 in $F_B^{(N)}$ satisfies $T_n 1 = 0$ and is cyclic for the operators ϕ_n , n=1,..,N. Therefore, by von Neumann's theorem, there exists a unitary map $U_N : F_B^{(N)} \to L^2(R^N)$ such that

$$U_N \phi_n U_N^{-1} = x_n, \qquad U_N \pi_n U_N^{-1} = -i \frac{\partial}{\partial x_n}$$

and

$$U_N 1 = \prod_{n=1}^{N} \left(\frac{\lambda_n}{2\pi} \right)^{1/4} e^{-\lambda_n x_n^2/4} .$$

Let

$$H_{OB}^{(N)} = H_{OB} \restriction F_B^{(N)} \cap D(H_{OB}) .$$

Then it follows that

$$U_N H_{OB}^{(N)} U_N^{-1} = \sum_{n=1}^{N} \left(- \frac{\partial^2}{\partial x_n^2} + \frac{1}{4} \lambda_n^2 x_n^2 - \frac{1}{2} \lambda_n \right).$$

Thus $H_{OB}^{(N)}$ is unitarily equivalent to the Hamiltonian of an N-dimensional harmonic oscillator. Then we can proceed quite analogously to the analysis in [17]. Namely, using the well-known Feynman-Kac formula in finite dimensional spaces (e.g.,[33,13]), we first establish the trace formula (2.23) on the subspace $F_B^{(N)}$. Then, using a limiting argument to take the limit $N \to \infty$, we get (2.23) (For convergence theorems on the trace, see, e.g., [15,32]). □

As a generalization of Theorem 2.2, we have

COROLLARY 2.3. Let $f_{ij} \in C^{\infty}(h_B)$, $i=0,1,2,..$, $j=1,2,..,k_i$, and $k_i=1,2,3,...$ Let $t_1,...,t_n$ be as in Theorem 2.2. Then, under the assumption in Theroem 2.2, we have

$$\frac{\mathrm{Tr}\left(\prod_{j=1}^{k_0} \phi(f_{0j})\, e^{-t_1 H_{OB}} \prod_{j=1}^{k_1} \phi(f_{1j})\, e^{-(t_2-t_1)H_{OB}} \cdots\cdots \prod_{j=1}^{k_n} \phi(f_{nj}) e^{-(\beta-t_n)H_{OB}} \right)}{\mathrm{Tr}\, e^{-\beta H_{OB}}}$$

$$= \int d\mu_{0,\beta} \prod_{j=1}^{k_0} \Phi_0(f_{0j}) \cdots\cdots \prod_{j=1}^{k_n} \Phi_{t_n}(f_{nj}). \qquad (2.24)$$

Remark. By estimate (2.20), we see that, for all $t > 0$, $[\prod_{j=1}^{k_i} \phi(f_{ij})]\exp(-tH_{0B})$ is bounded and trace class.

Proof. Take, for example, $G_0^B = \exp(\, is\phi(f)\,)$ in (2.23). Then, by the above remark, $(\exp(is\phi(f)\,) - 1)\exp(-t_1H_{0B})/s$ converges strongly to $i\phi(f) \times e^{-t_1H_{0B}}$ as $s \to 0$. Therefore, by Grümm's convergence theorem [15,32] and the Lebesgue dominated convergence theorem, we get (2.23) with G_0^B replaced by $\phi(f)$. Proceeding similarly, we get (2.24). □

Remark. Using Wick's theorem, one can also directly prove (2.24).

We next extend trace formulas (2.23) and (2.24) to the case where H_{0B} is replaced by a perturbed operator of the form $H_{0B} + V$ with V being a multiplication operator on $\mathcal{F}_B(\mathcal{H}_{-1})$.

Let v be a polynomially bounded continuous real function on R and bounded below. Let $(X, d\sigma(x)\,)$ be a finite measure space with X being compact Hausdorff (the Σ -field is suppressed) and ρ be an $\mathcal{H}$-valued strongly continuous function on X. Then it is easy to see that, for all $x \in X$ and $t \in [0,\beta]$, $v(\phi(\rho(x))\,)$ and $v(\Phi_t(\rho(x))\,)$ are in $L^2(Q, d\mu_0)$ and $L^2(\,\mathcal{H}_{-\gamma}\,, d\mu_{0,\beta}\,)$ respectively and that the Bochner integrals (e.g.,[37])

$$V(\phi) = \int_X d\sigma(x)\; v(\phi(\rho(x))\,) \quad \in \quad L^2(Q,\, d\mu_0) \tag{2.25}$$

and

$$V_t(\Phi) = \int_X d\sigma(x)\; v(\Phi_t(\rho(x))\,) \in L^2(\,\mathcal{H}_{-\gamma},\, d\mu_{0,\beta}\,) \tag{2.26}$$

are defined. For almost everywhere $\Phi \in \mathcal{H}_{-\gamma}$, $V_t(\Phi)$ is continuous in t. Since $\sigma(X)$ is finite, V and V_t are bounded below independently of t. Therefore, by a general theorem ([28],[29],[23, p.265, Theorem X.59]), the operator

$$H_B \equiv H_{OB} + V \tag{2.27}$$

is essentially self-adjoint on $C^\infty(H_{OB})\cap D(V)$ and bounded below.

THEOREM 2.4. Assume $(A)_B$. Let G_j^B, $G_j(t)$ and t_j be as in Theorem 2.2. Then,

$$\frac{\mathrm{Tr}(G_0^B e^{-t_1 H_B} G_1^B e^{-(t_2-t_1)H_B} G_2^B \cdots e^{-(t_n-t_{n-1})H_B} G_n^B e^{-(\beta-t_n)H_B})}{\mathrm{Tr}\, e^{-\beta H_{OB}}}$$

$$= \int d\mu_{0,\beta}\, G_0(0)G_1(t_1)\cdots G_n(t_n) \exp[-\int_0^\beta ds\, V_s] \tag{2.28}$$

Remark. Since V is bounded below, e^{-tV} is bounded for all $t > 0$. Hence, by the (generalized) Golden-Thompson inequality (e.g.,[32],[24]), e^{-tH_B} is trace class for all $t > 0$.

Proof. By the Trotter product formula, we have

$$e^{-tH_B} = \operatorname*{s-lim}_{N\to\infty} (e^{-tH_{OB}/N} e^{-tV/N})^N .$$

Then, using Grümm's convergence theorem [15] and Theorem 2.2, we can easily get (2.28) (cf. also [17]). □

By Segal's lemma([26],[23, X.9, Theorem X.57]), we have

$$\|e^{-tH_B}\| \le e^{-td} \|e^{-tH_{0B}}\|, \qquad t > 0.$$

with d = inf V. Therefore, by estimate (2.20), we see that, for all $f_1,..,f_n$ in $C^\infty(h_B)$, $e^{-tH_B}\phi(f_1)...\phi(f_n)$ is uniquely extended to a trace class operator on $F_B(H_{-1})$. Then, in the same way as in the proof of Corollary 2.3, we get

COROLLARY 2.5. Let f_{ij} and t_j be as in Corollary 2.3. Then, under the assumption in Theorem 2.4, we have

$$\frac{\mathrm{Tr}\left(\prod_{j=1}^{k_0} \phi(f_{0j})\, e^{-t_1 H_B} \prod_{j=1}^{k_1} \phi(f_{1j}) \cdots\cdots \prod_{j=1}^{k_n} \phi(f_{nj})\, e^{-(\beta - t_n)H_B} \right)}{\mathrm{Tr}\, e^{-\beta H_{0B}}}$$

$$= \int d\mu_{0,\beta} \prod_{j=1}^{k_0} \Phi_0(f_{0j}) \cdots \prod_{j=1}^{k_n} \Phi_{t_n}(f_{nj}) \exp[-\int_0^\beta ds\, V_s].$$

III. EUCLIDEAN FERMI FIELDS AT FINITE TEMPERATURE

The purpose of this section is to establish in an abstarct Fermion Fock space some trace formulas for quantities corresponding to those considered in the last section. To this end, we have to construct Euclidean Fermi fields at finite temperature.

3.1 Preliminaries

We first summarize some fundamental definitions and elementary facts in an abstract Fermion Fock space.

Let $\mathcal{K}$ be a complex separable Hilbert space with inner product $(\ ,\ )_{\mathcal{K}}$. Then the Fermion Fock space $\mathcal{F}_F(\mathcal{K})$ over $\mathcal{K}$ is defined as the completed infinite direct sum of the n-fold antisymmetric tensor product $\Lambda^n(\mathcal{K})$ of $\mathcal{K}$ with $\Lambda^0(\mathcal{K}) = C$:

$$\mathcal{F}_F(\mathcal{K}) = \bigoplus_{n=0}^{\infty} \Lambda^n(\mathcal{K}). \tag{3.1}$$

Let $\Lambda_0^n(\mathcal{K})$ be the linear subspace spanned by finite linear combinations of vectors of the form

$$f_1 \Lambda \ldots \Lambda f_n \cong \sum_{\pi\in\mathcal{P}_n} \varepsilon(\pi)\, f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}\,, \quad f_j \in \mathcal{K}\,,\ j=1,..,n, \tag{3.2}$$

where $\mathcal{P}_n$ denotes the group of all permutations on n letters and $\varepsilon(\pi)$ is the signature of $\pi \in \mathcal{P}_n$. Let

$$\mathcal{F}_{0F}(\mathcal{K}) = \bigoplus_{n=0}^{\infty} \Lambda_0^n(\mathcal{K}). \text{ (the incompleted direct sum)} \tag{3.3}$$

It is obvious that $\mathcal{F}_{0F}(\mathcal{K})$ is dense in $\mathcal{F}_F(\mathcal{K})$. The distinguished vector $\Omega_F \in \mathcal{F}_F(\mathcal{K})$ with $\Omega_F^{(0)} = 1$, $\Omega_F^{(n)} = 0$, n=1,2,3,..., is called the "Fock vacuum".

The number operator N_F is defined as the non-negative self-adjoint operator given by

$$(N_F\Psi)^{(n)} = n\,\Psi^{(n)}, \quad n=0,1,2,\dots. \tag{3.4}$$

For every f in $\mathcal{K}$, the creation operator $\psi^\dagger(f)$ is defined by

$$\begin{aligned} (\psi^\dagger(f)\Psi)^{(n)} &= n^{1/2}\,\mathrm{As}(f \otimes \Psi^{(n-1)}), \quad n=1,2,\dots, \\ (\psi^\dagger(f)\Psi)^{(0)} &= 0, \end{aligned} \tag{3.5}$$

where As is the antisymmetrization projection. The operator $\psi^\dagger(f)$ is a priori defined on $\mathcal{F}_{0F}(\mathcal{K})$ leaving it invariant and complex linear in f.

Let $J : \mathcal{K} \to \mathcal{K}$ be a conjugation, that is, J is anti-linear, anti-unitary and $J^2 = I$ (identity). We define a symmetric bilinear form $\langle\,,\,\rangle_{\mathcal{K}}$ on $\mathcal{K} \times \mathcal{K}$ by

$$\langle f, g \rangle_{\mathcal{K}} = (Jf, g)_{\mathcal{K}}, \quad f,g \in \mathcal{K}. \tag{3.6}$$

The annihilation operator $\psi(f)$ is then defined by the adjoint of $\psi^\dagger(Jf) \mid \mathcal{F}_{0F}(\mathcal{K})$:

$$\psi(f) = [\psi^{\dagger}(Jf)|\mathcal{F}_{0F}(\mathcal{K})]^{*}, \quad f \in \mathcal{K}, \tag{3.7}$$

which is well-defined on $\mathcal{F}_{0F}(\mathcal{K})$ with the action

$$(\psi(f)\Psi)^{(n)} = (n+1)^{1/2} \sum_{\pi\in\mathcal{P}} \varepsilon(\pi) < f , f_{\pi(1)}>_{\mathcal{K}} f_{\pi(2)}\otimes ..\otimes f_{\pi(n)}, \tag{3.8}$$

for vectors Ψ of the form

$$\Psi^{(n)} = f_1\wedge....\wedge f_n, \quad f_j \in \mathcal{K} \quad , j=1,..,n.$$

Clearly $\psi(f)$ is complex linear in f.

It is easy to see that the following anti-commutation relations hold on $\mathcal{F}_{0F}(\mathcal{K})$:

$$\{ \psi(f), \psi^{\dagger}(g)\} = <f , g >_{\mathcal{K}}, \tag{3.9}$$

$$\{ \psi(f), \psi(g) \} = 0 = \{\psi^{\dagger}(f),\psi^{\dagger}(g)\} , \qquad f,g \in \mathcal{K}, \tag{3.10}$$

where $\{ , \}$ denotes the anti-commutator :

$$\{A,B\} = AB + BA. \tag{3.11}$$

It follows from (3.9) that $\psi(f)$ and $\psi^{\dagger}(f)$ are bounded with the operator norm $||\psi^{\#}(f)|| = ||f||_{\mathcal{K}}$, where $\psi^{\#}$ denotes either ψ or $\psi^{\dagger}$. Therefore, $\psi^{\#}(f)$ extends uniquely to a bounded linear operator on $\mathcal{F}_{F}(\mathcal{K})$; We denote the extension by the same symbol. Let A be a self-adjoint operator acting in $\mathcal{K}$. Then, the second quantized operator $d\Gamma(A)$ is defined as the unique self-adjoint extension of the operator given by

$$d\Gamma(A)|\Lambda^n(K) = \underbrace{A \otimes I \otimes \ldots. \otimes I}_{n} + I \otimes A \otimes \; I \otimes \ldots \otimes I +$$

$$+\ldots. + I \otimes I \otimes \ldots. \otimes I \otimes A \tag{3.12}$$

(see, e.g., [22,23]). If A is non-negative, then so is $d\Gamma(A)$ and we have for all Re $z > 0$,

$$\psi(z,f) \equiv e^{-zd\Gamma(A)}\psi(f)e^{zd\Gamma(A)} = \psi(Je^{\bar{z}A}Jf), \qquad Jf \in D(e^{\bar{z}A}),$$

$$\psi^{\dagger}(z,f) \equiv e^{-zd\Gamma(A)}\,\psi^{\dagger}(f)e^{zd\Gamma(A)} = \psi^{\dagger}(e^{-zA}f), \qquad f \in K, \tag{3.13}$$

on $D(e^{zd\Gamma(A)})$.

Let A be a <u>strictly positive</u> self-adjoint operator and B a self-adjoint operator acting in K and commuting with A. Let

$$H = d\Gamma(A) + id\Gamma(B). \tag{3.14}$$

Suppose that exp (-A) is trace class on K. Then, $\exp(-d\Gamma(A))$ is trace class on $F_F(K)$. Therefore we can define an expectation value $< \;\; >_H$ by

$$< G >_H = \frac{\mathrm{Tr}(e^{-H}\, G)}{\mathrm{Tr}\; e^{-H}}, \tag{3.15}$$

where G is a bounded linear operator on $F_F(K)$.

<u>LEMMA 3.1</u>. For every f and g in K, we have

$$< \psi^{\dagger}(f)\psi(g) >_{H} = < (1 + e^{A+iB})^{-1} f, g >_{K} \tag{3.16}$$

$$< \psi(g)\psi^{\dagger}(f) >_{H} = < (1 + e^{-(A+iB)})^{-1} f, g >_{K} \tag{3.17}$$

$$< \psi(f)\psi(g) >_{H} = 0 , \tag{3.18}$$

$$< \psi^{\dagger}(f)\psi^{\dagger}(g) >_{H} = 0 , \tag{3.19}$$

$$< \psi^{\#}(f) >_{H} = 0. \tag{3.20}$$

Proof. Let f be in $D(e^{A})$ first. Using the anti-commutation relation (3.9), we have

$$\mathrm{Tr}(e^{-H} \psi^{\dagger}(f)\psi(g)) = < f, g >_{K} \mathrm{Tr}\, e^{-H} - \mathrm{Tr}(e^{-H} \psi(g)\psi^{\dagger}(f)).$$

By the cyclicity of the trace and (3.13), we have

$$\mathrm{Tr}(e^{-H} \psi(g)\psi^{\dagger}(f)) = \mathrm{Tr}(\psi^{\dagger}(f)e^{-H}\psi(g)) = \mathrm{Tr}(e^{-H}\psi^{\dagger}(e^{A+iB} f)\psi(g)).$$

Hence we get

$$< \psi^{\dagger}((1+e^{A+iB})f)\psi(g) >_{H} = < f, g >_{K}$$

Since A is strictly positive and commutes with B, -1 is in the resolvent set of e^{A+iB}. Then, replacing f by $(1+e^{A+iB})^{-1}f$ ($f \in K$), we obtain (3.16). Formulas (3.17)-(3.20) can be proved similarly. □

<u>LEMMA 3.2</u>. Let f_j, $j=1,..,$ be in K. Then,

$$< \psi^{\#}(f_1)\dots\psi^{\#}(f_{2n}) >_H$$

$$= \sum_{\text{comb}} \varepsilon_{i_1 j_1 \dots i_n j_n} < \psi^{\#}(f_{i_1})\psi^{\#}(f_{j_1}) >_H \dots < \psi^{\#}(f_{i_n})\psi^{\#}(f_{j_n}) >_H , \tag{3.21}$$

$$< \psi^{\#}(f_1)\dots\psi^{\#}(f_{2n-1}) >_H = 0, \quad n = 1,2,3,\dots\dots, \tag{3.22}$$

where $\sum_{\text{comb}}$ means the sum over all $(2n)!/2^n n!$ ways of writing $1,2,..,2n$ as n distinct pairs $(i_1,j_1),\dots,(i_n,j_n)$ ($i_1 < j_1,\dots, i_n < j_n$) and $\varepsilon_{i_1 j_1 \dots i_n j_n}$ the signature of the permutation $(1,..,2n) \to (i_1,j_1,..,i_n,j_n)$.

<u>Proof</u>. Let f_1 be in $D(e^A)$. In the same way as in the proof of Lemma 3.1, we have

$$\mathrm{Tr}(\, e^{-H}\, \psi^{\#}(f_1)\dots\psi^{\#}(f_{2n})\,)$$

$$= \sum_{j=2}^{2n} (-1)^j \{\psi^{\#}(f_1),\, \psi^{\#}(f_j)\}\, \mathrm{Tr}(e^{-H}\, \psi^{\#}(f_2)\dots\psi^{\#}(f_{j-1})\psi^{\#}(f_{j+1})\dots\psi^{\#}(f_{2n})\,)$$

$$-\, \mathrm{Tr}(\, \psi^{\#}(f_1)e^{-H}\, \psi^{\#}(f_2)\dots\psi^{\#}(f_{2n}),$$

where we have used the fact that $\{\psi^{\#}(f),\, \psi^{\#}(g)\}$ is a constant multiple of the identity (see (3.9) and (3.10)). On the other hand, we have

$$\mathrm{Tr}(\ \psi^{\#}(f_1)e^{-H}\ \psi^{\#}(f_2)\ldots\psi^{\#}(f_{2n})\ )$$

$$= \mathrm{Tr}(e^{-H}\ \psi^{\#}(Kf_1)\psi^{\#}(f_2)\ldots.\psi^{\#}(f_{2n})\ ),$$

where

$$\begin{aligned} K &= e^{A+iB} &&;\ \psi^{\#}(f_1) = \psi^{\dagger}(f_1), \\ &= Je^{-(A-iB)}J &&;\ \psi^{\#}(f_1) = \psi(f_1). \end{aligned}$$

Hence, replacing f_1 by $(1+K)^{-1}f_1$, we get

$$\mathrm{Tr}(e^{-H}\ \psi^{\#}(f_1)\ldots.\psi^{\#}(f_{2n})\ )$$

$$= \sum_{j=2}^{2n} (-1)^j\ \{\ \psi^{\#}(\ (1+K)^{-1}f_1\ ),\ \psi^{\#}(f_j)\}$$

$$\times\ \mathrm{Tr}(e^{-H}\ \psi^{\#}(f_2)\ldots\psi^{\#}(f_{j-1})\psi^{\#}(f_{j+1})\ldots\psi^{\#}(f_{2n})\ ).$$

Repeating this procedure and using (3.16)-(3.19), we obtain (3.21). Formula (3.22) follows from the above procedure and (3.20). □

We now take a special choice of H. Let h_F be a strictly positive self-adjoint operator acting in $\mathcal{K}$ such that (i) for all $t > 0$, e^{-th_F} is trace class on $\mathcal{K}$ and (ii) h_F commutes with J. Let

$$H_{0F} = d\Gamma(h_F). \qquad (3.23)$$

Then, $e^{-tH_{0F}}$ is trace class for all $t > 0$ with

$$\mathrm{Tr}\, e^{-tH_{0F}} = \det(1 + e^{-th_F}). \tag{3.24}$$

Let $\beta > 0$ and $\theta = 0$ or 1. Then, we introduce the expectation value

$$< G >_{\beta,\theta} = \frac{\mathrm{Tr}(e^{i\pi\theta N_F} e^{-\beta H_{0F}} G)}{\mathrm{Tr}(e^{i\pi\theta N_F} e^{-\beta H_{0F}})} \tag{3.25}$$

and define the "time ordered correlation functions" by

$$< T(\psi^{\#}(t_1,f_1)\ldots\psi^{\#}(t_n,f_n) >_{\beta,\theta}$$

$$= \varepsilon_{i_1\ldots i_n} < \psi^{\#}(t_{i_1},f_{i_1})\ldots\psi^{\#}(t_{i_n},f_{i_n}) >_{\beta,\theta} \quad \text{if} \quad 0 \le t_{i_1} < \ldots < t_{i_n} \le \beta. \tag{3.26}$$

Remark. The expectation value $< \cdot >_{\beta,0}$ (the case $\theta = 0$) is the usual state associated with the "free" Hamiltonian H_{0F}. In particular, it has a positivity in the sense that, for all bounded linear operators G on $F_F(K)$, $< G^*G >_{\beta,0} \ge 0$. However, in the expectation value $< \cdot >_{\beta,1}$ (the case $\theta = 1$), the positivity breaks down. It may be called an "alternating trace", since we have $\exp(i\pi N_F) | \Lambda^n(K) = (-1)^n$.

LEMMA 3.3. Let f and Jg be in $D(e^{\beta h_F})$. Then, for all $t,s \in [0,\beta]$ ($|t-s| \ne 0,\beta$), we have

$$< T(\psi(s,g)\psi^{\dagger}(t,f))>_{\beta,\theta} = \sum_{n \in Z} < g, (h_F + i\omega_\beta(n,\theta))^{-1} f >_K \frac{e^{i\omega_\beta(n,\theta)(t-s)}}{\beta}, \tag{3.27}$$

where

$$\begin{aligned} \omega_\beta(n,\theta) &= \frac{2n\pi}{\beta} \quad ; \quad \theta = 1 \\ &= \frac{(2n+1)\pi}{\beta} \quad ; \quad \theta = 0 \end{aligned} \tag{3.28}$$

Proof. Let $0 < |t-s| < \beta$. By definition of the time ordered correlation functions, (3.13) and Lemma 3.1, we have

$$< T(\psi(s,g)\psi^{\dagger}(t,f))>_{\beta,\theta} = \theta(t-s)F_\theta(t-s) - \theta(s-t)G_\theta(s-t), \tag{3.29}$$

where $\theta(t)$ is the Heaviside function and

$$F_\theta(\tau) = < (1-\alpha(\theta)e^{-\beta h_F})^{-1} e^{-\tau h_F} f, g >_K ,$$

$$G_\theta(\tau) = < (1-\alpha(\theta)e^{\beta h_F})^{-1} e^{\tau h_F} f, g >_K , \qquad 0 \le \tau \le \beta,$$

with

$$\begin{aligned} \alpha(\theta) &= 1 \ ; \ \theta = 1 \\ &= -1 \ ; \ \theta = 0. \end{aligned}$$

Let

$$f_n^\theta(t) = \beta^{-1/2} e^{i\omega_\beta(n,\theta)t} .$$

Then, $\{f_n^\theta\}_{n\in Z}$ forms a complete orthonormal system in $L^2(0,\beta)$. Using the spectral decomposition for h_F, we see that the Fourier coefficients of F_θ and G_θ are given by

$$\hat{F}_\theta(n) \equiv \int_0^\beta dt\, F_\theta(t)\, \overline{f_n^\theta(t)} = \langle (h_F + i\omega_\beta(n,\theta))^{-1} f, g \rangle_K \, \beta^{-1/2}$$

$$\hat{G}_\theta(n) = - \langle (h_F - i\omega_\beta(n,\theta))^{-1} f, g \rangle_K \, \beta^{-1/2}$$

On the other hand, it is easy to see that $F_\theta(\tau)$ and $G_\theta(\tau)$ are continuously differentiable in $(0,\beta)$. Therefore, by a standard theorem in Fourier analysis, we have for all $\tau \in (0,\beta)$

$$F_\theta(\tau) = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{F}_\theta(n) f_n^\theta(\tau),$$

$$G_\theta(\tau) = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{G}_\theta(n) f_n^\theta(\tau).$$

Substituting these equations into (3.29) and using the fact that $\theta(t-s) + \theta(s-t) = 1$, we get (3.27). $\square$

3.2 Euclidean Fermi fields and trace formulas

We next introduce the Euclidean Fermi fields in terms of which the time ordered correlation functions are represented. The idea is similar to that in [20](cf. also [34],[14],[21]).

Let

$$Z_{\beta,\theta} = \{ \omega_\beta(n,\theta) \mid n \in Z \} \tag{3.30}$$

and $\tilde{K}_{\beta,\theta}$ be the Hilbert space obtained by completing the space of K -valued sequences f on $Z_{\beta,\theta}$ with respect to the norm $||f||_{\tilde{K}_{\beta,\theta}}$ given by

$$||f||^2_{\tilde{K}_{\beta,\theta}} \equiv \sum_{\omega\in Z_{\beta,\theta}} ||(h_F+i\omega)^{1/2}(h_F^2 + \omega^2)^{-1/2} f(\omega)||^2_K < \infty . \tag{3.31}$$

Put

$$K_{\beta,\theta} = \tilde{K}_{\beta,\theta} \oplus \tilde{K}_{\beta,\theta}. \tag{3.32}$$

Let $\Psi^\dagger(F)$ and $\Psi(F)$, $F \in K_{\beta,\theta}$, be the creation and the annihilation operator on the Fermion Fock space $F_F(K_{\beta,\theta})$ over $K_{\beta,\theta}$ respectively (As the conjugation to define $\Psi(F)$ from $\Psi^\dagger(F)$, we take the obvious extension of J to $K_{\beta,\theta}$). For $\varepsilon > 0$, $t \in [0,\beta]$ and $f \in K$, we define a vector $f^\theta_{t,\varepsilon} \in \tilde{K}_{\beta,\theta}$ by

$$f^{\theta}_{t,\varepsilon}(\omega) = \beta^{-1/2} e^{-it\omega-\varepsilon|\omega|}\ f, \qquad \omega \in Z_{\beta,\theta}\ . \tag{3.33}$$

Let

$$\Psi^{\theta}_{1,\varepsilon}(t,f) = \Psi(f^{\theta}_{-t,\varepsilon} \oplus 0) + i\Psi^{\dagger}(0 \oplus f^{\theta}_{t,\varepsilon}), \tag{3.34}$$

$$\Psi^{\theta}_{2,\varepsilon}(t,f) = i\Psi(0 \oplus f^{\theta}_{-t,\varepsilon}) + \Psi^{\dagger}(f^{\theta}_{t,\varepsilon} \oplus 0), \tag{3.35}$$

where 0 denotes the zero vector. Then it follows from the anti-commutation relations of $\Psi^{\#}$ that, for all ε, $\varepsilon' > 0$, $t,s \in [0,\beta]$ and $f,g \in K$,

$$\{ \Psi^{\theta}_{j,\varepsilon}(t,f),\ \Psi^{\theta}_{k,\varepsilon'}(s,g) \} = 0, \quad j,k = 1,2. \tag{3.36}$$

We shall denote by Ω the Fock vacuum in $F_F(K_{\beta,\theta})$. We have

$$\Psi(F)\Omega = 0, \qquad F \in K_{\beta,\theta}\ . \tag{3.37}$$

LEMMA 3.4. Let $D_{\beta,\theta}$ be the subspace algebraically spanned by vectors of the form $\Psi^{\dagger}(F_1)\ldots\Psi^{\dagger}(F_n)\Omega$, $F_j \in \ell^2(Z_{\beta,\theta}; K \oplus K)$ (the Hilbert space of $K \oplus K$ -valued, square summable sequences on $Z_{\beta,\theta}$), $n = 0,1,2,\ldots\ldots$

Then, for all $t_1,..,t_n \in [0,\beta]$ ($|t_i - t_j| \neq 0,\ \beta,\ i \neq j$), $i,j = 1,2,..,n$, $f_1,..,f_n \in D(e^{\beta h_F})$ and $j_k = 1,2$, $k=1,2,..,n$, the limit

$$\lim_{\varepsilon_1,\ldots,\varepsilon_n \to 0} \Psi^{\theta}_{j_1,\varepsilon_1}(t_1,f_1)\ldots\ldots\Psi^{\theta}_{j_n,\varepsilon_n}(t_n,f_n) \equiv \Psi^{\theta}_{j_1}(t_1,f_1)\ldots\ldots\Psi^{\theta}_{j_n}(t_n,f_n) \quad (3.38)$$

exists in the sense of sesquilinear form on $D_{\beta,\theta} \times D_{\beta,\theta}$. Further, the limiting sesquilinear form $\Psi^{\theta}_{j_1}(t_1,f_1)\ldots\ldots\Psi^{\theta}_{j_n}(t_n,f_n)$ on $D_{\beta,\theta} \times D_{\beta,\theta}$ is continuous in t_j's on the set

$$\{ (t_1,\ldots,t_n) \in [0,\beta]^n \mid |t_i-t_j| \neq 0,\beta, \ i \neq j \}$$

and the discontinuity at $t_i=t_j$ or $|t_i-t_j| = \beta$ is the first kind.

Proof. It is sufficient to consider the sesquilinear form applied to vectors of the form

$$\Psi^{\dagger}(F_1)\ldots\Psi^{\dagger}(F_n)\Omega, \quad F_j \in \ell^2(Z_{\beta,\theta};\, K \oplus K), \quad n=0,1,2,\ldots$$

By Wick's theorem, it turns out that the problem is reduced to that of the limit $\varepsilon \downarrow 0$ ($\varepsilon' \downarrow 0$) of the following two kinds of quantities :

$$B_\varepsilon(t) \equiv (\Omega, \Psi^{\theta}_{j,\varepsilon}(t,f)\, \Psi^{\dagger}(F)\, \Omega),$$

$$C_{\varepsilon,\varepsilon'}(t,s) \equiv (\Omega, \Psi^{\theta}_{1,\varepsilon}(t,f)\, \Psi^{\theta}_{2,\varepsilon'}(s,g)\, \Omega), \quad f,g \in D(e^{\beta h_F}).$$

As for $B_\varepsilon(t)$, we have

$$B_\varepsilon(t) = \sum_{\omega\in Z_{\beta,\theta}} \beta^{-1/2}\, e^{\,it\omega-\varepsilon|\omega|} \langle (h_F - i\omega)^{-1}f,\ F_1(\omega)\rangle_K ,$$

where $F = F_1 \oplus F_2$ and, for simplicity, we consider the case j=1 (The case j=2 can be treated similarly). The norm of the summand on the right hand side is dominated by

$$\beta^{-1/2}||f||_K \, (\lambda_0^2 + \omega^2)^{-1/2} \, ||F_1(\omega)||_K$$

with $\lambda_0 > 0$ being the infimum of the spectrum of h_F, which, by the Schwarz inequality, is summable with respect to $\omega \in Z_{\beta,\theta}$. Therefore, by the dominated convergence theorem, the limit

$$\lim_{\varepsilon \downarrow 0} B_\varepsilon(t) = \sum_{\omega \in Z_{\beta,\theta}} \beta^{-1/2} e^{it\omega} < (h_F - i\omega)^{-1} f, \, F_1(\omega) >_K$$

exists and is continuous in $t \in [0,\beta]$.

Concerning $C_{\varepsilon,\varepsilon'}(t,s)$, we have

$$C_{\varepsilon,\varepsilon'}(t,s) = \sum_{\omega \in Z_{\beta,\theta}} \beta^{-1} e^{i(t-s)\omega-(\varepsilon+\varepsilon')|\omega|} < (h_F-i\omega)^{-1} f, \, g >_K .$$

For $|t-s| \neq 0,\beta$, the sum

$$G_{\beta,\theta}(t,s;f,g) \equiv \sum_{\omega \in Z_{\beta,\theta}} \beta^{-1} e^{i(t-s)\omega} < (h_F-i\omega)^{-1} f, \, g >_K \tag{3.39}$$

converges (not absolutely) and hence, by the Abelian theorem, we get

$$\lim_{\varepsilon,\varepsilon' \downarrow 0} C_{\varepsilon,\varepsilon'}(t,s) = G_{\beta,\theta}(t,s;f,g). \tag{3.40}$$

To prove the continuity of $G_{\beta,\theta}(t,s;f,g)$ in $t,s \in [0,\beta]$, $|t-s| \neq 0, \beta$, we note that $G_{\beta,\theta}(t,s;f,g)$ is written as

$$G_{\beta,\theta}(t,s;f,g) = \langle T(\psi(t,f)\psi^{\dagger}(s,g)\rangle_{\beta,\theta} \quad , \tag{3.41}$$

which follows from Lemma 3.3. By (3.16), (3.17) and (3.13), the right hand side of (3.41) is continuous in t, $s \in [0,\beta]$, $|t-s| \neq 0, \beta$, and the discontinuity at $|t-s| = 0, \beta$ is the first kind. Thus we get the desired result. □

LEMMA 3.5. Let $0 < t_1 < t_2 < \ldots < t_n < \beta$ and $f_1,\ldots,f_n$ be in $D(e^{\beta h_F})$. Then,

$$\frac{\mathrm{Tr}(e^{i\pi\theta N_F} e^{-t_1 H_{0F}} \psi^{\#}(f_1)\, e^{-(t_2-t_1)H_{0F}} \psi^{\#}(f_2)\ldots e^{-(t_n-t_{n-1})H_{0F}} \psi^{\#}(f_n) e^{-(\beta-t_n)H_{0F}})}{\mathrm{Tr}(e^{i\pi\theta N_F} e^{-\beta H_{0F}})}$$

$$= (\Omega,\ \Psi^{\theta}_{\#}(t_1,f_1)\Psi^{\theta}_{\#}(t_2,f_2)\ldots\ldots\ \Psi^{\theta}_{\#}(t_n,f_n)\Omega), \tag{3.42}$$

where the correspondence between $\psi^{\#}(f)$ and $\Psi^{\theta}_{\#}(t,f)$ is given by

$$\psi^{\dagger}(f) \longleftrightarrow \Psi^{\theta}_2(t,f), \qquad \psi(f) \longleftrightarrow \Psi^{\theta}_1(t,f).$$

Proof. The left hand side of (3.42) is written as

$$< T(\psi^{\#}(K_{t_1}^{\#} f_1)\ldots\psi^{\#}(K_{t_n}^{\#} f_n))>_{\beta,\theta}$$

where

$$K_t^{\dagger} = e^{-th_F}, \quad K_t = e^{th_F} .$$

From (3.39)-(3.41), we have

$$< T(\psi^{\#}(s,g)\psi^{\#}(t,f))>_{\beta,\theta} = (\Omega,\ \Psi_{\#}^{\theta}(s,g)\Psi_{\#}^{\theta}(t,f)\Omega).$$

Therefore, by Lemma 3.2 and Wick's theorem, we get (3.42). □

As a generalization of Lemma 3.5, we have

<u>THEOREM 3.6</u>. Let $t_1,..,t_n$ be as in Lemma 3.5 and f_{ij} , $i=1,..,n$, $j=1,..,k_i$, $k_i=1,2,3,..$, be in $D(e^{\beta h_F})$. Then,

$$\frac{\mathrm{Tr}(e^{i\pi\theta N_F} e^{-t_1 H_{0F}} \prod_{j=1}^{k_1} \psi^{\#}(f_{1j})\, e^{-(t_2-t_1)H_{0F}} \ldots e^{-(t_n-t_{n-1})H_{0F}} \prod_{j=1}^{k_n} \psi^{\#}(f_{nj}) e^{-(\beta-t_n)H_{0F}})}{\mathrm{Tr}(e^{i\pi\theta N_F} e^{-\beta H_{0F}})}$$

$$=(\Omega,\ [\prod_{j=1}^{k_1} \Psi_{\#}^{\theta}(t_1,f_{1j})][\prod_{j=1}^{k_2} \Psi_{\#}^{\theta}(t_2,f_{2j})]\ldots[\prod_{j=1}^{k_n} \Psi_{\#}^{\theta}(t_n,f_{nj})]\Omega), \qquad (3.43)$$

where the symbol $\prod_{j=1}^{n} \Psi_{\#}^{\theta}(t,f_j)$ on the right hand side is taken as

$$\lim_{\substack{\varepsilon_{n-1}\downarrow 0 \\ 0<\varepsilon_1<..<\varepsilon_{n-1}}} \Psi_{\#}^{\theta}(t,f_1)\Psi_{\#}^{\theta}(t+\varepsilon_1,f_2)\ldots\Psi_{\#}^{\theta}(t+\varepsilon_{n-1},f_n)$$

in the sense of sesquilinear form.

Proof. By virtue of Grümm's convergence theorem ([15,32]), we can write the left hand side of (3.43) as a limit of a function of the form given on the left hand side of (3.42), where each of some clusters of t_k's tends decreasingly to each point t_j. Then, Lemma 3.5 implies (3.43). □

Finally we consider a perturbation of H_{0F}. Let $(X, d\sigma)$ be as in Section II and χ be an $\mathcal{K}$-valued strongly continuous function on X with property $J\chi(x) = \chi(x)$. Let P be a real polynomial in one variable. Then, the Bochner integral

$$\hat{P} = \int_X d\sigma(x)\ P(\psi^{\dagger}(\chi(x))\psi(\chi(x))\ ) \tag{3.44}$$

defines a bounded self-adjoint operator on $\mathcal{F}_F(\mathcal{K})$. Therefore the operator

$$H_F = H_{0F} + \hat{P} \tag{3.45}$$

is self-adjoint on $D(H_{0F})$ and bounded below. Further, for all $t > 0$, e^{-tH_F} is trace class on $\mathcal{F}_F(\mathcal{K})$.

COROLLARY 3.7. Suppose that, for some $\alpha \in (0,1)$, $\hat{P}H_{0F}^{-\alpha} | \{\Omega_F\}^{\perp}$ is trace class, Then, for all $\beta > 0$,

$$\frac{\mathrm{Tr}(e^{i\pi\theta N_F} e^{-\beta H_F})}{\mathrm{Tr}(e^{i\pi\theta N_F} e^{-\beta H_{0F}})}$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{\beta} dt_n \int_0^{t_n} dt_{n-1} \cdots\cdots\cdots \int_0^{t_2} dt_1$$

$$\times \int d\sigma(x_n)\ldots d\sigma(x_1)\ (\Omega, P(\Psi_2^{\theta}(t_1,\chi(x_1))\Psi_1^{\theta}(t_1,\chi(x_1))\ )\ldots P(\Psi_2^{\theta}(t_n,\chi(x_n))\Psi_1^{\theta}(t_n,\chi(x_n))\Omega)$$

(3.46)

Proof. This follows from the lemma in Appendix A and Theorem 3.6. $\square$

In the same manner as above, we can get more general trace formulas associated with $e^{-\beta H_F}$, but, here, we do not go into the details.

IV. INFINITE DIMENSIONAL KÄHLER-DIRAC OPERATORS AND PATH INTEGRAL REPRESENTATION OF THEIR INDEX

We now proceed to the main part of this article. We first recall some definitions and facts on exterior differential calculus on infinite dimensional linear manifolds (cf.[2],[27]). Let H be as in Section II and H^c be the complexification of H. We consider the following Boson-Fermion Fock space :

$$F = F_B(H_{-1}) \otimes F_F(H^c). \tag{4.1}$$

By a general theorem on tensor products of Hilbert spaces (e.g.,[22, §II.4]), F is identified as

$$F = L^2(Q,\ d\mu_0\ ;\ F_F(H^c)\ ), \tag{4.2}$$

the Hilbert space of $F_F(H^c)$-valued square integrable functions on $(Q,\ d\mu_0)$. Then, F is decomposed as

$$F = \bigoplus_{p=0}^{\infty} F^p \tag{4.3}$$

with

$$F^p = L^2(Q, d\mu_0;\ \Lambda^p(H^c)\ ). \tag{4.4}$$

As the conjugation on H^c to define the annihilation operator on $F_F(H^c)$, we take the operator $J: H^c \to H^c$ defined by

$$J(f,g) = (f,-g), \qquad (f,g) \in H^c,$$

where the complex structure is given by $i(f,g) = (-g,f)$ and H is identified as $\{\ (f,0) \in H^c\ \}$.

Let $F_0^p \subset F^p$ be the subspace algebraically spanned by vectors of the form

$$\Psi = P_n(\phi(f_1),\ldots,\phi(f_n))g_1\wedge \ldots\wedge g_p, \quad n=1,2,3,\ldots, f_1,\ldots,f_n,\ g_1,\ldots,g_p \in C^\infty(h_B) \tag{4.5}$$

where P_n's are polynomials in n-variables. It is obvious that, for all $p \geq 0$, F_0^p is dense in F^p. Let $\Psi \in F_0^p$ be given by (4.5). Then we define the exterior differential operator $d_p : F_0^p \to F_0^{p+1}$ by

$$d_p\,\Psi(\phi) = \sum_{j=1}^{n} \frac{\partial P_n}{\partial x_j}(\phi(f_1),\ldots,\phi(f_n))(\psi^\dagger(f_j)g_1 \wedge \ldots\wedge g_p)^{(p+1)} \tag{4.6}$$

and extending it by linearity to F_0^p, where $\partial P_n/\partial x_j$ denotes the partial derivative of $P_n(x_1,\ldots,x_n)$ with respect to x_j. It follows immediately from the definition that, for all $p \geq 0$,

$$d_{p+1}d_p = 0 \tag{4.7}$$

on F_0^p. Since d_p is densely defined in F^p, its adjoint d_p^* exists as a linear operator from F^{p+1} to F^p. By integration by parts with respect to $d\mu_0$, we see that

$$d_{p-1}^*\Psi(\phi) = p^{-1/2} \sum_{k=1}^{p}(-1)^{k-1}\Big\{ P_n(\phi(f_1),\ldots,\phi(f_n))\phi(h_B g_k)$$

$$- \sum_{j=1}^{n} \frac{\partial P_n}{\partial x_j}(\phi(f_1),\ldots,\phi(f_n))(f_j,g_k)_{H^c}\Big\}\, g_1\wedge\ldots\wedge\hat{g}_k\wedge\ldots\wedge g_p\ , \tag{4.8}$$

where Ψ is given by (4.5) and $\hat{g}_k$ indicates the omission of g_k. Therefore,

d^*_{p-1} is densely defined with $D(d^*_{p-1}) \supset F^p_0$ and hence d_p is closable ; We denote the closure by the same symbol. It follows from (4.7) that, for all $p \geq 1$,

$$d^*_{p-1} d^*_p = 0, \tag{4.9}$$

on $D(d^*_p)$.

The Laplace-Beltrami operator Δ_p associated with the pair (d_p, d^*_p) is defined by

$$\Delta_p = d^*_p d_p + d_{p-1} d^*_{p-1} , \tag{4.10}$$

on $D(d^*_p d_p) \cap D(d_{p-1} d^*_{p-1}) \supset F^p_0$ (For p=0, the second term on the right hand side is set to be identically zero).

<u>LEMMA 4.1</u>. Let A and B be closed linear operators acting in a Hilbert space M with $D(A) \cap D(B) \neq \phi$. Suppose that, for all f and g in $D(A) \cap D(B)$,

$$(Af, Bg)_M = 0,$$

Then, A + B is closed on $D(A) \cap D(B)$.

<u>Proof</u>. An easy exercise. □

<u>LEMMA 4.2</u>. For all $p \geq 0$, the operator Δ_p given by (4.10) is closed symmetric on $D(d^*_p d_p) \cap D(d_{p-1} d^*_{p-1})$. In particular, Δ_0 is self-adjoint on $D(d^*_0 d_0)$.

Proof. By (4.9), we have for all Ψ and Φ in $D(d_p^* d_p) \cap D(d_{p-1} d_{p-1}^*)$

$$(d_p^* d_p \Psi,\ d_{p-1} d_{p-1}^* \Phi) = 0.$$

On the other hand, it follows from a general theorem (von Neumann's theorem)(e.g.,[23, §X.3, Theorem X.25]) that $d_p^* d_p$ and $d_{p-1} d_{p-1}^*$ are self-adjoint on $D(d_p^* d_p)$ and $D(d_{p-1} d_{p-1}^*)$ respectively and hence closed. Therefore, by Lemma 4.1, Δ_p is closed on $D(d_p^* d_p) \cap D(d_{p-1} d_{p-1}^*)$. The symmetricity of Δ_p is obvious. □

Remark. The operator Δ_p is clearly non-negative. Therefore, it has a self-adjoint extension as the Friedrichs extension (e.g.,[23]). In the following, we shall prove that the self-adjoint extension is unique , identifying it with a definite operator.

A simple computation gives

$$\Delta_p \Psi = \sum_{j=1}^{n} :\phi(f_1)\ldots\phi(f_{j-1})\phi(h_B f_j)\phi(f_{j+1})\ldots\phi(f_n): g_1 \wedge \ldots \wedge g_p$$

$$+ \sum_{k=1}^{p} :\phi(f_1)\ldots\phi(f_n): g_1 \wedge \ldots \wedge g_{k-1} \wedge h_B g_k \wedge g_{k+1} \wedge \ldots \wedge g_p$$

with the vector

$$\Psi = :\phi(f_1)\ldots\phi(f_n): g_1 \wedge \ldots \wedge g_p \in \mathcal{F}_0^p.$$

Let

$$H_0 = d\Gamma(h_B) \otimes I + I \otimes d\Gamma(h_B). \tag{4.11}$$

Then, H_0 is non-negative self-adjoint on its natural domain and reduced by $\mathcal{F}^p$; We denote by $H_{0,p}$ the reduced part of H_0 to $\mathcal{F}^p$.

PROPOSITION 4.3. The operator Δ_p is essentially self-adjoint on $\mathcal{F}_0^p$ and we have

$$\Delta_p = H_{0,p} \tag{4.12}$$

as operator equality.

Proof. By (4.8), we have (4.12) on $\mathcal{F}_0^p$. On the other hand, it is well-known that $H_{0,p}$ is essentially self-adjoint on $\mathcal{F}_0^p$ (e.g.,[22]). Combining this fact with Lemma 4.2, we get (4.12) as operator equality. □

Remarks. (1) Our Laplacian Δ_p is a slight generalization of that in [27] where h_B is taken as the identity ($h_B = I$) and hence $\mathcal{H}_{-1} = \mathcal{H}$. For applications to supersymmetric quantum field theories, our Laplacian is required.

(2) By (4.12) and the known spectral properties of $d\Gamma(h_B)$, we have

$$\dim \operatorname{Ker} \Delta_p = 0, \quad p=1,2,3,\ldots\ldots,$$

$$\dim \operatorname{Ker} \Delta_0 = 1.$$

(3) A decomposition theorem of De Rham-Hodge-Kodaira's type as in [27] (cf. also [2]) holds also in our case.

The operators d_p and d_p^* can be lifted to operators acting in the total space $\mathcal{F}$: For $\Psi = \{ \Psi^{(p)} \} \in \mathcal{F}$, we define $d\Psi \in \mathcal{F}$ by

$$
\begin{aligned}
(d\Psi)^{(0)} &= 0, \\
(d\Psi)^{(p)} &= d_{p-1}\Psi^{(p-1)}, \qquad p \geq 1.
\end{aligned}
\tag{4.13}
$$

The operator d is closed on its natural domain and its adjoint is given by

$$
(d^*\Psi)^{(p)} = d_p^* \, \Psi^{(p+1)}, \qquad p \geq 0. \tag{4.14}
$$

Let

$$
\mathcal{F}_0 = \bigoplus_{p=0}^{\infty} \mathcal{F}_0^p \qquad \text{(the incompleted direct sum)} \tag{4.15}
$$

Then, d and d^* leave $\mathcal{F}_0$ invariant satisfying

$$
d^2 = (d^*)^2 = 0 \tag{4.16}
$$

on $\mathcal{F}_0$. Further, by (4.9), we have

$$
(d^*\Psi, d\Phi) = 0, \quad \Psi \in D(d^*), \quad \Phi \in D(d). \tag{4.17}
$$

As in the case of finite dimensional manifolds, we define Kähler-Dirac operators associated with d and d^* by

$$Q_1 = d + d^*, \tag{4.18}$$

$$Q_2 = i(d - d^*) \tag{4.19}$$

on $D(Q_1)=D(Q_2)= D(d) \cap D(d^*)$.

<u>LEMMA 4.4</u>. (a) Each Q_i leaves F_0 invariant and the following equalities hold on F_0 :

$$Q_1^2 = Q_2^2 = dd^* + d^*d,$$

$$\{ Q_1, Q_2 \} = 0.$$

(b) Each Q_i is closed symmetric.

<u>Proof</u>. Part (a) easily follows from the definition of Q_i and properties of d and d^*. As for part (b), the symmetricity of Q_i is obvious. The closedness follows from (4.17) and Lemma 4.1. □

<u>PROPOSITION 4.5</u>. Each Q_i is self-adjoint and essentially self-adjoint on any core of H_0. Further, we have

$$H_0 = Q_1^2 = Q_2^2 \tag{4.20}$$

as operator equality.

Proof. By Proposition 4.3 and Lemma 4.4 (a), we have (4.20) on $\mathcal{F}_0$. Hence there exists a constant $c > 0$ such that, for all $\Psi \in \mathcal{F}_0$,

$$||Q_i \Psi|| \leq c||(H_0 + I)\Psi||$$

We have also $[Q_i, H_0] = 0$ on $\mathcal{F}_0$. On the other hand, H_0 is essentially self-adjoint on $\mathcal{F}_0$. Therefore, by the Glimm-Jaffe-Nelson commutator theorem ([23, §X.5],[13, §19.4]), Q_i is essentially self-adjoint on $\mathcal{F}_0$ and hence, by Lemma 4.4 (b), Q_i is self-adjoint. The commutator theorem implies also that any core of H_0 is a core of Q_i. Thus, the first half of the proposition is proved. Since Q_i is self-adjoint, Q_i^2 also is self-adjoint. Thus, we get (4.20) as operator equality. □

We now proceed to the main subject in this section, that is, the index problem. We first note that the total space $\mathcal{F}$ is decomposed as

$$\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^- \tag{4.21}$$

with

$$\mathcal{F}^+ = \bigoplus_{p=0}^{\infty} \mathcal{F}^{2p}, \qquad \mathcal{F}^- = \bigoplus_{p=1}^{\infty} \mathcal{F}^{2p-1}. \tag{4.22}$$

Let $P^{\pm}$ be the orthogonal projection onto $F^{\pm}$ and put

$$P_F = P^+ - P^-. \tag{4.23}$$

LEMMA 4.6. The operator P_F leaves $D(Q_i)$ invariant and the anti-commutation relation

$$\{ P_F, Q_i \} = 0 \tag{4.24}$$

holds on $D(Q_i)$.

Proof. From the definition of Q_i, we see that Q_i maps $F^{\pm} \cap F_0$ $F^{\mp} \cap F_0$. Therefore we have

$$(Q_i\Psi, P_F\Phi) + (P_F\Psi, Q_i\Phi) = 0 \tag{4.25}$$

for all $\Psi, \Phi \in F_0$. Since F_0 is a core of Q_i (Proposition 4.5), (4.25) extends to all $\Psi, \Phi \in D(Q_i)$. Hence, for all Ψ in $D(Q_i)$, $P_F\Psi$ is in $D(Q_i)$ and we have (4.24). □

Lemma 4.4, Proposition 4.5, (4.21) and Lemma 4.6 imply that the quadruple $\{ F, \{Q_1, Q_2\}, H_0, P_F \}$ is a supersymmetric quantum theory in the sense of [3] (cf. Appendix B in this paper). Therefore, as in Appendix B, we can define the index $\Delta(Q_i)$ associated with the decomposition (4.21).

PROPOSITION 4.7. Let $Q_{i+} : D(Q_i) \cap F^+ \to F^-$ be the restriction of Q_i to $D(Q_i) \cap F^+$ (see Appendix B). Then, for each i =1,2, Q_{i+} is Fredholm and its Fredholm index is equal to $\Delta(Q_i)$. Further, we have

$$\Delta(Q_i) = 1, \quad i=1,2. \tag{4.26}$$

Proof. In general, it is shown that a densely defined closed linear operator A from a Hilbert space to another Hilbert space is Fredholm if and only if A^* (or A^*A) is ([7]). Therefore, one has the criterion ([7]) that A is Fredholm if and only if $\sigma_{ess}(A^*A) > 0$, where $\sigma_{ess}(\cdot)$ denotes the essential spectrum. In the case of Q_{i+}, we have

$$Q_{i+}^* Q_{i+} = H_0 |\, F^+ \equiv H_0^+ .$$

On the other hand, the strict positivity of h_B implies that $\sigma_{ess}(H_0^+) > 0$. Therefore, we conclude that Q_{i+} is Fredholm. In this case, by definition, $\Delta(Q_i)$ coincides with the Fredholm index of Q_{i+}. Equality (4.26) follows from the fact that

$$\Delta(Q_i) = \dim \operatorname{Ker} H_0^+ \; - \; \dim \operatorname{Ker} H_0^-$$

(see Appendix B) and

$$\operatorname{Ker} H_0^+ = \{\text{const. } 1 \otimes \Omega_F \}, \quad \operatorname{Ker} H_0^- = \{0\},$$

where Ω_F is the Fock vacuum in $F_F(H^c)$ and $H_0^- = H_0 |\, F^-$

We next consider perturbations of Q_i in such a way that the perturbed operators also satisfy (4.24). Since Q_{i+} is Fredholm as we have seen above, we have some stability theorems on the index (e.g.,[18]). From our view point, such a case is not so interesting. Thus, we try to give perturbations under which the index may change.

Let X be a bounded closed rectangle in R^ν ($\nu=1,2,..$) and ρ be an $D(e^{\beta h_B}) \cap \mathcal{H}$-valued strongly continuous function such that so is $e^{\beta h_B}\rho(x)$. Let P be a real polynomial in one variable. Then, $P(\phi(\rho(x))) \otimes \psi^{\#}(\rho(x))$ is strongly continuous in $x \in X$ on $L^4(Q,d\mu_0;\mathcal{F}_F(\mathcal{H}^c))$. Therefore, the operator

$$U^{\#} = \int_X dx\, P(\phi(\rho(x))) \otimes \psi^{\#}(\rho(x)) \tag{4.27}$$

is defined as the strong Riemann integral on $L^4(Q,d\mu_0;\mathcal{F}_F(\mathcal{H}^c))$ and closable. We denote the closure by the same symbol. Henceforth, for notational simplicity, we write as

$$\phi(\rho(x)) = \phi_x, \quad \psi^{\#}(\rho(x)) = \psi^{\#}_x , \quad \phi(h_B\rho(x)) = h_B\phi_x , \tag{4.28}$$

and, if there exists no possibility of confusion, we drop the symbol $\otimes$ in operator tensor products. For a functional $F(\phi)$ on $\mathcal{F}_B(\mathcal{H}_{-1})$, we define the directional derivative $D_fF(\phi)$ in the direction $f \in \mathcal{H}_{-1}$ by

$$D_fF(\phi) = \lim_{\varepsilon\to 0} \frac{F(\phi+\varepsilon f) - F(\phi)}{\varepsilon} , \tag{4.29}$$

provided that the right hand side exists.

LEMMA 4.8. For all $\Psi \in F_0$, $U^{\#}\Psi$ is in $D(d) \cap D(d^*)$ and the following equalities hold on F_0 :

$$dU^\dagger = -U^\dagger d, \tag{4.30}$$

$$d^* U = -Ud^*, \tag{4.31}$$

$$d^* U^\dagger = -U^\dagger d^* + \int_X dx \quad [P(\phi_x)h_B\phi_x - P(\phi_x)D_{\rho(x)} - P'(\phi_x)\psi_x\psi_x^\dagger], \tag{4.32}$$

$$dU = -Ud + \int_X dx \quad [P'(\phi_x)\psi_x^\dagger\psi_x + P(\phi_x)D_{\rho(x)}], \tag{4.33}$$

$$U^2 = U^{\dagger 2} = 0, \tag{4.34}$$

$$\{ U, U^\dagger \} = || \int_X dx \quad P(\phi_x)\rho(x)||_H^2 \quad . \tag{4.35}$$

Proof. By direct computations using (4.6) and (4.8), one can prove the following equalities on F_0 :

$$d^* P(\phi_x)\psi_x = - P(\phi_x)\psi_x d^*,$$

$$dP(\phi_x)\psi_x^\dagger = - P(\phi_x)\psi_x^\dagger d,$$

$$d^* P(\phi_x)\psi_x^\dagger = - P(\phi_x)\psi_x d^* + P(\phi_x)h_B\phi_x - P'(\phi_x)\psi_x\psi_x^\dagger - P(\phi_x)D_{\rho(x)},$$

$$dP(\phi_x)\psi_x = - P(\phi_x)\psi_x d + P'(\phi_x)\psi_x^\dagger\psi_x + P(\phi_x)D_{\rho(x)}.$$

The right hand sides of these equations applied to vectors in F_0 are strongly Riemann integrable with respect to dx (Notice that $\sup_{x\in X} ||\rho(x)||_H < \infty$ because of the strong continuity of $\rho(x)$). Therefore, the closedness of d and d^* implies that $U^{\#} : F_0 \to D(d) \cap D(d^*)$ and (4.30)-(4.33) hold on F_0. (4.34) and (4.35) can be proved similarly. □

Let

$$d(P) = d + U^{\dagger} \tag{4.36}$$

on F_0. Then its adjoint is given by

$$d(P)^* = d^* + U \tag{4.37}$$

on F_0. Hence, d(P) is closable and we denote the closure of $d(P)|F_0$ by the same symbol.

Remark. Formally we can write as

$$d(P) = e^{-V} d\, e^{V}, \qquad d(P)^* = e^{V} d^* e^{-V},$$

where

$$V = \int_X dx\Big[\int_0^{\phi_x} dy\, P(y)\Big].$$

This kind of perturbations in the case of finite dimensional manifolds has been discussed in [35] and [19].

<u>LEMMA 4.9.</u> For each $p \geq 0$, $d(P)$ (resp. $d(P)^*$) maps F_0^p (resp. F_0^{p+1}) to F^{p+1} (resp. F^p) and

$$d(P)^2 = 0, \quad d(P)^{*2} = 0 \tag{4.38}$$

on F_0. Further, we have

$$(d(P)\Psi, d(P)^*\Phi) = 0, \quad \Psi \in D(d(P)), \quad \Phi \in D(d(P)^*). \tag{4.39}$$

<u>Proof</u>. The first half follows from the definition of $d(P)^{\#}$ and properties of $d^{\#}$ and $\psi^{\#}$ ((4.16)). Equality (4.39) follows from (4.38) and a limiting argument. □

The perturbed Kähler-Dirac operators corresponding to Q_1 and Q_2 are defined by

$$\begin{aligned} Q_1(P) &= d(P) + d(P)^* \\ Q_2(P) &= i(d(P) - d(P)^*) \end{aligned} \tag{4.40}$$

on $D(d(P)) \cap D(d(P)^*)$. By (4.39) and Lemma 4.1, $Q_i(P)$ is closed symmetric with $D(Q_i) = D(d(P)) \cap D(d(P)^*)$.

<u>LEMMA 4.10</u>. For each $i=1,2$, P_F leaves $D(Q_i(P))$ invariant and the anti-commutation relation

$$\{ P_F, Q_i(P)\} = 0 \tag{4.41}$$

holds on $D(Q_i(P))$.

Proof. It is easy to see that, for all Ψ in $D(d(P)^*)$ and Φ in F_0,

$$(P_F\Psi,\ d(P)\Phi) \ + \ (d(P)^*\Psi,\ P_F\Phi) = 0. \tag{4.42}$$

By a limiting argument, we can extend (4.42) to all Φ in $D(d(P))$. Then, (4.42) implies that P_F leaves $D(d(P)^{\#})$ invariant and

$$\{ P_F,\ d(P)^{\#}\} = 0 \tag{4.43}$$

on $D(d(P)^{\#})$. Thus the desired result follows. □

We define the perturbed Laplace-Beltrami operator H by

$$H = Q_1(P)^2 \tag{4.44}$$

in the sense of sesquilinear form on $D(Q_1(P)) \times D(Q_1(P))$, so that H is non-negative self-adjoint with $D(H^{1/2}) = D(Q_1(P))$. This is possible, because $Q_1(P)$ is closed as we have already noticed. By (4.39), we have also

$$H = Q_2(P)^2 \tag{4.45}$$

in the sense of sesquilinear form on $D(Q_2(P)) \times D(Q_2(P))$. Using Lemmas 4.8 and 4.9, we can write down the explicit action of H on F_0. Namely, for all Ψ in F_0, we have

$$H\Psi = Q_i(P)^2\,\Psi = (H_0 + W + H_I)\Psi\ , \tag{4.46}$$

with

$$W(\phi) = \left\|\int_X dx\, P(\phi_x)\rho(x)\right\|_H^2 + \int_X dx[P(\phi_x)h_B\phi_x - P'(\phi_x)\|\rho(x)\|_H^2], \qquad (4.47)$$

$$H_I = 2 \int_X dx\, P'(\phi_x)\psi_x^\dagger\psi_x, \qquad (4.48)$$

where we have used the fact that $\{\psi_x, \psi_x^\dagger\} = \|\rho(x)\|_H^2$.

LEMMA 4.11. Let r be the degree of P. Then, we have

$$D(H_0^{2r}) \subset D(H_0) \cap D(W) \cap D(H_I). \qquad (4.49)$$

Further, for all $\Psi \in D(H_0^{2r})$, $Q_i(P)\Psi$ is in $D(Q_i(P))$ and

$$H\Psi = Q_i(P)^2\Psi = (H_0 + W + H_I)\Psi. \qquad (4.50)$$

Proof. By the Schwarz inequality, we have

$$|W| \le \left(\int_X dx\, P(\phi_x)^2\right)\left(\int_X dx\|\rho(x)\|_H^2\right) + \int_X dx[|P(\phi_x)h_B\phi_x| + |P'(\phi_x)|\|\rho(x)\|_H^2].$$

Therefore, using estimate (2.20), we get

$$\|W\Psi\| \le \text{const.}\ \|(H_0 + I)^{2r}\Psi\|, \quad \Psi \in F_0. \qquad (4.51)$$

Noting that $||\psi_x^\dagger\psi_x|| \le ||\rho(x)||_H^2$, we have also

$$||H_I\Psi|| \le \text{const.}||(H_0 + I)^{2r}\Psi||, \quad \Psi \in F_0. \tag{4.52}$$

On the other hand, we know from a standard argument (e.g.,[22,23]) that F_0 is a core of $(H_0 + I)^{2r}$. Therefore, (4.51) and (4.52) extend to all Ψ in $D(H_0^{2r})$. Thus (4.49) is proved. To prove the second half, we note that

$$||d^\#\Psi|| \le ||H_0^{1/2}\Psi||, \quad \Psi \in D(H_0^{1/2})$$

and

$$||U^\#\Psi|| \le \text{const.}||(H_0 + I)^r\Psi||, \ \Psi \in D(H_0^r).$$

Hence, in particular, we have

$$||\, Q_i(P)\Psi|| \le \text{const.}||(H_0 + I)^{2r}\Psi||, \quad \Psi \in D(H_0^{2r}). \tag{4.52)'$$

Since F_0 is a core of $(H_0 + I)^{2r}$, for every Ψ in $D(H_0^{2r})$, we can take a sequence $\{\Psi_n\}$, $\Psi_n \in F_0$, such that $\Psi_n \xrightarrow{s} \Psi$ and $(H_0 + I)^{2r}\Psi_n \xrightarrow{s} (H_0 + I)^{2r}\Psi$. By (4.46), we have

$$Q_i(P)^2\Psi_n = (H_0 + W + H_I)\Psi_n.$$

By (4.51) and (4.52), the right hand side converges strongly to $(H_0 + W + H_I)\Psi$. On the other hand, we have from (4.52)'that $Q_i(P)\Psi_n \xrightarrow{s} Q_i(P)\Psi$. Thus, by the closedness of $Q_i(P)$, $Q_i\Psi$ is in $D(Q_i(P))$ and (4.50) holds. □

LEMMA 4.12. Suppose that W is bounded below. Then, $H_0 + W$ is essentially self-adjoint on $C^\infty(H_0) \cap D(W)$ and bounded below.

Proof. It is obvious that W is in $L^2(Q,d\mu_0)$. Since h_B is strictly positive, $\exp(-tH_{0B})$ forms a hypercontractive semi-group(e.g.,[28,23]). Therefore, by a general theorem (e.g.,[23, §X.9, Theorem X.59]), $H_{0B} + W$ is essentially self-adjoint on $C^\infty(H_{0B}) \cap D(W)$ as an operator acting in $L^2(Q,d\mu_0)$. Since $H_0 = H_{0B} \otimes I + W \otimes I + I \otimes d\Gamma(h_B)$, H_0 is essentially self-adjoint on $C^\infty(H_{0B}) \cap D(W) \otimes C^\infty(d\Gamma(h_B)) \subset C^\infty(H_0) \cap D(W \otimes I)$. Thus, the desired result follows. □

We denote the closure of $(H_0 + W)|(C^\infty(H_0) \cap D(W))$ by the same symbol.

We now state our main result :

THEOREM 4.13. Assume $(A)_B$ in Section II. Suppose that :

(i) W is bounded below.

(ii) For some $\alpha \in (0,1)$ and a sufficiently large constant $c > 0$,

$H_I(H_0 + W + c)^{-\alpha}$ is trace class on $\mathcal{F}$.

Then, $Q_i(P)$ is self-adjoint and the index $\Delta(Q_i(P))$ (see Appendix B) is given by

$$\Delta(Q_i(P)) = \sum_{n=0}^{\infty} (-2)^n \int_0^\beta dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \int_X dx_1 \ldots dx_n$$

$$\times \int_{\mathcal{H}_{-\gamma}} d\mu_{0,\beta}(\Phi) \exp[-\int_0^\beta ds\, W(\Phi_s)]\, P'(\Phi_{t_1}(\rho(x_1))) \ldots P'(\Phi_{t_n}(\rho(x_n)))$$

$$\times (\Omega, \Psi_2^1(t_1,\rho(x_1))\Psi_1^1(t_1,\rho(x_1)) \ldots \Psi_2^1(t_n,\rho(x_n))\Psi_1^1(t_n,\rho(x_n))\Omega)_{\mathcal{F}(\mathcal{H}^c_{\beta,1})},$$

(4.53)

independently of $\beta>0$, where Φ_s is the Gaussian mean zero process on $[0,\beta]$ introduced in Section II, $\Psi^1_{\#}(t,\cdot)$ is the Euclidean Fermi field on $F_F(H^c_{\beta,1})$ (the case $\theta=1$) (see Section III) and Ω is the Fock vacuum in $F_F(H^c_{\beta,1})$.

Proof. Let

$$L = H_0 + W.$$

By assumption (i) and Lemma 4.12, L is self-adjoint and bounded below. Then, by assumption (ii) and by the proof of the lemma in Appendix A, we conclude that $L + H_I$ is self-adjoint on D(L) and essentially self-adjoint on any core of L. By Lemma 4.12, L is essentially self-adjoint also on $D(H_0^{2r})$. Therefore, by Lemma 4.11 and the self-adjointness of H, we have

$$H = L + H_I$$

as operator equality. Then, in the same way as in the proof of Proposition 4.5, we can show that $Q_i(P)$ is essentially self-adjoint on $D(H_0^{2r})$ and hence self-adjoint on $D(Q_i(P))$., Thus, the first half is proved. By condition (i) and the generalized Golden-Thompson inequality (e.g.,[32]), e^{-tL} is trace class for all $t > 0$. Then, it follows from assumption (ii) and the lemma in Appendix A that, for all $t > 0$, e^{-tH} is trace class. Therefore, by the lemma in Appendix B, we have

$$\Delta(Q_i(P)) = \mathrm{Tr}(P_F e^{-\beta H})$$

independently of $\beta > 0$. Noting that

P_F is written as

$$P_F = e^{i\pi N_F},$$

we have

$$\Delta(Q_i(P)) = \mathrm{Tr}(\, e^{i\pi N_F} e^{-\beta H}\,).$$

By the lemma in Appendix A and a limiting argument, we get

$$\Delta(Q_i(P)) = \sum_{n=0}^{\infty} (-2)^n \int_0^{\beta} dt_n \int_0^{t_n} dt_{n-1} \;\ldots\; \int_0^{t_2} dt_1 \int_X dx_1 \ldots dx_n$$

$$\times\, \mathrm{Tr}[e^{i\pi N_F} e^{-t_1 H_{OF}} \psi^\dagger_{x_1} \psi_{x_1} e^{-(t_2-t_1)H_{OF}} \ldots\ldots \psi^\dagger_{x_n} \psi_{x_n} e^{-(\beta-t_n)H_{OF}}]$$

$$\times\, \mathrm{Tr}[e^{-t_1 L_B} P'(\phi_{x_1}) e^{-(t_2-t_1)L_B} P'(\phi_{x_2}) \ldots P'(\phi_{x_n}) e^{-(\beta-t_n)L_B}],$$

where $H_{OF} = d\Gamma(h_B)$ acting in $\mathcal{F}_F(\mathcal{H}^c)$ and

$$L_B = H_{OB} + W$$

acting in $\mathcal{F}_B(\mathcal{H}_{-1})$ (L_B is self-adjoint and e^{-tL_B} is trace class for all $t > 0$). Then, Corollary 2.4 and Theorem 3.6 together with the fact that

$$\mathrm{Tr}(e^{i\pi N_F} e^{-\beta H_{OF}})\mathrm{Tr}(e^{-\beta H_{OB}}) = 1$$

(see (2.21) and (3.24)) imply (4.53). □

Finally we give an alternative representation of the right hand side of (4.53). To this end, we introduce the kernel

$$K_\Phi(t,x;s,y) \equiv -2P'(\Phi_t(\rho(x)))\ (\Omega,\Psi_2^1(t,\rho(x))\Psi_1^1(s,\rho(y))\Omega)_{F_F(H^c_{\beta,1})}$$

$$= 2P'(\Phi_t(\rho(x)) \sum_{\omega\in Z_{\beta,1}} \beta^{-1}e^{-i(t-s)\omega} < (h_B-i\omega)^{-1}\rho(x),\rho(y)>_{H^c}\ , \tag{4.54}$$

which is well-defined for $t,s \in (0,\beta)$, $|t-s| \neq 0$, $x,\ y \in X$, and almost everywhere $\Phi \in H_{-\gamma}$. We define also

$$K_\Phi(t,x;t,y) \equiv \lim_{\varepsilon\downarrow 0} K_\Phi(t,x;t+\varepsilon,y) \tag{4.55}$$

for $t \in (0,\beta)$. By the theory in Section III, we can express $K_\Phi(t,x;s,y)$ and $K_\Phi(t,x;t,y)$ explicitly in terms of quantities in the Fermion Fock space $F_F(H^c)$. Using these expressions, we see that the integral operator K_Φ defined by

$$(K_\Phi f)(t,x) = \int_0^\beta ds \int_X dy\ K_\Phi(t,x;s,y)f(s,y), \quad f \in L^2([0,\beta] \times X), \tag{4.56}$$

is Hilbert-Schmidt on $L^2([0,\beta] \times X)$. Further, the trace-like quantity

$$\tilde{\mathrm{Tr}}\ K_\Phi \equiv \int_0^\beta dt \int_X dx\ K_\Phi(t,x;t,x)$$

$$= -2 \int_0^\beta dt \int_X dx\, P'(\Phi_t(\rho(x))) < (1 - e^{\beta h_B})^{-1}\rho(x),\rho(x) >_{H^c}$$

(4.57)

exists (Note that $K_\Phi(t,x;t,y)$ is continuous in (t,x,y) for almost everywhere Φ and hence $K_\Phi(t,x\, ;\, t,x)$ is unambiguously defined).

COROLLARY 4.14. Suppose that, for some constant $\varepsilon > 0$,

$$\exp[- \int_0^\beta ds\, W(\Phi_s) + (1+\varepsilon)|\tilde{\mathrm{Tr}}\, K_\Phi| + \frac{1}{2}(1+\varepsilon)^2 ||K_\Phi||^2_{HS}\,] \in L^1(H_{-\gamma}, d\mu_{0,\beta}), \quad (4.58)$$

where $||\cdot||_{HS}$ denotes the Hilbert-Schmidt norm. Then, under the same assumption as in Theorem 4.13, we have

$$\Delta(Q_i(P)) = \int d\mu_{0,\beta}\, \det_2(I + K_\Phi)\, \exp[- \int_0^\beta ds\, W(\Phi_s) + \tilde{\mathrm{Tr}}\, K_\Phi\,], \quad (4.59)$$

where $\det_2(I + K_\Phi)$ is the regularized determinant of $I + K_\Phi$ (e.g., [31,32]).

Remark. Formally, we have

$$\det_2(I + K_\Phi)\, \exp[\, \tilde{\mathrm{Tr}}\, K_\Phi] = \det\,(I + K_\Phi),$$

the unregularized determinant.

Proof. Let

$$K_{\Phi}^{(n)}(t_1,x_1;t_2,x_2;\dots;t_n,x_n)$$

$$= (-2)^n P'(\Phi_{t_1}(\rho(x_1)))\dots\dots P'(\Phi_{t_n}(\rho(x_n)))$$

$$\times\ (\Omega,\Psi_2^1(t_1,\rho(x_1))\Psi_1^1(t_1,\rho(x_1))\dots\dots\ \Psi_2^1(t_n,\rho(x_n))\Psi_1^1(t_n,\rho(x_n)\ \Omega)_{F_F(H_{\beta,1}^c)}\ .$$

Then, $K_{\Phi}^{(n)}(t_1,x_1;\dots;t_n,x_n)$ is symmetric with respect to every permutation of $(t_1,x_1),\dots,(t_n,x_n)$. Therefore, one can write as

$$\int_0^{\beta} dt_n \int_0^{t_n} dt_{n-1}\ \dots\dots \int_0^{t_2} dt_1 \quad \int_X dx_1\dots dx_n\ K_{\Phi}^{(n)}(t_1,x_1;\dots;t_n,x_n)$$

$$= \frac{1}{n!}\int_0^{\beta} dt_1\dots dt_n \quad \int_X dx_1..dx_n\ K_{\Phi}^{(n)}(t_1,x_1;\dots;t_n,x_n).$$

It is easy to see that

$$K_{\Phi}^{(n)}(t_1,x_1;\dots;t_n,x_n) = \det\ (\ K_{\Phi}(t_i,x_i;t_j,x_j)\ )_{1\le i,j\le n}\ ,$$

from which it follows that

$$|K_{\Phi}^{(n)}(t_1,x_1;\dots;t_n,x_n)| \le C_{\beta}^n\ n!\ \prod_{j=1}^{n}[|P'(\Phi_{t_j}(\rho(x_j)))|\ \ ||\rho(x_j)||_{H}^{2}].$$

with a constant $C_{\beta} > 0$. Therefore, we see that the integral with respect to $dt_1\dots dt_n dx_1\dots dx_n$ and that with respect to $d\mu_{0,\beta}$ in (4.53) are interchangeable. Further, one can show that, for all $\lambda \in C$,

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^\beta dt_1 \ldots dt_n \int_X dx_1 \ldots dx_n \; K_\Phi^{(n)}(t_1,x_1;\ldots;t_n,x_n)$$

$$= \det{}_2(\, I + \lambda K_\Phi) \exp[\lambda \, \tilde{\mathrm{Tr}} \, K_\Phi].$$

(see, e.g.,[31,32]). It is well-known [31] that

$$|\det{}_2(I + \lambda K_\Phi)| \leq \exp[\, \tfrac{1}{2} \lambda^2 ||K_\Phi||_{HS}^2 \,].$$

By Cauchy's estimate, we have for all $N \geq 1$ and any $\varepsilon > 0$,

$$\sum_{n=0}^{N} \frac{1}{n!} \left| \int_0^\beta dt_1 \ldots dt_n \int_X dx_1 \ldots dx_n \; K_\Phi^{(n)}(t_1,x_1;\ldots;t_n,x_n) \right|$$

$$\leq [(1+\varepsilon)/\varepsilon] \exp[(1+\varepsilon)|\tilde{\mathrm{Tr}} \, K_\Phi| + \tfrac{1}{2}(1+\varepsilon)^2 ||K_\Phi||_{HS}^2 \,].$$

Then, by condition (4.58) and the Lebesgue dominated convergence theorem, we may change the sum $\sum_{n=0}^{\infty}$ and the integral $d\mu_{0,\beta}$ in (4.53) and hence we get (4.59). □

APPENDIX A. A TRACE LEMMA

LEMMA. Let A and B be linear operators acting in a separable Hilbert space $\mathcal{H}$ and suppose that :

(i) A is strictly positive self-adjoint and, for all $t > 0$, e^{-tA} is trace class.

(ii) B is symmetric such that, for some $\alpha \in (0,1)$, $BA^{-\alpha}$ is trace class.

Then, A+B is self-adjoint with domain D(A) and bounded below. Furthermore, for all $t > 0$, $e^{-t(A+B)}$ is trace class and we have

$$e^{-t(A+B)} = \sum_{n=0}^{\infty} (-1)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, e^{-t_n A} B e^{-(t_{n-1}-t_n)A} B \ldots \times e^{-(t_1-t_2)A} B e^{-(t-t_1)A} \tag{A.1}$$

in the trace norm.

Proof. By condition (ii), B is relatively compact with respect to A and hence is relatively bounded with respect to A with the relative bound infinitesimally small. Therefore, by the Kato-Rellich theorem (e.g.,[18], [23]), A+B is self-adjoint on D(A) and bounded below (cf. also [24, §XIII.4]). To prove the second half, we note that the following formula holds (Duhammel's formula) :

$$e^{-t(A+B)} = e^{-tA} - \int_0^t ds\, e^{-(t-s)(A+B)} B e^{-sA} .$$

This is easily seen by showing that both sides applied to a vector in D(A) solve the same first order differential equation. By change of variable $t-s \to s$ and by iteration, we have

$$e^{-t(A+B)} = \sum_{n=0}^{N} (-1)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \, e^{-t_n A} Be^{-(t_{n-1}-t_n)A} B \ldots\ldots$$

$$\times e^{-(t_1-t_2)A} Be^{-(t-t_1)A}$$

$$+ R_N, \tag{A.2}$$

where

$$R_N = (-1)^{N+1} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_N} dt_{N+1} \, e^{-t_{N+1}(A+B)} Be^{-(t_N-t_{N+1})A} B \ldots e^{-(t-t_1)A} .$$

Writing as $Be^{-sA} = s^{-\alpha}BA^{-\alpha}[(sA)^{\alpha}e^{-sA}]$ and using condition (ii), we see that, for all $s > 0$, Be^{-sA} is trace class with bound

$$||Be^{-sA}||_1 \le s^{-\alpha}||BA^{-\alpha}||_1 C_\alpha ,$$

where $||\ \ ||_1$ denotes the trace norm and

$$C_\alpha = ||A^{\alpha}e^{-A}|| \qquad (\text{ operator norm }).$$

Therefore it follows that every term on the right hand side of (A.2) is trace class and hence that so is $e^{-t(A+B)}$. Further, we have

$$||R_N||_1 \leq e^{t|d|}(C_\alpha ||BA^{-\alpha}||_1)^N \; t^{(1-\alpha)(N+1)} \; \Gamma(1-\alpha)^{N+1} \; \Gamma(\;(1-\alpha)(N+1)+1)^{-1},$$

where d is the infimum of the spectrum of A+B and $\Gamma(z)$ is the gamma function. Since $\Gamma(z) \sim \sqrt{2\pi}\; e^{-z}\; z^{z-(1/2)}$ as $|z| \to \infty$, we have

$$\lim_{N \to \infty} ||R_N||_1 = 0.$$

Thus, we get (A.1) in the trace norm. □

APPENDIX B. INDEX OF A SELF-ADJOINT OPERATOR IN A GRADED HILBERT SPACE

Let $\mathcal{H}$ be a Hilbert space with inner product (,) and Q be a self-adjoint operator acting in $\mathcal{H}$. Suppose that $\mathcal{H}$ is decomposed into two mutually orthogonal closed subspaces $\mathcal{H}^{\pm}$:

$$\mathcal{H} = \mathcal{H}^{+} \oplus \mathcal{H}^{-}. \tag{B.1}$$

Let $P^{\pm}$ be the orthogonal projection onto $\mathcal{H}^{\pm}$ and put

$$K = P^{+} - P^{-}.$$

<u>Assume</u> that, for all f,g in D(Q),

$$(Qf,Kg) + (Kf,Qg) = 0.$$

Then, it is easy to see [3] that K leaves D(Q) invariant and, for all f in D(Q),

$$(KQ + QK)f = 0. \tag{B.2}$$

In particular, Q maps $D(Q) \cap \mathcal{H}^{\pm}$ to $\mathcal{H}^{\mp}$ and hence there exists a unique densely defined closed linear operator Q_{+} from $\mathcal{H}^{+}$ to $\mathcal{H}^{-}$ with $D(Q_{+}) = D(Q) \cap \mathcal{H}^{+}$ such that Q is written as

$$Q \begin{pmatrix} f^{+} \\ f^{-} \end{pmatrix} = \begin{pmatrix} 0 & Q_{+}^{*} \\ Q_{+} & 0 \end{pmatrix} \begin{pmatrix} f^{+} \\ f^{-} \end{pmatrix} \qquad f^{\pm} \in \mathcal{H}^{\pm} \cap D(Q). \tag{B.3}$$

We define the index $\Delta(Q)$ of Q associated with the decomposition (B.1) by

$$\Delta(Q) = \dim \operatorname{Ker} Q_+ - \dim \operatorname{Ker} Q_+^* , \tag{B.4}$$

provided that both Ker Q_+ and Ker Q_+^* are finite dimensional. We remark that, if Ran Q_+ is closed in addition, then Q_+ is Fredholm and $\Delta(Q)$ is the Fredholm index of Q_+ [18].

Let

$$H = Q^2.$$

Then, using (B.2), one can easily show that H is reduced by $\mathcal{H}^{\pm}$ [3]. We denote by $H^{\pm}$ the reduced part of H to $\mathcal{H}^{\pm}$, so that we have

$$H^+ = Q_+^* Q_+, \quad H^- = Q_+ Q_+^*.$$

It is obvious that Ker H^+ = Ker Q_+ and Ker H^- = Ker Q_+^* and hence we get

$$\Delta(Q) = \dim \operatorname{Ker} H^+ - \dim \operatorname{Ker} H^-. \tag{B.5}$$

LEMMA. Let Q be as above and suppose that $\exp(-tQ^2)$ is trace class with $t > 0$. Then,

$$\Delta(Q) = \operatorname{Tr}(Ke^{-tQ^2}) \tag{B.6}$$

independently of t.

Proof. One can write as

$$\mathrm{Tr}(Ke^{-tQ^2}) = \mathrm{Tr}\, e^{-tH^+} - \mathrm{Tr}\, e^{-tH^-}.$$

On the other hand, it follows from a general theorem [11] (cf. also [12]) that $\sigma_p(H^+)\setminus\{0\} = \sigma_p(H^-)\setminus\{0\}$ with the dimension of their corresponding eigenspaces coinciding, where σ_p denotes the point spectrum.
From this fact and (B.5), we get (B.6). □

Remarks. (1) For the case where $\exp(-tQ^2)$ is not necessarily trace class, $\Delta(Q)$ may be computed as

$$\Delta(Q) = \lim_{t\to\infty} \mathrm{Tr}(e^{-tH^+} - e^{-tH^-}),$$

provided that $\mathcal{H}^+ = \mathcal{H}^-$ and $e^{-tH^+} - e^{-tH^-}$ is trace class for all sufficiently large $t > 0$ with some suitable regularities [7,5]. Also a resolvent regularization can be used to compute the index [8,7,5].

(2) The quadruple $\{\mathcal{H}, Q, H, K\}$ given as above is a supersymmetric quantum theory in the sense of [3]. In this context, physisists call $\Delta(Q)$ the Witten index associated with the supersymmetric system (e.g., [35,36,9]).

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