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Analytic Functions

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Notes On Interpolation By Bounded Analytic Functions

By

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Let $\{z_n\}$ be a sequence in the open unit disc and write
 $\rho_n = \prod_{m; m \neq n} |(z_n - z_m)(1 - \bar{z}_m z_n)^{-1}|$. In the case of $|w_n| \leq \rho_n$
for all n , the interpolation problems are considered.

§1. Theorems

Let H^∞ be the Hardy space of bounded analytic functions in the unit disc D with boundary values in $L^\infty = L^\infty(d\theta/2\pi)$. Let $\{z_n\}$ be a sequence of distinct points in D and $\{w_n\}$ be a bounded sequence of complex numbers. Our notes concern the interpolation problem

$$f(z_n) = w_n, \quad n = 1, 2, \dots$$

for f in H^∞ . Put

$$\rho_n = \prod_{m; m \neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right|.$$

Carleson [1] proved that every interpolation problem has a solution if and only if $\inf_n \rho_n > 0$. Such a sequence is called uniformly separated. We wish to consider the interpolation problem when $\inf_n \rho_n = 0$. Gleason has observed (unpublished) that Earl's proof of Carleson's theorem yields a solution of the interpolation problem whenever $|w_n| \leq \rho_n^2$ for all n (cf. [3]). Moreover Garnett [3] shows that interpolation is possible if we have $|a_n| \leq \rho_n(1 + \log 1/\rho_n)^{-2}$ but interpolation is sometimes impossible if $|a_n| = \rho_n(1 + \log 1/\rho_n)^{-1}$.

In this paper we show the following two theorems. If $\{z_n\}$ is a finite union of interpolating sequences, then theorem 1 says ρ_n is the slowest possible rate of decay in $|w_n|$ for interpolation to occur and theorem 2 shows that if $|w_n|$ decays at a faster rate,

then the interpolant of minimal norm is unique and an inner function.

Theorem 1. $\{z_n\}$ is the union of a finite number of uniformly separated sequences if and only if for $|w_n| \leq \rho_n$ for all n , there exists a function in H^∞ such that $f(z_n) = w_n$ for all n .

The referee kindly pointed out us the following : If $\{z_n\}$ is a finite union of interpolating sequences, then there is a constant M so that if $|w_n| \leq \rho_n$ for all n , then there exists an f in H^∞ such that $f(z_n) = w_n$ and $\|f\|_\infty \leq M$. This is a little surprising, since there are interpolating sequences $\{z_n\}$ and sequences $\{w_n\}$ with $|w_n| \leq \rho = \inf \rho_n$ with $M \geq C/(\log 1/\rho)$.

The similar theorem for H^1 is not true. For when

$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$, we can show that if $\sum_{n=1}^{\infty} \rho_n^{-1} |w_n| < \infty$ then there exists a function f in H^1 such that $(1 - |z_n|) f(z_n) = w_n$ for all n .

Theorem 2. Let $\{z_n\}$ be the union of a finite number of uniformly separated sequences and $\rho_n^{-1} w_n \rightarrow 0$. Then there exists a unique f in H^∞ of minimal norm such that $f(z_n) = w_n$ for all n . This function is a complex constant times an inner function and has analytic continuation across $\partial D \setminus \overline{\{z_n\}}$.

When $\{z_n\}$ is uniformly separated, Øyma [6] proved theorem 2.

§2. Proof of Theorem 1.

In order to prove the theorem we need two well known lemmas.

Let $B_j(z) = \prod_{n=1}^j \frac{z - z_n}{1 - \bar{z}_n z}$, $B_{jn}(z) = B_j(z) \frac{1 - \bar{z}_n z}{z - z_n}$ and $b_{jn} = B_{jn}(z_n)$ ($1 \leq n \leq j$). Define

$$m_j(w) = \inf \{ \|f_j + B_j g\|_\infty ; g \in H^\infty \}$$

where $f_j(z) = \sum_{n=1}^j \frac{-1}{b_{jn}} w_n B_{jn}(z)$.

Lemma 1. Let $w = \{w_n\}$, then

$$m_j(w) = \sup \left\{ \left| \sum_{n=1}^j \frac{w_n}{b_{jn}} f(z_n) (1 - |z_n|^2) \right| ; f \in H^1 \text{ and } \|f\|_1 \leq 1 \right\}.$$

The proof is in [4, p197 ~ p198].

Lemma 2. $\{z_n\}$ is the union of a finite number of uniformly separated sequences if and only if the measure $\sum (1 - |z_n|) \delta_{z_n}$ is a Carleson measure, where δ_{z_n} denotes point mass at z_n .

The proof is in [5].

The proof of Theorem 1. For the part of 'only if', put $\ell = (w = \{w_n\}; |w_n| \leq \delta_n, n = 1, 2, \dots)$. By Lemma 1,

$$\begin{aligned}
& \sup_{w \in \ell} m_j(w) \\
&= \sup_{w \in \ell} \sup_f \left| \sum_{n=1}^j \frac{w_n}{b_{jn}} f(z_n)(1 - |z_n|^2) \right| \\
&= \sup_{w \in \ell} \sup_f \left| \sum_{n=1}^j \frac{w_n}{\delta_n} \frac{\delta_n}{b_{jn}} f(z_n)(1 - |z_n|^2) \right| \\
&\leq \sup_f \sum_{n=1}^j \left| \frac{\delta_n}{b_{jn}} \right| |f(z_n)| (1 - |z_n|^2) \\
&\leq \sup_f \sum_{n=1}^j |f(z_n)| (1 - |z_n|^2).
\end{aligned}$$

By Lemma 2, $\sup_j \sup_{w \in \ell} m_j(w) < \infty$ and this finishes the proof of 'only if' (see [5, p197]).

For the part of 'if', by [5, p197], $\sup_j \sup_{w \in \ell} m_j(w) < \infty$.

By Lemma 1,

$$\begin{aligned}
& \sup_j \sup_f \sum_{n=1}^j \left| \frac{\delta_n}{b_{jn}} \right| |f(z_n)| (1 - |z_n|^2) \\
&= \sup_j \sup_{w \in \ell} \sup_f \left| \sum_{n=1}^j \frac{w_n}{\delta_n} \frac{\delta_n}{b_{jn}} f(z_n)(1 - |z_n|^2) \right| < \infty.
\end{aligned}$$

Put $\mu_j = \sum_{n=1}^j \left| \frac{\delta_n}{b_{jn}} \right| (1 - |z_n|) \delta_{z=z_n}$, then for any $f \in H^1$

and all j there exists a finite positive constant γ

$$\int_D |f| d\mu_j \leq \gamma \int_0^{2\pi} |f(e^{i\theta})| d\theta / 2\pi$$

and $\|\mu_j\| \leq \gamma$. Let μ be the weak-* cluster point of $\{\mu_j\}$, then μ is a measure on the closed unit disc \bar{D} and $\|\mu\| \leq \gamma$. Since for any continuous function u on \bar{D} that is analytic in D

$$\begin{aligned} & \sum_{n=1}^j \left| \frac{\delta_n}{b_{jn}} \right| (1 - |z_n|) |u|^2(z_n) \\ &= \int_{\mathbb{D}} |u|^2 d\mu_j \leq \int_0^{2\pi} |u(e^{i\theta})|^2 d\theta / 2\pi, \end{aligned}$$

$$\int_{\mathbb{D}} |u|^2 d\mu = \sum_{n=1}^{\infty} (1 - |z_n|) |u|^2(z_n) \leq \gamma \int_0^{2\pi} |u(e^{i\theta})|^2 d\theta / 2\pi \quad \text{and}$$

$\mu|_{\mathbb{D}} = \sum_{n=1}^{\infty} (1 - |z_n|) \delta_{z=z_n}$. This implies $\sum_{n=1}^{\infty} (1 - |z_n|) \delta_{z=z_n}$ is a

Carleson measure and this finishes the proof of 'if' by Lemma 2.

§3. Proof of Theorem 2

Let Q denote the orthogonal projection from L^2 onto $e^{-i\theta}H^2$. For ϕ in L^∞ let H_ϕ denote the Hankel operator on H^2 defined by $H_\phi x = Q(\phi x)$. Let ℓ^∞ be the space of all bounded sequences of complex numbers and ℓ_0^∞ the subspace of ℓ^∞ of sequences tending to zero. Let $\{z_n\}$ be a sequence of distinct points in D and b a Blaschke product with zeros $\{z_n\}$. If f is in H^∞ and $H_{\bar{b}f}$ is compact then $\{f(z_n)\}$ is in ℓ_0^∞ [2]. Clark [2] showed that when $\{z_n\}$ is uniformly separated, if $\{f(z_n)\}$ is in ℓ_0^∞ then $H_{\bar{b}f}$ is compact. The following lemma is a generalization of the Clark's theorem and we need it to prove theorem 2.

Lemma 3. Suppose $\{z_n\}$ is the union of a finite number of uniformly separated sequences. If $\{\delta_n^{-1}f(z_n)\}$ is in ℓ_0^∞ then $H_{\bar{b}f}$ is compact.

Proof. It is Hartman's theorem (cf.[7, p6]) that $H_{\bar{b}f}$ is compact if and only if $\bar{b}f \in H^\infty + C$ where C denotes the space of continuous complex valued functions on ∂D . We shall show that $\wedge^{\text{if}} \{\delta_n^{-1}f(z_n)\}$ is in ℓ_0^∞ then $\bar{b}f \in H^\infty + C$. There is a factorization $b = b_1 b_2 \dots b_\ell$ such that b_j ($1 \leq j \leq \ell$) is a Blaschke product of $\{z_n^{(j)}\}$ where $\{z_n^{(j)}\}$ is uniformly separated and $\bigcup_j \{z_n^{(j)}\} = \{z_n\}$. Let $b'_j = \prod_{k \neq j} b_k$ then

$\{b'_j(z_n^{(j)})^{-1}f(z_n^{(j)})\} \in \ell_0^\infty$. Since $\{z_n^{(j)}\}$ is uniformly separated, by Carleson's theorem there exist a function f in H^∞ such that $f_j(z_n^{(j)}) = b'_j(z_n^{(j)})^{-1}f(z_n^{(j)})$ for all n . Set

$$g = \sum_{j=1}^{\ell} b_j f_j ,$$

then $g(z_n) = f(z_n)$ for all n and so $H_{\bar{b}g} = H_{\bar{b}f}$. By Clark's theorem, $\bar{b}_j f_j \in H^\infty + C$ for each j , and hence $\bar{b}g \in H^\infty + C$. Since $\bar{b}(g - f) \in H^\infty$, we conclude $\bar{b}f \in H^\infty + C$.

The proof of Theorem 2. Let b be a Blaschke product with zeros $\{z_n\}$. Then by Nehari's theorem (cf. [7, p6]) $\|H_{\bar{b}f}\| = \|\bar{b}f + H^\infty\|$. By Theorem 2, $H_{\bar{b}f}$ is compact and so by Hartman's theorem (cf. [7, p6]), $\bar{b}f \in H^\infty + C$. Suppose $f(z_n) = w_n$ for all n , then we may assume that f is of minimal norm, that is, $\|f + bH^\infty\| = \|f\|_\infty$. The $\bar{b}f$ defines a continuous linear functional on $e^{i\theta}H^1$. Since $\bar{b}f \in H^\infty + C$, there exist a function $g \in e^{i\theta}H^1$ such that $\int \bar{b}fg \, d\theta/2\pi = \|\bar{b}f + H^\infty\|$ and $\|g\|_1 = 1$. This implies that f is a desired inner function and unique.

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