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Notes On Interpolation By Bounded Analytic Functions

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Let $\{z_n\}$ be a sequence in the open unit disc and write $\rho_n = \prod_{m;m\neq n} |(z_n - z_m)(1 - \bar{z}_m z_n)^{-1}|$. In the case of $|w_n| \leq \rho_n$ for all n, the interpolation problems are considered.

§1. Theorems

Let H^∞ be the Hardy space of bounded analytic functions in the unit disc D with boundary values in $L^\infty = L^\infty(d\theta/2\pi)$. Let $\{z_n\}$ be a sequence of distinct points in D and $\{w_n\}$ be a bounded sequence of complex numbers. Our notes concern the interpolation problem

$$f(z_n) = w_n, \quad n = 1,2,\ldots$$

for f in H^∞. Put

$$\rho_n = \prod_{m;m\neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right|.$$

Carleson [1] proved that every interpolation problem has a solution if and only if $\inf_n \rho_n > 0$. Such a sequence is called uniformly separated. We wish to consider the interpolation problem when $\inf_n \rho_n = 0$. Gleason has observed (unpublished) that Earl's proof of Carleson's theorem yiels a solution of the interpolation problem whenever $|w_n| \leq \rho_n^2$ for all n (cf. [3]). Moreover Garnett [3] shows that interpolation is possible if we have $|a_n| \leq \rho_n(1 + \log 1/\rho_n)^{-2}$ but interpolation is sometimes impossible if $|a_n| = \rho_n(1 + \log 1/\rho_n)^{-1}$.

In this paper we show the following two theorems. If $\{z_n\}$ is a finite union of interpolating sequences, then theorem 1 says ρ_n is the slowest possible rate of decay in $|w_n|$ for interpolation to occur and theorem 2 shows that if $|w_n|$ decays at a faster rate, then the interpolant of minimal norm is unique and an inner function.

Theorem 1. $\{z_n\}$ is the union of a finite number of uniformly separated sequences if and only if for $|w_n| \leq \rho_n$ for all n, there exists a function in H^∞ such that $f(z_n) = w_n$ for all n.

The referee kindly pointed out us the following : If $\{z_n\}$ is a finite union of interpolating sequences, then there is a constant M so that if $|w_n| \leq \rho_n$ for all n, then there exists an f in H^∞ such that $f(z_n) = w_n$ and $\| f \|_\infty \leq M$. This is a little surprising, since there are interpolating sequences $\{z_n\}$ and sequences $\{w_n\}$ with $|w_n| \leq \rho = \inf \rho_n$ with $M \geq C/(\log 1/\rho)$.

The similar theorem for H^1 is not true. For when $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$, we can show that if $\sum_{n=1}^{\infty} \rho_n^{-1} |w_n| < \infty$ then there exists a function f in H^1 such that $(1 - |z_n|)\, f(z_n) = w_n$ for all n.

Theorem 2. Let $\{z_n\}$ be the union of a finite number of uniformly separated sequences and $\rho_n^{-1} w_n \longrightarrow 0$. Then there exists a unique f in H^∞ of minimal norm such that $f(z_n) = w_n$ for all n. This function is a complex constant times an inner function and has analytic continuation across $\partial D \setminus \overline{\{z_n\}}$.

When $\{z_n\}$ is uniformly seporated, Øyma [6] proved theorem 2.

§2. Proof of Theorem 1.

In order to prove the theorem we need two well known lemmas. Let $B_j(z) = \prod_{n=1}^{j} \frac{z - z_n}{1 - \bar{z}_n z}$, $B_{jn}(z) = B_j(z) \frac{1 - \bar{z}_n z}{z - z_n}$ and $b_{jn} = B_{jn}(z_n)$ $(1 \leq n \leq j)$. Define

$$m_j(w) = \inf \{ \| f_j + B_j g \|_\infty ; g \in H^\infty \}$$

where $f_j(z) = \sum_{n=1}^{j} b_{jn}^{-1} w_n B_{jn}(z)$.

Lemma 1. Let $w = \{w_n\}$, then

$$m_j(w) = \sup \{ | \sum_{n=1}^{j} \frac{w_n}{b_{jn}} f(z_n)(1 - |z_n|^2) | ; f \in H^1 \text{ and } \|f\|_1 \leq 1 \}.$$

The proof is in [4, p197 ~ p198].

Lemma 2. $\{z_n\}$ is the union of a finite number of uniformly separated sequences if and only if the measure $\Sigma(1 - |z_n|)\delta_{z_n}$ is a Carleson measure, where δ_{z_n} denotes point mass at z_n.

The proof is in [5].

The proof of Theorem 1. For the part of 'only if', put $\ell = (w = \{w_n\};\ |w_n| \leq \delta_n,\ n = 1,2,\ldots)$. By Lemma 1,

$$\sup_{w \in \ell} m_j(w)$$

$$= \sup_{w \in \ell} \sup_{f} \left| \sum_{n=1}^{j} \frac{w_n}{b_{jn}} f(z_n)(1 - |z_n|^2) \right|$$

$$= \sup_{w \in \ell} \sup_{f} \left| \sum_{n=1}^{j} \frac{w_n}{\delta_n} \frac{\delta_n}{b_{jn}} f(z_n)(1 - |z_n|^2) \right|$$

$$\leq \sup_{f} \sum_{n=1}^{j} \left|\frac{\delta_n}{b_{jn}}\right| |f(z_n)|(1 - |z_n|^2)$$

$$\leq \sup_{f} \sum_{n=1}^{j} |f(z_n)|(1 - |z_n|^2).$$

By Lemma 2, $\sup_{j} \sup_{w \in \ell} m_j(w) < \infty$ and this finishes the proof of 'only if' (see [5, p197]).

For the part of 'if', by [5, p197], $\sup_{j} \sup_{w \in \ell} m_j(w) < \infty$. By Lemma 1,

$$\sup_{j} \sup_{f} \sum_{n=1}^{j} \left|\frac{\delta_n}{b_{jn}}\right| |f(z_n)| (1 - |z_n|^2)$$

$$= \sup_{j} \sup_{w \in \ell} \sup_{f} \left| \sum_{n=1}^{j} \frac{w_n}{\delta_n} \frac{\delta_n}{b_{jn}} f(z_n)(1 - |z_n|^2) \right| < \infty.$$

Put $\mu_j = \sum_{n=1}^{j} \left|\frac{\delta_n}{b_{jn}}\right| (1 - |z_n|)\delta_{z=z_n}$, then for any $f \in H^1$ and all j there exists a finite positive constant γ

$$\int_D | f | d\mu_j \leq \gamma \int_0^{2\pi} |f(e^{i\theta})| d\theta/2\pi$$

and $\| \mu_j \| \leq \gamma$. Let μ be the weak-* cluster point of $\{\mu_j\}$, then μ is a measure on the closed unit disc $\bar{D}$ and $\| \mu \| \leq \gamma$. Since for any continuous function u on $\bar{D}$ that is analytic in D

$$\sum_{n=1}^{j} \left|\frac{\delta_n}{b_{jn}}\right| (1 - |z_n|)\ |u|^2(z_n)$$

$$= \int_{\bar{D}} |u|^2 d\mu_j \leq \int_0^{2\pi} |u(e^{i\theta})|^2 d\theta/2\pi,$$

$\int_{\bar{D}} |u|^2 d\mu = \sum_{n=1}^{\infty} (1 - |z_n|)|u|^2(z_n) \leq \gamma \int_0^{2\pi} |u(e^{i\theta})|^2 d\theta/2\pi$ and

$\mu|D = \sum_{n=1}^{\infty} (1 - |z_n|)\delta_{z=z_n}$. This implies $\sum_{n=1}^{\infty} (1 - |z_n|)\delta_{z=z_n}$ is a Carleson measure and this finishes the proof of 'if' by Lemma 2.

§3. Proof of Theorem 2

Let Q denote the orthogonal projection from L^2 onto $e^{-i\theta}\bar{H}^2$. For ϕ in L^∞ let H_ϕ denote the Hankel operator on H^2 defined by $H_\phi x = Q(\phi x)$. Let ℓ^∞ be the space of all bounded sequences of complex numbers and ℓ_0^∞ the subspace of ℓ^∞ of sequences tending to zero. Let $\{z_n\}$ be a sequence of distinct points in D and b a Blaschke product with zeros $\{z_n\}$. If f is in H^∞ and $H_{\bar{b}f}$ is compact then $\{f(z_n)\}$ is in ℓ_0^∞ [2]. Clark [2] showed that when $\{z_n\}$ is uniformly separated, if $\{f(z_n)\}$ is in ℓ_0^∞ then $H_{\bar{b}f}$ is compact. The following lemma is a generalization of the Clark's theorem and we need it to prove theorem 2.

Lemma 3. Suppose $\{z_n\}$ is the union of a finite number of uniformly separated sequences. If $\{\delta_n^{-1}f(z_n)\}$ is in ℓ_0^∞ then $H_{\bar{b}f}$ is compact.

Proof. It is Hartman's theorem (cf.[7, p6]) that $H_{\bar{b}f}$ is compact if and only if $\bar{b}f \in H^\infty + C$ where C denotes the space of continuous complex valued functions on ∂D. We shall show that if $\{\delta_n^{-1}f(z_n)\}$ is in ℓ_0^∞ then $\bar{b}f \in H^\infty + C$. There is a factorization $b = b_1 b_2 \dots b_\ell$ such that b_j $(1 \leq j \leq \ell)$ is a Blaschke product of $\{z_n^{(j)}\}$ where $\{z_n^{(j)}\}$ is uniformly separated and $\bigcup_j \{z_n^{(j)}\} = \{z_n\}$. Let $b_j' = \prod_{k \neq j} b_k$ then

$\{b_j'(z_n^{(j)})^{-1}f(z_n^{(j)})\} \in \ell_0^\infty$. Since $\{z_n^{(j)}\}$ is uniformly separated, by Carleson's theorem there exist a function f in H^∞ such that $f_j(z_n^{(j)}) = b_j'(z_n^{(j)})^{-1}f(z_n^{(j)})$ for all n. Set

$$g = \sum_{j=1}^{\ell} b_j f_j \ ,$$

then $g(z_n) = f(z_n)$ for all n and so $H_{\bar{b}g} = H_{\bar{b}f}$. By Clark's theorem, $\bar{b}b_j f \in H^\infty + C$ for each j, and hence $\bar{b}g \in H^\infty + C$. Since $\bar{b}(g - f) \in H^\infty$, we conclude $\bar{b}f \in H^\infty + C$.

The proof of Theorem 2. Let b be a Blaschke product with zeros $\{z_n\}$. Then by Nehari's theorem (cf. [7, p6]) $\| H_{\bar{b}f} \| = \| \bar{b}f + H^\infty \|$. By Theorem 2, $H_{\bar{b}f}$ is compact and so by Hartman's theorem (cf. [7, p6]), $\bar{b}f \in H^\infty + C$. Suppose $f(z_n) = w_n$ for all n, then we may assume that f is of minimal norm, that is, $\| f + bH^\infty \| = \| f \|_\infty$. The $\bar{b}f$ defines a continuous linear functional on $e^{i\theta}H^1$. Since $\bar{b}f \in H^\infty + C$, there exist a function $g \in e^{i\theta}H^1$ such that $\int \bar{b}fg \, d\theta/2\pi = \| \bar{b}f + H^\infty \|$ and $\| g \|_1 = 1$. This implies that f is a desired inner function and unique.

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