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A Spectral Dilation Of Some Non-Dirichlet Algebra

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Abstract. A Dirichlet algebra has always a spectral dilation for any operator representation. We don't know the examples of non-Dirichlet algebras which have spectral dilations for any operator representations. In this paper we give an example of such an algebra.

Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Let H be a complex Hilbert space and $L(H)$ the algebra of all bounded linear operators on H. I is the identity operator in H. An algebra homomorphism $f \longrightarrow T_f$ of A in $L(H)$, which satisfies

$$T_1 = I$$

and

$$\|T_f\| \leq \|f\|$$

is called a representation of A on H. A representation $\phi \longrightarrow U_\phi$ of $C(X)$ on a Hilbert space K is called a spectral dilation of the representation $f \longrightarrow T_f$ of A on H if H is a Hilbert subspace of K and

$$T_f x = PU_f x \quad \text{for} \quad f \in A \quad \text{and} \quad x \in H$$

where P is the orthogonal projection of K on H.

If A is a Dirichlet algebra on X and $f \longrightarrow T_f$ a representation of A on H, then there exists a spectral dilation. This was proved by Foias and Suciu (cf. [3, Theorem 8.7.]). However it is unknown whether any representation of a non-Dirichlet algebra has a spectral dilation. In this paper we give an example of a uniform algebra which has a spectral dilation for any operator representation and is a subalgebra of a disc algebra, of codimension one.

If $f \longrightarrow T_f$ is a representation of A on a Hilbert space H with the inner product (x,y) $(x,y \in H)$, then there are measures $\mu_{x,y}$ $(x,y \in H)$ such that $\|\mu_{x,y}\| \leq \|x\| \; \|y\|$ for $x,y \in H$ and

$$(T_f x,y) = \int f d\mu_{x,y} \quad \text{for} \quad f \in A \quad \text{and} \quad x,y \in H.$$

(see [3, p173]). Let τ be in the maximal ideal space of A and G the Gleason part of τ. We say that the representation $f \longrightarrow T_f$ of A is G-continuous (G-singular) if there exists a system of finite measures $\{\mu_{x,y}\}$ such that $\mu_{x,y}$ is G-absolutely continuous (G-singular) and $(T_f x,y) = \int f d\mu_{x,y}$ for all $f \in A$ and all $x,y \in H$ (cf. [2, p182]). We need the following three lemmas to give a theorem. The first one is a theorem of Mlak [2, Theorem 2.3] and the second one is one result of Foias and Suciu (cf. [3, p173]).

Lemma 1. Let $f \longrightarrow T_f$ be a representation of A on H. Then $f \longrightarrow T_f$ is a unique orthogonal sum $T_f = T_f^a \oplus T_f^s$ where the representation $f \longrightarrow T_f^a$ $(f \longrightarrow T_f^s)$ of A is G-absolutely contenuous (G-singular).

Lemma 2. Let $f \longrightarrow T_f$ be a representation of A on H. Then there are measures $\mu_{x,y}$ $(x,y \in H)$ such that $\|\mu_{x,y}\| \leq \|x\| \; \|y\|$ for $x,y \in H$ and

$$((T_f + T_g^*)x,y) = \int f + \bar{g}d\mu_{x,y}$$

for $f, g \in A$ and $x,y \in H$.

A family $\lambda_{x,y}$ $(x,y \in H)$ of measures on X is called semispectral if it satisfies the following properties:

$$(1)\ \lambda_{\alpha x + \beta y, z} = \alpha\lambda_{x,z} + \beta\lambda_{y,z}\ ,$$

$$(2)\ \int \phi d\lambda_{x,y} = \overline{\int \bar{\phi} d\lambda_{y,x}} \quad (\phi \in C(X)),$$

$$(3)\ \lambda_{x,x} \geq 0\ ,$$

$$(4)\ |\lambda_{x,y}| \leq \gamma\ \|x\|\ \|y\|$$

where α and β are complex numbers, and γ is a positive number.

Now we can give an example of a uniform algebra which has a spectral dilation for any operator representation and is not a Dirichlet algebrta. Let $\mathcal{T}$ be the unit circle and A the algebra of those continuous functions on $\mathcal{T}$ which have analytic extensions $\tilde{f}$ to the interior such that $\tilde{f}(0) = f(1)$. Then A is a uniform algebra on $\mathcal{T}$ and $\mathcal{T}$ is the Shilov boundary of A. The complex homomorphism τ on A is defined by $\tau(f) = \tilde{f}(o) = f(1)$. Both $d\theta/2\pi$ and the unit point mass δ_1 at 1 represent the same linear functional τ on A. Therefore A is not a logmodular algebra and hence not a Dirichlet algebra on $\mathcal{T}$ (cf, [1, p38]).

Lemma 3. If μ is an annihilating measure on $\mathcal{T}$ for $A + \overline{A}$ then $d\mu = c(d\theta/2\pi - d\delta_1)$ for some constant c.

Proof. We may assume that μ is a real memasure on $\mathcal{T}$. If μ annihilates A then

$$\int z d\mu = \int z^2 d\mu = \int z^3 d\mu = \cdots$$

because the functions $z - z^2$, $z^2 - z^3$, $z^3 - z^4$, $\cdots$ are all in A. Hence for any positive integer n

$$\int z^n (d\mu - c_1 d\delta_1) = 0 ,$$

where $c_1 = \int z d\mu$. By a theorem of F. and M.Riesz (cf. [1, p45]), $d\mu - c_1 d\delta_1 = h d\theta/2\pi$ for some h in the usual Hardy space H^1. The absolutely continuous part of μ with respect to $d\theta/2\pi$ is a real measure and coincides with $h d\theta/2\pi$. Since H^1 has not nonconstant real functions , h is constant. Thus $d\mu = c d\theta/2\pi + c_1 d\delta_1$ and $c = - c_1$ because $\int 1 \, d\mu = 0$.

Theorem. Let $f \longrightarrow T_f$ be a representation of A on a Hilbert space H. There exists a spectral dilation $\phi \longrightarrow U_\phi$ of $f \longrightarrow T_f$.

Proof. By Lemma 1 we may assume that the representation $f \longrightarrow T_f$ of A is G-continuous or G-singular, where G is the Gleason part of τ in the maximal ideal space of A. Suppose the representation is G-continuous. By Lemma 2 there are measures $\mu_{x,y}$ $(x,y \in H)$ such that $\|\mu_{x,y}\| \leq \|x\| \|y\|$ and $((T_f + T_g^*)x,y) = \int f + \bar{g} d\mu_{x,y}$ for $f,g \in A$ and $x,y \in H$. Since the representation of A is G-continuous, by the definition $\mu_{x,y}$ is absolutely continuous with respect to $d\theta/2\pi + d\delta_1$. Hence

$$d\mu_{x,y} = h_{x,y} d\theta/2\pi + c_{x,y} d\delta_1$$

where $h_{x,y}$ is in the usual Lebesgue space $L^1(d\theta/2\pi)$ and $c_{x,y}$ is constant.

Put

$$d\lambda_{x,y} = (h_{x,y} + c_{x,y})\, d\theta/2\pi .$$

We shall prove that the family $\lambda_{x,y}$ $(x,y \in H)$ of measures on T is semispectral, that is, it satisfies (1) ~ (4). (4) is clear. $d\mu_{\alpha x+\beta y,z} - (\alpha d\mu_{x,z} + \beta d\mu_{y,z})$ annihilates $A + \bar{A}$. Therefore by Lemma 3 for some constant $a_{x,y,z}$

$$d\mu_{\alpha x+\beta y,z} - (\alpha d\mu_{x,z} + \beta d\mu_{y,z}) = a_{x,y,z}(d\theta/2\pi - d\delta_1),$$

consequently

$$h_{\alpha x+\beta y,z} - (\alpha h_{x,z} + \beta h_{y,z}) = a_{x,y,z}$$

and

$$c_{\alpha x+\beta y,z} - (\alpha c_{x,z} + \beta c_{y,z}) = -a_{x,y,z} .$$

This implies (1). $d\mu_{x,y} - d\bar{\mu}_{y,x}$ annihilates $A + \bar{A}$.

Therefore by Lemma 3 for some constant $b_{x,y}$

$$d\mu_{x,y} - d\bar{\mu}_{y,x} = b_{x,y} ,$$

consequently

$$h_{x,y} - \bar{h}_{y,x} = b_{x,y}$$

and

$$c_{x,y} - \bar{c}_{y,x} = -b_{x,y} .$$

This implies (2). By Proposition 7.8 in [3], if $f \in A$ and $\mathrm{Re} f \geq 0$ then $\mathrm{Re}\, T_f \geq 0$. Hence if $u \in A + \bar{A}$ and $u \geq 0$ then $\int u d\mu_{x,x} \geq 0$. Thus for $u \in A + \bar{A}$

with $u \geq 0$

$$\int u d\lambda_{x,x} = \int u(h_{x,x} + c_{x,x}) d\theta/2\pi = \int uh_{x,x} d\theta/2\pi + c_{x,x} \int u d\theta/2\pi$$

$$= \int uh_{x,x} d\theta/2\pi + c_{x,x} \int u d\delta_1 = \int u d\mu_{x,x} \geq 0$$

By the Riemann-Lebesgue lemma we know that $z^n \longrightarrow 0$ in the weak* topology of $L^\infty(d\theta/2\pi)$. Hence the functions $z, z^2, z^3, \cdots$ are all in the weak*-closure of A because $z^k = (z^k - z^{k-1}) + \cdots$

$+ (z^n - z^{n-1}) - z^n$ for $n > k$. Therefore for $u \in C(\mathcal{T})$ with $u \geq 0$ $\int u d\lambda_{x,x} \geq 0$ and this implies (3).

Since the family $\lambda_{x,y}$ $(x,y \in H)$ of measures on $\mathcal{T}$ is semispectral, there is a positive definite map $\phi \longrightarrow T'_\phi$ of $C(\mathcal{T})$ in $L(H)$ (cf. [3, Theorem 7.1]). By a dilation theorem of Naimark (cf. [3, Theorem 7.5]), we obtain a representation $\phi \longrightarrow U_\phi$ of $C(\mathcal{T})$ on a Hilbert space K which is a spectral dilation of $\phi \longrightarrow T'_\phi$. If $f \in A_0$ then $\int f\, d\theta/2\pi = \int f\, d\delta_1 = 0$ and hence

$$\begin{aligned}(T'_f x,y) &= \int f d\lambda_{x,y} = \int f\, h_{x,y} d\theta/2\pi \\ &= \int f\, d\mu_{x,y} = (T_f x,y)\ ,\ (x,y \in H).\end{aligned}$$

Thus $T'_f = T_f$ if $f \in A$ and the representation $\phi \longrightarrow U_\phi$ is the spectral dilation of $f \longrightarrow T_f$.

If the representation is G-singular the family $\mu_{x,y}$ $(x,y \in H)$ in the case of the proof of G-continous is singular with respect to $d\theta/2\pi + d\delta_1$. Then Lemma 3 implies that it is semispectral immediatly.

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