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Parametrization of a Singular Lagrangian Variety

by Goo ISHIKAWA

Dedicated to Professor Hiroshi Toda on his 60th birthday

Abstract: We give stabilization and parametrization theorems for a class of singular varieties in the space of polynomials of one variable and generalize the results of Arnol'd and Givental'. The class contains the open swallowtails and the open Whitney umbrella. The parametrization is associated with the singularity of a stable mapping (in the sense of Thom and Mather) of kernel rank one.

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§0. Introduction

There exist intimate relations between certain singular Lagrangean varieties and certain stable mappings.

Singular Lagrangian varieties arise, for instance, in the following reduction process (*): the set of characteristics through an isotropic manifold I^{n-1} in a hypersurface of a symplectic manifold M^{2n} is a Lagrangian variety in the reduced symplectic manifold N^{2n-2} at least locally.

Related to the variational problem, for instance, the problem of bypathing obstacles, the theory of symplectic triads is developed by Arnol'd and Givental' [2], and an important object "open swallowtail" is obtained.

Open swallowtails are generic singularities of a singular Lagrangian variety associated to a Lagrangian manifold simply tangent to a hypersurface along an isotropic submanifold I^{n-1} . (See [1;(5.3.23)] and [11,Lecture 3], for non-singular case.) As in [2], [4], an open swallowtail is constructed in a polynomial space of one variable and it is described as a "stable object" in some sense.

On the other hand, Givental' [5] shows that, in the problem of Lagrangian immersions of surfaces into $R^4 = T^*R^2$, the singularity "open Whitney umbrella" defined by $(x,t) \rightarrow (q_1, q_2, p_2, p_1) = (t^2/2, t^3/3, x, -xt)$ appears stably.

Open Whitney umbrella is parametrized by a minimal system of generators of the R -algebra,

$$\{h \in R[x,t] \mid (\partial h / \partial t)(0) = 0\}.$$

In this paper we remark that these objects can be constructed inclusively, by (*), in a polynomial space of one variable (Theorem 1 and Proposition 1). We treat an isotropic submanifold \mathbb{I}^{n-1} which is not necessarily an intersection of a Lagrangian submanifold and a hypersurface. As a consequence, we have a multidimensional analogue, which is denoted by $W_{2n}(4n+1)$ in this paper, of the open Whitney umbrella.

Explicite parametrizations of open swallowtails are given in [4, §3]. We show that more general objects are parametrized in a similar way (Theorem 2).

These parametrizations are related to the paper [7], where we associate ring sheaves with singularities of mappings and discuss the finite generatedness of associated ring sheaves.

Above all in this paper, we emphasize that, for a certain stable mapping (in the sense of Thom and Mather), a minimal system of generators of associated ring sheaf in the sense of [7] turns out a parametrization of certain singular Lagrangian variety (Theorems 3 and 2). This observation seems to give one of connections of symplectic geometry, singularity theory and differential analysis.

We also remark that there are important papers, for instance, [8], treating singular Lagrangian varieties on a different viewpoint from this paper.

In §1, after some preliminaries, we state Theorems 1, 2 and 3

and Proposition 1 .

We prove Theorem 3 in §2, and prove Theorems 1 and 2 in §3.

As a corollary of Theorem 2, we give a regular stratification of singular varieties treated in this paper, in §4.

In the last section, we recall symplectic geometry in polynomial spaces of one variable and prove Proposition 1.

The author would like to thank Professor S. Izumi for valuable comments.

§1. Statements of results.

Let K denote R or C . Set

$$H_n = \{F(t) = \sum_{0 \leq i \leq n} a_i t^{n-i}/(n-i)! \mid a_i \in K\},$$

the K -vector space of polynomials of one variable t with degree $\leq n$. Furthermore set

$$M_n = \{F \in H_n \mid a_0 = 1\} \quad \text{and} \quad V_n = \{F \in M_n \mid a_1 = 0\}.$$

Define a linear map $D^{-\ell}: H_n \rightarrow H_{n+\ell}$ ($-n \leq \ell$) by

$$D^{-\ell}(t^j/j!) = \begin{cases} t^{j+\ell}/(j+\ell)!, & (0 \leq j+\ell), \\ 0 & , \text{ (otherwise),} \end{cases}$$

and linearity. ($D^{-(-1)} = D$ is just the differential by t .)

Define a polynomial map $\pi: M_n \rightarrow V_{n-1}$ by

$$\pi(F(t)) = D(F(t-a_1(F))) = (DF)(t-a_1(F)).$$

For fixed positive integers n, k and m , consider the affine subspace $I = I_{n,k}^{(m)}$ of M_{m+1} , ($m \geq (2k-1)n$), defined by

$$I_{n,k}^{(m)} = D^{-(m+1-n)} M_n \oplus \left(\bigoplus_{1 \leq j \leq k-1} D^{-(m-(2j+1)n)} H_{n-1} \right),$$

in H_{m+1} .

In another way, $I_{n,k}^{(m)}$ is illustrated by the diagram:

(Please insert Diagram 1 here.)

where I is defined by setting the coordinates under the oblique lines to be zero.

Theorem 1. The sequence induced from differentiations by t ,

$$\cdots \rightarrow \pi(I_{n,k}^{(m+1)}) \xrightarrow{D} \pi(I_{n,k}^{(m)}) \xrightarrow{D} \pi(I_{n,k}^{(m-1)}) \rightarrow \cdots,$$

is stabilized at stage $m \geq 2kn + 1$. Precisely, there exist

polynomial section $s: V_m \rightarrow V_{m+1}$, ($m \geq 2kn + 1$), of $D = d/dt: V_{m+1} \rightarrow V_m$, mapping $\pi(I_{n,k}^{(m)})$ to $\pi(I_{n,k}^{(m+1)})$.

Remark 1. In the case $K = \mathbb{C}$, $\pi(I_{n,k}^{(m)})$ is an algebraic variety, and D is an isomorphism of algebraic varieties if $m \geq 2kn + 1$. In the case $K = \mathbb{R}$, $\pi(I_{n,k}^{(m)})$ is a semi-algebraic set.

Especially, we are interested in the case $k = 1$ and $k = 2$. Denote the variety $\pi(I_{n,1}^{(m)})$, (resp. $\pi(I_{n,2}^{(m)})$) $\subset V_m$ by $S_n(m)$, (resp. $W_{2n}(m)$).

Let A be an n -dimensional stratified subset of a $2n$ -dimensional complex (resp. real) symplectic manifold (M, ω) . We call A a Lagrangian variety if any holomorphic (resp. C^∞) mapping $h: N \rightarrow M$ with $h(N) \subset A$ is an isotropic map.

Proposition 1. The "stable objects" $S_n(2n+1) \subset V_{2n+1}$ and $W_{2n}(4n+1) \subset V_{4n+1}$ are Lagrangian varieties with respect to the natural symplectic structure of V_m for odd m . (See §5.)

The result in Theorem 1 for $S_n(m)$ is due to Arnol'd and Givental' ([4],[2]). The variety $S_n(2n+1)$ is called the n -dimensional open swallowtail. The variety $W_2(5)$ is isomorphic to the open Whitney umbrella by a symplectic diffeomorphism.

We seek a concrete parametrization of $\pi(I_{n,k}^{(m)})$.

For a polynomial $F(t)$, set

$$\begin{aligned}
F_{(i)}(t) &= (-1)^{i+1} D^{-1}((t^i/i!)F(t)), \\
&= (-1)^{i+1} \int_0^t (u^i/i!)F(u)du, \quad (i = 0, 1, 2, \dots).
\end{aligned}$$

For fixed $y \in K$ and $j \in N$, set

$$\kappa_j(F, y) = (F_{(0)}(y), \dots, F_{(j)}(y)) \in K^{j+1}.$$

We identify V_n with K^{n-1} by coordinates a_2, \dots, a_n and H_{n-1} with K^n by a_0, a_1, \dots, a_{n-1} . For example, $K^{kn} = V_n \times (\prod_{k-1} H_{n-1}) \times K$ and $K^{m-1} = V_n \times (\prod_{k-1} H_{n-1}) \times K^{m-kn} = V_m$.

Define $Q_{n,k,m}: K^{kn} \rightarrow K^{m-1}$ by

$$Q_{n,1,m}(F, y) = (F, \kappa_{m-n-1}(F, y)) \in V_n \times K^{m-n}.$$

$$\begin{aligned}
Q_{n,k,m}(F, G_1, \dots, G_{k-1}, y), \quad (k \geq 2), &= (F, \kappa_n(F, y), G_1, \kappa_{n-1}(G_1, y), \\
&\dots, G_{k-2}, \kappa_{n-1}(G_{k-2}, y), G_{k-1}, \kappa_{m-(2k-1)n-2}(G_{k-1}, y)),
\end{aligned}$$

where $F \in V_n$, and $G_i \in H_{n-1}$, $(i = 1, \dots, k-1)$.

Then we have

Theorem 2. Let $\psi: I_{n,k}^{(m)} \rightarrow V_m$, $(m \geq (2k-1)n)$, denote the restriction of $\pi: M_{m+1} \rightarrow V_m$.

(1) If $k = 1$, then there exists a diffeomorphism $\sigma: I_{n,1}^{(m)} \rightarrow K^n$ such that $\psi = Q_{n,1,m} \circ \sigma$.

(2) If $k \geq 2$, then there exist diffeomorphisms $\sigma: I_{n,k}^{(m)} \rightarrow K^{kn}$ and $\tau: K^{m-1} \rightarrow V_m$ such that $\psi = \tau \circ Q_{n,k,m} \circ \sigma$.

Remark 2. Let $k = 1$ or 2 , and $m = 2kn + 1$. Then ψ and Q are isotropic maps. Furthermore if $k = 2$, then we can choose τ a symplectic diffeomorphism at least locally.

The variety $\pi(I_{n,k}(2kn+1))$ is closely related to the stable mapping $f = f_{n,k}: K^{kn} \rightarrow K^{kn+k-1}$ defined by

$$f(F, G_1, \dots, G_{k-1}, t) = (F, G_1, \dots, G_{k-1}, F_{(0)}(t), G_{1(0)}(t), \dots, G_{k-1(0)}(t)),$$

where $F \in V_n$ and $G_i \in H_{n-1}$, ($i = 1, \dots, k-1$).

The mapping $f_{n,k}$ is a stable unfolding of the mapping $K \rightarrow K^k$ defined by $t \rightarrow (t^{n+1}/(n+1)!, 0, \dots, 0 \text{ (k-1-times)})$, ([61]).

The mapping Q (resp. f) can be extended naturally to \tilde{Q} :
 $M_n \times (\prod_{k-1} H_{n-1}) \times K \rightarrow M_n \times (\prod_{k-1} H_{n-1}) \times K^{m-kn}$, (resp. $\tilde{f}: M_n \times (\prod_{k-1} H_{n-1}) \times K \rightarrow M_n \times (\prod_{k-1} H_{n-1}) \times K^k$).

In general, let $f: M \rightarrow N$ be a mapping. If $K = \mathbb{C}$ (resp. $K = \mathbb{R}$), then f is assumed to be polynomial or holomorphic (resp. polynomial, C^ω or C^∞), according to the situation.

Associated to f , define a ring sheaf D_f by

$$D_f(U) = \{h \in O_M(U) \mid h_x - h(x) \in f^*_{m_{f(x)}} \cdot O_{M,x}, (x \in U)\},$$

where $U \subset M$ is an open subset, \mathcal{O}_M is the structure sheaf of M in each case, $\mathcal{O}_{M,x}$ is the stalk of \mathcal{O}_M at x , $f^*: \mathcal{O}_{N,f(x)} \rightarrow \mathcal{O}_{M,x}$ is the induced local homomorphism by f , $\mathfrak{m}_{f(x)} \subset \mathcal{O}_{N,f(x)}$ is the unique maximal ideal and h_x is the germ of h at x .

Then we have

Theorem 3. The ring sheaf $D_{\tilde{f}_{n,k}}$ (resp. $D_{f_{n,k}}$) is generated by all components of $\tilde{Q}_{n,k,2kn+1}$ (resp. $Q_{n,k,2kn+1}$), that is, $D_{\tilde{f}_{n,k}} = \tilde{Q}_{n,k,2kn+1} *_{\mathcal{O}_{K^{2kn+1}}}$ (resp. $D_{f_{n,k}} = Q_{n,k,2kn+1} *_{\mathcal{O}_{K^{2kn}}}$).

Furthermore these form a minimal system of generators of $D_{\tilde{f}_{n,k}}$ (resp. $D_{f_{n,k}}$) at the origin.

§2. Minimal system of generators

We use notations in §1.

For the proof of Theorems 1, 2 and 3, we use the following

Lemma 1. There exist $A_{m,i} \in \mathbb{Q}[x_0, x_1, \dots, x_n]$, ($0 \leq i \leq n$, $m = 0, 1, 2, \dots$), such that, for any $F = t^n/n! + \sum_{1 \leq i \leq n} a_i t^{n-i}/(n-i)! \in M_n$,

$$F_{(m)} = \sum_{1 \leq i \leq n} A_{m,i}(F_{(0)}, a_1, \dots, a_n) F_{(i)} + A_{m,0}(F_{(0)}, a_1, \dots, a_n),$$

($m = 0, 1, 2, \dots$), and that $\text{ord}_0 A_{m,i} \rightarrow \infty$ as $m \rightarrow \infty$.

Proof. There are rational numbers r_i , ($0 \leq i \leq n$), such that, for any $F \in M_n$,

$$F_{(n+m+1)} = r_0 F_{(0)} F_{(m)} + \sum_{1 \leq i \leq n} r_i a_i F_{(n+m+1-i)}, \quad (m = 0, 1, 2, \dots).$$

In fact, it is sufficient to set $r_0 = 1 / \left\{ \binom{n+m}{m} + \binom{n+m+1}{m} \right\}$ and $r_i = -r_0 \left\{ \binom{n+m-i}{m} + \binom{n+m+1-i}{m} \right\}$, ($1 \leq i \leq n$). Thus $F_{(m)}$, ($m \geq n+1$), is a linear homogeneous function of $F_{(m-1)}, \dots, F_{(m-n-1)}$ with coefficients in $\mathbb{Q}[F_{(0)}, a_1, \dots, a_n]$ with order ≥ 1 at the origin. Using this relation iteratively, we have required $A_{m,i}$.

We need the following lemma for the proof of Theorem 3 in the holomorphic and C^ω cases.

Lemma 2. Let $h: (K^n, 0) \rightarrow (K^p, 0)$ be an analytic and finite map-germ and $\hat{g} \in F_0 = K[[y_1, \dots, y_p]]$ be a formal power series. If $\hat{g} \circ h$ is analytic, then there exists an analytic $g \in O_0 = K\{(y_1, \dots, y_p)\}$ such that $g \circ h = \hat{g} \circ h$.

Proof. In general, let $\varphi: (A, m) \rightarrow (B, n)$ be a local and unitary homomorphism of local rings. Assume B is a finite A -module via φ . Then $\text{Im } \varphi$ is closed with respect to n -adic topology. (This is kindly communicated to the author by

S. Izumi.)

In fact, by Artin-Rees lemma, $\text{Im } \varphi$ is closed with respect to m -adic topology ([9, (3.13 (iii))]). Since $n^k B \subset m B \subset n B$ for some $k \in \mathbb{N}$, m -adic topology of B coincides with n -adic one.

The closure of $\text{Im } \varphi$ is equal to $\text{Im } \hat{\varphi} \cap B$, where $\hat{\varphi}: \hat{A} \rightarrow \hat{B}$ is the completion of φ . Thus we see $\text{Im } \varphi = \text{Im } \hat{\varphi} \cap B$.

Applying this to $\varphi = h^*: \mathcal{O}_{K^p,0} \rightarrow \mathcal{O}_{K^n,0}$, we have Lemma 2.

Proof of Theorem 3, (First Half). The first half of Theorem 3 for the original \mathcal{Q} and f follows easily from that for the extended $\tilde{\mathcal{Q}}$ and \tilde{f} .

We also treat complex and real formal cases. Then the structure sheaf of K^p is defined by $\mathcal{O}(U) = \prod_{x \in U} F_x$, where $F_x = E_x / (\bigcap_{1 \leq i} m^i) \cong K[[y_1, \dots, y_p]]$, E_x is the ring of germs of C^∞ -functions at x and $U \subset K^p$ is the open subset.

Let, Λ_u , ($u = 0, 1, 2, \dots$), denote the set of $(F, G_1, \dots, G_{k-1}) \in M_n \times (\prod_{k-1} H_{n-1}) \times K = K^{kn+1}$ satisfying

$$(D^j F)(t) = (D^j G_1)(t) = \dots = (D^j G_{k-1})(t) = 0, \quad (0 \leq j \leq u-1).$$

Define an ideal sheaf I_f by

$$I_f(U) = \{k \in \mathcal{O}(U) \mid D^u k \text{ vanishes on } \Lambda_{u+1}, (u = 0, 1, 2, \dots)\},$$

where $U \subset K^{kn}$ is an open subset.

Then, by the proof of Prop.3.1 of [7], we see

$$D_f(U) = \{k \in O(U) \mid Dk \in I_f(U)\}.$$

(This is valid in all cases which we are treating.)

We remark that I_f is generated by F, G_1, \dots, G_{k-1} , considered as elements of $O(K^{kn})$. In fact, since F, G_1, \dots, G_{k-1} can be taken as a part of coordinates of K^{kn} , the set of sections over U of the ideal sheaf generated by F, G_1, \dots, G_{k-1} is equal to $\{k \in O(U) \mid k \text{ vanishes on } \Lambda_1\}$, which contains $I_f(U)$.

In the polynomial case, for any $k \in D_{f,x}$,

$$Dk = A_0 F + A_1 (F + G_1) + \dots + A_{k-1} (F + G_{k-1}),$$

for some polynomials $A_i \in O_x$. Then,

$$k = k(x) + D^{-1}(A_0 F) + \sum_{1 \leq i \leq k-1} D^{-1}(A_i (F + G_i)),$$

using the notation in §1. By Lemma 1, each term in the right hand side is a polynomial of components of $Q_{n,k,2kn+1}$ and $G_{i(n)}$ ($1 \leq i \leq k-1$).

Furthermore, setting $R(t) = F(t) - t^n/n!$,

$$\begin{aligned} G_{i(n)} &= (-1)^{n+1} \int_0^t (u^n/n!) G_i(u) du \\ &= (-1)^{n+1} \int_0^t \{F(u) G_i(u) + R(u) G_i(u)\} du. \end{aligned}$$

Therefore $G_{i(n)}$ is the linear combination of $F_{(0)}, \dots, F_{(n)}, G_{i(0)}, \dots, G_{i(n-1)}$. Thus we have required result in the polynomial case.

For the C^∞ , complex formal and real formal cases, we follow the proof of Prop. 2.2 of [7]. The different point is only the usage of Lemma 1 instead of Lemma 2.6 of [7].

By the finiteness of Q and by Lemma 2, the result in the holomorphic (resp. C^ω) case follows from that in the complex (resp. real) formal case.

Proof of Theorem 3, (Second Half). Set $\tilde{Q} = \tilde{Q}_{n,k,2kn+1}$. Let $\tau = \tau(a_i, b_{ji}, c_\ell, d_{jm} \mid 1 \leq i \leq n, 1 \leq j \leq k-1, 0 \leq \ell \leq n, 0 \leq m \leq n-1)$ be a formal power series with coefficients in K . Assume $\tau \circ \tilde{Q} = 0$. Then it is sufficient to show $\text{ord}_0 \tau \geq 2$.

First, restricting the relation $\tau \circ \tilde{Q} = 0$ to the subspace $\{t = 0\}$, we see $\partial\tau/\partial a_i(0) = \partial\tau/\partial b_{ji}(0) = 0$, $(1 \leq i \leq n, 1 \leq j \leq k-1)$.

Second, for fixed j_0 , $(1 \leq j_0 \leq k-1)$, restrict $\tau \circ \tilde{Q} = 0$ to the subspace $\{a = 0, b_{ji} = 0 \text{ if } (j,i) \neq (j_0,n)\}$. Since $b_{j_0,n}, t^{n+1}, \dots, t^{2n+1}, b_{j_0,n}t, \dots, b_{j_0,n}t^n$ has no formal relation of order ≤ 1 , we see $\partial\tau/\partial c_\ell(0) = \partial\tau/\partial d_{j_0,m}(0) = 0$, $(0 \leq \ell \leq n, 0 \leq m \leq n-1)$.

Thus $\text{ord}_0 \tau \geq 2$.

Similarly we have that $\tau \circ Q = 0$ implies $\text{ord}_0 \tau \geq 2$.

This completes the proof of Theorem 3.

§3. Stabilization and Parametrization

For $F \in H_m$ and $y \in K$, set $\Phi_y(F(t)) = F(t+y)$. Then $\Phi_y: H_m \rightarrow H_m$ is a linear automorphism preserving M_m .

We need the following lemma, which is an easy consequence of integral by part:

Lemma 3. Let $F \in H_n$, and $\ell > 0$. Then

$$\Phi_{-y} \circ D^{-\ell-1} \circ \Phi_y F = D^{-\ell-1} F - \int_0^y \{(t-u)^\ell / \ell!\} F(u) du,$$

for any $y \in K$.

Define $\Phi: M_m \times K \rightarrow M_m$ by $\Phi(F, y) = \Phi_{-y} F$. Then, $\Phi^{-1}(V_m) = \{(F, y) \in M_m \times K \mid y = a_1(F)\}$. Set $i(F) = (F, a_1(F)) \in M_m \times K$.

Then $\pi: M_{m+1} \rightarrow V_m$ is factorized as $\Phi \circ i \circ D$.

Notice that $DI_{n,k}^{(m)} = I_{n,k}^{(m-1)}$. Let $\varphi: I_{n,k}^{(m-1)} \times K \rightarrow M_m$ be the restriction of Φ . (This is a similar construction as [1, (5.3.29)].)

Proof of Theorems 1 and 2, ($k = 1$). First we will parametrize φ . Define $\Sigma: I_{n,1}^{(m-1)} \times K \rightarrow M_n \times K$ by $\Sigma(G, y) = (\Phi_{-y} D^{m-n} G, y)$. Then Σ is a polynomial diffeomorphism. Define $Q_m': M_n \times K \rightarrow M_m$ by $Q_m'(F, y) = \Phi_{-y} \circ D^{-(m-n)} \circ \Phi_y(F)$. Then we have (a): $\varphi = Q_m' \circ \Sigma$.

Thus $\psi = \pi|_{I_{n,1}^{(m)}} = Q_m' \circ \Sigma \circ i \circ D|_{I_{n,1}^{(m)}}$. Set $\sigma = \Sigma \circ i \circ D|_{I_{n,1}^{(m)}}: I_{n,1}^{(m)} \rightarrow V_n \times K$. Then σ is a polynomial

diffeomorphism.

We will show (b): $Q_{n,1,m} = Q_m' | V_n \times K$.

By Lemma 3, for $(F,y) \in M_n \times K$,

$$\begin{aligned} Q_m'(F,y) &= D^{-(m-n)}F - \int_0^y \{(t-u)^{m-n-1}/(m-n-1)!\} F(u) du \\ &= D^{-(m-n)}F + \sum_{0 \leq i \leq m-n-1} F_{(i)}(y) t^{m-n-1-i}/(m-n-1-i)!. \end{aligned}$$

Identifying M_m with $M_n \times K^{m-n}$, we can write

$$Q_m'(F,y) = (F, F_{(0)}(y), F_{(1)}(y), \dots, F_{(m-n-1)}(y)).$$

Hence we have (b). This shows Theorem 2.(1).

Next we will show that (c): for $2n+1 \leq \ell \leq m$, there exists a polynomial map $s_{m,\ell}: M_\ell \rightarrow M_m$ such that $Q_m' = s_{m,\ell} \circ Q_\ell'$ and that $D^{m-\ell} \circ s_{m,\ell} = \text{id}_{M_\ell}$.

By Lemma 1 or Theorem 3, for $\ell < j \leq m$, there exists a polynomial $A_j \in K[a_1, \dots, a_n; x_0, \dots, x_n] = K[M_{2n+1}]$ such that

$F_{(j-n-1)}(y) = A_j(F; F_{(0)}(y), \dots, F_{(n)}(y))$. Then, to see (c), it is sufficient to define $s_{m,\ell}$ by

$$s_{m,\ell}(G) = (G, A_{\ell+1}(D^{\ell-(2n+1)}G), \dots, A_m(D^{\ell-(2n+1)}G)).$$

Set $s = s_{m+1,m} | V_m: V_m \rightarrow V_{m+1}$. Then s is a polynomial section of D , and $s(\pi(I_{n,1}(m))) = s \circ Q_m'(V_n \times K) = Q_{m+1}'(V_n \times K) = \pi(I_{n,1}(m+1))$, by (a). This shows Theorem 1, ($k = 1$).

Proof of Theorems 1 and 2 for general k . Define $Q_{k,m}' : M_n \times (\prod_{k-1} H_{n-1}) \times K \rightarrow M_m$ by

$$Q_{k,m}'(F, G_1, \dots, G_{k-1}, y) = \Phi_{-y} \circ D^{-(m-n)} \circ \Phi_y F + \sum_{1 \leq j \leq k-1} \Phi_{-y} \circ D^{-(m-(2j+1)n)} \circ \Phi_y G_j.$$

There is a natural isomorphism $\lambda : M_n \times (\prod_{k-1} H_{n-1}) \times K \rightarrow I_{n,k}^{(m-1)}$,

$$\text{that is, } \lambda(F, G_1, \dots, G_{k-1}) = D^{-(m-n)} F + \sum_{1 \leq j \leq k-1} D^{-(m-(2j+1)n-1)} G_j.$$

Define $\Sigma : I_{n,k}^{(m-1)} \times K \rightarrow M_n \times (\prod_{k-1} H_{n-1}) \times K$ by

$$\Sigma(G, y) = (\Phi_{-y} F, \Phi_{-y} G_1, \dots, \Phi_{-y} G_{k-1}, y),$$

where $\lambda^{-1}G = (F, G_1, \dots, G_{k-1})$.

Then we see (a): $\varphi = Q_{k,m}' \circ \Sigma$.

By Lemma 3, the coefficient a_ν of $t^{m-\nu}/(m-\nu)!$ of $Q_{k,m}'(F, G_1, \dots, G_{k-1}, y) \in M_m$ is equal to

$$F_{(\nu-n-1)}(y) + \sum_{1 \leq j \leq k-1} G_{j(\nu-(2j+1)n-1)}(y),$$

for $2kn+1 \leq \nu \leq m$. Then by Theorem 3 (polynomial case), there exists $A_\nu \in K[M_{2kn+1}]$ such that $a_\nu = A_\nu \circ Q_{k,2kn+1}'$.

We see (c): for $2kn+1 \leq \ell \leq m$, there exists a polynomial map

$s_{m,\ell}: M_\ell \rightarrow M_m$ such that $Q_{k,m}' = s_{m,\ell} \circ Q_{k,\ell}'$ and that $D^{m-\ell} \circ s_{m,\ell} = \text{id}_{M_\ell}$.

In fact, it is sufficient to set, for $G \in M_\ell$,

$$s_{m,\ell}(G) = (G, A_{\ell+1}(D^{\ell-(2kn+1)}G), \dots, A_m(D^{\ell-(2kn+1)}G)).$$

Setting $s = s_{m+1,m}|_{V_m}$, we have Theorem 1 in general.

Next we will show (b): there exists a polynomial diffeomorphism $T: M_m \rightarrow M_m$ preserving V_m such that $Q_{k,m}' = T \circ Q_{n,k,m}'$.

Set $\xi = Q_{k,m}'(F, G_1, \dots, G_{k-1}, y)$ and $\eta = Q_{n,k,m}'(F, G_1, \dots, G_{k-1}, y)$. Then, by Lemma 3 and Theorem 3 (polynomial case), each coefficient $a_\nu(\xi - \eta)$, $(2n+1 < \nu)$, of the difference $\xi - \eta$ is a polynomial of preceding coefficients of η ; $a_\nu(\xi - \eta) = A_\nu(a_1(\eta), a_2(\eta), \dots, a_{\nu-1}(\eta))$.

Then define T by

$$T(a_1, \dots, a_m) = (a_1, \dots, a_{2n+1}, a_{2n+2} + A_{2n+2}(a), \dots, a_m + A_m(a)).$$

We easily see T satisfies the required property of (b).

By (b), $\psi = T \circ Q_{n,k,m}' \circ \Sigma \circ i \circ D|_{I_{n,k}^{(m)}}$.

Set $\sigma = \Sigma \circ i \circ D|_{I_{n,k}^{(m)}}: I_{n,k}^{(m)} \rightarrow V_n \times (\prod_{k-1} H_{k-1}) \times K$ and $\tau =$

$T|_{V_m}$. Then we have $\psi = \tau \circ Q_{n,k,m}' \circ \sigma$. Thus Theorem 2, (2) is proved.

This completes the proof of Theorems 1 and 2.

Remark 3. In the above proof, though Q_m' can be extended

naturally to $Q_m': H_n \times K \rightarrow H_m$, the statement (c) does not hold for extended Q_m' .

§4. Stratification

We deduce from Theorem 2 a property of the stable object $\pi(I_{n,k}(2kn+1))$.

Lemma 4. Let $m \geq 2n+1$ if $k = 1$, and $m \geq (2k-1)n+1$ if $k \geq 2$. Then $\psi: I_{n,k}^{(m)} \rightarrow V_m$ is a homeomorphism onto the image.

Proof. By Theorem 2, it is sufficient to show that $Q = Q_{n,k,m}$ is a homeomorphism onto the image.

First Q is continuous and injective. In fact, let $Q(F, G_1, \dots, G_{k-1}, t) = Q(F', G_1', \dots, G_{k-1}', t')$. Then $F = F'$, $G_j = G_j'$, $(1 \leq j \leq k-1)$, and $F_{(i)}(t) = F_{(i)}(t')$, $(0 \leq i \leq n)$. By Lemma 2.7. of [7], we see $t = t'$.

Furthermore Q is proper, since already the mapping $(F, G_1, \dots, G_{k-1}, t) \rightarrow (F, G_1, \dots, G_{k-1}, F_{(0)}(t))$ is proper.

Thus we have Lemma 4.

Two stratified sets $(A, \mathcal{A}) \subset K^r$ and $(B, \mathcal{B}) \subset K^s$ are called isomorphic if there is a homeomorphism $\sigma: A \rightarrow B$ mapping diffeomorphically each stratum of \mathcal{A} onto a stratum of \mathcal{B} .

Corollary 1. The variety $\pi(I_{n,k}(2kn+1))$ has a Whitney

regular stratification \mathcal{T} isomorphic to the stratification of K^{kn} by the Thom-Boardman stratification of $f_{n,k}: K^{kn} \rightarrow K^{kn+k-1}$.

The dimension of each stratum of \mathcal{T} is a multiple of k .

See [12, §19] and [10, §2] for the Whitney regularity.

Proof. By Theorem 2, $\pi(I_{n,k}(2kn+1))$ is parametrized by $Q = Q_{n,k,2kn+1}$. The decomposition of K^{kn} by the Thom-Boardman singularities of Q coincides with that of $f = f_{n,k}$:

$$K^{kn} = \bigcup_{0 \leq i \leq n} \Sigma_{(i)}(Q) = \bigcup_{0 \leq i \leq n} \Sigma_{(i)}(f),$$

where $\Sigma_{(i)} = \Sigma^{1,1,\dots,1}$, (i -times). (See [3].)

Further we have $\Sigma_{(i)}(Q) = \{(F, G_1, \dots, G_{k-1}) \in K^{kn} \mid t \text{ is a common root of } F, G_1, \dots, G_{k-1} \text{ of multiplicity } \geq i\}$.

We see $\Sigma_{(i)}(Q)$ is a submanifold of codimension ki .

By Lemma 4, Q is a homeomorphism onto $\pi(I_{n,k}(2kn+1))$.

Therefore $Q|_{\Sigma_{(i)}(Q)}$ is a diffeomorphism onto the image, by the definition of Thom-Boardman singularity.

Set $\mathcal{T} = \{Q(\Sigma_{(i)}(Q))\}_{0 \leq i \leq n}$.

Let $x \in \Sigma_{(i)}(f)$. Then the germ $f: K^{kn}, x \rightarrow K^{kn+k-1}$ is right-left equivalent to the germ of $f_{n-i,k} \times \text{id}_{K^{ki}}$ at 0. Thus

D_f is isomorphic to $D_{f_{n-i,k}} \times \text{id}_{K^{ki}}$. Then we see the germ of Q at x is right-left equivalent to the germ at 0 of

$(Q_{n-i,k,2k(n-i)+1} \times \text{id}_{K^{ki}}, 0): K^{kn} \rightarrow K^{2kn-ki} \times K^{ki}$, where $0: K^{kn} \rightarrow$

K^{ki} is the constant zero map.

Therefore, for each $y \in Q(\Sigma_{(i)}Q)$, there exists a germ of diffeomorphism $K^{2kn}, y \rightarrow K^{2k(n-i)} \times K^{2ki}, 0$ mapping \mathcal{T} to $\mathcal{T}' \times K^{ki} \times 0$ and $Q(\Sigma_{(i)}Q)$ to $0 \times K^{ki} \times 0$, where \mathcal{T}' is the stratification associated to $f_{n-i,k}$.

Thus \mathcal{T} is Whitney regular along $Q(\Sigma_{(i)}Q)$, ($0 \leq i \leq n$).

§5. Lagrangian property

We recall the symplectic structures on H_{2n+1} , M_{2n} and V_{2n-1} , ([2, §11]).

Set $\Phi_y(F(t)) = F(t+y)$, ($y \in K$, $F \in H_{2n+1}$). Then $\Phi_y: H_{2n+1} \rightarrow H_{2n+1}$ is a linear symplectic transformation with respect to the symplectic structure $\omega = \sum_{0 \leq i \leq n} dp_i \wedge dq_i$, where $q_i = a_{n-i}$, $p_i = (-1)^{i+1} a_{n+i+1}$, ($0 \leq i \leq n$).

The flow $\{\Phi_y\}_{y \in K}$ is a Hamiltonian flow with the Hamiltonian

$$\begin{aligned} h &= (1/2) \sum_{0 \leq i \leq 2n} (-1)^{n+i} a_i a_{2n+i}, \\ &= q_0^2/2 + \sum_{0 \leq i \leq n-1} p_i q_{i+1}. \end{aligned}$$

In fact, $X = \sum_{1 \leq i \leq 2n+1} a_{i-1} \partial/\partial a_i$ is the infinitesimal transformation of $\{\Phi_y\}_{y \in K}$, and $X \lrcorner \omega = -dh$.

Notice that h is independent of a_{2n+1} .

The field $\partial/\partial a_{2n+1}$ is a characteristic field on the

hypersurface $\{a_0 = 1\}$. Then the space of characteristics is identified with M_{2n} and the induced symplectic structure on M_{2n} is $\omega = \sum_{0 \leq i \leq n-1} dp_i \wedge dq_i$. The induced function

$$h = q_0^2/2 + p_0 q_1 + \dots + p_{n-2} q_{n-1} + p_{n-1}$$

is a Hamiltonian of the induced flow $\{\Phi_y\}_{y \in K}$ on M_{2n} .

The space of characteristics of hypersurface $\{h = 0\}$ is identified with V_{2n-1} by the mapping $\pi: \{h = 0\} \subset M_{2n} \rightarrow V_{2n-1}$, (cf. §1). Then the induced symplectic structure on V_{2n-1} is $\omega = \sum_{0 \leq i \leq n-2} dp_i \wedge dq_i$.

Lemma 5. If $k = 1$ or 2 , then $\psi: I_{n,k}(2kn+1) \rightarrow V_{2kn+1}$ is isotropic, that is, $\psi^* \omega = 0$.

Proof. If $k = 1$ or 2 , then $I_{n,k}(2kn+1) \subset \{h = 0\}$.

Furthermore, the symplectic form $\omega_{M_{2kn+2}}$ vanishes on $I_{n,k}(2kn+1)$.

Thus $\psi^* \omega_{V_{2kn+1}} = \omega_{M_{2kn+2}} | I_{n,k}(2kn+1) = 0$.

Corollary 2. Each stratum of $(S_n(2n+1), \mathcal{T})$, (resp. $(W_{2n}(4n+1), \mathcal{T})$) is a isotropic manifold.

Proof of Proposition 1. Let $h: N \rightarrow M$ satisfy $h(N) \subset S_n(2n+1)$ (resp. $W_{2n}(4n+1)$). Denote by U_i the interior of $h^{-1}Q(\Sigma_{(i)}Q)$. Then $h^* \omega | U_i = 0$, $(0 \leq i \leq n)$, and $N = \bigcup_{0 \leq i \leq n} \bar{U}_i$. Thus

$$h^*\omega = 0.$$

Remark 4. $I_{n,1}(2n+1) \subset M_{2n+2}$ is an intersection of the Lagrangian manifold $I_{n+1,1}(2n+1)$ and the hypersurface $\{h = 0\}$. On the other hand, $I_{n,2}(4n+1) \subset M_{4n+2}$ is not an intersection of any Lagrangian manifold and $\{h = 0\}$.

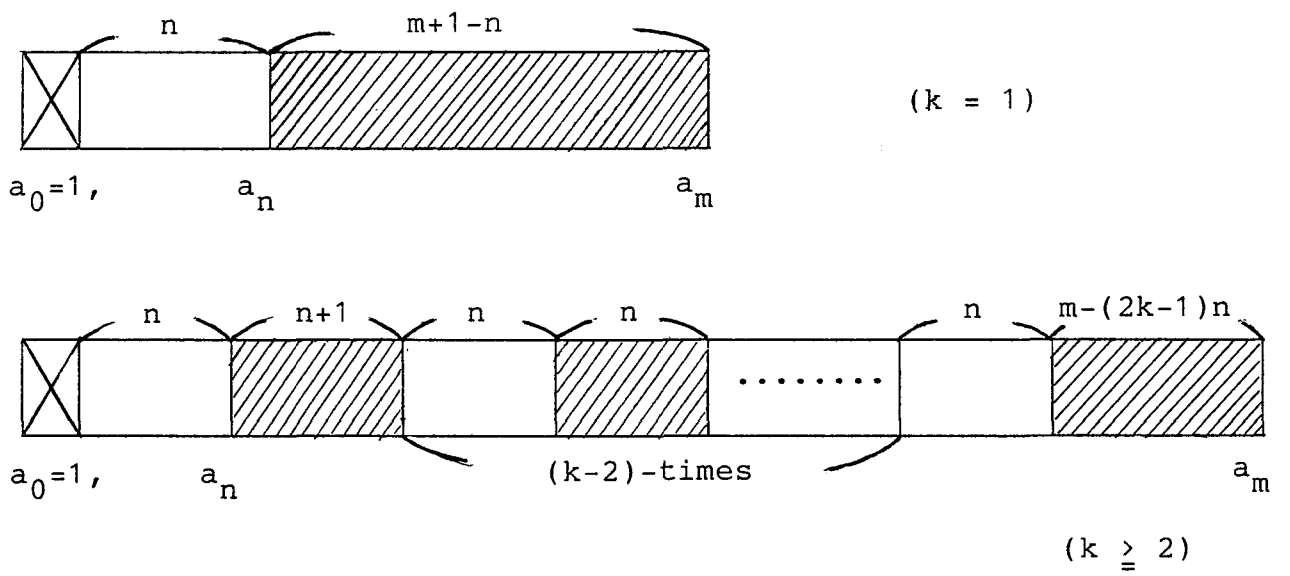
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- Diagram 1 -



(Please insert Diagram 1 in page 5.)