21st Century COE Program:
Mathematics of Nonlinear Structure via Singularity

16th COE Lecture Series

CHAOS IN TRAVELING WAVES IN LATTICE SYSTEMS OF UNBOUNDED MEDIA

COE Researcher
HUNSEOK KANG

February 4th, 2008 (Mon), February 8th, 2008 (Fri)

Series #132. March, 2008

Publication of this series is partly supported by Grant-in-Aid for formation of COE.
'Mathematics of Nonlinear Structures via Singularities' (Hokkaido University)
URL:http://coe.math.sci.hokudai.ac.jp

HOKKAIDO UNIVERSITY
TECHNICAL REPORT SERIES IN MATHEMATICS

#110 K. Matsumoto, 超幾何関数早春学校, 87 pages. 2006.

#111 T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura, Y. Tonegawa, K. Tsutaya and T. Sakajo, The 31th Sapporo Symposium on Partial Differential Equations, 91 pages. 2006.

#112 H. Okamoto, D. Sheen, Z. Shi, T. Ozawa, T. Sakajo and Y. Chen, Book of Abstracts of the First China-Japan-Korea Joint Conference on Numerical Mathematics & The Second East Asia SIAM Symposium, 78 pages. 2006.

#113 N. Ishimura, T. Ishiwata, T. Sakajo, T. Sakurai, M. Nagayama, T. Nara, K. Hayami, D. Furihata and T. Matsuo, 応用数理サマーセミナー「確率微分方程式」, 116 pages. 2006.

#114 T. Abe, 第 8 回 COE 研究員連続講演会『超平面配置と対数的ベクトル場の幾何』, 23 pages. 2006.

#115 H. Kubo and T. Ozawa, Sapporo Guest House Symposium on Mathematics 22 "Nonlinear Wave Equations", 67 pages. 2006.

#116 S. Okabe, 第 9 回 COE 研究員連続講演会『ある束縛条件下における平面弾性閉曲線のダイナミクス』, 31 pages. 2006.

#117 A. Suzuki, T. Ito, N. Sato, K. Shibuya, D. Hirose, Y. Maekawa and K. Matsumoto, 第 3 回数学総合若手研究集会 –他分野との学際的交流を目指して, 264 pages. 2007.

#118 T. Yamamoto, Y. Sato, N. Kataoka and H. Takagi, Proceedings of the conference on New Aspects of High-dimensional Nonlinear Dynamics, 73 pages. 2007.

#119 T. Yamamoto and T. Nakazi, 第 15 回関数空間セミナー報告集, 82 pages. 2007.

#120 K. Hirata, 第 10 回 COE 研究員連続講演会『正値調和関数に対する Martin 積分表現と非線形楕円型方程式の正値解の存在』, 25 pages. 2007.

#121 K. Izuchi, 第 12 回 COE 研究員連続講演会『 I）Segal-Bargmann 空間の巡回ベクトルについて II）2 重単位開円板上の逆シフト不変部分空間上のクロス交換子について』, 18 pages. 2007.

#122 T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura, Y. Tonegawa, K. Tsutaya and T. Sakajo, The 32nd Sapporo Symposium on Partial Differential Equations, 73 pages. 2007.

#123 H. Kubo, T. Ozawa, H. Takamura and K. Tsutaya, Nonlinear Wave Equations, 53 pages. 2007.

#124 N. Ishimura, T. Sakajo, T. Sakurai, M. Nagayama, T. Nara, D. Furihata and T. Matsuo, 応用数理サマーセミナー 2007「計算ホモロジーとその応用」, 110 pages. 2007.

#125 S. Hattori, 第 15 回 COE 研究員連続講演会『分岐理論と有限平坦 Galois 表現』, 28 pages. 2008.

#126 T. Miyaguchi, M. Kon, R. Kamijo, S. Kawano, Y. Sato and D. Hirose, 第 4 回数学総合若手研究集会 –他分野との学際的交流を目指して, 298 pages. 2008.

#127 K. Saji, 第 13 回 COE 研究員連続講演会『波面とその特異点』, 15 pages. 2008.

#128 T. Nakazi and S. Miyajima, 第 16 回関数空間セミナー報告集, 105 pages. 2008.

#129 M. Kon, 学位論文 Doctoral thesis "Minimal submanifolds immersed in a complex projective space", 77 pages. 2008.

#130 S. Minabe, 第 14 回 COE 研究員連続講演会『開ミラー対称性における最近の進展』, 8 pages. 2008

#131 H. Kubo and T. Ozawa, Sapporo Guest House Symposium, Final "Nonlinear Partial Differential Equations", 73 pages. 2008.

21st Century COE Program:
Mathematics of Nonlinear Structure via Singularity

16th COE Lecture Series

CHAOS IN TRAVELING WAVES IN LATTICE SYSTEMS OF UNBOUNDED MEDIA

COE Researcher
HUNSEOK KANG

February 4th, 2008 (Mon) 14:45-16:15 3-311
February 8th, 2008 (Fri) 16:30-18:00 3-311

CHAOS IN TRAVELING WAVES IN LATTICE SYSTEMS OF UNBOUNDED MEDIA

HUNSEOK KANG

Abstract. The goal of this report is to describe the chaotic dynamics of the local maps of coupled map lattices(CML) from scientific models: the Maginu model of morphogenesis, the Turing Model of morphogenesis, and the Brusselator model of Belousov-Zhabotinsky reaction. I present the dynamical system approach to the analysis of the global behavior of solutions of CML, which is aimed at establishing spatio-temporal chaos associated with the set of traveling wave solutions of CML and describing the dynamics of the evolution operator on this set.

1. Introduction

In this section, we go over *coupled map lattices*(CML). Then we introduce reaction-diffusion equations and their discrete version as a source of coupled map lattices, and we finally give three examples of CMLs obtained from a discretization of reaction-diffusion equations.

1.1. Coupled map lattices.

Coupled map lattices (CML) form a special class of infinite-dimensional dynamical systems. They are described by the equation of the form

$$u_j(n+1) = f(u_j(n)) + \epsilon g_j(\{u_i(n)\}_{|i-j|\leq s}). \tag{1.1}$$

Here $n \in \mathbb{Z}$ is the discrete time coordinate, j, the discrete space coordinate, and $u(\bar{j}, n) = u_{\bar{j}}(n)$ is a characteristic of the medium (for example, its density, or distribution of the temperature, etc.). Furthermore, $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g_{\bar{j}} : (\mathbb{R}^d)^{(2s+1)} \to \mathbb{R}^d$ are smooth functions; f is called the *local map* and g the *interaction* of size s (where s can be infinite). Finally, ϵ is a parameter which is assumed to be sufficiently small.

In this report, we assume that the medium of CML is *unbounded* and *spatially homogeneous*. We also assume that every solution of CML is uniquely determined if an initial condition $u_j(0)$ and a boundary condition are fixed. We consider the case (up to) when solutions grow at infinity with an exponential rate. The main achievement is the description of the global behavior of solutions of CML for local maps exhibiting hyperbolic or invariant properties.

1.2. Reaction-diffusion equations.

A natural source of CMLs is discrete versions of partial differential equations of evolution type. In general, no information on the global behavior of solutions of a partial differential equation can be derived from the study of its discrete versions even when steps of discretization are small.

Reaction-diffusion equations are a special member of evolutionary PDEs, described by

$$u_t = h(u) + \mathcal{A}\kappa_i D_x u,$$

where $u = u(x,t)$ is a function with values in $\mathbb{R}^d$, $x \in \mathbb{R}$ and $t \in \mathbb{R}$ are space and time coordinates, resp., $\mathcal{A}$ is the $d \times d$ coupling matrix, and κ_i are diffusion coefficients.

When discretized, a reaction-diffusion equation form an equation

$$u(x,t+\gamma) = u(x,t) + \gamma h(u(x,t)) + \mathcal{A}\gamma\kappa_i(\text{discretization of } D_x u),$$

which is in the same form as CMLs (1.1).

1.3. **Examples of CMLs.** The Turing model of morphogenesis is described by a reaction-diffusion equation

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= -(Au_1^2 + Bu_1u_2) + C + \kappa_1\frac{\partial^2 u_1}{\partial x^2},\\ \frac{\partial u_2}{\partial t} &= (Au_1^2 + Bu_1u_2) - Du_2 + E + \kappa_2\frac{\partial^2 u_2}{\partial x^2}.\end{aligned}$$

Its corresponding CML has a local Map

$$f(u_1,u_2) = (u_1 - au_1^2 - bu_1u_2 + c, au_1^2 + bu_1u_2 + (1-d)u_2 + e).$$

The Maginu model of morphogenesis is a simplification of the Turing model above. This model is described by a reaction-diffusion equation, called FitzHugh-Nagumo Equation,

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= -au_1(u_1-\theta)(u_1-1) - bu_2 + \kappa_1\frac{\partial^2 u_1}{\partial x^2},\\ \frac{\partial u_2}{\partial t} &= cu_1 - du_2 + \kappa_2\frac{\partial^2 u_2}{\partial x^2}.\end{aligned}$$

Its discrete version gives us a CML whose local map is

$$f(u_1,u_2) = (u_1 - Au_1(u_1-\theta)(u_1-1) - \alpha u_2, \beta u_1 + \gamma u_2).$$

Finally, we discuss the Brusselator model of Belousov-Zhabotinsky reaction. This is a famous model of chemical reactions with oscillations, described by

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= A - (B+1)u_1 + u_1^2u_2 + \kappa_1\frac{\partial^2 u_1}{\partial x^2},\\ \frac{\partial u_2}{\partial t} &= Bu_1 - u_1^2u_2 + \kappa_2\frac{\partial^2 u_2}{\partial x^2}.\end{aligned}$$

The corresponding CML has the local map

$$f(u_1,u_2) = (a + (1-\gamma-b)u_1 + \gamma u_1^2u_2,\ u_2 + bu_1 - \gamma u_1^2u_2).$$

2. Traveling Waves in Lattice Systems

In this section, we introduce traveling waves as a tool to analyze CMLs, and then we deal with a special type of CMLs which is related to spatio-temporal chaos, and we finally show how to use a traveling wave map to study the dynamics of CMLs.

2.1. **Operators on a Phase Space.** We first define a weighted norm by

$$\|u\|_{q_1,q_2} = \sum_{j<0} \frac{|u_j|}{q_1^{-j}} + \sum_{j\geq 0} \frac{|u_j|}{q_2^j},$$

for $q_1 > 1$ and $q_2 > 1$. Let $\mathcal{M}_{q_1,q_2}$ be the phase space in the direct product of finite-dimensional Euclidean spaces $\otimes_{\mathbb{Z}}\mathbb{R}^d = (\mathbb{R}^d)^{\mathbb{Z}}$ with the norm $\|\cdot\|_{q_1,q_2}$:

$$\mathcal{M}_{q_1,q_2} = \{u = (u_j) \in (\mathbb{R}^d)^{\mathbb{Z}} : \|u\|_{q_1,q_2} < \infty\}.$$

We define the nonlinear evolution operator $\Phi = \Phi_\epsilon$ by

$$(\Phi u)_j(n) = f(u_j(n)) + \epsilon g_j(\{u_i(n)\}_{|i-j|\leq s}).$$

Then $\mathcal{M}_{q_1,q_2}$ corresponds to solutions of the CML which satisfy the initial condition $(u_j(0)) \in \mathcal{M}_{q_1,q_2}$ and boundary condition $\|u\|_{q_1,q_2} < \infty$. In [1], it is shown that $(\mathcal{M}_{q_1,q_2}, \Phi_\epsilon)$ is a *well-defined dynamical system* under the following assumptions:

A1. There exists $M > 0$ such that for $k = 1, 2$,

$$\sup_{x\in\mathbb{R}^d} \|D^k f_x\| \leq M, \qquad \sup_{1\leq i\leq 2s+1} \sup_{x_i\in\mathbb{R}^d} \left\| \frac{\partial^k g_j}{\partial x_i^k} \right\| \leq M.$$

A2. For any $x = (x_i) \in \mathbb{R}^{d(2s+1)}$,

$$\det\left(\frac{\partial g_j(x)}{\partial x_i}\right) \neq 0, \qquad \sup_{x\in\mathbb{R}^{d(2s+1)}} \det\left(\frac{\partial g_j(x)}{\partial x_i}\right)^{-1} < \infty.$$

On the other hand, we define another operator on the phase space related to spatial coordinates. A spatial translations $S^k : \mathcal{M}_{q_1,q_2} \to \mathcal{M}_{q_1,q_2}$ is given by

$$S^k(u)_i = u_{i+k},$$

where i and k are integers. We additionally assume that

A3. Φ_ϵ^i and S^k commute for any $i, k \in \mathbb{Z}^+$.

If g_j does not depend on j, i.e. $g_j = g$, then the assumption A3 always holds. This is the case when the lattice system is obtained as a discretization of a PDE.

2.2. **Coupled Map Lattices of Hyperbolic Types.** In [10], Y. Pesin and Y. Sinai suggested a definition of spatio-temporal chaos in lattice dynamical systems. A lattice dynamical system is said to display:

(1) *temporal chaos* if there exists a measure μ invariant under the $\mathbb{Z}$-action $\{\Phi^j\}$ which is mixing;
(2) *spatial chaos* if there exists a measure μ invariant under the $\mathbb{Z}$-action $\{S^k\}$ which is mixing;
(3) *spatio-temporal chaos* if there exists a measure μ invariant under the $\mathbb{Z}^2$-action $\{\Phi^j, S^k\}$ which is mixing.

In [4], Jiang and Pesin established the existence and uniqueness of *Gibbs distributions* for arbitrary CML whose *local map possesses a hyperbolic set.* They proved

(1) *the existence of Gibbs distributions* (that implies equilibrium measures for CML are invariant) and;
(2) *the uniqueness of Gibbs distributions* (that implies equilibrium measures for CML are mixing).

Therefore, the equilibrium measure for CML is the measure μ for the definition of *spatio-temporal chaos.*

2.3. **Traveling Wave Maps.** Spatio-temporal chaos can be physically observed by *traveling wave maps*. To observe this, we consider the traveling wave map $F_\epsilon : (\mathbb{R}^d)^{ls+m} \to (\mathbb{R}^d)^{ls+m}$ given by

$$F_\epsilon(x_1, ..., x_{ls+m}) = (x_2, ..., x_{ls+m}, f(x_{ls+1}) + \epsilon g(x_{p(i)})_{i=-s}^{s}),$$

where $p(i) = l(s + i) + 1$. In [1], the authors proved the following theorem.

Theorem 2.1 (See [1]). *There exist $q_1^{(0)} > 1$, $q_2^{(0)} > 1$ such that for any $q_1 > q_1^{(0)}$, $q_2 > q_2^{(0)}$, the traveling wave map F_ϵ are smoothly conjugate to the evolution operator Φ_ϵ; in other words, there exists a smooth embedding χ of $(\mathbb{R}^d)^{ls+m}$ into $\mathcal{M}_{q_1,q_2}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} (\mathbb{R}^d)^{ls+m} & \xrightarrow{\chi} & \mathcal{A}_{q_1,q_2} \subset \mathcal{M}_{q_1,q_2} \\ {\scriptstyle F_\epsilon{}^m}\downarrow & & \downarrow{\scriptstyle \Phi_\epsilon} \\ (\mathbb{R}^d)^{ls+m} & \xrightarrow{\chi} & \mathcal{A}_{q_1,q_2} \subset \mathcal{M}_{q_1,q_2} \end{array}$$

where $\mathcal{A}_{q_1,q_2} = \chi((\mathbb{R}^d)^{ls+m})$ is finite-dimensional.

Theorem 2.1 tells us that the dynamics of the evolution operator Φ_ϵ on the set of traveling wave solutions is completely determined by the traveling wave map F_ϵ.

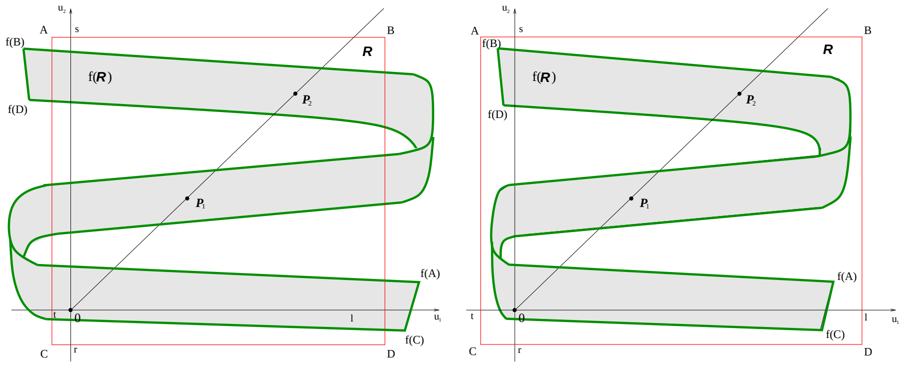
3. Dynamics of Discrete Models

In this section, we deal with the application of sections 1 and 2 to the real models in sciences. First we build the relationship between the traveling wave map F_ϵ and the local map f under the assumption that the interaction of CML is sufficiently small. Then we analyze the dynamics of the local maps of CMLs corresponding the Maginu model of morphogenesis, the Turing model of morphogenesis, and the Brusselator model of Belousov-Zhabotinsky reaction.

3.1. **Dynamics of Local Maps.** Now we see that, in the case that the interactions ϵ are sufficiently small, most hyperbolic and ergodic properties of dynamics of the traveling wave map are dominated by the behavior of the local map f. More precisely, it has been proven that

(1) if it possesses a locally maximal hyperbolic set then so does the traveling wave map(See [1]);
(2) if the local map is of a Morse-Smale type then so is the traveling wave map(See [2]);
(3) if the local map possesses a strictly forward-invariant region then so does the traveling wave map(See [6]).

In particular, if the local map is hyperbolic in a strong sense (i.e., it possesses a hyperbolic set; every trajectory in this set is highly unstable) then so are the traveling wave map and the restrictions of space and time translations to the submanifold of traveling wave solutions. This implies that the CML displays chaotic behavior of the highest degree, i.e., there exists a measure invariant under space and time translations which is supported on the set of traveling wave solutions and has ergodic properties of higher order. In the last case, we may predict asymptotic behaviors of trajectories; where trajectories tend to head and how they behave after a sufficiently large number of iterations by the local map f.



(a) Smale horseshoe. (b) Trapping region.

FIGURE 1. Rectangular regions and their images under the map f. Depending one the condition of parameters, each of them shows a smale horseshoe construction and bounded trapping region, respectively.

3.2. **Maginu Model of Morphogenesis.** As seen in 1.3, the Maginu model of morphogenesis is described by the FitzHugh-Nagumo equation, and its discrete version forms a CML whose local map is

$$f(u_1, u_2) = (u_1 - Au_1(u_1 - \theta)(u_1 - 1) - \alpha u_2, \beta u_1 + \gamma u_2).$$

This local map has the following interesting dynamics:

(1) for all sufficiently large values of A, f is hyperbolic, i.e., f has a *locally maximal hyperbolic set*.
(2) for intermediate values of A, f has a *trapping region* and a strange attractor.
(3) for all sufficiently small values of A, the compactification map $\tilde{f}$ of f is of *Morse-Smale type*.

The detailed conditions on parameters are given in [8]. In the case that f has a trapping region, there may be a henon-like strange attractor.

If the local map of a CML is hyperbolic in a strong sense (i.e., it possesses a locally maximal hyperbolic set) then so is the traveling wave map F_ϵ (for sufficiently small $\epsilon > 0$) and in turn, space and time translations considered on the set of traveling wave solutions. In this case the dynamics of the CML is chaotic on a finite-dimensional smooth submanifold in the infinite-dimensional phase space $\mathcal{M}_{q_1,q_2}$ (endowed with the metric $\| \cdot \|_{q_1,q_2}$). An example is the two-dimensional local map for this FitzHugh-Nagumo equation in some range of parameters. If the local map of a CML is Morse-Smale (i.e., it is hyperbolic in a weak sense) then so are the traveling wave map F_ϵ (for sufficiently small $\epsilon > 0$) and space and time translations restricted to the set of traveling wave solutions. In this case the dynamics of the CML is not chaotic and the topological behavior of individual trajectories can be completely described.

3.3. **Turing Model of Morphogenesis.** The local map of the CML corresponding the Turing model of morphogenesis is

$$f(u_1, u_2) = (u_1 - au_1^2 - bu_1u_2 + c, au_1^2 + bu_1u_2 + (1 - d)u_2 + e).$$

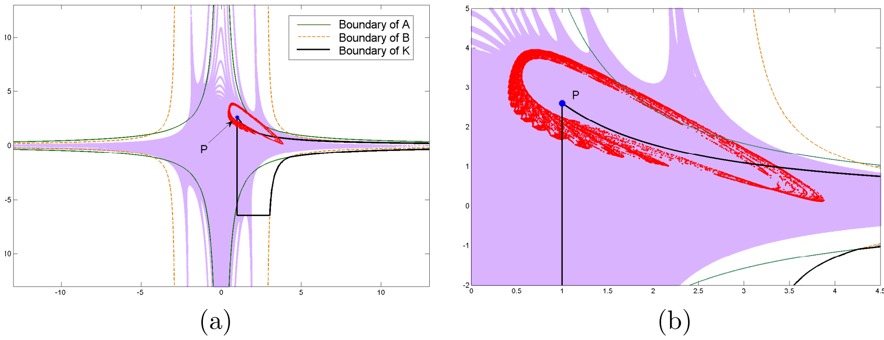


FIGURE 2. Julia set and strange attractor for $a = 0.25$, $b = 0.65$, $\gamma = 0.25$. (a) Shown are the Julia set J, the region K bounded by the thick curves, boundaries for A and B, and the fixed point $P(1, 2.6)$. (b) Zoom-in region around the fixed point.

Its asymptotic behaviors are

(1) for all values of parameters, there exists a region where every trajectory originating in R escapes to infinity,
(2) either for a sufficiently large $a > 0$ or for a sufficiently large $b > 0$ there exists the Julia set, the set of all bounded trajectories, and
(3) for $d > 0$ sufficiently close to 1, f has the locally maximal hyperbolic set, which implies spatio-temporal chaos of this system as seen in Section 2.

We apply a traditional *cone technique* to show the existence of the locally maximal hyperbolic set of the local map. Detailed theories and proofs will be published soon.

3.4. **Brusselator model of Belousov-Zhabotinsky reaction.** Lastly, we discuss the dynamics of the local map obtained from the discretization of the reaction-diffusion equation corresponding to the Brusselator model for Belousov-Zhabotinsky reaction, mainly describing the asymptotic behaviors of trajectories of the local map. The local map f is given by

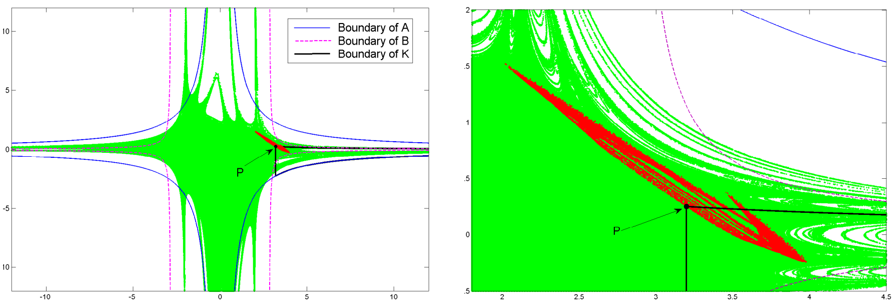
$$f(u_1, u_2) = (a + (1 - \gamma - b)u_1 + \gamma u_1^2 u_2,\ u_2 + bu_1 - \gamma u_1^2 u_2).$$

In particular, we study the dynamics of two disjoint components of trajectories whose union is the entire set: bounded trajectories and unbounded trajectories.

3.4.1. *Behavior of Unbounded Trajectories.* We describe a region R in the u_1u_2-plane such that any trajectory in R escapes to infinity, i.e., $|f^n(u_1, u_2)| \to \infty$ as $n \to \infty$, and then describe how unbounded trajectories escape to infinity in the region R. Finally, we show that the union of all backward images of R by f is the maximal set of trajectories that escape to infinity. This implies the existence of the Julia set, which has interesting topological properties.

We first define two regions A and B in the plane $\mathbb{R}^2$:

$$A = \{(u_1, u_2) : \gamma u_1^2 |u_2| - m|u_1| - a \geq 0\},$$
$$B = \{(u_1, u_2) : |(\gamma u_1^2 - 2)u_2| - b|u_1| \geq 0,\ \gamma u_1^2 \geq 2\},$$



(a) (b)

FIGURE 3. Julia set and attractor for $a = 0.8$, $b = 0.2$, $\gamma = 0.25$. (a) Shown are the Julia set J, the region K bounded by the thick curves, boundaries for A and B, and the fixed point $P(3.2, 0.25)$. (b) Zoom-in region around the fixed point.

where $m := \max(|2-\gamma-b|, \gamma+b)$. Let $R = A \cap B$. Then for any point $(u_1, u_2) \in A$, $|f_1(u_1, u_2)| > |u_1| + 2a$, while for any point $(u_1, u_2) \in B$, $|f_2(u_1, u_2)| > |u_2|$. Using these facts, we prove

Theorem 3.1 (See [7])**.** *For all parameters $a > 0$, $b > 0$, and $\gamma > 0$, there exists a forward-invariant region $R = R(a, b, \gamma)$ such that for any point $(u_1, u_2) \in R$, the trajectory $\{f^n(u_1, u_2)\} \subset R$ escapes to infinity.*

The region R in Theorem 3.1 is symmetric with respect to the origin and does not intersect u_1-axis nor u_2-axis. Let R_i be the the partition of R in the i^{th}-quadrant in the plane $\mathbb{R}^2$, for $i = 1, 2, 3, 4$, respectively. Then

$$f(R_1) \subset R_4, \quad f(R_3) \subset R_2, \quad f(R_2) \subset R_4, \quad \textit{and} \quad f(R_4) \subset R_2.$$

We now find the maximal region K so that every point in K escapes to infinity while trajectories of points outside K are bounded. Then, clearly, the set $\mathbb{R}^2 \setminus K$ is the Julia set of the system and thus we find various Julia sets under a certain condition on values of parameters.

Theorem 3.2 (See [7])**.** *Assume that $a < 1$ and $\gamma + b < 1$. Let $\{f^n(u_1, u_2)\}$ be a trajectory that originates in $\mathbb{R}^2 \setminus R$. This trajectory is unbounded if and only if there exists a positive integer $m = m(a, b, \gamma, u_1, u_2)$ such that $f^m(u_1, u_2) \in R$. In particular, the trajectory escapes to infinity via the region R.*

The proof of Theorem 3.2 can be obtained by carefully examining the behavior of trajectories originating in $\mathbb{R}^2 \setminus R$ whose initial values are sufficiently large.

Theorem 3.2 provides the following explicit description of the Julia set

$$J = \mathbb{R}^2 \setminus \bigcup_{n \geq 0} f^{-n}(R).$$

The set J is *nonempty* as it contains one unique fixed point $P(p_1, p_2) = (a/\gamma, b/a)$. Furthermore, J has a *positive area* in $\mathbb{R}^2$ for some values of parameters a, b, and γ, because there are bounded trapping region in J. Additionally, J is *closed* and *unbounded* in $\mathbb{R}^2$. Also, there exists connected component in J which reaches to the infinity along to either u_1- or u_2-axis.

To generate images of the Julia set for a sufficiently large $N > 0$ we generate the set of points x for which

$$f^n(x) \in \mathbb{R}^2 \setminus R,$$

for all $n = 0, 1, 2, ..., N$. One can observe that branches of the Julia set are separated by white regions which are backward images of R. Each branch is further split into more branches and this goes on giving a way to a Cantor-like construction whose limit set is the desired set J.

Let us stress that bounded trajectories are the one that are interesting from the physical point of view. We believe that periodic trajectories are dense in the Julia set and that this can be used to justify the existence of oscillating solutions in the Belousov-Zhabotinsky reaction.

We found a region $R \subset \mathbb{R}^2$ such that every trajectory in R escapes to infinity and showed that the region R is strictly forward-invariant under the local map f. As seen in Section 3.1, the traveling wave map F_ϵ corresponding to f also possesses a strictly forward-invariant set $\mathcal{R} = \otimes_{i=1}^{ls+m} R$. Also, the Julia set is obtained by using the maximal region of R that is the union of all backward images of R. Similarly, one can obtain the Julia set $\mathcal{J}_\epsilon$ of the traveling wave map F_ϵ. The Julia set $\mathcal{J}_\epsilon$ is nonempty since the traveling wave map F_ϵ has a fixed point, which is contained in $\mathcal{J}_\epsilon$. In fact, the Julia set $\mathcal{J}_\epsilon$ is also unbounded.

3.4.2. *Behavior of Bounded Trajectories.* In this section we continue to study the dynamics of the local map focusing on the behavior of bounded trajectories inside the Julia set, while we focused on the behaviors of trajectories which escape to infinity in Section 3.4. In addition, we establish the relationship between the dynamics of the CML corresponding to the Brusselator model and its local map with respect to the evolution operator and the traveling wave map.

Under the same assumption on the parameters as in Theorem 3.2, we shall establish existence of an *eventually trapping region* $\tilde{J} \subset J$, i.e., a bounded region in J which every trajectory, originating in J, enters and stays in forever. In [6], we give the way to construct the set $\tilde{J}$ and prove the following theorem.

Theorem 3.3 (See [6]). *The set $\tilde{J} \subset J$ satisfies:*

(1) *$\tilde{J}$ is bounded and compact;*
(2) *for each $u \in \tilde{J}$, $f(u) \in \tilde{J}$, i.e. $f(\tilde{J}) \subset \tilde{J}$;*
(3) *for each $u \in J$, there exists $n = n(u)$ such that $f^n(u) \in \tilde{J}$.*

Theorem 3.3 tells us that the set $\tilde{J} := J \cap E$ is a forward-invariant bounded region where all the bounded trajectories of this system, which originate in J, are trapped in $\tilde{J}$ forever once they enter in E. Furthermore, by Theorem 3.3, for sufficiently small coupling constant ϵ, the traveling wave map of CML also possesses this eventually trapping region, and hence, so does its evolution operator on the set of traveling wave solutions.

As stated in Section 2.2, the Julia set J is unbounded, so that there exist trajectories in J which begins from infinity. More precisely, every trajectory in J enters in the bounded region $\tilde{J}$ and is trapped inside forever no matter how large its initial values are.

Inside the eventually trapping region $\tilde{J}$ in Section 3.1, we find a *visiting region* $K \subset \tilde{J}$, i.e., a subset of $\tilde{J}$ which every trajectory, originating in J, visits infinitely many times or converges to the fixed point P of f.

Theorem 3.4 (See [6]). *There exists a region K in $\tilde{J}$ such that every trajectory $\{f^n(u) : u \in J\}$ satisfies one of the following two statements:*

(1) *there exists $n_0 = n_0(u) > 0$ such that $f^{n_0}(u) \in K \cap J$,*
(2) *$f^n(u)$ converges to the fixed point $P(a/\gamma, b/a) \in K$ as $n \to \infty$.*

Theorem 3.4 implies that the set $K \cap J$ is the *visiting region.* One can apply Theorem 3.4 as follows. Every periodic orbit can be viewed as a bounded trajectory, and it does not converges to the fixed point. So by Theorem 3.4, any periodic orbit must visit K periodically, and hence, all the periodic points starts from K. In fact, it is easy to show that there exists at least one periodic orbit of period two of the local map f for all values of parameters. So the periodic orbit of period two begins at a point in the visiting region $K \cap J$.

References

[1] V. Afraimovich and Ya. Pesin, *Travelling Waves in Lattice Models of Multi-dimensional and Multi-component Media : I. General Hyperbolic Properties*, Nonlinearity, **6** (1993), 429–455.
[2] M. Brin and Ya. Pesin, *On Morse-Smale Endomorphisms*, Amer. Math. Soc. Transl. (2), **171** (1996), 35–43.
[3] G. H. Choe, "Computational Ergodic Theory," Springer, New York, 2005.
[4] M. Jiang and Ya. B. Pesin, *Equilibrium Measures for Coupled Map Lattices: Existence, Uniqueness, and Finite-Dimensional Approximations*, Comm. Math. Phys., **193:3** (1998), 675–711.
[5] K. Kaneko, "Theory and Applications of Coupled Map Lattices," Wiley, New York, 1993.
[6] H. Kang, *Dynamics of Local Map of a Discrete Brusselator Model: Eventually Trapping Region and Strange Attractors*, DCDS-A, **20:4** (2008), 939–959.
[7] H. Kang and Ya. B. Pesin, *Dynamics of a Discrete Brusselator Model: Escape to Infinity and Julia Set*, Milan J. of Math., **73** (2005), 1–17.
[8] D. R. Orendovici and Ya. B. Pesin, *Chaos in Traveling Waves of Lattice Systems of Unbounded Media*, IMA Vol. Math. Appl., **119** (2000), 327–358.
[9] I. Prigogine and R. Lefever, *Symmetry-breaking instabilities in dissipative systems*, J. Chem. Phys., **48** (1968), 1695–1700.
[10] Ya. Pesin and Ya. Sinai, *Space-time chaos in chains of weakly hyperbolic mappings*, Advances in Soviet Math., **3** (1991), 165–198.
[11] S. H. Strogatz, "Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry and Engineering," Addison-Wesley, 1994.
[12] J. J. Tyson, "The Belousov-Zhabotinskii Reaction," Springer-Verlag, Heidelberg, 1976.

Department of Mathematics, Hokkaido University, Sapporo 060-0810
E-mail address: kang@math.sci.hokudai.ac.jp