Introduction to Variational Models in Image Processing

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Course description: This lecture note is a short introduction to problems arising in image processing and computer vision such as image restoration and image segmentation. They have applications in the areas such as medical imaging, computer animation, just to name a few. Topics include the direct method of the calculus of variations, anisotropic diffusions, functions of bounded variation(BV), numerical methods for solving PDEs, cartoon-texture image decomposition $f = u + v$, Yves Meyer's G-norm, the characterization of the ROF, level set method, geometric energy functionals and geodesic active contours, and Mumford-Shah image segmentation problem.

Image restoration by variational regularization methods
Ω : open, bounded subset of $\mathbb{R}^2$, $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$.
$u : (x, y) \longmapsto u(x, y)$: light intensity value at (x, y).
Image restoration: To reconstruct an original image u of a real scene from an observed image u_0(a degradation of u). The transformation (degradation) connecting u and u_0 is in general the result of two phenomena: deterministic and random.
Deterministic: is related to the mode and image acquisition, e.g. defects in the imaging system, motion, atmospheric turbulence, etc.
Random: Noise coming from signal transmission.

The simplest image degradation model accounting for both phenomena is,

$$u_0 = Ru + \eta; \text{ linear additive noise model.}$$

R: a linear operatorrepresenting a blur. Usually a convolution, η: random noise. (e.g. Gaussian, Uniform Gamma etc.)

$$u_0 = R * u + \eta. \tag{1}$$

Examples (Image degradation model)**.**

1. $u_0 = (Ru)\zeta$: multiplicative noise model.
2. $u_0 = (Ru)\zeta + \eta$: composite noise model.

Recovering u from u_0 knowing (1) is an example of inverse problem. It is not an easy task since we know little about η, a random variable.

$(x_1, x_2, \ldots, x_n)$: random sample data.
$f(x_1, x_2, \ldots, x_n | \theta_1, \theta_2, \ldots, \theta_k)$: probability function of observing $x_1, x_2, \ldots, x_n$ given the parameters $\theta_1, \theta_2, \ldots, \theta_k$.
$p(\theta_1, \theta_2, \ldots, \theta_k | x_1, x_2, \ldots, x_n)$: likelihood function of $\theta_1, \theta_2, \ldots, \theta_k$ given data $x_1, x_2, \ldots, x_n$.

Maximum likelihood Estimation (MLE): R.Fisher

It looks for parameters $\theta_1, \theta_2, \ldots, \theta_k$ that maximizes the likelihood function p given observed data $x_1, x_2, \ldots, x_n$.

Example. Suppose $x_1, x_2, \ldots, x_n \sim$ *i.i.d.* $N(\mu, \sigma^2)$. Suppose that σ^2 is known. Which μ maximizes $p(\mu, \sigma^2 | x_1, x_2, \ldots, x_n)$?

$$\begin{aligned}
\mu^{MLE} &= \operatorname*{argmax}_{\mu} p(\mu, \sigma^2 | x_1, x_2, \ldots, x_n) \\
&= \operatorname*{argmax}_{\mu} f(x_1, x_2, \ldots, x_n | \mu, \sigma^2) \\
&= \operatorname*{argmax}_{\mu} \prod_{i=1}^{n} f(x_i | \mu, \sigma^2) \qquad \text{by } i.i.d \\
&= \operatorname*{argmax}_{\mu} \sum_{i=1}^{n} \log f(x_i | \mu, \sigma^2) \quad \text{by monotonicity of log} \\
&= \operatorname*{argmax}_{\mu} \sum_{i=1}^{n} (x_i - \mu)^2 \qquad : \text{least squares problem.}
\end{aligned}$$

We formulate $u_0 = Ru + \eta$ into

$$\inf_{u} \int_{\Omega} |u_0 - Ru|^2 dxdy.$$

Assumption on the noise η.
η: zero mean, Gaussian.

Variational calculus

Let X be a Banach space. $F : X \to \mathbb{R} \cup \{\infty\}$. $u, y \in X$.

DEFINITION. *The* Gâteaux derivative *of F at u in the direction y is*

$$F'(u)y := \lim_{t \to 0^+} \frac{F(u + ty) - F(u)}{t}.$$

If there is $\tilde{u} \in X'$, the (topological) dual of X such that

$$F'(u)y = \tilde{u}(y) \qquad \forall y \in X,$$

then F is Gâteaux differentiable *at u and $F'(u) = \tilde{u}$.*

u is a critical point *of F if*

$$F'(u) = 0. \tag{2}$$

(2) *is called the* Euler-Lagrange equation *for F.*

Note. If the problem $\inf\limits_u F(u)$ has a minimizer, then $F'(u) = 0$.
Example.

$$\inf_{u \in L^2(\Omega)} \int_\Omega |u_0 - Ru|^2 dxdy, \;\; F = \int_\Omega |u_0 - Ru|^2 dxdy.$$

Let $g(t) = F(u + tv)$, $v \in L^2(\Omega)$, $t \in \mathbb{R}$. Suppose u is a minimizer of $F(u)$. Then $F'(u) = 0$, $g'(0) = 0$: Euler-Lagrange equation for F.

$$\begin{aligned}
g(t) &= \int_\Omega |u_0 - R(u+tv)|^2 dxdy \\
g'(t) &= \int_\Omega -2(u_0 - R(u+tv))Rvdxdy \\
g'(0) &= \int_\Omega -2(u_0 - R(u))Rvdxdy = 0 \quad , \forall v \in L^2(\mathbb{R}) \\
&\Leftrightarrow \int_\Omega R^*(u_0 - R(u))vdxdy = 0. \\
& R^*\text{: the adjoint of } R, \;\; < u, Rv >=< R^*u, v > . \\
\therefore \;\; & R^*(u_0 - Ru) = 0 \text{ a.e.} \\
& R^*Ru = R^*u_0.
\end{aligned}$$

Note: R^*R is not one-to-one in general. Even if R^*R has an inverse, its eigenvalues may be small, causing numerical instability.

To overcome ill-posed minimizing problems, Tikhonov(1977) proposed to consider

$$\inf_\mu \underbrace{\int_\Omega |u_0 - Ru|^2 dxdy}_{\text{data fitting term}} + \lambda \underbrace{\int_\Omega \phi(|\Delta u|)dxdy}_{\text{regularization}}.$$

$\phi : \mathbb{R} \to \mathbb{R}^+$.
Role of ϕ: a priori smoothness assumption on u.
$\lambda > 0$: balance between the data term and the regularization term.

LEMMA. *Suppose that X is a Banach space. $F : X \to \mathbb{R}$ is Gâteaux differentiable. Suppose $\bar{x} \in X$ such that $F(\bar{x}) = \inf\{F(x)|x \in X\}$. Then*

$$F'(\bar{x})y = 0 \qquad \textit{for } \forall y \in X.$$

Proof. Let $g(t) = F(\bar{x} + ty),\ y \in X, t \in \mathbb{R}$. Then g is differentiable since $\bar{x}$ is a minimizer for F.
$g(0) = \min\{g(t) : t \in \mathbb{R}\}$

$$\begin{aligned} 0 = \frac{d}{dt} g(t)|_{t=0} &= \lim_{\varepsilon \to 0} \frac{g(t+\varepsilon) - g(t)}{\varepsilon}|_{t=0} \\ &= \lim_{\varepsilon \to 0} \frac{F(\bar{x} + (t+\varepsilon)y) - F(\bar{x} + ty)}{\varepsilon}|_{t=0} \\ &= \lim_{\varepsilon \to 0} \frac{F(\bar{x} + ty + \varepsilon y) - F(\bar{x} + ty)}{\varepsilon}|_{t=0} \\ &= \lim_{\varepsilon \to 0} \frac{F(\bar{x} + \varepsilon y) - F(\bar{x})}{\varepsilon} \\ &= F'(\bar{x})y. \end{aligned}$$

LEMMA. *Let $F : X \to \mathbb{R}$ be Gâteaux differentiable and convex. Assume $\bar{x} \in X$ such that $F'(\bar{x})y = 0$ for $\forall y \in X$. Then*

$$F(\bar{x}) = \inf\{F(x) | x \in X\}.$$

Proof. Since F is convex, $F(y) \geq F(\bar{x}) + \underbrace{\langle y - \bar{x}; F'(\bar{x}) \rangle}_{F'(\bar{x})(y-\bar{x})=0}\ \ \forall y \in X$.

$$\begin{aligned} F((1-\lambda)\bar{x} + \lambda y) &\leq (1-\lambda)F(\bar{x}) + \lambda F(y),\ \lambda \in (1,0) \\ F(\bar{x} + \lambda(y - \bar{x})) &\leq F(\bar{x}) + \lambda(F(y) - F(\bar{x})) \\ \lim_{\lambda \to 0} \frac{1}{\lambda}(F(\bar{x} + \lambda(y - \bar{x}) - F(\bar{x})) &\leq F(y) - F(\bar{x}) \\ F'(\bar{x})(y - \bar{x}) + F(\bar{x}) &\leq F(y) \\ F(\bar{x}) &\leq F(y) \\ \therefore F(\bar{x}) &= \inf\{F(x) | x \in X\}. \end{aligned}$$

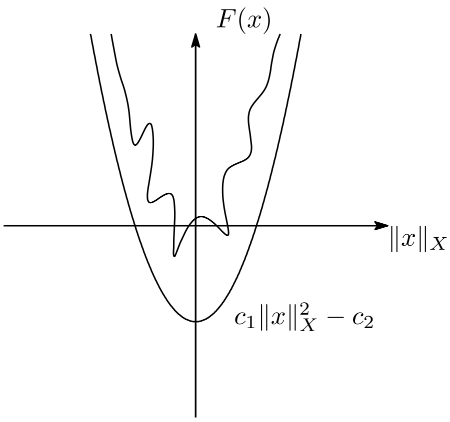
Many problems arising in geometry, analysis or applied mathematics can be formulated as $\min_{u \in A} F(x, u(x), \nabla u(x))$.

Note. In $\mathbb{R}^n$, Weierstrass theorem states that a lower semicontinuous real valued function attains a minimum in a sequentially compact set.

The closed unit ball of a normed space is compact for the norm topology iff the space is finite dimensional. In an infinite dimensional Banach space, there are nearly no compact sets. Compact sets have the empty interior and are not much of interest.

The direct methods in the calculus of variation.
DEFINITION. *X: a Banach space, $F : X \to \mathbb{R}$. F is* coercive *if*

$$F(x) \geq c_1 \|x\|_X^2 - c_2,\ c_1 > 0,\ c_2 \text{ are constants.}$$



DEFINITION. *A sequence x_n in X is a* minimizing sequence *if*

$$\lim_{n\to\infty} F(x_n) = \inf F(x).$$

DEFINITION. *A sequence x_n in X* convergences weakly *to x in X if*

$$\langle x_n, x^* \rangle \longrightarrow \langle x, x^* \rangle \qquad \textit{for } \forall x^* \in X'.$$

DEFINITION. *F is* weak lower semicontinuous *if for any $x_n \longrightarrow x$*

$$F(x) \leq \liminf_{n\to\infty} F(x_n).$$

Existence of a minimizer of $\inf F(x, u(x), \nabla u(x))$**.**

1. Show that F is coercive.
2. Construct a minimizing sequence $\{x_n\}$ monotonically decreasing to $\inf F$. Then $\|x_n\|_X < K$.
3. Compactness if X is reflexive. We can find a weakly convergent subsequence $\{x_{n_j}\}$ converging to $x_0 \in X$.
4. Show that F is lower semicontinuous at x_0. $(F(x_0) \leq \liminf_{n_j\to\infty} F(x_{n_j}))$

i.e. x_0 is a minimizer.

Example.

$$\inf_{u\in A} \left\{ F = \frac{1}{2} \int_\Omega |\nabla u|^2 dx \right\} \tag{P}$$

Ω : open, bounded set in $\mathbb{R}^1$.
$A = \{u \in W^{1,2}(\Omega) | u = 0 \text{ on } \partial\Omega\}$

1. F is coercive.

$$\|u\|_{W^{1,2}(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$$
$$\|u\|_{W^{1,2}(\Omega)} \leq c|\nabla u|_{L^2(\Omega)} \qquad \text{(Poincaré)}$$

2. Construct a minimizing sequence $\{u_n\}$. We may assume

$$F(u_n) \leq m+1 \qquad (m = \inf F).$$

3. We can extract a weakly convergent subsequence of $\{u_n\}$, $u_{n_j} \rightharpoonup u_0$.

$$\begin{aligned} |\nabla u_{n_j}|^2 &= |\nabla u_0|^2 - 2\nabla u_0 \cdot (\nabla u_{n_j} - \nabla u_0) + |\nabla u_{n_j} - \nabla u_0|^2 \\ &\geq |\nabla u_0|^2 - 2\nabla u_0 \cdot (\nabla u_{n_j} - \nabla u_0) \end{aligned}$$
$$\int_\Omega |\nabla u_{n_j}|^2 dx \geq \int_\Omega |\nabla u_0|^2 dx - 2\int_\Omega \nabla u_0 \cdot (\nabla u_{n_j} - \nabla u_0) dx$$

Since $\nabla u_0 \in L^2(\Omega)$, $\nabla u_{n_j} - \nabla u_0 \rightharpoonup 0$ in L^2. By weak convergence in L^2,

$$\lim_{n_j \to \infty} \int_\Omega \nabla u_0 (\nabla u_{n_j} - \nabla u_0) dx = 0$$
$$\liminf_{n_j \to \infty} \int_\Omega |\nabla u_{n_j}|^2 dx \geq \int_\Omega |\nabla u_0|^2 dx$$
$$\liminf_{n_j \to \infty} F(u_{n_j}) \geq F(u_0)$$

i.e. F is lower semicontinuous at u_0.

Combining 1-3, u_0 is a minimizer of (P).

Euler-Lagrange equation

$$\begin{aligned} \frac{d}{d\varepsilon} F(u+\varepsilon v)|_{\varepsilon=0} &= \frac{d}{d\varepsilon} \int_\Omega \frac{1}{2} |\nabla(u+\varepsilon v)|^2 dx|_{\varepsilon=0} \\ &= \int_\Omega \nabla u \cdot \nabla v dx \end{aligned}$$
$$\frac{d}{d\varepsilon} F(u+\varepsilon v)|_{\varepsilon=0} = 0, \qquad \forall v \in A$$
$$\int_\Omega \nabla u \cdot \nabla v dx = 0.$$

By Hölder inequality,

$$\int_\Omega \nabla u \cdot \nabla v dx \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} < \infty$$
$$F(u+v) - F(u) = \int_\Omega \nabla u \nabla v dx + \frac{1}{2} \int_\Omega |\nabla v|^2 dx \geq F'(u)v.$$

With equality if and only if $v = 0$. i.e. F is strictly convex.

u_0 is a minimizer of (P). Suppose $u \in A$ is a minimizer. Let $y := u - u_0$

$$\begin{aligned} F(y+u_0) - F(u_0) &\geq F'(u_0)y \qquad \text{with equality if and only if } y = 0 \\ F(u) - F(u_0) &\geq F'(u_0)y \end{aligned}$$

u_0 is the unique minimizer.

Let Ω be bounded open set in $\mathbb{R}^n$.
Let $f \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. $f = f(x, u, \xi)$ satisfies

H1) $(u, \xi) \longrightarrow f(x, u, \xi)$ is convex for all $x \in \bar{\Omega}$

H2) there is $p > 1$, $\alpha_1 > 0$, $\alpha_2 \in \mathbb{R}$ such that

$$f(x, u, \xi) \geq \alpha_1 |\xi|^p + \alpha_2, \ (x, u, \xi) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$$

H3) there exists a constant $\beta > 0$ such that for every $(x, u, \xi) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$

$$|f_u(x, u, \xi)|, \ |f_\xi(x, u, \xi)| \leq \beta(1 + |u|^{p-1} + |\xi|^{p-1})$$
$$\left(f_\xi = (f_{\xi_1}, f_{\xi_2}, \ldots, f_{\xi_n}), \ f_{\xi_i} = \frac{\partial f}{\partial \xi_i}, \ f_u = \frac{\partial f}{\partial u} \right).$$

LEMMA. Consider:

$$\inf_{u \in A} \left\{ F(u) = \int_\Omega f(x, u, \nabla u) dx \right\} = m \tag{Q}$$

$A = \{u \in W^{1,p} | u = u_0 \text{ on } \partial\Omega\}$, where $u_0 \in W^{1,p}$, $F(u_0) < +\infty$. i.e. $u - u_0 \in W_0^{1,p}(\Omega)$. Then there exists a minimizer u of (Q)
Proof. Step1. By assumption $-\infty < m \leq F(u_0) < +\infty$. Let $\{u_n\} \in A$ be a minimizing sequence.

$$\lim_{n \to \infty} F(u_n) = \inf F(u) = m$$
$$m + 1 \geq F(u_n) \geq \alpha_1 \|\nabla u_n\|_{L^p}^p - |\alpha_2||\Omega|, \ \alpha_1 > 0, \ \alpha_2 \in \mathbb{R}$$
$$\therefore \|\nabla u_n\|_{L^p} \leq \alpha_3.$$

By Poincaré inequality, $\exists \alpha_4, \ \alpha_5$ such that

$$\alpha_3 \geq \|\nabla u_n\|_{L^p} \geq \alpha_4 \|u_n\|_{W^{1,p}} - \alpha_5 \|u_0\|_{W^{1,p}}.$$

There exists α_6 such that

$$\|u_n\|_{W^{1,p}} \leq \alpha_6.$$

By reflexity of $W^{1,p}$ there is a subsequence $\{u_n\}$ converging weakly to $\bar{u} \in A$.

Step2. By convexity of $f(x, u, \xi)$,

$$f(x, u_n, \nabla u_n) \geq f(x, \bar{u}, \nabla \bar{u}) + f_u(x, \bar{u}, \nabla \bar{u})(u_n - \bar{u}) + \langle f_\xi(x, \bar{u}, \nabla \bar{u}); \nabla u_n - \nabla \bar{u} \rangle . \tag{3}$$

We want to show that $f_u, f_\xi \in L^{p'}, \frac{1}{p} + \frac{1}{p'} = 1$. Using (H3)

$$\begin{aligned}\int_\Omega |f_u|^{p'} dx &\leq \gamma_1(1 + |u|^{p-1} + |\xi|^{p-1})^{\frac{p}{p-1}} \\ &\leq \gamma_2(1 + \|u\|^p_{W^{1,p}}) < +\infty.\end{aligned}$$

Similarly, $f_\xi \in L^{p'}$. By Hölder, $f_u(x, \bar{u}, \nabla\bar{u})(u_n - \bar{u}) \in L^1$, $f_\xi(x, \bar{u}, \nabla\bar{u})(\nabla u_n - \nabla\bar{u}) \in L^1$. By integrating (3),

$$F(x, u_n, \nabla u_n) \geq \int_\Omega f(x, \bar{u}, \nabla\bar{u}) + f_u(x, \bar{u}, \nabla\bar{u})(u_n - \bar{u}) + \langle f_\xi(x, \bar{u}, \nabla\bar{u}); \nabla u_n - \nabla\bar{u}\rangle\, dx.$$

Since $u_n - \bar{u} \rightharpoonup 0$ weakly in $W^{1,p}$

$$\lim_{n\to\infty} \int_\Omega f_u(u_n - \bar{u}) = \lim_{n\to\infty} \int \langle f_\xi(x, \bar{u}, \nabla\bar{u}); \nabla u_n - \nabla\bar{u}\rangle = 0$$
$$\therefore F(x, u_n, \nabla u_n) \geq F(x, \bar{u}, \nabla\bar{u}).$$

So, F is lower-semicontinuous.

Combing the two steps, since $\{u_n\}$ is a minimizing sequence of (Q) and for such a sequence we have lower semicontinuity, we deduce that there is a minimizer $\bar{u} \in A$.

We prove that for $u, \phi \in W^{1,p}(\Omega), \varepsilon \in \mathbb{R}$

$$\lim_{\varepsilon\to 0} \frac{F(u + \varepsilon\phi) - F(u)}{\varepsilon} = \int_\Omega (f_u(x, u, \nabla u)\phi + \langle f_\xi(x, u, \nabla u); \nabla\phi\rangle)\, dx$$
$$f(x, u, \xi) = f(x, 0, 0) + \int_0^1 \frac{d}{dt} f(x, tu, t\xi) dt \quad , (x, u, \xi) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n.$$

By (H3)

$$|f(x, u, \xi)| \leq \gamma_1(1 + |u|^p + |\xi|^p).$$

Define $g(x, \varepsilon) := f(x, u + \varepsilon\phi, \nabla u + \varepsilon\nabla\phi)$, then $F(u + \varepsilon\phi) = \int_\Omega g(x, \varepsilon) dx$. Since f is C^1, for a.e.$x \in \Omega$, $\varepsilon \longmapsto g(x, \varepsilon)$ is C^1.

There exists $\theta \in [-|\varepsilon|, |\varepsilon|]$, $\theta = \theta(x)$ such that $g(x, \varepsilon) - g(x, 0) = g_\varepsilon(x, \theta)\varepsilon$ where

$$g_\varepsilon(x, \theta) = f_u(x, u + \theta\phi, \nabla u + \theta\nabla\phi)\phi + \langle f_\xi(x, u + \theta\phi, \nabla u + \theta\nabla\phi); \nabla\phi\rangle$$
$$\left|\frac{g(x, \varepsilon) - g(x, 0)}{\varepsilon}\right| = |g_\varepsilon(x, \theta)| \equiv G(x)$$
$$\leq \gamma_2(1 + |u|^p + |\varphi|^p + |\nabla u|^p + |\nabla\phi|^p)$$

$G(x) \in L^1$, $g(x, \varepsilon) \in L^1$, $g(x, 0) \in L^1$

$$\frac{g(x, \varepsilon) - g(x, 0)}{\varepsilon} \longrightarrow g_\varepsilon(x, 0) \quad \text{a.e.in } \Omega$$

using Dominated convergence theorem

$$\lim_{\varepsilon\to 0}\frac{F(u+\varepsilon\phi)-F(u)}{\varepsilon}=\int_\Omega (f_u(x,u,\nabla u)\phi+\langle f_\xi(x,u,\nabla u);\nabla\phi\rangle)dx.$$

Anisotropic diffusions In scale-space theory, an image u_0 is embeded into an evolution process, denoted by $u(t,\cdot)$ such that $\{u(t,\cdot)\}_{t\geq 0}$ is a family of gradually smoother versions of the initial image u_0.

As t increases, $u(t,\cdot)$ changes into a more and more simplied image without creating spurions structures.

"Axioms and fundamental equations in image processing" Alvarez, Guichard, Lions and Moral.

The scale-spaces are governed by PDEs with the original image u_0 as initial condition.

Ω: bounded, open subset in $\mathbb{R}^2$.

Consider

$$\begin{cases}\partial_t u+F(x,u(x,t),\nabla u(x,t),\nabla^2 u(x,t))=0 & (0,T)\times\Omega\\ \frac{\partial u}{\partial N}=0 \quad (0,T)\times\partial\Omega\\ u(x,0)=u_0.\end{cases}$$

Example. The heat equation in 2-D

$$\begin{cases}u_t=\Delta u\\ u(x,0)=u_0(x)\end{cases}$$

$$u(x,t)=\begin{cases}u_0 & t=0\\ G_{\sqrt{2t}}*u_0(x) & t>0\end{cases}$$

, where $G_\sigma(x)=\dfrac{1}{2\pi\sigma^2}e^{-\frac{|x|^2}{2\sigma^2}}$ in $\mathbb{R}^2$, $G_{\sqrt{2t}}*u_0=\displaystyle\int_{\mathbb{R}^2}G_{\sqrt{2t}}(x-y)u_0(y)dy.$

This gives correspondence between the time variable t and the scale parameter σ of the Gaussian Kernal G.

Adaptive smoothing attempt to smooth u_0 while preserving the image features (edges, lines, textures,...).

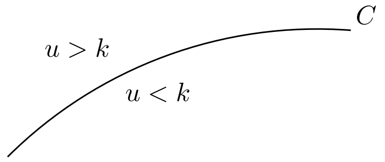
1) Control the amount of smoothing i.e. less smoothing at the locations with strong image features, and more smoothing at the locations with weak image features.

2) Control the direction of smoothing i.e. less smoothing in the direction across the image features, and more smoothing in the direction along the image features.

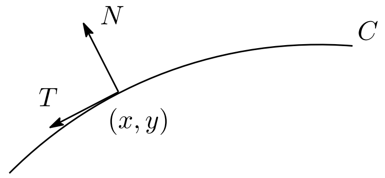
Suppose $u(x,y)$ is regular and let C be a level contour of u

$$C=\{(x,y)|u(x,y)=k,\ k\in[0,255]\}.$$

C separates the two homogeneous regions $\{(x,y)|u(x,y)>k\} \cup \{(x,y)|u(x,y)<k\}$.



For (x,y) with $|\nabla u(x,y)| \neq 0$, $N = \frac{\nabla u}{|\nabla u|}$, $T \perp N$, $|T| = 1$.



Denoted by u_{NN} and u_{TT} respectively, the second derivatives of u in the direction of T and N are

$$u_{TT} = T^t D^2 u T = \frac{u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy}}{|\nabla u|^2}$$

$$u_{NN} = N^t D^2 u N = \frac{u_x^2 u_{xx} + u_y^2 u_{yy} + 2u_x u_y u_{xy}}{|\nabla u|^2}$$

$$D^2 u = \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix}; \quad \text{Hessian matrix}$$

$$u_{TT} + u_{NN} = \Delta u \quad \text{isotropic}$$

$$\kappa = \text{mean curvarute} = \frac{u_{TT}}{|\nabla u|}.$$

Consider

$$\min F(u) = \frac{1}{2}\int_\Omega |u_0 - Ru|^2 dx + \lambda \int_\Omega \phi(|\nabla u|) dx.$$

Euler-Lagrange equation

$$R^* Ru - \lambda \text{div}\left(\frac{\phi'(|\nabla u|)\nabla u}{|\nabla u|}\right) = R^* u_0.$$

$\text{div}\left(\frac{\phi'(|\nabla u|)\nabla u}{|\nabla u|}\right) = \frac{\phi'(|\nabla u|)}{|\nabla u|} u_{TT} + \phi''(|\nabla u|) u_{NN} = \alpha u_{TT} + \beta u_{NN}$: decompose it into the weighted sum of u_{NN} and u_{TT}.

1) At locations where $|\nabla u|$ is small, isotropic smoothing is preferred. i.e. we impose

$$\phi'(0) = 0, \lim_{s \to 0^+} \frac{\phi'(s)}{s} = \lim_{s \to 0^+} \phi''(s) = \phi''(0) = m > 0.$$

2) At location where $|\nabla u|$ is large, diffusion only in the direction of T is preferred. i.e. we impose

$$\lim_{s\to\infty} \phi''(s) = 0, \quad \lim_{s\to\infty} \frac{\phi'(s)}{s} = n > 0.$$

But these two conditions are incompatible.

We need to compromise

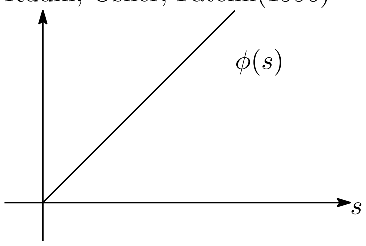
$$\lim_{s\to\infty} \phi''(s) = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{\phi'(s)}{s} = 0, \lim_{s\to\infty} \frac{\phi''(s)}{\phi'(s)/s} = 0.$$

In order to use the direct method of the calculus of variation, we suppose that $\lim\limits_{s\to\infty} \phi(s) = +\infty$. The growth to ∞ must not be too strong because we don't want to penalize strong gradient (edges, lines, ...). Suppose ϕ has a linear growth at ∞, i.e. there are $a_i > 0$, b_i such that $a_1 s + b_1 < \phi(s) < a_2 s - b_2$. The solution space $v = \{u \in L^2(\Omega),\ \nabla u \in L^1(\Omega)\}$.

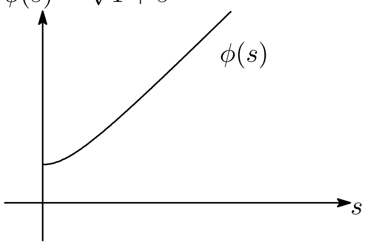
Example of $\phi(s)$ with linear growth at ∞.

1. $\phi(s) = s$

   Rudin, Osher, Fatemi(1990)



2. $\phi(s) = \sqrt{1+s^2}$



Remark. A better edge preservation behaving of ϕ would be

$$\lim_{s\to\infty} \phi(s) \approx c > 0, \quad \lim_{s\to 0} \phi(s) = s^2.$$

But ϕ is quadratic near 0 so it has a nonconvex shape. However we don't have existence of a minimizer.

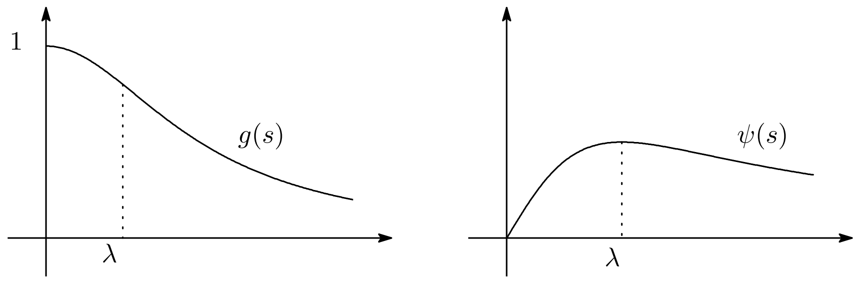
Example of nonconvex ϕ.

$$\phi(s) = \frac{s^2}{1+s^2}$$

Numerically nonconvex ϕ yields better (sharper edges) than the convex potential.

Perona, Malik (1990) proposed a nonlinear diffusion method which decreases the diffusivity function g at the locations with high likelihood to be edges.
g: nonnegative, monotonically decreasing, $g(0) = 1$, $\lim_{s\to+\infty} g(s) = 0$.
Example of g.

$$g(s) = \frac{1}{1+(s/\lambda)^2} \qquad ,\lambda > 0$$
$$\psi(s) = \frac{s}{1+(s/\lambda)^2} = g(s)s;\ \text{flux function}$$
$$\psi'(s) \geq 0\ (s \leq \lambda),\ \psi'(s) < 0\ (s > \lambda).$$



Consider

$$\begin{aligned} u_t &= \mathrm{div}(g(|\nabla u|^2)\nabla u) \\ &= 2(u_{xx}u_x^2 + u_{yy}u_y^2 + 2u_xu_yu_{xy})g'(|\nabla u|^2) + g(|\nabla u|^2)(u_{xx}+u_{yy}) \\ &= 2|\nabla u|^2 g'(|\nabla u|^2)u_{NN} + g(|\nabla u|^2)(u_{TT}+u_{NN}) \\ &= g(|\nabla u|^2)u_{TT} + [g(|\nabla u|^2) + 2|\nabla u|^2 g'(|\nabla u|^2)]u_{NN} \\ &= g(|\nabla u|^2)u_{TT} + \psi'(|\nabla u|^2)u_{NN} \end{aligned}$$

Define $b(s) = g(s) + 2sg'(s)$. Then $u_t = g(|\nabla u|^2)u_{TT} + b(|\nabla u|^2)u_{NN}$.

The Perona-Malik is of forward parabolic if $|\partial_x u| \leq \lambda$ and of backward parabolic if $|\partial_x u| > \lambda$. For nonmonotone flux, there is no mathematical theory which guarantees the well-posedness.

Kichenassamy (1997)

1-D backward heat equation

$$\begin{cases} u_t(t,x) = -u_{xx} & \text{on } (0,T)\times\mathbb{R} \\ u(0,x) = u_0(x). \end{cases} \tag{4}$$

Change of variable $\tau = T - t$, let $v(\tau, x) = u(T - t, x)$

$$\begin{cases} v_\tau(\tau,x) = v_{xx}(\tau,x) & \text{on } (0,T)\times\mathbb{R} \\ v(T,x) = u_0(x). \end{cases} \tag{5}$$

If $u(t,x)$ solves (4), $v(\tau,x) = u(T-t,x)$ solves (5).

But according to the regularizing property of the heat equation $u_0(x)$ should necessarily be infinitely differentiable. If not, (4) does not have a (weak) solution. If the initial data $u_0(x)$ is not infinitely differentiable, then there is no weak solution. "The Perona-Malik Paradox".

Catte'et.al.

$$\begin{cases} u_t = \operatorname{div}(|\nabla G_\sigma * u|^2 \nabla u) \\ u(0,x) = u_0(x). \end{cases} \tag{6}$$

They proved that (6) is well-posed.
Existence. Sauder's fixed point theorem.

Hessian method

It uses higher order derivatives to extract image feature direction i.e. it considers the direction of the maximal second derivative to be the direction across the image features,

$$\begin{gathered} \nabla u^T \cdot v \qquad , v = \nabla u \\ H = \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix} : \text{Hessian matrix} \\ \max_v v^T H v \qquad \text{s.t. } v^T v = 1 \\ \max_v v^T H v + \lambda v^T v, \ \lambda\text{: Lagrange multiplier} \\ Hv + \lambda v = 0 \\ Hv = -\lambda v. \end{gathered}$$

So the eigenvector corresponding to the largest (in magnitude) eigenvalue is considered to be the direction across the image features.

The two eigenvalues of H denoted by λ_1 and λ_2 are given by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[(u_{xx}+u_{yy}) + \sqrt{(u_{xx}-u_{yy})^2 + 4(u_{xy})^2}\right] \\ \lambda_2 &= \frac{1}{2}\left[(u_{xx}+u_{yy}) - \sqrt{(u_{xx}-u_{yy})^2 + 4(u_{xy})^2}\right]. \end{aligned}$$

Let $\lambda_N = \max\{|\lambda_1|, |\lambda_2|\}$, $\lambda_T = \min\{|\lambda_1|, |\lambda_2|\}$.

Let T denote the eigenvector corresponding to λ_T.
Let N denote the eigenvector corresponding to λ_N.
Note: $u_{NN} = \lambda_N,\ u_{TT} = \lambda_T$
The Hessian method: $u_t = \alpha\lambda_T + \beta\lambda_N$
$\alpha,\ \beta > 0$: constants, controlling the unevenness of the smoothing between the T and N direction

Applications of eigenvalues of the Hessian.
"Multiscale blood vessel enhancement filtering" Fragi et.al.
"Exploiting the Hessian matrix for context-based retrieval of volume-data-features" Hladuvka et.al.
The Gabor method. The *Gabor method* extras the image feature directions using space-frequency analysis. The *Gabor transform* of a given image u(x,y) is given by

$$\mathcal{F}(x, y; \omega, \theta) = \int u(r, s)G_\sigma(r - x, s - y)\Omega(r, s, \omega, \theta)drds.$$

G_σ: Gaussian Kernel, $\Omega(r, s, \omega, \theta) = e^{-i\omega C(r\cos\theta + s\sin\theta)}$

$\mathcal{F}(\omega, \theta)$ is the response to the frequence ω is the direction θ. It has information about oscillations at all frequences in all directions in the neighborhood of the location (x, y).

The image feature direction can be found using the spectral energy, the squared magnitude of the $\mathcal{F}(\omega, \theta)$. The spectral energy is accumulated in all directions

$$\theta_N = \operatorname*{argmax}_\theta\{\int |\mathcal{F}(\omega, \theta)|^2 P(\omega)\omega d\omega\}$$

θ: direction with maximum spectral energy.
$P(\omega)$: weight function e.g. $P(\omega) = \omega^3$ discontinuity decays at $\frac{1}{\omega^2}$.
$\arg(T)\perp\theta_N$.
Weickert's approach. Fick's law

$$j = -D \cdot \nabla u.$$

j: flux, ∇u: concentration gradient. D: diffusion tensor, a symmetric positive definite matrix.

The continuity equation says diffusion only transports mass without destroying it or creating new one. Diffusion process tries to equilibrate concentration differences.

$$u_t = -\operatorname{div} j$$
$$u_t = \operatorname{div}(D \cdot \nabla u)$$

In certain applications, it is desirable to smooth along the local coherence direction (enhance local coherence), e.g. finger prints, images with oriented flow-like structures.

Weickert: consider the vector-valued structure descriptor ∇u_σ within a matrix frame work

$$\nabla u_\sigma \otimes \nabla u_\sigma = \nabla u_\sigma \cdot \nabla u_\sigma^t, \qquad u_\sigma = G_\sigma * u.$$

It is symmetric positive definite. This has an eigenbasis consisting of v_1 and v_2 with $v_1 \parallel u_\sigma$ and $v_2 \perp \nabla u_\sigma$, The corresponding eigenvalues are $\mu_1 = |\nabla u_\sigma|^2$ and $\mu_2 = 0$. Structure tensor denoted by $J_\sigma(\nabla u_\sigma)$.

$$J_\sigma(\nabla u_\sigma) = G_\sigma * (\nabla u_\sigma \otimes \nabla u_\sigma) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$$

$*$: componentwise convolution

$J_\sigma(\nabla u_\sigma)$ is symmetric positive definite. It has eigenvalues μ_1 and μ_2 such that

$$\mu_{1,2} = \frac{j_{11} + j_{22} \pm \sqrt{(j_{11} - j_{22})^2 + 4j_{12}^2}}{2}, \qquad \mu_1 \le \mu_2$$

μ_1 and μ_2 measure the average gray value variation in the eigendirections. v_1 is the direction with the largest gray value fluctuations. v_2 is the local coherence direction. Then consider $(\mu_1 - \mu_2)^2$: coherence measure

$$\begin{aligned} \mu_1 &\approx \mu_2 : \text{constant regions} \\ \mu_1 &\gg \mu_2 = 0 : \text{straight lines} \\ \mu_1 &\ge \mu_2 \gg 0 : \text{corners.} \end{aligned}$$

In order to enhance the local coherence structures, smooth mainly along the coherence direction v_2 with diffusivity λ_2. Constant D such that it has the same eigenvectors v_1, v_2 as $J_\sigma(\nabla u_\sigma)$ and choose its corresponding eigenvalues λ_1 and λ_2 as

$$\begin{aligned} \lambda_1 &= \alpha \\ \lambda_2 &= \begin{cases} \alpha & \text{if} \quad \mu_1 \approx \mu_2 \\ \alpha + (1-\alpha) e^{\frac{1}{|\mu_1 - \mu_2|^2}} & \text{otherwise}, \end{cases} \qquad \alpha \in (0,1). \end{aligned}$$

Finite difference methods for solving 1-D heart equation $u_t = u_{xx}$. Look for pure exponential solution

$$\begin{aligned} u(x,t) &= G(t) e^{ikx} \\ G' e^{ikx} &= -k^2 G e^{ikx} \\ G &= e^{-k^2 t} \\ u(x,t) &= e^{-k^2 t} e^{ikx}. \end{aligned}$$

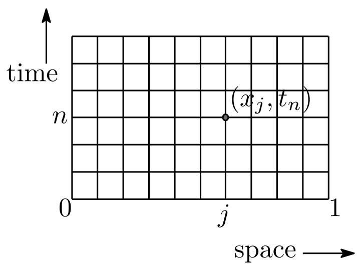
Discontinuities are immediately smoothed out. Energy decays. $E = \int |u|^2 dx$.

Consider

$$\begin{cases} v_t(t,x) = v_{xx}(t,x) \\ v(0,x) = v_0(x) \\ v(t,0) = v(t,1) = 0. \end{cases}$$

A finite difference method typically involves the following steps.

1. Generate a grid.
2. Substitute the derivatives in the PDE with finite difference schemes. It becomes a system of algebraic equations.
3. Solve the system of algebraic equations.
4. Do error analysis.



m : number of intervals in the spatial domain

$\Delta x : \frac{1}{m}$, $\quad \Delta t$: time step

$x_j = j\Delta x, \quad j = 0, 1, 2, \ldots, m$

$t_n = n\Delta t, \quad n = 0, 1, \ldots$

Taylor expansion of v about (x_j, t_n)

$$v(t_n + \Delta t, x_j) = v(t_n, x_j) + \Delta t \frac{\partial v}{\partial t}(t_n, x_j) + \frac{\Delta t^2}{2}\frac{\partial^2 v}{\partial t^2}(t_n, x_j) + O(\Delta t^3)$$

$$v(t_n, x_j + \Delta x) = v(t_n, x_j) + \Delta x \frac{\partial v}{\partial x}(t_n, x_j) + \frac{\Delta x^2}{2}\frac{\partial^2 v}{\partial x^2}(t_n, x_j) + O(\Delta x^3)$$

$$v(t_n, x_j - \Delta x) = v(t_n, x_j) - \Delta x \frac{\partial v}{\partial x}(t_n, x_j) + \frac{\Delta x^2}{2}\frac{\partial^2 v}{\partial x^2}(t_n, x_j) + O(\Delta x^3)$$

Approximate the derivatives using the following finite differences.

Forward (time) finite difference : $\dfrac{u_j^{n+1} - u_j^n}{\Delta t} \approx \dfrac{\partial u}{\partial t}$

Forward (domain) finite difference : $\dfrac{u_{j+1}^n - u_j^n}{\Delta x} \approx \dfrac{\partial u}{\partial x}$

Backward finite difference : $\dfrac{u_j^n - u_{j-1}^n}{\Delta t} \approx \dfrac{\partial u}{\partial x}$

Central finite difference : $\dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \approx \dfrac{\partial u}{\partial x}$

1-D heat equation can be approximated by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$: Forward in time Central in space (FTCS).

Approximate $\frac{\partial^2 u}{\partial x^2}$

$$\text{Central finite difference}: \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \approx \frac{\partial^2 u}{\partial x^2}$$

Local truncation error: the difference of the differential equation and the finite difference scheme

$$\tau_n = u_j^n - v(t_n, x_j)$$

plugging in the analytic solution into the finite difference scheme.
Example: τ_n for $v_t = v_{xx}$, FTCS

$$\frac{v(t_n + \Delta t, x_j) - v(t_n, x_j)}{\Delta t} = \frac{v(t_n, x_j + \Delta x) - 2v(t_n, x_j) + v(t_n, x_j - \Delta x)}{\Delta x^2}$$

$$\frac{v(t_n, x_j) + \Delta t \frac{\partial v}{\partial t}(t_n, x_j) + \frac{\Delta t^2}{2}\frac{\partial^2 v}{\partial t^2}(t_\xi, x_j) - v(t_n, x_j)}{\Delta t}$$

$$= \frac{1}{\Delta x^2}\Bigg[v(t_n, x_j) + \Delta x \frac{\partial v}{\partial x}(t_n, x_j) + \frac{\Delta x^2}{2}\frac{\partial^2 v}{\partial x^2}(t_n, x_j) + \frac{\Delta x^3}{6}\frac{\partial^3 v}{\partial x^3}(t_n, x_j) + \frac{\Delta x^4}{24}\frac{\partial^4 v}{\partial x^4}(t_n, x_\eta) - 2v(t_n, x_j) + v(t_n, x_j) - \Delta x \frac{\partial v}{\partial x}(t_n, x_j) + \frac{\Delta x^2}{2}\frac{\partial^2 v}{\partial x^2}(t_n, x_j) - \frac{\Delta x^3}{6}\frac{\partial^3 v}{\partial x^3}(t_n, x_j) + \frac{\Delta x^4}{24}\frac{\partial^4 v}{\partial x^4}(t_n, x_\varphi)\Bigg]$$

for some $t_\xi \in (t_n, t_{n+1}),\ x_\eta \in (x_j, x_{j+1}), x_\varphi \in (x_{j-1}, x_j)$

$$\frac{1}{\Delta t}\left[\Delta t \frac{\partial v}{\partial t}(t_n, x_j) + \frac{\Delta t^2}{2}\frac{\partial^2 v}{\partial t^2}(t_\xi, x_j)\right]$$

$$= \frac{1}{\Delta x^2}\left[\Delta x^2 \frac{\partial^2 v}{\partial x^2}(t_n, x_j) + \frac{\Delta x^4}{24}\frac{\partial^4 v}{\partial x^4}(t_n, x_\eta) + \frac{\Delta x^4}{24}\frac{\partial^4 v}{\partial x^4}(t_n, x_\varphi)\right]$$

$$\tau_n(\Delta t, \Delta x) = \frac{\Delta t}{2}\frac{\partial v}{\partial t}(t_\xi, x_j) - \frac{\Delta x^2}{24}\frac{\partial^4 v}{\partial x^4}(t_n, x_\eta) - \frac{\Delta x^2}{24}\frac{\partial^4 v}{\partial x^4}(t_n, x_\varphi)$$

$$\lim_{\substack{\Delta t \to 0 \\ \Delta x^2 \to 0}} \tau_n = 0$$

We say that a finite different scheme is consistent if the local truncation error tends to 0 as Δt and Δx approach 0. The FTCS is of order $(\Delta t, \Delta x^2)$.

Stability of finite difference schemes.
It is concerned with the growth of errors introduced at each time step.
The Von Neumann criterion: to analyze the stability of a finite difference scheme, plugging in the discrete Fourier mode $\xi^n e^{ikj\Delta x}$ into the finite difference scheme. The necessary condition for the stability is $|\xi| \le 1$.

Example:

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2},\quad r=\frac{\Delta t}{\Delta x^2}$$
$$u_j^{n+1}-u_j^n=r(u_{j+1}^n-2u_j^n+u_{j-1}^n)$$
$$\xi^{n+1}e^{ikj\Delta x}-\xi^n e^{ikj\Delta x}=r\left(\xi^n e^{ik(j+1)\Delta x}-2\xi^n e^{ikj\Delta x}+\xi^n e^{ik(j-1)\Delta x}\right)$$
$$\xi^n e^{ikj\Delta x}(\xi-1)=r\xi^n e^{ikj\Delta x}\left(e^{ik\Delta x}-2+e^{ik\Delta x}\right)$$
$$\xi-1=r(2\cos k\Delta x-2)=-4r\sin^2\frac{k\Delta x}{2}$$
$$\xi=1-4r\sin^2\frac{k\Delta x}{2},\ |\xi|\le 1\Rightarrow r\le\frac{1}{2}\text{ or }\Delta t\le\frac{\Delta x^2}{2}$$

$\therefore$ we cannot take a big time step.

Implicit finite difference scheme for solving $u_t=u_{xx}$

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}$$

implicit FTCS of $(\Delta t,\Delta x^2)$.

Explicit FTCS

$n+1$ • $u(t_{n+1},x_j)$

n • • •

Implicit FTCS

$n+1$ • • • $u(t_{n+1},x_j)$

n •

$$u_j^{n+1}-u_j^n=r(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1})$$
$$(2r+1)u_j^{n+1}-ru_{j+1}^{n+1}-ru_{j-1}^{n+1}=u_j^n$$

In matrix form

$$\begin{bmatrix} 2r+1 & -r & & & & \\ -r & 2r+1 & -r & & \text{\Large 0} & \\ & \ddots & \ddots & \ddots & & \\ \text{\Large 0} & & -r & 2r+1 & -r \\ & & & -r & 2r+1 \end{bmatrix}\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{m-1}^{n+1}\end{bmatrix}=\begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{m-1}^n\end{bmatrix}$$

It's a tridiagonal system.

Implicit finite difference scheme for 1-D heat equation $u_t=u_{xx}$.

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}$$

$u_j^n=u(t_n,x_j)$, $x_j=j\cdot\Delta x$, $j=0,1,\ldots,m$, $t_n=n\cdot\Delta t$

To update u_j^{n+1}, we need to solve a tridiagonal system.

To check the stability, we plug in the discrete *Fourier* mode $\xi^n e^{ikj\Delta x}$ into the finite difference scheme.

$$\frac{\xi^{n+1}e^{ikj\Delta x} - \xi^n e^{ikj\Delta x}}{\Delta t} = \frac{\xi^{n+1}e^{ik(j+1)\Delta x} - 2\xi^{n+1}e^{ikj\Delta x} + \xi^{n+1}e^{ik(j-1)\Delta x}}{\Delta x^2}, \ r = \frac{\Delta t}{\Delta x^2}$$

$$\xi^n e^{ikj\Delta x}(\xi - 1) = r\xi^{n+1}e^{ikj\Delta x}(e^{ik\Delta x} - 2 + e^{-ik\Delta x})$$

$$\xi - 1 = -2r\xi \cdot 2\sin^2\frac{k\Delta x}{2}$$

$$\xi = \frac{1}{1 + 4r\sin^2\frac{k\Delta x}{2}} \qquad \Rightarrow \ |\xi| \leq 1.$$

It is unconditionally stable. No time-step restriction.

The θ-method for solving $u_t = u_{xx}$.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \theta\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}\right] + (1-\theta)\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right]$$

$\theta = 1 \Rightarrow$ fully implicit scheme, order$(\Delta t, \Delta x^2)$
$\theta = 0 \Rightarrow$ explicit scheme, order$(\Delta t, \Delta x^2)$
$\theta = \frac{1}{2} \Rightarrow$ Crank-Nicolson scheme, order$(\Delta t^2, \Delta x^2)$.

Stability analysis for Crank-Nicolson

$$\begin{aligned}\xi^{n+1}e^{ikj\Delta x} - \xi^n e^{ikj\Delta x} &= \frac{r}{2}\Big[\xi^{n+1}e^{ik(j+1)\Delta x} - 2\xi^{n+1}e^{ikj\Delta x} + \xi^{n+1}e^{ik(j-1)\Delta x} \\ &\qquad + \xi^n e^{ik(j+1)\Delta x} - 2\xi^n e^{ikj\Delta x} + \xi^n e^{ik(j-1)\Delta x}\Big]\end{aligned}$$

$$\xi^n e^{ikj\Delta x}(\xi - 1) = \frac{r}{2}\xi^n e^{ikj\Delta x}\left[\xi e^{ik\Delta x} - 2\xi + \xi e^{-ik\Delta x} + e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right]$$

$$\xi\left(1 + 2r\sin^2\frac{k\Delta x}{2}\right) = 1 - 2r\sin^2\frac{k\Delta x}{2}$$

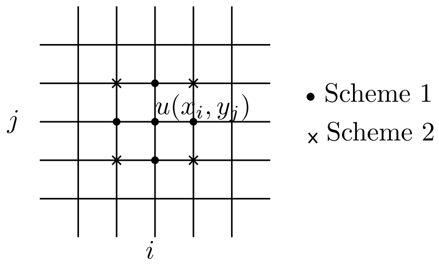
$$\xi = \frac{1 - 2r\sin^2\frac{k\Delta x}{2}}{1 + 2r\sin^2\frac{k\Delta x}{2}}$$

$|\xi| \leq 1$ Crank-Nicolson is unconditionally stable.

5-points finite difference scheme for solving $u_t = u_{xx} + u_{yy}$.

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \frac{u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n - 4u_{i,j}^n}{h^2} \qquad \text{(Scheme1)}$$

$h = \frac{1}{m}$, m : number of intervals in x and y
$u_{i,j}^n = u(t_n, x_i, y_j)$



It is of order $(\Delta t, h^2, h^2)$.

Note : The 2-D Δ is rotationally invariant. Consider a binary image representing a vertical edge and the same image after a rotation of $\frac{\pi}{4}$ radians.

case1) Scheme1 yields 1

case2) Scheme1 yields 2

$$\Delta u \approx \frac{u^n_{i+1,j+1} + u^n_{i-1,j+1} + u^n_{i+1,j-1} + u^n_{i-1,j-1} - 4u^n_{i,j}}{2h^2} \qquad \text{(Scheme2)}$$

case1) Scheme2 yields 2

case2) Scheme2 yields 1

Aubert and Kornprobst proposed to consider

$$\lambda \frac{u^n_{i+1,j} + u^n_{i-1,j} + u^n_{i,j+1} + u^n_{i,j-1} - 4u^n_{i,j}}{h^2} + (1-\lambda)\frac{u^n_{i+1,j+1} + u^n_{i-1,j+1} + u^n_{i+1,j-1} + u^n_{i-1,j-1} - 4u^n_{i,j}}{2h^2}.$$

Assume $h = 1$.
case1)

$$\lambda \cdot 1 + \frac{1-\lambda}{2} \cdot 2 \qquad (7)$$

case2)

$$\lambda \cdot 2 + \frac{1-\lambda}{2} \cdot 1 \qquad (8)$$

We want (7)=(8).

$$\lambda + (1-\lambda) = 2\lambda + \frac{1-\lambda}{2} \quad \Rightarrow \quad \lambda = \frac{1}{3}$$

9-points scheme for $u_{xx} + u_{yy}$.

$$
\begin{aligned}
\Delta u &\approx \frac{1}{3} \frac{u^n_{i+1,j} + u^n_{i-1,j} + u^n_{i,j+1} + u^n_{i,j-1} - 4u^n_{i,j}}{h^2} \\
&\qquad + \frac{2}{3} \frac{u^n_{i+1,j+1} + u^n_{i-1,j+1} + u^n_{i+1,j-1} + u^n_{i-1,j-1} - 4u^n_{i,j}}{2h^2} \\
&= \frac{1}{3h^2} \left[u^n_{i+1,j} + u^n_{i-1,j} + u^n_{i,j+1} + u^n_{i,j-1} \right. \\
&\qquad \left. + u^n_{i+1,j+1} + u^n_{i-1,j+1} + u^n_{i+1,j-1} + u^n_{i-1,j-1} - 8u^n_{i,j} \right] \\
u_t &= \operatorname{div}(g(|\nabla u|)\nabla u).
\end{aligned}
$$

Rewrite it as

$$
u_t = \frac{\partial}{\partial x}\left(g(|\nabla u|) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(g(|\nabla u|) \frac{\partial u}{\partial y} \right).
$$

Introduce

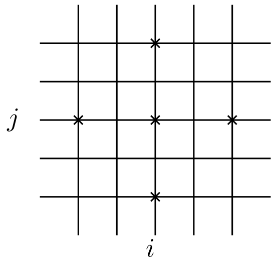
$$
\begin{aligned}
D^+_x u^n_{i,j} &= \frac{u^n_{i+1,j} - u^n_{i,j}}{h} : \text{ Forward difference} \\
D^-_x u^n_{i,j} &= \frac{u^n_{i,j} - u^n_{i-1,j}}{h} : \text{ Backward difference} \\
D^0_x u^n_{i,j} &= \frac{u^n_{i+1,j} - u^n_{i-1,j}}{2h} : \text{ Central difference.}
\end{aligned}
$$

Similarly, $D^+_y u^n_{i,j}$, $D^-_y u^n_{i,j}$, $D^0_y u^n_{i,j}$.

1. Central difference approximations

$$
\begin{aligned}
\operatorname{div}(g \cdot \nabla u) &\approx D^0_x (g_{i,j} D^0_x u^n_{i,j}) + D^0_y (g_{i,j} D^0_y u^n_{i,j}) \\
&= D^0_x \left(g_{i,j} \frac{u^n_{i+1,j} - u^n_{i-1,j}}{2h} \right) + D^0_y \left(g_{i,j} \frac{u^n_{i,j+1} - u^n_{i,j-1}}{2h} \right) \\
&= \frac{1}{2h} \left[g_{i+1,j} \frac{u^n_{i+2,j} - u^n_{i,j}}{2h} - g_{i-1,j} \frac{u^n_{i,j} - u^n_{i-2,j}}{2h} \right] \\
&\quad + \frac{1}{2h} \left[g_{i,j+1} \frac{u^n_{i,j+2} - u^n_{i,j}}{2h} - g_{i,j-1} \frac{u^n_{i,j} - u^n_{i,j-2}}{2h} \right] \\
&= \frac{1}{4h^2} \left[g_{i+1,j} u^n_{i+2,j} + g_{i-1,j} u^n_{i-2,j} + g_{i,j+1} u^n_{i,j+2} \right. \\
&\quad \left. + g_{i,j-1} u^n_{i,j-2} - (g_{i+1,j} + g_{i-1,j} + g_{i,j+1} + g_{i,j-1}) u^n_{i,j} \right].
\end{aligned}
$$

Central finite difference uses the values of u at $i \pm 2$, $j \pm 2$ and (i,j).



2. Forward and backward finite difference approximations

$$\mathrm{div} \approx D_x^+(g_{i,j}D_x^- u) + D_y^+(g_{i,j}D_y^- u)$$

$$u_t = \mathrm{div}(g|\nabla u|)\nabla u)$$

1.centered differences

Uses values of u at $((i \pm 2)h, (j \pm 2)h)$.

2.forward and backward

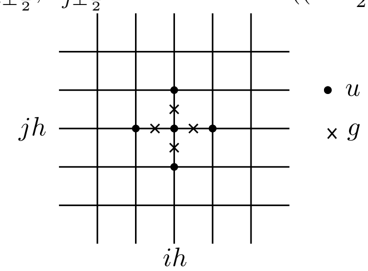
$$\begin{aligned}
&D_x^+(g_{i,j}D_x^- u_{i,j}) + D_y^+(g_{i,j}D_y^- u_{i,j}) \\
&= D_x^+\left(g_{i,j}\frac{u_{i,j}-u_{i-1,j}}{h}\right) + D_y^+\left(g_{i,j}\frac{u_{i,j}-u_{i,j-1}}{h}\right) \\
&= \frac{g_{i+1,j}\frac{u_{i+1,j}-ui,j}{h} - g_{i,j}\frac{u_{i,j}-u_{i-1,j}}{h}}{h} + \frac{g_{i,j+1}\frac{u_{i,j+1}-ui,j}{h} - g_{i,j}\frac{u_{i,j}-u_{i,j-1}}{h}}{h} \\
&= \frac{1}{h^2}[g_{i+1,j}u_{i+1,j} + g_{i,j}u_{i-1,j} + g_{i,j+1}u_{i,j+1} + g_{i,j}u_{i,j-1} \\
&\qquad\qquad - (g_{i+1,j} + 2g_{i,j} + g_{i,j+1})u_{i,j}].
\end{aligned}$$

Note. Use values of g at $((i+1)h, j)$, $(ih, (j+1)h)$, and (ih, jh).

3.introduce

$$\delta_x^* u_{i,j} = \frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{h}, \delta_y^* u_{i,j} = \frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{h}$$

$u_{i\pm\frac{1}{2}}, u_{j\pm\frac{1}{2}}$: values of u at $((i \pm \frac{1}{2})h, (j \pm \frac{1}{2})h)$



Approximate $\frac{\partial}{\partial x}(g_{i,j}\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(g_{i,j}\frac{\partial u}{\partial y})$ by

$$
\begin{aligned}
&\delta_x^*(g_{i,j}\delta_x^* u_{i,j}) + \delta_y^*(g_{i,j}\delta_y^* u_{i,j}) \\
&= \delta_x^*\left(g_{i,j}\frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{h}\right) + \delta_y^*\left(g_{i,j}\frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{h}\right) \\
&= \frac{g_{i+\frac{1}{2},j}\frac{u_{i+1,j}-u_{i,j}}{h} - g_{i-\frac{1}{2},j}\frac{u_{i,j}-u_{i-1,j}}{h}}{h} + \frac{g_{i,j+\frac{1}{2}}\frac{u_{i,j+1}-u_{i,j}}{h} - g_{i,j-\frac{1}{2}}\frac{u_{i,j}-u_{i,j-1}}{h}}{h} \\
&= \frac{1}{h^2}[g_{i+\frac{1}{2},j}u_{i+1,j} + g_{i-\frac{1}{2},j}u_{i-1,j} + g_{i,j+\frac{1}{2}}u_{i,j+1} + g_{i,j-\frac{1}{2}}u_{i,j-1} \\
&\qquad\qquad - (g_{i+\frac{1}{2},j} + g_{i-\frac{1}{2},j} + g_{i,j+\frac{1}{2}} + g_{i,j+\frac{1}{2}} + g_{i,j-\frac{1}{2}})u_{i,j}].
\end{aligned}
$$

Note: It requires interpolation for g.

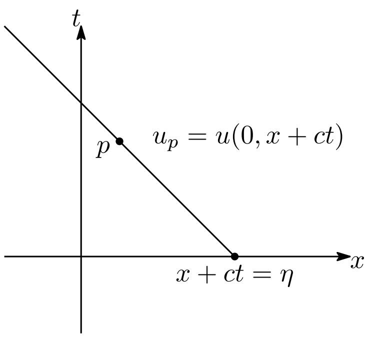
1-D advection equation $u_t = cu_x$, , c is a nonzero constant. Assume that $u(0,x) = e^{ikx}$, no boundary. $u(t,x)$ is a multiple of e^{ikx}.

$$u(t,x) = g(k,t)e^{ikx}$$

plugging in $u_t = cu_x$, $g'e^{ikx} = cike^{ikx}g$,

$$u(t,x) = e^{ikx+ickt} = e^{ik(x+ct)} = u(0,x+ct).$$

The solution travels along characteristic line $x + ct = \eta$ with speed c.



Finite difference scheme for solving $u_t = cu_x$.

Upwind scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = c\frac{u_{j+1}^n - u_j^n}{\Delta x}$$

It is of 1st order.

The Courant-Friedrichs-Leny(CFL) condition: for a stable numerical scheme, the numerical domain of dependence must contain the analytical domain of dependence.

Note: This is only necessary condition, not a sufficient condition. $c\frac{\Delta t}{\Delta x} \leq 1$ is the CFL condition for upwind scheme. Characteristic line: $x + ct = \eta$. When $t = n\Delta t$, the numerical domain of dependence reach up to $x + n\Delta x$.

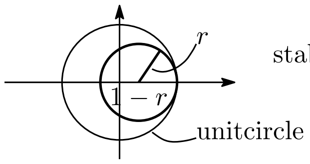
The CFL condition: $x + n\Delta x \geq x + cn\Delta t$

$$c\frac{\Delta t}{\Delta x} \leq 1$$

stability: plug in the discrete Fourier mode $\xi^n e^{ikj\Delta x}$ into $\frac{u_j^{n+1}-u_j^n}{\Delta t} = cu_{j+1}^n - u_j^n \Delta x$.

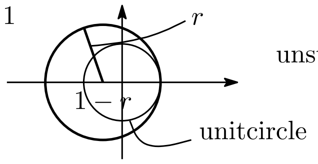
$$\xi = 1 - r + re^{ik\Delta x}$$

if $r \leq 1$



stable

if $r > 1$



unstable

Note: If $c < 0$, upwind scheme will not converge. In that case, we need to use wind down scheme instead.

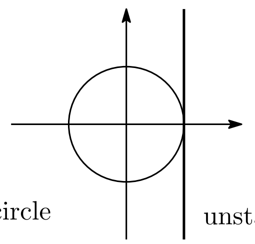
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = c\frac{u_j^n - u_{j-1}^n}{\Delta x}.$$

2.Centered difference scheme.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

stability: Amplification factor

$$\begin{aligned} \xi &= 1 + \frac{r}{2}e^{ik\Delta x} - \frac{r}{2}e^{-ik\Delta x} \\ &= 1 + ir\sin k\Delta x \end{aligned}$$



unit circle

unstable for any $r = C\frac{\Delta t}{\Delta x}$

Note: This shows that the CFL condition is only a necessary condition.

3.Lax Friedrichs

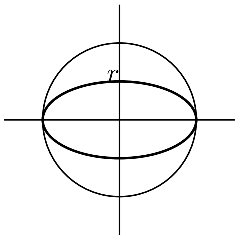
$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\Delta t} = c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

1st order scheme. stability:

$$\xi = \frac{1+r}{2}e^{ik\Delta x} + \frac{1-r}{2}e^{-ik\Delta x} = \cos k\Delta x + ir\sin k\Delta x$$
$$|\xi|^2 = \cos^2 k\Delta x + r^2 \sin^2 k\Delta x$$

Note: This scheme work for both $c > 0$ and $c < 0$.

if $r^2 \leq 1$



4.Lax Wendroff

$$u(t+\Delta t, x) = u(t,x) + \Delta t\frac{\partial u}{\partial t}(t,x) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(t,x) + \ldots$$
$$\frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} = \frac{\partial u}{\partial t}(t,x) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(t,x) + \ldots$$

Note: Some $u_t = cu_x$, $u_{tt} = c^2 u_{xx}$, use centered differences to approximate $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + c^2\frac{\Delta t}{2\Delta x^2}[u_{j+1}^n - 2u_j^n + u_{j-1}^n]$$

2nd order accurate.
stability: $u_j^n = \xi^n e^{ikj\Delta x}$

$$\xi = (1-r^2) + \frac{1}{2}(r^2+r)e^{ik\Delta x} + \frac{1}{2}(r^2-r)e^{-ik\Delta x}$$
$$= (1-r^2) + r^2\cos k\Delta x + ir\sin k\Delta x$$

Watch out when $k\Delta x$ is a multiple of π.

$$\xi = 1 - r^2 - r^2 = 1 - 2r^2$$
$$|\xi|^2 = (1-2r^2)^2 \leq 1 \Leftrightarrow r^2 \leq 1$$

It is stable if $r^2 \leq 1$.

Lax equivalence theorem: stability is equivalent to convergence, for a consistent approximation $L_h u_h$ to a well-posed linear problem $Lu = f$.

Suppose that the following conditions hold

1. Consistency. $f_h \to f$, $L_h u \to Lu$ for smooth solution u.

2. Stability. The inverse L_h^{-1} stays uniformly bounded

$$\|L_h^{-1} f_h\| \leq c_1 \|f_h\|.$$

Under these conditions, the approximate solution u_h will approach u as $h \to 0$

$$u - u_h = L_h^{-1}(L_h u - Lu) + L_h^{-1}(f - f_h) \longrightarrow 0 \text{ as } h \to 0.$$

Space of bounded variation(BV).

"Measure theory and fine properties of functions" C. Evans, R.Gariepy

"Minimal surfaces and functions of bounded variation" E.Giusti

Ω: open, bounded subset in $\mathbb{R}^N$

$C_c(\Omega; \mathbb{R}^N)$: space of continuous functions with compact support in Ω

Radon measure μ on Ω: Borel regular and $\mu(K) < \infty$ for each compact $K \subset \Omega$

Signed measure υ on Ω: υ is called a signed measure if there is a Radon measure μ on Ω and μ-summable function f such that $\upsilon(K) = \int_K f d\mu$ for all compact set $K \subset \Omega$.

Riesz representation theorem.

Let $L : C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying. $\sup\{L(\phi) | \phi \in C_c(\mathbb{R}^n; \mathbb{R}^m), \|\phi\|_{L^\infty} \leq 1, \mathrm{supp}(\phi) \subset K\} < \infty$. for each compact set $K \subset \mathbb{R}^n$

Then there exists a Random measure μ on $\mathbb{R}^n$ and μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that

i) $|\sigma(x)| = 1$ for μ a.e.x

ii) $L(\phi) = \int_{\mathbb{R}^n} \phi \cdot \sigma d\mu$ for all $\phi \in C_c(\mathbb{R}^n; \mathbb{R}^m)$.

DEFINITION. A function $u \in L^1(\Omega)$ has a *bounded variation* if

$$\sup\left\{\int_\Omega u\mathrm{div}\psi dx : \psi \in C_c^1(\Omega; \mathbb{R}^N), \psi = (\psi_1, \psi_2, \ldots, \psi_N), |\psi|_{L^\infty(\Omega)} \leq 1\right\} < \infty$$

Example: $u \in C^1(\Omega), \int_\Omega u\mathrm{div}\psi dx = -\int_\Omega \nabla u \cdot \psi dx \leq \int_\Omega |\nabla u| dx$ e.g. $\psi = \frac{\nabla u}{|\nabla u|}$

Example: $u \in W^{1,1}(\Omega), \int_\Omega u\mathrm{div}\psi dx = -\int_\Omega Du \cdot \psi dx \leq \int_\Omega |Du| dx$

Example:Let $\Omega = (-1, 1), u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$ $< Du, \psi >= \psi(0)$. The distributional derivative of u is Dirac mass at $x = 0$, which does not belong to $W^{1,1}(\Omega)$.

DEFINITION. *An* $\mathcal{L}^N$ *measurable subset* $E \subset \mathbb{R}^N$ *has* finite perimeter *if* $\chi_E \in BV(\mathbb{R}^N)$.

DEFINITION. $BV(\Omega) := \{u \in L^1(\Omega) | |Du| < \infty\}$

Structure theorem for BV functions

If $u \in BV(\Omega)$, there exists a Radon measure μ on Ω and μ-measurable function $\sigma : \Omega \to \mathbb{R}^N$ such that

i) $|\sigma(x)| = 1\mu$ a.e.x

ii) $\int_\Omega u\text{div}\psi dx = -\int_\Omega \psi \cdot \sigma d\mu$.
for all $\psi \in C_c^1(\Omega; \mathbb{R}^N)$

Let $L : C_c^1(\Omega; \mathbb{R}^N) \to \mathbb{R}$ be a linear function.

$$L(\psi) = \int_\Omega u\text{div}\psi dx, \psi \in C_c^1(\Omega; \mathbb{R}^N)$$

$$\text{since } u \in BV(\Omega), \sup \left\{ \int_\Omega u\text{div}\psi dx, |\psi|_{L^\infty(\Omega} \leq 1 \right\} = c(\Omega, u) < \infty$$

$$\text{for } \psi \in C_c^1(\Omega; \mathbb{R}^N), |L(\psi)| \leq c|\psi|_{L^\infty}$$

Let $K \subset \Omega$ be a compact subset and $\psi \in C_c^\infty(\Omega; \mathbb{R}^N)$ with $\text{supp}(\psi) \subset K$, we can find $\psi_n \to \psi$ uniformly, $\psi_n \in C_c^1(\Omega; \mathbb{R}^N)$. Define $\bar{L}(\psi) = \lim_{n\to\infty} L(\psi_n)$. This limit exists, independent of ψ_n. $\bar{L}(\psi)$ extends uniquely to

$$\bar{L}(\psi) : C_c(\Omega; \mathbb{R}^N) \to \mathbb{R}.$$

By the Riesz representation theorem, there exists a Random measure μ on Ω and μ-measurable function $\sigma : \Omega \to \mathbb{R}^N$ such that

i) $|\sigma(x)| = 1 \ \ \mu$-a.e.x

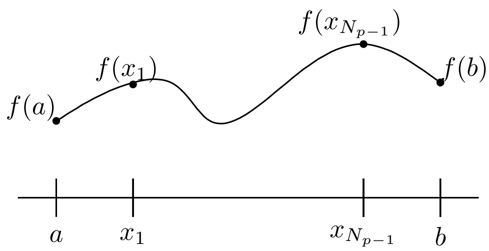
ii) $\int_\Omega u\text{div}\psi dx = -\int_\Omega \psi \cdot \sigma d\mu$, for $\psi \in C_c^1(\Omega; \mathbb{R}^N)$.

Notation. Du is a vector valued measure $\sigma d\mu$.

DEFINITION. *The* total variation *TV of a real valued function f on an interval $[a, b] \subset \mathbb{R}$ is given by*

$$TV(f) = \sup_{p\in\mathcal{P}} \left\{ \sum_{i=0}^{N_p-1} |f(x_{i+1}) - f(x_i)| \right\}$$

The supremum is taken over all $\mathcal{P} = \{p = (x_0, x_1, \ldots, x_{N_p}) : p$ is a partition of $[a, b]\}$.



If f is differentiable and integrable on $[a, b]$

$$TV(f) = \int_a^b |f'(x)|dx.$$

DEFINITION. *Let $u \in L^1_{loc}(\Omega)$. u is said to be a* function of bounded variation *if the distributional derivative is an $\mathbb{R}^N$-valued finite Radon measure in Ω.*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega \phi d D_i u, \text{ for all } \phi \in C^1_c(\Omega), i = 1, 2, \ldots, N$$
$$Du = (D_1 u, D_2 u, \ldots, D_N u)$$

Example. Suppose $E \subset \Omega$ has C^2 boundary and define the characteristic function of E by

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in \Omega \setminus E \end{cases}$$

$$\begin{aligned} \int_\Omega \chi_E \operatorname{div}\phi dx &= \int_E \operatorname{div}\phi dx \\ &= -\int_{\partial E} \phi \cdot \nu \mathcal{H}^{N-1}, \quad \nu \text{ is the outward unit normal on } \partial E \\ &\leq \int_{\partial E} \mathcal{H}^{N-1} \\ &= \mathcal{H}^{N-1}(\partial E \cap \Omega), \quad \phi \in C^1_c(\Omega) \end{aligned}$$

i.e. $\chi_E \in BV(\Omega)$

Radon-Nikodym: Let μ be a positive measure and ν be a $\mathbb{R}^N$-valued measure on$(\Omega, B(\mathbb{R}^N))$. Assume μ is a σ-finite. Then there exists a function $f \in L^1(\Omega, \mu; \mathbb{R}^N)$ and a measure ν^S, singular with respect to μ, such that $\nu = f\mu + \nu^S$.

Note.

(i) If $\nu << \mu$, then $\nu = f\mu$ for some $f \in L^1(\Omega, \mu, \mathbb{R}^N)$

(ii) Since $\nu << |\nu|$, $\nu = \sigma|\nu|$, $\quad \sigma \in L^1(\Omega, \mu; \mathbb{R}^N)$

$$|\nu| = |\sigma|\nu|| = |\sigma||\nu|, \qquad \text{thus, } |\sigma| = 1, \quad \mu \text{ a.e.}$$

$C^1_c(\Omega)$ is the space of all continuous functions with compact support. $C_0(\Omega)$ is its completion with respect to the sup-norm.

For all $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$, the set of the $\mathbb{R}^N$-valued Borel measures.

$$L_\mu(\phi) := \int_\Omega \phi d\mu = \sum_{i=1}^N \int_\Omega \phi_i d\mu_i, \ \phi_i \in L^1_{loc}(\Omega, \mu; \mathbb{R}^N)$$

L_μ is linear.

$$\begin{aligned}
\|L_\mu\| &= \sup\left\{\int_\Omega \phi d\mu,\ \phi \in C_0(\Omega, \mathbb{R}^N),\ |\phi| \leq 1\right\} \\
&= \sup\left\{\int_\Omega \phi \cdot \sigma d|\mu|,\ \phi \in C_0(\Omega, \mathbb{R}^N),\ |\phi| \leq 1\right\} \\
&= \int_\Omega d|\mu| \\
&= |\mu|(\Omega)
\end{aligned}$$

LEMMA. $|\mu|$ is lower semicontinuity of $|Du|$.
Proof. Let $\varphi \in C_c^1(\Omega; \mathbb{R}^n), |\varphi| \leq 1$. Suppose $u_n \to u$ in $L^1_{loc}(\Omega)$.

$$\begin{aligned}
\int_\Omega u \mathrm{div}\varphi = \lim_{n\to\infty} \int_\Omega u_n \mathrm{div}\varphi = - \lim_{n\to\infty} \int_\Omega \varphi \cdot \sigma_k d|Du_n| \\
\leq \liminf_{n\to\infty} |Du_n| \\
|Du| = \sup\left\{\int_\Omega u \mathrm{div}\varphi : \varphi \in C_c^1(\Omega; \mathbb{R}^N)\right\} \\
\leq \liminf_{n\to\infty} |Du_n|
\end{aligned}$$

LEMMA. The space $BV(\Omega)$ equipped with

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|$$

is a Banach space.
Proof. Let $\{u_n\}$ be a Cauchy sequence in $BV(\Omega)$. Then $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega)$. By the completeness of $L^1(\Omega)$, there is $u \in L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$. Since $\{u_n\}$ is a Cauchy sequence in $BV(\Omega)$, $|Du_n|$ is bounded. By the semicontinuity property of $|Du|$,

$$|Du| \leq \liminf_{n\to\infty} |Du_n|, \text{so } u \in BV(\Omega)$$

For $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that

$$\int |Du_p - Du_q| < \varepsilon, \quad \text{for all } p, q \geq N_\varepsilon$$

$$u_p - u_q \to u - u_q \text{ in } L^1(\Omega) \text{ as } p \to \infty.$$

By the lower semicontinuity property,

$$|D(u - u_q)| \leq \liminf_{p\to\infty} |D(u_p - u_q)| \leq \varepsilon, \quad \text{for } q \geq N_\varepsilon$$

$$\lim_{q\to\infty} |D(u - u_q)| = 0.$$

So $u_n \to u$ in $BV(\Omega)$.

DEFINITION. *Let* $M(\Omega, \mathbb{R}^N)$ *be the* topological dual *of* $C_0(\Omega, \mathbb{R}^N)$. *A sequence* $\{\mu_n\} \subset M(\Omega, \mathbb{R}^N)$ converges weakly-$*$ *to* $\mu \in M(\Omega, \mathbb{R}^N)$ *if*
$\lim\limits_{n\to\infty} \int_\Omega \varphi d\mu_n = \int_\Omega \varphi d\mu$, for all $\varphi \in C_0(\Omega, \mathbb{R}^N)$

Banach-Alaoglu: Bounded sets of the dual of a separable Banach space are sequentially relatively compact with respect to the weak-$*$ topology.

DEFINITION. weak-$*$ compactness:
If $\{\mu_n\}$ is a sequence of finite Radon measures on Ω with $\sup\{|\mu_n|(\Omega)\} < \infty$, then it has a weakly-$*$ converging subsequence $\{\mu_{n_j}\}$.

DEFINITION. convolution:
If f, g are functions defined on $\mathbb{R}^N$, *the* convolution of f and g *is defined by*

$$f * g(x) = \int_{\mathbb{R}^N} f(x-y)g(y)dy.$$

Example. If $f \in L^1$, $g \in L^p$, $p \in [1, \infty]$,

$$\|f * g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}.$$

If $f \in L^1_{loc}$, $g \in C_c$, $f * g$ is defined and it is continuous.
If $f \in L^1_{loc}$, $g \in C_c^\infty$, $f * g$ is in $C^\infty(\mathbb{R}^N)$.

$$\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g$$

DEFINITION. *A collection of functions* $\{\rho_\varepsilon\}_{\varepsilon>0}$ *is called a* mollifier *if*

$$\rho_\varepsilon(x) = \varepsilon^{-N} \rho\left(\frac{x}{\varepsilon}\right), \quad \text{where}$$

$$\rho(x) \ge 0, \ \int_{\mathbb{R}^N} \rho(x)dx = 1, \ \rho(x) = \rho(-x), \rho \in C_c^\infty(\mathbb{R}^N),$$
$$\text{for} \ \ x \in \mathbb{R}^N \ \ \text{and} \ \ \operatorname{supp} \rho \subset \overline{B(0,1)}$$

Example.

$$\rho(x) := \begin{cases} C \cdot e^{-\frac{1}{1-|x|^2}}, & \text{if } \ |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

C is chosen such that $\int_{\mathbb{R}} \rho(x)dx = 1$

DEFINITION. *Let* $u \in L^1_{loc}(\mathbb{R}^N)$. *Define*

$$u_\varepsilon = \rho_\varepsilon * u(x) = \int_{\mathbb{R}^N} \rho_\varepsilon(x-y)u(y)dy.$$

u_ε *is called the* mollification *of* u.

Properties of u_ε

i) $u_\varepsilon \in C^\infty(\mathbb{R}^N)$, $\varepsilon > 0$

ii) if u is continuous, u_ε converges uniformly to u on compact sets of $\mathbb{R}^N$

iii) if $u \in L^p(\mathbb{R}^N)$, $1 \le p < \infty$, then $u_\varepsilon \in L^p(\mathbb{R}^N)$, and $\|u_\varepsilon\|_{L^p(\mathbb{R}^N)} \le \|u\|_{L^p(\mathbb{R}^N)}$

iv) $D(\rho_\varepsilon * u) = (D\rho_\varepsilon) * u$

Analogously, we can define the convolution between ρ_ε and Radon measures. Let μ be an $\mathbb{R}^N$-valued Radon measure on Ω in $\mathbb{R}^N$

$$\rho_\varepsilon * \mu(x) = \int \rho_\varepsilon(x-y)d\mu(y) = \varepsilon^{-N} \int \rho\left(\frac{x-y}{\varepsilon}\right) d\mu(y)$$

Properties of $\rho_\varepsilon * \mu$

$$\Omega_\varepsilon = \{x \in \Omega |\, \mathrm{dist}(x, \partial\Omega) > \varepsilon\},$$

$E \subset \Omega_\varepsilon$, a Borel set;

$$I_\varepsilon(E) = \varepsilon\text{-neighborhood of } E = \{x \in \Omega |\, \mathrm{dist}(x, E) < \varepsilon\}$$
$$\int_E |\rho_\varepsilon * \mu|(x)dx \le |\mu|(I_\varepsilon(E))$$

Partition of unity

$A \subset \mathbb{R}^N$, arbitrary subset of $\mathbb{R}^N$. $\{\Omega_i\}_{i=0}^\infty$, a countable collection of open sets which cover A. A family of functions $\{\varphi_i\}_{i=0}^\infty$ is called a *partition od unity subordinate* to $\{\Omega_i\}_{i=0}^\infty$ if it satisfies,

i) $0 \le \varphi(x) \le 1$

ii) $\varphi_i \in C_c^\infty(\Omega_i)$

iii) $\sum_{i=0}^\infty \varphi_i(x) = 1$, for all $x \in A$

iv) Each $x \in A$ has a neighborhood in which all but a finite number of the φ_i is zero.

Approximation of functions in BV by smooth functions.

THEOREM. *Let* $u \in BV(\Omega)$. *There exists a sequence* $\{u_i\}$ *in* $BV(\Omega) \cap C^\infty(\Omega)$ *such that* $\lim\limits_{i\to\infty} \int_\Omega |u_i - u|dx = 0$ *and* $\lim\limits_{i\to\infty} \int_\Omega |Du_i|(\Omega)dx = \int_\Omega |Du|(\Omega)dx$.

Proof. Consider a collection $\{\Omega_i\}_{i=0}^\infty$ of open subsets in Ω such that

$$|Du|(\Omega \setminus \Omega_0) < \varepsilon,\ \Omega_i \Subset \Omega_{i+1} \text{ and } \Omega = \bigcup_{i=0}^\infty \Omega_i.$$

Construct the open covering $\{V_i\}_{i=0}^{\infty}$ as follows. Let $v_0 = \Omega_1$ and $v_i = \Omega_{i+1} - \bar{\Omega}_{i-1}$ for $i = 1, 2, \ldots$. Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{v_i\}_{i=0}^{\infty}$. i.e. $\varphi_i \in C_c^{\infty}(v_i)$, $0 \le \varphi_i \le 1$, $\sum_{1=0}^{\infty} \varphi_i(x) = 1$.

For each i, choose $\varepsilon_i > 0$ such that $\mathrm{supp}(\rho_{\varepsilon_i} * \varphi_i u) \subset v_i$

$$\int_\Omega |\rho_{\varepsilon_i} * \varphi_i u - \varphi_i u| < \varepsilon 2^{-(i+1)}$$
$$\int_\Omega |\rho_{\varepsilon_i} * u D\varphi_i - u D\varphi_i| < \varepsilon 2^{-(i+1)}.$$

Define $u_\varepsilon = \sum\limits_{i=0}^{\infty} \rho_{\varepsilon_i} * \varphi_i u$.

Then $u_\varepsilon \in C^\infty(\Omega)$ and $u = \sum\limits_{i=0}^{\infty} \phi_i u$

$$\int_\Omega |u_\varepsilon - u| dx \le \sum_{i=0}^{\infty} \int_\Omega |\rho_{\varepsilon_i} * \varphi_i u - \varphi_i u| dx < \varepsilon.$$

In the distributional sense, $D(\varphi_i u) = \varphi_i Du + u D\varphi_i \mathcal{L}\lfloor_\Omega$

$$\begin{aligned} Du_\varepsilon &= \sum_{i=0}^{\infty} \rho_{\varepsilon_i} * \varphi_i Du + \sum_{i=0}^{\infty} \rho_{\varepsilon_i} * u D\varphi_i \\ &= \sum_{i=0}^{\infty} \rho_{\varepsilon_i} * \varphi_i Du + \sum_{i=0}^{\infty} \rho_{\varepsilon_i} * u D\varphi_i - \sum_{i=0}^{\infty} u D\varphi_i. \end{aligned}$$

Note $\sum\limits_{i=0}^{\infty} \varphi_i = 1$

$$\begin{aligned} \int_\Omega |Du_\varepsilon| dx &\le \sum_{i=0}^{\infty} \int_\Omega |\rho_{\varepsilon_i} * \varphi_i Du| dx + \sum_{i=0}^{\infty} \int_\Omega |\rho_{\varepsilon_i} * u D\varphi_i - u D\varphi_i| dx \\ &\le \sum_{i=0}^{\infty} \int_\Omega |\rho_{\varepsilon_i} * \varphi_i Du| dx + \varepsilon \\ &\le \sum_{i=0}^{\infty} |Du|(v_i) + \varepsilon \\ &\le |Du|(\Omega) + \sum_{i=0}^{\infty} |Du|(v_i) + \varepsilon. \end{aligned}$$

Since each $x \in \Omega$ belongs to at most two v_i's

$$\begin{aligned} &\le |Du|(\Omega) + 2|Du|(\Omega - \Omega_0) + \varepsilon \\ &\le |Du|(\Omega) + 3\varepsilon. \end{aligned}$$

DEFINITION. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $BV(\Omega)$ and $u \in BV(\Omega)$. We say that u_n converges to u in the sense of the* intermediate convergence *if $u_n \to u$ in $L^1(\Omega)$ and $\|Du_n\| \to \|Du\|$.*

Note. This definition is due to Temam.

Last time we show that the space $C^\infty(\Omega) \cap BV(\Omega)$ is dense in $BV(\Omega)$ equipped with the intermediate convergence.

We extend the Sobolev embedding theorem on the space $W^{1,1}(\Omega)$ to $BV(\Omega)$.

THEOREM. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $\partial\Omega$ is smooth. For all p, $1 \leq p \leq \frac{N}{N-1}$, the embedding $BV(\Omega) \hookrightarrow L^p(\Omega)$ is continuous. i.e. There exists a constant C which depends on (Ω, p, N) such that $\|u\|_{L^p} \leq C\|u\|_{BV(\Omega)}$.*

Proof. Let $u \in BV(\Omega)$ and choose $\{u_n\}_{n\in\mathbb{N}} \subset BV(\Omega) \cap C^\infty(\Omega)$ converges to u for the intermediate convergence. For $1 \leq p \leq \frac{N}{N-1}$, the embedding $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous. Therefore, there exists a constant $C = C(N, p, \Omega)$ such that

$$\left(\int |u_n|^p dx\right)^{\frac{1}{p}} < C\left(\int |u_n| dx + \|Du_n\|\right) < +\infty$$

According to the weakly lower semicontinuity of the norm of $L^p(\Omega)$

$$\begin{aligned}
\left(\int |u|^p dx\right)^{\frac{1}{p}} &\leq \liminf_{n\to\infty}\left(\int |u_n|^p dx\right)^{\frac{1}{p}} \\
&\leq \liminf_{n\to\infty} C\left(\int |u_n| dx + \|Du_n\|\right) \\
&= C\|u\|_{BV(\Omega)}
\end{aligned}$$

THEOREM. *Let Ω be an open, bounded subset of $\mathbb{R}^N$, $\partial\Omega$ is sufficiently regular for the Rellich Theorem to hold region. For all p, $1 \leq p < \frac{N}{N-1}$, the embedding $BV(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset BV(\Omega)$ such that $\|u_n\|_{BV(\Omega)} \leq 1$. For each n, we can choose $v_n \in BV(\Omega) \cap C^\infty(\Omega)$ such that $(\int |u_n - v_n|^p)^1 p \leq \frac{1}{n}$ and $\|Dv_n\| \leq 2$, $\|v_n\|_{W^{1,1}(\Omega)} \leq 4$. Since for $1 \leq p < \frac{N}{N-1}$, the embedding $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, there exists $\{v_{n_k}\}$ and $u \in L^p(\Omega)$ such that

$$v_{n_k} \to u \text{ in } L^p(\Omega).$$

Thus,

$$u_{n_k} \to u \text{ in } L^p(\Omega).$$

According to $u_n \to u$ in $L^1(\Omega)$ and the lower semicontinuity of $\|Du\|$,

$$
\begin{aligned}
|u|_{L^1(\Omega)} + \|Du\| &\le \liminf_{n\to\infty} |u_n|_{L^1(\Omega)} + \liminf_{n\to\infty} \|Du\| \\
&\le \liminf_{n\to\infty} (|u_n|_{L^1(\Omega)} + \|Du\|) \\
&\|u_n\|_{BV(\Omega)} \le 1.
\end{aligned}
$$

Note. This result is sharp.
Example. Let $\Omega = \{x \in \mathbb{R}^N | |x| < 2\}$

$$
U_k = k^{N-1}\chi_k, \quad \chi_k(x) = \begin{cases} 1, & \text{if } |x| < \frac{1}{k} \\ 0, & \text{otherwise} \end{cases}
$$

$\{u_k\}$ is unbounded in $L^p(\Omega)$ for

$$
p > \frac{N}{N-1}.
$$

"Analysis of bounded variation penalty methods for ill-posed problems." R.Acar, C.Vogel

Poincaré inequality: Ω: open connected bounded in $\mathbb{R}^N$. $\partial\Omega$: Lipschitz. There is a constant $C = C(\Omega, N)$ such that

$$
\int |u - u_\Omega| dx \le C\|Du\|(\Omega), \quad \forall u \in BV(\Omega), \ u_\Omega := \frac{1}{|\Omega|}\int u dx.
$$

Proof. Argue by contradiction.

$$
\int |u_k - u_{\Omega,k}| dx > k\|Du_k\| \quad \text{for all } k \in \mathbb{N}
$$

Let

$$
v_k := \frac{u_k - u_{\Omega,k}}{\|u_k - u_{\Omega,k}\|_{L^1}}.
$$

Then

$$
v_k \in BV(\Omega)
$$

$$
\int_\Omega v_k dx = 0, \quad \int_\Omega |v_k| dx = 1, \ \|Dv_k\| = \frac{\|Du_k\|}{\int |u_k - u_{\Omega,k}| dx}
$$

$$
\frac{\|Du_k\|}{\int |u_k - u_{\Omega,k}| dx} < \frac{1}{k}
$$

$$
\|Dv_k\| < \frac{1}{k} \quad \text{for all } k
$$

$$
\|v_k\|_{BV(\Omega)} < M \quad \text{a constant.}
$$

There is $\{v_{k_j}\}$ and $v \in BV(\Omega)$ such that

$$v_{k_j} \to v \text{ in } L^1(\Omega)$$
$$\|Dv\| \leq \liminf_{j\to\infty} \|Dv_{k_j}\|.$$

Thus, $\|Du\| = 0 \Rightarrow v$ is constant in Ω, which is connected

$$\int_\Omega v dx = 0, \quad \int_\Omega |v| = 1.$$

Therefore there is $C = C(N, \Omega)$ such that $\int |u - u_l| dx \leq C\|Du\|(\Omega)$.

DEFINITION. *The product of the strong topology of $L^1(\Omega)$ and of the weak$*$ topology of measures for Du is called the* weak$*$ topology *of $BV(\Omega)$, and is denoted by BV-ω^**

$$u_n \rightharpoonup u \Leftrightarrow \begin{cases} u_n \to u \\ Du_n \to Du \quad \Omega) = \text{set of Radon measures} \end{cases}$$

Every bounded sequence in $BV(\Omega)$ admits a subsequence converging in BV-ω^*. This sequence is relatively compact in $L^p(\Omega)$ for $p = \frac{N}{N-1}$ and $N \geq 1$, and relatively weakly compact in $L^p(\Omega)$ for$p = \frac{N}{N-1}$ and $N \geq 2$.

The Poincaré-Wirtinger inequality

Let $u \in BV(\Omega)$ and define, $u_\Omega := \frac{\int_\Omega u(x)dx}{|\Omega|}$. There exists μ such that

$$|u - u_\Omega|_{L^p} \leq M|Du|(\Omega), p = \frac{N}{N-1} \text{ for } N > 1, p = +\infty \text{ for } N = 1.$$

$$\inf \left\{ F(u) = \int_\Omega |Ku - f|^2 dx + \lambda \int_\Omega \varphi(|Du|) dx \right\}$$

Assumptions on φ and K to ensure the existence of a minimizer in $BV(\Omega)$ are:

A1. $\varphi : \mathbb{R} \to \mathbb{R}^+$, even convex and nondecreasing in $\mathbb{R}^+$

- i) $\varphi(0) = 0$ (without loss of generality)
- ii) There exists $C > 0$ and $b \geq 0$ such that

$$Cz - b \leq \varphi(z) \leq Cz + b, \ \forall z \in \mathbb{R}^+$$

A2. $K : L^2 \to L^2$ is a linear and continuous operator

A3. $K\chi_\Omega \neq 0$

A4. K is injective

Note. A3 implies K does not annihilate constant functions. Since ϕ is convex and finite, it is continuous moreover its asymptote function φ^∞ exists

$$\varphi^\infty(z) := \lim_{t\to 0} \frac{\varphi(tz)}{t}.$$

By A1(ii),

$$\lim_{t\to 0} \frac{\varphi(t)}{t} = C,\ \varphi^\infty(z) = C \cdot z \cdot \mathrm{sign}(z).$$

Note. We define the approximate upper limit $u^+(x)$ and the approximate lower limit $u^-(x)$ and

$$u^+(x) = \inf\left\{ t \in [-\infty, \infty] \middle| \lim_{r\to 0} \frac{\mathcal{L}^N\{u > t\} \cap B(x,t)}{r^N} = 0 \right\}$$

$$u^-(x) = \sup\left\{ t \in [-\infty, \infty] \middle| \lim_{r\to 0} \frac{\mathcal{L}^N\{u < t\} \cap B(x,t)}{r^N} = 0 \right\}$$

$$B(x,r) = \text{Ball of radius } r \text{ centered at } x$$

$u \in L^1(\Omega)$, $x \in \Omega$ is called a Lebesgue point of u if

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(x) - u(y)| dy = 0$$

$$u(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y)| dy.$$

At such point,

$$u(x) = u^+(x) = u^-(x).$$

The jumpset S_u is the complement, up to a set of $\mathcal{H}^{N-1}$ measure zero, of the set of Lebesgue points.

$$S_u = \{x \in \Omega | u^-(x) < u^+(x)\}$$

Then S_u is countable rectifiable, and for $\mathcal{H}^{N-1}$ a.e.x, we can define a normal $n_u(x)$. By the Lebesgue decomposition,

$$Du = \nabla u \mathcal{L}^N + D_s u.$$

L.Ambrosio

$$Du = \nabla u \mathcal{L}^N + \underbrace{(u^+ - u^-) n_u \mathcal{H}^{N-1}|_{S_u}}_{\text{jump part of } D_s u} + \underbrace{C_u}_{\text{cantor part of } D_s u}$$

C_u is singular with respect to the Lebesgue measure. $C_u(s) = 0$ for every set of $\mathcal{H}^{N-1}(s) < \infty$. It is possible to defined convex functions of measure.

$$\varphi(|Du|) = \varphi(|\nabla u|)\mathcal{L}^N + \varphi^\infty(1)|D_s u|$$

Functional

$$J(u) = \int_\Omega \varphi(|\nabla u|)dx + \varphi^\infty(1)\int_\Omega |D_s u|$$

$J(u)$ is weak$*$ lower semicontinuous on $\mu(\Omega)$
$\Omega \subset \mathbb{R}^2$, $\partial\Omega$: Lipschitz boundary

Consider:

$$\inf\left\{\hat{F} = \int_\Omega |Ku - f|^2 dx + \int_\Omega \varphi(|\nabla u|)dx + C\int_\Omega |D_s u|\right\} \qquad (*)$$

Proposition. Let $f \in L^2(\Omega)$. Under the assumptions A1-A4. There is a minimizer $u \in BV(\Omega)$ of $(*)$.
Proof. Let $\{u_n\}$ be a minimizing sequence.
Acart. Vogel: $\hat{F}$ is coercive in $BV(\Omega)$.
Chombolle, Lions: Passing to the limit in the minimizing sequence.
From A1(ii), $|Du_n| < M$

$$|Du_n| = |\nabla u_n|\mathcal{L}^N + |D_s u_n|$$

we need to show $|\int u_n dx| < M$.
Let $\omega_n = \frac{1}{|\Omega|}\int_\Omega u_n dx \chi_\Omega,\ v_n = u_n - \omega_n$

$$\int_\Omega v_n dx = 0,\ Dv_n = Du_n$$

$$\begin{aligned} M &\geq \|Ku_n - f\|_{L^2}^2 \geq \|Kv_n - K\omega_n - f\|_{L^2}^2 \geq (\|Kv_n - f\|_{L^2} - \|K\omega_n\|_{L^2})^2 \\ &\geq \|K\omega_n\|_{L^2}(\|K\omega_n\|_{L^2} - 2(\|K\|\|v_n\|_{L^2} + \|f\|_{L^2})) \end{aligned}$$

Let $a_n := \|K\|\|v_n\|_{L^2} + \|f\|_{L^2}$

$$\begin{aligned} \chi_n &:= \|K\omega_n\|_{L^2} \\ &\geq \chi_n(\chi_n - 2a_n) \end{aligned}$$

$$0 \leq \chi_n \leq a_n + \sqrt{a_n^2 + M} \leq M'$$

$$\|K\omega_n\|_{L^2} \leq M',\ \frac{|\int u_n(x)dx|}{|\Omega|}\|K\chi_\Omega\|_{L^2} \leq M',$$

$$\left|\int u_n(x)dx\right| < M''$$

The Poincaré-Wiritinger

$$|u_n - u_\Omega|_{L^2} < C \cdot M$$

$$\|u_n\|_{L^2} = \|u_n - \frac{\int u_n dx}{|\Omega|} + \frac{\int u_n dx}{|\Omega|}\|_{L^2} \leq \|u_n - u_\Omega\|_{L^2} + \left|\frac{\int u_n dx}{|\Omega|}\right| \leq M'''$$

In particular, u_n is bounded in $L^1(\Omega)$. So u_n is bounded in $BV(\Omega)$. There is a subsequence $u_{n_j} \xrightarrow[Bv-\omega^*]{} u$

$$\inf\left\{F(u)=\int_\Omega |Ku-f|^2dx+\lambda\int_\Omega \phi(|\nabla u|)dx\right\} \tag{9}$$

Assumptions on ϕ and K to ensure the existence and uniqueness of minimizer of (9).

A1) $\phi:\mathbb{R}\to\mathbb{R}^+$ is even, convex and nondecreasing in $\mathbb{R}^+$

(i) $\phi(0)=0$

(ii) There exist $C>0$ and $b\geq 0$ such that $Cz-b\leq\phi(z)\leq Cz+b$ for $z\in\mathbb{R}^+$

A2) $K: L^2\to L^2$ is linear and continuous operator

A3) $K\chi_\Omega\neq 0$

A4) K is injective

$F(u)$ is not lower semicontinuous with respect to the BV-ω^* topology.

Relaxed energy functional of (9) for BV-ω^* topology

$$\inf\left\{\hat{F}(u)=\int_\Omega |Ku-f|^2dx+\varphi(|Du|)\Omega\right\} \tag{$\sharp$}$$
$$\varphi(|Du|)=\varphi(|\nabla u|)\mathcal{L}_N+\varphi^\infty(9)|D_s u|$$

Proposition. There exists a unique minimizer u of ($\sharp$). From A1(ii)

$$\begin{cases}|Du_n|\leq M, & M \text{ is a strictly positive constant}\\ \int_\Omega |Ku-f|^2\leq M\end{cases}$$

Let

$$\omega_n := \frac{\int_\Omega u_n dx}{|\Omega|}\chi_\Omega,$$

then

$$\int_\Omega v_n dx=0,\ |Dv_n|=|Du_n|\leq M.$$

The P-W inequality: $\|v_n\|_{L^2}\leq\bar{M}$. We want to show that $|\int_\Omega u_n dx|$ is bounded.

$$M\geq\int_\Omega |Ku-f|^2dx$$
$$\|K\omega_n\|_{L^2}\leq|\int u_n dx|\frac{\|K\omega_n\|_{L^2}}{|\Omega|}\leq M'\Rightarrow|\int u_n dx|\leq M''$$
$$\|u_n\|_{L^2}\leq\|u_n-\frac{\int_\Omega u_n dx}{|\Omega|}\|+|\int_\Omega u_n dx|\leq M''' \text{ for all } n\geq 1$$

Thus, $\{u_n\}_{n\geq 1}$ is bounded in $L^2(\Omega)$ and in particular in $L^1\Omega)$. Since $|Du_n|$ is also bounded, up to a subsequence, there exists a $u \in BV(\Omega)$ such that $u_n \rightharpoonup u$ in BV-ω^*. From the weak lower semicontinuity property of the convex functions of measures and the lower semicontinuity property of the L^2-norm,

$$\int \phi(Du) \leq \liminf_{n\to\infty} \int \phi(Du_n)$$

$$\int |Ku - f|^2 dx \leq \int |Ku_n - f|^2 dx$$

$$\hat{F}(u) \leq \liminf_{n\to\infty} \hat{F}(u_n) = \inf_v \hat{F}(v)$$

i.e. u is a minimizer.

uniqueness: Let u and v be two solutions of $(*)$.

Suppose $Ku \neq Kv$ $Ku \mapsto \int_\Omega |Ku - f|^2 dx$ is strictly convex

$$\hat{F}\left(\frac{1}{2}u + \frac{1}{2}v\right) < \frac{1}{2}\hat{F}(u) + \frac{1}{2}\hat{F}(v) = m := \inf \hat{F}$$

because $\hat{F}$ is the sum of the two convex functions with independent variables Ku and Du. The first one is strictly convex. This inequality cannot hold if u and v are minimizers.

Thus, $Ku = Kv$. Since K is injective, $u = v$.

The BV seminorm of u is denoted by $|u|_{BV}$

$$|u|_{BV} = \sup\left\{\int_\Omega u\mathrm{div}\varphi : \varphi \in (C_c^1(\Omega))^2, \|\varphi\|_{L^\infty} \leq 1\right\}$$

Rudin, Osher, Fatemi (ROF): Given an image F,

$$\inf\left\{F(u) = \lambda \int_\Omega |u - f|^2 dx + |u|_{BV}\right\}$$

$$f = u + v$$

f: observed, noisy version of the true known image
u: restored image
v: noise

$$(u, v) \in BV(\Omega) \times L^2(\Omega),\ f = u + v$$

Cartoon-texture decomposition

Decompose f into $u + v$

$$f = u + v$$

u: "cartoon" or geometric part of f which is a sketch of f
v: osciatory part of f which contains texture and noise

Mumford, Gidas, Quart, Appl, Math 59 (2001)

They showed that natural images can be seen as samples of scale invariant probability distributions that are supported on distributions only.

Cartoon-texture decomposition via variational energy minimization

$$\inf\{K(u,v) = F_1(u) + \lambda F_2(v) : f = u + v\}$$
$$(u,v) \in X_1 \times X_2$$

$F_1, F_2 \geq 0$ are functionals
X_1, X_2 are spaces of functions or distributions such that

$$X_1 = \{u : F_1(u) < \infty\},\ X_2 = \{v : F_2(v) < \infty\}$$

λ is a tunning parameter; Assume $f \in X_1 + X_2$ usually.

F_1 and F_2 are norms or seminorms of function spaces arising in image processing.

A good model for K is given by a choice of X_1 and X_2 such that for $f = u+v$, $F_1(u) = \|u\|_{X_1}$ and $F_2(v) = \|v\|_{X_2}$ are small.

Example. ROF: $F_1(u) = |u|_{BV}$, $F_2(v) = \|v\|_{L^2(\Omega)}$ given $f = u + v$

Mumford and Shah segmentation model

$f \in L^\infty(\Omega) \subset L^2(\Omega)$ is split into $u \in SBV$ (a piecewise smooth function with its discontinuity set J_n composed of a union of curves with finite total Length) and $v = f - u \in L^2(\Omega)$ represents noise and texture.

$$\inf_{(u,v)\in SBV(\Omega)\times L^2(\Omega)} \left\{ \int_{\Omega\setminus J_n} \underbrace{|\nabla u|^2 + \alpha\mathcal{H}^1(J_n)}_{F_1(u)} + \underbrace{\beta\|v\|_{L^2}^2}_{F_2(v)} : f = u + v \right\}$$

Y.Meyer. "Oscillating Patterns in image processing and nonlinear evolution equations."

He investigated the properties of the solution (u,v) to the ROF model. He showed some limitations of the ROF model. He proposed to replace the $L^2(\Omega)$ by a weaker norm for the oscillatory part v by using generalized function spaces.

Meyer proposed to use $(BV(\Omega)'$, the dual of the space $BV(\Omega)$. We have the following inclusions

$$BV(\Omega) \subset L^2(\Omega) \subset (BV(\Omega))'.$$

However, there is no integral representation of continuous linear functionals on $BV(\Omega)$.

DEFINITION. $G(\mathbb{R}^2)$ *is the Banach space* composed of *distributions* f *which can be written as*

$$f = \mathrm{div}(\vec{g}) = \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2},\ \vec{g} = (g_1, g_2) \in L^\infty(\mathbb{R}^2).$$

The space $G(\mathbb{R}^2)$ *is endowed with the following norm*

$$\|f\|_{G(\mathbb{R}^2)} = \inf\left\{ \left\| \sqrt{g_1^2 + g_2^2} \right\|_{L^\infty(\mathbb{R}^2)} : f = \mathrm{div}\vec{g}, \vec{g} = (g_1, g_2), g_1, g_2 \in L^\infty(\mathbb{R}^2) \right\}$$

$G(\mathbb{R}^2)$ is the dual space of the closure in $BV(\mathbb{R}^2)$ of the schwartz class. So it is related to the dual of $BV(\mathbb{R}^2)$. This is the motivation to decompose f on $BV(\mathbb{R}^2) + G(\mathbb{R}^2)$.

The invariance and the scale nature of the parameter λ

$$E(u) = \frac{1}{2}\int_\Omega |u - f|^2 dx + \lambda \int_\Omega \phi(|\nabla u|) dx$$

Assume $f \in L^2(\Omega)$. Let $u(x, \lambda)$ be the unique minimizer of $E(u)$. For all $v \in W^{1,1}(\Omega) \cap L^2(\Omega)$,

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u(x,\lambda) - f(x)|^2 dx + \lambda \int_\Omega \phi(|\nabla(u(x,\lambda))|) dx \qquad (**) \\ \leq &\frac{1}{2}\int_\Omega |v(x,\lambda) - f(x)|^2 dx + \lambda \int_\Omega \phi(|\nabla(v(x,\lambda))|) dx \end{aligned}$$

$u(x, \lambda)$ must satisfy the Euler-Lagrange equations

$$\begin{cases} u(x,\lambda) - f(x) = \lambda \mathrm{div}\left(\frac{\phi'(|\nabla u(x,\lambda)|)}{|\nabla u(x,\lambda)|} \cdot \nabla u(x,\lambda)\right) \text{ in } \Omega \\ \frac{\phi'(|\nabla u(x,\lambda)|)}{|\nabla u(x,\lambda)|} \cdot \frac{\partial u}{\partial N}(x,\lambda) = 0 \text{ on } \partial\Omega \end{cases}$$

Property 1. The L^2-norm of $u(\cdot, \lambda)$ is bounded by a constant independent of λ.
Proof. Let $v = 0$ in $(**)$. Since $\phi(0) = 0$,

$$\int_\Omega |u(x,\lambda) - f(x)|^2 dx \leq \int_\Omega |f(x)|^2 dx$$
$$\|u(x,\lambda)\|_{L^2} \leq \|u(x,\lambda) - f(x)\|_{L^2} + \|f(x)\|_{L^2} \leq 2\|f\|_{L^2}$$

The invariance and the scale nature of the parameter λ

$$E(u) = \frac{1}{2}\int_\Omega |u - f|^2 dx + \lambda \int_\Omega \phi(|\nabla u|) dx$$

Property 2. For every λ, we have $\int_\Omega u(x,\lambda) dx = \int_\Omega f(x) dx$.
Proof. From the E.L. equation,

$$\int_\Omega u(x,\lambda) dx = \int_\Omega f(x) dx + \int_\Omega \lambda \mathrm{div}\left(\frac{\phi'(|\nabla u(x,\lambda)|)}{|\nabla u(x,\lambda)|} \cdot \nabla u(x,\lambda)\right) dx.$$

Using the Green's formula

$$\int_{\Omega} \operatorname{div}\left(\frac{\phi'(|\nabla u(x,\lambda)|)}{|\nabla u(x,\lambda)|}\cdot \nabla u(x,\lambda)\right) dx = \int_{\partial\Omega} \frac{\phi'(|\nabla u(x,\lambda)|)}{|\nabla u(x,\lambda)|}\cdot \frac{\partial u}{\partial N}(x,\lambda) ds$$
$$= 0, \text{ by boundary condition.}$$

$$\therefore \int_{\Omega} u(x,\lambda)dx = \int_{\Omega} f(x)dx.$$

Property 3 $u(\cdot,\lambda)$ converges in $L^1(\Omega)$ strong to the average of the initial data f as $\lambda \to +\infty$. i.e. $\lim\limits_{\lambda\to+\infty} \inf_{\Omega} |u(\cdot,\lambda) - f_{\Omega}|dx = 0$

Proof. Let $v = 0$.

$$0 \le \lambda \int_{\Omega} \phi(|\nabla u(x,\lambda)|)dx \le \frac{1}{2}\int_{\Omega} |f(x)|^2 dx$$
$$0 \le \lim_{\lambda\to+\infty} \int_{\Omega} \phi(|\nabla u(x,\lambda)|)dx \le \lim_{\lambda\to+\infty} \frac{1}{2\lambda}\int_{\Omega} |f(x)|^2 dx = 0$$

The invariance and the scale nature of the parameter λ

$$E(u) = \frac{1}{2}\int_{\Omega} |u-f|^2 dx + \lambda \int_{\Omega} \phi(|\nabla u|)dx$$

$$\lim_{\lambda\to+\infty} \int_{\Omega} \phi(|\nabla u(x,\lambda)|)dx = 0 \text{ since } \phi \text{ is convex and has a growth.}$$

$$\lim_{\lambda\to+\infty} |\nabla u(x,\lambda)|_{L^1} = 0$$

By the P-W inequality,

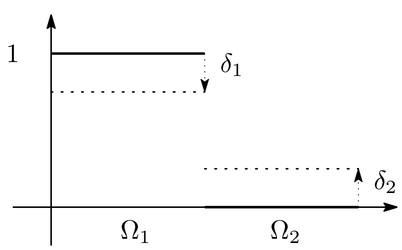
$$\int_{\Omega} |u(x,\lambda) - f_{\Omega}|dx \le C|\nabla u(x,\lambda)|_{L^1},\ f_{\Omega} = \frac{1}{|\Omega|}\int_{\Omega} f(x)dx.$$

Thus,

$$\lim_{\lambda\to+\infty} \int_{\Omega} |u(x,\lambda) - f_{\Omega}|dx = 0.$$

David Strong, Tony Chan, "Exact solutions to total variation regularization problems."

Effects of total variation on piecewise constant functions.



—— : u_0 noise-free 1D signal with a single discontinuity

$$u_0(x) = \begin{cases} 1 & \text{if } x \in \Omega_1 \\ 0 & \text{if } x \in \Omega_2 \end{cases}$$

$$\min_u \|u - u_0\|_{L^2(\Omega)} + \alpha |u|_{TV}$$

Look for $u(x) = \begin{cases} 1 - \delta_1 \text{ if } x \in \Omega_1 \\ 0 + \delta_2 \text{ if } x \in \Omega_2 \end{cases}$

$$\min_{\delta_1, \delta_2} \frac{1}{2}(|\Omega_1|\delta_1^2 + |\Omega_2|\delta_2^2) + \alpha(1 - \delta_1 - \delta_2)$$

Differentiating with respect to δ_i, $i = 1, 2$

$$\delta_i = \frac{\alpha}{|\Omega_i|}, \ i = 1, 2$$

THEOREM. *Suppose $u_0(x)$ is defined such that*

$$\frac{\int_{\Omega_1} u_0(x)dx}{|\Omega_1|} = 1, \ \frac{\int_{\Omega_2} u_0(x)dx}{|\Omega_2|} = 0, \ \textit{and} \ \max_{x \in \Omega_2} u_0(x) \leq \min_{x \in \Omega_1} u_0(x).$$

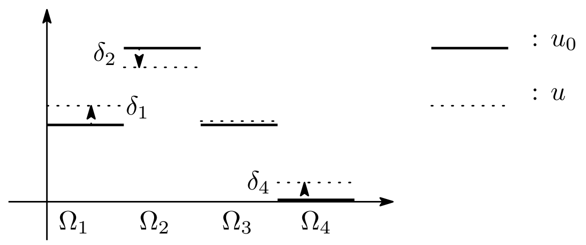
If we assume $\max_{x \in \Omega_2} u_0(x) \leq 0 + \frac{\alpha}{|\Omega_2|} \leq 1 - \frac{\alpha}{|\Omega_1|} \leq \min_{x \in \Omega_1} u_0(x)$ *then the unique solution u is given by*

$$u(x) = \begin{cases} 1 - \frac{\alpha}{|\Omega_1|} & x \in \Omega_1 \\ 0 + \frac{\alpha}{|\Omega_2|} & x \in \Omega_2. \end{cases}$$

α-*condition*: A value of α which meets the α-*condition*

1) α is sufficiently small that all jumps in u_0 are present in u
2) α is sufficiently large so that noise is completely removed, resulting in a regularized function that is exactly piecewise constant with reduced contrast.

General piecewise constant functions in $\mathbb{R}^1$



THEOREM. *Suppose the function u_0 is as shown above and the α-condition holds. Then the unique solution u to the min problem*

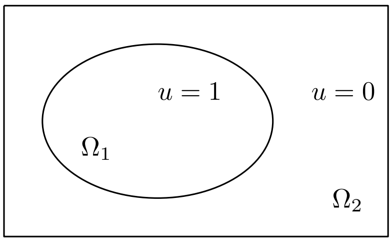
$$\min_u \frac{1}{2}\|u - u_0\|_{L^2} + \alpha|u|_{TV} \text{ is shown above,}$$

where the change in intensity is given by

$$\delta_i = \frac{2}{|\Omega_i|}\alpha \quad \text{"extremun" region } (\Omega_2)$$
$$\delta_i = \frac{1}{|\Omega_i|}\alpha \quad \text{"boundary" region } (\Omega_1, \Omega_4)$$
$$\delta_i = \frac{0}{|\Omega_i|}\alpha = 0 \quad \text{"step" region } (\Omega_3).$$

Note. The change in intensity depends only on $|\Omega_i|$ of the feature and on α but each δ_i is independent of the original function u_0. δ_i's are mutually independent as long as α is not too large to flatten out all jumps between regions. TV regularization is somewhat local.

Piecewise constant functions in $\mathbb{R}^2$



$$u_0(x) = \begin{cases} 1, & x \in \Omega_1 \\ 0, & x \in \Omega_2 \end{cases}$$

Assume

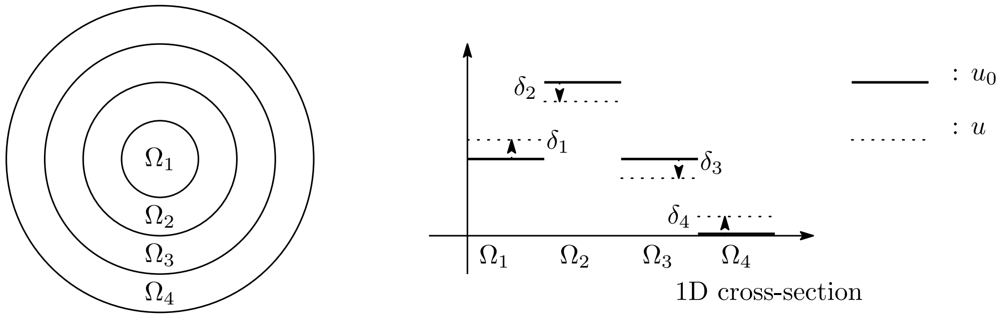
$$u(x) = \begin{cases} 1 - \delta_1, & x \in \Omega_1 \\ 0 + \delta_2, & x \in \Omega_2 \end{cases}$$
$$\|u - u_0\|_{L^2}^2 = |\Omega_1|\delta_1^2 + |\Omega_2|\delta_2^2$$
$$|u|_{TV} = (1 - \delta_1 - \delta_2)|\partial\Omega_{1,2}|$$
$$\min_{\delta_1,\delta_2} \frac{1}{2}(|\Omega_1|\delta_1^2 + |\Omega_2|\delta_2^2) + \alpha(1 - \delta_1 - \delta_2)|\partial\Omega_{1,2}|.$$

Then,

$$\delta_i = \frac{|\partial\Omega_{1,2}|}{|\Omega_i|}\alpha, \quad i = 1, 2.$$

Let $s = \frac{|\Omega|}{|\partial\Omega|}$, then $\delta = \frac{\alpha}{s}$, s: scale.

Radially symmetric piecewise constant functions in $\mathbb{R}^2$



THEOREM. *Given the initial function u_0 as above. The change in intensity for each region is given by*

$$\delta_i = \frac{|\partial\Omega_{i,i-1}| + |\partial\Omega_{i,i+1}|}{|\Omega_i|}\alpha \quad \text{"extremum" region}$$

$$\delta_i = \frac{|\partial\Omega_{i,i-1}|}{|\Omega_i|}\alpha \quad \text{"boundary" region}$$

$$\delta_i = \frac{|\partial\Omega_{i,i+1}| - |\partial\Omega_{i,i-1}|}{|\Omega_i|}\alpha \quad \text{"step" region.}$$

Modeling very oscillating patterns

Given an image f, decompose it into $u+v$ via energy minimization problem

$$\inf_{(u,v)\in X\times X}\{K(u,v) = F_1(u) + \lambda F_2(v) | f = u+v\}$$

u: contains geometric part of f and sharp boundaries
v: contains oscillatory part of f e.g. textures and noise

F_1, F_2 are norms or seminorms of function spaces arising in image processing. X_1, X_2 are spaces of functions or distributions.

A good model is given by choices of X_1 and X_2 such that $\|u\|_{X_1}$ and $\|v\|_{X_2}$ are small.

Examples. In ROF image restoration algorithm, $F_1(u) = |u|_{BV},\ F_2(v) = \|v\|_{L^2}$. In Mumford Shah image segmentation model, $u \in SBV,\ v \in L^2$.

Y.Meyer: interpreted the ROF algorithm as a texture separation algorithm and showed that pieces which belong to the true objects contained in the image are wiped out by the ROF algorithm and viewed as belonging to the texture components.

Devore, Lucier: replaced the Banach space BV by the homogeneous Besov space $B = \dot{B}_1^{1,1}$ and considered $\inf\limits_{(u,v)}\{J(u) = |u|_{BV} + \lambda\|v\|_{L^2}\}$.

Drawback: characteristic functions of smooth domains do not belong to $\dot{B}_1^{1,1}$.

Y.Meyer: suggested to replace the L^2-norm by $(BV)'$, the dual of BV. He showed that there exists a norm $\|\cdot\|_*$ and a threshold $\frac{1}{2\lambda}$ such that an image f with norm $\|f\|_* \leq \frac{1}{2\lambda}$ is seen as textures and an image f with $\|f\|_* > \frac{1}{2\lambda}$ is reduced by a fixed amount.

Since BV is not separable, its dual is not a function space. However, there exists a closed subspace of BV which has a simple dual.

DEFINITION. $\mathcal{BV} = \{f \in BV | \nabla f \in L^1\}$

DEFINITION. *The space of texture G is defined as the dual of $\mathcal{BV}$. $G = \mathcal{BV}^*$. The space G is endowed with the dual norm, denoted by $\|\cdot\|_*$. The dual norm can be estimated by duality.*

DEFINITION. $G(\mathbb{R}^2)$ *is a Banach space* consisting *of distributions f which can be written as $f = \partial_1 g_1 + \partial_2 g_2 = \mathrm{div}\vec{g},\ \vec{g} = (g_1, g_2),\ g_1, g_2 \in L^\infty(\mathbb{R}^2)$. The space $G(\mathbb{R}^2)$ is endowed with the following norm.*

$$\|f\|_* = \inf_{\vec{g}} \{\|g_1^2 + g_2^2\|_{L^\infty(\mathbb{R}^2)} : f = \mathrm{div}\vec{g},\ \vec{g} = (g_1, g_2),\ g_1, g_2 \in L^\infty(\mathbb{R}^2)\}$$

$L^\infty(\mathbb{R}^2)$ is a dual space, and there is such an optimal decomposition which provides the norm. For a bounded domain Ω in $\mathbb{R}^2$, we use the followings definition of $G(\Omega)$.

DEFINITION. $G(\Omega)$ *is the subspace of $W^{-1,\infty}(\Omega)$ defined by*

$$G(\Omega) = \{f \in L^2(\Omega) | f = \mathrm{div}\vec{g},\ \vec{g} = (g_1, g_2),\ g_1, g_2 \in L^\infty(\Omega),\ \vec{g} \cdot N = 0 \text{ on } \partial\Omega\}.$$

Note.

$$W^{-k,p}(\Omega) = (W_0^{k,q}(\Omega))^*,\ \frac{1}{p} + \frac{1}{q} = 1$$

$$W^{-k,p}(\Omega) = \left\{ u \in D'(\Omega) : u = \sum_{|\alpha| \leq k} \partial^\alpha u_\alpha \text{ for some } u_\alpha \in L^p(\Omega) \right\}$$

$W^{-k,p}(\Omega)$ is a Banach space with a norm

$$\|u\|_{W^{-k,p}(\Omega)} = \sup_{\substack{\mathrm{v} \in \mathrm{W}^{\mathrm{k,p}} \\ \|v\|_{W^{k,q}} \neq 0}} \frac{|< u, v >|}{\|v\|_{W^{k,q}}},\ \ < u, v >= \sum_{|\alpha| \leq k} < \partial^\alpha u_\alpha, v >$$

Note. If for a given function f, no vector field g exists, then we set $\|f\|_* = \infty$. However, this pathological case is not import, as any function f in $L^2(\Omega)$ can be redefined to have finite G-norm by subtracting off a constant.

LEMMA.
$$G(\Omega) = \{f \in L^2(\Omega) | \int_\Omega f dx = 0\}$$

Proof. Let
$$H(\Omega) = \{f \in L^2(\Omega) | \int_\Omega f dx = 0\}$$
$$\int_\Omega f dx = \int_\Omega \mathrm{div}\vec{g} dx = \int_{\partial\Omega} \vec{g} \cdot N = 0, \text{ so } f \in H(\Omega)$$

Let $f \in G(\Omega)$. Suppose $f \in H(\Omega)$. Then there exists $\vec{g} \in C^0(\bar{\Omega}, \mathbb{R}^2) \cap W^{1,2}(\Omega, \mathbb{R}^2)$ such that $f = \mathrm{div}\vec{g}$, $\vec{g} = 0$ on $\partial\Omega$. (Bourgain, Brezis "on the equation $\mathrm{div}\vec{g} = f$ and appl. to control of phases.") so $g \in L^\infty(\Omega; \mathbb{R}^2)$ and $g \cdot N = 0$ on $\partial\Omega$.

LEMMA. *If f belongs to BV and g to $L^2(\Omega)$, then*
$$\left| \int_\Omega f(x)g(x)dx \right| \leq |f|_{BV} \|g\|_*.$$

Proof. By duality, the inequality holds if $f \in \mathcal{BV}$. When $f \in BV$, we use an approximation of identity $\varphi_n(x) = n^2\varphi(nx)$ where $\varphi > 0$ is regular, and $\int \varphi(x)dx = 1$.

Let $f_n = f * \varphi_n$. Then $f_n \in \mathcal{BV}$, $\|f_n - f\|_{L^2} \to 0$, $|f_n|_{BV} \to |f|_{BV}$ when $n \to \infty$. Applying lemma to $f_n \in \mathcal{BV}$, $|\int f_n(x)g(x)dx| \leq |f_n|_{BV}\|g\|_*$. By letting $n \to \infty$, $\int f_n(x)g(x)dx \xrightarrow[\text{weak}]{} \int f(x)g(x)dx$. Since $g \in L^2$, we can pass to the weak limit.

Example. Let $a > 0$, $n > 0$ be fixed. Let φ be the smooth cutoff function such that
$$\varphi(x) = \begin{cases} a \text{ if } |x| \leq n \\ 0 \text{ if } |x| > n+1. \end{cases}$$

Let $m > 0$ and $f(x) = \frac{1}{m}\varphi'(x)\sin(mx) + \varphi(x)\cos(mx)$. Let $g(x) = \frac{\varphi(x)}{m}\sin(mx) + C$, C is an arbitrary conatant.
$$f = \mathrm{div} g$$
$$\|f\|_* = \frac{a}{m} \to 0 \text{ as } m \to \infty$$
$$\|f\|_{L^2}^2 \geq 2a^2 \int_0^n (\cos(mx))^2 dx = a^2(n + \frac{1}{2m}\sin(2mn)) \to a^2 n > 0 \text{ as } m \to \infty.$$

DEFINITION. *Two functions $u \in BV$, $v \in L^2$ are called an* extremal pair *(u, v) if*
$$\int uv dx = |u|_{BV}\|v\|_*.$$

Y.Meyer showed that there is a norm $\|\cdot\|_*$ and a threshold $\frac{1}{2\lambda}$ such that an image f with $\|f\|_* > \frac{1}{2\lambda}$ is reduced by a fixed amount.

The Characterization of the ROF decomposition

Proposition. Let f belong to $L^2(\Omega)$ and $(u, v) \in BV \times L^2$. If $\|f\|_* \leq \frac{1}{2\lambda}$, then the image f is seen as a texture. i.e. $(u, v) = (0, f)$.

Proof. Suppose $\|f\|_* \leq \frac{1}{2\lambda}$. For any $h \in BV$,

$$\begin{aligned}|h|_{BV} + \lambda\|f-h\|_{L^2}^2 &= |h|_{BV} + \lambda\|f\|_{L^2}^2 - 2\lambda\int fhdx + \lambda\|h\|_{L^2}^2 \\ &\geq |h|_{BV} + \lambda\|f\|_{L^2}^2 - |h|_{BV} + \lambda\|h\|_{L^2}^2 \\ &\geq \lambda\|f\|_{L^2}^2 \quad \text{ for all } h \in BV.\end{aligned}$$

Conversely, if $(u, v) = (0, f)$ is the ROF decomposition, then for any $h \in BV$,

$$|h|_{BV} + \lambda\|f-h\|_{L^2}^2 \geq \lambda\|f\|_{L^2}^2$$
$$|h|_{BV} + \lambda\|f\|_{L^2}^2 - 2\lambda\int fhdx + \lambda\|h\|_{L^2}^2 \geq \lambda\|f\|_{L^2}^2.$$

Let $h \mapsto \varepsilon h,\ \varepsilon \in \mathbb{R}$

$$|\varepsilon h|_{BV} + \lambda\|\varepsilon h\|_{L^2}^2 \geq 2\lambda\int f\varepsilon hdx,\ \forall h \in BV.$$

Let $\varepsilon \to 0$

$$|h|_{BV} \geq 2\lambda\left|\int fhdx\right|,\ \forall h \in BV.$$

By lemma, we have $\|f\|_* \leq \frac{1}{2\lambda}$.

Proposition. If $\|f\| > \frac{1}{2\lambda}$, then the ROF decomposition $f = u + v$ is characterized by the following two conditions.

(i) $\|v\|_* = \frac{1}{2\lambda}$

(ii) $\int uvdx = \frac{1}{2\lambda}|u|_{BV}$

Proof. Suppose that the two conditions (i) and (ii) hold. For any $h \in BV$, $\varepsilon \in \mathbb{R}$,

$$\begin{aligned}|u+\varepsilon h|_{BV} + \lambda\|v-\varepsilon h\|_{L^2}^2 &\geq 2\lambda\int(u+\varepsilon h)vdx + \lambda\|v\|_{L^2}^2 - 2\lambda\int\varepsilon vhdx + \lambda\|\varepsilon h\|_{L^2}^2 \\ &= 2\lambda\int uvdx + \lambda\|v\|_{L^2}^2 + \lambda\|\varepsilon h\|_{L^2}^2 \\ &= |u|_{BV} + \lambda\|v\|_{L^2}^2 + \lambda\|\varepsilon h\|_{L^2}^2 \\ &\geq |u|_{BV} + \lambda\|v\|_{L^2}^2 \quad \text{ for any } h \in BV.\end{aligned}$$

Conversely, if for any $h \in BV$, $|u+\varepsilon h|_{BV} + \lambda\|v-\varepsilon h\|_{L^2}^2 \geq |u|_{BV} + \lambda\|v\|_{L^2}^2$. LHS is bounded by $|u|_{BV} + |\varepsilon||h|_{BV} + \lambda\|v-\varepsilon h\|_{L^2}^2$. using the same calculation as in the previous proposition, we have

$$\|v\|_* \leq \frac{1}{2\lambda}.$$

Take $h = u$ if $\varepsilon \in (-1, 1)$

$$2\lambda\varepsilon \int uvdx \leq \varepsilon |u|_{BV}$$

$$\text{if } \varepsilon \geq 0,\ 2\lambda \int uvdx \leq |u|_{BV}$$

$$\text{if } \varepsilon > 0,\ 2\lambda \int uvdx \geq |u|_{BV}.$$

Thus, $\int uvdx = \frac{1}{2\lambda}|u|_{BV}$. Hence, $\|v\|_* = \frac{1}{2\lambda}$.

Image segmentation

It consists of partitioning an image into its constituent and disjoint subregions.

Two main approaches in variational image segmentation

1. Edge based segmentation. Detects boundaries of objects.
2. Region based segmentation. Partitions an image into piecewise smooth regions, separated by piecewise regular boundaries.

Kass-Witkins-Terzopoulos. "Active contours" or "Snakes"

Deform an initial contour towards the boundary of the object to be detected. The deformation is obtained by minimizing a functional designed so that its local minimum is obtained at the boundary of the object.

Let $C(p) : [0, 1] \to \mathbb{R}^2$ be a parameterized planer curve.

Let $I : \Omega \to \mathbb{R}^+$ be a given image

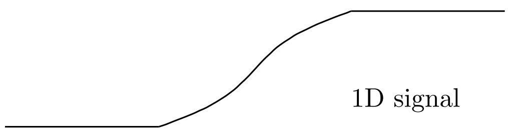
$$\inf_C E(C) = \alpha \int_0^1 |C'(p)|^2 dp + \beta \int_0^1 |C''(p)|^2 dp - \lambda \int_0^1 |\nabla I(C(p))| dp \ \alpha, \beta, \gamma \in \mathbb{R}^+$$

The first two terms control the smoothness of the contours C. The third term will drive the contours towards the object boundaries in the image I.

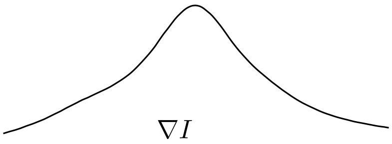
Remark. $E(C)$ is not intrinsic since it depends on the parameterization of the contours C. It does not change topology.

Edge detection: The process of labeling the locations in the image where the gray level's rate of change is relatively high.

Example.

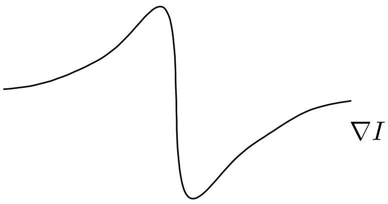


Sobel, Prewitt, other gradient estimator



local max of the image gradient magnitude

Marr Hildreth edge detector



zero crossing curves of ΔI

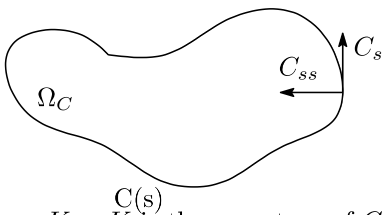
Edge integration: The process of combing or grouping local and perhaps sparse and non contiguous pieces into meaningful, long edge curves (or closed curves) for segmentation.

Let $C(p) = (x(p), y(p))$ be a planer, oriented curve in $\mathbb{R}^2$.

The normal vector $n = \frac{(-y_p, x_p)}{|C_p|}$, $|C_p| = \sqrt{x_p^2 + y_p^2}$

If C is regular, then one can parameterize it by arclengths

$$ds = \sqrt{dx^2 + dy^2} = |C_p|dp,\ C_s = \frac{C_p}{|C_p|}$$



$C_{ss} = Kn$, K is the curvature of C.

Let $\nabla I(x, y) = (I_x, I_y)$ be the image gradient vector field. The alignment of ∇I and n is measured by the inner product $< n, \nabla I >$. We look for the curves whose normals best align with the image gradient field. We consider some geometric functionals defined by curves.

Example. $E(C) := \int_0^L g(C(s))ds$ e.g.$g = \frac{1}{1+|\nabla G * I(x,y)|^2}$, $G = \frac{1}{2\pi\sigma^2} e^{\frac{-|x|^2}{2\sigma^2}}$

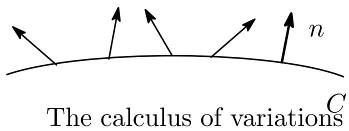
$E(C) := \iint_{\Omega_C} f(x, y)dxdy$; weighted area functional

The extremes of the functionals $E(C)$ can be identified by the Euler-Lagrange equations.

$\frac{\delta E}{\delta C}(C) = 0$ A dynamic process known as gradient descent takes an arbitrary curve towards a min/max of $E(C)$.

$\frac{\partial C}{\partial t}$, t is an artificial parameter.

We will see that the Marr-Hildreth edge detector, which computes the zero crossings of the image Laplacian, can be seen as optimal edge integration curves solving a geometric variational problem.



$< \nabla I, n >$: alignment measure

The calculus of variations for geometric measures. Given a curve integral of the general form

$$E(C) = \oint_C L(C, C_p)dp,$$

we compute the first variation by

$$\frac{\delta E(C)}{\delta C} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{d}{dp}\frac{\partial}{\partial x_p} \\ \frac{\partial}{\partial y} & \frac{d}{dp}\frac{\partial}{\partial y_p} \end{pmatrix} L(C, C_p)$$

LEMMA 1. *Given the vector field* $\vec{v}(x, y) = (u(x, y), v(x, y))$*, we have the alignment measure* $E_A(C) = \oint_C < \vec{v}, n > ds$ *for which the first variation is given by* $\frac{\delta E_A(C)}{\delta C} = \mathrm{div}(\vec{v})n$.

Proof.

$$\begin{aligned} E_A(C) &= \int_0^L < \vec{v}, n > ds \\ &= \int_0^L < (u, v), n > ds \\ &= \int_o^1 \left\langle (u, v), \frac{(-y_p, x_p)}{|C_p|} \right\rangle |C_p| dp \\ &= \int_0^1 (-uy_p + vx_p)dp \\ \frac{\delta E_A}{\delta x}(C) &= \left(\frac{\partial}{\partial x} - \frac{d}{dp}\frac{\partial}{\partial x_p} \right)(-uy_p + vx_p) \\ &= -u_x y_p + v_x x_p - \frac{d}{dp} v \\ &= -u_x y_p + v_x x_p - v_x x_p - v_y y_p \\ &= -y_p(u_x + v_y) \\ &= -y_p \mathrm{div}\vec{v}. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\delta E_A}{\delta y}(C) &= x_p(u_x + v_y) \\ &= x_p \mathrm{div}\vec{v}. \end{aligned}$$

By freedom of parameterization,

$$\frac{\delta E_A}{\delta C}(C) = \text{div}\vec{v}n.$$

Note. If $\vec{v} = \nabla I$, we have $\frac{\delta E_A(C)}{\delta C} = \Delta I \cdot n$, the Marr-Hildreth edge detector
LEMMA 2. *The robust alignment is given by* $E_{AR}(C) = \oint_C | < \vec{v}, n > | ds$ *yield the first variation* $\frac{\delta E_{AR}}{\delta C} = \text{sign} < \vec{v}, n > \text{div}(\vec{v})n$.
Proof.

$$\begin{aligned}
E_{AR}(C) = \int_0^L | < \vec{v}, n > | ds &= \int_0^1 \left| \left\langle (u, v), \frac{(-y_p, x_p)}{|C_p|} \right\rangle \right| |C_p| dp \\
&= \int_0^1 \sqrt{(vx_p - uy_p)^2} dp \\
\frac{\delta E_{AR}}{\delta x}(C) &= \left(\frac{\partial}{\partial x} - \frac{d}{dp}\frac{\partial}{\partial x_p} \right) \sqrt{(vx_p - uy_p)^2} \\
&= -y_p \, \text{sign}(vx_p - uy_p)(u_x + v_y) \\
&= -y_p \, \text{sign}(vx_p - uy_p)\text{div}\vec{v} \\
&= -y_p \, \text{sign} < \vec{v}, n > \text{div}\vec{v}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\delta E_{AR}}{\delta y}(C) &= x_p \, \text{sign}(vx_p - uy_p)(u_x + v_y) \\
&= x_p \, \text{sign}(vx_p - uy_p)\text{div}\vec{v} \\
&= x_p \, \text{sign} < \vec{v}, n > \text{div}\vec{v}.
\end{aligned}$$

Therefore,

$$\frac{\delta E_{AR}}{\delta C} = \text{sign} < \vec{v}, n > \text{div}\vec{v}n.$$

LEMMA 3. *The geodesic active contour model is* $E_{GAC}(C) = \oint_C g(C(s))ds$ *for which the first variation is given by* $\frac{\delta E_{gAC}}{\delta C}(C) = -(\kappa g - < \nabla g, n >)n$.

Proof.

$$\begin{aligned}
E_{GAC}(C) &= \int_0^L g(C(x(s), y(s)))ds \\
&= \int_0^1 g(C(p))|C_p|dp \\
&= \int_0^1 g(C(p))\sqrt{x_p^2 + y_p^2}dp \\
\frac{\delta E_{GAC}(C)}{\delta x} &= \left(\frac{\partial}{\partial x} - \frac{d}{dp}\frac{\partial}{\partial x_p}\right) g(C(p))\sqrt{x_p^2 + y_p^2} \\
&= g_x|C_p| - \frac{d}{dp}\left[g(C(p))\frac{x_p}{|C_p|}\right] \\
&= g_x|C_p| - (g_x x_p + g_y y_p)\frac{x_p}{|C_p|} - g \cdot \frac{x_{pp}|C_p| - x_p \cdot \frac{x_p x_{pp} + y_p y_{pp}}{|C_p|}}{|C_p|^2} \\
&= \frac{g_x|C_p|^2 - g_x x_p^2 - g_y x_p y_p}{|C_p|} - g\frac{x_{pp}|C_p|^2 - x_p(x_p x_{pp} + y_p y_{pp})}{|C_p|^3} \\
&= \frac{g_x y_p^2 - g_y x_p y_p}{|C_p|} - y_p g\kappa
\end{aligned}$$

where $\kappa = \frac{x_p y_{pp} - x_{pp} y_p}{|C_p|^3}$,

$$= y_p(- < \nabla g, n > + g\kappa).$$

Similarly,

$$\frac{\delta E_{GAC}(C)}{\delta y} = -x_p(- < \nabla g, n > + g\kappa)$$

so,

$$\frac{\delta E_{GAC}}{\delta C}(C) = -(g\kappa - < \nabla g, n >)n.$$

Note. If $g = 1$, $\frac{\delta E_{gAC}}{\delta C}(C) = -\kappa n$, mean curvature flow. $C_t = \kappa n = C_{ss}$, the geometric heat equation.

LEMMA 4. *The weight area function $E_w(C) = \iint_{\Omega_C} f(x,y)dx$ yields the first variation $\frac{\delta E_w(C)}{\delta C} = -f(x,y)n$.*

Proof. Define the two functions $P(x,y)$ and $Q(x,y)$ such that

$$P_y(x,y) = -\frac{1}{2}f(x,y), \text{ and } Q_x(x,y) = \frac{1}{2}f(x,y)$$

$$
\begin{aligned}
E_w(C) &= \iint_{\Omega_C} f(x,y)dxdy \\
&= \iint_{\Omega_C} (Q_x - P_y)dxdy \\
&= \int_C Pdx + Qdy \\
&= \int_C (Px_s + Qy_s)ds \\
&= \int_C < (-Q,P), n > ds.
\end{aligned}
$$

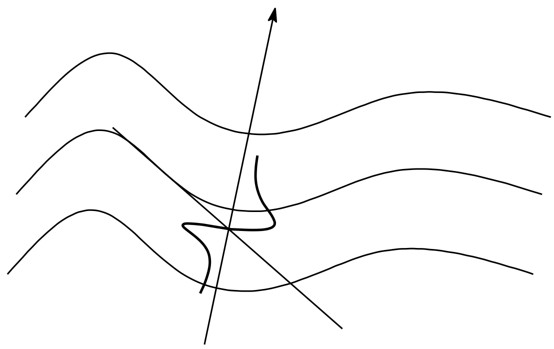
Let $\vec{v}(-Q,P)$. From lemma 1,

$$\frac{\delta E_w(C)}{\delta C} = \mathrm{div}\vec{v}n = (-Q_x + P_y)n = -f(x,y)n.$$

Note. Harralick/Canny/Deriche et.al.

$$\Delta I = I_{TT} + I_{NN}$$

$I_{NN} = 0$. i.e. The second derivative along the gradient direction.



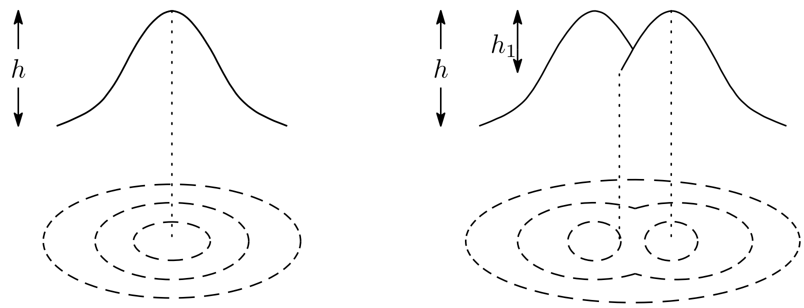
$$I_{NN} = \Delta I - I_{TT} = \Delta I - \mathrm{div}\left(\frac{\nabla I}{|\nabla I|}\right)|\nabla I| = \Delta I - \kappa|\nabla I|$$

Geometric functional that yields $I_{TT}n = 0$

$$\iint_{\Omega_C} I_{TT}dxdy = \iint_{\Omega_C} \kappa_I|\nabla I|dxdy = \int_{\mathbb{R}}\int_{\{I=t\}} \kappa_I dsdt$$

For a closed level set contour $\oint \kappa_I ds = 2\pi$

$\iint_{\Omega_C} I_{TT}dxdy$ measures the topological complexity of I over Ω_C.



$\iint_{\Omega_C} I_{TT} = 2\pi \cdot h$ $\qquad$ $\iint_{\Omega_C} I_{TT} = 2\pi(h + h_1)$

Edge integration

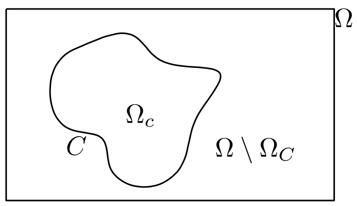
	$\frac{\delta E}{\delta C}$
$E_A(C) = \oint_C < \nabla I, n > ds$	$\Delta I n$
$E_{AR}(C) = \oint_C \vert < \nabla I, n > \vert ds$	$\text{sign}(< \nabla I, n >)\Delta I n$
$E_{gAR}(C) = \oint_C g(C(s))ds$	$(\kappa g - < \nabla g, n >)n$
$E_w(C) = \iint_{\Omega_C} f(x, y)dxdy$	$-f(x, y)n$

LEMMA 5. *The minimal variance is given by*

$$E_{MV}(C, d_1, d_2) = \frac{1}{2}\iint_{\Omega_C}(I - d_1)^2 dxdy + \frac{1}{2}\iint_{\Omega\setminus\Omega_C}(I - d_2)^2 dxdy$$

for which the variations are

$$\begin{aligned}
\frac{\delta E_{MV}}{\delta C} &= (d_2 - d_1)\left(I - \frac{d_1 + d_2}{2}\right)n \\
\frac{\delta E_{MV}}{\delta d_1} &= -\iint_{\Omega_1} I dxdy + d_1 \iint_{\Omega_C} dxdy \\
\frac{\delta E_{MV}}{\delta d_2} &= -\iint_{\Omega\setminus\Omega_C} I dxdy + d_2 \iint_{\Omega\setminus\Omega_C} dxdy.
\end{aligned}$$



Proof. By lemma 4,

$$\begin{aligned}
\frac{\delta E_{MV}}{\delta C} &= \frac{1}{2}((I-d_1)^2-(I-d_2)^2)n \\
&= \frac{1}{2}(I^2-2Id_1+d_1^2-I^2+2Id_2-d_2^2)n \\
&= (d_2-d_1)\left(I-\frac{d_1+d_2}{2}\right)n \\
\frac{\delta E_{MV}}{\delta d_1} &= \iint_{\Omega_C}(I-d_1)(-1)dxdy \\
&= -\iint_{\Omega_C} Idxdy+\iint_{\Omega_C} d_1dxdy \\
\frac{\delta E_{MV}}{\delta d_1} &= 0 \;\Rightarrow\; d_1=\frac{\iint_{\Omega_C} Idxdy}{\iint_{\Omega_C} dxdy}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\delta E_{MV}}{\delta d_2} &= -\iint_{\Omega\setminus\Omega_C} Idxdy+\iint_{\Omega\setminus\Omega_C} d_2dxdy \\
\frac{\delta E_{MV}}{\delta d_2} &= 0 \;\Rightarrow\; d_2=\frac{\iint_{\Omega\setminus\Omega_C} Idxdy}{\iint_{\Omega\setminus\Omega_C} dxdy}.
\end{aligned}$$

LEMMA 6. *The minimal total variation is given by* $E_{Md}(C,d_1,d_2)$

$$E_{Md}(C,d_1,d_2)=\iint_{\Omega_C}|I-d_1|dxdy+\iint_{\Omega\setminus\Omega_C}|I-d_2|dxdy$$

for which the first variations are

$$\begin{aligned}
\frac{\delta E_{Md}}{\delta C} &= (|I-d_1|-|I-d_2|)n \\
\frac{\delta E_{Md}}{\delta d_1} &= -\iint_{\Omega_C}\mathrm{sign}(I-d_1)dxdy \\
\frac{\delta E_{Md}}{\delta d_2} &= -\iint_{\Omega\setminus\Omega_C}\mathrm{sign}(I-d_2)dxdy.
\end{aligned}$$

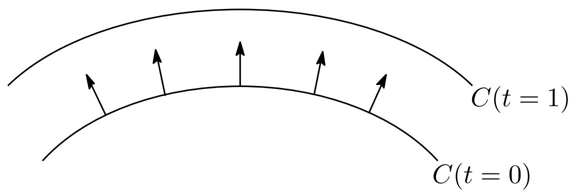
Proof. By lemma 4,

$$
\begin{aligned}
\frac{\delta E_{Md}}{\delta C} &= (|I-d_1| - |I-d_2|)n \\
\frac{\delta E_{Md}}{\delta d_1} &= \frac{\partial}{\partial d_1}\iint_{\Omega_C}\sqrt{(I-d_1)^2}dxdy \\
&= \iint_{\Omega_C} -\frac{I-d_1}{|I-d_1|}dxdy \\
&= -\iint_{\Omega_C}\operatorname{sign}(I-d_1)dxdy \\
\frac{\delta E_{Md}}{\delta d_1} &= 0 \;\Rightarrow\; \iint_{\Omega_C}\operatorname{sign}(I-d_1)dxdy = 0,\; d_1 = \operatorname*{median}_{\Omega_C} I(x,y).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{\delta E_{Md}}{\delta d_2} &= -\iint_{\Omega\backslash\Omega_C}\operatorname{sign}(I-d_2)dxdy \\
\frac{\delta E_{Md}}{\delta d_1} &= 0 \;\Rightarrow\; d_2 = \operatorname*{median}_{\Omega\backslash\Omega_C} I(x,y).
\end{aligned}
$$

Moving curves/surfaces

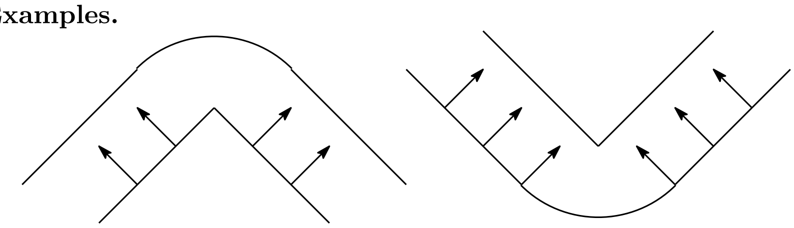


curve propagating with speed F in normal direction

Marker-Particle method ("Lagrangian")

Follow the marker particles on the curves/surfaces

Examples.



Rarefaction Shock

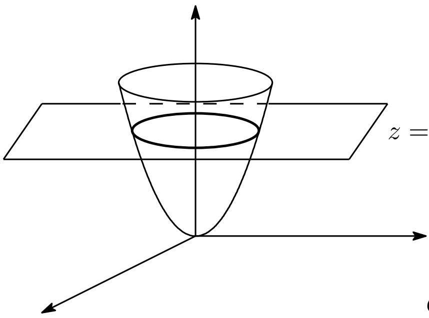
Lagragian method: accurate for small scale motions of interface

Drawbacks: instabilities and topological limitations

Level set method: Osher, Sethian

A closed curve in $\mathbb{R}^2$ can be represented as the zero level set of a function in high dimension.
Example. $\phi(x, y) = x^2 + y^2$

$$z = \phi(x, y) = x^2 + y^2$$



C: level set of $\phi(x, y)$

Let $\phi : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz continuous function such that

$$\phi(t, x(t, p)) = 0.$$

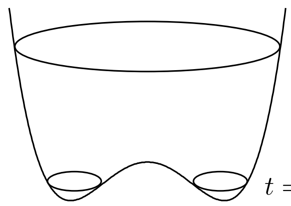
Differentiating ϕ with respect to t, we have

$$\phi_t + \nabla\phi \cdot \vec{\nu} = 0, \quad \vec{\nu} = \nu_N \vec{N} + \nu_T \vec{T}.$$

Since the motion is only in the normal direction

$$\begin{aligned} &\vec{\nu} = \nu_N \vec{N} \\ &\phi_t + \nabla\phi \cdot \nu_N \vec{N} = 0, \quad \vec{N} = \tfrac{\nabla\phi}{|\nabla\phi|} \\ &\phi_t + \nu_N |\nabla\phi| = 0 \qquad \text{"Level set equation"}. \end{aligned}$$

Note. It handles topological changes such as merging and splitting
Example.



	$E(C)$	$\frac{\delta E}{\delta C}$	Level set form
weight area	$\iint_{\Omega_C} f(x, y)dxdy$	$-f(x, y)n$	$-f(x, y)\lvert\nabla\phi\rvert$
GA	$\oint g(C(s))ds$	$(<\nabla g, u> -\kappa g)n$	$-\mathrm{div}(g\frac{\nabla\phi}{\lvert\nabla\phi\rvert})\lvert\nabla\phi\rvert$

The Munford-Shah image segmentation problem.

Ω: bounded open set of $\mathbb{R}^2$

f: given gray scale image

Munford, Shah proposed an image segmentation model in which they look for a simplified approximation u of f such that u has slow variations on $\Omega \setminus K$ and u has discontinuous or jumps on the singular set K

$$F_{MS}(u,K) = \int_{\Omega\setminus K} |u-f|^2 + \alpha \int_{\Omega\setminus K} |\nabla u|^2 + \beta \int_K d\sigma,$$

$$\alpha, \beta > 0, \int_K d\sigma \text{ is the length of } K$$

Conjecture: There exists a minimizer of F_{MS} such that the edges (the discontinuity set K) are the finite union of C^1 curves.

Moreover, each curve can meet $\partial\Omega$ perpendicularly or end either as a crack tip or in a triple junction with $\frac{3\pi}{2}$ angle between each pair.

If we want to extend the notion of length to nonsmooth sets, we rewrite F_{MS} as

$$F_{MS}(u,K) = \int_{\Omega\setminus K} |u-f|^2 + \alpha \int_{\Omega\setminus K} |\nabla u|^2 + \beta \mathcal{H}^{N-1}(K)$$

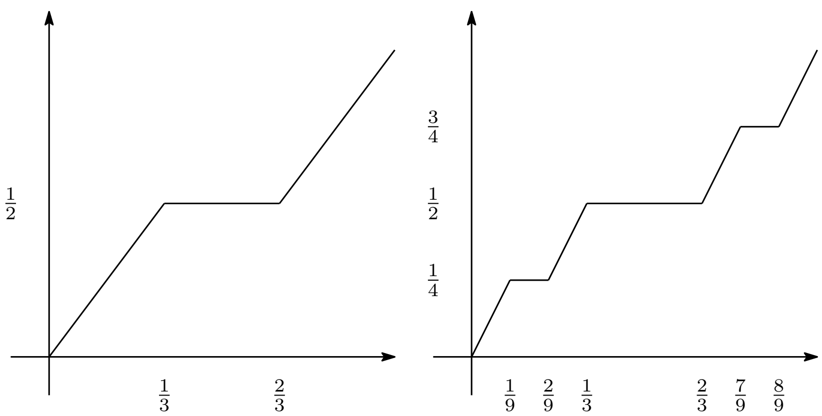
In order to apply the direct method of the Calculus of variations, we reformulate F_{MS} such that it involves functions only. De Giorgi, Carriero and Leaci proposed to identify the set of discontinuities K with the jump set S_u of u, which allows to eliminate K.

$$F(u) = \int_{\Omega} (u-f)^2 + \alpha \int_{\Omega\setminus S_u} |\nabla u|^2 + \beta \mathcal{H}^{N-1}(S_u)$$

Existence of a minimizer was established.

But $BV(\Omega)$ contains nonconstant continuous functions whose approximate differential equals zero almost everywhere.

Example. The cantor vitali function



u_1 u_2

For these functions, F reduces to $\int_\Omega (u-f)^2$ and does not depend on the distributional derivatives of u.

DEFINITION. *The space of special functions of bounded variation SBV is defined as the space of bounded variation $BV(\Omega)$ such that $Cu = 0$.*

The weak formulation of F_{MS}

$$\inf_{u\in SBV(\Omega)} \left\{ F(u) = \int_\Omega (u-f)^2 + \alpha \int_{\Omega\setminus S} |\nabla u|^2 + \beta \mathcal{H}^{N-1}(S_u) \right\}$$

Ambroisio, Fusco, Pallara: existence of a solution $u \in SBV(\Omega)$ with $\mathcal{H}^{N-1}(S_u) < \infty$ using compactness in $SBV(\Omega)$ and lower semicontinuity theorems.

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