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Proceedings of the 33rd Sapporo Symposium  
on Partial Differential Equations

Edited by  
T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura,  
Y. Tonegawa, K. Tsutaya, and T. Sakajo

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# Proceedings of the 33rd Sapporo Symposium on Partial Differential Equations

Edited by

T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura,  
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Sapporo, 2008

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## PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 26 through August 28 in 2008 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 30 years ago. Professor Kôji Kubota and Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura,  
Y. Tonegawa, K. Tsutaya, and T. Sakajo

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# The 33rd Sapporo Symposium on Partial Differential Equations (第33回偏微分方程式論札幌シンポジウム)

組織委員: 小澤 徹, 儀我 美一, 神保 秀一, 中村 玄, 利根川 吉廣, 津田谷 公利, 坂上 貴之  
Organizers: T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura, Y. Tonegawa, K. Tsutaya, T. Sakajo

Period (期間) August 26 - 28 , 2008

Venue (場所) Room 203, Faculty of Science Building #5, Hokkaido University  
北海道大学 理学部 5号館大講義室 (203号室)

URL <http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program.html>

August 26, 2008 (Tuesday)

9:30-9:40 Opening Session

9:40-10:40 磯崎 洋 (筑波大学) Hiroshi Isozaki (University of Tsukuba)  
Spectral theory and inverse problems on asymptotically hyperbolic manifolds

11:10-12:10 Bendong Lou (Tongji University)  
Curved traveling waves of a mean curvature flow in heterogeneous media

14:00-14:30 Free discussion with speakers in the tea room

14:30-15:00 加藤 正和 (東北大学) Masakazu Kato (Tohoku University)  
Large time behavior of solutions to the Burgers-Poisson equations

15:15-15:45 関 行宏 (東京大学) Yukihiro Seki (University of Tokyo)  
Mean curvature flow closes open ends of noncompact surface of rotation

16:00-16:30 河野 真士 (北海道大学) Shinji Kawano (Hokkaido University)  
A remark on the uniqueness of positive solutions to semilinear elliptic equations  
with double power nonlinearities

16:30-17:00 Free discussion with speakers in the tea room

August 27, 2008 (Wednesday)

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Asymptotic behavior of global solutions to some systems of semilinear wave equations
- 11:00-12:00 田沼 一実 (群馬大学) Kazumi Tanuma (Gunma University)  
Perturbation and dispersion of Rayleigh waves in anisotropic elasticity  
(joint work with Chi-Sing Man (University of Kentucky),  
Gen Nakamura (Hokkaido University) and Shengzhang Wang (Fudan University))
- 14:00-14:30 Free discussion with speakers in the tea room
- 14:30-15:30 新居 俊作 (九州大学) Shunsaku Nii (Kyushu University)  
A topological index for eigenvalue problems: a  $\pi_2$ -theory of operators
- 15:45-16:15 渡辺 道之 (東京理科大学) Michiyuki Watanabe (Tokyo University of Science)  
Inverse boundary value problem for elliptic equation with complex coefficient
- 16:15-16:45 Free discussion with speakers in the tea room
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The variational problem for a certain action functional defined on closed curves
- 18:00-20:00 Reception at Enreiso (懇親会, エンレイソウ)

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- 9:30-10:30 森田 善久 (龍谷大学) Yoshihisa Morita (Ryukoku University)  
Entire solutions associated with front waves to reaction-diffusion equations
- 11:00-12:00 高木 泉 (東北大学) Izumi Takagi (Tohoku University)  
Collapse of patterns in some reaction-diffusion systems
- 12:00-13:00 Free discussion with speakers in the tea room

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# SPECTRAL THEORY AND INVERSE PROBLEMS ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

HIROSHI ISOZAKI

## 1. INTRODUCTION

Given non-compact Riemannian manifolds, can one identify the metric by observing the waves at infinity? More precisely, we send waves from infinities of the manifold and observe the scattered waves at infinities. From the observed data, we try to identify or reconstruct the metric or the manifold itself. For this to be possible, the geometric structure of infinity is important. The model of the manifold will thus be a composition of a bounded part and a neighborhood of infinity, on which we equip with a metric slightly perturbed from the standard one. Let  $H_0$  and  $H$  be the unperturbed and perturbed Laplace-Beltrami operators of this manifold. The waves are created by the Schrödinger equation,  $i\partial_t u = Hu$ , or the wave equation,  $\partial_t^2 u + Hu = 0$ . The wave operator  $W_\pm$  is defined by

$$W_\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}$$

for the Schrödinger equation, and by

$$W_\pm = s - \lim_{t \rightarrow \pm\infty} e^{it\sqrt{H}} e^{-it\sqrt{H_0}}$$

for the wave equation. We used the same notation  $W_\pm$  for both of Schrödinger, and wave equations, since it is known that they usually coincide. The scattering operator is then defined by

$$S = (W_+)^* W_-,$$

which assigns to the far field pattern of the remote past,  $f_-$ , the far field pattern of the remote future,  $f_+$ , since for  $e^{-itH_0} f_-$  there exists  $f$  such that

$$e^{-itH} f - e^{-itH_0} f_\pm \rightarrow 0, \quad t \rightarrow \pm\infty.$$

This also holds with  $H, H_0$  replaced by  $\sqrt{H}, \sqrt{H_0}$ , i.e. for the wave equation. The scattering operator  $S$  is believed to contain all the information of the manifold, even though it is apparently related with only the infinity of the manifold. Our problem is thus stated as follows : *From the scattering operator, reconstruct the Riemannian metric, or the manifold itself.*

Few is known about this general setting. Even when the metric on the end is asymptotically Euclidean, this problem is still open. The only exception is the recent result of Sa Barreto [SaBa05], which deals with the asymptotically hyperbolic metric in the framework of Melrose' scattering metric. However, it is also known that the problem becomes more accessible when we allow only the local perturbation of the metric. Namely, if two metrics are asymptotic to the standard one, and they coincide near infinity, they actually coincide everywhere if their scattering operators

coincide. We address to this modest problem, allowing the metric perturbation as general as possible.

To formulate this problem in a proper mathematical setting, we need to develop spectral theories on non-compact Riemannian manifolds, especially the theory of generalized Fourier transformations. We shall announce some recent results in the case of asymptotically hyperbolic metric. Details will be seen in [IsKu08].

## 2. CLASSIFICATION OF 2-DIMENSIONAL HYPERBOLIC MANIFOLDS

Before entering into our framework, let us recall the classical example. The hyperbolic manifold is, by definition, a complete Riemannian manifold with all sectional curvatures equal to  $-1$ . General hyperbolic manifolds are constructed by the action of discrete groups on the upper-half space. The resulting quotient manifold is either compact, or non-compact but finite volume, or non-compact with infinite volume. In the latter two cases, the manifold can be split into bounded part and unbounded part, this latter being called the end. To study the general structure of ends is beyond our scope. We briefly look at the 2-dimensional case.

Recall that  $\mathbf{C}_+ = \{z = x + iy; y > 0\}$  is a 2-dimensional hyperbolic space equipped with the metric

$$(2.1) \quad ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}.$$

Let  $\partial\mathbf{C}_+ = \partial\mathbf{H}^2 = \{(x, 0); x \in \mathbf{R}\} \cup \infty = \mathbf{R} \cup \infty$ . For a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$$

the Möbius transformation is defined by

$$(2.2) \quad \mathbf{C}_+ \ni z \rightarrow \gamma z := \frac{az + b}{cz + d},$$

which is an isometry on  $\mathbf{H}^2$ . This transformation  $\gamma$  is classified into 3 categories :

*elliptic*  $\iff$  there is only one fixed point in  $\mathbf{C}_+$

$$\iff |\operatorname{tr} \gamma| < 2,$$

*parabolic*  $\iff$  there is only one degenerate fixed point in  $\partial\mathbf{C}_+$

$$\iff |\operatorname{tr} \gamma| = 2,$$

*hyperbolic*  $\iff$  there are two fixed points in  $\partial\mathbf{C}_+$

$$\iff |\operatorname{tr} \gamma| > 2.$$

Let  $\Gamma$  be a discrete subgroup of  $SL(2, \mathbf{R})$ , which is usually called a *Fuchsian* group. Let  $\mathcal{M} = \Gamma \backslash \mathbf{H}^2$  be the fundamental domain for the action (2.2).  $\Gamma$  is said to be *geometrically finite* if  $\mathcal{M}$  is chosen to be a finite-sided convex polygon. The sides are then geodesics of  $\mathbf{H}^2$ . The geometric finiteness is equivalent to that  $\Gamma$  is finitely generated.

As a simple example, consider the cyclic group  $\Gamma$  generated by the action  $z \rightarrow z + 1$ . This is parabolic with fixed point  $\infty$ . The associated fundamental domain is

$$\mathcal{M} = (-1/2, 1/2] \times (0, \infty),$$

which is a hyperbolic manifold with metric (2.1). It has two infinities :  $(-1/2, 1/2] \times \{0\}$  and  $\infty$ . The part  $(-1/2, 1/2] \times (0, 1)$  has an infinite volume, and is called the

thick part. The part  $(-1/2, 1/2] \times (1, \infty)$  has a finite volume, and is called the cusp. Note that the sides  $x = \pm 1/2$  are geodesics.

Another simple example is the cyclic group generated by the action  $z \rightarrow \lambda z$ ,  $\lambda > 1$ , which is hyperbolic. The sides of the fundamental domain are semi-circles perpendicular to  $y = 0$ , which are geodesics. The quotient manifold is diffeomorphic to  $S^1 \times (-\infty, \infty)$ . It is parametrized by  $(t, r)$ , where  $t \in \mathbf{R}/\log \lambda \mathbf{Z}$  and  $r$  is the signed distance from the segment  $\{(0, t); 1 \leq t \leq \lambda\}$ . The metric is then written as

$$(2.3) \quad ds^2 = (dr)^2 + \cosh^2 r (dt)^2.$$

The part  $x > 0$  (or  $x < 0$ ) is called the funnel. Letting  $y = 2e^{-r}$ , one can rewrite (2.3) as

$$ds^2 = \left(\frac{dy}{y}\right)^2 + \left(\frac{1}{y} + \frac{y}{4}\right)^2 (dt)^2.$$

Let  $\Lambda(\Gamma)$  be the set of all limit points of the orbit  $\{\gamma z; \gamma \in \Gamma\}$ . Since  $\Gamma$  acts discontinuously on  $\mathbf{C}_+$ ,  $\Lambda(\Gamma) \subset \partial \mathbf{H}^2$ . If  $\Lambda(\Gamma)$  is a finite set,  $\Gamma$  is said to be *elementary*. In this case,  $\mathcal{M}$  is either  $\mathbf{H}^2$ , or the quotient manifold by hyperbolic, or parabolic cyclic groups. For non-elementary case, we have the following theorem (see [Bo07]).

**Theorem 2.1.** *Let  $\mathcal{M} = \Gamma \backslash \mathbf{H}^2$  be a non-elementary geometrically finite hyperbolic manifold. Then there exists a compact subset  $\mathcal{K}$  such that  $\mathcal{M} \setminus \mathcal{K}$  is a finite disjoint union of cusps and funnels.*

### 3. ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

**3.1. Assumptions on ends.** With the above example in mind, we state our assumptions on the manifold. We consider an  $n$ -dimensional connected Riemannian manifold  $\mathcal{M}$ , which is written as a union of open sets:

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N.$$

The following assumptions are imposed:

(A.1)  $\bar{\mathcal{K}}$  is compact.

(A.2)  $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ ,  $i \neq j$ .

(A.3) Each  $\mathcal{M}_i$ ,  $i = 1, \dots, N$ , is diffeomorphic either to  $\mathcal{M}_0 = M \times (0, 1)$  or to  $\mathcal{M}_\infty = M \times (1, \infty)$ ,  $M$  being a compact Riemannian manifold of dimension  $n - 1$ . Here the manifold  $M$  is allowed to be different for each  $i$ .

(A-4) On each  $\mathcal{M}_i$ , the Riemannian metric  $ds^2$  has the following form

$$ds^2 = y^{-2} \left( (dy)^2 + h(x, dx) + A(x, y, dx, dy) \right),$$

$$A(x, y, dx, dy) = \sum_{i,j=1}^{n-1} a_{ij}(x, y) dx^i dx^j + 2 \sum_{i=1}^{n-1} a_{in}(x, y) dx^i dy + a_{nn}(x, y) (dy)^2,$$

where  $h(x, dx) = \sum_{i,j=1}^{n-1} h_{ij}(x) dx^i dx^j$  is a positive definite metric on  $M$ , and  $a_{ij}(x, y)$ ,  $1 \leq i, j \leq n$ , satisfies the following condition

$$(3.1) \quad |\tilde{D}_x^\alpha D_y^m a(x, y)| \leq C_{\alpha m} (1 + |\log y|)^{-|\alpha| - m - 1 - \epsilon_0}, \quad \forall \alpha, m$$

for some  $\epsilon_0 > 0$ . Here  $\tilde{D}_x = \tilde{y}(y) \partial_x$ ,  $\tilde{y}(y) \in C^\infty((0, \infty))$  such that  $\tilde{y}(y) = y$  for  $y > 2$  and  $\tilde{y}(y) = 1$  for  $0 < y < 1$ .

Let us note that the above model in particular contains  $\mathbf{H}^n$ . In fact, we take  $\mathcal{M}$  to be  $\mathcal{K} \cup \mathcal{M}_1$ , where  $\mathcal{M}_1$  is diffeomorphic to  $S^{n-1} \times (1, \infty)$  equipped with the metric  $(dr)^2 + \sinh^2 r (d\theta)^2$ , the hyperbolic metric written by geodesic polar coordinates. Taking  $e^r = 2/y$ , we arrive at the above model.

The 2nd important remark is that if  $\mathcal{M}_i$  is diffeomorphic to  $M \times (0, 1)$ , one can transform the above metric into the form

$$(3.2) \quad ds^2 = y^{-2} \left( (dy)^2 + h(x, dx) + \sum_{i,j=1}^{n-1} a_{ij}(x, y) dx^i dx^j \right)$$

with  $a_{ij}(x, y)$  satisfying the condition (3.1). Therefore in the following we consider the metric of the form (3.2) for such ends.

Compared with the example in §2, the infinity of  $\mathcal{M}_0$  corresponds to the funnel or the thick part, and that of  $\mathcal{M}_\infty$  to the cusp. In this paper, the former part is called regular infinity, and the latter is said to be cusp.

Let  $\Delta_g$  be the Laplace-Beltrami operator on  $\mathcal{M}$ . Let  $V$  be a 1st order differential operator on  $\mathcal{M}$  with  $C^\infty$ -coefficients such that  $H = -\Delta_g - (n-1)^2/4 + V$  satisfies the following conditions.

(A.5)  $H$  is formally self-adjoint. Namely

$$(H\varphi, \psi) = (\varphi, H\psi), \quad \forall \varphi, \psi \in C_0^\infty(\mathcal{M}),$$

where  $(\cdot, \cdot)$  is the inner product of  $L^2(\mathcal{M})$ , i.e.,

$$(\varphi, \psi) = \int_{\mathcal{M}} \varphi \bar{\psi} d\mathcal{M},$$

$d\mathcal{M}$  being the measure induced from the metric on  $\mathcal{M}$ .

(A.6)  $V$  is short-range on each  $\mathcal{M}_i$  ( $1 \leq i \leq N$ ). Namely if  $V$  is represented as

$$V = \sum_{|\alpha| \leq 1} a_\alpha(x, y) D^\alpha, \quad D = (D_x, D_y) = (y\partial_x, y\partial_y),$$

there exists a constant  $\epsilon > 0$  such that

$$|\tilde{D}_x^\beta D_y^k a_\alpha(x, y)| \leq C_{\beta, k} (1 + |\log y|)^{-1-\epsilon}, \quad \forall \beta, \quad \forall k.$$

We use the following partition of unity. Fix  $x_0 \in \mathcal{K}$  arbitrarily, and pick  $\chi_0 \in C_0^\infty(\mathcal{M})$  such that

$$\chi_0(x) = \begin{cases} 1, & \text{dis}(x, x_0) < R, \\ 0, & \text{dis}(x, x_0) > R + 1. \end{cases}$$

Taking  $R$  large enough, we define  $\chi_j \in C^\infty(\mathcal{M})$  such that

$$\chi_j(x) = \begin{cases} 1 - \chi_0(x), & x \in \mathcal{M}_j, \\ 0, & x \notin \mathcal{M}_j. \end{cases}$$

Then we have

$$\begin{cases} \sum_{j=0}^N \chi_j = 1, \\ \text{supp } \chi_j \subset \mathcal{M}_j, \quad 1 \leq j \leq N, \\ \chi_0 = 1 \quad \text{on } \mathcal{K}. \end{cases}$$

For  $1 \leq j \leq N$ , we construct  $\tilde{\chi}_j \in C^\infty(\mathcal{M})$  such that

$$\text{supp } \tilde{\chi}_j \subset \mathcal{M}_j, \quad \tilde{\chi}_j = 1 \quad \text{on } \text{supp } \chi_j.$$

**3.2. The Besov type space.** To study the resolvent estimates, the following Besov type space plays a key role. Let  $\mathbf{H}$  be a Hilbert space endowed with norm  $\|\cdot\|$ . We decompose  $(0, \infty)$  into  $(0, \infty) = \cup_{k \in \mathbf{Z}} I_k$ , where

$$I_k = \begin{cases} (\exp(e^{k-1}), \exp(e^k)], & k \geq 1 \\ (e^{-1}, e], & k = 0 \\ (\exp(-e^{|k|}), \exp(-e^{|k|-1})], & k \leq -1. \end{cases}$$

We fix an integer  $n \geq 2$  and put

$$d\mu(y) = \frac{dy}{y^n}.$$

Let  $\mathcal{B}$  be the space of  $\mathbf{H}$ -valued function on  $(0, \infty)$  satisfying

$$\|f\|_{\mathcal{B}} = \sum_{k \in \mathbf{Z}} e^{|k|/2} \left( \int_{I_k} \|f(y)\|^2 d\mu(y) \right)^{1/2} < \infty.$$

The dual space  $\mathcal{B}^*$  is identified with the space equipped with norm

$$\|u\|_{\mathcal{B}^*} = \left( \sup_{R > e} \frac{1}{\log R} \int_{\frac{1}{R} < y < R} \|u(y)\|_{\mathbf{H}}^2 d\mu \right)^{1/2} < \infty.$$

For  $s \in \mathbf{R}$ , we define the space  $L^{2,s}$  by

$$u \in L^{2,s} \iff \|u\|_s^2 = \int_0^\infty (1 + |\log y|)^{2s} \|u(y)\|_{\mathbf{H}}^2 d\mu(y) < \infty.$$

For  $s > 1/2$ , we have the following inclusion relations :

$$L^{2,s} \subset \mathcal{B} \subset L^{2,1/2} \subset L^2 \subset L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}.$$

In the upper half-space model  $\mathbf{R}_+^n$ , we represent a point of  $\mathbf{R}_+^n$  as  $(x, y), x \in \mathbf{R}^{n-1}, y > 0$ , and put  $\mathbf{H} = L^2(\mathbf{R}^{n-1})$ .

For our manifold  $\mathcal{M}$ , the spaces  $L^{2,s}$ ,  $\mathcal{B}$ ,  $\mathcal{B}^*$  are defined in the same way as above using the partition of unity.

### 3.3. Resolvent estimates.

**Theorem 3.1.** (1)  $H|_{C_0^\infty(\mathcal{M})}$  is essentially self-adjoint.

(2)  $\sigma_e(H) = [0, \infty)$ .

**Theorem 3.2.** (1) If one of the  $\mathcal{M}_i$ 's is diffeomorphic to  $\mathcal{M}_0$ ,  $\sigma_p(H) \cap (0, \infty) = \emptyset$ .

(2) If all of the  $\mathcal{M}_i$ 's are diffeomorphic to  $\mathcal{M}_\infty$ , then  $\sigma_p(H) \cap (0, \infty)$  is discrete with finite multiplicities, whose possible accumulation points are 0 and  $\infty$ .

We put  $\sigma_\pm(\lambda) = \frac{n-1}{2} \mp i\sqrt{\lambda}$ . We say that a solution  $u \in \mathcal{B}^*$  of the equation

$$(H - \lambda)u = f \in \mathcal{B}$$

satisfies the outgoing radiation condition, when  $\mathcal{M}_i$  has a regular infinity

$$(3.3) \quad \lim_{R \rightarrow \infty} \frac{1}{\log R} \int_{1/R}^{1/2} \|(D_y - \sigma_+(\lambda))u(\cdot, y)\|_{L^2(\mathbf{E}_i)}^2 \frac{dy}{y^n} = 0,$$

and when  $\mathcal{M}_i$  has a cusp

$$(3.4) \quad \lim_{R \rightarrow \infty} \frac{1}{\log R} \int_2^R \|(D_y - \sigma_-(\lambda))u(\cdot, y)\|_{L^2(\mathbf{E}_i)}^2 \frac{dy}{y^n} = 0.$$

The incoming radiation condition is defined similarly with  $\sigma_+(\lambda)$  replaced by  $\sigma_-(\lambda)$ .

**Theorem 3.3.** For  $\lambda \in \sigma_e(H) \setminus \sigma_p(H)$ , there exists a limit

$$\lim_{\epsilon \rightarrow 0} R(\lambda \pm i\epsilon) \equiv R(\lambda \pm i0) \in \mathbf{B}(\mathcal{B}; \mathcal{B}^*)$$

in the weak  $*$ -sense. Moreover for any compact interval  $I \subset \sigma_e(H) \setminus \sigma_p(H)$  there exists a constant  $C > 0$  such that

$$\|R(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad \lambda \in I.$$

For  $f \in \mathcal{B}$ , we put  $u = R(\lambda \pm i0)f$ . Then  $u$  is a unique solution to the equation  $(H - \lambda)u = f$  satisfying the outgoing (for the case +), incoming (for the case -) radiation condition. For  $f, g \in \mathcal{B}$ ,  $(R(\lambda \pm i0)f, g)$  is continuous with respect to  $\lambda > 0$ .

**3.4. Fourier transforms associated with  $H$ .** Let  $H_{0j} = -\Delta_j$  be the Laplace-Beltrami operator on  $M_j \times (0, \infty)$ , and  $\chi_j$  the partition of unity. Letting

$$R_{0j}(z) = (H_{0j} - z)^{-1},$$

we have

$$\begin{aligned} \chi_j R(\lambda \pm i0) &= R_{0j}(\lambda \pm i0)\chi_j \\ &+ R_{0j}(\lambda \pm i0) ([H_{0j}, \chi_j] - \chi_j V) R(\lambda \pm i0). \end{aligned}$$

We assume that  $\mathcal{M}_j$ ,  $1 \leq j \leq M$  has regular infinity, and  $\mathcal{M}_j$ ,  $M+1 \leq j \leq N$  has cusp.

*Definition of  $\mathcal{F}_{0j}^{(\pm)}(k)$ .* We define  $\mathcal{F}_{0j}^{(\pm)}(k)$  as follows. Let  $\lambda_{j,1}, \lambda_{j,2}, \dots$  be the eigenvalues of the Laplace-Beltrami operator on  $M_j$  and  $\varphi_{j,1}, \varphi_{j,2}, \dots$  be the associated eigenvectors. For  $f(x, y)$ , let  $\widehat{f}_m(y) = \langle f(\cdot, y), \varphi_m(\cdot) \rangle$  be its Fourier coefficient.

(i) For  $1 \leq j \leq M$  (the case of regular infinity)

$$(\mathcal{F}_{0j}^{(\pm)}(k)f)(x) = \sum_{m \geq 0} C_m^{(\pm)}(k) \varphi_{j,m}(x) F_{0,m}^{(\pm)}(k) \widehat{f}_m(\cdot),$$

where the right-hand side is defined by

$$(F_{0m}f)(k) = \frac{(2k \sinh(k\pi))^{1/2}}{\pi} \int_0^\infty y^{(n-1)/2} K_{ik}(\sqrt{\lambda_m} y) f(y) \frac{dy}{y^n},$$

$$F_0^{(\pm)}f(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty y^{\frac{n-1}{2} \pm ik} f(y) \frac{dy}{y^n},$$

$$F_{0m}^{(\pm)} = \begin{cases} F_{0m}(\lambda_m \neq 0), \\ F_0^{(\pm)}(\lambda_m = 0), \end{cases}$$

$$c_m^{(\pm)}(k) = \begin{cases} \left(\frac{\sqrt{\lambda_m}}{2}\right)^{\mp ik}, & (\lambda_m \neq 0), \\ \frac{\pm i}{k\omega_\pm(k)} \sqrt{\frac{\pi}{2}}, & (\lambda_m = 0), \end{cases}$$

$$\omega_\pm(k) = \frac{\pi}{(2k \sinh(k\pi))^{1/2} \Gamma(1 \mp ik)}.$$

(ii) For  $M + 1 \leq j \leq N$  (the case of cusp)

$$\mathcal{F}_{0j}^{(\pm)}(k)f = \frac{1}{\sqrt{|\mathbf{E}_j|}} F_{00}^{(\mp)}(k) \widehat{f}_0(\cdot).$$

We put

$$(3.5) \quad \mathcal{F}_{jm}^{(\pm)}(k) = F_{0m}^{(\pm)}(k) (\chi_j + ([H_{0j}, \chi_j] - \chi_j V) R(k^2 \pm i0)).$$

*Definition of  $\mathcal{F}^{(\pm)}(k)$ .* The Fourier transformation associated with  $H$  is defined by

$$\mathcal{F}^{(\pm)}(k) = (\mathcal{F}_1^{(\pm)}(k), \dots, \mathcal{F}_N^{(\pm)}(k)),$$

where for  $1 \leq j \leq M$

$$\begin{aligned} \mathcal{F}_j^{(\pm)}(k) &= \mathcal{F}_{0j}^{(\pm)}(k) [\chi_j + ([H_{0j}, \chi_j] - \chi_j V) R(k^2 \pm i0)] \\ &= \sum_{m \geq 0} C_m^{(\pm)}(k) \varphi_{j,m}(x) \mathcal{F}_{jm}^{(\pm)}(k), \end{aligned}$$

and for  $M + 1 \leq j \leq N$

$$(3.6) \quad \begin{aligned} \mathcal{F}_j^{(\pm)}(k) &= \mathcal{F}_{0j}^{(\pm)}(k) [\chi_j + ([H_{0j}, \chi_j] - \chi_j V) R(k^2 \pm i0)] \\ &= \frac{1}{\sqrt{|M_j|}} \mathcal{F}_{j0}^{(\pm)}(k). \end{aligned}$$

For functions  $f, g \in \mathcal{B}^*$  on  $\mathcal{M}$ , by  $f \simeq g$  we mean that on each end

$$\begin{aligned} \lim_{R \rightarrow \infty} \frac{1}{\log R} \int_{1 < y < R} \|f(y) - g(y)\|_{L^2(M_i)}^2 \frac{dy}{y^n} &= 0, \\ \lim_{R \rightarrow \infty} \frac{1}{\log R} \int_{\log(1/R) < y < 1} \|f(y) - g(y)\|_{L^2(M_i)}^2 \frac{dy}{y^n} &= 0. \end{aligned}$$

**Theorem 3.4.** *Let  $f \in \mathcal{B}$ ,  $k^2 \in \sigma_\varepsilon(H) \setminus \sigma_p(H)$ , and  $\chi_j$  the partition of unity. Then we have*

$$\begin{aligned} R(k^2 \pm i0)f &\simeq \omega_\pm(k) \sum_{j=1}^M \chi_j y^{(n-1)/2 \mp ik} \mathcal{F}_j^{(\pm)}(k) f \\ &\quad + \omega_\pm^{(c)}(k) \sum_{j=M+1}^N \chi_j y^{(n-1)/2 \pm ik} \mathcal{F}_j^{(\pm)}(k) f. \end{aligned}$$

We put

$$\mathbf{h}_\infty = (\oplus_{i=1}^M L^2(M_i)) \oplus (\oplus_{i=M+1}^N \mathbf{C}),$$

and for  $\varphi, \psi \in \mathbf{h}_\infty$  we define the inner product by

$$(\varphi, \psi)_{\mathbf{h}_\infty} = \sum_{i=1}^M (\varphi_i, \psi_i)_{L^2(M_i)} + \sum_{j=M+1}^N \varphi_j \overline{\psi_j} |M_j|.$$

We also put

$$\mathcal{F}^{(\pm)}(k)f = (\mathcal{F}_1^{(\pm)}(k)f, \dots, \mathcal{F}_N^{(\pm)}(k)f),$$

$$\widehat{\mathcal{H}} = L^2((0, \infty); \mathbf{h}_\infty; dk).$$

**Theorem 3.5.** We define  $(\mathcal{F}^{(\pm)}f)(k) = \mathcal{F}^{(\pm)}(k)f$  for  $f \in \mathcal{B}$ . Then  $\mathcal{F}^{(\pm)}$  is uniquely extended to a bounded operator from  $L^2(\mathcal{M})$  to  $\widehat{\mathcal{H}}$  with the following properties.

- (1)  $\text{Ran } \mathcal{F}^{(\pm)} = \widehat{\mathcal{H}}$ .
- (2)  $\|f\| = \|\mathcal{F}^{(\pm)}f\|$  for  $f \in \mathcal{H}_{ac}(H)$ .
- (3)  $\mathcal{F}^{(\pm)}f = 0$  for  $f \in \mathcal{H}_p(H)$ .
- (4)  $(\mathcal{F}^{(\pm)}Hf)(k) = k^2(\mathcal{F}^{(\pm)}f)(k)$  for  $f \in \text{Dom } H$ .
- (5)  $\mathcal{F}^{(\pm)}(k)^* \in \mathbf{B}(\mathbf{h}_\infty; \mathcal{B}^*)$  and  $(H - k^2)\mathcal{F}^{(\pm)}(k)^* = 0$  for  $k^2 \in (0, \infty) \setminus \sigma_p(H)$ .
- (6) For  $f \in \mathcal{H}_{ac}(H)$ , the inversion formula holds:

$$f = \left(\mathcal{F}^{(\pm)}\right)^* \mathcal{F}^{(\pm)}f = \sum_{j=1}^N \int_0^\infty \mathcal{F}_j^{(\pm)}(k)^* \left(\mathcal{F}_j^{(\pm)}f\right)(k) dk.$$

### 3.5. $S$ matrix.

**Theorem 3.6.** If  $k^2 \notin \sigma_p(H)$ , we have

- (1)  $\mathcal{F}^{(\pm)}(k)\mathcal{B} = \mathbf{h}_\infty$ .
- (2)  $\{u \in \mathcal{B}^*; (H - k^2)u = 0\} = \mathcal{F}^{(\pm)}(k)^*\mathbf{h}_\infty$ .

Let  $V_q$  be the differential operator defined by

$$V_q = -[H_{0q}, \chi_q] + V\chi_q \quad (1 \leq q \leq N).$$

For  $1 \leq p \leq M$ ,  $1 \leq q \leq N$ , we define

$$\widehat{S}_{pq}(k) = \delta_{pq}J_p(k) - \frac{\pi i}{k} \mathcal{F}_p^{(+)}(k)V_q^* \left(\mathcal{F}_{0q}^{(-)}(k)\right)^*,$$

$$J_p(k)\psi = \sum_{m \geq 1} \left(\frac{\sqrt{\lambda_{p,m}}}{2}\right)^{-2ik} \varphi_{p,m}(x)\widehat{\psi}_m \quad (1 \leq p \leq M).$$

For  $M+1 \leq p \leq N$ ,  $1 \leq q \leq N$ , we define

$$\widehat{S}_{pq}(k) = -\frac{\pi i}{k} \mathcal{F}_p^{(+)}(k)V_q^* \left(\mathcal{F}_{0q}^{(-)}(k)\right)^*.$$

We define an operator-valued  $N \times N$  matrix  $\widehat{S}(k)$  by

$$\widehat{S}(k) = \left(\widehat{S}_{pq}(k)\right),$$

and call it  $S$ -matrix. This is a bounded operator on  $\mathbf{h}_\infty$ .

**Theorem 3.7.** (1) For any  $u \in \mathcal{B}^*$  satisfying  $(H - k^2)u = 0$ , there exists a unique  $\psi^{(\pm)} \in \mathbf{h}_\infty$  such that

$$\begin{aligned} u &\simeq \omega_-(k) \sum_{p=1}^M \chi_p y^{(n-1)/2+ik} \psi_p^{(-)} \\ &\quad + \omega_-(k) \sum_{p=M+1}^N \chi_p y^{(n-1)/2-ik} \widehat{\psi}_{p0}^{(-)} \\ &\quad - \omega_+(k) \sum_{p=1}^M \chi_p y^{(n-1)/2-ik} \psi_p^{(+)} \\ &\quad - \omega_+^{(c)}(k) \sum_{p=M+1}^N \chi_p y^{(n-1)/2+ik} \psi_p^{(+)}. \end{aligned}$$

(2) For any  $\psi^{(-)} \in \mathbf{h}_\infty$ , there exists a unique  $\psi^{(+)} \in \mathbf{h}_\infty$  and  $u \in \mathcal{B}^*$  satisfying  $(H - k^2)u = 0$ , for which the expansion (1) holds. Moreover

$$\psi^{(+)} = \widehat{S}(k)\psi^{(-)}.$$

**Theorem 3.8.** (1)  $\widehat{S}(k)$  is unitary on  $\mathbf{h}_\infty$ .

(2)  $\mathcal{F}^{(+)}(k) = \widehat{S}(k)\mathcal{F}^{(-)}(k)$ .

#### 4. INTRODUCTION TO INVERSE SCATTERING

Suppose we are given two asymptotically hyperbolic metrics which differ only on a compact set. If the associated scattering operators coincide, these two metrics coincide up to a diffeomorphism. This result can be extended to manifolds with asymptotically hyperbolic ends when two metrics coincide on one end having a regular infinity.

**4.1. Local problem on  $\mathbf{H}^n$ .** In the geodesic polar coordinates, the metric on  $\mathbf{H}^n$  takes the form

$$ds^2 = (dr)^2 + \sinh^2 r (d\theta)^2,$$

where  $(d\theta)^2$  is the standard metric on  $S^{n-1}$ . Let  $\mathcal{K} = \mathbf{H}^n \cap \{r < 2\}$ ,  $\mathcal{M}_1 = \mathbf{H}^n \cap \{r > 1\}$ . Letting  $y = 2e^{-r}$  and  $x = \theta$ , one can rewrite the above metric as

$$ds^2 = \left(\frac{dy}{y}\right)^2 + \left(\frac{1}{y} - \frac{y}{4}\right)^2 (dx)^2.$$

Suppose this metric is perturbed so that

$$ds^2 = \frac{(dy)^2 + (dx)^2 + A(x, y, dx, dy)}{y^2},$$

with  $A(x, y, dx, dy)$  satisfying the assumption (A-4) of §3.

**Theorem 4.1.** Suppose we are given two Riemannian metrics  $G^{(p)}$ ,  $p = 1, 2$ , on  $\mathbf{H}^n$  satisfying the above assumption. Suppose their scattering operators coincide. Suppose furthermore  $G^{(1)}$  and  $G^{(2)}$  coincide except for a compact set. Then  $G^{(1)}$  and  $G^{(2)}$  are isometric.

The proof is done by the following steps. Let  $K \subset \mathbf{H}^n$  be a compact set such that  $G^{(1)} = G^{(2)}$  outside  $K$ . We first take a geodesic sphere  $S \subset \mathbf{H}^n \setminus K$  and consider the boundary value problem for the Laplace-Beltrami operators in the interior domain  $\Omega$ . Then the associated Dirichlet-Neumann map (or Neumann-Dirichlet map) coincide. We use the boundary control method of Belishev-Kurylev to show that  $G^{(1)}$  and  $G^{(2)}$  are isometric in  $\Omega$ .

**4.2. Inverse scattering at regular ends.** Let  $\mathcal{M}$  be a manifold satisfying the assumptions (A.1) ~ (A.4) in §3 with ends of number  $N \geq 2$ . Since  $\mathcal{M}$  has  $N$ -ends, the S-matrix for  $\mathcal{M}$  is an  $N \times N$ -matrix:

$$\widehat{S}(k) = \left( \widehat{S}_{ij}(k) \right)_{1 \leq i, j \leq N}.$$

Now, suppose we are given two metrics  $G^{(j)}$ ,  $j = 1, 2$ , on  $\mathcal{M}$ , and the operator  $H^{(j)}$  associated with  $G^{(j)}$ .

**Theorem 4.2.** *Let  $\mathcal{M}$  be a manifold satisfying the assumptions (A.1) ~ (A.4) in §3. We assume that one of the ends has a regular infinity, and denote it by  $\mathcal{M}_1$ . Suppose we are given two metrics  $G^{(j)}$ ,  $j = 1, 2$ , on  $\mathcal{M}$ . Assume that  $G^{(1)} = G^{(2)}$  on  $\mathcal{M}_1$ . If  $\widehat{S}_{11}(k) = \widehat{S}_{11}(k)$  for all  $k > 0$ , then  $G^{(1)}$  and  $G^{(2)}$  are isometric on  $\mathcal{M}$ .*

We can actually prove a stronger version of Theorem 4.2, i.e. it is valid for two manifolds whose number of ends are not known a-priori.

**Theorem 4.3.** *Let  $\mathcal{M}^{(p)}$ ,  $p = 1, 2$ , be manifolds satisfying the assumptions (A.1) ~ (A.4) in §3 endowed with metric  $G^{(p)}$ ,  $p = 1, 2$ . We assume that for both of  $\mathcal{M}^{(1)}$  and  $\mathcal{M}^{(2)}$  one of the ends has a regular infinity, and denote it by  $\mathcal{M}_1^{(p)}$ ,  $p = 1, 2$ . Assume that  $\mathcal{M}_1^{(1)}$  and  $\mathcal{M}_1^{(2)}$  are isometric, and  $\widehat{S}_{11}(k) = \widehat{S}_{11}(k)$  for all  $k > 0$ . Then  $\mathcal{M}^{(1)}$  and  $\mathcal{M}^{(2)}$  are isometric.*

Let us remark that Theorem 4.2 is also true for the upper-half space model. Namely, if two metrics coincide for  $y < \epsilon$  and the scattering matrices coincide, then these two metrics are isometric.

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# Curved Traveling Waves of a Mean Curvature Flow in Heterogeneous Media

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## 1 Introduction

Consider

$$(1) \quad V = a(x, y)\kappa + b(x, y) \quad \text{for } (x, y) \in \Gamma_t \subset \mathbb{R}^2,$$

where  $\Gamma_t$  is a simple plane curve,  $V$  denotes the normal velocity on  $\Gamma_t$ ,  $\kappa$  is the curvature,  $a, b$  are positive functions. This equation is an example of mean curvature flows in heterogeneous media. It can be used to model the motion of an interface. Generally, one can expect to get more information about the interface by studying mean curvature flows instead of reaction diffusion equations. Especially, in the case where an interface reduces to a simple plane curve, the shape of the interface, the relation between the propagation speed and the shape are expected to be described clearly.

In [2], [3], [8], [9], [10] etc., the authors studied traveling waves of  $V = a_0\kappa + b_0$ : the homogeneous version of (1). We are concerned with *traveling waves* of (1).

In the last two decades, many authors studied traveling waves of reaction diffusion equations in heterogeneous media (cf. [1], [7], [12] and references therein). Most of the study are concerned with traveling waves with planar-like front (that is, the level set of the front – interface – is bounded oscillation of a hyperplane). Recently, [4], [5], [11] etc. studied traveling waves with curved fronts, but for homogeneous equations. As far as we know, no much is known about traveling waves with curved fronts for heterogeneous equations.

In this talk, we are concerned with graphic curves, that is, for each  $t > 0$ ,  $\Gamma_t$  is the graph of a function  $y = u(x, t)$ . In this case, equation (1) is equivalent to

$$(2) \quad u_t = a(x, u) \frac{u_{xx}}{1 + u_x^2} + b(x, u) \sqrt{1 + u_x^2}.$$

In this talk, we assume that both of  $a(x, y)$  and  $b(x, y)$  are smooth, positive and almost periodic in both variables.

### Definitions of Traveling waves

(i) Assume that  $a$  and  $b$  are independent of  $u$ . A solution of (2) with the form  $u(x, t) = v(x) + ct$  (for some  $v$  and  $c > 0$ ) is called a *traveling wave* (solution).

(ii) If  $a$  and  $b$  are 1-periodic in  $y$ , then we call a solution  $u(x, t)$  of (2) as a *periodic traveling wave* with *average speed*  $c$  if  $u(x, t + 1/c) = u(x, t) + 1$ .

(iii) If  $a$  and  $b$  are almost periodic in  $y$ , then we call a solution  $u(x, t)$  of (2) as an *almost periodic traveling wave* with *average speed*  $c$  if there exist a continuous map  $W : \mathcal{H}_a \times \mathcal{H}_b \rightarrow L_{loc}^\infty(\mathbb{R})$  and an increasing function  $\xi(t)$  such that

$$(3) \quad u(\cdot, t) - \xi(t) = W\left(a(\cdot, y + \xi(t)), b(\cdot, y + \xi(t))\right)$$

and

$$[\xi(t + T) - \xi(t)]/T \rightarrow c \quad \text{as } T \rightarrow \infty.$$

Here  $\mathcal{H}_a := \overline{\{a(x, y + s) \mid s \in \mathbb{R}\}}^{L^\infty(\mathbb{R}^2)}$  denotes the hull of  $a$ . Note that this definition is indeed the analogue of Matano's definition for traveling wave of reaction diffusion equation in heterogeneous media ([7]).

(iv) A traveling wave is called a curved one if  $u(x, t) - u(0, t)$  is unbounded. Especially, if the graph of  $u(x, t) - u(0, t)$  lies in a bounded neighborhood of a line for  $x > 0$  and lies in a bounded neighborhood of another line for  $x < 0$ , then we call it a “V”-like traveling wave.

Our main purpose in this talk is to study the existence of “V”-like traveling waves of (2), as well as the relation among the traveling speed, the shape of the profile and the spatially heterogeneity.

Denote  $a_* := \inf a(x, y)$ ,  $a^* := \sup a(x, y)$ ,  $b_* := \inf b(x, y)$ ,  $b^* := \sup b(x, y)$ .

## 2 Traveling waves of $V = a(x)\kappa + b(x)$

In this part we assume that  $a$  and  $b$  are independent of  $y$ , and are almost periodic in  $x$ . In this case a traveling wave  $u(x, t) = v(x) + ct$  satisfies

$$c = a(x) \frac{v''}{1 + v'^2} + b(x) \sqrt{1 + v'^2}, \quad x \in \mathbb{R}.$$

After a further transformation  $\varphi(x) := \arctan v'(x)$ , the equation is equivalent to

$$(4) \quad \varphi'(x) = \frac{c}{a(x)} - \frac{b(x)}{a(x) \cos \varphi(x)}, \quad \varphi(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \forall x \in \mathbb{R}.$$

For some of our results, we need to restrict  $\alpha$  to the range

$$(5) \quad \alpha \in \left(\arccos \frac{b_*}{b^*}, \frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, -\arccos \frac{b_*}{b^*}\right) \quad \left(\Leftrightarrow b_* > b^* \cos \alpha\right).$$

**Lemma 1.** *For any  $\alpha$  in the range given by (5), there exists a unique pair  $(c, \varphi) \in \mathbb{R} \times C^1(\mathbb{R})$  such that (i)  $(c, \varphi)$  satisfies (4), (ii)  $\varphi$  is almost periodic, and (iii) the arithmetic mean of  $\tan \varphi$  is  $\tan \alpha$ .*

In addition, as a function of  $\alpha$ , the unique solution  $(c, \varphi)$  satisfies the following estimates

$$(6) \quad \arccos \frac{b_M \cos \alpha}{b_m} \leq \operatorname{sgn}(\alpha) \varphi(x) \leq \arccos \frac{b_m \cos \alpha}{b_M} \quad \text{for } x \in \mathbb{R},$$

$$(7) \quad 0 \leq \operatorname{sgn}(\alpha) \frac{dc}{d\alpha} \leq \frac{3a_M b_M^3}{a_m b_m^2 \cos^4 \alpha}.$$

(In the case,  $a$  and  $b$  are periodic,  $\varphi$  is also periodic).

**Theorem A.** Assume that  $\alpha_1 > 0$  and  $\alpha_2 < 0$  satisfy (5). Also assume that  $(c, \varphi_1)$  and  $(c, \varphi_2)$  are solutions of (4), that  $\varphi_1$  and  $\varphi_2$  are almost periodic and  $\mathcal{M}[\tan \varphi_i] = \tan \alpha_i$ ,  $i = 1, 2$ . Then, for any small initial value  $\varphi(0)$ , problem (4) (with the same  $c$ ) admits a unique solution  $\varphi$  and the solution satisfies the following (see Figure 1):

- (i)  $\varphi_2(x) < \varphi(x) < \varphi_1(x)$  for all  $x$ ;
- (ii) There exist positive constants  $L$  and  $\nu$  such that

$$\varphi_1(x) - Le^{-\nu x} \leq \varphi(x) \leq \varphi_2 + Le^{\nu x} \quad \forall x \in \mathbb{R}.$$

- (iii) There exist a unique  $x_0$  and a unique  $S_0 > 0$  such that the functions defined by

$$v_1(x) := \int_{x_0}^x \tan \varphi_1(x) dx, \quad v_2(x) := \int_{x_0}^x \tan \varphi_2(x) dx, \quad v(x) := \int_{x_0}^x \tan \varphi(x) dx + S_0$$

satisfy, for some  $\hat{L} > 0$  and the same  $\nu$  as above,

$$\max\{v_1(x), v_2(x)\} < v(x) < \max\{v_1(x), v_2(x)\} + \hat{L}e^{-\nu|x|} \quad \forall x \in \mathbb{R}.$$

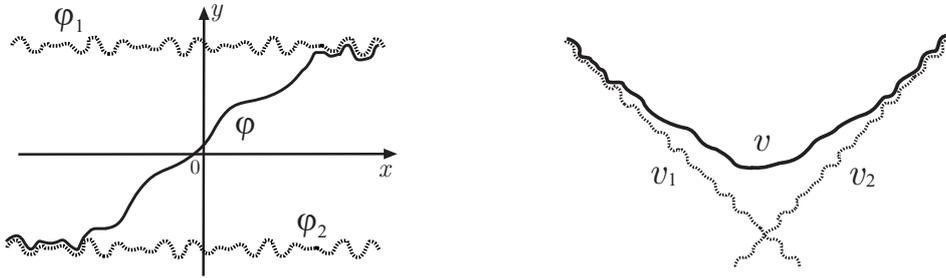


Fig. 1 (a)  $\varphi(x)$  approaches  $\varphi_i(x)$  as  $x \rightarrow \pm\infty$ ;

(b) “V”-like profile

In order to give an explicit estimate for the traveling speed  $c$  in terms of  $\alpha$ , we consider a homogenization problem:

$$\frac{d}{dx} \varphi^\varepsilon(x) = \frac{c^\varepsilon}{a^\varepsilon(x)} - \frac{b^\varepsilon(x)}{a^\varepsilon(x) \cos \varphi^\varepsilon(x)}, \quad \varphi^\varepsilon(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \forall x \in \mathbb{R} \quad (4)_\varepsilon$$

where  $a_\varepsilon(x) = a(x/\varepsilon)$ ,  $b_\varepsilon(x) = b(x/\varepsilon)$ . For an almost periodic function  $h(x)$  we use  $\mathcal{M}[h]$  to denote its arithmetic mean. Denote

$$A = \left( \mathcal{M} \left[ \frac{1}{a(x)} \right] \right)^{-1}, \quad B = \mathcal{M} \left[ \frac{b(x)}{a(x)} \right] A.$$

**Theorem B.** For any  $\alpha$  satisfying (5), let  $(c^\varepsilon, \varphi^\varepsilon)$  be the solution of  $(4)_\varepsilon$  as in Lemma 1. Then

$$(8) \quad \lim_{\varepsilon \searrow 0} \left\| \varphi^\varepsilon - \alpha \right\|_{L^\infty(\mathbb{R})} = 0, \quad \lim_{\varepsilon \searrow 0} c^\varepsilon = \frac{B}{\cos \alpha}.$$

If in addition assume that for some  $L_1, L_2 > 0$ ,

$$(9) \quad \left| \int_0^x \left( \frac{1}{a(x)} - \frac{1}{A} \right) dx \right| \leq L_1, \quad \left| \int_0^x \left( \frac{b(x)}{a(x)} - \frac{B}{A} \right) dx \right| \leq L_2 \quad \text{for } x \in \mathbb{R},$$

then there exists a positive constant  $Q$  depending only on  $L_1, L_2, a_m, a_M, b_m, b_M$  and  $\alpha$  such that

$$(10) \quad \left\| \varphi^\varepsilon(\cdot) - \alpha \right\|_{L^\infty(\mathbb{R})} + \left| c^\varepsilon - \frac{B}{\cos \alpha} \right| \leq Q\varepsilon \quad \forall \varepsilon > 0.$$

### 3 Traveling waves of $V = a(x, y)\kappa + b$

In this part, for some technical reasons, we study curvature flow  $V = a(x, y)\kappa + b$ , where  $a$  is almost periodic in  $y$ ,  $b$  is a positive constant.

**Theorem C.** For any given  $c > b$ , equation (2) (with  $b(x, u) \equiv b$ ) has an almost periodic traveling wave  $U(x, t) \in C^{2+\nu, 1+\nu/2}(\mathbb{R}^2)$  ( $\nu \in (0, 1)$ ). For each  $t \in \mathbb{R}$ , the graph of  $U(x, t)$  is a “V”-like curve:

$$(11) \quad \varphi(x, a_*) \leq U(x, t) - ct \leq \varphi(x, a^*) + (a^* - a_*)S \quad \text{for } x \in \mathbb{R},$$

where  $S = S(b, c) > 0$  is a constant, both  $\varphi(x, a_*)$  and  $\varphi(x, a^*) + (a^* - a_*)S$  approach the same line  $x\sqrt{c^2 - b^2}/b - a_*S$  as  $x \rightarrow \infty$ , approach  $-x\sqrt{c^2 - b^2}/b - a_*S$  as  $x \rightarrow -\infty$  (see Figure 2).

(When  $a(x, y)$  is periodic in  $y$ , the traveling wave is also a periodic one).

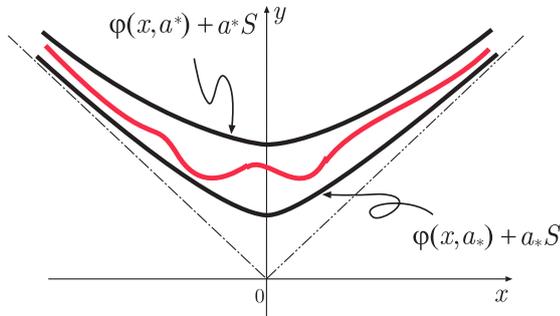


Fig. 2 “V”-like profile with asymptotic straight wings

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# Large time behavior of solutions to the Burgers-Poisson equations

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## 1 Introduction

In this talk, we consider large time behavior of global solutions to the Burgers-Poisson equations:

$$(1.1) \quad u_t + (u^2)_x + q_x = 0, \quad t > 0, \quad x \in \mathbb{R},$$

$$(1.2) \quad -q_{xx} + q + u_x = 0, \quad t > 0, \quad x \in \mathbb{R},$$

$$(1.3) \quad u(x, 0) = u_0(x),$$

where  $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$ . For an integer  $s \geq 2$ ,  $H^s(\mathbb{R})$  denotes the space of functions  $u = u(x)$  such that  $\partial_x^l u$  are  $L^2$ -functions on  $\mathbb{R}$  for  $0 \leq l \leq s$ , endowed with norm  $\|\cdot\|_{H^s}$ . The system is a mathematical model describing the behavior of the high-temperature gas, where  $u(x, t)$  corresponds to the velocity of the gas and  $q(x, t)$  to the radiative heat-flux, respectively (see [1, 5]).

In Kawashima, Nikkuni and Nishibata [4], it was shown that if  $E_s = \|u_0\|_{H^s} + \|u_0\|_{L^1}$  is suitably small, then there exist uniquely a pair of solutions  $u \in C^0([0, \infty); H^s) \cap C^1([0, \infty); H^{s-1})$  and  $q \in C([0, \infty); H^{s+1})$  of the problem (1.1), (1.2) and (1.3) verifying

$$(1.4) \quad \|\partial_x^l u(\cdot, t)\|_{H^{s-1-2l}} + \|\partial_x^l q(\cdot, t)\|_{H^{s-1-2l}} \leq CE_s(1+t)^{-\frac{1}{4}-\frac{l}{2}}, \quad t \geq 0,$$

$$(1.5) \quad \|\partial_x^l q(\cdot, t)\|_{H^{s-3-2l}} \leq CE_s(1+t)^{-\frac{3}{4}-\frac{l}{2}}, \quad t \geq 0,$$

where  $s \geq 2$  and  $0 \leq 2l \leq s-1$  in (1.4), and  $0 \leq 2l \leq s-3$  in (1.5). Moreover, they proposed the following viscous Burgers equation as a reduced equation for the Burgers-Poisson equations:

$$(1.6) \quad v_t + (v^2)_x = v_{xx}, \quad t > 0, \quad x \in \mathbb{R},$$

$$(1.7) \quad v(x, 0) = u_0(x).$$

If  $E_s$  is suitably small, then we have

$$(1.8) \quad \|\partial_x^k v(\cdot, t)\|_{L^2} \leq CE_s(1+t)^{-\frac{3}{4}-\frac{k}{2}}$$

where  $0 \leq k \leq s$  (see e.g. [2]). Observe that the equation (1.6) follows from (1.1) and (1.2) by neglecting the second order derivative  $q_{xx}$ , formally. In fact, they proved that if

$E_s$  is suitably small depending on any fixed  $\epsilon > 0$ , then

$$(1.9) \quad \|\partial_x^l(u - v)(\cdot, t)\|_{H^{s-4-3l}} \leq CE_s(1+t)^{-\frac{3}{4}-\frac{l}{2}+\epsilon}, \quad t \geq 0,$$

$$(1.10) \quad \|\partial_x^l(q + v_x)(\cdot, t)\|_{H^{s-7-3l}} \leq CE_s(1+t)^{-\frac{5}{4}-\frac{l}{2}+\epsilon} \quad t \geq 0,$$

hold, where  $s \geq 5$  and  $0 \leq 3l \leq s - 4$  in (1.9), and  $0 \leq 3l \leq s - 7$  in (1.10). However, this result on the asymptotic behavior leads us to a natural question whether it is possible to take  $\epsilon = 0$ . Our main result is that we can improve these estimates as follows.

**Theorem 1.1.** *If  $E_s$  is suitably small, then we have*

$$(1.11) \quad \|\partial_x^l(u - v)(\cdot, t)\|_{H^{s-4-3l}} \leq CE_s(1+t)^{-\frac{3}{4}-\frac{l}{2}}, \quad t \geq 0,$$

$$(1.12) \quad \|\partial_x^l(q + v_x)(\cdot, t)\|_{H^{s-7-3l}} \leq CE_s(1+t)^{-\frac{5}{4}-\frac{l}{2}}, \quad t \geq 0,$$

where  $s \geq 5$  and  $0 \leq 3l \leq s - 4$  in (1.11), and  $0 \leq 3l \leq s - 7$  in (1.12).

In order to show this theorem, the explicit representation formula and the weighted decay estimates of the solution for the linearized equation play an important role.

## 2 Preliminaries

In order to prove the basic estimates given by Proposition 3.1 and Proposition 3.2, we prepare the following two lemmas. Those are concerned with the decay estimates for semigroup  $e^{t\Delta}$  associated with heat equation. For the proof, see Kawashima [3].

**Lemma 2.1.** *Let  $l$  be a nonnegative integer and  $p \in [1, 2]$ . Then the estimate*

$$(2.1) \quad \|\partial_x^l e^{t\Delta} f\|_{L^2} \leq Ce^{-t} \|\partial_x^l f\|_{L^2} + C(1+t)^{-\gamma(p)-\frac{l-k}{2}} \|\partial_x^k f\|_{L^p}$$

holds for any  $0 \leq k \leq l$ , where  $\gamma(p) = (1/p - 1/2)/2$  and  $f(x)$  is a function such that the norms on the right hand side of (2.1) are finite.

We set

$$H_p^l \equiv \{f \in L_{loc}^1(\mathbb{R}) \mid \sum_{m=0}^l \|\partial_x^m f\|_{L^p} < \infty\}.$$

**Lemma 2.2.** *Let  $l$  be a positive integer,  $p \in [1, 2]$  and  $t_1 \leq t_2$ . Suppose  $f \in C^0(0, \infty; H^{l-1}) \cap C^0(0, \infty; H_p^l)$ . Then the estimate*

$$\begin{aligned} & \|\partial_x^l \int_{t_1}^{t_2} e^{(t_2-\tau)\Delta} [f(\tau)] d\tau\|_{L^2} \\ & \leq C \left( \int_{t_1}^{t_2} e^{-(t_2-\tau)} \|\partial_x^{l-1} f(\tau)\|_{L^2}^2 d\tau \right)^{1/2} + C \int_{t_1}^{t_2} (1+t_2-\tau)^{-\gamma(p)-\frac{l-k}{2}} \|\partial_x^k f(\tau)\|_{L^p} d\tau \end{aligned}$$

holds for any  $0 \leq k \leq l$ , where  $\gamma(p) = (1/p - 1/2)/2$ .

We deal with the following linearized equation which corresponds to (3.4), (3.5) below:

$$(2.2) \quad z_t = z_{xx} - (2vz)_x, \quad t > 0, \quad x \in \mathbb{R},$$

$$(2.3) \quad z(x, 0) = z_0(x).$$

If we put

$$r(x, t) = \int_{-\infty}^x z(y, t) dy,$$

then we see from (2.2), (2.3) that  $r(x, t)$  satisfies

$$\begin{aligned} r_t &= r_{xx} - 2vr_x, \quad t > 0, \quad x \in \mathbb{R}, \\ r(x, 0) &= \int_{-\infty}^x z_0(y) dy. \end{aligned}$$

Then a direct computation yields

$$\left( \frac{r(x, t)}{\eta(x, t)} \right)_t = \left( \frac{r(x, t)}{\eta(x, t)} \right)_{xx},$$

where

$$(2.4) \quad \eta(x, t) = \exp \left( \int_{-\infty}^x -v(y, t) dy \right).$$

The explicit representation formula (2.6) below plays a crucial role in our analysis. For the proof, see [2].

**Lemma 2.3.** *If we set*

$$(2.5) \quad U[\rho](x, t, \tau) = \int_{\mathbb{R}} \partial_x(G(x-y, t-\tau)\eta(x, t)) \frac{1}{\eta(y, \tau)} \left\{ \int_{-\infty}^y \rho(\xi) d\xi \right\} dy, \\ 0 \leq \tau < t, \quad x \in \mathbb{R}$$

and  $G(x, t)$  is the heat kernel

$$G(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right),$$

then the solutions for (2.2) and (2.3) is given by

$$(2.6) \quad z(x, t) = U[z_0](x, t, 0), \quad t > 0, \quad x \in \mathbb{R}.$$

It is easy to deduce the following estimates from (2.4) and (1.8). Here and after, we assume that  $E_s \leq 1$ .

**Lemma 2.4.** *Let  $s$  be a positive integer. Then, if  $E_s$  is suitably small, we have*

$$(2.7) \quad \left\| \left( \frac{1}{\eta} \right) (\cdot, t) \right\|_{L^\infty} + \|\eta(\cdot, t)\|_{L^\infty} \leq C,$$

$$(2.8) \quad \left\| \partial_x^k \left( \frac{1}{\eta} \right) (\cdot, t) \right\|_{L^2} + \|\partial_x^k \eta(\cdot, t)\|_{L^2} \leq CE_s (1+t)^{-\frac{1}{4} - \frac{k-1}{2}}, \quad 1 \leq k \leq s,$$

$$(2.9) \quad \left\| \partial_x^k \left( \frac{1}{\eta} \right) (\cdot, t) \right\|_{L^\infty} + \|\partial_x^k \eta(\cdot, t)\|_{L^\infty} \leq CE_s (1+t)^{-\frac{k}{2}}, \quad 1 \leq k \leq s-1.$$

### 3 Proof of Theorem 1.1

We shall mention how to deduce the decay estimate (1.11) and (1.12). Once we have (1.11), it is easy to show (1.12). In fact, from (1.2), (1.11) and (1.5), we have

$$\begin{aligned} \|\partial_x^l(q + v_x)(\cdot, t)\|_{H^{s-7-3l}} &\leq \|\partial_x^{l+1}(u - v)(\cdot, t)\|_{H^{s-4-3(l+1)}} + \|\partial_x^{l+2}q(\cdot, t)\|_{H^{s-3-2(l+2)}} \\ &\leq CE_s(1+t)^{-\frac{3}{4}-\frac{l+1}{2}} + CE_s(1+t)^{-\frac{3}{4}-\frac{l+2}{2}} \\ &\leq CE_s(1+t)^{-\frac{5}{4}-\frac{l}{2}}. \end{aligned}$$

Therefore our task is reduced to (1.11). Using Lemmas 2.1, 2.2 and 2.4, (1.4), and (1.5), we derive the following.

**Proposition 3.1.** *Let  $s \geq 5$  be an integer and  $(u, q)$  be the global solution. Suppose  $l$  is a nonnegative integer satisfying  $3l \leq s - 4$ . Then, if  $E_s$  is suitably small, we have*

$$(3.1) \quad \|\partial_x^j \int_0^t U[q_{xxx}(\tau)](x, t, \tau) d\tau\|_{L^2} \leq CE_s(1+t)^{-\frac{3}{4}-\frac{1}{2}\min(j,l)},$$

where  $0 \leq j \leq s - 4 - 2l$ .

In order to deal with the nonlinear term, we introduce

$$(3.2) \quad N(T) = \sum_{j=0}^{s-4-2l} \sup_{0 \leq t \leq T} (1+t)^{\frac{3}{4}+\frac{1}{2}\min(j,l)} \|\partial_x^j w(\cdot, t)\|_{L^2}.$$

Using Lemmas 2.1, 2.2 and 2.4, we derive the following.

**Proposition 3.2.** *Under the same assumptions in Proposition 3.1, we have*

$$(3.3) \quad \|\partial_x^j \int_0^t U[(w^2)_x(\tau)](x, t, \tau) d\tau\|_{L^2} \leq CN(T)^2(1+t)^{-\frac{3}{4}-\frac{1}{2}\min(j,l)},$$

where  $0 \leq j \leq s - 4 - 2l$ .

From (1.2), we have  $q = -u_x + q_{xx}$ . Substituting this equation into (1.1), we obtain

$$u_t + (u^2)_x = u_{xx} - q_{xxx}.$$

If we put  $w(x, t) = u(x, t) - v(x, t)$ , then  $w(x, t)$  satisfies

$$(3.4) \quad w_t = w_{xx} - (2vw)_x - q_{xxx} - (w^2)_x, \quad t > 0, \quad x \in \mathbb{R},$$

$$(3.5) \quad w(x, 0) = 0.$$

Applying the Duhamel principle for the problem, we have

$$(3.6) \quad w(x, t) = - \int_0^t U[q_{xxx}(\tau) + (w^2)_x(\tau)](\cdot, t, \tau) d\tau.$$

Observe that (1.11) follows from

$$(3.7) \quad N(T) \leq CE_s$$

for all  $T \geq 0$ . Applying  $\partial_x^j$  to (3.6) and taking the  $L^2$ -norm, we obtain

$$(3.8) \quad \begin{aligned} \|\partial_x^j w(\cdot, t)\|_{L^2} &\leq \|\partial_x^j \int_0^t U[q_{xxx}(\tau)](\cdot, t, \tau) d\tau\|_{L^2} \\ &+ \|\partial_x^j \int_0^t U[(w^2)_x(\tau)](\cdot, t, \tau) d\tau\|_{L^2}, \end{aligned}$$

where  $0 \leq j \leq s - 4 - 2l$ . From Propositions 3.1 and 3.2, we have (3.7) provided  $E_s$  is suitably small. This completes the proof.

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# Mean curvature flow closes open ends of noncompact surface of rotation

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## 1 Introduction

This is a joint work with Yoshikazu Giga and Noriaki Umeda. We discuss motion of noncompact hypersurfaces  $\Gamma_t$  moved by mean curvature flow, whose initial surface  $\Gamma_0$  is rotationally symmetric to an axis, say the  $x_1$ -axis, and is represented by rotating the graph of a positive function  $u_0$  around the axis. Under the symmetric assumption, the mean curvature flow equation is equivalent with one-dimensional quasilinear parabolic equation

$$u_t = \frac{u_{xx}}{1 + (u_x)^2} - \frac{n-1}{u}, \quad x \in \mathbf{R}, \quad t > 0, \quad (1.1)$$

with initial data

$$u(x, 0) = u_0(x) > 0, \quad x \in \mathbf{R}. \quad (1.2)$$

### 1.1 Derivation of the equation

The surfaces  $\Gamma_t$  remain rotationally symmetric to the axis so long as they exist as is proved in [1, Theorem 4.3a]. Namely, we may assume that the hypersurfaces are given of the form

$$\Gamma_t = \{x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid r = u(x_1, t)\}$$

with some function  $u$ , where  $r = \left(\sum_{j=2}^n x_j^2\right)^{1/2}$  denotes the distance from the  $x_1$ -axis to  $\Gamma_t$ . We call these hypersurfaces **axisymmetric surfaces**.

Although the derivation of equation (1.3) was already done in [10, 2] by the radial distance  $u$  of the surface to its axis of rotation, we introduce another way to derive the equation based on level set method (c.f.[4]), which

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describes a hypersurface as the zero level set of an auxiliary function. Under the symmetric assumption, we may take the auxiliary function as

$$\phi(x, t) := -r + u(x_1, t),$$

so that the surfaces are represented as  $\Gamma_t = \{x \in \mathbf{R}^n \mid \phi(x, t) = 0\}$ . Here we observe that  $|\nabla\phi| = (1 + u_{x_1}^2)^{1/2}$  does not vanish on  $\Gamma_t$ . With this function, a unit normal vector field  $\mathbf{n}$  of  $\Gamma_t$  is given by  $\mathbf{n} = -\nabla\phi/|\nabla\phi|$ , so that we may compute  $V$  and  $H$  respectively as

$$\begin{aligned} V &= \frac{dx(t)}{dt} \cdot \mathbf{n} = \frac{\phi_t}{|\nabla\phi|} = \frac{u_t}{(1 + u_{x_1}^2)^{1/2}}, \\ H &= -\nabla \cdot \mathbf{n} = \nabla \cdot \left( \frac{\nabla\phi}{|\nabla\phi|} \right) = \frac{u_{x_1 x_1}}{(1 + u_{x_1}^2)^{3/2}} - \frac{1}{(1 + u_{x_1}^2)^{1/2}} \frac{n-1}{r}, \end{aligned}$$

where  $x(t)$  is a  $C^1$ -curve such that  $\phi(x(t), t) = 0$ . (Here and henceforth we do not take average of principal curvatures to define mean curvature.) Putting  $V = H$ , we get

$$u_t = \frac{u_{x_1 x_1}}{1 + (u_{x_1})^2} - \frac{n-1}{u}. \quad (1.3)$$

Since the spatial independent variable of unknown function  $u$  is essentially one dimension, we may and shall denote  $x_1$  by  $x$  in the following for simplicity. We now arrive at the Cauchy problem (1.1)-(1.2).

## 1.2 Quenching problem

The Cauchy problem (1.1)-(1.2) has a unique positive classical solution  $u$  locally in time, but the solution is forced to reach zero in finite time as long as bounded initial data are concerned. This fact is readily seen if one compares  $u$  with the explicit solution  $v_M(t) = \sqrt{2(n-1)(T(M) - t)}$  with  $M = \sup_{x \in \mathbf{R}} u_0(x)$  and  $T(M) = M^2/2(n-1)$ .

Once a solution reaches zero, the equation (1.1) does not make sense and hence the solution cannot be extended globally in time as a classical solution. For a given initial datum  $u_0$ , we set

$$T(u_0) = \sup\{t > 0; \inf_{x \in \mathbf{R}} u(x, t) > 0\} < \infty$$

and call it the *quenching time* of  $u$ . It is immediate that

$$\liminf_{t \nearrow T(u_0)} \inf_{x \in \mathbf{R}} u(x, t) = 0.$$

A point  $a \in \mathbf{R}$  is said to be a *quenching point* (or *pinching point*) of  $u$  if there exists a sequence  $\{(x_k, t_k)\} \subset \mathbf{R} \times (0, T(u_0))$  such that

$$x_k \rightarrow a, t_k \nearrow T(u_0) \text{ and } u(x_k, t_k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In other words, a point  $a \in \mathbf{R}^N$  is a quenching point if and only if  $u$  is not bounded away from zero. Quenching points of  $u$  correspond to positions of pinching necks of the surface  $\Gamma_t$  at  $t = T(u_0)$ .

### 1.3 Known results

We shall recall several known results on mean curvature flow equations for compact hypersurfaces.

#### General surface

1. Huisken [11].

If initial surface is smooth, compact and convex in  $\mathbf{R}^n, n \geq 3$ , then the hypersurface  $\Gamma_t$  remains smooth, compact and convex and shrinks to a “round point” in finite time.

2. Gage and Hamilton [3].

The result of [11] still holds for simple convex curves in  $\mathbf{R}^2$ .

3. Grayson [9].

Even if initial curve is not convex, the solution curve must become convex before it shrinks to a point.

#### Axisymmetric surface

4. Grayson [10].

When  $n \geq 3$ , this result of [9] can fail to hold in general. There is an example of surface whose neck pinches before it shrinks to a point. (A barbell-like surface: two spherical surfaces connected by a thin “neck”.)

5. Altschuler, Angenent and Giga [1].

There exists a finite sequence  $0 = t_0 < t_1 < \dots < t_\ell$  such that  $\Gamma_t$  is smooth for  $t_{j-1} < t < t_j$ . The number of components **can** change only at  $t = t_j, (j = 1, 2, \dots, \ell)$ .

### 1.4 Our aim

We would like to show that any noncompact axisymmetric hypersurface such that the quenching time is “minimal” has no pinching point on the axis of

rotation at the quenching time **no matter how thin** necks of the initial surface are, except for flat surfaces, and to characterize such hypersurfaces by initial data. We will give the definition that the quenching time is minimal in the next section.

## 2 Main results

If the initial data is a positive constant, then the solution of (1.1)-(1.2) coincides with the solution  $v_m(t)$  of the corresponding ordinary differential equation

$$v' = -\frac{n-1}{v}, \quad t > 0; \quad v(0) = m, \quad (2.1)$$

that is,

$$v_m(t) = \sqrt{2(n-1)(T(m)-t)} \quad \text{with} \quad T(m) = \frac{m^2}{2(n-1)}. \quad (2.2)$$

The function  $v_m$  provides the vanishing cylinder with diameter  $v_m(t)$  for each time  $t < T_m$ . In what follows,  $m$  is chosen as

$$m = \inf_{x \in \mathbf{R}} u_0(x).$$

A simple comparison argument shows that any solution  $u$  of (1.1)-(1.2) satisfies

$$u(x, t) \geq v_m(t) \quad \text{in } x \in \mathbf{R} \times (0, T_m).$$

We thus have, in general,

$$T(u_0) \geq T_m.$$

**Definition.** We say that a solution  $u$  of the Cauchy problem (1.1)-(1.2) has a *minimal quenching time* if

$$T(u_0) = T(m).$$

**Proposition 2.1.** *Suppose that a solution  $u$  of the Cauchy problem (1.1)-(1.2) quenches at minimal quenching time  $T(m)$ . Then*

$$\liminf_{x \rightarrow -\infty} u(x, t) = v_m(t) \quad \text{or} \quad \liminf_{x \rightarrow +\infty} u(x, t) = v_m(t)$$

for every  $t \in [0, T(m))$  and quenching occurs at space infinity in the sense that there exists a sequence  $\{(x_k, t_k)\} \subset \mathbf{R} \times (0, T(m))$  such that

$$t_k \nearrow T(m) \quad \text{and} \quad u(x_k, t_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We are now in the position to state our main results.

**Theorem 2.2.** *Let  $u$  be a solution of the Cauchy problem (1.1)-(1.2) having a minimal quenching time  $T(m)$ . If  $u_0 \not\equiv m$ , then there is no quenching point of  $u$ . Moreover, there exists a function  $u(\cdot, T(m)) \in C^\infty(\mathbf{R})$  such that  $u(\cdot, t) \rightarrow u(\cdot, T(m))$  in the Frechét space  $C^\infty(\mathbf{R})$  as  $t \nearrow T(m)$ , and it fulfills  $u(x, T(m)) > 0$  in the whole  $\mathbf{R}$ . Furthermore,  $\lim_{x \rightarrow -\infty} u(x, T(m)) = 0$  and/or  $\lim_{x \rightarrow +\infty} u(x, T(m)) = 0$ .*

We can actually obtain a necessary and sufficient condition on initial data for a solution of the Cauchy problem (1.1)-(1.2) to have a minimal quenching time, making use of the technique developed in [15, 14] for related blow-up problems. We shall consider the following conditions on initial data:

$$\begin{aligned} &\text{There exists a sequence } \{x_k\} \subset \mathbf{R} \text{ such that } x_k \rightarrow \infty \text{ and} \\ &u_0(x + x_k) \rightarrow m \quad \text{a.e. as } k \rightarrow \infty. \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\text{There exists a sequence } \{x_k\} \subset \mathbf{R} \text{ such that } x_k \rightarrow -\infty \text{ and} \\ &u_0(x + x_k) \rightarrow m \quad \text{a.e. as } k \rightarrow \infty. \end{aligned} \tag{2.4}$$

**Theorem 2.3.** *A solution of the Cauchy problem (1.1)-(1.2) has a minimal quenching time if and only if  $u_0$  satisfies conditions (2.3) or (2.4).*

### 3 Tools

We shall recall some basic tools obtained in [1] in the restricted form convenient to our aim. In what follows, the half interval  $(0, \infty)$  is denoted by  $\mathbf{R}_+$ .

**Lemma 3.1.**(Altschuler-Angenent-Giga [1]; gradient bound.) *Let  $u$  be a solution of (1.1) in  $(a, b) \times (0, T)$  for some  $-\infty < a < b < \infty$ . Then there is a function  $\sigma : \mathbf{R}_+ \times \mathbf{R}_+ \rightarrow \mathbf{R}$  such that*

$$|u_x(x, t)| \leq \sigma(t, u(x, t)) \tag{3.1}$$

*holds for all  $a < x < b, 0 < t < T$ . The function  $\sigma$  has the form  $\sigma(t, u) = \exp(\rho(u)/t)$  with a positive continuous function  $\rho$  on  $\mathbf{R}_+$  and depends only on  $\sup u(x, 0)$  and  $b - a$ . Moreover, if  $u$  solves the equation in  $\mathbf{R} \times (0, T)$ , then (3.1) holds in  $\mathbf{R} \times (0, T)$  and  $\sigma$  depends only on  $\sup u(x, 0)$ .*

**Lemma 3.2.**(Altschuler-Angenent-Giga [1, Single-Point Pinching Lemma]).  
Let  $u$  be a solution of (1.1) in  $(a, b) \times (0, T)$  for some  $-\infty < a < b < \infty$ . If the solution  $u$  of (1.1) is monotone increasing (or decreasing) with respect to  $x$  in  $(a, b)$ , then for any subinterval  $(c, d) \Subset (a, b)$ , there is a constant  $\delta > 0$  such that

$$u(x, t) \geq \delta \quad \text{in } (c, d) \times (0, T).$$

## 4 Related studies

Our problem is closely related with a blow-up problem ([7, 8, 16, 15, 14]) for nonlinear parabolic equation

$$u_t = \Delta\phi(u) + f(u), \quad x \in \mathbf{R}^N, \quad t > 0, \quad (4.1)$$

with initial data

$$u(x, 0) = u_0(x), \quad x \in \mathbf{R}^N. \quad (4.2)$$

Typical nonlinear terms are  $\phi(u) = u^m$  and  $f(u) = u^p$  with  $m > 0, p > 1$  being constants. Our results and proofs of the present study are the same with those of [15, 14] in spirit.

In the semilinear case  $\phi(u) = u$ , if  $u_0$  is not a constant and takes its maximum at infinity, then the solution of (4.1)-(4.2) blows up only at space infinity: See [12, 5] for one-dimensional problem; [7, 8, 16] for the Cauchy problem in  $\mathbf{R}^N$  (See also [6].) The notion of ‘‘blow-up direction’’ was originally introduced in [8]. For a solution  $u$  of (4.1)-(4.2) blowing up at  $t = T(u_0)$ , we say that a direction  $\psi \in S^{N-1}$  is a blow-up direction if there exists a sequence  $\{(x_n, t_n)\} \subset \mathbf{R}^N \times (0, T(u_0))$  such that

$$|x_n| \rightarrow \infty, \quad \frac{x_n}{|x_n|} \rightarrow \psi, \quad t_n \nearrow T(u_0), \quad \text{and} \quad u(x_n, t_n) \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.$$

It is shown in [8] that the blow-up directions are characterized by initial data. The authors of [15, 14] generalized the results of [7, 8] to the quasilinear case. The definition of ‘‘minimal blow-up time’’ (or ‘‘the least (possible) blow-up time’’) was originally given in [15]. The notion of minimal quenching time introduced in the present note is an analogue of this notion. The authors of [15, 14] obtained a necessary and sufficient condition on initial data for a solution to have a minimal blow-up time, which is close to Theorem 2.3.

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# A remark on the uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

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## Abstract

We consider the uniqueness of positive solutions to

$$\begin{cases} \Delta u - \omega u + u^p - u^{2p-1} = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (1)$$

It is known that for fixed  $p > 1$ , a positive solution to (1) exists if and only if  $\omega \in (0, \omega_p)$ , where  $\omega_p := \frac{p}{(p+1)^2}$ . We deduce the uniqueness in the case where  $\omega$  is close to  $\omega_p$ , from the argument in the classical paper by Peletier and Serrin [9], thereby recovering a part of the uniqueness result of Ouyang and Shi [8] for all  $\omega \in (0, \omega_p)$ .

## 1 Introduction

We shall consider a boundary value problem

$$\begin{cases} u_{rr} + \frac{n-1}{r}u_r - \omega u + u^p - u^{2p-1} = 0 & \text{for } r > 0, \\ u_r(0) = 0, \\ \lim_{r \rightarrow \infty} u(r) = 0, \end{cases} \quad (2)$$

where  $n \in \mathbb{N}$ ,  $p > 1$  and  $\omega > 0$ . The above problem arises in the study of

$$\begin{cases} \Delta u - \omega u + u^p - u^{2p-1} = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (3)$$

Indeed, the classical work of Gidas, Ni and Nirenberg [4, 5] tells us that any positive solution to (3) is radially symmetric. On the other hand, for a solution  $u(r)$  of (2),  $v(x) := u(|x|)$  is a solution to (3).

The condition to assure the existence of positive solutions to (3) (and so (2)) was given by Berestycki and Lions [1] and Berestycki, Lions and Peletier [2]: A solution to (2) with fixed  $p > 1$  exists if and only if  $\omega \in (0, \omega_p)$ , where

$$\omega_p = \frac{p}{(p+1)^2}.$$

We shall review what this  $\omega_p$  is for in Section 2. Throughout this paper, a *solution* means a classical solution.

Uniqueness of positive solutions to (2) had long remained unknown. Finally in 1998 Ouyang and Shi [8] proved uniqueness for (2) with all  $\omega \in (0, \omega_p)$ ,  $p > 1$ . See also Kwong and Zhang [6].

In this present paper, we prove that for  $\omega$  close to  $\omega_p$ , the uniqueness result is obtained directly from the classical result given by Peletier and Serrin [9] in 1983. For another attempt to obtain the uniqueness when  $\omega$  is close to  $\omega_p$ , see Mizumachi [7]. Our result of the present paper is the following:

**Theorem 1.** *Let  $n \in \mathbb{N}$ ,  $p > 1$  and  $\omega \in [a_p, \omega_p)$ , where  $a_p := \frac{p(7p-5)}{4(p+1)(2p-1)^2}$ . Then (2) has exactly one positive solution.*

**Remark 1.** Note that

$$0 < a_p < \omega_p = \frac{p}{(p+1)^2}, \quad p > 1.$$

In the next section we clarify the definitions of  $\omega_p$  and  $a_p$  from the point of view from [9].

## 2 Study of the nonlinearity as a function

In this section, we study the properties of the function  $f(u) := -\omega u + u^p - u^{2p-1}$  in  $(0, \infty)$ , where  $\omega > 0$  and  $p > 1$  are given constants.

First we define  $F(u) := \int_0^u f_{\omega,p}(s) ds$ , and by a direct calculation we have

$$\begin{aligned} F(u) &= -\frac{\omega}{2}u^2 + \frac{u^{p+1}}{p+1} - \frac{u^{2p}}{2p} \\ &= \frac{u^2}{2p(p+1)} \left[ -\omega p(p+1) + 2pu^{p-1} - (p+1)u^{2(p-1)} \right]. \end{aligned} \quad (4)$$

There are two cases of concern:

- (a)  $\omega < \omega_p \iff F$  has two zeros in  $(0, \infty)$ .
- (b)  $\omega \geq \omega_p \iff F$  has at most one zero in  $(0, \infty)$ .

The condition to assure the existence of positive solutions of (2) given in [1, 2] is the following;

**Lemma 1.** *The problem (2) has a positive solution if and only if both of the following hypotheses are fulfilled:*

- (H1)  $\lim_{u \rightarrow +0} \frac{f(u)}{u}$  exists and is negative,
- (H2)  $F(\delta) > 0$  for some positive constant  $\delta$ .

**Lemma 2.** *The problem (2) has a positive solution if and only if*

$$\omega \in (0, \omega_p)$$

for  $p > 1$ .

*Proof.* (H1) is equivalent to the condition  $\omega > 0$ . (H2) is equivalent to the condition (a) above.  $\square$

This is the origin of  $\omega_p$ . Next we turn to the exponent  $a_p$ .

As a preparation, we calculate the derivatives of  $f(u) = -\omega u + u^p - u^{2p-1}$ :

$$\begin{aligned} f'(u) &= -\omega + pu^{p-1} - (2p-1)u^{2(p-1)}, \\ f''(u) &= 2(p-1)(2p-1)u^{p-2} \left[ \frac{p}{2(2p-1)} - u^{p-1} \right]. \end{aligned}$$

We shall introduce four positive constants  $\alpha$ ,  $b$ ,  $c$  and  $\beta$ .

- Let  $\alpha$  denote the unique zero of  $f''$  in  $(0, \infty)$ :  $\alpha = \left[ \frac{p}{2(2p-1)} \right]^{\frac{1}{p-1}}$ .

- Let  $b$  denote the first zero of  $f$  in  $(0, \infty)$ :  $b = \left[ \frac{1 - \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}}$ .
- Let  $c$  denote the last zero of  $f$  in  $(0, \infty)$ :  $c = \left[ \frac{1 + \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}}$ .
- Let  $\beta$  denote the first zero of  $F$  in  $(0, \infty)$ :  $\beta = \left[ \frac{p}{p+1} \left( 1 - \sqrt{1 - \frac{(p+1)^2}{p}\omega} \right) \right]^{\frac{1}{p-1}}$ .

It is easy to check that

$$\beta \in (b, c) \tag{5}$$

either by observing the graphs or by a straightforward calculation. From (5) we deduce

$$f(\beta) > 0, \tag{6}$$

which will be used later.

We are not able to give a clear explanation on the relation between  $\alpha$  and  $\beta$ .

**Lemma 3.** *The condition  $\alpha \leq \beta$  is equivalent to  $\omega \geq a_p = \frac{p(7p-5)}{4(p+1)(2p-1)^2}$ .*

*Proof.* A simple calculation. □

This is where our  $a_p$  comes into play. In the next section, we see what this condition stands for.

### 3 Proof of Theorem 1.

First we state the result by Peletier and Serrin [9], which assures the uniqueness of solutions of (2).

**Lemma 4.** *Let  $f$  satisfy (H1-3), where (H1), (H2) are in Lemma 1., and (H3) is the following:*

(H3)  $G(u) := \frac{f(u)}{u - \beta}$  is nonincreasing in  $(\beta, c)$ .

*Then (2) has exactly one positive solution.*

Now we are in a position to prove Theorem 1.

*Proof of Theorem 1.* We shall see that for  $\omega \in [a_p, \omega_p)$ , (H1-3) are satisfied. It is enough to show that if  $\omega \geq a_p$ , then

$$k(u) := f'(u)(u - \beta) - f(u) \leq 0 \quad \text{in } (\beta, c). \tag{7}$$

To prove (7) we calculate the derivative of  $k(u)$

$$k'(u) = f''(u)(u - \beta),$$

and note that

$$\begin{aligned} f''(u) &> 0 && \text{in } (0, \alpha); \\ f''(u) &< 0 && \text{in } (\alpha, \infty). \end{aligned}$$

So if  $\alpha \leq \beta$  (i.e.  $\omega \geq a_p$ , see Lemma 3), then  $k'(u) < 0$  in  $(\beta, c)$ , i.e.  $k$  is decreasing in the interval. Therefore

$$k(u) < k(\beta) = -f(\beta) < 0 \quad \text{in } (\beta, c),$$

where the last inequality follows by (6).

This proves (7) and completes the proof. □

If  $\alpha > \beta$ , we need to check that  $k(\alpha) \leq 0$ , i.e.

$$\alpha - \frac{f(\alpha)}{f'(\alpha)} \leq \beta. \quad (8)$$

This condition provides an implicit relation between  $\omega$  and  $p$ . Besides,

**Remark 2.** The condition (8) does not cover all  $\omega \in (0, \omega_p)$ . That is for  $\omega$  close to zero,  $\alpha - \frac{f(\alpha)}{f'(\alpha)} > \beta$ .

*Proof.* The left hand side of (8) is estimated from below as

$$\begin{aligned} \alpha - \frac{f(\alpha)}{f'(\alpha)} &= \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{-\omega + p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}} \\ &> \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}} > 0, \end{aligned}$$

for all  $\omega \in (0, \omega_p)$ , whereas the right hand side  $\beta$  decreases to zero as  $\omega$  decreases to zero. □

When  $\omega$  is close to zero, a very delicate observation is needed. See Ouyang and Shi [8] for details.

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# ASYMPTOTIC BEHAVIOR OF GLOBAL SOLUTIONS TO SOME SYSTEMS OF SEMILINEAR WAVE EQUATIONS

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## 1. INTRODUCTION

We consider the Cauchy problem for the following system of semilinear wave equations

$$(1.1) \quad \begin{cases} \square u_i = F_i(u, \partial u) \text{ in } (0, \infty) \times \mathbb{R}^3 \ (i = 1, 2, \dots, N), \\ u(0, x) = \varepsilon f(x), \ (\partial_t u)(0, x) = \varepsilon g(x) \text{ for } x \in \mathbb{R}^3, \end{cases}$$

where  $\square = \partial_t^2 - \Delta_x$ ,  $\Delta_x = \sum_{k=1}^3 \partial_k^2$ ,  $u = (u_j)_{1 \leq j \leq N}$ , and  $\partial u = (\partial_a u_j)_{0 \leq a \leq 3, 1 \leq j \leq N}$ . Here we have used the notation

$$\partial_0 = \partial_t = \partial/\partial t, \quad \partial_k = \partial/\partial x_k \ (k = 1, 2, 3).$$

We assume  $f = (f_j)_{1 \leq j \leq N}$ ,  $g = (g_j)_{1 \leq j \leq N} \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^N)$ .  $\varepsilon$  is a positive and small parameter.

For simplicity, we suppose that each  $F_i$  ( $1 \leq i \leq N$ ) is a homogeneous polynomial of degree 2 in its arguments  $(u, \partial u)$ .

We say that *small data global existence* (or **(SDGE)**) holds if for any  $f, g \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^N)$ , there exists a positive constant  $\varepsilon_0$  such that (1.1) admits a global solution  $u \in C^\infty([0, \infty) \times \mathbb{R}^3; \mathbb{R}^N)$  for any  $\varepsilon \in (0, \varepsilon_0]$ .

Let us recall the known results. In general, the solution to (1.1) may blow up in finite time no matter how small  $\varepsilon$  is. In other words, (SDGE) does not hold in general. For example, if we consider a single equation  $\square u = F(u, \partial u)$  with either  $F = (\partial_t u)^2$ ,  $F = u(\partial_t u)$ , or  $F = u^2$ , then there exist  $f$  and  $g \in C_0^\infty(\mathbb{R}^3)$  such that the solution  $u$  blows up in finite time for any  $\varepsilon > 0$  (see [4] and [5]). Hence we need some restriction to obtain (SDGE). The null condition introduced by Klainerman [7] is one of sufficient conditions for (SDGE). For  $F_i = F_i(u, \partial u)$ , we define the *reduced nonlinearity*  $F_i^{\text{red}}$  by

$$(1.2) \quad F_i^{\text{red}}(\omega, X, Y) = F_i((X_j)_{1 \leq j \leq N}, (\omega_a Y_j)_{0 \leq a \leq 3, 1 \leq j \leq N})$$

for  $X = (X_j)_{1 \leq j \leq N}$ ,  $Y = (Y_j)_{1 \leq j \leq N} \in \mathbb{R}^N$ , and  $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$  with  $\omega_0 = -1$ . Here the right-hand side of (1.2) means that  $X_j$  and  $\omega_a Y_j$  are substituted in place of  $u_j$  and  $\partial_a u_j$ , respectively. We set  $F^{\text{red}}(\omega, X, Y) = (F_j^{\text{red}}(\omega, X, Y))_{1 \leq j \leq N}$ .

**Definition 1.1** (The null condition). *We say that  $F(u, \partial u) = (F_j(u, \partial u))_{1 \leq j \leq N}$  satisfies the null condition if*

$$(1.3) \quad F^{\text{red}}(\omega, X, Y) \equiv 0$$

for any  $X, Y \in \mathbb{R}^N$ , and any  $\omega \in S^2$ .

We define the *null forms*  $Q_0$  and  $Q_{ab}$  by

$$(1.4) \quad Q_0(\varphi, \psi) = (\partial_t \varphi)(\partial_t \psi) - \sum_{k=1}^3 (\partial_k \varphi)(\partial_k \psi),$$

$$(1.5) \quad Q_{ab}(\varphi, \psi) = (\partial_a \varphi)(\partial_b \psi) - (\partial_b \varphi)(\partial_a \psi), \quad 0 \leq a < b \leq 3.$$

Then we see that  $F = (F_j)_{1 \leq j \leq N}$  satisfies the null condition if and only if each  $F_j$  ( $1 \leq j \leq N$ ) is a linear combination of the null forms  $Q_0(u_j, u_k)$  and  $Q_{ab}(u_j, u_k)$  with  $1 \leq j, k \leq N$  and  $0 \leq a < b \leq 3$ .

Klainerman [7], and Christodoulou [2] independently proved that if  $F$  satisfies the null condition, then (SDGE) holds.

Investigating the proof in [7], we also find that, if  $F$  satisfies the null condition, then the global solution  $u$  to (1.1) is asymptotically free in the sense of the energy, that is to say, there exists a solution  $u^+$  to the free wave equation  $\square u^+ = 0$  such that  $\lim_{t \rightarrow \infty} \|(\partial u - \partial u^+)(t, \cdot)\|_{L^2(\mathbb{R}^3)} = 0$ .

Let us assume  $F = F(\partial u)$  for a while (thus we write  $F^{\text{red}} = F^{\text{red}}(\omega, Y)$ ). Note that the null condition implies that  $F = F(\partial u)$  (if all  $F_j$  are homogeneous polynomials of degree 2). Lindblad and Rodnianski [8] introduced the notion of the *weak null condition*, and conjectured that the weak null condition implies (SDGE). In connection to this, Alinhac introduced the following condition, which is stronger than the weak null condition, but still weaker than the null condition: **(A)** There exist  $\beta(\omega) = (\beta_j(\omega))_{1 \leq j \leq N} (\in \mathbb{R}^N)$  and  $M(\omega, Y) (\in \mathbb{R})$  such that

- (1)  $F_i^{\text{red}}(\omega, Y) = M(\omega, Y)\beta_i(\omega)$  for any  $Y \in \mathbb{R}^N$  and  $\omega \in S^2$  ( $1 \leq i \leq N$ ),
- (2)  $M(\omega, \beta(\omega)) = 0$  for any  $\omega \in S^2$ .

Moreover, there exist some numbers of bilinear forms  $h_j(\omega, Y)$  ( $1 \leq j \leq I$ ) in  $\omega$  and  $Y$ , and linear forms  $g_i^j(\omega, Y)$  ( $1 \leq i \leq N$ ,  $1 \leq j \leq I$ ) in  $Y$  (with smooth coefficients in  $\omega$ ) such that

- (3)  $F_i^{\text{red}}(\omega, Y) = \sum_{1 \leq j \leq I} g_i^j(\omega, Y)h_j(\omega, Y)$  for any  $Y \in \mathbb{R}^N$  and  $\omega \in S^2$  ( $1 \leq i \leq N$ ),
- (4)  $h_j(\omega, \beta(\omega)) = 0$  for any  $\omega \in S^2$  ( $1 \leq j \leq I$ ).

Alinhac [1] proved that if the condition (A) is satisfied, then (SDGE) holds. The typical example which satisfies the condition (A), but not the null condition is

$$(1.6) \quad \begin{cases} \square u_1 = (\partial_1 u_1)(\partial_2 u_1 - \partial_1 u_2), \\ \square u_2 = (\partial_2 u_1)(\partial_2 u_1 - \partial_1 u_2) \end{cases}$$

in  $(0, \infty) \times \mathbb{R}^3$ . By setting  $w = \partial_2 u_1 - \partial_1 u_2$ ,  $v_1 = u_1$  and  $v_2 = u_2$ , (1.6) is reduced to

$$(1.7) \quad \begin{cases} \square v_1 = w(\partial_1 v_1), \\ \square v_2 = w(\partial_2 v_1), \\ \square w = Q_{12}(v_1, w). \end{cases}$$

The right-hand side of the equation in  $w$  is written in terms of the null forms, while the equation for  $v = (v_1, v_2)$  is linear in  $v$ .

In general, if the condition (A) is satisfied, we can reduce the original system to another system with a similar structure to (1.7) in the region  $|x| \geq (1+t)/2$ , say. This is the key point for the proof of (SDGE) under the condition (A) in [1].

Concerning the asymptotic behavior, Katayama–Kubo [6] proved that the global solution  $u$  under the condition (A) is not necessarily asymptotically free in the sense of the energy. For example, it is proved that there exist  $f, g \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^2)$  and two positive constants  $\varepsilon_1, C$  such that

$$(1.8) \quad \|\partial u(t)\|_{L^2(\mathbb{R}^3)} \geq C(1+t)^{C\varepsilon},$$

provided that  $\varepsilon \in (0, \varepsilon_1]$ , where  $u$  is the global solution to (1.6). Note that (1.8) implies  $\lim_{t \rightarrow \infty} \|\partial u(t, \cdot)\|_{L^2(\mathbb{R}^3)} = \infty$ , while the energy must stay bounded when the solution is asymptotically free in the sense of the energy.

Now we are led to the following question: Is there a system of semilinear wave equations for which (SDGE) holds and whose global solution behaves differently from the free solutions, though the energy stays bounded? We would like to answer this question by investigating the pointwise behavior of the global solution for the large time.

## 2. PRELIMINARIES

First we recall Friedlander’s radiation field, which describes the asymptotic behavior of the solution to the homogeneous wave equation.

Let us consider the homogeneous linear wave equation

$$(2.1) \quad \begin{cases} \square u = 0 & \text{in } (0, \infty) \times \mathbb{R}^3, \\ \square u(0, x) = f(x), (\partial_t u)(0, x) = g(x) & \text{for } x \in \mathbb{R}^3. \end{cases}$$

We suppose that  $f, g \in C_0^\infty(\mathbb{R}^3)$  and  $\text{supp } f \cup \text{supp } g \subset \{x \in \mathbb{R}^3; |x| \leq M\}$  with some  $M > 0$ . Friedlander’s radiation field is defined by

$$(2.2) \quad \mathcal{F}(\sigma, \omega; f, g) = \frac{1}{4\pi} (\mathcal{R}(\sigma, \omega; g) - \partial_\sigma \mathcal{R}(\sigma, \omega; f))$$

for  $\sigma \in \mathbb{R}$  and  $\omega \in S^2$ , where

$$\mathcal{R}(\sigma, \omega; \varphi) = \int_{y \cdot \omega = \sigma} \varphi(y) dS(y).$$

Note that we have  $\mathcal{F}(\sigma, \omega; f, g) = 0$  for  $|\sigma| > M$ .

Let  $u$  be the solution to (2.1). Then we have

$$(2.3) \quad \begin{aligned} |ru(t, r\omega) - \mathcal{F}(r-t, \omega)| + \sum_{a=0}^3 |r\partial_a u(t, r\omega) - \omega_a(\partial_\sigma \mathcal{F})(r-t, \omega)| \\ \leq C(1+t+r)^{-1} \quad \text{for } r \geq (t/2) + 1 \text{ and } \omega = (\omega_1, \omega_2, \omega_3) \in S^2, \end{aligned}$$

where  $\omega_0 = -1$ , and  $\mathcal{F}(\sigma, \omega) = \mathcal{F}(\sigma, \omega; f, g)$  (see Hörmander [3] for example). Here  $C$  is a positive constant which may depend on  $M$ .

Our aim in this talk is to obtain a similar result to (2.3) for the semilinear system (1.1).

We introduce

$$S = \partial_t + x \cdot \nabla_x, \quad L_j = t\partial_j + x_j\partial_t \quad (j = 1, 2, 3), \quad \Omega = (\Omega_1, \Omega_2, \Omega_3) = x \times \nabla_x.$$

We have

$$[S, \square] = -2\square, [L_j, \square] = [\Omega_j, \square] = [\partial_a, \square] = 0 \quad (1 \leq j \leq 3, 0 \leq a \leq 3).$$

We set

$$\Gamma = (\Gamma_a)_{0 \leq a \leq 10} = (S, (L_j)_{1 \leq j \leq 3}, (\Omega_j)_{1 \leq j \leq 3}, (\partial_a)_{0 \leq a \leq 3}),$$

and using a multi-index  $\alpha = (\alpha_0, \dots, \alpha_{10})$ , we write  $\Gamma^\alpha = \Gamma_0^{\alpha_0} \cdots \Gamma_{10}^{\alpha_{10}}$ . For a nonnegative integer  $k$  and a smooth function  $\varphi$ , we define

$$(2.4) \quad |\varphi(t, x)|_k = \sum_{|\alpha| \leq k} |\Gamma^\alpha \varphi(t, x)|, \quad \|\varphi(t, \cdot)\|_k = \left\| |\varphi(t, \cdot)|_k \right\|_{L^2(\mathbb{R}^3)}.$$

We set  $r = |x|$ ,  $\omega = x/r$ , and  $\partial_r = \sum_{j=1}^3 (x_j/r) \partial_j$ . We also define  $L_r = \sum_{j=1}^3 (x_j/r) L_j = r\partial_t + t\partial_r$ , and  $\partial_\pm = \partial_t \pm \partial_r$ . Noting that  $S = t\partial_t + r\partial_r$ , we obtain

$$(2.5) \quad \partial_+ = \frac{1}{t+r}(S + L_r).$$

Since  $\partial_t = (\partial_- + \partial_+)/2$  and  $\partial_r = (-\partial_- + \partial_+)/2$ , we obtain

$$(2.6) \quad \left| \left( \partial_t - \frac{1}{2} \partial_- \right) \varphi(t, x) \right| + \left| \left( \partial_r + \frac{1}{2} \partial_- \right) \varphi(t, x) \right| \leq (1+t+r)^{-1} |\varphi(t, x)|_1,$$

in view of (2.5). Since  $\nabla_x = \omega \partial_r - (\omega/r) \times \Omega$ , and  $(\omega/r) \times \Omega = t^{-1} \omega \times (\omega \times L)$  with  $L = (L_1, L_2, L_3)$ , we have

$$(2.7) \quad |(\partial_j - \omega_j \partial_r) \varphi(t, x)| \leq C(1+t+r)^{-1} |\varphi(t, x)|_1 \quad \text{for } j = 1, 2, 3.$$

Now (2.6) and (2.7) leads to

$$(2.8) \quad \left| \left( \partial_a - \frac{-\omega_a}{2} \partial_- \right) \varphi(t, x) \right| \leq C(1+t+r)^{-1} |\varphi(t, x)|_1 \quad \text{for } 0 \leq a \leq 3.$$

Since  $tS - rL_r = (t^2 - r^2)\partial_t$  and  $tL_r - rS = (t^2 - r^2)\partial_r$ , we get

$$(2.9) \quad (1 + |t - r|) (|\partial_t \varphi(t, x)| + |\partial_r \varphi(t, x)|) \leq C |\varphi(t, x)|_1.$$

Equations (2.7) and (2.9) yield

$$(2.10) \quad (1 + |t - r|) |\partial \varphi(t, x)| \leq C |\varphi(t, x)|_1.$$

### 3. THE MAIN RESULT

For a matrix  $X$ , let  $X^T$  denote the transpose of  $X$ .

We divide the unknown  $u = (u_1, \dots, u_N)^T$  into two unknowns  $v$  and  $w$ :

$$u = (u_1, \dots, u_K, u_{K+1}, \dots, u_N)^T = (v_1, \dots, v_K, w_1, \dots, w_L)^T = (v^T, w^T)^T,$$

where  $K$  is some integer, and  $L = N - K$ . More precisely, we have set

$$v = (v_1, \dots, v_K)^T = (u_1, \dots, u_K)^T \quad \text{and} \quad w = (w_1, \dots, w_L)^T = (u_{K+1}, \dots, u_N)^T.$$

Motivated by (1.7), we consider the system

$$(3.1) \quad \begin{cases} \square v = \sum_{a=0}^3 A^a(w, \partial w) \partial_a v + N^{(v)}(\partial u) & \text{in } (0, \infty) \times \mathbb{R}^3, \\ \square w = N^{(w)}(\partial u) & \text{in } (0, \infty) \times \mathbb{R}^3, \\ u(0, x) = \varepsilon f(x), \quad (\partial_t u)(0, x) = \varepsilon g(x) & \text{for } x \in \mathbb{R}^3, \end{cases}$$

where  $A^a(w, \partial w)$  ( $0 \leq a \leq 3$ ) are  $K \times K$  matrix-valued functions whose components are homogeneous polynomials of degree 1 in  $(w, \partial w)$ , while  $N^{(v)}(\partial u)$  and  $N^{(w)}(\partial u)$  are  $\mathbb{R}^K$ -valued and  $\mathbb{R}^L$ -valued functions, respectively, whose components are linear combinations of the null forms  $Q_0(u_j, u_k)$  and  $Q_{ab}(u_j, u_k)$  with  $1 \leq j, k \leq N$  and  $0 \leq a < b \leq 3$ .

We set  $N(\partial u) = (N^{(v)}(\partial u)^T, N^{(w)}(\partial u)^T)^T$ . Writing  $f = (f_1, \dots, f_N)^T$  and  $g = (g_1, \dots, g_N)^T$ , we define

$$\begin{aligned} f^{(v)} &= (f_1, \dots, f_K)^T, & f^{(w)} &= (f_{K+1}, \dots, f_N)^T, \\ g^{(v)} &= (g_1, \dots, g_K)^T, & g^{(w)} &= (g_{K+1}, \dots, g_N)^T, \end{aligned}$$

so that we have  $v(0) = \varepsilon f^{(v)}$ ,  $(\partial_t v)(0) = \varepsilon g^{(v)}$ ,  $w(0) = \varepsilon f^{(w)}$  and  $(\partial_t w)(0) = \varepsilon g^{(w)}$ .

Concerning the existence of solutions to (3.1), using the method in [1] (see also [6]), we obtain the following:

**Proposition 3.1.** (SDGE) *holds for (3.1).*

*Moreover, if we fix  $k \in \mathbb{N}$ ,  $0 < \lambda \ll 1$  and  $0 < \rho < 1 - 4\lambda$ , there exist two positive constants  $\varepsilon_1$  and  $C$  such that*

$$(3.2) \quad |v(t, x)|_{k+1} \leq C\varepsilon(1+t+r)^{\lambda-1},$$

$$(3.3) \quad |w(t, x)|_{k+2} \leq C\varepsilon(1+t+r)^{-1}(1+|t-r|)^{-\rho}$$

for  $(t, x) \in [0, \infty) \times \mathbb{R}^3$ , provided that  $\varepsilon \in (0, \varepsilon_1]$ . We also have

$$(3.4) \quad \|\partial v(t, \cdot)\|_{2k} \leq C\varepsilon(1+t)^\lambda,$$

$$(3.5) \quad \|\partial w(t, \cdot)\|_{2k} \leq C\varepsilon$$

for  $t \geq 0$ , provided that  $\varepsilon \in (0, \varepsilon_1]$ .

From (3.2) and (3.3), in view of (2.10), we get

$$(3.6) \quad |\partial v(t, x)|_k \leq C\varepsilon(1+t+r)^{\lambda-1}(1+|t-r|)^{-1},$$

$$(3.7) \quad |\partial w(t, x)|_{k+1} \leq C\varepsilon(1+t+r)^{-1}(1+|t-r|)^{-1-\rho}.$$

For  $\psi(\sigma, \omega)^T = (\psi_j(\sigma, \omega))_{1 \leq j \leq L}$ , we define

$$(3.8) \quad A(\sigma, \omega; \psi) = \sum_{a=0}^3 \frac{-\omega_a}{2} A^a(\psi(\sigma, \omega), (\omega_b(\partial_\sigma \psi)(\sigma, \omega))_{0 \leq b \leq 3}),$$

where the right-hand side of (3.8) means that  $\psi_j$  and  $\omega_b \partial_\sigma \psi_j$  ( $1 \leq j \leq L$ ,  $0 \leq b \leq 3$ ) are substituted in place of  $w_j$  and  $\partial_b w_j$  ( $1 \leq j \leq L$ ,  $0 \leq b \leq 3$ ), respectively.

Now we are in a position to state the main result.

**Theorem 3.2.** *Suppose*

$$(3.9) \quad \text{supp } f \cup \text{supp } g \subset \{x \in \mathbb{R}^3; |x| \leq M\}.$$

*Let  $\varepsilon$  be sufficiently small, and  $u = (v^T, w^T)^T$  be the global solution to (3.1).*

*Then, there exists some  $\mathbb{R}^L$ -valued function  $W = W(\sigma, \omega)$  such that*

$$(3.10) \quad |rw(t, r\omega) - W(r-t, \omega)| \leq C\varepsilon(1+t+r)^{-1}(1+|t-r|)^{1-\rho},$$

$$(3.11) \quad |\partial_-(rw(t, r\omega)) - (-2)(\partial_\sigma W)(r-t, \omega)| \leq C\varepsilon(1+t+r)^{-1}(1+|t-r|)^{-\rho}$$

for  $r \geq (t/2) + M$  and  $\omega \in S^2$ , where  $\rho$  is from (3.3). We set a  $K \times K$  matrix-valued function  $\Theta$  by

$$\Theta(r, \sigma, \omega) = \left( \log \frac{r}{2M - \sigma} \right) A(\sigma, \omega; W).$$

Then there exists some  $\mathbb{R}^K$ -valued function  $V = V(\sigma, \omega)$  such that

$$(3.12) \quad \left| e^{-\Theta(r, r-t, \omega)} \partial_- (rv(t, r\omega)) - (-2)(\partial_\sigma V)(r-t, \omega) \right| \leq C\varepsilon(1+t+r)^{2\lambda+C\varepsilon-1}$$

for  $(t/2) + M \leq r \leq t + M$  and  $\omega \in S^2$ , where  $\lambda$  is from (3.2).

Moreover, setting

$$\mathcal{F}^{(v)}(\sigma, \omega) = \mathcal{F}(\sigma, \omega; f^{(v)}, g^{(v)}), \quad \mathcal{F}^{(w)}(\sigma, \omega) = \mathcal{F}(\sigma, \omega; f^{(w)}, g^{(w)})$$

with  $\mathcal{F}$  defined by (2.2), we have

$$(3.13) \quad |\partial_\sigma V(\sigma, \omega) - \varepsilon \partial_\sigma \mathcal{F}^{(v)}(\sigma, \omega)| \leq C\varepsilon^2(1+|\sigma|)^{2\lambda+C\varepsilon-1},$$

$$(3.14) \quad |\partial_\sigma^j W(\sigma, \omega) - \varepsilon \partial_\sigma^j \mathcal{F}^{(w)}(\sigma, \omega)| \leq C\varepsilon^2(1+|\sigma|)^{-\rho-j} \quad (j = 0, 1).$$

Here we remark that the assumption (3.9) implies that  $u(t, r\omega) = 0$  for  $r > t + M$ .

Note that, in view of (2.8), we also have

$$(3.15) \quad \left| re^{-\Theta(r, r-t, \omega)} (\partial_a v)(t, r\omega) - \omega_a (\partial_\sigma V)(r-t, \omega) \right| \leq C\varepsilon(1+t+r)^{2\lambda+C\varepsilon-1},$$

and

$$(3.16) \quad |r(\partial_a w)(t, r\omega) - \omega_a (\partial_\sigma W)(r-t, \omega)| \leq C\varepsilon(1+t+r)^{-1}(1+|t-r|)^{-\rho}$$

for  $(t/2) + M \leq r \leq t + M$  and  $\omega \in S^2$  with  $0 \leq a \leq 4$ .

We also note that we can get a similar result under the condition (A).

#### 4. EXAMPLES

First we consider (1.7). Since  $v_2$  does not play any essential role, we simplify (1.7) as

$$(4.1) \quad \begin{cases} \square v = w(\partial_1 v), \\ \square w = Q_{12}(v, w). \end{cases}$$

From Theorem 3.2, there exists  $W$  such that  $rw(t, r\omega) \sim W(r-t, \omega)$  for large  $t$  with  $\sigma = r-t$  being fixed. Using this  $W$ ,  $\Theta$  is defined by

$$\Theta(r, \sigma, \omega) = \left( \log \frac{r}{2M - \sigma} \right) \left( -\frac{\omega_1}{2} W(\sigma, \omega) \right).$$

Thus we find that there exists  $V(\sigma, \omega)$  such that

$$\partial_- (rv)(t, r\omega) \sim -2 \left( \frac{r}{2M - (r-t)} \right)^{-\omega_1 W(r-t, \omega)/2} (\partial_\sigma V)(r-t, \omega)$$

for large  $t$  with fixed  $\sigma = r-t$ . From this asymptotic behavior, we understand that the factor  $r^{-\omega_1 W(r-t, \omega)/2}$  makes the energy increase.

Next example is the following system:

$$(4.2) \quad \begin{cases} \square v_1 = (\partial_t w)(\partial_t v_2), \\ \square v_2 = -(\partial_t w)(\partial_t v_1), \\ \square w = Q_0(v_1, v_2). \end{cases}$$

It is easy to see that

$$(4.3) \quad \|\partial v_1(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\partial v_2(t)\|_{L^2(\mathbb{R}^3)}^2 = \|\partial v_1(0)\|_{L^2(\mathbb{R}^3)}^2 + \|\partial v_2(0)\|_{L^2(\mathbb{R}^3)}^2.$$

Proposition 3.1 implies that  $\|\partial w(t)\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon$ . Thus the energy stays bounded for the system (4.2).

From Theorem 3.2, there exists  $W$  such that  $r\partial_t w(t, r\omega) \sim -(\partial_\sigma W)(r-t, \omega)$  for large  $t$  with fixed  $\sigma = r-t$ .  $\Theta$  can be written as

$$\Theta(r, \sigma, \omega) = -\frac{1}{2} \left( \log \frac{r}{2M-\sigma} \right) (\partial_\sigma W)(\sigma, \omega) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then we find

$$e^{\Theta(r, \sigma, \omega)} = \begin{pmatrix} \cos \theta(r, \sigma, \omega) & -\sin \theta(r, \sigma, \omega) \\ \sin \theta(r, \sigma, \omega) & \cos \theta(r, \sigma, \omega) \end{pmatrix},$$

where

$$\theta(r, \sigma, \omega) = -\frac{1}{2} \left( \log \frac{r}{2M-\sigma} \right) (\partial_\sigma W)(\sigma, \omega).$$

Hence we find that there exists some  $V$  such that

$$\partial_-(rv)(t, r\omega) \sim -2 \begin{pmatrix} \cos \theta(r, r-t, \omega) & -\sin \theta(r, r-t, \omega) \\ \sin \theta(r, r-t, \omega) & \cos \theta(r, r-t, \omega) \end{pmatrix} (\partial_\sigma V)(r-t, \omega)$$

for large  $t$  with fixed  $\sigma = r-t$ . From this, we conclude that  $v$  behaves differently from free solutions at least for some data, because, in view of (3.13) and (3.14), we can choose the initial data such that  $(\partial_\sigma V)(\sigma, \omega) \neq 0$  and  $(\partial_\sigma W)(\sigma, \omega) \neq 0$  for all  $(\sigma, \omega)$  belonging to some open set, provided that  $\varepsilon$  is small enough.

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# Perturbation and Dispersion of Rayleigh Waves in Anisotropic Elasticity

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## 1 Introduction

Rayleigh waves are elastic surface waves which propagate along the traction-free surface with the phase velocity in the subsonic range and whose amplitude decays exponentially with depth below that surface. Such waves serve as a useful tool in nondestructive characterization of materials. The problem there is what material information we obtain if we could measure accurately Rayleigh waves propagating in any direction on the traction-free surface.

For definiteness, we choose a Cartesian coordinate system such that the material half-space occupies the region  $x_3 \leq 0$ , whereas the 1- and 2-axis are arbitrarily chosen. Then Rayleigh wave considered here can be described as a time-harmonic solution to the equation of motion with zero body force

$$\rho \frac{\partial^2}{\partial t^2} u_i = \sum_{j,k,l=1}^3 \frac{\partial}{\partial x_j} \left( C_{ijkl} \frac{\partial u_k}{\partial x_l} \right) \quad \text{in } x_3 < 0, \quad i = 1, 2, 3 \quad (1)$$

with the zero-traction boundary condition

$$\sum_{j,k,l=1}^3 C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j \Big|_{x_3=0} = 0, \quad i = 1, 2, 3. \quad (2)$$

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Here  $\rho > 0$  is the uniform mass density,  $t$  is the time,  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1, u_2, u_3)$  is the displacement at the place  $\mathbf{x} = (x_1, x_2, x_3)$  at time  $t$ ,  $(n_1, n_2, n_3) = (0, 0, 1)$  is the outward unit normal to the surface, and  $\mathbf{C} = \mathbf{C}(\mathbf{x}) = (C_{ijkl})_{i,j,k,l=1,2,3}$  is the elasticity tensor, which has the physically natural symmetries

$$C_{ijkl} = C_{jikl} = C_{klij}, \quad i, j, k, l = 1, 2, 3 \quad (3)$$

and satisfies the strong convexity condition

$$\sum_{i,j,k,l=1}^3 C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} > 0 \quad \left( (\varepsilon_{ij}) : \text{any nonzero } 3 \times 3 \text{ real symmetric matrix} \right)$$

at each  $\mathbf{x}$ .

First we consider Rayleigh waves propagating along the traction-free surface of a homogeneous elastic half-space. For isotropic elasticity, such waves are well known: Their phase velocity  $v_R^{\text{Iso}}$  is determined from the secular equation, which is a bi-cubic equation written in terms of the Lamé constants  $\lambda$  and  $\mu$  (see (6)).

Suppose that the elasticity tensor can be expressed as the sum of its isotropic and its perturbative part. We consider elastic media for which the perturbative part of the elasticity tensor is sufficiently small as compared with the isotropic part. The isotropic part of a given elasticity tensor is itself also an elasticity tensor, which we interpret as a comparative ‘unperturbed’ isotropic state. The perturbative part then gives the deviation of the elasticity tensor from the comparative isotropic state and represents the anisotropy that the elastic material carries. Here we do not put any restriction on the material symmetry of the perturbative part so that it has 21 independent components.

We investigate the perturbation of the phase velocity  $v_R$  of Rayleigh waves, i.e., the shift in  $v_R$  from its comparative isotropic value  $v_R^{\text{Iso}}$ , caused by the perturbative part. In Section 2 we present a perturbation formula for the phase velocity which is correct to within terms linear in the components of the perturbative part. This formula shows explicitly how the perturbative part, to first order of itself, affects the phase velocity of Rayleigh waves. We obtain these formulas by a consistent method on the basis of the Stroh formalism.

Second, we consider Rayleigh waves propagating along the traction-free surface of a vertically inhomogeneous elastic half-space. Here we assume that the elastic tensor depends smoothly only on the depth  $x_3$ . The purpose is to derive a high-frequency asymptotic formula for the velocity of Rayleigh waves propagating in various directions along the surface. We seek a time-harmonic solution

to (1) and (2) of the form

$$\mathbf{u} = (u_1, u_2, u_3) = e^{-\sqrt{-1}k(x_1\eta_1+x_2\eta_2-vt)} \mathbf{v}(\eta_1, \eta_2, x_3, v, k), \quad (4)$$

where  $k$  is a wave number,  $\eta = (\eta_1, \eta_2, 0)$  is the direction of wave propagation,  $v$  is phase velocity and  $\mathbf{v}$  is a complex vector function which decays exponentially as  $x_3 \rightarrow -\infty$ . In Section 3 we will develop a procedure with which, for each direction of propagation, we express each term of the asymptotic expansion of Rayleigh-wave velocity  $v_R$  for large  $k$  in terms of  $C_{ijkl}$  ( $1 \leq i, j, k, l \leq 3$ ) at  $x_3 = 0$  and their  $x_3$ -derivatives at  $x_3 = 0$ . This expresses the frequency-dependence of the Rayleigh-wave velocity, or the dispersion of the Rayleigh-wave velocity, caused by vertical inhomogeneity of the elasticity tensor. In nondestructive characterization of materials, by measuring the dispersion of the Rayleigh-wave velocity for various propagation directions, we obtain some information on  $C_{ijkl}$  and their  $x_3$ -derivatives at  $x_3 = 0$ .

The project in Section 3 is still in progress. As a partial result we give the first two terms of the asymptotic expansion of Rayleigh-wave velocity for large  $k$  when the material has an orthorhombic symmetry. Future extension is to study the perturbation of each term of the asymptotic expansion of Rayleigh-wave velocity caused by the deviation of the elasticity tensor from its comparative ‘unperturbed’ isotropic state.

## 2 Perturbation of Rayleigh-wave velocity

Suppose that the elasticity tensor  $\mathbf{C} = (C_{ijkl})_{i,j,k,l=1,2,3}$  is independent of  $\mathbf{x}$  and has the form

$$\mathbf{C} = \mathbf{C}^{\text{Iso}} + \mathbf{A}, \quad (5)$$

where  $\mathbf{C}^{\text{Iso}}$  is the isotropic part of  $\mathbf{C}$ ,

$$\mathbf{C}^{\text{Iso}} = (C_{ijkl}^{\text{Iso}})_{i,j,k,l=1,2,3}, \quad C_{ijkl}^{\text{Iso}} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj})$$

with the Lamé moduli  $\lambda$  and  $\mu$ , and  $\mathbf{A}$  is the perturbative part of  $\mathbf{C}$ ,

$$\mathbf{A} = (a_{ijkl})_{i,j,k,l=1,2,3}.$$

From the symmetries (3) of  $\mathbf{C}$  it follows that

$$a_{ijkl} = a_{jikl} = a_{klij}, \quad i, j, k, l = 1, 2, 3,$$

but we do not assume any other symmetry for  $\mathbf{A}$ . Hence the perturbative part  $\mathbf{A}$  is fully anisotropic and has 21 independent components.

**Theorem 1** ([4, 5]) *The phase velocity  $v_R$  of Rayleigh waves which propagate along the surface of the half-space  $x_3 \leq 0$  in the direction of the 2-axis can be written, to within terms linear in the perturbative part  $\mathbf{A} = (a_{ijkl})_{i,j,k,l=1,2,3}$ , as*

$$v_R = v_R^{\text{Iso}} - \frac{1}{2\rho v_R^{\text{Iso}}} \cdot \left[ \gamma_1(v_R^{\text{Iso}}) a_{2222} + \gamma_2(v_R^{\text{Iso}}) a_{2233} + \gamma_3(v_R^{\text{Iso}}) a_{3333} + \gamma_4(v_R^{\text{Iso}}) a_{2323} \right],$$

where

$$\begin{aligned} \gamma_1(v) &= \frac{(\lambda + 2\mu) \left[ -8\mu^2(\lambda + \mu) + 2\mu(5\lambda + 6\mu)V - (2\lambda + 3\mu)V^2 \right]}{D(v)}, \\ \gamma_2(v) &= \frac{4\lambda(\mu - V) \left[ 4\mu(\lambda + \mu) - (\lambda + 2\mu)V \right]}{D(v)}, \\ \gamma_3(v) &= \frac{(\lambda + 2\mu - V) \left[ -8\mu^2(\lambda + \mu) + 2\mu(5\lambda + 6\mu)V - (2\lambda + 3\mu)V^2 \right]}{D(v)} \\ &= \left( 1 - \frac{V}{\lambda + 2\mu} \right) \gamma_{22}(v), \\ \gamma_4(v) &= \frac{-8\mu(\lambda + 2\mu - V) \left[ 2\mu(\lambda + \mu) - (\lambda + 2\mu)V \right]}{D(v)}, \\ D(v) &= (\lambda + \mu) \left[ 8\mu^2(3\lambda + 4\mu) - 16\mu(\lambda + 2\mu)V + 3(\lambda + 2\mu)V^2 \right], \\ V &= \rho v^2, \end{aligned}$$

and  $v_R^{\text{Iso}}$  is the velocity of Rayleigh waves in the comparative isotropic medium defined by  $\mathbf{C} = \mathbf{C}^{\text{Iso}}$  and  $\mathbf{A} = \mathbf{O}$ , i.e.,  $V_R^{\text{Iso}} = \rho (v_R^{\text{Iso}})^2$  is the unique solution to the cubic equation

$$V^3 - 8\mu V^2 + \frac{8\mu^2(3\lambda + 4\mu)}{\lambda + 2\mu} V - \frac{16\mu^3(\lambda + \mu)}{\lambda + 2\mu} = 0 \quad (6)$$

in the range  $0 < V < \mu$ .

**Remarks** Only four components  $a_{2222}$ ,  $a_{2323}$ ,  $a_{2233}$  and  $a_{3333}$  of the perturbative part  $\mathbf{A}$  can influence the first order perturbation of the phase velocity  $v_R$ . The perturbation formula above do not agree totally with the result in [1]. In [5], where the initial stress is also taken into account, an argument is given to support our present result. According to our first-order formula above and the transformation formula for fourth-order tensors, we shall see that the anisotropy-induced velocity shifts of Rayleigh waves, taken in totality of all propagation directions

on the free surface, carry information only on 13 components of the perturbative part  $\mathbb{A}$  of the elasticity tensor [5].

In the homogeneous medium where the elasticity tensor  $\mathbf{C}$  is independent of  $\mathbf{x}$ , the surface-wave solution to (1) in the half-space  $x_3 \leq 0$  which decays exponentially as  $x_3 \rightarrow -\infty$  and has direction of propagation  $\eta = (\eta_1, \eta_2, 0)$ , phase velocity  $v$  and wave number  $k$  can be expressed in the form

$$\mathbf{u} = (u_1, u_2, u_3) = \sum_{\alpha=1}^3 e^{-\sqrt{-1}k(x_1\eta_1+x_2\eta_2+p_\alpha x_3-vt)} c_\alpha \mathbf{a}_\alpha(\eta_1, \eta_2, v), \quad (7)$$

where  $p_\alpha$  ( $\text{Im } p_\alpha > 0, \alpha = 1, 2, 3$ ) are Stroh's eigenvalues,  $\mathbf{a}_\alpha$  ( $\alpha = 1, 2, 3$ ) are linearly independent vectors in  $\mathbb{C}^3$  and  $c_\alpha$  ( $\alpha = 1, 2, 3$ ) are arbitrary complex constants. The boundary traction

$$\mathbf{t} = \left( \sum_{j,k,l=1}^3 C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j \right) \Big|_{i \downarrow 1,2,3} \Big|_{x_3=0} \quad (8)$$

pertaining to the solution (7) can be written in the form

$$\mathbf{t} = -\sqrt{-1}k \sum_{\alpha=1}^3 e^{-\sqrt{-1}k(x_1\eta_1+x_2\eta_2-vt)} c_\alpha \mathbf{l}_\alpha(\eta_1, \eta_2, v). \quad (9)$$

It follows that  $\begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{l}_\alpha \end{bmatrix} \in \mathbb{C}^6$  ( $\alpha = 1, 2, 3$ ) are linearly independent eigenvectors of Stroh's eigenvalue problem associated with the eigenvalues  $p_\alpha$  ( $\alpha = 1, 2, 3$ ) (see, for example, [4]).\*

The surface impedance matrix  $\mathbf{Z}(v, \eta)$ , which maps the boundary displacement  $\mathbf{u}|_{x_3=0}$  linearly to the boundary traction (8) is given by

$$\mathbf{Z}(v, \eta) = -\sqrt{-1} [\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3] [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]^{-1}, \quad (10)$$

where  $[\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3]$  and  $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$  denote  $3 \times 3$  matrices which consist of the column vectors  $\mathbf{l}_\alpha$  and  $\mathbf{a}_\alpha$  respectively. It is proved that  $\mathbf{Z}(v, \eta)$  is Hermitian.

From (2) it follows that the phase velocity  $v_R$  of Rayleigh waves in the homogeneous medium satisfies

$$\det \mathbf{Z}(v, \eta) = 0 \quad \text{at } v = v_R. \quad (11)$$

Applying the implicit function theorem to (11), we obtain Theorem 1.

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\*When  $\begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{l}_\alpha \end{bmatrix}$  becomes a generalized eigenvector of Stroh's eigenvalue problem for some  $\alpha$ , the forms (7) and (9) are slightly modified.

### 3 Dispersion of Rayleigh-wave velocity

Suppose that the elasticity tensor  $\mathbf{C} = (C_{ijkl})_{i,j,k,l=1,2,3}$  depends smoothly only on the depth  $x_3$  below the surface  $x_3 = 0$ . Our procedure will be divided into three steps.

1. Construct an asymptotic solution to (1) of the form (4) for large  $k$  by using the factorization of the principal symbol pertaining to the differential operator in  $x_3$  which is obtained from substitution of the form (4) into (1).
2. Let  $\mathbf{Z}(v, \eta, k)$  be the surface impedance matrix, which maps the boundary displacement  $\mathbf{u}|_{x_3=0}$  linearly to the boundary traction

$$\mathbf{t} = \left( \sum_{j,k,l=1}^3 C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j \right) \Big|_{i \downarrow 1,2,3} \Big|_{x_3=0}.$$

Determine  $3 \times 3$  surface impedance matrix  $\mathbf{Z}_n(v, \eta)$  ( $n = 0, 1, 2, \dots$ ) that appear in the asymptotic formula

$$\mathbf{Z}(v, \eta, k) = k\mathbf{Z}_0(v, \eta) + \mathbf{Z}_1(v, \eta) + k^{-1}\mathbf{Z}_2(v, \eta) + k^{-2}\mathbf{Z}_3(v, \eta) + \dots \quad (12)$$

for large  $k$ .

3. Apply the implicit function theorem to  $\det \mathbf{Z}(v, \eta, k) = 0$  to obtain the asymptotic formula for the phase velocity  $v_R$  of Rayleigh waves for large  $k$ :

$$v_R = v_R(\eta, k) = v_0(\eta) + v_1(\eta) k^{-1} + v_2(\eta) k^{-2} + \dots \quad (13)$$

$\mathbf{Z}_0(v, \eta)$  in (12) is the surface impedance matrix for the homogeneous half-space defined by (10) with  $\mathbf{C} = \mathbf{C}(0)$ . Then  $v_0(\eta)$  in (13) is identical to  $v_R$  in (11) under  $\mathbf{C} = \mathbf{C}(0)$ .

Now we shall give an equation for  $\mathbf{Z}_1(v, \eta)$  in (12). Define the  $3 \times 3$  matrices  $\mathbf{R}_0 = \mathbf{R}_0(\eta)$ ,  $\mathbf{T}_0$  by

$$\mathbf{R}_0 = \left( \sum_{j=1}^2 C_{ijk3}(0) \eta_j \right)_{i \downarrow k \rightarrow 1,2,3}, \quad \mathbf{T}_0 = (C_{i3k3}(0))_{i \downarrow k \rightarrow 1,2,3} = \mathbf{T}_0^T$$

and put

$$\mathbf{K}_0 = \mathbf{K}_0(v, \eta) = \mathbf{T}_0^{-1} \left( \mathbf{R}_0^T - \sqrt{-1} \mathbf{Z}_0(v, \eta) \right).$$

Define the  $3 \times 3$  matrices  $\mathbf{Q}'_0 = \mathbf{Q}'_0(\eta)$ ,  $\mathbf{R}'_0 = \mathbf{R}'_0(\eta)$ ,  $\mathbf{T}'_0$  by

$$\begin{aligned} \mathbf{Q}'_0 &= \left( \sum_{j,l=1}^2 C'_{ijkl}(0) \eta_j \eta_l \right)_{i \downarrow k \rightarrow 1,2,3} = \mathbf{Q}'_0{}^T, & \mathbf{R}'_0 &= \left( \sum_{j=1}^2 C'_{ijk3}(0) \eta_j \right)_{i \downarrow k \rightarrow 1,2,3}, \\ \mathbf{T}'_0 &= (C'_{i3k3}(0))_{i \downarrow k \rightarrow 1,2,3} = \mathbf{T}'_0{}^T, \end{aligned}$$

where superimposed primes ( $'$ ) on  $C_{ijkl}$  denote differentiation with respect to  $x_3$ .

**Theorem 2** *Hermitian matrix  $\mathbf{Z}_1 = \mathbf{Z}_1(v, \eta)$  is the unique solution to the linear system*

$$(\mathbf{K}_0^*)^2 \mathbf{Z}_1 - 2 \mathbf{K}_0^* \mathbf{Z}_1 \mathbf{K}_0 + \mathbf{Z}_1 (\mathbf{K}_0)^2 = \mathbf{Q}'_0 - \mathbf{R}'_0 \mathbf{K}_0 - \mathbf{K}_0^* (\mathbf{R}'_0)^T + \mathbf{K}_0^* \mathbf{T}'_0 \mathbf{K}_0, \quad (14)$$

where  $\mathbf{K}_0^*$  is the adjoint matrix of  $\mathbf{K}_0$ .

In what follows we assume that a half-space  $x_3 \leq 0$  is occupied by elastic materials whose elasticity tensor  $\mathbf{C} = \mathbf{C}(x_3) = (C_{ijkl})_{i,j,k,l=1,2,3}$  has an orthorhombic symmetry at each  $x_3$  and that the axes of the orthorhombic symmetry of the medium coincide with the 1-, 2-, and 3-axis of the Cartesian coordinate system. Then possibly non-zero components of  $\mathbf{C} = \mathbf{C}(x_3)$  are  $C_{1111}, C_{2222}, C_{3333}, C_{1122}, C_{1133}, C_{2233}, C_{2323}, C_{1313}$  and  $C_{1212}$  at each  $x_3$ . Under the setting above we consider Rayleigh waves which propagate along the surface of the half-space  $x_3 \leq 0$  in the direction of the 2-axis. Henceforth we set  $\eta = (0, 1, 0)$ .

It is well known [3] that  $v_0(\eta)$  in (13) is the unique solution to

$$\begin{aligned} &C_{3333}(0)C_{2323}(0)(C_{2222}(0) - V)V^2 \\ &- (C_{2323}(0) - V)(C_{3333}(0)(C_{2222}(0) - V) - C_{2233}^2(0))^2 = 0 \end{aligned} \quad (15)$$

in the subsonic range with  $V = \rho v_0(\eta)^2$ .<sup>†</sup>

For  $v_1(\eta)$  in (13) we have

**Corollary 3**

$$v_1(\eta) = - \left. \frac{x_{22} z_{33} + x_{33} z_{22} - 2z_{23} x_{23}}{\frac{\partial}{\partial v}(z_{22} z_{33} - z_{23}^2)} \right|_{v=v_0(\eta)}.$$

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<sup>†</sup>Equation (15) follows also from (11).

Here  $z_{22}$ ,  $z_{33}$  and  $\sqrt{-1} z_{23}$  are the (2,2), (3,3) and (2,3) components of  $\mathbf{Z}_0(v, \eta)$ , respectively, which are given by

$$\begin{aligned} z_{22} &= \sqrt{\frac{C_{2323}(0)}{C_{3333}(0)}} \frac{P_1}{P_1 + P_2} \sqrt{(P_1 + P_2)^2 - (C_{2233}(0) + C_{2323}(0))^2}, \\ z_{33} &= \sqrt{\frac{C_{3333}(0)}{C_{2323}(0)}} \frac{P_2}{P_1 + P_2} \sqrt{(P_1 + P_2)^2 - (C_{2233}(0) + C_{2323}(0))^2}, \\ z_{23} &= \frac{-1}{P_1 + P_2} (C_{2323}(0) P_1 - C_{2233}(0) P_2), \end{aligned}$$

with

$$P_1 = \sqrt{C_{3333}(0)(C_{2222}(0) - V)}, \quad P_2 = \sqrt{C_{2323}(0)(C_{2323}(0) - V)}, \quad V = \rho v^2$$

and  $x_{22}$ ,  $x_{33}$  and  $\sqrt{-1} x_{23}$  are the (2,2), (3,3) and (2,3) components of  $\mathbf{Z}_1(v, \eta)$ , respectively.

**Remarks** Writing down equation (14) explicitly, we see that  $x_{22}$ ,  $x_{33}$  and  $x_{23}$  are all real-valued and depend only on  $C_{2222}(0)$ ,  $C_{2233}(0)$ ,  $C_{3333}(0)$ ,  $C_{2323}(0)$  and the derivatives  $C'_{2222}(0)$ ,  $C'_{2233}(0)$ ,  $C'_{3333}(0)$ ,  $C'_{2323}(0)$ , from which we conclude that  $v_1(\eta)$  depends only on the boundary values of the four components of  $\mathbf{C}$  and their derivatives at the boundary. This corollary is an alternative expression of the result in [2].

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# $\pi_2$ -theory of operators

Shunsaku Nii\*

In this talk, a topological index theory which can be seen as  $\pi_2$ -theory of operators is introduced. This terminology is inspired by the one  $\pi_1$ -theory of operators by Sanson [6] referring to infinite dimensional Maslov index theory. This viewpoint begins by seeing the classical theory of Sturm-Liouville operators as  $\pi_1$ -theory of  $\mathbb{S}^1$ .

## 1 The theory of Sturm-Liouville operators: a $\pi_1$ -theory of $\mathbb{S}^1$

Consider the eigenvalue problem of a Sturm-Liouville operator:

$$-p'' + f(x)p = \lambda p, \quad \text{on } I = [-1, 1] \text{ or } \mathbb{R}$$

This equation is written as a system of first order equations:

$$\begin{cases} p' = q \\ q' = (f(x) - \lambda)p. \end{cases}$$

In the polar coordinate  $p = r \cos \theta$ ,  $q = r \sin \theta$ , this system becomes:

$$\begin{cases} r' = (1 - \lambda + f(x)) r \sin \theta \cos \theta \\ \theta' = 1 + (\lambda - f(x) - 1) \sin^2 \theta \end{cases}$$

Because the right hand side of the  $\theta$  equation is monotone in  $\lambda$ , the number of  $\theta(I)$  winds  $\mathbb{S}^1$  increase as  $\lambda$  does. Therefore for each eigenvalue  $\lambda$ ,  $\theta(I) \in \pi_1(\mathbb{S}^1)$  identifies the eigenfunction. That is, the eigenfunctions are ordered by the number of humps.

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## 2 Maslov index: a $\pi_1$ -theory of matrices

The first natural generalization of the above theory is what is called (Keller-)Maslov(-Arnol'd) index for the eigenvalue problem of a Schrödinger operator:

$$-p'' + M(x)p = \lambda p, \quad p \in \mathbb{R}^n, \quad \text{on } I = [-1, 1] \text{ or } \mathbb{R}$$

This system is equivalent to the following Hamiltonian system:

$$\begin{cases} p' = \frac{\partial H}{\partial q} \\ q' = -\frac{\partial H}{\partial p} \end{cases},$$

where the Hamiltonian is given by  $H(p, q) = \frac{1}{2} \{|q|^2 + {}^t p(\lambda I - M(x))p\}$ .

Because a Hamiltonian system preserves the symplectic structure, this system induces a flow on the Lagrangian Grassmannian manifold  $\Lambda(n) = Sp(n)(\mathbb{R}^n \times \{0\})$ , where  $Sp(n)$  is the symplectic group.

**Fact**  $\pi_1(\Lambda(n)) \cong \mathbb{Z}$

Therefore  $(p, q)(I) \in \pi_1(\Lambda(n))$  characterizes the eigenfunctions. ( $(p, q)(x)$  is not necessarily monotone.) This is what is called Maslov index.

## 3 Infinite dimensional Maslov index: a $\pi_1$ -theory of operators

There are several infinite dimensional generalization of Maslov index. One by Swanson [6] is among the earliest.

Let  $E = H \times H^*$  for a Hilbert space  $H$  and its dual  $H^*$ . Define a symplectic structure on  $E$  by  $\omega((e, \alpha), (f, \beta)) = \alpha \cdot f - \beta \cdot e$ . Then the Fredholm Lagrangian Grassmannian manifold  $\mathcal{F}\Lambda_H$  is defined by  $\mathcal{F}\Lambda_H = Sp_C(E)H$ , where  $Sp_C(E) := \{id + \text{compact} \mid \text{preserves } \omega\} \subset GL(E)$

**Fact**  $\pi_1(\mathcal{F}\Lambda_H) \cong \mathbb{Z}$

Swanson applied this fact for deformations of elliptic operators, and call his theory as  $\pi_1$ -theory of operators contrasting it to Fredholm index ( $\pi_0$ -theory of operators) which distinguishes connected components of operators.

Recently, Deng [2] reformulated this theory on  $E = H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$  for a star-shaped domain  $\Omega$  and applied to the boundary value problem of an elliptic operator.

## 4 The Stability index: a $\pi_2$ -theory of matrices

In spite of early development of the  $\pi_1$ -theory for selfadjoint operators, any analogous theory for non-selfadjoint operators has not appeared until recently. The obstacles were that the eigenvalues are not real and the systems are no longer Hamiltonian. The first step for this direction seems to be the Stability index theory by Alexander-Gardner-Jones [1, 5] explained below.

Consider the eigenvalue problem for a not-selfadjoint operator:

$$-p'' + M(x)p' + N(x)p = \lambda p, \quad p \in \mathbb{C}^n, \quad \text{on } I.$$

This system is equivalent to the following system on  $\mathbb{C}^{2n}$ :

$$\begin{cases} p' = q \\ q' = (N(x) - \lambda I)p + M(x)q \end{cases}.$$

This time, the system induces a flow on the complex Graßmannian manifold  $G_n(\mathbb{C}^{2n}) = GL(2n)(\mathbb{C}^n \times \{0\})$ . For a disc  $D \subset \mathbb{C}$ , this flow induces a map

$$\Phi: \mathbb{S}^2 \cong (D \times \partial I) \cup (\partial D \times I) \rightarrow G_n(\mathbb{C}^{2n})$$

**Fact**  $\pi_2(G_n(\mathbb{C}^{2n})) \cong \mathbb{Z}$

Then Alexander-Gardner-Jones proved sort of  $\pi_2$ -theory of matrices.

**Theorem (Alexander-Gardner-Jones [1], Gardner-Jones [5])**

$\Phi(\mathbb{S}^2) \in \pi_2(G_n(\mathbb{C}^{2n}))$  represents the number of eigenvalues in  $D$  including the multiplicity.

This theory is sometimes referred to as Alexander-Gardner-Jones bundle theory, as it is formulated by the terminology of line bundles and the Chern class.

## 5 The infinite dimensional Stability index: a $\pi_2$ -theory of operators

It is natural to think about infinite dimensional generalization of the Stability index from the viewpoints both in pure mathematics and in application. One such example is the following eigenvalue problem:

$$\begin{cases} u_{xx} + \Delta_y u + \beta(y)u_x + f(x, y)u = \lambda u, & (x, y) \in \mathbb{R} \times \Omega \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \mathbb{R} \times \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^m$  is a bounded domain. This equation can be written as an ordinary differential equation in  $x$ -variable on an appropriate Hilbert space  $H_\Omega$ .

Here is a difficulty:  $GL(H)$  is contractible for an infinite dimensional Hilbert space  $H$ . This means that a naive generalization of the Stability index becomes trivial and does not detect any information.

Fortunately, we can exploit compactness of the problem: Let  $GL_C(H) := \{ id + compact \mid invertible \} \subset GL(H)$  and fix a polarization  $H = H_- \oplus H_+$ , then the Fredholm Grassmannian manifold  $F(H_+)$  is the orbit of  $H_+$  under the action of  $GL_C(H)$  i.e.  $F(H_+) = GL_C(H)H_+$ .

Under this setting, the problem induces a system on  $F(H_+)$ .

**Remark** *In this case, the system does not generate a flow, as the problem is ill-posed.*

Then, for a disc  $D \subset \mathbb{C}$ , this system induces a map

$$\Phi: \mathbb{S}^2 \cong (D \times \partial I) \cup (\partial D \times I) \rightarrow F(H_+),$$

and we have the following theorem.

**Theorem (Deng-N. [3])**

$\Phi(\mathbb{S}^2) \in \pi_2(F(H_+))$  represents the number of eigenvalues in  $D$  including the multiplicity.

We also have a similar result for an elliptic operator posed on a bounded domain [4]

These results can be called  $\pi_2$ -theory of operators in the Swanson's expression.

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# Inverse boundary value problem for elliptic equation with complex coefficient

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Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^2$ . We consider the equation

$$\begin{cases} -\Delta u + Vu = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega. \end{cases} \quad (1)$$

We assume that  $V(x)$  is a complex valued  $L^p(\Omega)$  ( $p > 2$ ) function and that 0 is not a Dirichlet eigenvalue for the operator  $-\Delta + V$  on  $\Omega$ . Let  $\alpha = (p-2)/p$ . Then, for each  $f \in C^{1,\alpha}(\partial\Omega)$  there exists a unique solution  $u \in C^{1,\alpha}(\overline{\Omega})$  of (1) with the boundary value  $f$  on  $\partial\Omega$ . Here  $C^\alpha(\overline{\Omega})$  is a usual Hölder space and  $C^{1,\alpha}(\overline{\Omega})$  is the space of functions  $u \in C^1$  with  $\partial_j u \in C^\alpha$ .

The Dirichlet-to-Neumann map (DN map)  $\Lambda_V$  corresponding to  $V$  is defined by

$$\Lambda_V f := \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega},$$

where  $\nu$  is the outer unit normal on  $\partial\Omega$  and  $u$  is the solution of (1) with the Dirichlet data  $u = f$  on  $\partial\Omega$ .

**Inverse boundary value problem is:** determine  $V$  from the knowledge of  $\Lambda_V$ .

If  $V$  is uniquely determined by  $\Lambda_V$ , can we calculate  $V$  from the knowledge of  $\Lambda_V$ ? We call it the reconstruction problem.

In the multidimensional case  $\mathbb{R}^n$  ( $n \geq 3$ ), global uniqueness was proved by Sylvester and Uhlmann [18]. Nachman [17] gave a solution of the reconstruction problem.

In the two dimensional case, global uniqueness was resolved by Bukhgeim [3]. He also gave an inversion formula for smooth potentials in terms of boundary measurements with a special boundary condition.

In this talk, we shall discuss the reconstruction problem in the two dimensional case.

Let  $W^{m,p}(\Omega)$  be the usual Sobolev space of order  $m$  in  $L^p(\Omega)$ .

**Theorem 1.** *We assume that  $V \in W^{1,p}(\Omega)$  for some  $p > 2$ . Then there exists  $M = M(p, \Omega)$  such that if  $\|V\|_{W^{1,p}(\Omega)} \leq M$ , then there is a reconstruction scheme to identify  $V(x)$  for any  $x \in \overline{\Omega}$  from  $\Lambda_V$ .*

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# The variational problem for a certain action functional defined on closed curves

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Let  $\Gamma_0$  and  $\Gamma_1$  be planar closed curves and  $T$  be a positive constant. Let  $\gamma$  denote a planar closed curve and  $s$  be its arc length parameter. In this talk, we consider the following variational problem:

**Problem 1.** *Minimize*

$$(AF) \quad E(\gamma) := \int_0^T \oint_{\gamma(t)} \{ \kappa(t)^2 + v(t)^2 \} ds dt$$

over  $S = \{ \{ \gamma(t) \}_{t=0}^T \mid \gamma(0) = \Gamma_0, \gamma(T) = \Gamma_1 \}$ , where  $\kappa(t)$  and  $v(t)$  denote the curvature and the normal velocity of  $\gamma(t)$ , respectively.

We mention some related problems. Recently the action minimization problem that is formally associated to phase transformation in the stochastically perturbed Allen-Cahn equation is considered (e.g., [2], [3], [5], [6]). In particular, R. V. Kohn, F. Otto, M. G. Reznikoff, and E. Vanden-Eijnden ([2]) consider the problem for higher dimension. In [2] a space time functional

$$(1) \quad \int_0^T \int_{\Gamma(t)} (v_n + \kappa)^2 d\sigma dt$$

is appeared in the action functional and called a propagation cost, where  $\Gamma(t)$  is the interface at time  $t$ ,  $v_n$  is the normal velocity, and  $\kappa$  is the curvature (concerning on an action functional, see [1] for instance). It is possible to look on our problem as a kind of the variational problem for (1) in 2 dimensions.

First we consider Problem 1 in a radial symmetry class, i.e.,  $\gamma(t)$  is always a circle. Then, by an appropriate scaling argument, Problem 1 is reduced to the following variational problem:

**Problem 2.** *Minimize*

$$\mathcal{E}(R) = \int_0^1 \left\{ \frac{1}{R(t)^2} + R'(t)^2 \right\} R(t) \chi_{\{R>0\}} dt$$

under a condition  $R(0) = R_0$  and  $R(1) = R_1$ , where  $R_0$  and  $R_1$  are some positive constants.

Here,  $R(t)$  is an unknown scalar function which represents the radial of  $\gamma(t)$ , and  $\chi_{\{R>0\}}$  denotes a characteristic function. To begin with, making use of a certain replacement argument, we prove a positivity of minimizer of Problem 2 with data  $(R_0, R_1)$  satisfying  $R_0^2 + R_1^2 > 2$ . In this arguments, a mean curvature flow plays an important role. Since the positivity of minimizer allows us to apply a usual direct method and regularity arguments to Problem 2, we are able to prove the following:

**Theorem 1.** *Let an initial final data  $(R_0, R_1)$  satisfy the relation  $R_0^2 + R_1^2 > 2$ . Then, for each data  $(R_0, R_1)$ , there exists a unique minimizer of Problem 2 in the class  $C^2[0, 1]$ .*

And then, deriving the Euler-Lagrange equation and solving it, we classify the minimizer into four kinds for each initial final data  $(R_0, R_1)$ . In particular, there exists a minimizer  $H(t)$  governed by a mean curvature flow.

Next we focus on a non-radial symmetry case of Problem 1. In particular, we look for a non-radial symmetry critical point near  $H(t)$ . For this purpose, we parameterize  $\gamma$  as follows:

$$(2) \quad \gamma(\theta, t) := r(\theta, t)\omega(\theta),$$

where  $\theta \in \mathbb{R}/\mathbb{Z} := S^1$  and  $\omega(\theta) = (\cos \theta, \sin \theta)$ . By virtue of positivity of  $H(t)$ , the functional  $E(\gamma)$  is expressed in terms of  $r$  as follows:

$$(3) \quad E(\gamma(\theta, t)) = \int_0^1 \int_0^{2\pi} \left\{ r_t(\theta, t)^2 + \frac{(2r_\theta(\theta, t)^2 + r(\theta, t)^2 - r(\theta, t)r_{\theta\theta}(\theta, t))^2}{|\gamma_\theta|^6} \right\} |\gamma_\theta| d\theta dt,$$

where  $|\gamma_\theta| := \{r^2 + (r_\theta)^2\}^{1/2}$ . Moreover Euler-Lagrange equation for the functional (3) is written as

$$(4) \quad \begin{aligned} & -2(r_t |\gamma_\theta|)_t - \left( r_t^2 r_\theta |\gamma_\theta|^{-1} \right)_\theta + r_t^2 r |\gamma_\theta|^{-1} - 2 \left\{ (2r_\theta^2 + r^2 - rr_{\theta\theta}) r |\gamma_\theta|^{-5} \right\}_{\theta\theta} \\ & - \left\{ (2r_\theta^2 + r^2 - rr_{\theta\theta})(-2r_\theta^2 + 3r^2 + 5rr_{\theta\theta}) r_\theta |\gamma_\theta|^{-7} \right\}_\theta \\ & - (2r_\theta^2 + r^2 - rr_{\theta\theta})(6rr_\theta^2 + r^3 + 2r_\theta^2 r_{\theta\theta} - 3r^2 r_{\theta\theta}) |\gamma_\theta|^{-7} = 0. \end{aligned}$$

In what follows, let  $\mathcal{F}(r(\theta, t))$  denote the left hand side of (4), and  $S_I^1 = S^1 \times I$ , where  $I = (0, 1)$ . Finding a non-radial symmetry critical point near  $H$  is equivalent to looking for a non-trivial solution to the following initial final value problem:

$$(IFP) \quad \begin{cases} \mathcal{F}(H(t) + \hat{h}(\theta, t)) = 0 & \text{in } S_I^1, \\ \hat{h}(\theta, 0) = u_0(\theta), \quad \hat{h}(\theta, 1) = u_1(\theta). \end{cases}$$

In order to state our main result, we need some preparations. Let  $C^{(m+\alpha)}(\overline{S_I^1})$  denote the Banach space of  $\alpha$ -Hölder continuous functions in  $\overline{S_I^1}$ , together with all derivatives of the form  $\partial_t^p \partial_\theta^q$  for  $2p + q \leq m$ , where  $m \in \mathbb{N}$  and  $0 < \alpha < 1$ . We write its norm as  $|\cdot|_{S_I^1}^{(m+\alpha)}$ . (For the precise definition of this norm, see [4].) Whereas let  $|\cdot|_{S^1}^{m+\alpha}$  denote the norm of  $C^{m+\alpha}(S^1)$ . Let  $(W_2^{2m,m}(S_I^1), \|\cdot\|_{W_2^{2m,m}(S_I^1)})$  be the Banach space consisting of the elements of  $L^2(S_I^1)$  having distributed derivatives of the form  $\partial_t^p \partial_\theta^q$  for  $2p + q \leq 2m$ . Furthermore we denote  $W_2^{1,0}(S_I^1)$  the Banach space consisting of elements of  $L^2(S_I^1)$  having distributed derivative  $\partial_\theta$ . Using the spaces  $C^{(4+\alpha)}(\overline{S_I^1})$  and  $W_2^{1,0}(S_I^1)$ , we define a space  $\mathcal{M}$  consisting of the function such that

$$\|u\|_{\mathcal{M}} := |u|_{S_I^1}^{(4+\alpha)} + \|\partial_\theta^4 u\|_{W_2^{1,0}(S_I^1)} + \|\partial_t^2 u\|_{W_2^{1,0}(S_I^1)} < +\infty.$$

Let us consider an initial-final data  $u_0$  and  $u_1$  in the class  $C^{4+\alpha}(S^1) \cap H^5(S^1)$ , and set

$$\beta := \max \left\{ |u_0|_{S^1}^{4+\alpha}, |u_1|_{S^1}^{4+\alpha} \right\}, \quad \rho := \max \left\{ \|\partial_\theta^4 u_0\|_{H^1(S^1)}, \|\partial_\theta^4 u_1\|_{H^1(S^1)} \right\}.$$

Selecting a quadruplet  $(\varepsilon, \delta, \beta, \rho)$  of positive numbers appropriately, we shall construct a non trivial solution of (IFP) in the space  $\mathcal{N}_{\varepsilon, \delta}$  defined by

$$\begin{aligned} \mathcal{N}_{\varepsilon, \delta} = \{g \in \mathcal{M} \mid |g|_{S^1}^{(4+\alpha)} \leq \varepsilon, \max\{\|\partial_t^2 g\|_{W_2^{1,0}(S^1)}, \|\partial_\theta^4 g\|_{W_2^{1,0}(S^1)}\} \leq \delta, \\ g(\theta, 0) = u_0(\theta), g(\theta, 1) = u_1(\theta)\}. \end{aligned}$$

We are in a position to state our main result:

**Theorem 2.** *Fix a pair of sufficiently small positive numbers  $(\varepsilon, \delta)$  arbitrarily. For such pair  $(\varepsilon, \delta)$ , if we select a pair of positive numbers  $(\beta, \rho)$  sufficiently small, then the initial final value problem (IFP) has a unique solution  $\hat{h}(\theta, t)$  in the space  $\mathcal{N}_{\varepsilon, \delta}$ , where  $0 < \alpha \leq 1/2$ .*

We carry out the outline of proof of Theorem 2. To begin with, for each given  $v \in \mathcal{N}_{\varepsilon, \delta}$  and initial final data, we prove an existence of solution  $u \in \mathcal{N}_{\varepsilon, \delta}$  to the following problem:

$$\begin{cases} \mathcal{F}'[H](u) = -\mathcal{F}(v+H) + \mathcal{F}'[H](v) & \text{in } S^1, \\ u(\theta, 0) = u_0(\theta), \quad u(\theta, 1) = u_1(\theta). \end{cases}$$

Here  $\mathcal{F}'[H]$  is the linearized operator of  $\mathcal{F}$  at  $H$  written as follows:

$$\mathcal{F}'[H](\varphi) := \frac{d}{d\varepsilon} \mathcal{F}(H + \varepsilon\varphi) \Big|_{\varepsilon=0} = \frac{2}{H^3} \partial_\theta^4 \varphi + \frac{4}{H^3} \partial_\theta^2 \varphi - 2H \partial_t^2 \varphi - \frac{2}{H} \partial_t \varphi + \frac{4}{H^3} \varphi.$$

Next we define a map  $\Phi$  from  $\mathcal{N}_{\varepsilon, \delta}$  to itself by  $\Phi(v) = u$ , and show that the map  $\Phi$  is a contraction mapping. And then, the contraction mapping principle yields an existence of fixed point in  $\mathcal{N}_{\varepsilon, \delta}$ . The fixed point is nothing but a unique solution of initial-final value problem (IFP).

To do so, concerning the following initial final value problem for the linearized operator

$$(LIFP) \quad \begin{cases} \frac{1}{H^3} \partial_\theta^4 \phi + \frac{2}{H^3} \partial_\theta^2 \phi - H \partial_t^2 \phi - \frac{1}{H} \partial_t \phi + \frac{2}{H^3} \phi = f(\theta, t) & \text{in } S^1, \\ \phi(\theta, 0) \equiv 0, \quad \phi(\theta, 1) \equiv 0, \end{cases}$$

we need to prove the following: (i) the problem (LIFP) has a unique strong solution for each  $f \in L^2(S^1)$ ; (ii) the strong solution belongs to the class  $C^{(4+\alpha)}(\overline{S^1})$  for each  $f \in C^{(\alpha)}(\overline{S^1}) \cap W_2^{1,0}(S^1)$ .

To begin with, we mention the first question (i). Although the equation in (LIFP) looks like a kind of evolution equation, we are unable to apply well-known theories concerning evolution equations to our problem. For, the sign of main part of PDE in (LIFP), i.e.,  $-\partial_t^2 + \partial_\theta^4$ , is opposite to that of well-posed one. Moreover, since the problem is imposed not only an initial data but also a final data, the problem is not Cauchy problem. However, looking on the problem (LIFP) as a boundary value problem in  $S^1$  and using a method for elliptic equations, we can prove that (LIFP) has a unique strong solution for  $f \in L^2(S^1)$ . Indeed, we construct a weak solution  $\phi$  of (LIFP) in an appropriate Hilbert space  $W_2^{2,1}(S^1)$ . Here a key of our proof is to show a product

appeared in the definition of weak solution of (LIFP) becomes to the inner product of Hilbert space  $W_2^{2,1}(S_T^1)$ . And then, making use of usual regularity arguments, we proceed to prove that the weak solution  $\phi$  becomes to a strong solution of (LIFP).

We turn to the second question (ii). To solve the problem (IFP), we need Schauder type estimates of solution  $\phi$  of (LIFP) as we stated in (ii). However it is difficult to obtain such results by the standard regularity theory for elliptic equations, for the linearized operator is not elliptic. Hence, in order to prove the required regularity of  $\phi$ , we make use of a parabolic regularity theory. To do so, we first derive equivalent equations to that in (LIFP). For instance, one of the equivalent equation is expressed as follows:

$$(5) \quad (-\partial_t + H^{-2}\partial_\theta^2 + H^{-2})H(\partial_t + H^{-2}\partial_\theta^2 + H^{-2})\phi = f,$$

which is the linearized operator of the Euler-Lagrange equation for the functional

$$(6) \quad \int_I \oint_{\gamma(t)} (\kappa(t) - \nu(t))^2 ds dt.$$

With the aid of null Lagrangian, we are able to prove that the functional (6) is equivalent to (AF), i.e., the equation (5) is equivalent to that in (LIFP). Applying a well known regularity results on parabolic equations to the equation (5), we deduce the affirmative answer to the second question.

In this talk, we shall focus on the regularity arguments.

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# Entire solutions associated with front waves to reaction-diffusion equations

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## Abstract

We can observe propagation phenomena modeled by reaction-diffusion equations in various fields of materials science, biology and life science. Corresponding to a wave propagation, the equations allow a traveling wave that has a constant profile and a constant speed. Historically, starting from the pioneering work by [5, 15], there are enormous number of papers for the study of traveling wave solutions (for instance see [1], [2], [4], [7], [10], [11], [13], [12], [19], [18], [17], [20], [21], and references therein).

Here we consider a simple model equation, that is a scalar reaction-diffusion equation of one-space dimension

$$u_t = u_{xx} + f(u) \quad (x \in \mathbb{R}) \quad (1)$$

with the condition

$$f(0) = f(1) = 0, \quad f'(0) \neq 0, \quad f'(1) < 0. \quad (2)$$

The condition (2) tells that  $u = 0$  and  $u = 1$  are nondegenerate constant equilibria and that  $u = 1$  is asymptotically stable. In addition to (2), if  $f$  satisfies

$$f'(0) > 0, \quad f(u) \neq 0 \quad (u \neq 0, 1),$$

the equation (1) is called a monostable reaction-diffusion equation while if there is a number  $a \in (0, 1)$  such that

$$f'(0) < 0, \quad f(a) = 0, \quad f'(a) > 0, \quad f(u) \neq 0 \quad (u \neq 0, a, 1),$$

(1) is called a bistable reaction-diffusion equation. Typical examples of those cases are the Fisher-KPP equation with

$$f(u) = u(1 - u), \quad (3)$$

and the Nagumo equation with

$$f(u) = u(1 - u)(u - a), \quad 0 < a < 1, \quad (4)$$

respectively.

Putting  $u(x, t) = U(x + ct)$ ,  $z = x + ct$ , a monotone traveling wave (called front wave) is obtained by solving the equation

$$\begin{cases} U_{zz} - cU_z + f(U) = 0, & U(z) > 0 \quad (z \in \mathbb{R}), \\ U(-\infty) = 0, \quad U(\infty) = 0. \end{cases} \quad (5)$$

It is known that a monostable equation allows a family of traveling waves with speeds  $c \geq c_{min} > 0$ ; for instance  $c_{min} = 2$  if  $f(u) = u(1 - u)$ . On the other hand a bistable reaction-diffusion equation has a unique traveling wave up to translation. For the Nagumo equation the traveling wave is given explicitly as

$$\begin{aligned} U^{01}(x + ct) &:= \frac{\exp \left[ x/\sqrt{2} + (1/2 - a)t \right]}{1 + \exp \left[ x/\sqrt{2} + (1/2 - a)t \right]} \\ &= \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{x + ct}{2\sqrt{2}} \right), \quad c = \sqrt{2} \left( \frac{1}{2} - a \right). \end{aligned}$$

We note that the reflected one  $U^{10} := U^{01}(-x + ct)$  is also a traveling wave with a monotone decreasing profile.

As for the Nagumo equation there are other exact traveling wave solutions, which are given as

$$\begin{aligned} U^{0a}(x + c_1t) &:= \frac{a \exp \left[ ax/\sqrt{2} - (a - a^2/2)t \right]}{1 + \exp \left[ ax/\sqrt{2} - (a - a^2/2)t \right]} \\ &= \frac{a}{2} + \frac{a}{2} \tanh \left[ \frac{a(x + c_1t)}{2\sqrt{2}} \right], \quad c_1 = -\frac{2 - a}{\sqrt{2}}, \end{aligned}$$

and

$$\begin{aligned} U^{a1}(x + c_2t) &:= \frac{a + \exp \left[ (1 - a)x/\sqrt{2} + (1 - a^2)t/2 \right]}{1 + \exp \left[ (1 - a)x/\sqrt{2} + (1 - a^2)t/2 \right]} \\ &= \frac{a + 1}{2} + \frac{1 - a}{2} \tanh \left[ \frac{(1 - a)(x + c_2t)}{2\sqrt{2}} \right], \quad c_2 = \frac{1 + a}{\sqrt{2}}. \end{aligned}$$

We notice that the former one is a traveling wave connecting  $u = 0$  to  $u = a$  while the latter one connects  $u = a$  to  $u = 1$ . More interesting thing is that the Nagumo equation has the following exact solution:

$$u(x, t) = \frac{\exp \left[ x/\sqrt{2} + (1/2 - a)t \right] + a \exp \left[ ax/\sqrt{2} - (a - a^2/2)t \right]}{1 + \exp \left[ x/\sqrt{2} + (1/2 - a)t \right] + \exp \left[ ax/\sqrt{2} - (a - a^2/2)t \right]} \quad (6)$$

([14]). This solution is not a traveling wave with a constant profile. In fact it behaves as two traveling waves  $U^{0a}$  and  $U^{a1}$  propagate from the left axis and right axis respectively until they merge. Then the solution behaves like a single wave  $U^{01}$ .

We call an entire solution to (1) if it is a classical solution defined for all  $x$  and  $t$ . Although an equilibrium solution and a traveling wave solution are examples of entire solutions, the above solution (6) is a different example of an entire solution from those. The solution (6) suggests us an interesting problem how we can show the existence of an entire solution which behaves as two traveling waves for  $t \ll 0$ .

The aim of our talk is to introduce the existence theorem for some entire solutions [16]. Applying the theorem, we obtain the similar solution to (6) for a general cubic  $f$ . We can also obtain a different type of entire solution than (6).

Finally we note that there is an entire solution which behaves as two fronts  $U^{01}$  and  $U^{10}$  propagating from the both sides of  $x$ -axis and annihilating eventually. This kind of entire solutions are extensively studied in [3], [6], [8], [9], [22].

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# Collapse of Patterns in Some Reaction-Diffusion Systems

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## 1. Introduction.

In an attempt to explain the mechanism of pattern formation in early stages of developmental processes of life, A. M. Turing found the notion of “diffusion-driven instability” which says that when two chemicals with different diffusion rates interact and diffuse, the spatially homogeneous state may become unstable, as a result spatially nontrivial structure can be formed autonomously. Gierer and Meinhardt developed Turing’s idea and devised the following reaction-diffusion system to simulate the transplantation experiment on *hydra*:

$$(GM) \quad \begin{cases} \frac{\partial u}{\partial t} = \varepsilon^2 \Delta u - u + \rho_a \frac{u^p}{v^q(1 + \kappa u^p)} + \sigma_a & (x \in \Omega, t > 0), \\ \tau \frac{\partial v}{\partial t} = D \Delta v - v + \rho_h \frac{u^r}{v^s} + \sigma_h & (x \in \Omega, t > 0), \\ \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = \frac{\partial v}{\partial \nu} \Big|_{\partial \Omega} = 0 & (t > 0), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & (x \in \Omega). \end{cases}$$

Here,  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$  and  $\nu$  denotes the outer unit normal to  $\partial\Omega$ .  $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$  is the Laplace operator in  $\mathbb{R}^n$ . The coefficients  $\varepsilon$ ,  $\tau$ ,  $D$  are positive constants, whereas  $\kappa$  is a nonnegative constant. The basic production terms  $\sigma_a = \sigma_a(x)$ ,  $\sigma_h = \sigma_h(x)$  are nonnegative, and the interaction coefficients  $\rho_a = \rho_a(x)$ ,  $\rho_h = \rho_h(x)$  are positive over  $\bar{\Omega}$ . The initial data  $u_0(x)$ ,  $v_0(x)$  are positive on  $\bar{\Omega}$ .

The exponents  $p$ ,  $q$ ,  $r$ ,  $s$  are assumed to satisfy

$$(A) \quad p > 1, \quad q > 0, \quad r > 0, \quad s \geq 0, \quad \frac{p-1}{q} < \frac{r}{s+1}.$$

The unknowns  $u = u(x, t)$  and  $v = v(x, t)$  denote the concentrations at point  $x$  and time  $t$  of chemicals called an *activator* and an *inhibitor*, respectively. It is postulated that a change in cells occurs at the place where the activator concentration is high.

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### 1. Existence of Solutions of the Initial-Boundary Value Problem

To begin with, let us summarize the known results on the existence of solutions of the initial-boundary value problem (GM).

**Theorem I.** *In addition to the assumption (A), assume that the inequality*

$$(1.1) \quad p - 1 < r$$

*holds. Then the initial-boundary value problem (GM) has a unique solution for all  $t > 0$  and the following statements hold:*

- (i) *If  $\max_{x \in \overline{\Omega}} \sigma_a(x) > 0$ , then there exist positive constants  $m_a, M_a, m_h, M_h$  independent of the initial value  $(u_0(x), v_0(x))$  such that*

$$\begin{aligned} m_a &\leq \liminf_{t \rightarrow +\infty} \min_{x \in \overline{\Omega}} u(x, t) \leq \limsup_{t \rightarrow +\infty} \max_{x \in \overline{\Omega}} u(x, t) \leq M_a, \\ m_h &\leq \liminf_{t \rightarrow +\infty} \min_{x \in \overline{\Omega}} v(x, t) \leq \limsup_{t \rightarrow +\infty} \max_{x \in \overline{\Omega}} v(x, t) \leq M_h. \end{aligned}$$

- (ii) *If  $\sigma_a(x) \equiv 0$  and  $\max_{x \in \overline{\Omega}} \sigma_h(x) > 0$ , then there exist positive constants  $M_a, m_h, M_h$  independent of the initial value such that*

$$\begin{aligned} e^{-t} \min_{x \in \overline{\Omega}} u_0(x) &\leq \min_{x \in \overline{\Omega}} u(x, t) \text{ for all } t \geq 0, \quad \text{and} \quad \limsup_{t \rightarrow +\infty} \max_{x \in \overline{\Omega}} u(x, t) \leq M_a, \\ m_h &\leq \liminf_{t \rightarrow +\infty} \min_{x \in \overline{\Omega}} v(x, t) \leq \limsup_{t \rightarrow +\infty} \max_{x \in \overline{\Omega}} v(x, t) \leq M_h. \end{aligned}$$

- (iii) *If  $\sigma_a(x) \equiv 0$  and  $\sigma_h(x) \equiv 0$ , then there exist positive constants  $\lambda, \mu$  depending only on  $p, q, r, s, \tau$  and a positive constant  $C$  depending on the initial value such that*

$$e^{-t} \min_{x \in \overline{\Omega}} u_0(x) \leq u(x, t) \leq Ce^{\lambda t}, \quad e^{-t/\tau} \min_{x \in \overline{\Omega}} v_0(x) \leq v(x, t) \leq Ce^{\mu t}$$

*hold for all  $t > 0, x \in \overline{\Omega}$ .*

Results on the existence of global solutions appeared more than twenty years ago; see, e.g., [R], [MT]. In particular, [MT] proved the first assertion of Theorem I under the condition  $(p - 1)/r < N/(N + 2)$ . It was [LCQ] that proved Theorem I (i), while (ii) and (iii) were obtained recently by [J], [S], [ST].

On the other hand, in the case of  $p - 1 > r$  we have the following result.

**Proposition II.** *Assume that  $\kappa = 0$ , and that  $\rho_a(x)$ ,  $\rho_h(x)$ ,  $\sigma_a(x)$ ,  $\sigma_h(x)$  are all constants. If*

$$(1.2) \quad p - 1 > r$$

*then (GM) has solutions which blow up in finite time ([LCQ], [NST]).*

Obviously, for the systematic study of global behavior of solutions of (GM), it is important to know the behavior of solutions of the following kinetic system:

$$(K) \quad \begin{cases} \frac{du}{dt} = -u + \frac{u^p}{v^q} + \sigma_a, \\ \tau \frac{dv}{dt} = -v + \frac{u^r}{v^s} + \sigma_h. \end{cases}$$

Here we assume that  $\sigma_a, \sigma_h$  are both nonnegative constants. In this aspect, [NST] classified all the behavior of solution orbits in the case of  $\sigma_a = 0$  and  $\sigma_h = 0$ . The case  $\sigma_a > 0$  is treated in an on-going project [NS].

## 2. Collapse of Patterns

In some numerical simulations, it is observed that a solution starting from an almost uniform initial value develops localization in the activator concentration for a while, but it oscillates and eventually converges uniformly to the trivial state  $u \equiv 0$ . We call this kind of phenomenon the *collapse of patterns*. In this article we would like to understand the mechanism behind the collapse of patterns and to know when it occurs.

We consider the following reaction-diffusion system which generalizes slightly the Gierer-Meinhardt system:

$$(RD) \quad \begin{cases} \frac{\partial u}{\partial t} = \varepsilon^2 \Delta u - u + f(x, u, v) + \sigma_a, \\ \tau \frac{\partial v}{\partial t} = D \Delta v - v + g(x, u, v) + \sigma_h, \\ \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = \frac{\partial v}{\partial \nu} \Big|_{\partial \Omega} = 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x). \end{cases}$$

Here,  $f(x, u, v)$  and  $g(x, u, v)$  are continuous functions in  $x \in \bar{\Omega}$ ,  $0 \leq u < +\infty$ ,  $0 < v < +\infty$ , and locally Lipschitz continuous with respect to  $u$  and  $v$  (uniformly in  $x$ ); moreover, they satisfy the inequalities

$$(2.1) \quad 0 \leq f(x, u, v) \leq C_1 \frac{u^p}{v^q}, \quad 0 \leq g(x, u, v) \leq C_2 \frac{u^r}{v^s} \quad (x \in \bar{\Omega}, u > 0, v > 0),$$

where  $C_1, C_2$  are positive constants independent of  $(x, u, v)$ .

**Theorem 2.1.** *Assume that  $\sigma_a(x) \equiv 0$ . If*

$$(2.2) \quad \tau > \frac{q}{p-1}$$

and

$$(2.3) \quad \left( \min_{x \in \bar{\Omega}} v_0(x) \right)^q \leq \frac{C_1(p-1)}{p-1-\frac{q}{\tau}} \left( \max_{x \in \bar{\Omega}} u_0(x) \right)^{p-1},$$

then the solution  $(u(x, t), v(x, t))$  of the initial-boundary value problem (RD) satisfies

$$0 < \max_{x \in \bar{\Omega}} u(x, t) \leq C e^{-t}, \quad \max_{x \in \bar{\Omega}} |v(x, t) - H_0(x)| \leq C e^{-t/\tau},$$

in which  $C$  is a positive constant depending on  $(u_0(x), v_0(x))$ , and  $H_0$  is the solution of the boundary value problem

$$(2.4) \quad D\Delta H_0 - H_0 + \sigma_h(x) = 0 \quad (x \in \Omega), \quad \frac{\partial H_0}{\partial \nu} = 0 \quad (x \in \partial\Omega).$$

**Remark.** In addition to (GM), the activator-inhibitor system proposed by MacWilliams [M]

$$(2.5) \quad f(x, u, v) = u^p / (u^p + v^q), \quad g(x, u, v) = \alpha u^r / (u^r + \beta)$$

also satisfies the condition (2.1). Here,  $\alpha$  and  $\beta$  are positive constants. MacWilliams used this model to simulate the head-regeneration experiment on hydra.

To prove the theorem we follow the approach due to [WL] and make use of the following two lemmas:

**Lemma 2.2.**  $v(x, t) \geq \min_{x \in \bar{\Omega}} v_0(x) e^{-t/\tau}$ .

**Lemma 2.3.** Let  $w(t) = \min_{x \in \bar{\Omega}} v_0(x) e^{-t/\tau}$ , and let  $U(t)$  be the solution of the initial value problem

$$\frac{dU}{dt} = -U + C_1 \frac{U^p}{w(t)^q} \quad (t > 0), \quad U(0) = \max_{x \in \bar{\Omega}} u_0(x).$$

Then  $u(x, t) \leq U(t)$  for all  $x \in \bar{\Omega}$  and  $t \geq 0$  in the maximal existence interval of  $U(t)$ .

It is known ([S]) that Theorem I holds for  $g(x, u, v)$  satisfying

$$(2.6) \quad c_2 \frac{u^r}{v^s} \leq g(x, u, v) \leq C_2 \frac{u^r}{v^s} \quad (x \in \bar{\Omega}, u \geq 0, v > 0).$$

Thus we consider the behavior of solutions under the assumption that (2.6) and (1.2) are satisfied. Let  $\sigma_a(x) \equiv 0$ . First we consider the case  $\max_{x \in \bar{\Omega}} \sigma_h(x) > 0$ . Then by the proof of Theorem I (ii) ([S], [ST]) we see that for any  $\eta$  satisfying  $0 < \eta < 1$  there exists a positive number  $\delta$  such that

$$v(x, t) \geq \eta \left( \min_{x \in \bar{\Omega}} v_0(x) + \delta \right) \quad (x \in \bar{\Omega}, t > 0).$$

Thus putting

$$\gamma = \left[ \eta \left( \min_{x \in \bar{\Omega}} v_0(x) + \delta \right) \right]^{-q}$$

and letting  $U(t)$  be the solution of the initial value problem

$$\frac{dU}{dt} = -U + C_1 \gamma U^p, \quad U(0) = \max_{x \in \bar{\Omega}} u_0(x),$$

we obtain that if  $C_1 \gamma U(0)^{p-1} < 1$  then  $U(t)$  is monotone decreasing and satisfies the inequality

$$U(t) \leq C e^{-t}$$

for all  $t > 0$ . Here,  $C$  is a positive constant depending on  $C_1, \gamma, U(0)$ . From this follows that

$$\max_{x \in \bar{\Omega}} |v(x, t) - H_0(x)| \leq C e^{-t/\tau}$$

for all  $t > 0$ . Condition  $C_1 \gamma U(0)^{p-1} < 1$  is equivalent to

$$(2.7) \quad \left( \max_{x \in \bar{\Omega}} u_0(x) \right)^{p-1} < \frac{\eta^q}{C_1} \left( \min_{x \in \bar{\Omega}} v_0(x) + \delta \right)^q.$$

It is to be noted that (2.7) does not contain  $\tau$ , whereas (2.2) does.

This difference comes from the following fact.

**Proposition 2.2.** *Let  $f(x, u, v), g(x, u, v)$  be differentiable with respect to  $(u, v)$  in  $0 \leq u < +\infty, 0 < v < +\infty$ , and  $\partial f/\partial u, \partial f/\partial v, \partial g/\partial u, \partial g/\partial v$  are continuous in  $(x, u, v)$ . Assume that  $\sigma_a(x) \equiv 0, \max_{x \in \bar{\Omega}} \sigma_h(x) > 0$ . Then the stationary solution  $(u(x), v(x)) = (0, H_0(x))$  is asymptotically stable. Here the initial values  $u_0(x), v_0(x)$  are assumed to be positive.*

**Proof.** Since  $f_u(x, 0, H_0) = 0, f_v(x, 0, H_0) = 0$  and  $g_u(x, 0, H_0) = 0, g_v(x, 0, H_0) = 0$ , the linearized operator around the stationary solution  $(0, H_0(x))$  becomes

$$(2.8) \quad \mathcal{L} = \begin{pmatrix} \varepsilon^2 \Delta - 1 & 0 \\ g_u(x, 0, H_0)/\tau & (D\Delta - 1)/\tau \end{pmatrix}$$

The assertion is verified by showing that all the eigenvalues of  $\mathcal{L}$  have negative real part. q.e.d.

### 3. Concluding Remarks

From a view point of the possibility of collapse, the results may be summarized as in the table below:

Basic production terms	Collapse
$\sigma_a(x) \not\equiv 0$	never occurs.
$\sigma_a(x) \equiv 0, \sigma_h(x) \not\equiv 0$	occurs (for any $\tau > 0$ if $p - 1 < r$ ).
$\sigma_a(x) \equiv 0, \sigma_h(x) \equiv 0$	occurs for $\tau > q/(p - 1)$ .

Therefore, the case  $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$  may be viewed as a “regular” perturbation from the case  $\sigma_a(x) \equiv 0$  and  $\sigma_h(x) \not\equiv 0$ , but it is a “singular limit” as  $\max_{x \in \bar{\Omega}} \sigma_a(x) \downarrow 0$  of the case  $\sigma_a(x) \not\equiv 0$  and  $\sigma_h(x) \equiv 0$ .

**Remark 1.** For (GM) with  $\rho_a, \rho_h$  being constants, the condition (2.3) on the initial value of Theorem 2.1 may be relaxed in the case  $\varepsilon^2 = D/\tau$  and  $(p - 1)/q \geq 1$ .

**Remark 2.** It was Professor Niro Yangihara who, more than thirty years ago, found a solution of (GM) such that  $u(x, t) \rightarrow 0, v(x, t) \rightarrow 0$  as  $t \rightarrow +\infty$  in the case where both of  $\rho_a, \rho_h$  are constants,  $\sigma_a(x) \equiv 0, \sigma_h(x) \equiv 0, \kappa = 0$ , and  $(p, q, r, s) = (2, 1, 2, 0)$ .

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