ON DIFFERENTIAL GEOMETRY OF SURFACES IN ANTI DE SITTER 3-SPACE

by

Liang Chen

School of Mathematics and Statistics
Northeast Normal University
Changchun 130024, P. R. China

Abstract

We investigate the differential geometry of surfaces in Anti de Sitter 3-space as an application of the theory of singularities.

We first show six *Legendrian dualities* between pseudo-spheres in semi-Euclidean 4-space with index 2 which are basic tools for the study of extrinsic differential geometry of submanifolds in these pseudo-spheres.

Secondly, we apply the Legendrian dualities to investigate the geometric properties of non-degenerate surfaces in Anti de Sittter 3-space. We study the spacelike surfaces in Anti de Sitter 3-space as the first step of this research. We define the timelike Anti de Sitter Gauss images and timelike Anti de Sitter height functions on spacelike surfaces and investigate the geometric meanings of singularities of these mappings. We consider the contact of spacelike surfaces with models (so-called AdS-great-hyperboloids) as an application of Legendrian singularity theory.

Thirdly, we also use the Legendrian dualities to study the geometric properties of timelike surfaces in Anti de Sitter 3-space. We define two mappings associated to a timelike surface which are called *Anti de Sitter nullcone Gauss image* and *Anti de Sitter torus Gauss map.* We can prove that the nullcone Gauss image is a Legendrian map and the classification of its generic singularities is given. We investigate the relation between the Anti de Sitter nullcone Gauss image and the Anti de Sitter torus Gauss mapping. We prove that the Anti de Sitter torus Gauss mapping is a Lagrangian mapping, and that the Legendrian lift of the Anti de Sitter nullcone Gauss image is a covering of the Lagrangian lift of the Anti de Sitter torus Gauss mapping. We also define a family of functions named *Anti de Sitter null height function* on the timelike surface. We use this family of functions as a basic tool to investigate the geometric meanings of singularities of the Anti de Sitter nullcone Gauss image and the Anti de Sitter torus Gauss map.

At last, we study the geometric properties of degenerate surfaces, which are called *AdS null surfaces*, in Anti de Sitter 3-space from a contact viewpoint. These surfaces are associated to spacelike curves in Anti de Sitter 3-space. The geometry of these spacelike curves determines the behavior of the corresponding AdS null surfaces. We define a map which is called the *torus Gauss image.* It appears to be associated to the contacts of the spaclike curve with some model in Anti de Sitter 3-space. We also define two families of functions and use them to investigate the singularities of AdS null surfaces and torus Gauss images as applications of singularity theory of functions.

Keywords: Anti de Sitter 3-space; Legendrian dualities; Legendrian singularities; spacelike surface; timelike surface; AdS-null surface; spacelike curve; versal unfolding.

CONTANTS

Introduction

In this paper, we investigate the local differential geometric properties of surfaces in Anti de Sitter 3-space from the viewpoint of singularity theory. It is well known that Minkowski space is a flat Lorentzian space form and de Sitter space is the Lorentzian space form with positive curvature. There are several articles for the study of submanifolds in these two Lorentzian space forms [14, 16, 18, 19, 20, 21, 22, 23, 12]. The Lorentzian space form with the negative curvature is called Anti de Sitter space which is a very important subject in Physics (the theory of general relativity, the string theory and the brane world scenario etc [27, 40, 44]). Singularity theory tools, as illustrated by several papers which appeared so far ([2, 5, 6, 7, 8, 13, 14, 15, 25, 30, 31, 32, 33, 35, 38, 39, 41]), have proven to be useful in the description, of geometrical properties of submanifolds immersed in different ambient spaces, from both the local and global viewpoint. The natural connection between Geometry and Singularities relies on the basic fact that the contacts of a submanifold with the models (invariant under the action of a suitable transformation group) of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions, or equivalently, of their associated Lagrangian and/or Legendrian maps ([1, 34, 36]). However, there are not much results on submanifolds immersed in Anti de Sitter space, in particular from the view point of singularity theory. We remark that the causalities of Anti de Sitter space and de Sitter space are quite different.

On the other hand, A theorem of Legendrian dualities for pseudo-spheres in Minkowski space has been shown by Izumiya[17] which is now a fundamental tool for the study of extrinsic differential geometry on submanifolds in these pseudo-spheres from the view point of Singularity theory (cf., [17, 21, 24]). In the first part of this paper we consider similar Legedrian dualities between pseudo-spheres in semi-Euclidean 4-space with index 2. The main results (cf., Theorems 1.1 and 1.2) are simple analogies to the previous results in [17, 23]. However, there are some new applications and information. We can apply the Legendrian duality theorem to Anti de Sitter space and obtain some new geometric properties of submanifolds [10, 11]. For pseudo-spheres in the general semi-Euclidean space, we have shown the similar results on Legendrian dualities in [9]

In the second part of this paper, we study the differential geometry of non-degenerate surfaces in Anti de Sitter 3-space. As the first step of this research, we investigate the spacelike surfaces in Anti de Sitter 3-space from the view point of the theory of Legendrian singularities. As been well known, hypersurfaces in hyperbolic space have been studied in [15]. The basic notions and tools for the study of the differential geometry of hypersurfaces in hyperbolic space have been established. Especially, the hyperbolic Gauss indicatrix of a hypersurface in hyperbolic space has been explicitly described and the contact of hypersurfaces with model hypersurfaces has been systematically studied as an application of singulary theory to the hyperbolic Gauss indicatrix. Our aim in this part is to develop the analogous study for spacelike surfaces in Anti de Sitter 3-space. In §2.1 we develop local differential geometry of spacelike surfaces in Anti de Sitter 3-space and introduce the notion of timelike Anti de Sitter Gauss images of spacelike surfaces in Anti de Sitter 3-space. Corresponding to this notion we define the Anti de Sitter Gauss Kronecker (briefly, AdS-G-K) curvature and consider the geometric meaning of this curvature. One of our conclusions asserts that the AdS-G-K curvature describes the contact of spacelike surfaces with models (i.e., AdS-great-hyperboloids). We introduce the notion of timelike height functions in §2.2, named AdS-height function, which is useful to show that the TAdS-Gauss image has a singular point if and only if the AdS-G-K curvature vanishes at such point. In §2.3, we apply mainly the theory of Legendrian singularities for the study of the contact of spacelike surfaces with AdS-great-hyperboloids. In §2.4 we give the generic classification of singularities of TAdS-Gauss images. We introduce the notion of the AdS-Monge form of spacelike surfaces in Anti de Sitter 3-space in §2.5. As an application of this notion we give two examples.

In Part III, we mainly investigate the local differential geometric properties of timelike surfaces as an application of Legendrian singularity theory. We construct a basic framework for the study of timelike surfaces in Anti de Sitter 3-space here. As it was to be expected, the situation presents certain peculiarities when compared with the Minkowski case and the de Sitter case. For instance, in our case it is always possible to choose two lightlike normal directions along the timelike surface in the frame of its normal bundle. This is similar to the de Sitter case, but the normalized image is located in the Lorentzian torus T_1^2. For the de Sitter case, the normalized image of the lightlike normal is located in the spacelike sphere S_+^2. Moreover, there are no closed timelike surfaces in de Sitter space but there are such surfaces in Anti de Sitter space. In §3.1 We define the Anti de Sitter nullcone Gauss image (briefly, AdS-nullcone Gauss image) and Anti de Sitter torus Gauss map (briefly, AdS-torus Gauss map). We will find the AdS-nullcone Gauss image is more computable than the AdS-torus Gauss map. We also define the Anti de Sitter null Gauss-Kronecker curvature and Anti de Sitter torus Gauss-Kronecker curvature. We investigate their relations. We can prove that Anti de Sitter torus Gauss-Kronecker curvature is not a Lorentz invariant but it is an $SO(2) \times SO(2)$-invariant. Moreover, these two Gauss-Kronecker curvatures have the same zero points set. In §3.2 We introduce the notion of height functions on timelike surfaces, named AdS-null height function, which is useful to show that the AdS-nullcone Gauss image has a singular point if and only if the Anti de Sitter null Gauss-Kronecker curvature vanished at such point. We also apply mainly the Legendrian singularity theory to interpret the AdS-nullcone Gauss image as a Legendrian map. In §3.3 we define a surface, named Anti de Sitter torus cylindrical pedal, as a tool to study the relationship between the AdS-nullcone Gauss image and the AdS-torus Gauss map. We also study the contact of timelike surfaces with some model surfaces (i.e., AdS-horospheres) in §3.4. In §3.5 we give a generic classification of singularities of AdS-nullcone Gauss image and AdS-torus Gauss map. In the last section, §3.6, we also introduce the notion of the AdS-null Monge form of a timelike surface in Anti de Sitter 3-space and as an application of this notion we give two examples.

In the last part of this paper, we investigate the geometric properties of the degenerate surface which is called AdS null surface. This surface is a ruled surface along a spacelike curve in Anti de Sitter 3-space. All of the tangent planes of this ruled surface at regular points are degenerate plane. The geometry of these curves determines the behavior of the corresponding null surfaces. In §4.1 we first introduce the local differential geometry of spacelike curves in Anti de Sitter 3-space and then we use the spacelike curves to construct the AdS-null surfaces. We also define the torus Gauss image along a spacelike curve, which appears to be associated to the contacts of the curve with some model in Anti de Sitter 3-space. In §4.2 We define the torus height function and AdS distance-squared function on a spacelike curve. We apply the versal unfolding theory of functions to them and take advantage of these two families of functions to investigate the geometric properties of AdS null surfaces and torus Gauss image in Anti de Sitter 3-space. The main results in this part are Theorem 4.1.1 and Theorem 4.1.2. These theorems give the classifications of singularities of AdS null surfaces for the generic spacelike curves and the depiction of geometric meanings of these singularities. Moreover we introduce a new invariant $\sigma(s)$ for the spacelike curves. We can use it to characterize the contact of spacelike curves with some models (i.e., AdS nullcone) in Anti de Sitter 3-space. We also investigate the geometric meanings of these invariants in §4.3. In §4.4 we give the proof of Theorem 4.1.2.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

Part I. Basic notions and legendrian duality theorems

In this part we prove a useful tool named Legendrian duality theorems for studying the non-degenerate surfaces in Anti de Sitter 3-space.

First, we prepare basic notions on semi-Euclidean 4-space with index 2. For details of Lorentzian geometry, see [37].

Let $\mathbb{R}^4 = \{(x_1, \cdots, x_4) | x_i \in \mathbb{R}\ (i = 1, \cdots, 4)\ \}$ be a 4-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, \cdots, x_4)$ and $\boldsymbol{y} = (y_1, \cdots, y_4)$ in $\mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle, \rangle)$ a *smei-Euclidean 4-space with index 2* and write $\mathbb{R}^4_2$ instead of $(\mathbb{R}^4, \langle, \rangle)$.

We say that a non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^4_2$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^4_2$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We denote the signature of a vector $\boldsymbol{x}$ by

$$\operatorname{sign}(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \textit{ is spacelike} \\ 0 & \boldsymbol{x} \textit{ is null} \\ -1 & \boldsymbol{x} \textit{ is timelike} \end{cases}$$

For a vector $\boldsymbol{N} \in \mathbb{R}^4_2$ and a real number c, we define the *hyperplane with pseudo-normal* $\boldsymbol{N}$ by

$$HP(\boldsymbol{N}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_2 | \langle \boldsymbol{x}, \boldsymbol{N} \rangle = c\}.$$

We call $HP(\boldsymbol{N}, c)$ a *Lorentz hyperplane*, a *semi-Euclidean hyperplane of index 2* or a *null hyperplane* if $\boldsymbol{N}$ is *timelike, spacelike or null* respectively.

We now define *Anti de Sitter 3-space* (briefly, *AdS 3-space*) by

$$H^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

a *unit pseudo 3-sphere with index 2* by

$$S^3_2 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},$$

and a *closed nullcone* by

$$\Lambda^3 = \{\boldsymbol{x} \in \mathbb{R}^4_2 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We also define the *Lorentz torus* by

$$T^2_1 = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \Lambda^3 | x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\}.$$

If non-zero vector $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \Lambda^3$, we have

$$\tilde{\boldsymbol{x}} = \pm \frac{1}{\sqrt{x_1^2 + x_2^2}} (x_1, x_2, x_3, x_4) = \pm \frac{1}{\sqrt{x_1^2 + x_2^2}} \boldsymbol{x} \in T^2_1.$$

For any $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \in \mathbb{R}^4_2$, we define a vector $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ by

$$\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical basis of $\mathbb{R}^4_2$ and $\boldsymbol{X}_i = (x_i^1, x_i^2, x_i^3, x_i^4)$. We can easily check that $\langle \boldsymbol{X}, \boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 \rangle = \det(\boldsymbol{X}, \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3)$, so that $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ is pseudo-orthogonal to any $\boldsymbol{X}_i$ (for $i = 1, 2, 3$).

On the other hand, we now give a brief review on contact manifolds and Legendrian submanifolds. For some detailed results on contact geometry, please refer to [1, 3]. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α is a *contact form*. Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds

(N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if $\dim L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. For any $p \in E$, it is known that there is a local coordinate system $(x_1, \ldots, x_m, p_1, \ldots, p_m, z)$ around p such that $\pi(x_1, \ldots, x_m, p_1, \ldots, p_m, z) = (x_1, \ldots, x_m, z)$ and the contact structure is given by the 1-form $\alpha = dz - \sum_{i=1}^{m} p_i dx_i$.

Moreover, let $\pi : PT^*M \longrightarrow M$ be the projective cotangent bundle. This fibration is a Legendrian fibration with the canonical contact structure K. We now review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*M \longrightarrow PT^*M$ and differential map $d\pi : TPT^*M \longrightarrow TM$ of π. For any $X \in TPT^*M$, there exits an element $\alpha \in T^*M$ such that $\tau(X) = [\alpha]$. For an element $V \in T_xM$, the property $\alpha(V) = 0$ does not depend on the choice of the representative of the class $[\alpha]$. Thus we can define the canonical contact structure on PT^*M by

$$K = \{X \in TPT^*M \mid \tau(X)(d\pi(X)) = 0\}.$$

For a local coordinate neighborhood $(U, (x_1, \cdots, x_n))$ on M, we have a trivialization $PT^*U \cong U \times P(\mathbb{R}^{n-1})^*$ and we call $((x_1, \cdots, x_n), [\xi_1 : \cdots : \xi_n])$ homogeneous coordinates, where $[\xi_1 : \cdots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{n-1})^*$. It is easy to show that $X \in K_{(x,\xi)}$ if and only if $\sum_{i=1}^{n} \mu_i \xi_i = 0$, where $d\pi(X) = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i}$. This means that the contact form α on the affine coordinates $U_j = \{(\boldsymbol{x}, [\xi]) | \xi_j \neq 0\} \subset PT^*U$ is given by $\alpha = \sum_{i=1}^{n} (\xi_i/\xi_j) dx_i$. An immersion $i : L \longrightarrow PT^*M$ is said to be a *Legendrian immersion* if $\dim L = n - 1$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ a *Legendrian map* and the set

$W(i)$ =image $\pi \circ i$ the *wave front* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$.

We now consider the following double fibrations:

(1) (a) $H^3_1 \times S^3_2 \supset \Delta_1 = \{(\mathbf{v}, \mathbf{w}) \in H^3_1 \times S^3_2 \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \ \}$,

(b) $\pi_{11} : \Delta_1 \longrightarrow H^3_1, \quad \pi_{12} : \Delta_1 \longrightarrow S^3_2$,

(c) $\theta_{11} = \langle d\mathbf{v}, \mathbf{w} \rangle | \Delta_1, \quad \theta_{12} = \langle \mathbf{v}, d\mathbf{w} \rangle | \Delta_1$.

(2) (a) $H^3_1 \times \Lambda^3 \supset \Delta_2 = \{(\mathbf{v}, \mathbf{w}) \in H^3_1 \times \Lambda^3 \mid \langle \mathbf{v}, \mathbf{w} \rangle = -1 \ \}$,

(b) $\pi_{21} : \Delta_2 \longrightarrow H^3_1, \quad \pi_{22} : \Delta_2 \longrightarrow \Lambda^3$,

(c) $\theta_{21} = \langle d\mathbf{v}, \mathbf{w} \rangle | \Delta_2, \quad \theta_{22} = \langle \mathbf{v}, d\mathbf{w} \rangle | \Delta_2$.

(3) (a) $\Lambda^3 \times S^3_2 \supset \Delta_3 = \{(\mathbf{v}, \mathbf{w}) \in \Lambda^3 \times S^3_2 \mid \langle \mathbf{v}, \mathbf{w} \rangle = 1 \ \}$,

(b) $\pi_{31} : \Delta_3 \longrightarrow \Lambda^3, \quad \pi_{32} : \Delta_3 \longrightarrow S^3_2$,

(c) $\theta_{31} = \langle d\mathbf{v}, \mathbf{w} \rangle | \Delta_3, \quad \theta_{32} = \langle \mathbf{v}, d\mathbf{w} \rangle | \Delta_3$.

(4) (a) $\Lambda^3 \times \Lambda^3 \supset \Delta_4 = \{(\mathbf{v}, \mathbf{w}) \in \Lambda^3 \times \Lambda^3 \mid \langle \mathbf{v}, \mathbf{w} \rangle = -2 \ \}$,

(b) $\pi_{41} : \Delta_4 \longrightarrow \Lambda^3, \quad \pi_{42} : \Delta_4 \longrightarrow \Lambda^3$,

(c) $\theta_{41} = \langle d\mathbf{v}, \mathbf{w} \rangle | \Delta_4, \quad \theta_{42} = \langle \mathbf{v}, d\mathbf{w} \rangle | \Delta_4$.

Here,

$$\pi_{i1}(\mathbf{v}, \mathbf{w}) = \mathbf{v}, \quad \pi_{i2}(\mathbf{v}, \mathbf{w}) = \mathbf{w}, \ i = 1, 2, 3, 4.$$

$$\langle d\mathbf{v}, \mathbf{w} \rangle = -w_1 dv_1 - w_2 dv_2 + w_3 dv_3 + w_4 dv_4$$

$$\langle \mathbf{v}, d\mathbf{w} \rangle = -v_1 dw_1 - v_2 dw_2 + v_3 dw_3 + v_4 dw_4.$$

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i. The basic duality theorem is the following theorem:

Theorem 1.1 *Under the same notations as the previous paragraph, each* (Δ_i, K_i) $(i = 1, 2, 3, 4)$ *is a contact manifold and both of* π_{ij} $(j = 1, 2)$ *are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic each other.*

Proof. By definition we can easily show that each Δ_i $(i = 1, 2, 3, 4)$ is a smooth submanifold in $\mathbb{R}^4_2 \times \mathbb{R}^4_2$ and each π_{ij} $(i = 1, 2, 3, 4; j = 1, 2)$ is a smooth fibration. It also follows from the definition of θ_{ij} that each fibre of π_{ij} is an integral submanifold of K_i $(i = 1, 2, 3, 4)$.

Firstly, we show that (Δ_1, K_1) is a contact manifold. For any $\boldsymbol{v} = (v_1, \cdots, v_4) \in H^3_1$, we have $v_1^2 + v_2^2 \neq 0$. Therefore $(v_1, v_2) \neq (0, 0)$. We consider a coordinate neighborhood $V_1^+ = \{\boldsymbol{v} = (v_1, \ldots, v_4) \in H^3_1 \mid v_1 > 0\}$ on which we have $v_1 = \sqrt{-v_2^2 + \sum_{i=3}^4 v_i^2 + 1}$. Therefore, we regard that (v_2, v_3, v_4) is the local coordinates on V_1^+.

For any $\boldsymbol{w} = (w_1, \ldots, w_4) \in S^3_2$, we have $\sum_{i=3}^4 w_i^2 \neq 0$, so that $(w_3, w_4) \neq (0, 0)$. We also consider a coordinate neighborhood $W_3^+ = \{\boldsymbol{w} \in S^3_2 \mid w_3 > 0\}$. Then $V_1^+ \times W_3^+$ is one of the local coordinate of $H^3_1 \times S^3_2$. We now define a mapping $\Phi : \Delta_1 \cap (V_1^+ \times W_3^+) \longrightarrow PT^*H^3_1|V_1^+$ by

$$\Phi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, [(w_1v_2 - w_2v_1) : (-w_1v_3 + w_3v_1) : (-w_1v_4 + w_4v_1)])$$

Let $(v_2, v_3, v_4, [\xi_2 : \xi_3 : \xi_4])$ be homogeneous coordinates of $PT^*H^3_1|V_1^+ \equiv V_1^+ \times P(\mathbb{R}^{n-1})^*$. We have the canonical contact form $\alpha = \sum_{i=2}^4 (\xi_i/\xi_j)dv_i$ on $PT^*H^3_1$ over $V_1^+ \times U_j$, where $U_j = \{[\xi] \mid \xi_j \neq 0\}$. It follows that

$$\begin{aligned} \Phi^*\alpha &= \frac{\pm v_1}{w_jv_1 - w_1v_j}\left(-\sum_{i=1}^{2} w_i dv_i + \sum_{i=3}^{4} w_i dv_i\right) \\ &= \frac{\pm v_1}{w_jv_1 - w_1v_j}\langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_1 = \frac{\pm v_1}{w_jv_1 - w_1v_j}\theta_{11}, \end{aligned}$$

where $\pm$ depends on j of $\Phi^{-1}(V_1^+ \times U_j)$. Since

$$\Delta_1 \cap (V_1^+ \times W_3^+) = \bigcup_{j=2}^{4} \Phi^{-1}(V_1^+ \times U_j),$$

θ_{11} is a contact form on $\Delta_1 \cap (V_1^+ \times W_3^+)$ such that Φ is a contact morphism. We also have the similar calculation as the above on the other coordinate neighborhoods. Thus $(\Delta_1, \theta_{11}^{-1}(0))$ is a contact manifold. For the other Δ_i

$(i=2,3,4)$ we define smooth mappings $\Psi_{1i}:\Delta_1\longrightarrow\Delta_i$ by

$$\Psi_{12}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v},\boldsymbol{v}+\boldsymbol{w}),$$
$$\Psi_{13}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v}-\boldsymbol{w},-\boldsymbol{w}),$$
$$\Psi_{14}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v}-\boldsymbol{w},\boldsymbol{v}+\boldsymbol{w}).$$

We can construct the converse mappings defined by

$$\Psi_{21}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v},\boldsymbol{w}-\boldsymbol{v}),$$
$$\Psi_{31}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v}-\boldsymbol{w},-\boldsymbol{w})$$
$$\Psi_{41}(\boldsymbol{v},\boldsymbol{w})=\left(\frac{\boldsymbol{v}+\boldsymbol{w}}{2},\frac{\boldsymbol{w}-\boldsymbol{v}}{2}\right).$$

Therefore, Ψ_{1i} are diffeomorphisms. Moreover, we have

$$\begin{aligned}\Psi_{12}^*\theta_{21} &= \langle d\boldsymbol{v},\boldsymbol{v}+\boldsymbol{w}\rangle|\Delta_1=\langle d\boldsymbol{v},\boldsymbol{v}\rangle|\Delta_1+\langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta_1\\ &= \langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta_1=\theta_{11}.\end{aligned}$$

This means that (Δ_2,K_2) is a contact manifold such that Ψ_{12} is a contact diffeomorphism. For Δ_i $(i=3,4)$, we have the similar calculation, so that (Δ_i,K_i) $(i=3,4)$ are contact manifolds such that Ψ_{1i} are contact diffeomorphisms. This completes the proof. $\square$

We can also give contact diffeomorphisms $\Psi_{ij}:\Delta_i\longrightarrow\Delta_j$ for other pairs (i,j) as follows:

$$\begin{aligned}
&\Psi_{23}(\boldsymbol{v},\boldsymbol{w})=(2\boldsymbol{v}-\boldsymbol{w},\boldsymbol{v}-\boldsymbol{w}), && \Psi_{32}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v}-\boldsymbol{w},\boldsymbol{v}-2\boldsymbol{w});\\
&\Psi_{24}(\boldsymbol{v},\boldsymbol{w})=(2\boldsymbol{v}-\boldsymbol{w},\boldsymbol{w}), && \Psi_{42}(\boldsymbol{v},\boldsymbol{w})=\left(\frac{\boldsymbol{v}+\boldsymbol{w}}{2},\boldsymbol{w}\right);\\
&\Psi_{34}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v},\boldsymbol{v}-2\boldsymbol{w}), && \Psi_{43}(\boldsymbol{v},\boldsymbol{w})=\left(\boldsymbol{v},\frac{\boldsymbol{v}-\boldsymbol{w}}{2}\right).
\end{aligned}$$

We now explain the situation by a "mandala of Legendrian dualities" as the following commutative diagram:

$$
\begin{tikzcd}
 & & H_1^3 \times S_2^3 & & \\
 & & \cup & & \\
 & & \Delta_1 \arrow[dddd, ll, shift right, "\Psi_{12}"'] \arrow[d, shift left, "\Psi_{14}"] \arrow[dddd, rr, shift left, "\Psi_{13}"] & & \\
 & & \Delta_4 \arrow[u, shift left, "\Psi_{41}"] \arrow[ddd, ll, shift right, "\Psi_{42}"'] \arrow[ddd, rr, shift left, "\Psi_{43}"] & & \\
 & & \cap & & \\
 & & \Lambda^3 \times \Lambda^3 & & \\
\Delta_2 \arrow[rrrr, shift left, "\Psi_{23}"] \arrow[uuuu, rr, shift right, "\Psi_{21}"'] \arrow[uuu, rr, shift right, "\Psi_{24}"'] & & & & \Delta_3 \arrow[llll, shift left, "\Psi_{32}"] \arrow[uuuu, ll, shift left, "\Psi_{31}"] \arrow[uuu, ll, shift left, "\Psi_{34}"] \\
\cap & & & & \cap \\
H_1^3 \times \Lambda^3 & & & & \Lambda^3 \times S_2^3
\end{tikzcd}
$$

Fig. 1.1. The Mandala of Legendrian Dulaities

We can also consider the following two extra double fibrations:

(5) (a) $S_2^3 \times S_2^3 \supset \Delta_5 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\ \}$,

(b) $\pi_{51} : \Delta_5 \longrightarrow S_2^3$, $\pi_{52} : \Delta_5 \longrightarrow S_2^3$,

(c) $\theta_{51} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_5$, $\theta_{52} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_5$.

(6) (a) $H_1^3 \times H_1^3 \supset \Delta_6 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\ \}$,

(b) $\pi_{61} : \Delta_6 \longrightarrow H_1^3$, $\pi_{62} : \Delta_6 \longrightarrow H_1^3$,

(c) $\theta_{61} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_6$, $\theta_{62} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_6$.

We have the following theorem.

Theorem 1.2 *Under the same notations as the above, each* (Δ_i, K_i) $(i = 5, 6)$ *is a contact manifold and both of* π_{ij} $(j = 1, 2)$ *are Legendrian fibrations.*

The proof of the theorem is almost the same as that for (Δ_1, K_1) in

Theorem 1.1. We can show that (Δ_5, K_5) (respectively, (Δ_6, K_6)) is locally diffeomorphic to the projective cotangent bundle $\pi : PT^*H^3_1 \longrightarrow H^3_1$ (respectively, $\pi : PT^*S^3_2 \longrightarrow S^3_2$) which sends K_i to the canonical contact structure. We remark that these contact manifolds (Δ_j, K_j) $(j = 5, 6)$ are not canonically contact diffeomorphic to (Δ_i, K_i) $(i = 1, 2, 3, 4)$. Therefore we cannot add these contact manifolds to the mandala of Legendrian dualities. By definition, S^3_0 is a unit sphere in Euclidean space $\mathbb{R}^4_0$, so that (Δ_5, K_5) is the well known classical spherical duality in this case. Finally we remark that $\Delta_6 = \emptyset$ in $H^3_0 \times H^3_0$.

Part II. SPACELIKE SURFACES IN ANTI DE SITTER 3-SPACE

From this part we investigate the geometric properties of non-degenerate surfaces in Anti de Sitter 3-space from the viewpoint of singularity theory.

Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a regular surface (i.e., an embedding), where $U \subset \mathbb{R}^2$ is an open subset. We denote $M = \boldsymbol{X}(U)$ and identify M with U through the embedding $\boldsymbol{X}$. The embedding $\boldsymbol{X}$ is said to be non-degenerate if the induced metric $\mathbf{I}$ of M is non-degenerate. Locally we have two kinds of surfaces in this case which are called spacelike surfaces and timelike surfaces if the induced metric $\mathbf{I}$ of M is positive definite or Lorentizian respectively. We define a vector $\boldsymbol{N}(u)$ by

$$\boldsymbol{N}(u) = \frac{\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)}{\|\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)\|}.$$

By definition, we have

$$\langle \boldsymbol{N}(u), \boldsymbol{X}(u)\rangle \equiv \langle \mathbb{N}(u), \boldsymbol{X}_{u_i}(u)\rangle \equiv 0,$$

$$\langle \boldsymbol{X}(u), \boldsymbol{X}_{u_i}(u)\rangle \equiv 0 \text{ (for } i = 1, 2).$$

This means that $\boldsymbol{X}(u), \boldsymbol{N}(u) \in N_pM$, where N_pM is the normal space of M at p, $u = (u_1, u_2) \in U$ and $p = \boldsymbol{X}(u) \in M$.

We now consider a surface given by the intersection of H^3_1 with hyperplane $HP(\boldsymbol{n}, c)$. We denote it by $AH(\boldsymbol{n}, c) = H^3_1 \cap HP(\boldsymbol{n}, c)$ and call it a *AdS-hyperboloid*, a *AdS-pseudohyperboloid*, a *AdS-pseudosphere with index* 1 (briefly, *AdS-pseudosphere*) or a *AdS-horosphere* if $\boldsymbol{n}$ is timelike and $\|\boldsymbol{n}\| > |c|$, spacelike, timelike and $\|\boldsymbol{n}\| < |c|$ or null respectively. Especially, we call $AH(\boldsymbol{n}, 0)$ a *AdS-small pseudohyperboloid* or a *AdS-great-hyperboloid* if $\boldsymbol{n}$ is spacelike or timelike respectively.

In this part we mainly study the geometric properties of spacelike surfaces in Anti de Sitter 3-space.

2.1. Local differential geometry of spacelike surfaces

Let $\boldsymbol{X}(U) = M$ be a spacelike surfaces in Anti de Sitter 3-space. Since the embedding is spacelike and $\boldsymbol{X}(u) \in H_1^3$, $\boldsymbol{N}(u)$ is timelike. Therefore $\langle \boldsymbol{N}(u), \boldsymbol{N}(u) \rangle \equiv -1$. It follows that $\boldsymbol{N}(u) \in H_1^3 \cap N_pM$. Thus we can define a map

$$\mathbb{T} : U \longrightarrow H_1^3 \text{ by } \mathbb{T}(u) = \boldsymbol{N}(u).$$

We call it the *timelike Anti de Sitter Gauss image* (briefly, *TAdS-Gauss image*) of $\boldsymbol{X}$(or M).

We now consider the geometric meanings of the TAdS-Gauss image of a spacelike surface. We have the following proposition.

Proposition 2.1.1 *Let $\boldsymbol{X} : U \longrightarrow H_1^3$ be a spacelike surface in Anti de Sitter 3-space. If the TAdS-Gauss image $\mathbb{T}$ is constant, then the spacelike surface $\boldsymbol{X}(U) = M$ is a part of a AdS-great-hyperboloid.*

Proof. We consider the set $V = \{\boldsymbol{y} \in \mathbb{R}_2^4 | \langle \boldsymbol{y}, \boldsymbol{N} \rangle = 0\}$. Since $\mathbb{T} = \boldsymbol{N}$ is constant, the set $V = HP(\boldsymbol{N}, 0)$ is a Lorentz hyperplane. We also have $\langle \boldsymbol{X}, \boldsymbol{N} \rangle \equiv 0$, so $\boldsymbol{X}(U) = M \subset V \cap H_1^3$. $\square$

It is easy to show that $\mathbb{T}_{u_i}$ $(i = 1, 2)$ are tangent vectors of M. Therefore we have a linear transformation $W_p = -d\mathbb{T}(u) : T_pM \longrightarrow T_pM$ which is called the *Anti de Sitter shape operator* (briefly, *AdS-shape operator*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We denote the eigenvalue of W_p by $k_i(p)$ $(i = 1, 2)$.

The *Anti de Sitter Gauss-Kronecker curvature* (briefly, *AdS-G-K curvature*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$ is defined to be

$$K_{AdS}(u) = \det W_p = k_1(p) \cdot k_2(p).$$

We say that a point $p = \boldsymbol{X}(u)$ is an *Anti de Sitter parabolic point* (or, briefly an *AdS-parabolic point*) of $\boldsymbol{X} : U \longrightarrow H^3_1$ if $K_{AdS}(u) = 0$. We say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is an *umbilic point* if $W_p = k(p) id_{T_pM}$. We also say that $M = \boldsymbol{X}(U)$ is *totally umbilic* if all points on M are umbilic. Then we have the following proposition.

Proposition 2.1.2 *Suppose that $M = \boldsymbol{X}(U)$ is totally umbilic. Then $k(p)$ is constant k. Under this condition, we have the following classification.*

(1) *If $k \neq 0$ then M is a part of a AdS-hyperboloid $HP(\boldsymbol{n}, -1) \cap H^3_1$, where $\boldsymbol{n} = \boldsymbol{X} + \frac{1}{k}\boldsymbol{N}$ is a constant timelike vector.*

(2) *If $k = 0$ then M is a part of a AdS-great-hyperboloid $HP(\boldsymbol{n}, 0) \cap H^3_1$, where $\boldsymbol{n} = \boldsymbol{N}$ is a constant timelike vector.*

The proof is also given by direct calculations, so that we omit it.

Since $\boldsymbol{X}_{u_1}$ and $\boldsymbol{X}_{u_2}$ are spacelike vectors, we first introduce the Riemannian metric $ds^2 = \sum_{i,j=1}^{2} g_{ij} du_i du_j$ on $M = \boldsymbol{X}(U)$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$ for any $u \in U$. We also define the *Anti de Sitter second fundamental invariant* by $h_{ij}(u) = \langle -\mathbb{T}_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$ for any $u \in U$. We can also show the following results by exactly the same arguments as those of [15].

Proposition 2.1.3 *With the above notation, we have the following Anti de Sitter Weingarten formula:*

$$\mathbb{T}_{u_i} = -\sum_{j=1}^{2} h_i^j \boldsymbol{X}_{u_j},$$

where $(h_i^j) = (h_{ik})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$. $\square$

As a corollary of the above proposition, we have an explicit expression for the AdS-G-K curvature by Riemannian metric and the Anti de Sitter second fundamental invariant.

Corollary 2.1.4 *With the same notations as in the above Proposition, we have*:

$$K_{AdS}(u) = \frac{\det(h_{ij}(u))}{\det(g_{\alpha\beta}(u))}. \qquad \square$$

Since ds^2 is a Riemannian metric, we have the sectional curvature K_I of M, which we call an intrinsic Gaussian curvature. By B. O'Neil [37] (Page 107 Corollary 20), we remark that $K_{AdS} = -1 - K_I$.

2.2. Timelike height functions on spacelike surfaces

In this section we define a family of functions on a spacelike surface in Anti de Sitter 3-space which is useful for the study of singularities of TAdS-Gauss image.

Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a spacelike surface. We define a family of functions

$$H : U \times H^3_1 \longrightarrow \mathbb{R}$$

by $H(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle$. We call H *a timelike Anti de Sitter height function* (or, a *AdS-height function*) on $M = \boldsymbol{X}(U)$. We denote the *Hessian matrix* of the AdS-height function $h_{v_0}(u) = H(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(h_{v_0})(u_0)$. Then we have the following proposition.

Proposition 2.2.1 *Let $M = \boldsymbol{X}(U)$ be a spacelike surface in H^3_1 and $H : U \times H^3_1 \longrightarrow \mathbb{R}$ be a AdS-height function. Then we have the following assertions*:

(1) $H(u, \boldsymbol{v}) = \frac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = 0$ *(for $i = 1, 2$)* $\Longleftrightarrow \boldsymbol{v} = \pm \boldsymbol{N}(u) = \pm \mathbb{T}(u)$;

(2) *Let* $\boldsymbol{v}_0 = \boldsymbol{N}(u_0)$, *then* $\det\mathrm{Hess}(h_{v_0})(u_0) = 0 \Longleftrightarrow K_{AdS}(u_0) = 0$.

Proof. (1) Since $\{\boldsymbol{X}(u), \boldsymbol{N}(u), \boldsymbol{X}_{u_1}(u), \boldsymbol{X}_{u_2}(u)\}$ is a basis of the vector space $T_p\mathbb{R}^4_2$ where $p = \boldsymbol{X}(u)$, there exist real numbers $\lambda, \eta, \alpha_1, \alpha_2$ such that

$$\boldsymbol{v} = \lambda\boldsymbol{X}(u) + \eta\boldsymbol{N}(u) + \alpha_1\boldsymbol{X}_{u_1}(u) + \alpha_2\boldsymbol{X}_{u_2}(u).$$

Therefore $H(u, \boldsymbol{v}) = 0$ if and only if $\lambda = -\langle\boldsymbol{X}(u), \boldsymbol{v}\rangle = 0$. Since

$$0 = \frac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = \langle\boldsymbol{X}_{u_i}(u), \boldsymbol{v}\rangle = \sum_{j=1}^{2} g_{ij}(u)\alpha_i$$

and (g_{ij}) is non-degenerate, we have $\alpha_i = 0$ (for $i = 1, 2$). Therefore we have $\boldsymbol{v} = \eta\boldsymbol{N}$. Then from a straight forward calculation, we have $\eta = \pm 1$.

(2) By definition, we have

$$\mathrm{Hess}(h_{v_0})(u_0) = (\langle\boldsymbol{X}_{u_iu_j}(u_0), \mathbb{T}(u_0)\rangle) = (-\langle\boldsymbol{X}_{u_i}(u_0), \mathbb{T}_{u_j}(u_0)\rangle).$$

By the AdS-Weingarten formula, we have

$$-\langle\boldsymbol{X}_{u_i}, \mathbb{T}_{u_j}\rangle = \textstyle\sum_{\alpha=1}^{2} h_i^\alpha\langle\boldsymbol{X}_{u_\alpha}, \boldsymbol{X}_{u_j}\rangle = \sum_{\alpha=1}^{2} h_i^\alpha g_{\alpha j} = h_{ij}.$$

Therefore we have

$$K_{AdS} = \frac{\det(h_{i,j})}{\det(g_{\alpha\beta})} = \frac{\det\mathrm{Hess}(h_{v_0})(u_0)}{\det(g_{\alpha\beta}(u_0))}.$$

Then we complete the proof. □

As an application of the above proposition, we have the following.

Corollary 2.2.2 *Let $H : U \times H^3_1 \longrightarrow \mathbb{R}$, with $H(u, \boldsymbol{v}) = h_v(u)$ be a AdS-height function on spacelike surface $M = \boldsymbol{X}(U)$ and $\mathbb{T}$ be the TAdS-Gauss image, $p = \boldsymbol{X}(u)$. Then the following conditions are equivalent*:

(1) *There exists $\boldsymbol{v} \in H^3_1$, such that $p \in M$ is a degenerate singular point of AdS-height function h_v ;*

(2) *There exists $\boldsymbol{v} \in H^3_1$, such that $p \in M$ is a singular point of TAdS-Gauss image $\mathbb{T}$;*

(3) $K_{AdS}(u) = 0$. □

On the other hand, we can naturally interpret the TAdS-Gauss image $\mathbb{T}$ of M as a Legendrian map in the framework of Legendrian singularity theory. We give a brief review on Legendrian singularity theory [1]. The main tool of Legendrian singularities theory is the notion of generating families. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* if the mapping

$$\Delta^* F = (F, \tfrac{\partial F}{\partial q_1}, \cdots, \tfrac{\partial F}{\partial q_k}) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \cdots, q_k, x_1, \cdots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$. In this case we have a smooth $(n-1)-$dimensional submanifold,

$$\Sigma_*(F) = \{(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) | F(q, x) = \tfrac{\partial F}{\partial q_1}(q, x) = \cdots = \tfrac{\partial F}{\partial q_k}(q, x) = 0\}$$

and the map germ $\Phi_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\Phi_F(q, x) = (x, [\tfrac{\partial F}{\partial x_1}(q, x) : \cdots : \tfrac{\partial F}{\partial x_n}(q, x)])$$

is a Legendrian immersion germ. Then we have the following fundamental theorem [1, 45].

Proposition 2.2.3 *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a *generating family* of $\Phi_F(\Sigma_*(F))$. Therefore the corresponding wave front is

$$W(\Phi_F) = \{x \in \mathbb{R}^n | \text{there exists } q \in \mathbb{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0\}.$$

We sometimes denote $\mathcal{D}_F = W(\Phi_F)$ and call it the *discriminant set* of F.

Now we can apply the above arguments to our situation. We first have the following principle property with respect to the AdS-height function.

Proposition 2.2.4 *The AdS-height function $H : U \times H_1^3 \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces $h_v^{-1}(0)_{v \in H_1^3}$.*

Proof. For any $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in H_1^3$, we have $v_1 \neq 0$ or $v_2 \neq 0$. Without loss of the generality, we might assume that $v_1 > 0$, then $v_1 = \sqrt{1 + v_3^2 + v_4^2 - v_2^2}$. It follows that

$$H(u,\boldsymbol{v}) = -x_1(u)\sqrt{1+v_3^2+v_4^2-v_2^2} - x_2(u)v_2 + x_3(u)v_3 + x_4(u)v_4$$

where $\boldsymbol{X}(u) = (x_1(u), x_2(u), x_3(u), x_4(u))$. We have to prove the mapping

$$\Delta^* H = (H, \tfrac{\partial H}{\partial u_1}, \tfrac{\partial H}{\partial u_2})$$

is non-singular at any point. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$\begin{pmatrix} \langle \boldsymbol{X}_{u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_2}, \boldsymbol{v}\rangle & x_1\frac{v_2}{v_1} - x_2 & -x_1\frac{v_3}{v_1} + x_3 & -x_1\frac{v_4}{v_1} + x_4 \\ \langle \boldsymbol{X}_{u_1u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_1u_2}, \boldsymbol{v}\rangle & x_{1u_1}\frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1} + x_{4u_1} \\ \langle \boldsymbol{X}_{u_2u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_2u_2}, \boldsymbol{v}\rangle & x_{1u_2}\frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1} + x_{4u_2} \end{pmatrix}.$$

We claim that it will suffice to show that the determinant of the matrix

$$A = \begin{pmatrix} x_1\frac{v_2}{v_1} - x_2 & -x_1\frac{v_3}{v_1} + x_3 & -x_1\frac{v_4}{v_1} + x_4 \\ x_{1u_1}\frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1} + x_{4u_1} \\ x_{1u_2}\frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1} + x_{4u_2} \end{pmatrix},$$

does not vanish at $(u,\boldsymbol{v}) \in \Delta^* H^{-1}(\boldsymbol{0})$. In this case, $\boldsymbol{v} = \mathbb{T}(u)$ and we denote

$$\boldsymbol{b}_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ x_{1u_2} \end{pmatrix},\ \boldsymbol{b}_2 = \begin{pmatrix} x_2 \\ x_{2u_1} \\ x_{2u_2} \end{pmatrix},\ \boldsymbol{b}_3 = \begin{pmatrix} x_3 \\ x_{3u_1} \\ x_{3u_2} \end{pmatrix},\ \boldsymbol{b}_4 = \begin{pmatrix} x_4 \\ x_{4u_1} \\ x_{4u_2} \end{pmatrix}.$$

Then we have

$$\begin{aligned} \det A = &-\frac{v_1}{v_1}\det(\boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4) + \frac{v_2}{v_1}\det(\boldsymbol{b}_1\ \boldsymbol{b}_3\ \boldsymbol{b}_4) \\ &-\frac{v_3}{v_1}\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \ \boldsymbol{b}_4) + \frac{v_4}{v_1}\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} = (&-\det(\boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4),\ \det(\boldsymbol{b}_1\ \boldsymbol{b}_3\ \boldsymbol{b}_4), \\ &\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_4),\ -\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3)) \end{aligned}$$

Therefore we have

$$\det A = \langle (-\frac{v_1}{v_1}, -\frac{v_2}{v_1}, -\frac{v_3}{v_1}, -\frac{v_4}{v_1}), \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2}\rangle$$

$$= -\frac{1}{v_1}\langle \mathbb{T}, \| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \| \boldsymbol{e}\rangle$$
$$= \frac{\| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \|}{v_1} \neq 0. \qquad \square$$

We now define a mapping

$$\mathcal{L}_6 : U \longrightarrow \Delta_6$$

by $\mathcal{L}_6(u) = (\boldsymbol{X}(u), \mathbb{T}(u))$. Since $\langle \boldsymbol{X}(u), \mathbb{T}(u)\rangle = \langle d\boldsymbol{X}(u), \mathbb{T}(u)\rangle = 0$, the mapping $\mathcal{L}$ is a Legendrian embedding. We now show that H is a generating family of $\mathcal{L}_6(U) \subset \Delta_6$.

Proposition 2.2.5 *For any spacelike surfaces $\boldsymbol{X} : U \longrightarrow H^3_1$, the AdS-height function $H : U \times H^3_1 \longrightarrow \mathbb{R}$ of $\boldsymbol{X}$ is a generating family of the Legendrian embedding $\mathcal{L}_6$.*

Proof. We consider coordinate neighborhoods $V_1^+ = \{\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in H^3_1 \mid v_1 > 0\}$ and $W_1^+ = \{\boldsymbol{w} = (w_1, w_2, w_3, w_4) \in H^3_1 \mid w_1 > 0\}$. Then $V_1^+ \times W_1^+$ is one of the local coordinate of $H^3_1 \times H^3_1$. We now define a mapping $\Phi : \Delta_6 \cap (V_1^+ \times W_1^+) \longrightarrow PT^*H^3_1 \mid W_1^+$ by

$$\Phi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{w}, [v_1w_2 - v_2w_1 : -v_1w_3 + v_3w_1 : -v_1w_4 + v_4w_1]).$$

Let

$$((w_2, w_3, w_4), [\xi_2 : \xi_3 : \xi_4])$$

be homogeneous coordinates of $PT^*H^3_1 \mid W_1^+ \equiv W_1^+ \times P(\mathbb{R}^2)^*$. We have the canonical contact form $\alpha = \sum_{i=2}^4 (\xi_i/\xi_j)dw_i$ on $PT^*H^3_1$ over $W_1^+ \times U_j$, where $U_j = \{[\xi] | \xi_j \neq 0\}$. It follows that

$$\begin{aligned}\Phi^*\alpha &= \frac{\varepsilon}{v_jw_1 - v_1w_j}((v_1w_2 - v_2w_1)dw_2 + \sum_{i=3}^{4}(-v_1w_i + v_iw_1)dw_i) \\ &= \frac{\varepsilon w_1}{v_jw_1 - v_1w_j}\langle \boldsymbol{v}, d\boldsymbol{w}\rangle \mid \Delta_6 = \frac{\varepsilon w_1}{v_jw_1 - v_1w_j}\theta_2,\end{aligned}$$

where $\varepsilon = -1$ if $j = 2$ or $\varepsilon = 1$ if $j = 3, 4$. Since

$$\Delta_6 \cap (V_1^+ \times W_1^+) = \bigcup_{j=2}^{4} \Phi^{-1}(W_1^+ \times U_j),$$

θ_{62} is a contact form on $\Delta_6 \cap (V_1^+ \times W_1^+)$ such that Φ is a contact diffeomorphism. We also have the similar calculation as the above on the other coordinate neighborhoods.

Since AdS-height function H is a Morse family, we have a Legendrian immersion $\mathcal{L}_H : \Sigma_*(H) \mid (U \times W_1^+) \longrightarrow PT^*H_1^3 \mid W_1^+$ defined by

$$\mathcal{L}_H(u, \boldsymbol{w}) = (\boldsymbol{w}, [\frac{\partial H}{\partial w_2} : \frac{\partial H}{\partial w_3} : \frac{\partial H}{\partial w_4}]).$$

By Proposition 2.2.1, we have

$$\Sigma_*(H) = \{(u, \mathbb{T}(u)) \in U \times H_1^3 \mid u \in U\}.$$

Since $\boldsymbol{w} = \mathbb{T}(u)$ and $w_1 = \sqrt{-w_2^2 + w_3^2 + w_4^2 + 1}$, we have

$$\frac{\partial H}{\partial w_2}(u, \mathbb{T}(u)) = x_1(u)\frac{w_2(u)}{w_1(u)} - x_2(u),$$

$$\frac{\partial H}{\partial w_3}(u, \mathbb{T}(u)) = x_3(u) - x_1(u)\frac{w_3(u)}{w_1(u)},$$

$$\frac{\partial H}{\partial w_4}(u, \mathbb{T}(u)) = x_4(u) - x_1(u)\frac{w_4(u)}{w_1(u)},$$

where $\boldsymbol{X} = (x_1, x_2, x_3, x_4)$ and $\mathbb{T} = (w_1, w_2, w_3, w_4)$.We now define a smooth mapping $\mathcal{T} : U \longrightarrow PT^*H_1^3$ by

$\mathcal{T}(u) = (\mathbb{T}(u), [(x_1(u)w_2(u) - x_2(u)w_1(u)) : (-x_1(u)w_3(u) + x_3(u)w_1(u)) : (-x_1(u)w_4(u) + x_4(u)w_1(u))]).$

It follows that

$$\begin{aligned}\mathcal{L}_H(u, \mathbb{T}(u)) = (\mathbb{T}(u), [&x_1(u)w_2(u) - x_2(u)w_1(u) : \\ &-x_1(u)w_3(u) + x_3(u)w_1(u) : -x_1(u)w_4(u) + x_4(u)w_1(u)]) = \mathcal{T}(u).\end{aligned}$$

Therefore we have $\Phi \circ \mathcal{L}_6(u) = \mathcal{T}(u)$ on W_1^+. We also have the same relation as the above on the other local coordinates. This means that H is a generating family of $\mathcal{L}_6(U) \subset \Delta_6$. $\square$

Therefore we conclude that the TAdS-Gauss image $\mathbb{T}$ can be regarded as a Legendrian map and $\mathbb{T}(U)$ can be regarded as a wave front set of $\mathcal{L}_6$.

2.3. Contact with AdS-great-hyperboloids

In this section we consider the geometric meaning of the singularities of the TAdS-Gauss image of spacelike surface $M = \boldsymbol{X}(U)$ in H^3_1. We consider the contact of spacelike surfaces with AdS-great-hyperboloids. We now briefly review the theory of contact due to Montaldi [34]. Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact* of X_1 and Y_1 at y_1 is the same type as the *contact* of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [34], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 2.3.1 *Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, \mathbf{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(\mathbf{0}), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

For the definition of the $\mathcal{K}$-equivalent, See Martinet [29]. We now consider a function $\mathcal{H}$: $H^3_1 \times H^3_1 \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$. For any $\boldsymbol{v}_0 \in H^3_1$, we denote $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v}_0)$ and we have the AdS-great-hyperboloid $\mathfrak{h}_{v_0}^{-1}(0) = H^3_1 \cap HP(\boldsymbol{v}_0, 0) = AH(\boldsymbol{v}_0, 0)$. For any $u_0 \in U$, we consider the timelike vector $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. Then we have

$$\mathfrak{h}_{v_0} \circ \boldsymbol{X}(u_0) = \mathcal{H} \circ (\boldsymbol{X} \times id_{H^3_1})(u_0, \boldsymbol{v}_0) = H(u_0, \mathbb{T}(u_0)) = 0.$$

We also have relations

$$\frac{\partial \mathfrak{h}_{v_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, \mathbb{T}(u_0)) = 0,$$

for $i = 1, 2$. This means that the AdS-great-hyperboloid $AH(\boldsymbol{v}_0, 0)$ is tangent to $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$. In this case, we call $AH(\boldsymbol{v}_0, 0)$ the *tangent AdS-great-hyperboloid* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$ (or, u_0), which we write $AH(\boldsymbol{X}, u_0)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be timelike vectors. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent, then $HP(\boldsymbol{v}_1, 0)$ and $HP(\boldsymbol{v}_2, 0)$ are equal. Therefore, AdS-great-hyperboloids $AH(\boldsymbol{v}_1, 0) = AH(\boldsymbol{v}_2, 0)$. Then we have the following simple lemma.

Lemma 2.3.2 *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a spacelike surface. Consider two points $u_1, u_2 \in U$. Then*:

$$\mathbb{T}(u_1) = \mathbb{T}(u_2) \text{ if and only if } AH(\boldsymbol{X}, u_1) = AH(\boldsymbol{X}, u_2). \qquad \Box$$

We now consider the contact of M with tangent AdS-great-hyperboloid at $p \in M$ as an application of Legendrian singularity theorey. We introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves fibres of π and that $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^n$ at a point is said to be *Legendrian stable* if for every map with the given germ there are a neighbourhood in the space of Legendrian immersion (in the Whitney $C^\infty-$topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has, in the second neighbourhood, a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following proposition due to Zakalyukin[46].

Proposition 2.3.3 *Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$, $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i$ and $\pi \circ i'$ respectively are dense. Then i and i' are Legemndrian equivalent if and only if wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.*

We remark that the assumption in the above proposition is a generic condition for i and i'. In particular, if i and i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathcal{M}_n = \{h \in \mathcal{E}_n | h(\mathbf{0}) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that F and G are P–$\mathcal{K}$ *equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(q, x) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+n}}) = \langle G\rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a $\mathcal{K}$*-versal deformation* of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \cdots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{\mathbf{0}\}\rangle_{\mathbb{R}},$$

where $T_e(\mathcal{K})(f) = \langle \frac{\partial f}{\partial q_1}, \cdots, \frac{\partial f}{\partial q_k}, f\rangle_{\mathcal{E}_k}$. The main result in the theory of Legendrian singularities [1, 45] is the following:

Theorem 2.3.4 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Then*

(1) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent if and only if F and G are*

$P-\mathcal{K}$ equivalent;

(2) *$\mathcal{L}_F$ is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k\times \{\mathbf{0}\}$.*

By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, Proposition 2.3.3 and Theorem 2.3.4, we have the following classification result of Legendrian stable germs in the appendix of [15]. For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define the *local ring* of f by $Q(f) = \mathcal{E}_n/f^*(\mathcal{M}_p)\mathcal{E}_n$. Then we have the following proposition.

Proposition 2.3.5 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Suppose that $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian stable. Then the following conditions are equivalent:*

(1) *$(W(\mathcal{L}_F), \mathbf{0})$ and $(W(\mathcal{L}_G), \mathbf{0})$ are diffeomorphic as germs;*

(2) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent;*

(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|\mathbb{R}^k\times\{\mathbf{0}\}$ and $g = G|\mathbb{R}^k\times\{\mathbf{0}\}$.*

We have the tools for study of the contact of spacelike surfaces with AdS-great-hyperboloids. Let $\mathbb{T}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{v}_i)$ (for $i = 1, 2$) be TAdS-Gauss image germs of spacelike surface germs $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$. We say that $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-*equivalent* if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (H^3_1, \boldsymbol{v}_1) \longrightarrow (H^3_1, \boldsymbol{v}_2)$ such that $\Phi \circ \mathbb{T}_1 = \mathbb{T}_2 \circ \phi$. Suppose the regular set of $\mathbb{T}_i$ is dense in (U, u_i) for each $i = 1, 2$. It follows from Proposition 2.3.3 that $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian embedding germs ${\mathcal{L}_6}^1 : (U, u_1) \longrightarrow (\Delta_6, \mathbf{z}_1)$ and ${\mathcal{L}_6}^2 : (U, u_2) \longrightarrow (\Delta_6, \mathbf{z}_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are $P-\mathcal{K}$-equivalent by Theorem 2.3.4. Here,

$H_i : (U \times H^3_1, (u_i, \boldsymbol{v}_i)) \longrightarrow \mathbb{R}$ is the corresponding AdS-height function germ of $\boldsymbol{X}_i$.

On the other hand, we denote $h_{i,v_i} = H_i(u, \boldsymbol{v}_i)$; then we have $h_{i,v_i}(u) = \mathfrak{h}_{v_i} \circ \boldsymbol{X}_i(u)$. By Theorem 2.3.1,

$$K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2)$$

if and only if h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent. Therefore, we can apply the above arguments to our situation. We denote by $Q(\boldsymbol{X}, u_0)$ the local ring of the function germ $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. We remark that we can write the local ring explicitly as follows:

$$Q(\boldsymbol{X}, u_0) = \frac{C^\infty_{u_0}(U)}{\langle\langle \boldsymbol{X}(u), \mathbb{T}(u_0)\rangle\rangle_{C^\infty_{u_0}(U)}},$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathcal{M}_{u_0}(U)$.

Theorem 2.3.6 *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$ (for $i = 1, 2$) be spacelike surface germs such that the corresponding Legendrian embedding germs $\mathcal{L}_6{}^i : (U, u_i) \longrightarrow (\Delta_6, \boldsymbol{z}_i)$ are Legendrian stable. Then the following conditions are equivalent:*

(1) *TAdS-Gauss image germs $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent;*

(2) *H_1 and H_2 are $P{-}\mathcal{K}$-equivalent;*

(3) *h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent;*

(4) *$K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2)$;*

(5) *$Q(\boldsymbol{X}_1, u_1)$ and $Q(\boldsymbol{X}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.* □

For a spacelike surface germ

$$\boldsymbol{X} : (U, u_0) \longrightarrow (H^3_1, \boldsymbol{X}(u_0)),$$

we call $\boldsymbol{X}^{-1}(AH(\mathbb{T}(u_0), 0), u_0)$ the *tangent AdS-great-hyperboloidic indica-*

trix germ of $\boldsymbol{X}$. In general we have the following proposition:

Proposition 2.3.7 *Let* $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$*(for* $i = 1, 2$*) be spacelike surface germs such that their AdS-parabolic sets have no interior points as subspaces of* U*. If TAdS-Gauss image germs* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *are* $\mathcal{A}$*-equivalent, then*

$$K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2).$$

In this case, $\boldsymbol{X}_1^{-1}(AH(\mathbb{T}_1(u_1), 0), u_1)$ *and* $\boldsymbol{X}_2^{-1}(AH(\mathbb{T}_2(u_2), 0), u_2)$ *are diffeomorphic as set germs.*

Proof. The AdS-parabolic set is the set of singular points of the TAdS-Gauss image. So the corresponding Legendrian embedding $\mathcal{L}^i$ satisfy the hypothesis of Proposition 2.3.3. If TAdS-Gauss image germs $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent, then $\mathcal{L}^1$ and $\mathcal{L}^2$ are Legendrian equivalent, so that H_1 and H_2 are P–$\mathcal{K}$-equivalent. Therefore, h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent. By Theorem 2.3.6, this condition is equivalent to the condition that

$$K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2).$$

Moreover, we have $\boldsymbol{X}_i^{-1}(AH(\mathbb{T}_i(u_i), 0), u_i) = (h_{i,v_i}^{-1}(0), u_i)$. It follows from this fact that $\boldsymbol{X}_1^{-1}(AH(\mathbb{T}_1(u_1), 0), u_1)$ and $\boldsymbol{X}_2^{-1}(AH(\mathbb{T}_2(u_2), 0), u_2)$ are diffeomorphic as set germs because the $\mathcal{K}$-equivalent preserves the zero level sets. □

From the above proposition, the diffeomorphism type of the tangent AdS-great-hyperboloidic indicatrix germ is an invariant of $\mathcal{A}$-classification of the TAdS-Gauss image germ of $\boldsymbol{X}$. Moreover, we can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for a function germ. The local ring of a function is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension of a function germ is a numerical $\mathcal{K}$-invariant of function germs. We denote

$$\text{AdS-ord}(\boldsymbol{X}, u_0) = \dim \frac{C^\infty_{u_0}(U)}{\langle h_{v_0}, \partial h_{v_0}/\partial u_i \rangle_{C^\infty_{u_0}(U)}},$$

where $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. Usually $\text{AdS-ord}(\boldsymbol{X}, u_0)$ is called the $\mathcal{K}$-codimension of h_{v_0}. However, we call it the *order of contact with tangent AdS-great-hyperboloid* at $\boldsymbol{X}(u_0)$. We also have the notion of *corank* of function germs:

$$\text{AdS-corank}(\boldsymbol{X}, u_0) = 2 - \text{rankHess}(h_{v_0})(u_0),$$

where $\boldsymbol{v}_0 = \mathbb{T}(u_0)$.

By Proposition 2.2.1, $\boldsymbol{X}(u_0)$ is an AdS-parabolic point if and only if AdS-corank $(\boldsymbol{X}, u_0) \geq 1$. On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \boldsymbol{a}) \longrightarrow \mathbb{R}$ has the A_k−type singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $\text{AdS-corank}(\boldsymbol{X}, u_0) = 1$, the AdS-height function h_{v_0} has the A_k−type singularity at u_0 and is generic. In this case we have $\text{AdS-ord}(\boldsymbol{X}, u_0) = k$. This number is equal to the order of contact in the classical sense (cf., [4]). This is the reason why we call $\text{AdS-ord}(\boldsymbol{X}, u_0)$ the order of contact with the AdS-great-hyperboloid at $\boldsymbol{X}(u_0)$.

2.4. Classification of singularities of TAdS-Gauss images

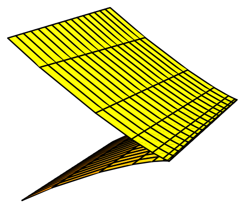
In this section we give the generic classification of singularities of TAdS-Gauss images. We have almost the same arguments as those of [15], so that we omit the details. We consider the space of spacelike embeddings $\text{Emb}_S(U, H^3_1)$ with the Whitney C^∞-topology. By the classification of stable Legendrian singularities of $n = 3$ and the transversality theorem of [15] (Proposition 7.1), we have the following theorem.

Theorem 2.4.1 *There exists an open dense subset* $\mathcal{O} \subset \text{Emb}_S(U, H^3_1)$ *such that for any* $\boldsymbol{X} \in \mathcal{O}$ *the following conditions hold.*

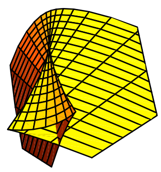
(1) *The AdS-parabolic set* $K_{AdS}^{-1}(0)$ *is a regular curve. We call such a curve the AdS-parabolic curve.*

(2) *The TAdS-Gauss image* $\mathbb{T}$ *along the AdS-parabolic curve is a cuspidal edge except at isolated points. At such the point* $\mathbb{T}$ *is the swallowtail.*

Here, a map germ $f : (\mathbb{R}^2, \boldsymbol{a}) \longrightarrow (\mathbb{R}^3, \boldsymbol{b})$ *is called a cuspidal edge if it is* $\mathcal{A}$*-equivalent to the germ* (u_1, u_2^2, u_2^3) *and a swallowtail if it is* $\mathcal{A}$*-equivalent to the germ* $(3u_1^4 + u_1^2 u_2, 4u_1^3 + 2u_1 u_2, u_2)$. (cf., Fig 2.4.1).



Cuspital edge



Swallowtail

Fig 2.4.1

The assertion of Theorem 2.4.1 can be interpreted as saying that the Legendrian embedding $\mathcal{L}$ of the TAdS-Gauss image $\mathbb{T}$ of $\boldsymbol{X}$ is Legendrian stable at each point. Following the terminology of Whitney [43], we say that a spacelike surface $\boldsymbol{X} : U \longrightarrow H_1^3$ has the *excellent TAdS-Gauss image* $\mathbb{T}$ if $\mathcal{L}$ is a stable Legendrian immersion germ at each point. In this case, the TAdS-Gauss image $\mathbb{T}$ has only cuspidal edges and swallowtails as singularities. Theorem 2.4.1 assert that a spacelike surface with the excellent TAdS-Gauss image is generic in the space of all spacelike surfaces in H_1^3.

We now consider the geometric meanings of cuspidal edges and swallowtails of the TAdS-Gauss image. We have the following results analogous to the results of Banchoff *et al*. [2].

Theorem 2.4.2 *Let* $\mathbb{T} : (U, u_0) \longrightarrow (H_1^3, \boldsymbol{v}_0)$ *be the excellent TAdS-Gauss image germ of a spacelike surface* $\boldsymbol{X}$ *and* $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ *be the AdS-*

height function germ at $\boldsymbol{v}_0 = \mathbb{T}(u_0)$*. Then we have the following.*

(1) *The point* u_0 *is an AdS-parabolic point of* $\boldsymbol{X}$ *if and only if* AdS-corank $(\boldsymbol{X}, u_0)= 1$.

(2) *If* u_0 *is an AdS-parabolic point of* $\boldsymbol{X}$*, then* h_{v_0} *has the* A_k-*type singularity for* $k = 2, 3$.

(3) *Suppose that* u_0 *is an AdS-parabolic point of* $\boldsymbol{X}$*. Then the following conditions are equivalent:*

(a) $\mathbb{T}$ *has the cuspidal edge at* u_0*;*

(b) h_{v_0} *has the* A_2-*type singularity;*

(c) AdS-order$(\boldsymbol{X}, u_0) = 2$*;*

(d) *the tangent AdS-great-hyperboloidic indicatrix germ is an ordinary cusp, where a curve* $C \subset \mathbb{R}^2$ *is called an ordinary cusp if it is diffeomorphic to the curve given by* $\{(u_1, u_2)|u_1^2 - u_2^3 = 0\}$.

(4) *Suppose that* u_0 *is an AdS-parabolic point of* $\boldsymbol{X}$*. Then the following conditions are equivalent:*

(a) $\mathbb{T}$ *has the swallowtail at* u_0*;*

(b) h_{v_0} *has the* A_3-*type singularity;*

(c) AdS-order$(\boldsymbol{X}, u_0) = 3$*;*

(d) *the tangent AdS-great-hyperboloidic indicatrix germ is an point or a tachnodal, where a curve* $C \subset \mathbb{R}^2$ *is called a tachnodal if it is diffeomorphic to the curve given by* $\{(u_1, u_2)|u_1^2 - u_2^4 = 0\}$.

(e) *for each* $\varepsilon > 0$*, there exit two points* $u_1, u_2 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2$*, neither of* u_1 *nor* u_2 *is an AdS-parabolic point and the tangent AdS-great-hyperboloids to* $M = \boldsymbol{X}(U)$ *at* u_1 *and* u_2 *are equal.*

Proof. By the Proposition 2.2.1, we have shown that u_0 is an AdS-parabolic point if and only if AdS-corank$(\boldsymbol{X}, u_0) \geq 1$. Since $n = 3$, we have AdS-corank$(\boldsymbol{X}, u_0) \leq 2$. Since AdS-height function germ H :

$(U \times H^3_1, (u_0, \boldsymbol{v}_0)) \longrightarrow \mathbb{R}$ can be considered as a generating family of the Legendrian embedding germ $\mathcal{L}$, h_{v_0} has only the A_k-type singularities ($k = 1, 2, 3$). This means that the corank of the Hessian matrix of the h_{v_0} at an AdS-parabolic point is 1. The assertion (2) also follows. For the same reason, the conditions $(3)\{(a), (b), (c)\}$(respectively, $(4)\{(a), (b), (c)\}$) are equivalent.

On the other hand, if the AdS-height function germ h_{v_0} has the A_2-type singularity, it is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 + u_2^3$. Since the $\mathcal{K}$-equivalence preserves the zero level sets, the tangent AdS-great-hyperboloidic indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is the ordinary cusp. The normal form for the A_3-type singularity is given by $\pm u_1^2 + u_2^4$, so the tangent AdS-great-hyperboloidic indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^4 = 0$. This means that the condition $(3)\{(d)\}$(respectively, $(4)\{(d)\}$) is also equivalent to the other conditions.

For the swallowtail point u_0, there is a self-intersection curve approaching u_0. On this curve, there are two distinct points u_1 and u_2 such that $\mathbb{T}(u_1) = \mathbb{T}(u_2)$. By Lemma 2.3.2, this means that the tangent AdS-great-hyperboloids to $M = \boldsymbol{X}(U)$ at u_1 and u_2 are equal. Since there are no other singularities in this case, the condition $(4)\{(e)\}$ characterizes a swallowtail point of $\mathbb{T}$. This completes the proof. $\square$

2.5. AdS-monge form

The notion of the Monge form of a surface in Euclidean 3-space is one of the powerful tools for the study of local properties of the surface from the view point of differential geometry. In this section we consider the analogous notion for a spacelike surface in H^3_1.

We now consider a function $f(u_1, u_2)$ with $f(0) = f_{u_i}(0) = 0$. Then we have a spacelike surface in H_1^3 defined by

$$\boldsymbol{X}_f(u_1, u_2) = (\sqrt{1 + u_1^2 + u_2^2 - f^2(u_1, u_2)}, f(u_1, u_2), u_1, u_2).$$

We can easily calculate $\boldsymbol{N}(0) = (0, 1, 0, 0)$; therefore $\mathbb{T}(0) = (0, 1, 0, 0)$. We call $\boldsymbol{X}_f$ a *Anti de Sitter Monge form* (briefly, *AdS-Monge form*). Then we have the following proposition.

Proposition 2.5.1 *Any spacelike surface in H_1^3 is locally given by the AdS-Monge form.*

Proof. Let $\boldsymbol{X} : U \longrightarrow H_1^3$ be a spacelike surface. We consider Lorentzian motion of H_1^3 which is a transitive action. Therefore, without loss of the generality, we assume that $p = \boldsymbol{X}(0) = (1, 0, 0, 0)$. We denote $M = \boldsymbol{X}(U)$, we have a basis $\{\boldsymbol{X}(0), \boldsymbol{N}(0), \boldsymbol{X}_{u_1}(0), \boldsymbol{X}_{u_2}(0)\}$ of $T_p\mathbb{R}_2^4$ such that $T_pM = \langle \boldsymbol{X}_{u_1}(0), \boldsymbol{X}_{u_2}(0)\rangle_{\mathbb{R}}$. Applying the Gram-Schmidt procedure we have a pseudo-orthonormal basis $\{\boldsymbol{X}(0), \boldsymbol{N}(0), \boldsymbol{e}_1, \boldsymbol{e}_2\}$ of $T_p\mathbb{R}_2^4$ such that $T_pM = \langle \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathbb{R}}$. In particular, $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is an orthonormal basis of T_pM. Since $p = (1, 0, 0, 0)$, T_pM is considered to be a subspace of ${}_0\mathbb{R}_1^3 = \{(0, x_1, x_2, x_3) | x_i \in \mathbb{R}\}$. By a rotation of the space ${}_0\mathbb{R}_1^3$, we might assume that $T_pM = \{(0, 0, u_1, u_2) | u_i \in \mathbb{R}\} \subset \mathbb{R}_2^4$. Then the germ (M, p) might be written in the form

$$(f_0(u_1, u_2), f(u_1, u_2), u_1, u_2)$$

with function germs $f_0(u_1, u_2), f(u_1, u_2)$. Since $M \subset H_1^3$, we have the relation

$$f_0(u_1, u_2) = \sqrt{1 + u_1^2 + u_2^2 - f^2(u_1, u_2)}.$$

Since we have $T_pM = \{(0, 0, u_1, u_2) | u_i \in \mathbb{R}\}$, the condition $f(0) = 0,\ f_{u_i}(0) = 0$ are automatically satisfied. $\square$

For the timelike vector $\boldsymbol{v}_0 = (0, 1, 0, 0)$, we consider the AdS-great-hyperboloid $\mathrm{AH}(\boldsymbol{v}_0, 0)$. Then we have the AdS-Monge form of $\mathrm{AH}(\boldsymbol{v}_0, 0)$:

$$\boldsymbol{a}(u_1, u_2) = (\sqrt{1 + u_1^2 + u_2^2}, 0, u_1, u_2).$$

Here, we can easily check the relation $\langle \boldsymbol{a}(u), \boldsymbol{v}_0 \rangle = 0$.

On the other hand, $\boldsymbol{a}(0) = (1, 0, 0, 0) = p$ and $\boldsymbol{a}_{u_i}(0)$ is equal to the x_{i+2}-axis for $i = 1, 2$. This means that $T_pM = T_p(\boldsymbol{a}(U))$. Therefore $\boldsymbol{a}(U)$ $=\mathrm{AH}(\boldsymbol{v}_0, 0)$ is the tangent AdS-great-hyperboloid of $M = \boldsymbol{X}_f(U)$ at $p = \boldsymbol{X}_f(0)$. It follows from this fact that the tangent AdS-great-hyperboloidic indicatrix of the AdS-Monge form germ $(\boldsymbol{X}_f, 0)$ is given as follows:

$$\boldsymbol{X}_f^{-1}(AH(\boldsymbol{v}_0, 0)) = \{(u_1, u_2) | f(u_1, u_2) = 0\}.$$

Since the height function of $\boldsymbol{X}_f$ at $\boldsymbol{v}_0$ is

$$h_{v_0}(u) = \langle \boldsymbol{X}_f(u), \boldsymbol{v}_0 \rangle = f(u_1, u_2),$$

we can calculate the Hessian matrix; then we have $\mathrm{Hess}(h_{v_0})(0) = \mathrm{Hess}(f)(0)$. Thus we conclude that AdS-corank$(\boldsymbol{X}_f, 0) = 2 - \mathrm{rankHess}(f)(0)$.

On the other hand, since $f(0) = f_{u_i}(0) = 0$, we may write

$$f(u_1, u_2) = \frac{1}{2}\bar{k_1}u_1^2 + \frac{1}{2}\bar{k_2}u_2^2 + g(u_1, u_2)$$

where $g \in \mathcal{M}_2^3$ and $\bar{k_1}, \bar{k_2}$ are eigenvalues of $(f_{u_1u_2}(0))$. Under this representation, we can easily calculate $\boldsymbol{X}_{f,u_1u_2}(0) = (\delta_{ij}, f_{u_1u_2}(0), 0, 0)$. It follows from this fact that

$$h_{ij}(0) = \langle \boldsymbol{N}(0), \boldsymbol{X}_{f,u_iu_j}(0) \rangle = f_{u_iu_j}(0) = \delta_{ij}\bar{k_i},$$

and

$$g_{ij}(0) = \langle \boldsymbol{X}_{f,u_i}(0), \boldsymbol{X}_{f,u_j}(0) \rangle = \delta_{ij}.$$

Therefore, we have $k_i(0) = \bar{k_i}$ and

$$K_{AdS}(0) = k_1(0)k_2(0) = \bar{k_1}\bar{k_2}.$$

The tangent AdS-great-hyperboloidic indicatrix is given by

$$\begin{aligned}\boldsymbol{X}_f^{-1}(AH(\boldsymbol{v}_0,0)) &= \{(u_1,u_2)| \pm \frac{1}{2}\bar{k}_1 u_1^2 \pm \frac{1}{2}\bar{k}_2 u_2^2 \pm g(u_1,u_2) = 0\} \\ &= \{(u_1,u_2)| \pm k_1(0)u_1^2 \pm k_2(0)u_2^2 \pm 2g(u_1,u_2) = 0\}.\end{aligned}$$

If we try to draw picture of the TAdS-Gauss image, it might be very hard to give a parameterization. However, by the AdS-Monge form of the tangent AdS-great-hyperboloidic indicatrix germ, we can easy to detect the type of singularities of the TAdS-Gauss image $\mathbb{T}$.

Example 2.5.1 *Consider the function given by*

$$f(u_1,u_2) = 2u_1^2 - 3u_2^3.$$

Then $\bar{k}_1 = 4$, $\bar{k}_2 = 0$. *We have* $k_1 = 4$, $k_2 = 0$, *so that the origin is an AdS- parabolic point. The tangent AdS-great-hyperboloidic indicatrix germ at the origin is the ordinary cusp. By Theorem 2.4.2,* $\mathbb{T}(0)$ *is the cuspidal edge.*

Example 2.5.2 *Consider the function given by*

$$f(u_1,u_2) = 2u_1^2 - 4u_2^4.$$

Then $\bar{k}_1 = 4$, $\bar{k}_2 = 0$. *We have* $k_1 = 4$, $k_2 = 0$, *so that the origin is an AdS- parabolic point. The tangent AdS-great-hyperboloidic indicatrix germ at the origin is the tachnodal. By Theorem 2.4.2,* $\mathbb{T}(0)$ *is the swallowtail.*

Part III. TIMELIKE SURFACES IN ANTI DE SITTER 3-SPACE

In this part we investigate the geometric properties of timelike surfaces in Anti de Sitter 3-space from the viewpoint of Legendrian singularity theory.

3.1. LOCAL DIFFERENTIAL GEOMETRY OF TIMELIKE SURFACES

Let $\boldsymbol{X}(U) = M$ be a timelike surfaces in Anti de Sitter 3-space. Since the embedding is timelike and $\boldsymbol{X}(u) \in H^3_1$, $\boldsymbol{N}(u)$ is spacelike. Therefore $\langle \boldsymbol{N}(u), \boldsymbol{N}(u) \rangle \equiv 1$. It follows that $\boldsymbol{X}(u) \pm \boldsymbol{N}(u) \in \Lambda^3 \cap N_pM$ and $\widetilde{\boldsymbol{X}(u) \pm \boldsymbol{N}}(u) \in T^2_1 \cap N_pM$. Thus we can define a map

$$\mathbb{G}^{\pm}_n : U \longrightarrow \Lambda^3 \text{ by } \mathbb{G}^{\pm}_n(u) = \boldsymbol{X}(u) \pm \boldsymbol{N}(u).$$

This map is analogous to the hyperbolic Gauss indicatrix of hypersurfaces in $H^n_+(-1)$ which was defined in [15]. Here, we call it the *Anti de Sitter nullcone Gauss image* (briefly, *AdS-nullcone Gauss image*) of $\boldsymbol{X}$(or M). We also define a map

$$\widetilde{\mathbb{G}^{\pm}_n} : U \longrightarrow T^2_1 \text{ by } \widetilde{\mathbb{G}^{\pm}_n}(u) = \widetilde{\boldsymbol{X}(u) \pm \boldsymbol{N}}(u) = \tfrac{1}{\xi(u)}\mathbb{G}^{\pm}_n(u),$$

where $\xi(u) = \pm\sqrt{(x_1(u) \pm n_1(u))^2 + (x_2(u) \pm n_2(u))^2}$. We call it the *Anti de Sitter torus Gauss map* (or, *AdS-torus Gauss map*) of $\boldsymbol{X}$.

We remak that the map $\mathbb{G}^{\pm}_n(u)$ was used by S. Lee [26] to study the timelike sufaces of constant mean curvature ± 1 in Anti de Sitter 3-space. He called $\mathbb{G}^{\pm}_n(u)$ the *hyperbolic Gauss map*. By a direct calculation we know that $\mathbb{G}^{\pm}_n$ is constant if and only if $\widetilde{\mathbb{G}^{\pm}_n}$ is constant.

It is easy to show that $\boldsymbol{N}_{u_i}$ $(i = 1, 2)$ are tangent vectors of M. Therefore we have a linear transformation $S^{\pm}_p = -d\mathbb{G}^{\pm}_n(u) = -(d\boldsymbol{X}(u) \pm d\boldsymbol{N}(u)) : T_pM \longrightarrow T_pM$ which is called the *Anti de Sitter null shape operator* (briefly, *AdS-null shape operator*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. Under the identification of U and M, the derivation $d\boldsymbol{X}(u)$ can be identified with the identity mapping id_{T_pM}, this means that $S_p^{\pm} = -d\mathbb{G}_n^{\pm}(u) = -(\mathrm{id}_{T_pM} \pm d\mathbb{N}(u))$. We have another linear mapping

$$d\widetilde{\mathbb{G}_n^{\pm}}(u) : T_pM \longrightarrow T_p\mathbb{R}^4_2 = T_pM \oplus N_pM.$$

If we consider the orthogonal projection $\pi^T : T_pM \oplus N_pM \longrightarrow T_pM$, then we have

$$\widetilde{S_p^{\pm}} = -(d\widetilde{\mathbb{G}_n^{\pm}}(u))^T = -\pi^T \circ d\widetilde{\mathbb{G}_n^{\pm}}(u) : T_pM \longrightarrow T_pM$$

and call it the *Anti de Sitter torus shape operator* (briefly, *AdS-torus shape operator*)of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We remark that $S_p^{\pm}$ (resp., $\widetilde{S_p^{\pm}}$) does not always have real eigenvalues. If the eigenvalues are real numbers, we denote it by $k_i^{\pm}$ (resp., $\widetilde{k_i^{\pm}}$) (for $i = 1, 2$).

We define $K^{\pm}_{AdSn}(u) = \det S_p^{\pm} = k_1^{\pm} \cdot k_2^{\pm}$ and $\widetilde{K^{\pm}_{AdSt}}(u) = \det\widetilde{S_p^{\pm}} = \widetilde{k_1^{\pm}} \cdot \widetilde{k_2^{\pm}}$. We respectively call $K^{\pm}_{AdSn}(u)$ the *Anti de Sitter null Gauss-Kronecker curvature* (briefly, *AdS-null G-K curvature*) and $\widetilde{K^{\pm}_{AdSt}}(u)$ the *Anti de Sitter torus Gauss-Kronecker curvature* (briefly, *AdS-torus G-K curvature*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We say that a point $p = \boldsymbol{X}(u)$ is a (*positive or negative*) *Anti de Sitter horospherical parabolic point* (briefly, *AdSh*$^{\pm}$-*parabolic point*) (resp. *positive or negative Anti de Sitter torus parabolic point, briefly, AdSt*$^{\pm}$*-parabolic point*) of $M = \boldsymbol{X}(U)$ if $K^{\pm}_{AdSn}(u) = 0$(resp. $\widetilde{K^{\pm}_{AdSt}}(u) = 0$). By a straightforward calculation we have the relation $S_p^{\pm} = \xi(u)\widetilde{S_p^{\pm}}$, so that we have $k_i^{\pm}(p) = \xi(u)\widetilde{k_i^{\pm}}(p)$ and $K^{\pm}_{AdSn}(u) = \xi^2(u)\widetilde{K^{\pm}_{AdSt}}(u)$. Then we have the following relations:

$$\begin{cases} k_i^{\pm}(p) = 0 \Longleftrightarrow \widetilde{k_i^{\pm}}(u) = 0 \\ K^{\pm}_{AdSn}(u) = 0 \Longleftrightarrow \widetilde{K^{\pm}_{AdSt}}(u) = 0. \end{cases}$$

We say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is an *umbilic point* if $S_p^{\pm} = k^{\pm}(p)id_{T_pM}$. We also say that $M = \boldsymbol{X}(U)$ is *totally umbilic* if all points on M are umbilic.

We now consider the geometric meaning of the AdS-nullcone Gauss image of a timelike surface. We have the following proposition.

Proposition 3.1.1 *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a timelike surface in Anti de Sitter 3-space. If the AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$ is constant, then the timelike surface $\boldsymbol{X}(U) = M$ is a part of a AdS-horosphere.*

Proof. We consider the set $V = \{\boldsymbol{y} \in \mathbb{R}^4_2 | \langle \boldsymbol{y}, \boldsymbol{X} \pm \boldsymbol{N} \rangle = -1\}$. Since $\mathbb{G}^{\pm}_n = \boldsymbol{X} \pm \boldsymbol{N}$ is constant, the set $V = HP(\mathbb{G}^{\pm}_n, -1)$ is a null hyperplane. We also have $\langle \boldsymbol{X}, \mathbb{G}^{\pm}_n \rangle \equiv -1$, so $\boldsymbol{X}(U) = M \subset V \cap H^3_1$. $\square$

We also have the following classification theorem on umbilic points.

Proposition 3.1.2 *Suppose that $M = \boldsymbol{X}(U)$ is totally umbilic. Then $k^{\pm}(p)$ is constant $k^{\pm}$. Under this condition, we have the following classification.*

(1) *Suppose* $k^{\pm} \neq 0$.

  (a) *If* $0 < |k^{\pm} + 1| < 1$, *then M is a part of a AdS-pseudohyperboloid;*

  (b) *If* $|k^{\pm} + 1| > 1$, *then M is a part of a AdS-pseudosphere;*

  (c) *If* $k^{\pm} = -1$, *then M is a part of a AdS-small pseudohyperboloid.*

(2) *Suppose* $k^{\pm} = 0$ *then M is a part of a AdS-horosphere.*

The proof is almost the same as that of Proposition 2.3 in [15], so that we omit it. We also call a point $p \in M$ the *Anti de Sitter horospherical point* (briefly, *AdS-horospherical point*) if $k^{\pm}_i(p) = 0$ $(i = 1, 2)$.

We now introduce the pseudo-Riemannian metric $ds^2 = \sum_{i,j=1}^{2} g_{ij} du_i du_j$ on $M = \boldsymbol{X}(U)$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$ for any $u \in U$. We also define the *Anti de Sitter null second fundamental invariant* by $h^{\pm}_{ij}(u) = \langle -(\mathbb{G}^{\pm}_n)_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$, *Anti de Sitter torus second fundamental invariant* by $\widetilde{h^{\pm}_{ij}}(u) = \langle -(\widetilde{\mathbb{G}^{\pm}_n})_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle = \frac{1}{\xi(u)} h^{\pm}_{ij}(u)$ for any $u \in U$. We can also show the following Weingarten formulas by exactly the same arguments as those of [9, 15, 23].

Proposition 3.1.3 *With the above notations the following hold*

(1) *The Anti de Sitter null Weingarten formula*:

$$(\mathbb{G}_n^{\pm})_{u_i} = -\sum_{j=1}^{2}(h^{\pm})_i^j \boldsymbol{X}_{u_j},$$

where $((h^{\pm})_i^j) = (h_{ik}^{\pm})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$.

(2) *The Anti de Sitter torus Weingarten formula*:

$$((\widetilde{\mathbb{G}_n^{\pm}})_{u_i})^T = \pi^T \circ (\widetilde{\mathbb{G}_n^{\pm}})_{u_i} = -\textstyle\sum_{j=1}^{2}(\widetilde{h^{\pm}})_i^j \boldsymbol{X}_{u_j} = -\dfrac{1}{\xi(u)}\displaystyle\sum_{j=1}^{2}(h^{\pm})_i^j \boldsymbol{X}_{u_j},$$

where $((\widetilde{h^{\pm}})_i^j) = (\widetilde{h_{ik}^{\pm}})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$. □

As a corollary of the above proposition, we have the following expression of the AdS-null G-K curvature and AdS-torus G-K curvature .

Corollary 3.1.4 *With the same notations as in the above Proposition, we have*:

$$K_{AdSn}^{\pm} = \frac{det(h_{ij}^{\pm})}{det(g_{ij})} = \xi^2\frac{det(\widetilde{h_{ij}^{\pm}})}{det(g_{ij})} = \xi^2\widetilde{K_{AdSt}^{\pm}}. \qquad \square$$

3.2. Height functions on timelike surfaces

In this section we define two families of functions on a timelike surface in Anti de Sitter 3-space which are useful for the study of singularities of AdS-nullcone Gauss image and AdS-torus Gauss map.

Let $\boldsymbol{X} : U \longrightarrow H_1^3$ be a timelike surface. We define a family of functions

$$H : U \times \Lambda^3 \longrightarrow \mathbb{R},\ H(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v}\rangle + 1.$$

We call H an *Anti de Sitter null height function* (or, *AdS-null height function*) on $M = \boldsymbol{X}(U)$. We denote the *Hessian matrix* of the AdS-null height function $h_{v_0}(u) = H(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(h_{v_0})(u_0)$. Then we have the following proposition.

Proposition 3.2.1 *Let $M = \boldsymbol{X}(U)$ be a timelike surface in H^3_1 and $H : U \times \Lambda^3 \longrightarrow \mathbb{R}$ be an AdS-null height function. Then we have the following:*

(1) *$H(u_0, \boldsymbol{v}) = \frac{\partial H}{\partial u_i}(u_0, \boldsymbol{v}) = 0$ (for $i = 1, 2$) if and only if $\boldsymbol{v} = \boldsymbol{X}(u_0) \pm \boldsymbol{N}(u_0) = \mathbb{G}^{\pm}_n(u_0)$;*

(2) *Let $\boldsymbol{v}^{\pm}_0 = \boldsymbol{X}(u_0) \pm \boldsymbol{N}(u_0)$, then $p = \boldsymbol{X}(u_0)$ is an AdSh$^{\pm}-$parabolic point if and only if* $\det\mathrm{Hess}\,(h_{v^{\pm}_0})(u_0) = 0$;

(3) *Let $\boldsymbol{v}^{\pm}_0 = \boldsymbol{X}(u_0) \pm \boldsymbol{N}(u_0)$, then $p = \boldsymbol{X}(u_0)$ is an AdS-horospherical point if and only if* $\mathrm{rank}\mathrm{Hess}(h_{v^{\pm}_0})(u_0) = 0$.

Proof. (1) Since $\{\boldsymbol{X}(u), \boldsymbol{N}(u), \boldsymbol{X}_{u_1}(u), \boldsymbol{X}_{u_2}(u)\}$ is a basis of the vector space $T_p\mathbb{R}^4_2$ where $p = \boldsymbol{X}(u)$, there exist real numbers $\lambda, \eta, \alpha_1, \alpha_2$ such that

$$\boldsymbol{v} = \lambda\boldsymbol{X}(u) + \eta\boldsymbol{N}(u) + \alpha_1\boldsymbol{X}_{u_1}(u) + \alpha_2\boldsymbol{X}_{u_2}(u).$$

Therefore $H(u, \boldsymbol{v}) = 0$ if and only if $\lambda = -\langle \boldsymbol{X}(u), \boldsymbol{v}\rangle = 1$. Since $0 = \frac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{v}\rangle = \sum_{j=1}^{2} g_{ij}\alpha_i$ and (g_{ij}) is non-degenerate, we have $\alpha_i = 0$ (for $i = 1, 2$). Therefore we have $\boldsymbol{v} = \boldsymbol{X}(u) + \eta\boldsymbol{N}(u)$. From the fact that $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$, we have $\eta = \pm 1$.

(2) By definition, we have

$$\mathrm{Hess}(h_{v^{\pm}_0})(u_0) = (\langle \boldsymbol{X}_{u_iu_j}(u_0), \mathbb{G}^{\pm}_n(u_0)\rangle) = (-\langle \boldsymbol{X}_{u_i}(u_0), \mathbb{G}^{\pm}_{n\,u_j}(u_0)\rangle).$$

From the AdS-null Weingarten formula, we have

$$-\langle \boldsymbol{X}_{u_i}, (\mathbb{G}^{\pm}_n)_{u_j}\rangle = \sum_{\alpha=1}^{2}(h^{\pm})^{\alpha}_{i}\langle \boldsymbol{X}_{u_\alpha}, \boldsymbol{X}_{u_j}\rangle = \sum_{\alpha=1}^{2}(h^{\pm})^{\alpha}_{i}g_{\alpha j} = h^{\pm}_{ij}.$$

Therefore we have

$$K^{\pm}_{AdSn}(u_0) = \frac{\det(h^{\pm}_{i,j}(u_0))}{\det(g_{ij}(u_0))} = \frac{\det\mathrm{Hess}(h_{v^{\pm}_0})(u_0)}{\det(g_{ij}(u_0))}.$$

Then assertion (2) is satisfied.

(3) By the AdS-null Weingarten formula, p is an umbilic point if and only if there exists an orthogonal matrix A such that $A^t((h^{\pm})_i^l)A = k^{\pm}I$. Therefore, we have $((h^{\pm})_i^l) = Ak^{\pm}IA^t = k^{\pm}I$. Then we have

$$\mathrm{Hess}(h_{v_0^{\pm}})(u_0) = (h_{ij}^{\pm}(u_0)) = ((h^{\pm})_i^l(u_0))(g_{lj}(u_0)) = k^{\pm}(g_{ij}(u_0)).$$

Thus, $p = \boldsymbol{X}(u_0)$ is a AdS-horospherical point $\Leftrightarrow$ $\mathrm{rankHess}(h_{v_0^{\pm}})(u_0) = 0$.
$\square$

As an application of the above proposition, we have the following direct corollary.

Corollary 3.2.2 *Let $H : U \times \Lambda^3 \longrightarrow \mathbb{R}$, with $H(u, \boldsymbol{v}) = h_v(u)$ be an AdS-null height function on a timelike surface $M = \boldsymbol{X}(U)$ and $\mathbb{G}_n^{\pm}$ be the AdS-nullcone Gauss image, $p = \boldsymbol{X}(u)$. Suppose $\boldsymbol{v}^{\pm} = G_n^{\pm}(u)$, then the following conditions are equivalent*:

(1) *$p \in M$ is a degenerate singular point of AdS-null height function $h_{v^{\pm}}$*

(2) *$p \in M$ is a singular point of AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$;*

(3) *$K_{AdSn}^{\pm}(u) = 0$.* $\square$

We can also define another family of functions

$$\widetilde{H} : U \times T_1^2 \longrightarrow \mathbb{R},\ \widetilde{H}(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle.$$

We call $\widetilde{H}$ an *Anti de Sitter torus height function* (briefly, *AdS-torus height function*) on $\boldsymbol{X}$. We denote the *Hessian matrix* of the AdS-torus height function $\widetilde{h_{v_0}}(u) = \widetilde{H}(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(\widetilde{h_{v_0}})(u_0)$. We remark that this family satisfies the same properties as those stated in Proposition 3.2.1 and Corollary 3.2.2.

On the other hand, we can naturally interpret the AdS-nullcone Gauss image of M as a Legendrian map by the Legendrian duality. The argument is almost the same as that of Proposition 2.3 in §2.3, so that we omit the details.

First, we have the following principle property with respect to the AdS-null height function H.

Proposition 3.2.3 *The AdS-null height function $H : U \times \Lambda^3 \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces $h_v^{-1}(0)_{v \in \Lambda^3}$.*

Proof. For any $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \Lambda^3$, we have $\boldsymbol{v} \neq \mathbf{0}$. Without loss of the generality, we might assume that $v_1 > 0$, then $v_1 = \sqrt{v_3^2 + v_4^2 - v_2^2}$. So that $H(u, \boldsymbol{v}) = -x_1(u)\sqrt{1 + v_3^2 + v_4^2 - v_2^2} - x_2(u)v_2 + x_3(u)v_3 + x_4(u)v_4 + 1$, where $\boldsymbol{X}(u) = (x_1(u), x_2(u), x_3(u), x_4(u))$. We have to prove the mapping

$$\Delta^* H = (H, \frac{\partial H}{\partial u_1}, \frac{\partial H}{\partial u_2})$$

is non-singular at any point. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$\begin{pmatrix} \langle \boldsymbol{X}_{u_1}, \boldsymbol{v} \rangle & \langle \boldsymbol{X}_{u_2}, \boldsymbol{v} \rangle & x_1 \frac{v_2}{v_1} - x_2 & -x_1 \frac{v_3}{v_1} + x_3 & -x_1 \frac{v_4}{v_1} + x_4 \\ \langle \boldsymbol{X}_{u_1 u_1}, \boldsymbol{v} \rangle & \langle \boldsymbol{X}_{u_1 u_2}, \boldsymbol{v} \rangle & x_{1u_1} \frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1} \frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1} \frac{v_4}{v_1} + x_{4u_1} \\ \langle \boldsymbol{X}_{u_2 u_1}, \boldsymbol{v} \rangle & \langle \boldsymbol{X}_{u_2 u_2}, \boldsymbol{v} \rangle & x_{1u_2} \frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2} \frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2} \frac{v_4}{v_1} + x_{4u_2} \end{pmatrix}.$$

We claim that it will suffice to show that the determinant of the matrix

$$A = \begin{pmatrix} x_1 \frac{v_2}{v_1} - x_2 & -x_1 \frac{v_3}{v_1} + x_3 & -x_1 \frac{v_4}{v_1} + x_4 \\ x_{1u_1} \frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1} \frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1} \frac{v_4}{v_1} + x_{4u_1} \\ x_{1u_2} \frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2} \frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2} \frac{v_4}{v_1} + x_{4u_2} \end{pmatrix},$$

does not vanish at $(u, \boldsymbol{v}) \in \Delta^* H^{-1}(\mathbf{0})$. In this case, $\boldsymbol{v} = \mathbb{G}_n^{\pm}(u)$ and we denote

$$\boldsymbol{b}_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ x_{1u_2} \end{pmatrix}, \ \boldsymbol{b}_2 = \begin{pmatrix} x_2 \\ x_{2u_1} \\ x_{2u_2} \end{pmatrix}, \ \boldsymbol{b}_3 = \begin{pmatrix} x_3 \\ x_{3u_1} \\ x_{3u_2} \end{pmatrix}, \ \boldsymbol{b}_4 = \begin{pmatrix} x_4 \\ x_{4u_1} \\ x_{4u_2} \end{pmatrix}.$$

Then we have

$$
\begin{aligned}
\det A = & -\frac{v_1}{v_1}\det(\boldsymbol{b}_2,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4) + \frac{v_2}{v_1}\det(\boldsymbol{b}_1,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4) \\
& - \frac{v_3}{v_1}\det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_4) + \frac{v_4}{v_1}\det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_3).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} = & (-\det(\boldsymbol{b}_2,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4),\ \det(\boldsymbol{b}_1,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4), \\
& \det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_4),\ -\det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_3)).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\det A &= \langle (-\frac{v_1}{v_1}, -\frac{v_2}{v_1}, -\frac{v_3}{v_1}, -\frac{v_4}{v_1}), \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \rangle \\
&= -\frac{1}{v_1} \langle \mathbb{G}_n^{\pm}, \| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \| \boldsymbol{N} \rangle \\
&= \mp \frac{\| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \|}{v_1} \neq 0.
\end{aligned}
$$

□

We now define a map

$$
\mathcal{L}_2 : U \longrightarrow \Delta_2,\ \mathcal{L}_2(u) = (\boldsymbol{X}(u), \mathbb{G}_n^{\pm}(u)).
$$

Since $\langle \boldsymbol{X}(u), \mathbb{G}_n^{\pm}(u) \rangle = -1, \langle d\boldsymbol{X}(u), \mathbb{G}_n^{\pm}(u) \rangle = 0$, by Theorem 1.1, $\mathcal{L}_2$ is a Legendrian embedding. We now show that H is a generating family of $\mathcal{L}_2(U) \subset \Delta_2$.

Proposition 3.2.4 *For any timelike surfaces $\boldsymbol{X} : U \longrightarrow H_1^3$, the AdS-null height function $H : U \times \Lambda^3 \longrightarrow \mathbb{R}$ of $\boldsymbol{X}$ is a generating family of the Legendrian embedding $\mathcal{L}_2$.*

Proof. We consider coordinate neighborhoods $V_1^+ = \{\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in H_1^3 \mid v_1 > 0\}$ and $W_1^+ = \{\boldsymbol{w} = (w_1, w_2, w_3, w_4) \in \Lambda^3 \mid w_1 > 0\}$. Then $V_1^+ \times W_1^+$ is one of the local coordinate of $H_1^3 \times \Lambda^3$. We now define a mapping $\Phi : \Delta_2 \cap (V_1^+ \times W_1^+) \longrightarrow PT^*\Lambda^3|W_1^+$ by

$$
\Phi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{w}, [(w_2v_1 - w_1v_2) : (-w_3v_1 + w_1v_3) : (-w_4v_1 + w_1v_4)])
$$

Let $(v_2, v_3, v_4, [\xi_2 : \xi_3 : \xi_4])$ be homogeneous coordinates of $PT^*\Lambda^3|W_1^+ \equiv W_1^+ \times P(\mathbb{R}^{n-1})^*$. We have the canonical contact form $\alpha = \sum_{i=2}^4 (\xi_i/\xi_j)dw_i$ on $PT^*\Lambda^3$ over $W_1^+ \times U_j$, where $U_j = \{[\xi] \,|\xi_j \neq 0\}$. It follows that

$$\begin{aligned}\Phi^*\alpha &= \frac{\pm w_1}{v_j w_1 - v_1 w_j}\left(-\sum_{i=1}^{2} v_i dw_i + \sum_{i=3}^{4} v_i dw_i\right) \\ &= \frac{\pm w_1}{v_j w_1 - v_1 w_j}\langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_2 = \frac{\pm w_1}{v_j w_1 - v_1 w_j}\theta_{22},\end{aligned}$$

where $\pm$ depends on j of $\Phi^{-1}(W_1^+ \times U_j)$. Since

$$\Delta_1 \cap (V_1^+ \times W_3^+) = \bigcup_{j=2}^{4} \Phi^{-1}(V_1^+ \times U_j),$$

θ_{11} is a contact form on $\Delta_1 \cap (V_1^+ \times W_3^+)$ such that Φ is a contact morphism. We also have the similar calculation as the above on the other coordinate neighborhoods. Since AdS-null height function H is a Morse family, we have a Legendrian immersion

$$\mathcal{L}_H : \Sigma_*(H) \mid (U \times W_1^+) \longrightarrow PT^*\Lambda^3 \mid W_1^+$$

defined by

$$\mathcal{L}_H(u, \boldsymbol{w}) = (\boldsymbol{w}, [\frac{\partial H}{\partial w_2} : \frac{\partial H}{\partial w_3} : \frac{\partial H}{\partial w_4}]).$$

By Proposition 3.2.1, we have

$$\Sigma_*(H) = \{(u, \mathbb{G}_n^\pm(u)) \in U \times \Lambda^3 \mid u \in U\}.$$

Since $\boldsymbol{w} = \mathbb{G}_n^\pm(u)$ and $w_1 = \sqrt{-w_2^2 + w_3^2 + w_4^2}$, we have

$$\frac{\partial H}{\partial w_2}(u, \mathbb{G}_n^\pm(u)) = x_1(u)\frac{w_2(u)}{w_1(u)} - x_2(u),$$

$$\frac{\partial H}{\partial w_3}(u, \mathbb{G}_n^\pm(u)) = x_3(u) - x_1(u)\frac{w_3(u)}{w_1(u)},$$

$$\frac{\partial H}{\partial w_4}(u, \mathbb{G}_n^\pm(u)) = x_4(u) - x_1(u)\frac{w_4(u)}{w_1(u)},$$

where $\boldsymbol{X} = (x_1, x_2, x_3, x_4)$ and $\mathbb{G}_n^\pm = (w_1, w_2, w_3, w_4)$. We now define a smooth mapping $\mathcal{G}^{\pm}: U \longrightarrow PT^*H^3_1$ by

$$\mathcal{G}^{\pm}(u) = (\mathbb{G}_n(u),\ [(x_1(u)w_2(u) - x_2(u)w_1(u)) : (-x_1(u)w_3(u) + x_3(u)w_1(u)) : (-x_1(u)w_4(u) + x_4(u)w_1(u))]).$$

It follows that

$$\mathcal{L}_H(u, \mathbb{G}^{\pm}_n(u)) = (\mathbb{G}^{\pm}_n(u),\ [x_1(u)w_2(u) - x_2(u)w_1(u) : - x_1(u)w_3(u) + x_3(u)w_1(u) : -x_1(u)w_4(u) + x_4(u)w_1(u)]) = \mathcal{G}^{\pm}(u).$$

Therefore we have $\Phi \circ \mathcal{L}_2(u) = \mathcal{G}^{\pm}(u)$ on W^+_1. We also have the same relation as the above on the other local coordinates. This means that H is a generating family of $\mathcal{L}_2(U) \subset \Delta_2$. $\square$

Therefore we conclude that the AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$ can be regarded as a Legendrian map and $\mathbb{G}^{\pm}_n(U)$ can be regarded as a wave front set of $\mathcal{L}_2$.

3.3. The AdS-torus cylindrical pedals of timelike surfaces

In this section we consider a surface associate to $M = \boldsymbol{X}(U)$, whose singular points set is diffeomophism to those of AdS-nullcone Gauss image. We can use this surface to investigate the relationship between the AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$ and the AdS-Gauss map $\widetilde{\mathbb{G}^{\pm}_n}$ of a timelike surface in the Anti de Sitter 3−space. For any timelike surface $\boldsymbol{X}: U \longrightarrow H^3_1$, we define a smooth mapping $ACP_M : U \longrightarrow T^2_1 \times \mathbb{R}^*$ by

$$ACP_M(u) = (\widetilde{\mathbb{G}^{\pm}_n}(u), -\langle \boldsymbol{X}(u), \widetilde{\mathbb{G}^{\pm}_n}(u)\rangle) = (\widetilde{\mathbb{G}^{\pm}_n}(u), \tfrac{1}{\xi(u)}).$$

We call it the *AdS-torus cylindrical pedal* of $M = \boldsymbol{X}(U)$, where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. We define a diffeomorphism $\phi : T^2_1 \times \mathbb{R}^* \longrightarrow \Lambda^3$ by $\phi(\boldsymbol{v}, \lambda) = \lambda^{-1}\boldsymbol{v}$. It is easy to check that $\phi(ACP_M(u)) = \mathbb{G}^{\pm}_n(u)$, this means that the singular points sets of $\mathbb{G}^{\pm}_n$ and ACP_M are diffeomorphism.

We now consider a family functions $\overline{H} : U \times T^2_1 \times \mathbb{R}^* \longrightarrow \mathbb{R}$ defined by

$$\overline{H}(u, \boldsymbol{v}, \lambda) = \langle \boldsymbol{X}(u), \boldsymbol{v}\rangle + \lambda = \widetilde{H}(u, \boldsymbol{v}) + \lambda,$$

we call it the *extended AdS-torus height function* on $M = \boldsymbol{X}(U)$. By the similar calculation to the proof of Proposition 3.1(1), we have

$$\mathcal{D}_{\overline{H}} = \{(\widetilde{\mathbb{G}_n^{\pm}}(u), \tfrac{1}{\xi(u)})|u \in U\} = \{ACP_M(u)|u \in U\}.$$

On the other hand, we consider the canonical projection $\pi_1 : T_1^2 \times \mathbb{R}^* \longrightarrow T_1^2$. Then we have $\pi_1|\mathcal{D}_{\overline{H}}$ can be identified with the AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$ of X. Since

$$\mathbb{G}_n^{\pm}(u) = -\frac{1}{\langle \boldsymbol{X}(u), \widetilde{\mathbb{G}_n^{\pm}}(u)\rangle}\widetilde{\mathbb{G}_n^{\pm}}(u) = \xi(u)\widetilde{\mathbb{G}_n^{\pm}}(u),$$

we have $\phi(\mathcal{D}_{\overline{H}}) = \{\mathbb{G}_n^{\pm}(u)|u \in U\} = \mathcal{D}_H$. Therefore, we may say that the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ is a *lift* of the AdS-Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$. In fact, we also have

$$\Sigma_*(\overline{H}) = \{(u,\ \widetilde{\mathbb{G}_n^{\pm}}(u),\ -\langle \boldsymbol{X}(u), \widetilde{\mathbb{G}_n^{\pm}}(u)\rangle)|u \in U\}.$$

We remark that similar discussions apply to the extended AdS-torus height function $\overline{H}$ and AdS-torus height function $\widetilde{H}$, we have $\overline{H}$ and $\widetilde{H}$ are Morse family.

On the other hand, for any $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in T_1^2$, we consider a coordinate neighborhood

$$U_{24}^{+} = \{\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in T_1^2 | v_2 > 0 \text{ and } v_4 > 0\},$$

then

$$\overline{H}(u, \boldsymbol{v}, \lambda) = \widetilde{H}(u, \boldsymbol{v}) + \lambda = -x_1v_1 - x_2\sqrt{1 - v_1^2} + x_3v_3 + x_4\sqrt{1 - v_3^2} + \lambda.$$

We now consider smooth mappings $\mathcal{L}_{\overline{H}} : \widetilde{\mathbb{G}_n^{\pm}}^{-1}(U_{24}^{+}) \longrightarrow T^*(T_1^2) \times \mathbb{R}^*$ defined by

$$\mathcal{L}_{\overline{H}}(u) = (\widetilde{\mathbb{G}_n^{\pm}}(u),\ [\frac{\partial \overline{H}}{\partial v_1} : \frac{\partial \overline{H}}{\partial v_3} : \frac{\partial \overline{H}}{\partial \lambda}],\ \frac{1}{\xi(u)}) = (\widetilde{\mathbb{G}_n^{\pm}}(u),\ \frac{\partial \overline{H}}{\partial v_1},\ \frac{\partial \overline{H}}{\partial v_3},\ \frac{1}{\xi(u)})$$

and $L_{\widetilde{H}} : \widetilde{\mathbb{G}_n^{\pm}}^{-1}(U_{24}^{+}) \longrightarrow T^*(T_1^2)$ defined by

$$L_{\widetilde{H}}(u) = (\widetilde{\mathbb{G}_n^{\pm}}(u),\ \frac{\partial \widetilde{H}}{\partial v_1},\ \frac{\partial \widetilde{H}}{\partial v_3}) = (\widetilde{\mathbb{G}_n^{\pm}}(u),\ \frac{\partial \overline{H}}{\partial v_1},\ \frac{\partial \overline{H}}{\partial v_3}).$$

According by these definitions we have $\mathcal{L}_{\overline{H}}$ is a Legendrian embedding whose generating family is the extended AdS-torus height function $\overline{H}$ and $L_{\widetilde{H}}$ is a Lagrangian embedding whose generating family is the AdS-torus height function $\widetilde{H}$. The details on Lagrangian singularities can be found in [1, 45]. We now consider the canonical projection

$$\pi : T^*(T_1^2) \times \mathbb{R}^* \longrightarrow T^*(T_1^2),\ \pi(\boldsymbol{v}, \lambda) = \boldsymbol{v},$$

then $\pi(\mathcal{L}_{\overline{H}}) = L_{\widetilde{H}}$. We remark that if we adopt other local coordinates on T_1^2, exactly the same results hold. Therefore we have the following proposition.

Proposition 3.3.1 *Under the same assumptions as in the above arguments, we have the following:*

(1) *The AdS-Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$ is a Lagrangian map. The corresponding Lagrangian embedding $L_{\widetilde{H}}$ is called the Lagrangian lift of the AdS-Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$;*

(2) *The Legendrian lift $\mathcal{G}^{\pm}$ of the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ is a covering of the Lagrangian lift $L_{\widetilde{H}}$ of the AdS-Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$.*

Proof. The assertion (1) follows from the above arguments.

On the other hand, for any $\boldsymbol{v} \in T_1^2$, without loss of the generality, we can assume that $v_2 > 0$ and $v_4 > 0$. Then we have $v_2 = \sqrt{1 - v_1^2}$, $v_4 = \sqrt{1 - v_3^2}$, so we can regard (v_1, v_3) as the coordinate system of T_1^2. Therefore, the homogeneous coordinates of $PT^*(T_1^2 \times \mathbb{R}^*)$ can be expressed as $(v_1, v_3, \lambda, [\varsigma_1 : \varsigma_2 : \varsigma])$. Moreover, if $\varsigma \neq 0$, we have

$$(v_1, v_3, \lambda, [\varsigma_1 : \varsigma_2 : \varsigma]) = (v_1, v_3, \lambda, [\frac{\varsigma_1}{\varsigma} : \frac{\varsigma_2}{\varsigma} : 1]),$$

so that we can adopt the corresponding affine coordinates $(v_1, v_3, \lambda, \rho_1, \rho_2)$, where $\rho_i = \varsigma_1/\varsigma$. By the above argument we can naturally regard $T^*(T_1^2) \times$

$\mathbb{R}^*$ as the affine part of $PT^*(T_1^2 \times \mathbb{R}^*)$. We also have the following relation:

$$H \circ (id_U \times \phi)(u, \boldsymbol{v}, \lambda) = H(u, \lambda^{-1}\boldsymbol{v}) = \lambda^{-1}\overline{H}(u, \boldsymbol{v}, \lambda).$$

This means that $H \circ (id_U \times \phi)$ and $\overline{H}$ are $\mathcal{C}$-equivalent in the sense of Mather[28]. So that these generating families correspond to the same Legendrian submanifold (cf., [1, 45]. Then we have a unique contact diffeomorphism $\Phi : PT^*(T_1^2 \times \mathbb{R}^*) \longrightarrow PT^*\Lambda^3$ covering $\phi : T_1^2 \times \mathbb{R}^* \longrightarrow \Lambda^3$ such that $\Phi \circ \mathcal{L}_{\overline{H}} = \mathcal{G}^{\pm}$. Therefore, $\mathcal{G}^{\pm}$ is a covering of $L_{\widetilde{H}}$. $\square$

3.4. Contact with ads-horospheres

In this section we consider the geometric meaning of the singularities of the AdS-nullcone Gauss image of a timelike surface $M = \boldsymbol{X}(U)$ in H_1^3. We consider the contact of timelike surfaces with AdS-horosphere type surfaces.

We now consider a function $\mathcal{H}$: $H_1^3 \times \Lambda^3 \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 1$. For any $\boldsymbol{v}_0 \in \Lambda^3$, we denote $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v}_0)$ and we define the AdS-horosphere by $\mathfrak{h}_{v_0}^{-1}(0) = H_1^3 \cap HP(\boldsymbol{v}_0, -1)$. We write $AH(\boldsymbol{v}_0, -1) = H_1^3 \cap HP(\boldsymbol{v}_0, -1)$. For any $u_0 \in U$, we consider the null vector $\boldsymbol{v}_0^{\pm} = \mathbb{G}_n^{\pm}(u_0)$. Then we have

$$\mathfrak{h}_{v_0^{\pm}} \circ \boldsymbol{X}(u_0) = \mathcal{H} \circ (\boldsymbol{X} \times id_{\Lambda^3})(u_0, \boldsymbol{v}_0) = H(u_0, \mathbb{G}_n^{\pm}(u_0)) = 0.$$

We also have relations

$$\frac{\partial \mathfrak{h}_{v_0^{\pm}} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, \mathbb{G}_n^{\pm}(u_0)) = 0,$$

for $i = 1, 2$. This means that the AdS-horosphere $AH(\boldsymbol{v}_0^{\pm}, -1)$ is tangent to $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$. In this case, we call $AH(\boldsymbol{v}_0^{\pm}, -1)$ the *tangent AdS-horosphere* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$ (or, u_0), which we write $AH^{\pm}(\boldsymbol{X}, u_0)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be null vectors. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent, then $HP(\boldsymbol{v}_1, -1)$ and $HP(\boldsymbol{v}_2, -1)$ are parallel. Therefore, we say that AdS-horosphere $AH(\boldsymbol{v}_1, -1)$ and $AH(\boldsymbol{v}_2, -1)$ are *parallel*, if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent. Then we have the following lemma.

Lemma 3.4.1 *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a timelike surface. Consider two points $u_1, u_2 \in U$. Then we have the following assertions*:

(1) $\mathbb{G}^{\pm}_n(u_1) = \mathbb{G}^{\pm}_n(u_2)$ *if and only if* $\mathrm{AH}^{\pm}(\boldsymbol{X}, u_1) = \mathrm{AH}^{\pm}(\boldsymbol{X}, u_2)$.

(2) $\widetilde{\mathbb{G}^{\pm}_n}(u_1) = \widetilde{\mathbb{G}^{\pm}_n}(u_2)$ *if and only if* $\mathrm{AH}^{\pm}(\boldsymbol{X}, u_1)$ *and* $\mathrm{AH}^{\pm}(\boldsymbol{X}, u_2)$ *are parallel.* $\square$

We now consider the contact of M with tangent AdS-horosphere at $p \in M$ as an application of Legendrian singularity theory. Let $\mathbb{G}^{\pm}_{n\,i} : (U, u_i) \longrightarrow (\Lambda^3, \boldsymbol{v}^{\pm}_i)$ (for $i = 1, 2$) be AdS-nullcone Gauss image germs of timelike surface germs $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$. We say that $\mathbb{G}^{\pm}_{n\,1}$ and $\mathbb{G}^{\pm}_{n\,2}$ are $\mathcal{A}$-*equivalent* if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (H^3_1, \boldsymbol{v}^{\pm}_1) \longrightarrow (H^3_1, \boldsymbol{v}^{\pm}_2)$ such that $\Phi \circ \mathbb{G}^{\pm}_{n\,1} = \mathbb{G}^{\pm}_{n\,2} \circ \phi$. Suppose the regular set of $\mathbb{G}^{\pm}_{n\,i}$ is dense in (U, u_i) for each $i = 1, 2$. It follows from Proposition 2.3.3 that $\mathbb{G}^{\pm}_{n\,1}$ and $\mathbb{G}^{\pm}_{n\,2}$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian embedding germs $\mathcal{G}^{\pm}_1 : (U, u_1) \longrightarrow (\Delta_1, \mathbf{z}_1)$ and $\mathcal{G}^{\pm}_2 : (U, u_2) \longrightarrow (\Delta_1, \mathbf{z}_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are P-$\mathcal{K}$ equivalent by Theorem 2.3.4. Here, $H_i : (U \times \Lambda^3, (u_i, \boldsymbol{v}^{\pm}_i)) \longrightarrow \mathbb{R}$ is the corresponding AdS-null height function germ of $\boldsymbol{X}_i$.

On the other hand, we denote $h_{i,v^{\pm}_i} = H_i(u, \boldsymbol{v}^{\pm}_i)$; then we have $h_{i,v^{\pm}_i}(u) = \mathfrak{h}_{v^{\pm}_i} \circ \boldsymbol{X}_i(u)$. By Theorem 2.3.1,

$$K(\boldsymbol{X}_1(U), \mathrm{AH}^{\pm}(\boldsymbol{X}_1, u_1), \boldsymbol{v}^{\pm}_1) = K(\boldsymbol{X}_2(U), \mathrm{AH}^{\pm}(\boldsymbol{X}_2, u_2), \boldsymbol{v}^{\pm}_2)$$

if and only if $h_{1,v^{\pm}_1}$ and $h_{2,v^{\pm}_2}$ are $\mathcal{K}$-equivalent. Therefore, we can apply the above arguments to our situation. We denote by $Q^{\pm}(\boldsymbol{X}, u_0)$ the local ring of the function germ $h_{\boldsymbol{v}_0^\pm} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{v}_0^\pm = \mathbb{G}_n^\pm(u_0)$. We remark that we can write the local ring explicitly as follows:

$$Q^\pm(\boldsymbol{X}, u_0) = \frac{C^\infty_{u_0}(U)}{\langle\langle \boldsymbol{X}(u), \mathbb{G}_n^\pm(u_0)\rangle + 1\rangle_{C^\infty_{u_0}(U)}}$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathcal{M}_{u_0}(U)$.

Theorem 3.4.2 *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$ (for $i = 1, 2$) be timelike surface germs such that the corresponding Legendrian embedding germs $\mathcal{G}_i^\pm : (U, u_i) \longrightarrow (\Delta_1, \boldsymbol{z}_i)$ are Legendrian stable. Then the following conditions are equivalent:*

(1) *AdS-nullcone Gauss image germs $\mathbb{G}^\pm_{n\,1}$ and $\mathbb{G}^\pm_{n\,2}$ are $\mathcal{A}$-equivalent;*

(2) *H_1 and H_2 are $P{-}\mathcal{K}$-equivalent;*

(3) *$h_{1,v_1^\pm}$ and $h_{2,v_2^\pm}$ are $\mathcal{K}$-equivalent;*

(4) $K(\boldsymbol{X}_1(U), AH^\pm(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1^\pm) = K(\boldsymbol{X}_2(U), AH^\pm(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2^\pm)$

(5) *$Q^\pm(\boldsymbol{X}_1, u_1)$ and $Q^\pm(\boldsymbol{X}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.* □

For a timelike surface germ $\boldsymbol{X} : (U, u_0) \longrightarrow (H^3_1, \boldsymbol{X}(u_0))$, we call $\boldsymbol{X}^{-1}(AH(\mathbb{G}_n^\pm(u_0), -1), u_0)$ the *tangent Anti de Sitter horospherical indicatrix germ* (briefly, *tangent AdS-horospherical indicatrix germ*) of $\boldsymbol{X}$. In general we have the following proposition:

Proposition 3.4.3 *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$ (for $i = 1, 2$) be timelike surface germs such that their AdSh$^\pm$-parabolic sets have no interior points as subspaces of U. If AdS-nullcone Gauss image germs $\mathbb{G}^\pm_{n\,1}$ and $\mathbb{G}^\pm_{n\,2}$ are $\mathcal{A}$-equivalent, then*

$$K(\boldsymbol{X}_1(U), AH^\pm(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1^\pm) = K(\boldsymbol{X}_2(U), AH^\pm(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2^\pm).$$

In this case, $\boldsymbol{X}_1^{-1}(AH(\mathbb{G}^\pm_{n\,1}(u_1), -1), u_1)$ and $\boldsymbol{X}_2^{-1}(AH(\mathbb{G}^\pm_{n\,2}(u_2), -1), u_2)$

are diffeomorphic as set germs. □

From the above proposition, the diffeomorphism type of the tangent AdS-horospherical indicatrix germ is an invariant of $\mathcal{A}$-classification of the AdS-nullcone Gauss image germ of $\boldsymbol{X}$. Moreover, we can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for a function germ. The local ring of a function is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension of a function germ is a numerical $\mathcal{K}$-invariant of function germs. We denote

$$\mathrm{Ah}^{\pm}\text{-ord}(\boldsymbol{X}, u_0) = \dim \frac{C^{\infty}_{u_0}(U)}{\langle h_{v_0^{\pm}}(u_0), \partial h_{v_0^{\pm}}(u_0)/\partial u_i \rangle_{C^{\infty}_{u_0}(U)}},$$

where $\boldsymbol{v}_0^{\pm} = \mathbb{G}_n^{\pm}(u_0)$. Usually $\mathrm{Ah}^{\pm}\text{-ord}(\boldsymbol{X}, u_0)$ is called the $\mathcal{K}$-codimension of $h_{v_0^{\pm}}$. However, We call it the *order of contact with tangent AdS-horosphere* at $\boldsymbol{X}(u_0)$. We also have the notion of *corank* of function germs:

$$\mathrm{Ah}^{\pm}\text{-corank}(\boldsymbol{X}, u_0) = 2 - \mathrm{rankHess}(h_{v_0^{\pm}})(u_0),$$

By Proposition 3.2.1, $\boldsymbol{X}(u_0)$ is an $\mathrm{AdSh}^{\pm}$-parabolic point if and only if $\mathrm{Ah}^{\pm}$-corank $(\boldsymbol{X}, u_0) \geq 1$ and $\boldsymbol{X}(u_0)$ is an AdS-horospherical point if and only if $\mathrm{Ah}^{\pm}$-corank $(\boldsymbol{X}, u_0) = 2$. On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \boldsymbol{a}) \longrightarrow \mathbb{R}$ has the A_k-singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $\mathrm{Ah}^{\pm}\text{-corank}(\boldsymbol{X}, u_0) = 1$, the AdS-null height function $h_{v_0^{\pm}}$ has the A_k-singularity at u_0 and is generic. In this case we have $\mathrm{Ah}^{\pm}\text{-ord}(\boldsymbol{X}, u_0) = k$. This number is equal to the order of contact in the classical sense (cf., [4]). This is the reason why we call $\mathrm{Ah}^{\pm}\text{-ord}(\boldsymbol{X}, u_0)$ the order of contact with the AdS-horosphere at $\boldsymbol{X}(u_0)$.

3.5. Classification of singularities of AdS-nullcone Gauss images

In this section we give the generic classification of singularities of AdS-nullcone Gauss images. We have almost the same arguments as those of [15], so that we omit the details. We consider the space of timelike embeddings $\mathrm{Emb}_T(U, H^3_1)$ with the Whitney C^∞-topology. By the classification of stable Legendrian singularities of $n = 3$ and the transversality theorem of [15] (Proposition 7.1), we have the following theorem.

Theorem 3.5.1 *There exists an open dense subset* $\mathcal{O}\subset\mathrm{Emb}_T(U, H^3_1)$ *such that for any* $\boldsymbol{X} \in\mathcal{O}$ *the following conditions hold.*

(1) *The AdSh*$^\pm$*-parabolic set* ${K^{\pm}_{AdSn}}^{-1}(0)$ *is a regular curve. We call such a curve the AdSh*$^\pm$*-parabolic curve.*

(2) *The AdS-nullcone Gauss image* $\mathbb{G}^\pm_n$ *along the AdSh*$^\pm$*-parabolic curve is a cuspidal edge except at isolated points. At such the point* $\mathbb{G}^\pm_n$ *is the swallowtail.*

(3) *The cuspidal edge points* (*swallowtail points*) *of AdS-nullcone Gauss image* $\mathbb{G}^\pm_n$ *correspond to fold points* (*cusp points*) *of AdS-torus Gauss map.*

Here, a map germ $f : (\mathbb{R}^2, \boldsymbol{a}) \longrightarrow (\mathbb{R}^3, \boldsymbol{b})$ *is called a cuspidal edge if it is* $\mathcal{A}$*-equivalent to the germ* (u_1, u_2^2, u_2^3) *and a swallowtail if it is* $\mathcal{A}$*-equivalent to the germ* $(3u_1^4 + u_1^2u_2, 4u_1^3 + 2u_1u_2, u_2)$*. A map germ* $f : (\mathbb{R}^2, \boldsymbol{a}) \longrightarrow (\mathbb{R}^2, \boldsymbol{b})$ *is called a fold if it is* $\mathcal{A}$*-equivalent to the germ* (u_1, u_2^2) *and a cusp if it is* $\mathcal{A}$*-equivalent to the germ* $(u_1, u_2^3 + u_1u_2)$ (cf., Fig 2.4.1).

Cuspital edge Swallowtail

Fig 2.4.1

Following the terminology of Whitney[43], we say that a timelike surface $\boldsymbol{X} : U \longrightarrow H^3_1$ has the *excellent AdS-nullcone Gauss image* $\mathbb{G}^{\pm}_n$, the AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$ has only cuspidal edges and swallowtails as singularities.

We now consider the geometric meanings of cuspidal edges and swallowtails of the AdS-nullcone Gauss image. We have the following results analogous to the results of [15].

Theorem 3.5.2 *Let* $\mathbb{G}^{\pm}_n : (U, u_0) \longrightarrow (\Lambda^3, \boldsymbol{v}^{\pm}_0)$ *be the excellent AdS-nullcone Gauss image germ of a timelike surface* $\boldsymbol{X}$ *and* $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ *be the AdS-null height function germ at* u_0*, where* $\boldsymbol{v}^{\pm}_0 = \mathbb{G}^{\pm}_n(u_0)$*. Then we have the following.*

(1) *The point* u_0 *is an AdSh*$^{\pm}$*-parabolic point of* $\boldsymbol{X}$ *if and only if* Ah$^{\pm}$-corank$(\boldsymbol{X}, u_0)= 1$.

(2) *If* u_0 *is an AdSh*$^{\pm}$*-parabolic point of* $\boldsymbol{X}$*, then* $h_{v^{\pm}_0}$ *has the* A_k-*singularity for* $k = 2, 3$.

(3) *Suppose that* u_0 *is an AdSh*$^{\pm}$*-parabolic point of* $\boldsymbol{X}$*. Then the following conditions are equivalent:*

(a) $\mathbb{G}^{\pm}_n$ *has the cuspidal edge at* u_0*;*

(b) $h_{v^{\pm}_0}$ *has the* A_2-*singularity;*

(c) Ah$^{\pm}$-order$(\boldsymbol{X}, u_0) = 2$*;*

(d) *the tangent AdS-horospherical indicatrix is an ordinary cusp, where a curve* $C \subset \mathbb{R}^2$ *is called an ordinary cusp if it is diffeomorphic to the curve given by* $\{(u_1, u_2)|u_1^2 - u_2^3 = 0\}$.

(e) *for each* $\varepsilon > 0$*, there exit two points* $u_1, u_2 \in U$ *such that* $|u_0-u_i| < \varepsilon$ *for* $i = 1, 2$*, neither of* u_1 *nor* u_2 *is an AdSh*$^{\pm}$*-parabolic point and the tangent AdS-horospheres to* $M = \boldsymbol{X}(U)$ *at* u_1 *and* u_2 *are parallel.*

(4) *Suppose that* u_0 *is an AdSh*$^{\pm}$*-parabolic point of* $\boldsymbol{X}$*. Then the following conditions are equivalent:*

(a) $\mathbb{G}_n^{\pm}$ *has the swallowtail at* u_0*;*

(b) $h_{v_0^{\pm}}$ *has the* A_3-*singularity;*

(c) $\mathrm{Ah}^{\pm}$-order$(\boldsymbol{X}, u_0) = 3$*;*

(d) *the tangent AdS-horospherical indicatrix is an point or a tachnodal, where a curve* $C \subset \mathbb{R}^2$ *is called a tachnodal if it is diffeomorphic to the curve given by* $\{(u_1, u_2) | u_1^2 - u_2^4 = 0\}$.

(e) *for each* $\varepsilon > 0$*, there exit three points* $u_1, u_2, u_3 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2, 3$*, none of* u_1, u_2, u_3 *are an AdSh*$^{\pm}$*-parabolic points and the tangent AdS-horospheres to* $M = \boldsymbol{X}(U)$ *at* u_1, u_2 *and* u_3 *are parallel.*

(f) *for each* $\varepsilon > 0$*, there exit two points* $u_1, u_2 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2$*, neither of* u_1 *nor* u_2 *is an AdS-parabolic point and the tangent AdS-horospheres to* $M = \boldsymbol{X}(U)$ *at* u_1 *and* u_2 *are equal.*

Proof. By the Proposition 3.2.1, we have shown that u_0 is an AdSh$^{\pm}$-parabolic point if and only if $\mathrm{Ah}^{\pm}$-corank$(\boldsymbol{X}, u_0) \geq 1$. Since $n = 3$, we have $\mathrm{Ah}^{\pm}$-corank$(\boldsymbol{X}, u_0) \leq 2$. Since AdS-null height function germ $H : (U \times \Lambda^3, (u_0, \boldsymbol{v}_0^{\pm})) \longrightarrow \mathbb{R}$ can be considered as a generating family of the Legendrian embedding germ $\mathcal{G}^{\pm}$, $h_{v_0^{\pm}}$ has only the A_k-singularities ($k = 1, 2, 3$). This means that the corank of the Hessian matrix of the $h_{v_0^{\pm}}$ at an AdSh$^{\pm}$-parabolic point is 1. The assertion (2) also follows. For the same reason, the conditions $(3)\{(a), (b), (c)\}$(respectively, $(4)\{(a), (b), (c)\}$) are equivalent.

On the other hand, if the AdS-null height function germ $h_{v_0^{\pm}}$ has the A_2-singularity, it is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 + u_2^3$. Since the $\mathcal{K}$-equivalence preserves the zero level sets, the tangent AdS-horospherical indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is the ordinary cusp. The normal form for the A_3-singularity is given by $\pm u_1^2 + u_2^4$, so the tangent AdS-horospherical indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^4 = 0$. This means that the condition (3)$\{(d)\}$(respectively, (4)$\{(d)\}$) is also equivalent to the other conditions.

Suppose that u_0 is an AdSh$^{\pm}$-parabolic point, by Proposition 3.3.1, the AdS-torus Gauss map has only folds or cusps. If the point u_0 is a fold point, there is a neighborhood of u_0 on which the AdS-torus Gauss map is 2 to 1 except the AdSh$^{\pm}$-parabolic line (i.e, fold curve). By Lemma 3.4.1, the condition (3)(e) holds. If the point u_0 is a cusp, the critical value set is an ordinary cusp. By the normal form, we can understand that the AdS-Gauss map is 3 to 1 inside region of the critical value. Moreover, the point u_0 is in the closure of the region. This means that the condition (4)(e) is satisfied. We can also observe that near by the cusp point, there are 2 to 1 points which approach to u_0. However, one of those points are always AdSh$^{\pm}$-parabolic points. Since other singularities do not appear in this case, so that the condition (3)(e) (respectively, (4)(e)) characterizes a fold (respectively, a cusp).

For the swallowtail, point u_0, there is a self-intersection curve approaching u_0. On this curve, there are two distinct points u_1 and u_2 such that $\mathbb{G}_n^{\pm}(u_1) = \mathbb{G}_n^{\pm}(u_2)$. By Lemma 3.4.1, this means that the tangent AdS-horospheres to $M = \boldsymbol{X}(U)$ at u_1 and u_2 are equal. Since there are no other singularities in this case, the condition (4)$\{(f)\}$ characterizes a swallowtail point of $\mathbb{G}_n^{\pm}$. This completes the proof. $\square$

3.6. AdS-null Monge form

The notion of the Monge form of a surface in Euclidean 3-space is one of the powerful tools for the study of local properties of the surface from the view point of differential geometry.In this section we consider the analogous notion for a timelike surface in H^3_1.

We now consider a function $f(u_1,u_2)$ with $f(0)=f_{u_i}(0)=0$. Then we have a timelike surface in H^3_1 defined by

$$\boldsymbol{X}_f(u_1,u_2)=(\sqrt{1+\varepsilon_1u_1^2+\varepsilon_2u_2^2+f^2(u_1,u_2)},\ \frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},$$
$$f(u_1,u_2),\ \frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2}),$$

where $\varepsilon_i=\mathrm{sign}(\boldsymbol{X}_i)$ $(i=1,2)$. We can easily calculate

$$\boldsymbol{N}(0)=(0,0,\frac{\varepsilon_2-\varepsilon_1}{2},0);$$

therefore $\mathbb{G}_n^{\pm}(0)=(1,0,\pm1,0)$. We call $\boldsymbol{X}_f$ a *Anti de Sitter null Monge form* (briefly, *AdS-null Monge form*). Then we have the following proposition.

Proposition 3.6.1 *Any timelike surface in H^3_1 is locally given by the AdS-null Monge form.*

Proof. Let $\boldsymbol{X}:U\longrightarrow H^3_1$ be a timelike surface. We consider Lorentzian motion of H^3_1 which is a transitive action. Therefore, without loss of the generality, we assume that $p=\boldsymbol{X}(0)=(1,0,0,0)$. We denote $M=\boldsymbol{X}(U)$, we have a basis $\{\boldsymbol{X}(0),\boldsymbol{N}(0),\boldsymbol{X}_{u_1}(0),\boldsymbol{X}_{u_2}(0)\}$ of $T_p\mathbb{R}^4_2$ such that $T_pM=\langle\boldsymbol{X}_{u_1}(0),\boldsymbol{X}_{u_2}(0)\rangle_{\mathbb{R}}$. Applying the Gram-Schmidt procedure we have a pseudo-orthonormal basis $\{\boldsymbol{X}(0),\boldsymbol{N}(0),\ \boldsymbol{e}_1,\boldsymbol{e}_2\}$ of $T_p\mathbb{R}^4_2$ such that $T_pM=\langle\boldsymbol{e}_1,\boldsymbol{e}_2\rangle_{\mathbb{R}}$. In particular, $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ is an orthonormal basis of T_pM. Since $p=(1,0,0,0)$, T_pM is considered to be a subspace of ${}_0\mathbb{R}^3_1=\{(0,x_1,x_2,x_3)|x_i\in\mathbb{R}\}$. By a rotation of the space ${}_0\mathbb{R}^3_1$, we might assume that $T_pM=\{(0,\frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},0,\frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2})|u_i\in\mathbb{R}\}\subset\mathbb{R}^4_2$. Then the germ (M,p) might be written in the form

$$(f_0(u_1,u_2),\ \frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},f(u_1,u_2),\ \frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2})$$

with function germs $f_0(u_1,u_2), f(u_1,u_2)$. Since $M \subset H^3_1$, we have the relation

$$f_0(u_1,u_2)=\sqrt{1+\varepsilon_1 u_1^2+\varepsilon_2 u_2^2+f^2(u_1,u_2)}.$$

Since we have

$$T_pM=\{(0,\frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},0,\frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2})|u_i\in\mathbb{R}\},$$

the condition $f(0)=0,\ f_{u_i}(0)=0$ are automatically satisfied. $\square$

For the null vector $\boldsymbol{v}_0^{\pm}=(1,0,\pm1,0)$, we consider the AdS-horosphere type surface AH($\boldsymbol{v}_0^{\pm}$,-1). Then we have the AdS-null Monge form of AH($\boldsymbol{v}_0^{\pm}$, -1):

$$\begin{aligned}\boldsymbol{a}^{\pm}(u_1,u_2)=(&\frac{\varepsilon_1 u_1^2+\varepsilon_2 u_2^2}{2}+1,\ \frac{(1-\varepsilon_1)u_1+(1-\varepsilon_2)u_2}{2},\\ &\pm\frac{\varepsilon_1 u_1^2+\varepsilon_2 u_2^2}{2},\ \frac{(1+\varepsilon_1)u_1+(1+\varepsilon_2)u_2}{2}).\end{aligned}$$

Here, we can easily check the relation $\langle \boldsymbol{a}(u),\boldsymbol{v}_0^{\pm}\rangle=-1$.

On the other hand, $\boldsymbol{a}^{\pm}(0)=(1,0,0,0)=p$ and $\boldsymbol{a}^{\pm}_{u_i}(0)$ is equal to the $x_{3+\varepsilon_i}$-axis for $i=1,2$. This means that $T_pM=T_p(\boldsymbol{a}^{\pm}(U))$. Therefore $\boldsymbol{a}^{\pm}(U)=\mathrm{AH}(\boldsymbol{v}_0^{\pm},-1)$ is the tangent AdS-horosphere of $M=\boldsymbol{X}_f(U)$ at $p=\boldsymbol{X}_f(0)$. It follows from this fact that the tangent AdS-horospherical indicatrix of the AdS-null Monge form germ $(\boldsymbol{X}_f,0)$ is given as follows:

$$\boldsymbol{X}_f^{-1}(AH(\boldsymbol{v}_0^{\pm},0))=\{(u_1,u_2)|\pm 2f(u_1,u_2)=\varepsilon_1 u_1^2+\varepsilon_2 u_2^2\}.$$

On the other hand, since $f(0)=f_{u_i}(0)=0$, we may write

$$f(u_1,u_2)=\frac{1}{2}\bar{k}_1 u_1^2+\frac{1}{2}\bar{k}_2 u_2^2+g(u_1,u_2)$$

where $g\in\mathcal{M}_2^3$ and $\bar{k}_1,\bar{k}_2$ are eigenvalues of $(f_{u_1u_2}(0))$. Under this representation, we can easily calculate $\boldsymbol{X}_{f,u_1u_2}(0)=(\varepsilon_i\delta_{ij},0,\bar{k}_i,\delta_{ij},0)$. It follows from this fact that

$$h^{\pm}_{ij}(0)=\langle \mathbb{G}^{\pm}_n(0),\boldsymbol{X}_{f,u_iu_j}(0)\rangle=\varepsilon_i\delta_{ij}(-1\pm\bar{k}_i)$$

and

$$g_{ij}(0) = \langle \boldsymbol{X}_{f,u_i}(0), \boldsymbol{X}_{f,u_j}(0)\rangle = \varepsilon_i\delta_{ij}.$$

Therefore, we have $k_i^{\pm}(0) = -1 \pm \varepsilon_i \bar{k}_i$ and

$$K^{\pm}_{AdSn}(0) = k_1^{\pm}(0)k_2^{\pm}(0) = (-1 \pm \varepsilon_1\bar{k}_1)(-1 \pm \varepsilon_2\bar{k}_2).$$

The tangent AdS-horospherical indicatrix is given by

$$\begin{aligned}\boldsymbol{X}_f^{-1}(AH(\boldsymbol{v}_0^{\pm}, -1)) &= \{(u_1, u_2) | \pm\bar{k}_1 u_1^2 \pm \bar{k}_2 u_2^2 \pm 2g(u_1, u_2) - \varepsilon_1 u_1^2 - \varepsilon_2 u_2^2 = 0\} \\ &= \{(u_1, u_2) | \varepsilon_1 k_1^{\pm}(0) u_1^2 + \varepsilon_2 k_2^{\pm}(0) u_2^2 \pm 2g(u_1, u_2) = 0\}.\end{aligned}$$

If we try to draw picture of the AdS-nullcone Gauss image, it might be very hard to give a parameterization. However, by the AdS-null Monge form of the tangent AdS-horospherical indicatrix germ, we can easy to detect the type of singularities of the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ (or, AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$).

Example 3.6.1 *Consider the function given by*

$$f(u_1, u_2) = \frac{1}{2}u_1^2 + u_2^2 + \frac{1}{2}u_1^3.$$

Suppose that $\varepsilon_1 = -1, \varepsilon_2 = 1$ *Then* $\bar{k}_1 = 1$, $\bar{k}_2 = 2$. *We have* $k_1^+ = -2$, $k_2^+ = 1$, $k_1^- = 0, k_2^- = -3$. *So that the origin is an* $AdSh^-$*- parabolic point. The tangent AdS-horospherical indicatrix germ at the origin is the ordinary cusp* $u_2^2 = -\frac{1}{3}u_1^3$. *By Theorem 3.5.2,* $\mathbb{G}_n^-$ $(\widetilde{\mathbb{G}_n^-})$ *is the cuspidal edge (fold) at the origin.*

Example 3.6.2 *Consider the function given by*

$$f(u_1, u_2) = \frac{1}{2}u_1^2 + \frac{1}{4}u_2^2 + \frac{3}{2}u_1^4.$$

Suppose that $\varepsilon_1 = 1, \varepsilon_2 = -1$ *Then* $\bar{k}_1 = 1$, $\bar{k}_2 = 1/2$. *We have* $k_1^+ = 0$, $k_2^+ = -3/2$, $k_1^- = -2, k_2^- = -1/2$. *So that the origin is an* $AdSh^+$*-*

parabolic point. The tangent AdS-horospherical indicatrix germ at the origin is the tachnodal $u_2^2 = u_1^4$. *By Theorem 3.5.2,* $\mathbb{G}_n^-$ ($\widetilde{\mathbb{G}_n^-}$) *is the swallowtail* (*cusp*) *at the origin.*

Part IV. Null surfaces in anti de sitter 3-space

In this part, we study the geometric properties of the degenerate surface which is called AdS null surface. This surface is a ruled surface along a spacelike curve in Anti de Sitter 3-space. All of the tangent planes of this ruled surface at regular points are degenerate plane.

4.1. The construction of null surfaces by spacelike curves

We first introduce the local differential geometry of spacelike curves in H^3_1. Let $\boldsymbol{\gamma} : I \longrightarrow H^3_1$ be a regular curve (i.e., an embedding). The regular curve $\boldsymbol{\gamma}$ is said to be spacelike if $\dot{\boldsymbol{\gamma}}$ is a spacelike vector at any $t \in I$, where $\dot{\boldsymbol{\gamma}} = d\boldsymbol{\gamma}/dt$. Since $\boldsymbol{\gamma}$ is a spacelike regular curve, it may admit an arc length parametrization $s = s(t)$. Therefore, we can assume that $\boldsymbol{\gamma}(s)$ is a unit speed curve. Now we have the unit tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$. Since $\langle \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s) \rangle \equiv -1$, we have $\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s) \rangle \equiv 0$. From a direct calculation we have $\langle \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) \rangle = -1$. We now denote $\boldsymbol{N}(s) = \boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)$ and $\boldsymbol{E}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{N}(s)$. It is easy to check that $\boldsymbol{N}$ and $\boldsymbol{E}$ are normal vectors of spacelike curve $\boldsymbol{\gamma}$ in H^3_1. We also define the *curvature* by $k_g(s) = \|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|$. We can show that spacelike curve $\boldsymbol{\gamma}$ is a geodesic in H^3_1 if $k_g(s) = 0$ and $\boldsymbol{N}(s) = \boldsymbol{0}$. In the case when $k_g(s) \neq 0$, we can define unit vectors $\boldsymbol{n}(s)$ and $\boldsymbol{e}(s)$ by

$$\boldsymbol{n}(s) = \frac{\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)}{\|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|} = \frac{\boldsymbol{N}(s)}{\|\boldsymbol{N}(s)\|}, \quad \boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s).$$

Then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\mathbb{R}^4_2$ along $\boldsymbol{\gamma}$. By the standard arguments, we can give the following *Frenet-Serret type formula*:

$$
\begin{cases}
\boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\
\boldsymbol{t}'(s) = \boldsymbol{\gamma}(s) + k_g(s)\boldsymbol{n}(s) \\
\boldsymbol{n}'(s) = -\delta k_g(s)\boldsymbol{t}(s) + \delta\tau_g(s)\boldsymbol{e}(s) \\
\boldsymbol{e}'(s) = \delta\tau_g(s)\boldsymbol{n}(s)
\end{cases}
$$

where $\delta = \mathrm{sign}(\boldsymbol{n}(s))$ and $\tau_g(s) = \dfrac{1}{k_g^2(s)}\det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s))$.

We remark that if $k_g(s) = 0$ and $\boldsymbol{N}(s) \neq \boldsymbol{0}$, we have $\boldsymbol{N}(s)$ is a null vector. Since $\langle \boldsymbol{N}(s) \pm \boldsymbol{E}(s), \boldsymbol{N}(s)\rangle = 0$, it follows $\boldsymbol{N}(s) \pm \boldsymbol{E}(s)$ is also a null vector which parallel to the vector $\boldsymbol{N}(s)$.

Now we define a map

$$
\widetilde{\mathbb{G}^{\pm}} : I \longrightarrow T_1^2
$$

by $\widetilde{\mathbb{G}^{\pm}}(s) = \widetilde{\boldsymbol{N}(s) \pm \boldsymbol{E}}(s)$. We call it *torus Gauss image*. Under the assumption that $k_g(s) \neq 0$ we also define a ruled surface

$$
\Lambda D_{\gamma}^{\pm} : I \times \mathbb{R} \longrightarrow H_1^3
$$

by $\Lambda D_{\gamma}^{\pm}(s, \lambda) = \boldsymbol{\gamma}(s) + \lambda(\boldsymbol{n}(s) \pm \boldsymbol{e}(s))$. We call this ruled surface *Anti de Sitter null surface* associated to a spacelike curve $\boldsymbol{\gamma}$ (briefly, *AdS null surface*). In this part we mainly consider the Lorentzian geometric meaning of the singularities of the torus Gauss images and of the AdS null surfaces. We now introduce a new invariant σ of a spacelike curve in H_1^3 by

$$
\sigma(s) = k_g'(s) \mp k_g(s)\tau_g(s)
$$

We also consider the following set in H_1^3 which is given by the intersection of H_1^3 with a hyperplane $HP(\boldsymbol{v}, c)$. We denote it by $AH(\boldsymbol{v}, c) = H_1^3 \cap HP(\boldsymbol{v}, c)$ and call it a *Anti de Sitter nullcone* (briefly, *AdS nullcone*) with vertex $\boldsymbol{v}$, a *Anti de Sitter null hyperbolic cylinder* (briefly, *AdS null hyperbolic cylinder*) if $\boldsymbol{v} \in H_1^3$ and $c = -1$ or $\boldsymbol{v} \in \Lambda^3$ and $c = 0$ respectively.

On the other hand, let $F : H_1^3 \longrightarrow \mathbb{R}$ be a submersion and $\boldsymbol{\gamma} : I \longrightarrow H_1^3$ be a spacelike curve in H_1^3. We say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have *k-point contact* for $t = t_0$ provided the function g defined by

$$g(t) = F(\gamma_1(t), \cdots, \gamma_4(t)) = F(\boldsymbol{\gamma}(t))$$

satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0,\ g^{(k)}(t_0) \neq 0$. We also say that the *order of contact* is k. Dropping the condition $g^{(k)}(t_0) \neq 0$ we say that there is *at least k-point contact* (cf., [4]). The main result in this paper is the following:

Theorem 4.1.1 *Let $\boldsymbol{\gamma} : I \longrightarrow H_1^3$ be a unit speed spacelike curve in Anti de Sitter 3-space.*

(1) Suppose $\boldsymbol{N}(s) \neq 0$ and $\boldsymbol{v}_0 \in \Lambda^3$, we have the following:

(a) *The singular set of torus Gauss image $\widetilde{\mathbb{G}^{\pm}}$ is $\{s \in I | k_g(s) = 0\}$.*

(b) *$\boldsymbol{\gamma}$ and the AdS null hyperbolic cylinder $AH(\boldsymbol{v}_0, 0)$ have $3-$point contact for s_0 if and only if $\boldsymbol{v}_0 = \widetilde{\boldsymbol{N}(s_0) \pm \boldsymbol{E}}(s_0)$ and $k_g(s_0) = 0$. Under this condition, the torus Gauss image $\widetilde{\mathbb{G}^{\pm}}$ has a cusp point at s_0.*

(2) *Assume $k_g(s) \neq 0$ and $\boldsymbol{v}_0 \in H_1^3$, we have the following:*

(a) *The singular set of $\Lambda D^{\pm}_{\boldsymbol{\gamma}}$ is $\{(s, \lambda) | \lambda = \dfrac{1}{\delta k_g(s)},\ s \in I, \lambda \in \mathbb{R}\}$.*

(b) *$\boldsymbol{\gamma}$ and the AdS nullcone $AH(\boldsymbol{v}_0, -1)$ have $3-$point contact for s_0 if and only if*

$$\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{1}{\delta k_g(s_0)}(\boldsymbol{n} \pm \boldsymbol{e}) \text{ and } \sigma(s_0) \neq 0.$$

Under this condition, the germ of image $\Lambda D^{\pm}_{\boldsymbol{\gamma}}$ at $\boldsymbol{v}_0$ is diffeomophic to the cuspital edge $C \times \mathbb{R}$.

(c) *$\boldsymbol{\gamma}$ and the AdS nullcone $AH(\boldsymbol{v}_0, -1)$ have $4-$point contact for s_0 if and only if*

$$\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{1}{\delta k_g(s_0)}(\boldsymbol{n} \pm \boldsymbol{e}),\ \sigma(s_0) = 0 \text{ and } \sigma'(s_0) \neq 0.$$

Under this condition, the germ of image $\Lambda D^{\pm}_{\boldsymbol{\gamma}}$ at $\boldsymbol{v}_0$ is diffeomophic to the swallowtail SW.

Where $\quad C\times\mathbb{R}=\{(x_1,x_2)|x_1^2-x_2^3=0\}\times\mathbb{R}$ is the cuspital edge, $SW=\{(x_1,x_2,x_3)|x_1=3u^4+u^2v,\ x_2=4u^3+2uv,\ x_3=v\}$ is the swallowtail (cf., Figure 2.5.1)

We remark that the above theorem gives a classification of the singularities of the AdS null surface of a generic spacelike curve in Anti de Sitter 3-space and describe the contact between AdS nullcones and spacelike curves. Let $\mathrm{Emb}_S(I,H_1^3)$ be the space of spacelike embeddings $\boldsymbol{\gamma}:I\longrightarrow H_1^3$ with $k_g(s)\neq 0$ equipped with Whitney $C^\infty-$topology. The generic classification result is given as follows:

Theorem 4.1.2 *There exists an open and dense subset* $\mathcal{O}\subset$ *$Emb_S(I,H_1^3)$ such that for any $\boldsymbol{\gamma}\in\mathcal{O}$, the AdS null surface $\Lambda D_{\boldsymbol{\gamma}}^{\pm}$ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidal edge or the swallowtail at any singular point.*

4.2. Functions on spacelike curves

In this section we define two families of functions on a spacelike curve which are useful for the studying of singularities of torus Gauss images and of AdS null surfaces of the spacelike curves. For a unit speed spacelike curve $\boldsymbol{\gamma}:I\longrightarrow H_1^3$, we may define a function

$$H:I\times T_1^2\longrightarrow\mathbb{R}$$

by $H(s,\boldsymbol{v})=\langle\boldsymbol{\gamma}(s),\boldsymbol{v}\rangle$. We call H *a torus height function* on $\boldsymbol{\gamma}$. We denote that $h_v(s)=H(s,\boldsymbol{v})$ for any $\boldsymbol{v}\in T_1^2$. Then we have the following proposition.

Proposition 4.2.1 *Let $\boldsymbol{\gamma}:I\longrightarrow H_1^3$ be a unit spacelike curve with* $\boldsymbol{N}(s)\neq 0$*. Then we have the following assertions:*

(1) $h_v(s_0)=h_v'(s_0)=0$ *if and only if* $\boldsymbol{v}=\boldsymbol{N}(s_0)\pm\widetilde{\boldsymbol{E}}(s_0)$;

(2) $h_v(s_0) = h'_v(s_0) = h''_v(s_0) = 0$ *if and only if* $\boldsymbol{v} = \widetilde{\boldsymbol{N}(s_0) \pm \boldsymbol{E}}(s_0)$, $k_g(s_0) = 0$.

Proof. (1) Since $h_v(s_0) = h'_v(s_0) = 0$, it follows that $\boldsymbol{v} \perp \boldsymbol{\gamma}(s_0)$ and $\boldsymbol{v} \perp \boldsymbol{t}(s_0)$. On the other hand $\boldsymbol{v} \in T_1^2$, we have $\boldsymbol{v} = \widetilde{\boldsymbol{N}(s_0) \pm \boldsymbol{E}}(s_0)$.

(2) Since $h''_v(s_0) = \langle \boldsymbol{t}'(s_0), \boldsymbol{v} \rangle$, it follows that $h_v(s_0) = h'_v(s_0) = h''_v(s_0) = 0$ if and only if $\langle \boldsymbol{N}(s_0) + \boldsymbol{\gamma}(s_0), \widetilde{\boldsymbol{N}(s_0) \pm \boldsymbol{E}}(s_0) \rangle = 0$. This means that $\|\boldsymbol{N}(s_0)\| = 0$. Therefore $k_g(s_0) = 0$. □

We now define a function

$$D : I \times H_1^3 \longrightarrow \mathbb{R}$$

by $D(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s) - \boldsymbol{v}, \boldsymbol{\gamma}(s) - \boldsymbol{v} \rangle$. We call D *Anti de Sitter distance-squared function* (or, *AdS distance-squared function*) on $\boldsymbol{\gamma}$. We denote that $d_v(s) = D(s, \boldsymbol{v})$ for any $\boldsymbol{v} \in H_1^3$. Then we have the following proposition.

Proposition 4.2.2 *Let* $\boldsymbol{\gamma} : I \longrightarrow H_1^3$ *be a unit spacelike curve with* $k_g(s) \neq 0$. *Then we have the following assertions:*

(1) $d_v(s_0) = d'_v(s_0) = 0$ *if and only if there exists* $\lambda \in \mathbb{R}$ *such that*

$$\boldsymbol{v} = \boldsymbol{\gamma}(s_0) + \lambda(\boldsymbol{n}(s_0) \pm \boldsymbol{e}(s_0));$$

(2) $d_v(s_0) = d'_v(s_0) = d''_v(s_0) = 0$ *if and only if*

$$\boldsymbol{v} = \boldsymbol{\gamma}(s_0) + \frac{1}{\delta k_g(s_0)}(\boldsymbol{n}(s_0) \pm \boldsymbol{e}(s_0));$$

(3) $d_v(s_0) = d'_v(s_0) = d''_v(s_0) = d'''_v(s_0) = 0$ *if and only if*

$$\boldsymbol{v} = \boldsymbol{\gamma}(s_0) + \frac{1}{\delta k_g(s_0)}(\boldsymbol{n}(s_0) \pm \boldsymbol{e}(s_0)) \text{ and } \sigma(s_0) = 0;$$

(4) $d_v(s_0) = d'_v(s_0) = d''_v(s_0) = d'''_v(s_0) = d_v^{(4)}(s_0) = 0$ *if and only if*

$$\boldsymbol{v} = \boldsymbol{\gamma}(s_0) + \frac{1}{\delta k_g(s_0)}(\boldsymbol{n}(s_0) \pm \boldsymbol{e}(s_0)),\ \sigma(s_0) = 0 \text{ and } \sigma'(s_0) = 0.$$

Proof. (1) Since $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ is a pseudo orthonormal frame of $\mathbb{R}^4_2$ along $\boldsymbol{\gamma}$ and $\boldsymbol{v} \in H^3_1$, there exist $\eta, \alpha, \lambda, \beta \in \mathbb{R}$ with $-\eta^2 + \alpha^2 + \delta\lambda^2 - \delta\beta^2 = -1$ such that $\boldsymbol{v} = \eta\boldsymbol{\gamma}(s) + \alpha\boldsymbol{t}(s) + \lambda\boldsymbol{n}(s) + \beta\boldsymbol{e}(s)$. By the definition of $d_v(s)$ and the condition $d_v(s_0) = 0$ we have $\eta = 1$. Therefore, we have $\boldsymbol{v} = \boldsymbol{\gamma}(s_0) + \alpha\boldsymbol{t}(s_0) + \lambda\boldsymbol{n}(s_0) + \beta\boldsymbol{e}(s_0)$. Since $d'_v(s_0) = 0$, we have $\langle \boldsymbol{t}(s_0), \boldsymbol{\gamma}(s_0) - \boldsymbol{v}\rangle = 0$. This means $\alpha = 0$. It follows from the fact $\delta\lambda^2 - \delta\beta^2 = 0$ that $\beta = \pm\lambda$. So that $\boldsymbol{v} = \boldsymbol{\gamma}(s_0) + \lambda(\boldsymbol{n}(s_0) \pm \boldsymbol{e}(s_0))$.
(2) Since $d''_v(s_0) = 0$, we have $\langle \boldsymbol{t}'(s_0), \boldsymbol{\gamma}(s_0) - \boldsymbol{v}\rangle + \langle \boldsymbol{t}(s_0), \boldsymbol{t}(s_0)\rangle = 0$. This means $\lambda = \dfrac{1}{\delta k_g(s_0)}$.
(3) Since $d'''_v(s_0) = 0$, we have

$$\langle k'_g(s_0)\boldsymbol{n}(s_0)+k_g(s_0)\boldsymbol{n}'(s_0)+\boldsymbol{t}(s_0), \boldsymbol{\gamma}(s_0)-\boldsymbol{v}\rangle+\langle k_g(s_0)\boldsymbol{n}(s_0)+\boldsymbol{\gamma}(s_0), \boldsymbol{t}(s_0)\rangle = 0.$$

It follows that $\frac{k'_g(s_0)}{k_g(s_0)} \mp \tau_g(s_0) = 0$. This is equivalent to the condition $\sigma(s_0) = 0$.
(4) Since $d^{(4)}_v(s) = 0$, we have

$$\langle (1 - \delta k_g^2(s))\boldsymbol{t}(s) + k'_g(s)\boldsymbol{n}(s) + \delta k_g(s)\tau_g(s)\boldsymbol{e}(s), \boldsymbol{\gamma}(s) - \boldsymbol{v}\rangle' = 0.$$

This means $k''_g(s_0) \mp k'_g(s_0)\tau_g(s_0) \mp k_g(s_0)\tau'_g(s_0) = 0$. So that, we have $\sigma'(s_0) = 0$. $\square$

We now use some general results on the singularity theory for families of function germs to give a proof of our main result Theorem 4.1.1. Detailed descriptions at here can be found in [4, 14, 12]. Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, x_0)) \longrightarrow \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F(s, x_0)$. We say that f has an A_k*−singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has an $A_{\geq k}$*−singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ has A_k−singularity ($k \geq 1$) at s_0. We denote the

$(k-1)-$jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by

$$j^{k-1}(\tfrac{\partial F}{\partial x_i}(s, x_0))(s_0) = \textstyle\sum_{j=0}^{k-1} \alpha_{ji} s^j \text{ for } i = 1, \cdots, r.$$

According to J. W. Bruce and P. J. Giblin [4] (Page 149 Theorem 6.10), we have F is versal unfolding if and only if the $k \times r$ matrix of coefficients (α_{ji}) has rank k $(k \leq r)$.

We now introduce an important set concerning the unfoldings relative to the above notions. The *discriminant set* of F is the set

$$\mathcal{D}_F = \{x \in \mathbb{R}^r | \text{ there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x)\}.$$

Then we have the following well-known result (cf., [4]).

Theorem 4.2.3 *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \longrightarrow \mathbb{R}$ be an $r-$parameter unfolding of $f(s)$ which has an A_k-singularity at s_0. Suppose that F is a versal unfolding. Then we have the followings:*

(1) *If $k = 2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$;*

(2) *If $k = 3$, then $\mathcal{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

Now we can apply the above arguments to our case. Let $\boldsymbol{\gamma} : I \longrightarrow H_1^3$ be a unit speed spacelike curve in H_1^3, H be the torus height function on $\boldsymbol{\gamma}$ and D be the AdS distance-squared function on $\boldsymbol{\gamma}$. By Proposition 4.2.1 and Proposition 4.2.2, the discriminant sets of the torus height function H and of the AdS distance-squared function D are given respectively by

$$\mathcal{D}_H = \{\widetilde{\boldsymbol{N}(s) \pm \boldsymbol{E}}(s) | s \in I\}, \mathcal{D}_D = \{\boldsymbol{\gamma}(s) + \lambda(\boldsymbol{n} \pm \boldsymbol{e}) | s \in I, \lambda \in \mathbb{R}\}.$$

Then we have the following propositions.

Proposition 4.2.4 *Let $H : I \times T_1^2 \longrightarrow \mathbb{R}$ be the torus height function on a unit speed spacelike curve $\boldsymbol{\gamma}$ in H_1^3 with $\boldsymbol{N} \neq 0$. Suppose that $\boldsymbol{v} \in \mathcal{D}_H$ and $h_v(s) = H(s, \boldsymbol{v})$. If h_v has A_2-singularity at s, then H is a versal* *unfolding of h_v.*

Proof. Let $\boldsymbol{\gamma}(s) = (\gamma_1(s), \cdots, \gamma_4(s)) \in H_1^3$, $\boldsymbol{v} = (\cos\theta, \sin\theta, \cos\alpha, \sin\alpha) \in T_1^2$, we have

$$H(s, \boldsymbol{v}) = -\gamma_1\cos\theta - \gamma_2\sin\theta + \gamma_3\cos\alpha + \gamma_4\sin\alpha = H(s, \theta, \alpha).$$

Then we have

$$\frac{\partial H}{\partial \theta}(s, \theta, \alpha) = \gamma_1(s)\sin\theta - \gamma_2(s)\cos\theta,$$
$$\frac{\partial H}{\partial \alpha}(s, \theta, \alpha) = -\gamma_3(s)\sin\alpha + \gamma_4(s)\cos\alpha$$

and

$$\frac{\partial^2 H}{\partial s \partial \theta}(s, \theta, \alpha) = \gamma_1'(s)\sin\theta - \gamma_2'(s)\cos\theta,$$
$$\frac{\partial^2 H}{\partial s \partial v_i}(s, \theta, \alpha) = -\gamma_3'(s)\sin\alpha + \gamma_4'(s)\cos\alpha.$$

So that the $1-$jets of $\frac{\partial H}{\partial \theta}$ and $\frac{\partial H}{\partial \alpha}$ with constant terms at s_0 are given by

$$j^1(\frac{\partial H}{\partial \theta}(s, \theta, \alpha))(s_0) = \gamma_1(s)\sin\theta - \gamma_2(s)\cos\theta + (\gamma_1'(s)\sin\theta - \gamma_2'(s)\cos\theta)s,$$
$$j^1(\frac{\partial H}{\partial \alpha}(s, \theta, \alpha))(s_0) = \gamma_4(s)\cos\alpha - \gamma_3(s)\sin\alpha + (\gamma_4'(s)\cos\alpha - \gamma_3'(s)\sin\alpha)s.$$

Let

$$A = \begin{pmatrix} \gamma_1(s)\sin\theta - \gamma_2(s)\cos\theta & \gamma_4(s)\cos\alpha - \gamma_3(s)\sin\alpha \\ \gamma_1'(s)\sin\theta - \gamma_2'(s)\cos\theta & \gamma_4'(s)\cos\alpha - \gamma_3'(s)\sin\alpha \end{pmatrix}.$$

It is easy to show that

$$\det A = -\sin\theta\sin\alpha(\gamma_1(s)\gamma_3'(s) - \gamma_1'(s)\gamma_3(s)) + \sin\theta\cos\alpha(\gamma_1(s)\gamma_4'(s) - \gamma_1'(s)\gamma_4(s))$$
$$+\cos\theta\sin\alpha(\gamma_2(s)\gamma_3'(s) - \gamma_2'(s)\gamma_3(s)) - \cos\theta\cos\alpha(\gamma_2(s)\gamma_4'(s) - \gamma_2'(s)\gamma_4(s)).$$

We claim that it will suffice to show that the rank of the 2×2 matrix A is 2. This means that we have to prove that $\det A \neq 0$.

Since $\boldsymbol{v} \in \mathcal{D}_H$ is a singular point, we have $\boldsymbol{v} = \widetilde{\boldsymbol{N}(s) + \boldsymbol{E}}(s)$. It follows that

$$
\begin{aligned}
\boldsymbol{\gamma}\wedge\boldsymbol{t}\wedge\boldsymbol{v}=(\ &\mathrm{cos}\alpha(\gamma_2\gamma_4'-\gamma_2'\gamma_4)-\mathrm{sin}\theta(\gamma_3\gamma_4'-\gamma_3'\gamma_4)-\mathrm{sin}\alpha(\gamma_2\gamma_3'-\gamma_2'\gamma_3),\\
&\mathrm{cos}\theta(\gamma_3\gamma_4'-\gamma_3'\gamma_4)-\mathrm{cos}\alpha(\gamma_1\gamma_4'-\gamma_1'\gamma_4)+\mathrm{sin}\alpha(\gamma_1\gamma_3'-\gamma_1'\gamma_3),\\
&\mathrm{cos}\theta(\gamma_2\gamma_4'-\gamma_2'\gamma_4)-\mathrm{sin}\theta(\gamma_1\gamma_4'-\gamma_1'\gamma_4)+\mathrm{sin}\alpha(\gamma_1\gamma_2'-\gamma_1'\gamma_2),\\
&\mathrm{sin}\theta(\gamma_1\gamma_3'-\gamma_1'\gamma_3)-\mathrm{cos}\theta(\gamma_2\gamma_3'-\gamma_2'\gamma_3)-\mathrm{cos}\alpha(\gamma_1\gamma_2'-\gamma_1'\gamma_2)).
\end{aligned}
$$

We denote $\boldsymbol{w}=\boldsymbol{\gamma}\wedge\boldsymbol{t}\wedge\boldsymbol{v}$ and $\boldsymbol{v}_1=(\mathrm{cos}\theta,\mathrm{sin}\theta,-\mathrm{cos}\alpha,-\mathrm{sin}\alpha)$. Then we have $\boldsymbol{w}\perp\boldsymbol{v}$, $\boldsymbol{w}\perp\boldsymbol{t}$, it follows that $\boldsymbol{w}\in\Lambda^3$ and $\boldsymbol{w}$ is parallel to $\boldsymbol{v}$. So that $\langle\boldsymbol{v}_1,\boldsymbol{w}\rangle\neq 0$.

On the other hand, since $\mathrm{det}A=\frac{1}{2}\langle\boldsymbol{v}_1,\boldsymbol{w}\rangle\neq 0$. This completes the proof. $\square$

Proposition 4.2.5 *Let $D: I\times H_1^3\longrightarrow\mathbb{R}$ be the AdS distance-squared function on a unit speed spacelike curve $\boldsymbol{\gamma}$ in H_1^3 with $k_g(s)\neq 0$. Suppose that $\boldsymbol{v}\in\mathcal{D}_D$ and $d_v(s)=D(s,\boldsymbol{v})$. If d_v has A_k-singularity at s ($k=2,3$), then D is a versal unfolding of d_v.*

Proof. Let $\boldsymbol{\gamma}(s)=(x_1(s),\cdots,x_4(s))$ and $\boldsymbol{v}=(v_1,\cdots,v_4)\in H_1^3$, we have $v_1\neq 0$ or $v_2\neq 0$. Without loss of generality, we may assume that $v_1>0$, then $v_1=\sqrt{1-v_2^2+v_3^2+v_4^2}$. Therefore, we have

$$D(s,\boldsymbol{v})=2(x_1(s)\sqrt{1-v_2^2+v_3^2+v_4^2}+x_2(s)v_2-x_3(s)v_3-x_4(s)v_4-1).$$

Then we have

$$\frac{\partial D}{\partial v_2}(s,\boldsymbol{v})=-2(x_1(s)\frac{v_2}{v_1}-x_2(s)),$$

$$\frac{\partial D}{\partial v_i}(s,\boldsymbol{v})=2(x_1(s)\frac{v_i}{v_1}-x_i(s)),\ i=3,4$$

and

$$\frac{\partial^2 D}{\partial s\partial v_2}(s,\boldsymbol{v})=-2(x_1'(s)\frac{v_2}{v_1}-x_2'(s)),$$

$$\frac{\partial^2 D}{\partial s\partial v_i}(s,\boldsymbol{v})=2(x_1'(s)\frac{v_i}{v_1}-x_i'(s)),\ i=3,4$$

and

$$\frac{\partial^3 D}{\partial s^2 \partial v_1}(s,\boldsymbol{v}) = -2(x_1''(s)\frac{v_2}{v_1} - x_2''(s)),$$

$$\frac{\partial^3 D}{\partial s^2 \partial v_i}(s,\boldsymbol{v}) = 2(x_1''(s)\frac{v_i}{v_1} - x_i''(s)),\ i = 3, 4.$$

So that the 2−jet of $\frac{\partial D}{\partial v_i}(s, v)$ $(i = 2, 3, 4)$ at s_0 with constant is given by

$$\begin{aligned} j^2(\frac{\partial D}{\partial v_i}(s,\boldsymbol{v}))(s_0) &= \frac{\partial D}{\partial v_i}(s_0,\boldsymbol{v}) + \frac{\partial^2 D}{\partial s \partial v_i}(s_0,\boldsymbol{v})s + \frac{1}{2}\frac{\partial^3 D}{\partial s^2 \partial v_i}(s_0,\boldsymbol{v})s^2 \\ &= \alpha_{0i} + \alpha_{1i}s + \frac{1}{2}\alpha_{2i}s^2,\ i = 2, 3, 4. \end{aligned}$$

We claim that it will suffice to show that the rank of the 3×3 matrix A of coefficients (α_{ji}) $(0 \le j \le 2,\ 2 \le i \le 4)$ is 3, where

$$A = \begin{pmatrix} \alpha_{02} & \alpha_{03} & \alpha_{04} \\ \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix},$$

We denote

$$\boldsymbol{a}_1 = \begin{pmatrix} x_1 \\ x_1' \\ x_1'' \end{pmatrix},\ \boldsymbol{a}_2 = \begin{pmatrix} x_2 \\ x_2' \\ x_2'' \end{pmatrix},\ \boldsymbol{a}_3 = \begin{pmatrix} x_3 \\ x_3' \\ x_3'' \end{pmatrix},\ \boldsymbol{a}_4 = \begin{pmatrix} x_4 \\ x_4' \\ x_4'' \end{pmatrix}.$$

Then we have

$$\begin{aligned} \det A = &\frac{v_1}{v_1}\det(\boldsymbol{a}_2,\ \boldsymbol{a}_3,\ \boldsymbol{a}_4) - \frac{v_2}{v_1}\det(\boldsymbol{a}_1,\ \boldsymbol{a}_3,\ \boldsymbol{a}_4) \\ &+\frac{v_3}{v_1}\det(\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_4) - \frac{v_4}{v_1}\det(\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_3). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \boldsymbol{\gamma} \wedge \boldsymbol{\gamma}' \wedge \boldsymbol{\gamma}'' = (&-\det(\boldsymbol{a}_2,\ \boldsymbol{a}_3,\ \boldsymbol{a}_4),\ \det(\boldsymbol{a}_1,\ \boldsymbol{a}_3,\ \boldsymbol{a}_4), \\ &\det(\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_4),\ -\det(\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_3)). \end{aligned}$$

Therefore, we have

$$\det A = \langle (\frac{v_1}{v_1}, \frac{v_2}{v_1}, \frac{v_3}{v_1}, \frac{v_4}{v_1}), \boldsymbol{\gamma} \wedge \boldsymbol{\gamma}' \wedge \boldsymbol{\gamma}' \rangle$$
$$= \frac{1}{v_1} \langle \boldsymbol{v}, \boldsymbol{\gamma} \wedge \boldsymbol{\gamma}' \wedge \boldsymbol{\gamma}'' \rangle.$$

Since $\boldsymbol{v} \in \mathcal{D}_D$ is a singular point,

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) + \frac{1}{\delta k_g(s)}(\boldsymbol{n}(s) \pm \boldsymbol{e}(s)).$$

Moreover, we have

$$\boldsymbol{\gamma}(s) \wedge \boldsymbol{\gamma}'(s) \wedge \boldsymbol{\gamma}''(s) = k_g(s)\boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s) = k_g(s)\boldsymbol{e}(s).$$

This means that

$$\det A = \frac{1}{v_1} \langle \boldsymbol{v}, \boldsymbol{\gamma} \wedge \boldsymbol{\gamma}' \wedge \boldsymbol{\gamma}'' \rangle = \mp \frac{1}{v_1} \neq 0.$$

This completes the proof. □

Proof of Theorem 4.1.1 (1) Let $\boldsymbol{\gamma} : I \longrightarrow H_1^3$ be a unit speed spacelike curve with $\boldsymbol{N}(s) \neq 0$. We consider the torus height function $H : I \times T_1^2 \longrightarrow \mathbb{R}$. According to the implicit function theorem, under the condition $(\partial^2 H / \partial s \partial \boldsymbol{v})(s_0, \boldsymbol{v}_0) \neq 0$, $(\partial H / \partial s)(s_0, \boldsymbol{v}_0) = 0$ if and only if there exits a smooth function $f : I \longrightarrow T_1^2$ such that $(\partial H / \partial s)^{-1}(0) = \{(s, \boldsymbol{v}) | \boldsymbol{v} = f(s)\}$. By the Proposition 4.2.1, we have $f(s) = \mathbb{G}^{\pm}(s)$. Moreover, $(\partial^2 H / \partial s^2)(s_0, \boldsymbol{v}_0) = 0$ if and only if ${\mathbb{G}^{\pm}}'(s_0) = 0$. This means that s_0 is a singular point of the torus Gauss image $\mathbb{G}^{\pm}$, it follows that the assertion (a) holds. We now define a function $\mathcal{H}$: $H_1^3 \longrightarrow \mathbb{R}$ by $\mathcal{H}(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{v}_0 \rangle$, where $\boldsymbol{v}_0 \in T_1^2$. Then we have $\mathcal{H}(\boldsymbol{\gamma}(s)) = h_{v_0}(s)$. Since $\mathcal{H}^{-1}(0) = AH(\boldsymbol{v}_0, 0)$ and 0 is a regular value of $\mathcal{H}$, h_{v_0} has the A_k−singularity at s_0 if and only if $\boldsymbol{\gamma}$ and $AH(\boldsymbol{v}_0, 0)$ have $(k+1)$−point contact for s_0. By Proposition 4.2.1, Theorem 4.2.3 and Proposition 4.2.4, we have proved that the assertion (b) is right.

(2) Let $\boldsymbol{\gamma} : I \longrightarrow H_1^3$ be a unit speed spacelike curve with $k_g(s) \neq 0$. By the definition of $\Lambda D_{\boldsymbol{\gamma}}^{\pm}$ and simple calculation, we have

$$\frac{\partial \Lambda D^{\pm}_{\boldsymbol{\gamma}}}{\partial s} = (1-\lambda\delta k_g(s))\boldsymbol{t}(s) + \lambda\tau_g(s)\boldsymbol{e}(s) \pm \lambda\tau_g(s)\boldsymbol{n}(s),$$
$$\frac{\partial \Lambda D^{\pm}_{\boldsymbol{\gamma}}}{\partial \lambda} = \boldsymbol{n}(s) \pm \boldsymbol{e}(s).$$

Therefore the assertion (a) is right. We now define a function $\mathcal{G}$: $H^3_1 \longrightarrow \mathbb{R}$ by $\mathcal{G}(\boldsymbol{u}) = \langle \boldsymbol{u} - \boldsymbol{v}^{\pm}_0, \boldsymbol{u} - \boldsymbol{v}^{\pm}_0\rangle$, where $\boldsymbol{v}^{\pm}_0 \in \Lambda D^{\pm}_{\boldsymbol{\gamma}}$. Then we have $\mathcal{G}(\boldsymbol{\gamma}(s)) = d_{v^{\pm}_0}(s)$. Since $\mathcal{G}^{-1}(0) = AH(\boldsymbol{v}^{\pm}_0, -1)$ and 0 is a regular value of $\mathcal{G}$, $d_{v^{\pm}_0}$ has the A_k−singularity at s_0 if and only if $\boldsymbol{\gamma}$ and $AH(\boldsymbol{v}^{\pm}_0, -1)$ have $(k+1)$−point contact for s_0. By Proposition 4.2.2, Theorem 4.2.3 and Proposition 4.2.5, we have completed the proof of Theorem 4.1.1. □

4.3. Invariants of spacelike curves

In this section we try to explain the geometric meanings of the invariants $\sigma(s)$, $k_g(s)$ and $\tau_g(s)$ which are defined in §4.1. Let $\boldsymbol{w}$ is spacelike vector and $\boldsymbol{v} \in H^3_1$, we define a *Anti de Sitter null curve* (or,*AdS null curve*) in H^3_1 by $AH(\boldsymbol{v}, -1) \cap HP(\boldsymbol{w}, 0)$. We call it *Anti de Sitter null circle* (briefly, *AdS null circle*) or *Anti de Sitter null line* (briefly, *AdS null line*) if $\boldsymbol{w}$ is spacelike or timelike respectively. We have the following proposition.

Proposition 4.3.1 *For a unit speed spacelike curve $\boldsymbol{\gamma} : I \longrightarrow H^3_1$ with $k_g(s) \neq 0$. Let $\boldsymbol{v}^{\pm}(s) = \boldsymbol{\gamma}(s) + \dfrac{1}{\delta k_g(s)}(\boldsymbol{n}(s) \pm \boldsymbol{e}(s))$. We have the following:*

(1) *Suppose that $k_g(s)$ is not a constant. Then the following conditions are equivalent:*

(a) $\boldsymbol{v}^{\pm}(s)$ *is a constant vector*;

(b) $\sigma(s) \equiv 0$;

(c) $\boldsymbol{\gamma}$ *is located on a AdS nullcone.*

(2) *Suppose that $k_g(s)$ is a constant. Then the following conditions are equivalent:*

(a) $\boldsymbol{v}^{\pm}(s)$ *is a constant vector*;

(b) $\tau_g(s) \equiv 0$;

(c) $\boldsymbol{\gamma}$ *is a part of AdS null circle or AdS null line.*

Proof. (1) Since $k_g(s)$ is not a constant and

$$\boldsymbol{v}^{\pm}(s) = \boldsymbol{\gamma}(s) + \frac{1}{\delta k_g(s)}(\boldsymbol{n}(s) \pm \boldsymbol{e}(s)),$$

we have

$$\begin{aligned} \boldsymbol{v}^{\pm'}(s) &= \boldsymbol{t} + \frac{(\boldsymbol{n}'(s) \pm \boldsymbol{e}'(s))\delta k_g(s) - (\boldsymbol{n}(s) \pm \boldsymbol{e}(s))\delta k_g'(s)}{k_g^2(s)} \\ &= \frac{\pm\delta k_g(s)\tau_g(s) - \delta k_g'(s)}{k_g^2(s)}\boldsymbol{n}(s) + \frac{\delta k_g(s)\tau_g(s) \mp \delta k_g'(s)}{k_g^2(s)}\boldsymbol{e}(s). \end{aligned}$$

Therefore, $\boldsymbol{v}^{\pm'}(s) = 0$ if and only if $\sigma(s) = 0$. The conditions (a) and (b) of (1) are equivalent.

If $\boldsymbol{\gamma}$ is located on a AdS nullcone $AH(\boldsymbol{v}_0, -1)$, we have $D(s, \boldsymbol{v}_0) \equiv 0$, where $D(s, \boldsymbol{v})$ is a AdS distance-squared function on $\boldsymbol{\gamma}$. By the assertion (3) of Proposition 4.2.2, we have $\sigma(s) \equiv 0$. This means the condition (c) implies the condition (b). It is obviously that condition (a) implies the condition (c). So that the conditions of (1) are equivalent each other.

(2) Suppose that $k_g(s)$ is a constant c, we have

$$\boldsymbol{v}^{\pm}(s) = \boldsymbol{\gamma}(s) + \frac{1}{\delta c}(\boldsymbol{n}(s) \pm \boldsymbol{e}(s)).$$

Then

$$\boldsymbol{v}^{\pm'}(s) = \boldsymbol{t} + \frac{1}{\delta c}(-\delta c\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{e}(s) \pm \tau_g\boldsymbol{n}(s) = \frac{\tau_g}{\delta c}(\boldsymbol{e}(s) \pm \boldsymbol{n}(s)).$$

This means that (a) is equivalent to (b) in (2). We now consider a AdS null curve given by $AH(\boldsymbol{v}^{\pm}, -1) \cap \langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s)\rangle_{\mathbb{R}}$. Assume that $\boldsymbol{v}^{\pm}(s)$ is a constant vector, then $\tau_g(s) \equiv 0$. So that we have $\boldsymbol{e}(s)$ is a constant vector. This means that $AH(\boldsymbol{v}^{\pm}, -1) \cap \langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s)\rangle_{\mathbb{R}}$ is constant. Thus $\boldsymbol{\gamma}$ is a part of AdS null curve. If $\boldsymbol{e}(s)$ is spacelike we have $\boldsymbol{\gamma}$ is a part of AdS null circle. If $\boldsymbol{e}(s)$ is timelike we have $\boldsymbol{\gamma}$ is a part of AdS null line. Suppose that $\boldsymbol{\gamma}$ is a part of AdS null curve. According to the definition of AdS null curve, it follows that $\tau_g(s) \equiv 0$. This completes the proof of the assertion (2). $\square$

We remark that by the Theorem 4.1.1, the assertion (1) of the above proposition suggests that invariant $\sigma(s)$ depicts the contact between spacelike curves and AdS nullcones, the assertion (2) suggests that two invariants $k_g(s)$ and $\tau_g(s)$ describe the contact between spacelike curves and AdS null curves.

Proposition 4.3.2 *For a unit speed spacelike curve* $\boldsymbol{\gamma} : I \longrightarrow H^3_1$ *with* $\boldsymbol{N}(s) \neq 0$. *Then we have the following assertions*:

(1) $s_0 \in I$ *is a singular point of the torus Gauss image* $\mathbb{G}^{\pm}$ *if and only if* $k_g(s_0) = 0$;

(2) $k_g(s) \equiv 0$ *if and only if the torus Gauss image* $\mathbb{G}^{\pm}$ *is a constant.*

Proof. By the proof of Theorem 4.1.1, It follows that (1) is right.

The assertion (2) is a direct corollary of assertion (1). $\square$

4.4. Generic properties

In this section we consider generic properties of spacelike curves in H^3_1. The main tool is a kind of transversality theorem. Let $\mathrm{Emb}_S(I, H^3_1)$ be a space of spacelike embeddings $\boldsymbol{\gamma} : I \longrightarrow H^3_1$ with $k_g(s) \neq 0$ equipped with Whitney $C^{\infty}-$topology. We also consider the function $\bar{\mathcal{G}}$: $H^3_1 \times H^3_1 \longrightarrow \mathbb{R}$ which is given by $\bar{\mathcal{G}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}-\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle$. We claim that $\bar{\mathcal{G}}_v$ is a submersion for any $\boldsymbol{v} \in H^3_1$, where $\bar{\mathcal{G}}_v(\boldsymbol{u}) = \bar{\mathcal{G}}(\boldsymbol{u}, \boldsymbol{v})$. For any $\boldsymbol{\gamma} \in \mathrm{Emb}_S(I, H^3_1)$, we have $D = \bar{\mathcal{G}} \circ (\boldsymbol{\gamma} \times id_{H^3_1})$. We also have the $l-$jet extension

$$j^l_1 D : I \times H^3_1 \longrightarrow J^l(I, \mathbb{R})$$

defined by $j_1^l D(s, \boldsymbol{v}) = j^l d_v(s)$. We consider the trivialisation

$$J^l(I, \mathbb{R}) \equiv U \times \mathbb{R} \times J^l(1,1).$$

For any submanifold $Q \subset J^l(1,1)$, we denote $\widetilde{Q} = I \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 of Wassermann [42].(See also Izumiya *et al.*[14, 12])

Proposition 4.4.1 *Let Q be a submanifold of $J^l(1,1)$. Then the set*

$$T_Q = \{\boldsymbol{\gamma} \in \mathrm{Emb}_S(I, H_1^3) | j_1^l D \textit{ is transversal to } \widetilde{Q}\}$$

is a residual subset of $\mathrm{Emb}_S(U, H_1^3)$. *If Q is a closed subset, then T_Q is open.*

On the other hand, let $f : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be a function germ which has an A_k-singularity at 0. It is well known that there exists a diffeomorphism germ $\phi : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ such that $f \circ \phi = \pm s^{k+1}$. This is the classification of A_k-singularities (cf., [4]). For any $z = j^l(f)(0) \in J^l(1,1)$, we have the orbit $L^l(z)$ given by the action of the Lie group of $l-$jets of diffeomorphism germs. If f has A_k-singularity, then the codimension of the orbit is k. There is another characterisation of versal unfoldings as follows (cf., [14, 12, 29]):

Proposition 4.4.2 *Let $F : (\mathbb{R} \times \mathbb{R}^r,\ \mathbf{0}) \longrightarrow (\mathbb{R},\ 0)$ be an $r-$parameter unfolding of $f : (\mathbb{R},\ 0) \longrightarrow (\mathbb{R},\ 0)$ which has an A_k-singularity at* 0. *Then F is a versal unfolding if and only if $j_1^l F$ is transversal to the orbit $\widetilde{L^l(j^l f(0))}$ for $l \geq k+1$.*

Here, $j_1^l F : (\mathbb{R} \times \mathbb{R}^r,\ \mathbf{0}) \longrightarrow J^l(\mathbb{R}, \mathbb{R})$ is the $l-$jet extension of F given by $j_1^l F(s, x) = j^l F_x(s)$.

By the above arguments we can prove Theorem 4.1.2 as follows:

Proof of Theorem 4.1.2 For $l \geq 4$, we consider the decomposition of the jet space $J^l(1,1)$ into $L^l(1)$ orbits. We now define a semi-algebraic set by

$$\Sigma^l = \{z = j^l f(0) \in J^l(1,1) | f \text{ has an } A_{\geq 4}\text{−singularity}\}.$$

Then the codimension of Σ^l is 4. Therefore, the codimension of $\widetilde{\Sigma}^l = I \times \{0\} \times \Sigma^l$ is 5. We have the orbit decomposition of $J^l(1,1) - \Sigma^l$ into

$$J^l(1,1) - \Sigma^l = L_0^l \cup L_1^l \cup L_2^l \cup L_3^l,$$

where L_k^l is the orbit through an A_k−singularity. Thus, the codimension of $\widetilde{L_k^l}$ is $k+1$. We consider the l−jet extension $j_1^l D$ of the AdS distance-squared function D. By Proposition 4.4.1, There exists an open and dense subset $\mathcal{O} \subset \mathrm{Emb}_S(I, H_1^3)$ such that $j_1^l D$ is transversal to $\widetilde{L_k^l}(k = 0,1,2,3)$ and the submanifolds of $\widetilde{\Sigma}^l$. This means that $j_1^l D(I \times H_1^3) \cap \widetilde{\Sigma}^l = \emptyset$ and D is a versal unfolding of d_v at any points $(s_0, \boldsymbol{v}_0)$. By Theorem 4.2.3, the discriminant set of D (i.e., the AdS null surface of $\boldsymbol{\gamma}$) is locally diffeomorphic to the cuspidal edge or the swallowtail if the point is singular. $\square$

Acknowledgments

This paper was completed under the guidance of Professor Shyuichi Izumiya when the author was a joint PhD candidate at the Hokkaido University. He would like to thank Prof. Shyuichi Izumiya for his warmly help, not only in study but also in his living in Japan. He also would like to thank Prof. Goo Ishikawa, Prof. Keizo Yamaguchi, Dr. Kentaro Saji, Dr. Masatomo Takahashi, M.Sc Masaki Kasedou and other peoples in department of mathematics for their good advice, helpful discussions, warm encouragement and hospitality.

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