Proceedings of the 34th Sapporo Symposium on Partial Differential Equations

Edited by
T. Ozawa, Y. Giga, T. Sakajo, S. Jimbo, H. Takaoka,
K. Tsutaya, Y. Tonegawa, and G. Nakamura

Sapporo, 2009

Series #141. August, 2009

HOKKAIDO UNIVERSITY
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Partially supported by Grant-in-Aid for Scientific Research, the Japan Society for the Promotion of Science.
日本学術振興会科学研究費補助金　（基盤研究 S 課題番号 21224001）
日本学術振興会科学研究費補助金　（基盤研究 B 課題番号 80296748）
日本学術振興会科学研究費補助金　（基盤研究 B 課題番号 19340028）
日本学術振興会科学研究費補助金　（若手研究 A 課題番号 21684003）

PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 24 through August 26 in 2009 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 30 years ago. Professor Kôji Kubota and Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

T. Ozawa, Y. Giga, T. Sakajo, S. Jimbo,
H. Takaoka, K. Tsutaya, Y. Tonegawa,
and G. Nakamura

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（第３４回偏微分方程式論札幌シンポジウム）

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利根川 吉廣, 中村 玄

Organizers: T. Ozawa, Y. Giga, T. Sakajo, S. Jimbo, H. Takaoka, K. Tsutaya,
Y. Tonegawa, G. Nakamura

Period (期間): August 24 - 26, 2009

Venue (場所): Room 203, Faculty of Science Building #5, Hokkaido University
北海道大学 理学部 5 号館大講義室 (203 号室)

URL: http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program.html

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Nonlinear elliptic equations on a spherical domain and related topics

Catherine Bandle (Universität Basel)
Yoshitsugu Kabeya (Osaka Prefecture University)
Hirokazu Ninomiya (Meiji University)

1 Abstract

In this talk, we consider the nonlinear elliptic equation

$$\Lambda u + \lambda(|u|^{p-1}u - u) = 0 \quad \text{in } \Omega \subset \mathbb{S}^n \tag{1.1}$$

under homogeneous Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Here Λ denotes the Laplace-Beltrami operator on the standard unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and $p > 1$ is an arbitrary fixed number. We shall assume in addition that $n \geq 3$, $\lambda > 0$ and that $\Omega \subset \mathbb{S}^n$ is a geodesic ball, also called a "spherical cap", centered at the North Pole.

The corresponding problem in balls in $\mathbb{R}^n$ is well-understood and has a long history which can be traced back to the work of Fowler. In particular it can be shown by means of variational methods that if $1 < p < \frac{n+2}{n-2}$, the boundary value problem above possesses for every λ a positive solution. This solution is radial and monotone decreasing. If $p \geq \frac{n+2}{n-2}$ this technique fails because of lack of compactness of the Palais-Smale sequences. In fact for $p \geq \frac{n+2}{n-2}$ no solutions exist.

The situation is different on the sphere. Solutions also exist for all p provided the spherical cap contains the half-sphere and λ is sufficiently large. An explanation could be that the curvature has an effect on the existence of positive solutions. This is one of the reasons why the problem has received some attention in the recent years.

Problems on the sphere are often transofrmed to those on the Euclidean space via the stereographic projection, however, it is not so suitable for

investigating the bifurcation diagram. We treat the problem without using the projection.

Let $(y_1, y_2, \dots, y_{n+1})$ be the Cartesian coordinates in $\mathbb{R}^{n+1}$. We express the points of $\mathbb{S}^n$ in terms of polar coordinates:

$$\begin{cases} y_k = \Big(\prod_{j=1}^{k} \sin\theta_j\Big)\cos\theta_{k+1}, \quad k = 1, 2, \dots, n-2, \\ y_{n-1} = \Big(\prod_{j=1}^{n-1} \sin\theta_j\Big)\cos\phi, \\ y_n = \Big(\prod_{j=1}^{n-1} \sin\theta_j\Big)\sin\phi, \\ y_{n+1} = \cos\theta_1. \end{cases}$$

In polar coordinates the spherical caps are expressed as

$$\Omega = \Omega_\varepsilon := \Big\{(\theta_1, \theta_2, \dots, \theta_{n-1}, \phi)\,\Big|\quad 0 \le \theta_1 < \pi - \varepsilon, \\ 0 \le \theta_i \le \pi, (i = 2, 3, \dots, n-1),\ 0 \le \phi \le 2\pi\Big\}.$$

We shall consider the case where $\varepsilon > 0$ is small, i.e. the case where Ω_ε is close to the full sphere $\mathbb{S}^n$.

Observe that in polar coordinates Λ becomes

$$\Lambda u = (\sin\theta_1)^{1-n}\frac{\partial}{\partial\theta_1}\Big\{(\sin\theta_1)^{n-1}\frac{\partial u}{\partial\theta_1}\Big\} + \\ \sum_{k=2}^{n-1}(\sin\theta_1\dots\sin\theta_{k-1})^{-2}(\sin\theta_k)^{k-n}\frac{\partial}{\partial\theta_k}\Big\{(\sin\theta_k)^{n-k}\frac{\partial u}{\partial\theta_k}\Big\} + \Big(\prod_{k=1}^{n-1}\sin\theta_k\Big)^{-2}\frac{\partial^2 u}{\partial\phi^2}.$$

We shall consider (1.1) in the class of radial functions, that is, functions depending only on the azimuthal angle θ_1 ("latitude"). For such functions v, Λ reads as

$$\Lambda v = \frac{1}{\sin^{n-1}\theta_1}\frac{\partial}{\partial\theta_1}\Big\{\Big(\sin^{n-1}\theta_1\Big)\frac{\partial v}{\partial\theta_1}\Big\}$$

which in the sequel will be denoted by Λ_{θ_1}.

With this notation the boundary value problem (1.1), (1.2) for radial functions assumes the form

$$\begin{cases} \Lambda_{\theta_1} v + \lambda(|v|^{p-1}v - v) = 0, \quad 0 < \theta_1 < \pi - \varepsilon, \\ v(\pi - \varepsilon) = 0, \quad v_{\theta_1}(0) = 0. \end{cases} \tag{1.3}$$

Stingelin [18] computed for a fixed spherical cap, containing the upper hemi-sphere and for large $\lambda > 0$ numerically different types of positive solutions. He observed that the number of solutions increases with λ. Recently, Brezis and Peletier [4] studied Problem (1.3) for $n = 3$ and $p = 5$. They confirmed Stingelin's results [18]. Moreover, very recently, Bandle and Wei [6, 7, 8] studied intensively this subject from the singular perturbation points of view. They found similar concentration phenomena as in Ambrosetti, Malchiodi and Ni [1, 2] and Malchiodi, Ni and Wei [15].

Stingelin's diagrams looked very much like the one obtained in Kabeya, Morishita and Ninomiya [13] for the problem

$$\triangle u + \lambda(u^p - u) = 0 \quad \text{in} \quad \{|y| < 1\} \subset \mathbb{R}^n, \quad \frac{\partial u}{\partial \nu} + \epsilon u = 0 \quad \text{on} \quad \{|y| = 1\},$$

where $\partial/\partial\nu$ denotes the outer normal derivative.

This similarity motivated us to study Problem (1.3) with the methods of imperfect bifurcation developed in [13].

For our purpose, to construct solutions for λ near the eigenvalues of the *linearized problem in* Ω_ε

$$\begin{cases} \Lambda_{\theta_1} v + (p-1)\lambda v = 0 \quad \text{in} \quad (0, \pi - \epsilon) \\ v_{\theta_1}(0) = 0, \ v(\pi - \varepsilon) = 0, \end{cases} \tag{1.4}$$

and to determine their asymptotic behavior as $\varepsilon \to 0$.

Observe that (1.1) on $\mathbb{S}^n$ (limit case $\varepsilon = 0$) has a constant solution $v \equiv 1$ for any λ. Although this constant is never a solution to the boundary value problem in Ω_ε the analysis of the *linearized problem on* $\mathbb{S}^n$

$$\begin{cases} \Lambda_{\theta_1} v + (p-1)\lambda v = 0, \\ v_{\theta_1}(\pi) = v_{\theta_1}(0) = 0. \end{cases} \tag{1.5}$$

will be crucial. We are therefore led to a two-parameter problem, depending on the one hand on ε and on the other hand on the distance from λ to the next eigenvalue of (1.5).

In order to describe our main result we have to introduce some notation:

$$0 < \lambda_{1,\varepsilon} < \lambda_{2,\varepsilon} < \ldots < \lambda_{j,\varepsilon} < \ldots$$

denote the eigenvalues and $\varphi_{j,\varepsilon}$ the corresponding eigenfunctions of (1.4);

$$0 = \lambda_1 < \lambda_2 < \ldots \lambda_j < \ldots$$

are the eigenvalues and φ_j are the corresponding eigenfunctions of (1.5).

Notice that all eigenvalues are simple and that the corresponding eigenfunction vanishes $j-1$ times and φ_1 =const.

We define a function space. For $n \geq 3$, we choose a fixed q satisfying

$$\max\left\{\frac{n}{2}, \left(1-\frac{1}{p}\right)n\right\} \leq q < n, \tag{1.6}$$

(later on q will be very close to n) and set

$$\mathscr{W} := W^{1,q}_{0,\mathrm{az}}(\Omega_\varepsilon),$$

where $\mathscr{W}$ is the completion of $C_0^\infty(\Omega_\varepsilon)$-functions depending only on θ_1, with respect to the norm

$$\|\Phi\|_{\mathscr{W}} := \left(\int_{\Omega_\varepsilon} |\Phi_{\theta_1}|^q \, dS + \int_{\Omega_\varepsilon} |\Phi|^q \, dS\right)^{\frac{1}{q}}.$$

Note that for a function g depending only on θ_1, we have

$$\int_{\Omega_\varepsilon} g(\theta_1)\, dS = |\mathbb{S}^{n-1}| \int_0^{\pi-\varepsilon} g(\theta_1) \sin^{n-1}\theta_1 \, d\theta_1.$$

Because of the particular choice of q, Sobolev's embedding

$$W^{1,q}(\Omega_\varepsilon) \hookrightarrow L^{pq}(\Omega_\varepsilon) \quad W^{1,q}(\Omega_\varepsilon) \hookrightarrow L^{2q}(\Omega_\varepsilon)$$

holds. Moreover, we denote the orthogonal projection into the linear space $\langle \varphi_{j,\varepsilon} \rangle$, by $P_{j,\varepsilon}$ and the projection into its orthogonal space $\langle \varphi_{j,\varepsilon} \rangle^\perp$ by $Q_{j,\varepsilon}$.

Theorem 1.1 *Let $p > 1$ and $n \geq 3$. Then there exist two positive numbers ε_* and ζ_* sufficiently small, and a sequence of one-dimensional C^1-manifolds $\mathscr{S}_\varepsilon(j) \subset (\lambda_j - \zeta_*, \lambda_j + \zeta_*) \times W^{1,q}_{0,az}(\Omega_\varepsilon)$, for any $\varepsilon \in (0, \varepsilon_*)$, where $(p-1)\lambda_j = (j-1)(j+n-2)$ and $j = 2, 3, \ldots$ with the following property:*

(1) *the elements $v(\theta_1;\varepsilon)$ of $\mathscr{S}_\varepsilon(j)$ are solutions of* (1.1).

(2) *They are of the following form*

$$v(\theta_1;\varepsilon) = \rho_\varepsilon + w_\varepsilon + s(\varepsilon)\varphi_{j,\varepsilon} + h(s;\varepsilon),$$

where $\rho_\varepsilon \in C_0^\infty(0, \pi-\varepsilon)$ is such that $0 \leq \rho_\varepsilon \leq 1$ and $\rho_\varepsilon(\theta_1) = 1$ in $(0, \pi-2\varepsilon)$: $w_\varepsilon, h(s;\varepsilon) \in \mathscr{W}$ are small in the sense that $\|w_\varepsilon\|_{\mathscr{W}}, \|h(s;\varepsilon)\|_{\mathscr{W}} \to 0$ as $\varepsilon \to 0$ and $s \to 0$.

More precisely we have:

(3) $\|w_\varepsilon\|_{\mathscr{W}} = O(\varepsilon^{(n-q)/q})$, $w_\varepsilon(\theta_1) \to 1$ *locally uniformly on* $[0,\pi)$ *as* $\varepsilon \to 0$. *and* $\|h(s;\varepsilon)\|_{\mathscr{W}} = O(\varepsilon^{n\min\{p-1,1\}/pq}|s| + |s|^{\min\{p,2\}})$ *for* $|s| \le s^*(\varepsilon)$ *with some* $s^*(\varepsilon) > 0$.

(4) $s^* = o(\varepsilon^{(n-2)/\min\{2,p\}})$.

(5) *The relation between* s *and* λ *is determined implicitly from the equation* $H_\kappa(s,\lambda) = 0$ *which for small* ε *and* s *is expressed as*

$$H_\varepsilon(s,\kappa) = s\kappa + \eta(\varepsilon) + O(\varepsilon^{n-2}\kappa + \varepsilon^n|s| + |s|^{\min\{2,\ p\}}). \tag{1.7}$$

Here $\kappa = (p-1)(\lambda - \lambda_{j,\varepsilon})$ *varies within* $\kappa = O(\varepsilon^{(n-2)(\min\{2,p\}-1)/\min\{2,p\}})$, $\eta(\varepsilon)$ *depends only on* ε *and satisfies*

$$\eta(\varepsilon) = (-1)^j b_{n,j}\varepsilon^{n-2} + O(\varepsilon^{n-1})$$

with a positive constant $b_{n,j}$.

Remark 1.1 *The leading two terms of* (1.7) *indicate that the bifurcation diagram looks like hyperbolae near the eigenvalues of the whole sphere.*

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Recent Topics on the Boltzmann Equation

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The Boltzmann equation without Grad's angular cutoff assumption has been expected to exhibit various regularizing effects on the solutions. This talk is to introduce some recent results from the joint works with R. Alexandre, Z.-H. Huo, Y. Morimoto, C.-J. Xu, and T. Yang.

We are concerned with the Boltzmann equation of the form,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \quad x \in \mathbb{R}^n, \ v \in \mathbb{R}^n, t > 0$$

where the unknown $f = f(t,x,v) \geq 0$ stands for the number density of gas particles at position x with velocity v at time t while the *collision operator* Q is given by

$$Q(g,f)(v) = \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} B\left(v - v_*, \sigma\right) \left\{ g(v'_*) f(v') - g(v_*) f(v) \right\} d\sigma dv_*$$

where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma,$$

and $f(v_*) = f(t,x,v_*)$ etc. The *collision kernel* B is determined by the interaction law of two colliding particles. To capture the regularization properties, it is usual to assume the form

$$B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos\theta), \quad \cos\theta = \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$
$$\Phi(|v - v_*|) = (1 + |v - v_*|^2)^{\gamma/2}, \qquad \gamma \leq 1,$$
$$\sin^{n-2}\theta \, b(\cos\theta) \approx K\theta^{-1-2s} \quad \text{as } \theta \to 0, \qquad s \in (0,1).$$

Notice that B has a non-integrable singularity at $\theta = 0$, which entails that Q behaves like a fractional Laplacian $(-\Delta)^s$, or more precisely,

Proposition.[HMUY], [AMUXY] *Let $s \in (0,1)$ and $\gamma \in \mathbb{R}$.*
(1) *For any $m, \beta \in \mathbb{R}$, $\exists C > 0$ such that*

$$\|Q(g,\, h)\|_{H^m_v} \leq C \|g\|_{L^1_{v,\beta^+ + (\gamma+2s)^+}} \|h\|_{H^{m+2s}_{v,(\beta+\gamma+2s)^+}}.$$

(2) (Coercivity) *Let $g \geq 0, \not\equiv 0$, $g \in L^1_2 \bigcap L\log L$. Then, $\exists c_g > 0$, such that for any $h \in \mathcal{S}_v$*

$$\|\langle D_v \rangle^s \langle v \rangle^{\gamma/2} h\|^2_{L^2_v} \leq -\, c_g (Q(g,h), h)_{L^2_v} + C \|g\|_{L^1_{v,\max(\gamma^+, 2-\gamma^+)}} \|h\|^2_{L^2_{v,\gamma^+/2}},$$

where $r^+ = \max(r, 0)$ for $r \in \mathbb{R}$.

Thus, the v-regularization is expected on the solutions. First, consider the spatially homogeneous Cauchy problem

$$\partial_t f = Q(f,f) \quad (t>0), \qquad f|_{t=0} = f_0(v).$$

In [HMUY], the $H_v^{+\infty}$ and $\mathcal{S}_v$ regularizations have been proved on any weak solutions, and actually, they can be a more strong regularization, i.e., the Gevrey regularization, under an extra condition.

Theorem 1. [MU] *Suppose that $\gamma \in [0,1], s \in (0,1/2), \gamma + 2s < 1$ and let $f = f(t,v)$ be any smooth solution satisfying*

$$f \geq 0\,, \ \not\equiv 0, \qquad \exists \delta_0 > 0 \ \ s.t. \ \ e^{\delta_0 \langle v \rangle^2} f \in L^\infty([0,T];\, H_v^{+\infty}).$$

Then for any $t_0 \in (0,T)$, there exist $\rho > 0$ and $\delta, \kappa > 0$ with $\delta > \kappa T$ such that

$$\sup_{t\in[t_0,T]} \sup_\alpha \frac{\rho^{|\alpha|} \| e^{(\delta-\kappa t)\langle v\rangle^2} \partial_v^\alpha f(t)\|_{L_v^2}}{\{\alpha!\}^{2s}} < +\infty\,.$$

It is expected to remove the extra condition.

For the spatially inhomogeneous case, although the *transport term* $v \cdot \nabla_x$ is a hyperbolic operator, it produces the smoothing effect in x when coupled with the v-smoothing operator Q.

Theorem 2. [AMUXY] *Assume that $s \in (0,1/2)$ and $\gamma + 2s < 1$.*
(1) (Existence) *Let $f_0 \geq 0$ be such that*

$$\exists\, \rho_0 > 0 \ s.t. \ e^{\rho_0 \langle v\rangle^2} f_0 \in H^m_{x,v}, \qquad m \geq 4.$$

Then there exists $T_* > 0$ such that the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = Q(f,f) \quad (t>0), \qquad f|_{t=0} = f_0(x,v),$$

admits a unique non-negative solution f satisfying

$$\exists \rho > 0 \quad \text{s.t.} \quad e^{\rho\langle v\rangle^2} f \in C^0([0,T_*];\, H^m_{x,v}).$$

(2) (Full regularization) *Furthermore, if $m \geq 5$ and if there exists a compact set $K \subset \mathbb{R}^3_x$ such that*

$$\|f_0(x,\,\cdot)\|_{L^1(\mathbb{R}^3_v)} > 0, \quad \forall\, x \in K,$$

then there exist $\tilde{T}_0 \in (0,T_*)$ and a neighborhood V_0 of K in $\mathbb{R}^3_x$ such that

$$f \in C^\infty(]0,\tilde{T}_0[\times V_0;\, \mathcal{S}_v).$$

(3) *Moreover, if $\gamma \leq 0$, the non-negative solution f is unique in the space $C^0([0,T_*]\,;\, H^m_p(\mathbb{R}^6_{x,v}))$ for $m > 3/2 + 2s$, $p > 3/2 + 4s$.*

The local solutions are known to exist in the Gevrey class [S.U. '86] while (1) states the first existence theorem in Sobolev spaces. The global existence is a big open issue. The proof of (2) uses a generalized version of the uncertainty principle [MUXY]. The t-regularity comes from the equation itself. The Gevrey regularization is expected but is still open. More of related results and references will be presented in the course of the talk.

Mathematical Data Analysis for Magnetic Resonance Elastography

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1 Introduction

Recently, by using a phase contrast MRI (Magnetic Resonance Imaging) technique combined with some low frequency vibration equipment, a new technology called MRE (Magnetic Resonance Elastography) enables us to see the shear wave with attenuation generated in the soft tissues of a living body ([4]).

An early stage cancer doesn't have enough hydrogen atoms while MRI can only detect the density distribution of hydrogen atoms. However, MRE enables to detect an early stage cancer by identifying the stiffness and viscosity (i.e. viscoelasticity) of the soft tissues via measuring the shear wave by MRE.

MRE can measure all three components of the wave displacement field by repeating the experiment with MSG (motion-sensitizing gradient) along each of the three measurement directions. Since MRI is a slower imaging method than ultrasound, unlike ultrasound elastography, we can only use a time harmonic excitation on the boundary to generate a stationary displacement. But meanwhile the slow speed makes it possible to have a higher sensitivity and resolution than ultrasound elastography and to access deeper into the soft tissues of a living body, such as the brain.

Recovering the viscoelasticity of the soft tissues of a living body from the MRE measurement is an inverse problem. To solve it robustly, one needs to have a good model described as a partial differential equation abbreviated by PDE model which can describe wave motions inside the soft tissues of a living body, and based on this PDE model and the given measurement, inversion analysis must be done theoretically and numerically.

2 Mathematical Models of MRE

It is reasonable to consider the soft tissues of a living body as a viscoelastic body ([1, 3]). There are various models to describe the viscoelasticity of a living body. These models are described by using the density, elasticity tensor and viscosity tensor. Since the phase velocity of the shear wave depends on these physical coefficients and MRE can measure the displacement of shear wave, MRE may give us a way to identify them if we can identify the speed of the shear wave from measured MRE data.

Corresponding to MRE experiments done by M. Suga, we find the suitable partial differential equation with mixed boundary conditions to be the PDE model. The well-posedness of the boundary value problem for the PDE are proved, and a numerical simulation of the solution is given. Furthermore, we present how to formulate the phase velocity of the shear wave propagating in the soft tissues of a living body which is considered as an isotropic viscoelastic medium.

2.1 Kelvin-Voigt Model of Linear Viscoelasticity in Isotropic Medium

In order to describe a small deformation in the soft tissues of a living body, let

$$\mathbf{U}(t,x) = (U_1(t,x),\ U_2(t,x),\ U_3(t,x))$$

be the displacement vector of the deformation in the soft tissues of a living body at time $t \geq 0$ and a point x of a reference domain $\Omega \subset \mathbb{R}^3$ which is a bounded domain with Lipschitz continuous boundary $\partial\Omega$. If the reference was in an equilibrium state and there is not other exterior force except the force which kept the reference in the equilibrium state and a wave displacement field $\mathbf{U}(t,x)$ given on $\partial\Omega$, then $\mathbf{U}(t,x)$ satisfies the equations:

$$\rho(x)\partial_t^2 U_i(t,x) = \sum_{l=1}^{3} \frac{\partial}{\partial x_l}\sigma_{il}(\mathbf{U}) \quad (t \geq 0, x \in \Omega, 1 \leq i \leq 3). \tag{2.1}$$

where σ_{il} is stress tensor.

We consider the Kelvin-Voigt model of linear viscoelasticity in the soft tissues of a living body. The constitutive equation of this model is as follows:

$$\sigma_{il}(\mathbf{U}) = \sum_{k,m=1}^{3} \boldsymbol{\lambda}_{ilkm}\varepsilon_{km}(\mathbf{U}) + \sum_{k,m=1}^{3} \boldsymbol{\eta}_{ilkm}(\mathbf{U})\partial_t\varepsilon_{km} \quad (1 \leq i,\ l \leq 3), \tag{2.2}$$

with the strain tensor

$$\varepsilon_{ij}(\mathbf{U}) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \quad (1 \leq i, j \leq 3). \tag{2.3}$$

Here $0 < \rho(x) \in L^\infty(\Omega)$ (usually known as $\approxeq 1.0 \times 10^3$ kg/m^3), $\boldsymbol{\lambda}_{ilkm}(x) \in L^\infty(\Omega)$ and $\boldsymbol{\eta}_{ilkm}(x) \in L^\infty(\Omega)$ are density, elasticity tensor and viscosity tensor, respectively. It is physically natural to assume that the elasticity tensor and viscosity tensor satisfy the full symmetries:

$$\begin{cases} \boldsymbol{\lambda}_{ilkm}(x) = \boldsymbol{\lambda}_{likm}(x) = \boldsymbol{\lambda}_{ilmk}(x) = \boldsymbol{\lambda}_{kmil}(x), \\ \boldsymbol{\eta}_{ilkm}(x) = \boldsymbol{\eta}_{likm}(x) = \boldsymbol{\eta}_{ilmk}(x) = \boldsymbol{\eta}_{kmil}(x) \end{cases}$$

for any $x \in \overline{\Omega},\ i,l,k,m\,(1 \leq i,l,k,m \leq 3)$ and the strong convexity: there exists a constant $C > 0$ such that

$$\begin{cases} \displaystyle\sum_{i,l,k,m=1}^{3} \boldsymbol{\lambda}_{ilkm}(x)\zeta_{il}\zeta_{km} \geq C\sum_{i,l=1}^{3}\zeta_{il}^2, \\ \displaystyle\sum_{i,l,k,m=1}^{3} \boldsymbol{\eta}_{ilkm}(x)\zeta_{il}\zeta_{km} \geq C\sum_{i,l=1}^{3}\zeta_{il}^2 \end{cases} \tag{2.4}$$

for any $x \in \overline{\Omega},\ i,l,k,m\,(1 \leq i,l,k,m \leq 3)$ and symmetric matrix (ζ_{il}).

If these tensors are isotropic, that is the viscoelastic property does not depend on directions, then they are given as

$$\begin{cases} \boldsymbol{\lambda}_{ilkm} := \lambda(x)\delta_{il}\delta_{km} + \mu(x)(\delta_{ik}\delta_{lm} + \delta_{im}\delta_{lk}), \\ \boldsymbol{\eta}_{ilkm} := \zeta(x)\delta_{il}\delta_{km} + \eta(x)(\delta_{ik}\delta_{lm} + \delta_{im}\delta_{lk}) \end{cases}$$

with the Kronecker delta δ_{ij} for any $x \in \overline{\Omega},\ i,l,k,m\,(1 \leq i,l,k,m \leq 3)$. Here $\lambda(x)$ and $\mu(x)$ are the Lamé modulus, while $\zeta(x)$ and $\eta(x)$ are the viscosity coefficients. Especially, $\mu(x)$ and $\eta(x)$ are called shear modulus and shear viscosity respectively. Physically well known Possion's ratio ν is given by $\nu = \lambda/2(\lambda + \mu)$.

For simplicity, we assume that the soft tissues of a living body are isotropic. For the isotropic medium, the strong convexity (2.4) becomes

$$\begin{cases} \mu > \delta, \quad \eta > \delta, \\ 3\lambda + 2\mu > \delta, \quad 3\zeta + 2\eta > \delta \end{cases} \quad \text{(a.e. in } \Omega)$$

for some constant $\delta > 0$.

Consequently, in $(0, +\infty) \times \Omega$, (2.1), (2.2) and (2.3) become as follows:

$$\rho\partial_t^2 \mathbf{U} = \Big\{\nabla[\lambda\nabla \cdot \mathbf{U}] + \nabla \cdot [2\mu\varepsilon(\mathbf{U})]\Big\} + \Big\{\nabla[\zeta\nabla \cdot \partial_t\mathbf{U}] + \nabla \cdot [2\eta\varepsilon(\partial_t\mathbf{U})]\Big\}. \tag{2.5}$$

Based on the setup of MRE experiment done by M. Suga in Chiba University, Japan, we have to assume that the input has to be time harmonic and it is given on a part Γ_D of the surface $\partial\Omega$ of the living body while the rest of the part Γ_N is traction free. That is, let $\Gamma_D, \Gamma_N \subset \partial\Omega$ be open sets such that $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, $\Gamma_D \neq \emptyset$ and $\Gamma_D \cap \Gamma_N = \emptyset$, then the setup of the experiment given by the following mixed boundary conditions:

$$\begin{cases} \mathbf{U}(t, x) = \chi(t)e^{i\omega t}\mathbf{f}(x) & \text{on } (0, +\infty) \times \Gamma_D, \\ \partial_\nu \mathbf{U}(t, x) = 0 & \text{on } (0, +\infty) \times \Gamma_N \end{cases} \tag{2.6}$$

and free initial boundary condition:

$$\mathbf{U} = \partial_t\mathbf{U} = 0 \quad \text{on } \{0\} \times \Omega, \tag{2.7}$$

where ω is a given angular motion frequency (LF, ~50-1000 Hz), $\chi(t) \in C^\infty([0, \infty))$,

$$\chi(t) = \begin{cases} 0 & (0 \le t \le \frac{1}{2}), \\ 1 & (t \ge 1) \end{cases} \tag{2.8}$$

is a cutoff function and the conormal derivative along the outward unit normal vector $\boldsymbol{\nu}$ to $\partial\Omega$ is defined as

$$\partial_\nu \mathbf{U}(t, x) := \Big\{[\lambda(\nabla \cdot \mathbf{U}) + 2\mu\varepsilon(\mathbf{U})] + [\zeta(\nabla \cdot \partial_t\mathbf{U}) + 2\eta\varepsilon(\partial_t\mathbf{U})]\Big\}\boldsymbol{\nu}. \tag{2.9}$$

If $\lambda(x)$, $\mu(x)$, $\zeta(x)$ and $\eta(x)$ are piecewise smooth functions and discontinuous on some C^1 boundary ∂D of $D \subset \Omega$, then we have the transmission boundary condition on ∂D:

$$\begin{cases} \mathbf{U}^+ = \mathbf{U}^-, \\ \partial_\nu \mathbf{U}^+ = \partial_\nu \mathbf{U}^- \end{cases} \quad \text{on} \quad \partial D$$

Further, if the variation functions and $\rho(x)$ are very small except ∂D, then we can assume that these functions are piecewise constants. Later in this thesis, we are able to assume that these functions are locally piecewise constants for our inversion analysis.

2.2 Time Harmonic Motions in a Living Body

We will study the time harmonic waves in the soft tissues of a living body generated by the time harmonic input given as (2.6). Since we have the model (2.5) and (2.7), we state the following theorem to describe the time harmonic behaviors to the wave displacements field $\mathbf{U}(t, x)$. A similar result can also be found in [5].

Before we state our theorems, we would like to introduce the fractional Sobolev spaces $\dot{H}^s(\overline{\Gamma'_D})$ ($s = 1/2$ or $3/2$) to which the Dirichlet input $\mathbf{f}$ may belong. Here Γ'_D is a subset of Γ_D, its boundary is far away $\partial\Gamma_D$

and of C^2 class. Using the definitions in [2], for $s = 1/2$ or $3/2$, we write $\dot{H}^s(\overline{\Gamma'_D})$ for the set of distributions in usual fractional Sobolev space $H^s(\Gamma'_D)$. Furthermore, we introduce the spaces

$$V := \{\mathbf{u} \in H^1(\Omega) : \ \mathbf{u} = 0 \text{ on } \Gamma_D\}, \quad V_0 := \{\mathbf{u} \in V : \ \nabla \cdot \mathbf{u} = 0\}.$$

Associated with these, we define V^* and V_0^* as the dual spaces of V and V_0 in $L^2(\Omega)$. Hereafter, the norm $\|\cdot\|_V$, $\|\cdot\|_{V_0}$ and inner product $(\cdot,\cdot)$ of V, V_0 are those of $H^1(\Omega)$ and the norm of V^*, V_0^* are denoted by $\|\cdot\|_{V^*}$, $\|\cdot\|_{V_0^*}$. For some Sobolev space X the norm is defined as $\|f\|_{L^2((0,t);X)} := \int_0^t \|f\|_X dt$ $(t > 0)$.

The next theorem shows that the displacement $\mathbf{U}(t,x)$ of the wave converges exponentially .

Theorem 2.1. *1) For any given Dirichlet input $\mathbf{f}(x) \in \dot{H}^{3/2}(\overline{\Gamma'_D})$, there exists a unique solution $\mathbf{U}(t,x) \in C^1([0,+\infty); H^1(\Omega)) \cap C^2([0,+\infty); L^2(\Omega))$ to the following mixed problem (i.e. (2.5), (2.6) and (2.7)):*

$$\begin{cases} \rho\partial_t^2\mathbf{U} = \Big\{\nabla[\lambda\nabla\cdot\mathbf{U}] + \nabla\cdot[2\mu\varepsilon(\mathbf{U})]\Big\} + \Big\{\nabla[\zeta\nabla\cdot\partial_t\mathbf{U}] + \nabla\cdot[2\eta\varepsilon(\partial_t\mathbf{U})]\Big\} & \text{in } (0,+\infty)\times\Omega, \\ \mathbf{U}(t,x) = \chi(t)e^{i\omega t}\mathbf{f}(x) & \text{on } (0,+\infty)\times\Gamma_D, \\ \partial_\nu\mathbf{U}(t,x) = 0 & \text{on } (0,+\infty)\times\Gamma_N, \\ \mathbf{U} = \partial_t\mathbf{U} = 0 & \text{on } \{0\}\times\Omega. \end{cases}$$

where $\chi(t)$ and $\partial_\nu\mathbf{U}$ are defined in (2.8) and (2.9) respectively.

Moreover, this solution $\mathbf{U}(t,x)$ satisfies

$$\|\mathbf{U}(t)\|_{H^1(\Omega)} + \|\partial_t\mathbf{U}(t)\|_{L^2(\Omega)} \le C\|\mathbf{f}\|_{\dot{H}^{3/2}(\overline{\Gamma'_D})},$$

where C is a positive constant independent of $\mathbf{f}$.

2) There exist constants $M, \beta > 0$, such that

$$\|\mathbf{U}(t,x) - e^{i\omega t}\mathbf{u}(x)\|_{H^1(\Omega)} \le Me^{-\beta t},$$

where $\mathbf{u}(x) := (u_1(x), u_2(x), u_3(x)) \in H^1(\Omega)$ is the unique solution to the mixed boundary value problem for the stationary viscoelasticity equation (we call it **stationary viscoelasticity model** *for simplicity):*

$$\begin{cases} [\nabla(\lambda\nabla\cdot\mathbf{u}) + \nabla\cdot(2\mu\varepsilon(\mathbf{u}))] + i\omega[\nabla(\zeta\nabla\cdot\mathbf{u}) + \nabla\cdot(2\eta\varepsilon(\mathbf{u}))] + \rho\omega^2\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{f} & \text{on } \Gamma_D, \\ \partial_\nu\mathbf{u} := [\lambda\nabla\cdot\mathbf{u} + 2\mu\varepsilon(\mathbf{u})]\nu + i\omega[\zeta\nabla\cdot\mathbf{u} + 2\eta\varepsilon(\mathbf{u})]\nu = 0 & \text{on } \Gamma_N. \end{cases} \tag{2.10}$$

Remark. *By Theorem 2.1, $\mathbf{U}(t,x)$ converges to $\mathbf{u}(x)$ exponentially in time. Hence, it is reasonable to say that we know $\mathbf{u}(x)$ in $\overline{\Omega}$ if we know $\mathbf{U}(t,x)$ on $[0,T]\times\overline{\Omega}$ for some sufficiently large $T > 0$. Thus, the mixed boundary value problem (2.10) is the model of our MRE problem.*

2.3 Approximate Models In Soft Tissues

The incompressibility assumption of the medium used by the other MRE research teams can be recovered in our model to show the applicability of the corresponding reduced models. That is, by an asymptotic analysis, we can reduce the stationary isotropic viscoelasticity model to the modified Stokes model under some reasonable assumption for the soft tissues of a living body. All these will be used in inversion analysis for identifying the viscoelasticity.

3 Inverse Problem for MRE

We would say that in practical applications, it is very important to give reconstruction scheme to find the unknown parameters. Corresponding to this definition of inverse problem, we can set the partial differential equations to be the forward (direct) problem, an interior measurement of the wave displacement field is the knowing observations, then the goal is to find the coefficients (or parameters) of the PDE's, i.e. to recover the viscoelasticity.

Inverse Problem: By giving some proper PDE model and inputs $\mathbf{f}$'s on the boundary $\partial\Omega$ of a fixed domain Ω. Reconstruct the unknown $\mu(x)$ (if necessary and possible, η, λ, ζ and ρ also) from measured data $\mathbf{u}(x)$ in $\overline{\Omega}$, where $\mathbf{u}(x)$ is the solution to some model mentioned above.

3.1 Inversion Methods

For any practical inverse problem, the robustness and accuracy are required for any reconstruction scheme. By assuming the medium is locally homogeneous, we give several practical and numerical inversion methods for identifying the viscoelasticity: weak form method, regularized numerical differentiation method and wavelet analysis method based on Morlet wavelet. If the medium is not locally homogeneous. We give the least square method based on the projected gradient method as an inversion method. All the inversion methods are tested by using simulated data and real experimental data in both two dimensions and three dimensions. Finally, we give some brief discussion on the comparison of these inversion methods.

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The Gibbs-Thomson relation for anisotropic phase transitions

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1 Introduction and Main result

This is a joint work with Marco Cicalese in Università di Napoli Federico II and Giovanni Pisante in Seconda Universitá di Napoli.

In this talk we consider the Gibbs-Thomson relation between the coarse grained chemical potential and the non homogeneous and anisotropic mean curvature of a phase interface within the (non homogeneous and anisotropic) gradient theory of phase transitions thus proving a generalization of a conjecture stated by Gurtin and proved by Luckhaus and Modica in the isotropic case.

We consider the following energy functional

$$E_\varepsilon(u_\varepsilon) = \int_\Omega \varepsilon f(x, Du_\varepsilon) + \frac{W(u_\varepsilon)}{\varepsilon}\, dx, \tag{1}$$

where $W(s) : \mathbb{R} \to \mathbb{R}$ is a standard double-well potential which has exactly two zeros at ± 1 and $\varepsilon > 0$ is a parameter which denote the order of the width of the phase interface. We assume the function $f(x, p) \in C^2(\Omega \times (\mathbb{R}^n \setminus \{0\}))$, which denote non homogenuous anistropic surface tension, satisfies

$$f(x, tp) = t^2 f(x, p) \quad \text{for } \ x \in \Omega,\ p \in \mathbb{R}^n,\ t \in \mathbb{R} \tag{2}$$

and

$$c_1|p|^2 \le f(x, p) \le c_2|p|^2 \quad \text{for } \ x \in \Omega,\ p \in \mathbb{R}^n, \tag{3}$$

for positive constants $0 < c_1 \le c_2$. Also we assume $f^{\frac{1}{2}}(x, p)$ is strictly convex with respect to $p \in \mathbb{R}^n$.

We consider the variational problem of this energy under the volume constraint;

$$\min\left\{E_\varepsilon(u) \mid \int_\Omega u\, dx = m\right\} \tag{4}$$

for $-|\Omega| \le m \le |\Omega|$.

If we choose $f(x,p) = |p|^2$, that is, isotropic case, the energy functional is known as Ginzburg-Landau type and Modica proved that the energy functional converges to the the minimal surface area with fixed volume in [8], that is,

$$\int_\Omega \frac{\varepsilon|Du_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\,dx \to 2c_0\mathcal{H}^{n-1}(\Omega \cap \partial\{u_0 = -1\}) \tag{5}$$

as $\varepsilon \to 0$, where $c_0 = \int_{-1}^{+1} \sqrt{W(s)}\,ds$.

More generally Bouchitté proved that our energy functional (1) Γ-converges to the generalized perimeter of the limit interface in [6],

$$\int_\Omega \varepsilon f(x, Du_\varepsilon) + \frac{W(u_\varepsilon)}{\varepsilon}\,dx \to 2c_0 \int_{\Omega\cap\partial\{u_0=-1\}} f^{\frac{1}{2}}(x, n(x))\,d\mathcal{H}^{n-1} \tag{6}$$

as $\varepsilon \to 0$, where n is unit normal of $\partial\{u_0 = -1\}$.

We note that there is one more interesting model which Braides in [1] and [7] considered the phase transition in a periodic medium, that is, the case of $f(x,p)$ is δ-periodic with respect to x. δ represents the length scale of inhomogeneities in the medium. The energy functional Γ-converges to two different types of the generalized perimeter by the order of ε and δ.

Now we focus on the curvature of the limit interface. The minimizer u_ε satisfies the following Euler-Lagrange equation

$$-\varepsilon \sum_{i=1}^{n} D_i(f_{p_i}(x, Du_\varepsilon)) + \frac{W'(u_\varepsilon)}{\varepsilon} = \lambda_\varepsilon \quad \text{in } \Omega, \tag{7}$$

where λ_ε is Lagrange multiplier from the volume constraint. Luckhaus and Modica proved that in isotropic case, that is $f(x,p) = |p|^2$, the Lagrange multiplier λ_ε converges to the constant mean curvature of the limit interface in [8]. Moreover in the isotropic case, not only as the minimizer but also as the solution of the PDE, convergence to the curvature in suitable weak sense was proved, which is strongly related to the modified De Giorgi conjecture and recently studied in [10] and [11].

We consider this analogy for non homogenuous anistropic surface tension. Here we prove the Lagrange multiplier λ_ε converges to the anisotropic curvature for the energy functional (1). For the variational problem (4) the following fact holds.

Proposition 1.1. *Let $\{u_{\varepsilon_l}\}_{l=1}^\infty \subset C^1(\Omega)$ be a sequence of the minimizers of (4). Assume that u_{ε_l} converge to some $u_0 \in BV(\Omega)$ in $L^1(\Omega)$ as $\varepsilon_l \to 0$. Then we have*
(i) $u_0 = \pm 1$ for almost every $x \in \Omega$.
(ii) $E_0 = \{x \in \Omega | u_0 = -1\}$ is a solution of the variational problem;

$$\min\{P_\Omega^f(E) | E \subset \Omega, |E| = \frac{1}{2}(|\Omega| - m)\} \tag{8}$$

(iii)

$$\lim_{l\to\infty} E_{\varepsilon_l}(u_{\varepsilon_l}) = 2c_0 P_\Omega^f(E_0), \tag{9}$$

where P^f_Ω is a generalized perimeter defined by (17).

Now we show the convergence of the Lagrange multiplier. We remark that we assume that the regularity of the limit interface because even in isotropic case, it is known that in higher dimension ($n \geq 7$) minimal surface may have singularities.

Theorem 1.2. (Main Result) *Let $\{u_\varepsilon\} \subset C^1(\Omega)$ be a family of minimizers for the variational problem* (4) *and let $\{\lambda_\varepsilon\}$ be a family of Lagrange multiplier in* (7)*. In addition suppose that $\partial\{u_0 = -1\}$ is of class C^2. Then there exists subsequence $\{\varepsilon_l\}_{l=1}^\infty$ satisfying u_{ε_l} converges to $u_0 \in \mathrm{BV}(\Omega; \{-1, 1\})$ in $L^1(\Omega)$ and*

$$\lim_{l\to\infty} \lambda_{\varepsilon_l} = c_0\lambda_0 \tag{10}$$

where λ_0 is a constant anisotropic curvature of the limit interface $\partial E_0 \cap \Omega$ with $E_0 = \{x \in \Omega | u_0 = -1\}$ in the sense of (24).

2 Definitions and lemmata

We use the framework of Finsler metric to define the anisotropic curvature. Here, we state the perimeter and the first variation under the anisotropy, refer to [2], [3], [4] and [5].

Let ϕ be a non-negative function $\phi \in C^1(\Omega \times (\mathbb{R}^n \backslash \{0\}))$ satisfying

$$\phi(x, tp) = |t|\phi(x, p) \qquad \text{for} \quad x \in \Omega, \;\; p \in \mathbb{R}^n, \;\; t \in \mathbb{R}, \tag{11}$$

and

$$\lambda|p| \leq \phi(x, p) \leq \Lambda|p| \qquad \text{for} \quad x \in \Omega, \;\; p \in \mathbb{R}^n, \tag{12}$$

for positive constants λ and Λ. The dual function ϕ° of ϕ is defined by

$$\phi^\circ(x, p^\star) = \sup\{p^\star \cdot p \;\mid\; p \in B_\phi(x)\}, \tag{13}$$

where $B_\phi(x) = \{p \in \mathbb{R}^n | \; \phi(x, p) \leq 1\}$. Here we only treat $\phi^2(x, p)$ and ${\phi^\circ}^2(x, p)$ which are stricktly convex with respect to $p \in \mathbb{R}^n$. We apply the following terminology to

$$\phi^\circ(x, p) = \sqrt{f(x, p)}. \tag{14}$$

(ϕ-total variation) We define the gerneralized total variation by

$$\int_\Omega |Du|_\phi = \sup\left\{\int_\Omega u \,\mathrm{div} g \, dx | \; g \in \; C^1_0(\Omega;\, \mathbb{R}^n) \text{ and } g \in B_{\phi^\circ}(x)\right\} \tag{15}$$

for $u \in BV(\Omega)$. For a measurable set E in $\mathbb{R}^n$, we define the perimeter $P^\phi_\Omega(E)$ of E as

$$P^\phi_\Omega(E) = \int_\Omega |D\chi_E|_{\phi^\circ}. \tag{16}$$

Under the assumption of the convexity and (11) of ϕ, for smooth set E the definition (16) is equivalent to the following;

$$P_\Omega^\phi(E) = \int_{\Omega\cap\partial E} \phi^\circ(x,n)\, d\mathcal{H}^{n-1}, \tag{17}$$

where n is outer normal of ∂E.
(ϕ-normal vector and Cahn-Hoffman vector) Let n_ϕ^* and n_ϕ be

$$n_\phi^* = \frac{n}{\phi^\circ(x,n)} \quad \text{and} \quad n_\phi = \nabla_p \phi^\circ(x, n_\phi^*), \tag{18}$$

which n_ϕ^* is the normal vector with respect to anisotropy ϕ and n_ϕ is Cahn-Hoffman vector. For these vectors, the property

$$n_\phi^* \cdot n_\phi = 1 \tag{19}$$

holds. We define the signed distance function d_ϕ associated to ϕ

$$d_\phi(x) = \begin{cases} -\inf\{\delta_\phi(x,y) \mid y \in \mathbb{R}^n \backslash E\} & \text{if } x \in E, \\ \inf\{\delta_\phi(x,y) \mid y \in E\} & \text{if } x \in \mathbb{R}^n \backslash E, \end{cases} \tag{20}$$

where

$$\delta_\phi(x,y) = \inf\left\{ \int_0^1 \phi(\gamma,\, \gamma) dt \;\mid\; \gamma \in W^{1,1}([0,1];\Omega),\;\; \gamma(0) = x,\;\; \gamma(1) = y \right\}. \tag{21}$$

Thus on the boundary ∂E

$$n_\phi^* = Dd_\phi \quad \text{and} \quad n_\phi = \nabla_p \phi^\circ(x, Dd_\phi). \tag{22}$$

We can consider the extension of n_ϕ^* and n_ϕ on the neighborhood of ∂E by using this distance δ_ϕ.
(ϕ-tangential divergence) We define the ϕ-tangential divergence on ∂E by

$$\mathrm{div}_{\partial E}^\phi g = \mathrm{tr}[(\mathrm{Id} - n_\phi \otimes n_\phi^*) D\widetilde{g} + \phi_x^\circ(x, n_\phi^*) \otimes \widetilde{g}\,] \tag{23}$$

for $g \in C_0^\infty(\partial E; \mathbb{R}^n)$ where $\widetilde{g}$ is any smooth extension to $\mathbb{R}^n$. We notice that it holds independent of the extension.
(ϕ-anisotropic curvature) We define the ϕ-anisotropic curvature κ_ϕ by

$$\kappa_\phi = -\mathrm{div}_{\partial E}^\phi n_\phi. \tag{24}$$

Since $-n_\phi^* \cdot D_j n_\phi = 0$ by (19), we have

$$\mathrm{div}_{\partial E}^\phi n_\phi = \mathrm{div} n_\phi \quad \text{on } \partial E. \tag{25}$$

(first variation) Let $E_t := \{x + tg(x) | x \in E\}$ for $g \in C_0^\infty(\Omega; \mathbb{R}^n)$ and $t \in \mathbb{R}^n$. The first variation of the perimeter P_Ω^ϕ is calculated (Theorem 5.1 in [5])

$$\frac{d}{dt} P_\Omega^\phi(E_t)|_{t=0} = \int_{\Omega\cap\partial E} g \cdot n_\phi^*\, \kappa_\phi\, \phi^\circ(x,n)\, d\mathcal{H}^{n-1} \tag{26}$$

for $g \in C_0^\infty(\Omega;\mathbb{R}^n)$. For ϕ-tangential divergence, by the property (25) the following lemma holds.

Lemma 2.1. *If E is a C^2 set in $\mathbb{R}^n$ and $g \in C_0^\infty(\Omega;\mathbb{R}^n)$, then we have*

$$\int_{\Omega\cap\partial E} \mathrm{div}^\phi_{\partial E} g\ \phi^\circ(x,n)\ d\mathcal{H}^{n-1} = \int_{\Omega\cap\partial E} g\cdot n^*_\phi\ \mathrm{div}^\phi_{\partial E} n_\phi\ \phi^\circ(x,n)\ d\mathcal{H}^{n-1}. \quad (27)$$

Thus together with (26) and (27) for the first variation it follows that

$$\frac{d}{dt} P^\phi_\Omega(E_t)|_{t=0} = \int_{\Omega\cap\partial E} \mathrm{div}^\phi_{\partial E} g\ \phi^\circ(x,n)\ d\mathcal{H}^{n-1}. \quad (28)$$

(weak convergence) In order to justify the convergence, we consider the suitable form of Reshetnyak theorem. For this total variation (15), we can generalize the theorem of [8].

Lemma 2.2. *Let $\{v_\varepsilon\} \subset C^1(\Omega)$ and $v_0 \in BV(\Omega)$ satisfying v_ε converges to v_0 in $L^1(\Omega)$. If we assume*

$$Dv_\varepsilon \rightharpoonup Dv_0 \quad \text{in } \Omega \quad (29)$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega |Dv_\varepsilon|_\phi = \int_\Omega |Dv_0|_\phi, \quad (30)$$

then for the function $F(x,p) \in C(\Omega\times\mathbb{R}^n)$ satisfying

$$F(x,tp) = tF(x,p) \quad \text{for} \quad x\in\Omega,\ \ p\in\mathbb{R}^n,\ \ t\geq 0 \quad (31)$$

and

$$F(x,p) = 0 \quad \text{for} \quad x\notin K,\ \ p\in\mathbb{R}^n \quad (32)$$

with K is a fixed compact subset of Ω, we have

$$\lim_{\varepsilon\to 0}\int_\Omega F(x,Dv_\varepsilon)\ dx = \int_\Omega F(x,n^*_\phi)\ d|Dv_0|_\phi, \quad (33)$$

where $n^*_\phi = \frac{Dv_0}{|Dv_0|_\phi}$ of $\partial\{v_0 = -1\}$.

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Landau solutions for incompressible Navier-Stokes equations and applications

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This is a joint work with Tai-Peng Tsai (University of British Columbia).

We consider point singularities of very weak solutions of the 3D stationary Navier-Stokes equations in a finite region Ω in $\mathbb{R}^3$. The Navier-Stokes equations for the velocity $u : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ with external force $f : \Omega \to \mathbb{R}^3$ are

$$-\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \operatorname{div} u = 0, \quad (x \in \Omega). \tag{1}$$

A *very weak solution* is a vector function u in $L^2_{loc}(\Omega)$ which satisfies (1) in distribution sense:

$$\int -u \cdot \Delta\varphi + u_j u_i \partial_j \varphi_i = \langle f, \varphi \rangle, \quad \forall \varphi \in C^\infty_{c,\sigma}(\Omega), \tag{2}$$

and $\int u \cdot \nabla h = 0$ for any $h \in C^\infty_c(\Omega)$. Here the force f is allowed to be a distribution and

$$C^\infty_{c,\sigma}(\Omega) = \{\varphi \in C^\infty_c(\Omega, \mathbb{R}^3) : \operatorname{div} \varphi = 0\}. \tag{3}$$

In this definition the pressure is not needed. Denote $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ and $B^c_R = \mathbb{R}^3 \setminus B_R$ for $R > 0$.

We are concerned with the behavior of very weak solutions which solve (1) in the punctured ball $B_2 \setminus \{0\}$ with zero force, i.e., $f = 0$. There are a lot of studies on this problem [5, 10, 11, 4, 8]. A typical result is to show that, under some conditions, the solution is a very weak solution across the origin without singular forcing supported at the origin (removable singularity), and is regular, i.e., locally bounded, under possibly more assumptions (regularity). Dyer-Edmunds [5] proved removable singularity and regularity assuming both $u, p \in L^{3+\varepsilon}(B_2)$ for some $\varepsilon > 0$. Shapiro [10, 11] proved removable singularity and regularity assuming $u \in L^{3+\varepsilon}(B_2)$ for some $\varepsilon > 0$ and $u(x) = o(|x|^{-1})$ as $x \to 0$, without assumption on p. Choe and Kim [4] proved removable singularity assuming $u \in L^3(B_2)$ *or* $u(x) = o(|x|^{-1})$ as $x \to 0$, and regularity assuming $u \in L^{3+\varepsilon}(B_2)$ for some $\varepsilon > 0$. Kim and Kozono [8] recently proved removable singularity under the same assumptions as [4], and regularity assuming $u \in L^3(B_2)$ or u is small in weak L^3. As mentioned in [8], their result is optimal in the sense that if their assumption is replaced by

$$|u(x)| \le C_* |x|^{-1} \tag{4}$$

for $0 < |x| < 2$, then the singularity is not removable in general, due to the existence of *Landau solutions*, which is the family of explicit singular solutions calculated by

L. D. Landau in 1944 [6], and can be found in standard textbooks, see e.g., [7, p. 82] or [1, p. 206].

The purpose of this talk is to characterize the singularity and to identify the leading order behavior of very weak solutions satisfying the threshold assumption (4) when the constant C_* is sufficiently small. We show that it is given by Landau solutions.

We now recall Landau solutions in order to state our main theorems. Landau solutions can be parametrized by vectors $b \in \mathbb{R}^3$ in the following way: For each $b \in \mathbb{R}^3$ there exists a unique (-1)-homogeneous solution U^b of (1) together with an associated pressure P^b which is (-2)-homogeneous, such that U^b, P^b are smooth in $\mathbb{R}^3\backslash\{0\}$ and they solve

$$-\Delta u + (u \cdot \nabla)u + \nabla p = b\delta, \quad \operatorname{div} u = 0. \tag{5}$$

in $\mathbb{R}^3$ in the sense of distributions, where δ denotes the Dirac δ function. When $b = (0, 0, \beta)$, they have the following explicit formulas in spherical coordinates r, θ, ϕ with $x = (r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r\cos\theta)$:

$$U = \frac{2}{r}\left(\frac{A^2 - 1}{(A - \cos\theta)^2} - 1\right) e_r - \frac{2\sin\theta}{r(A - \cos\theta)} e_\theta, \quad P = \frac{-4(A\cos\theta - 1)}{r^2(A - \cos\theta)^2} \tag{6}$$

where $e_r = \frac{x}{r}$ and $e_\theta = (-\sin\theta\sin\phi,\ \sin\theta\cos\phi,\ \cos\theta)$. The parameters $\beta \geq 0$ and $A \in (1, \infty]$ are related by the formula

$$\beta = 16\pi\left(A + \frac{1}{2}A^2 \log\frac{A-1}{A+1} + \frac{4A}{3(A^2-1)}\right). \tag{7}$$

The formulas for general b can be obtained from rotation. One checks directly that $\|rU^b\|_{L^\infty}$ is monotone in $|b|$ and $\|rU^b\|_{L^\infty} \to 0$ (or ∞) as $|b| \to 0$ (or ∞). Recently Sverak [12] observed that Landau solutions were the only solutions of (1) in $\mathbb{R}^3\backslash\{0\}$ which are smooth and (-1)-homogeneous in $\mathbb{R}^3\backslash\{0\}$, without assuming axisymmetry. Hence Landau solutions can be regarded as the canonical family of the solutions for (1). See also [13, 2, 9] for related results.

If u, p is a solution of (1.1), we will denote by

$$T_{ij}(u, p) = p\delta_{ij} + u_i u_j - \partial_i u_j - \partial_j u_i \tag{8}$$

the momentum flux density tensor in the fluid, which plays an important role to determine the equation for (u, p) at 0. Our main result is the following.

Theorem 1 *For any $q \in (1, 3)$, there is a small $C_* = C_*(q) > 0$ such that, if u is a very weak solution of* (1) *with zero force in $B_2\backslash\{0\}$ satisfying* (4) *in $B_2\backslash\{0\}$, then*

there is a scalar function p satisfying $|p(x)| \le C|x|^{-2}$, unique up to a constant, so that (u,p) satisfies (5) in B_2 with $b_i = \int_{|x|=1} T_{ij}(u,p)n_j(x)$, and

$$\|u - U^b\|_{W^{1,q}(B_1)} + \sup_{x\in B_1} |x|^{3/q-1}|(u-U^b)(x)| \le CC_*, \tag{9}$$

where the constant C is independent of q and u.

The exponent q can be regarded as the degree of the approximation of u by U^b. The closer q gets to 3, the less singular $u - U^b$ is. But in our theorem, $C_*(q)$ shrinks to zero as $q \to 3_-$. Ideally, one would like to prove that $u - U^b \in L^\infty$. However, it seems quite subtle in view of the following model equation for a scalar function,

$$-\Delta v + cv = 0, \quad c = \Delta v/v. \tag{10}$$

If we choose $v = \log|x|$, then $c(x) \in L^{3/2}$ and $\lim_{|x|\to 0} |x|^2|c(x)| = 0$, but $v \notin L^\infty$. In equation for the difference $w = u - U^b$, there is a term $(w\cdot\nabla)U^b$ which has similar behavior as cv above.

In fact, we have the following stronger result. Denote by L^r_{wk} the weak L^r spaces. We claim the same conclusion as in Theorem 1 assuming only a small L^3_{wk} bound of u but not the pointwise bound (4).

Theorem 2 *There is a small $\varepsilon_* > 0$ such that, if u is a very weak solution of (1) with zero force in $\Omega = B_{2.1}\backslash\{0\}$ satisfying $\|u\|_{L^3_{wk}(\Omega)} =: \varepsilon \le \varepsilon_*$, then u satisfies $|u(x)| \le C_1\varepsilon|x|^{-1}$ in $B_2\backslash\{0\}$ for some C_1. Thus the conclusion of Theorem 1 holds if $C_1\varepsilon \le C_*(q)$.*

Our results are closely related to the *regularity problem* of very weak solutions, which could be considered when u is only assumed to be in L^2_{loc}. In fact, the problem with the assumption u being large in L^3_{wk} already exhibits a great difficulty. Recall the scaling property of (1): If (u,p) is a solution of (1), then so is

$$(u_\lambda, p_\lambda)(x) = (\lambda u(\lambda x), \lambda^2 p(\lambda x)), \quad (\lambda > 0). \tag{11}$$

The known methods are primarily perturbation arguments. Since L^3_{wk}-quasi-norm is invariant under the above scaling and does not become smaller when restricted to smaller regions, one would need to exploit the structure of the Navier-Stokes equations in order to get a positive answer. Compare the recent result [3] on axisymmetric solutions of nonstationary Navier-Stokes equations, which also considers a borderline case under the natural scaling.

This work is inspired by Korolev-Sverak [9] in which they study the asymptotic as $|x| \to \infty$ of solutions of (1) satisfying (4) in $\mathbb{R}^3\backslash B_1$. They show that the leading

behavior is also given by Landau solutions if C_* is sufficiently small. Our theorem can be considered as a dual version of their result. However, their proof is based on the unique existence in $\mathbb{R}^3$ of the equation for $v = \varphi(u - U^b) + \zeta$ where φ is a cut-off function supported near infinity and ζ is a suitable function chosen to make $\operatorname{div} v = 0$. If one tries the same approach for our problem, since one needs to remove the origin as well as the region $|x| \geq 2$ while extending $u - U^b$, one needs to choose a sequence φ_k with the supports of $1 - \varphi_k$ shrinking to the origin, which produce very singular force terms near the origin.

Instead, we first prove some lemma which gives the equation for (u, p) near the origin. Since the equation for u is same as U^b near the origin for $b = b(u)$, the δ-functions at the origin cancel in the equation for the difference. We then apply the approach of Kim-Kozono [8] to the difference equation, and prove its unique existence in $W_0^{1,r}(B_2)$ for $3/2 \leq r < 3$ and uniqueness in $W_0^{1,r} \cap L^3_{wk}(B_2)$ for $1 < r < 3/2$, which improves the regularity of the original difference. Above $W_0^{1,r}(B_2)$ is the closure of $C_c^\infty(B_2)$ in the $W^{1,r}(B_2)$-norm.

As an application, we give the following corollary. Recall u_λ for $\lambda > 0$ is defined in (11). A solution u on $B_2 \backslash \{0\}$ is called *discretely self-similar* if there is a $\lambda_1 \in (0, 1)$ so that $u_{\lambda_1} = u$. Such a solution is completely determined by its values in the annulus $B_1 \backslash B_{\lambda_1}$ since $u(\lambda_1^k x) = \lambda_1^{-k} u(x)$. They contain minus-one homogeneous solutions as a special subclass.

Corollary 3 *If u satisfies the assumptions of Theorem 1 and furthermore u is discretely self-similar in $B_2 \backslash \{0\}$, then $u \equiv U^b$.*

This corollary also follows from [9] (with domain $\mathbb{R}^3 \backslash B_1$ and $\lambda_1 > 1$). In the case of small C_*, this corollary extends the result of Sverak [12] on minus-one homogeneous solutions. The classification of discretely self-similar solutions with large C_* is unknown.

As another application, we consider a conjecture by Sverak [12, §5]:

Conjecture 4 *If u is a solution of the stationary Navier-Stokes equations* (1) *with zero force in $\mathbb{R}^3 \backslash \{0\}$ satisfying* (4) *with some $C_* > 0$. Then u is a Landau solution.*

We give a partial answer for this problem.

Corollary 5 *Conjecture 4 is true, provided the constant C_* is sufficiently small.*

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A singular weighted mean curvature flow in the plane
a Hamilton-Jacobi equation with an unusual free boundary

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joint work with **Yoshikazu Giga** Tokyo, **Przemysław Górka**, Talca & Warszawa

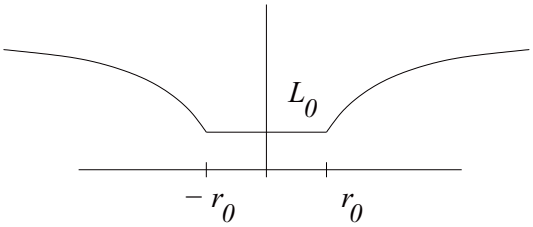
In many free boundary problems the modified Gibbs-Thomson law appears, it takes the form of a driven weighted mean curvature (wmc) flow

$$\beta V = \sigma + \kappa_\gamma \qquad \text{on } \Gamma(t). \tag{1}$$

Here κ_γ is the weighted mean curvature and it has be to carefully interpreted. Formally, it is defined as $\kappa_\gamma = -\mathrm{div}_S\,(\nabla_X \gamma)(X)|_{X=\mathbf{n}(x)}$, so the wmc flow is a second order parabolic equation. When we consider a well-justified anisotropy function γ given by the following formula

$$\gamma(p_1, p_2) = |p_1|\gamma_\Lambda + |p_2|\gamma_R,$$

then it turns out that the present definition of κ_γ does not make any sense. However, one may define it correctly through a variational principle. Subsequently, if $\sigma = 0$, then the resulting crystalline curvature flow has been well-studied by many authors. We consider here non-constant, positive σ which satisfies the so-called Berg's effect. One can see that for a piecewise linear γ as above the wmc degenerates to a first order problem. For the sake of simplicity, we restrict our attention to the evolution of a graph of an even function $\bar{d}$ over $\mathbb{R}$ like this one.



The distance of the central facet to the x_1-axis is denoted by $L_0(t)$. For such graphs equation (1) may interpreted as

$$\begin{aligned} \dot{L}_0 = \fint_0^{r_0(t)} \sigma(t, s, L_0)\, ds + \frac{\gamma_\Lambda}{r_0(t)} \quad &\text{on} \qquad [0, r_0(t)], \\ d_t = \sigma(t, s, d) m(d_x) \quad &\text{for} \qquad s \in (r_0(t), \infty), \end{aligned} \tag{2}$$

where m is the mobility coefficient $1/\beta(\mathbf{n})$. It is important to stress that $r_0(t)$, a half length of the central facet is a genuine free boundary. Its knowledge is necessary to close system (2).

We have tools to determine the sign of $\dot{r}_0(0)$ from the initial data. It turns out that more interesting is the case when $\dot{r}_0(0)$ is positive. We can try to write (2) as single first order PDE. We set

$$\bar{d}(t, x) = \begin{cases} L_0(t), & \text{if } x < r_0(t) \\ d(t, x), & \text{if } x \geq r_0(t), \end{cases}$$

$$\bar{H}(t, x, u, p) = \begin{cases} -\sigma(t, r_0^*(t), u) m(p), & \text{if } x < r_0(t) \\ -\sigma(t, x, u) m(p), & \text{if } x \geq r_0(t), \end{cases}$$

were r_0^* is chosen that $\sigma(t, r_0^*, u) = \fint_0^{r_0} \sigma(t, s, L_0)\, ds + \frac{\gamma_\Lambda}{r_0}$. Then (2) takes the following form,

$$\bar{d}_t + \bar{H}(t, x, d, d_x) = 0 \text{ in } (0, T_0) \times \mathbb{R} \quad \bar{d}(0, x) = \bar{d}_0(x), x \in \mathbb{R}, \tag{3}$$

It turns out that the case $\dot{r}_0(t) > 0$ is more interesting because then $r_0^*(t) < r_0(t)$ and $\bar{H}$ is discontinuous.

We notice that the obvious condition of continuity of $\bar{d}(t, \cdot)$, i.e.

$$d(r_0(t), t) = L_0(t) \tag{4}$$

(which we call the *matching condition*) is sufficient to determine the evolution of $r_0(t)$. We show that if $\dot{r}_0(0) > 0$, then (4) is a fixed point problem for $r_0(t)$. It has a solution for $t \in [0, T)$, where T is sufficiently small. Moreover, the solution is unique, provided that the right derivative of d_0 at $r_0(0)$ is positive.

Once we are given $r_0(t)$ we can close system (2) and prove that it has a unique solution. However, if the data d_0 for d in (2) is smooth, then uniqueness of the interfaces cannot be assured. For this reason we study viscosity solutions to (3) with initial data

$$\bar{d}_t + \bar{H}(t, x, d, d_x) = 0 \text{ in } (0, T_0) \times \mathbb{R} \quad \bar{d}(0, x) = \bar{d}_0^\epsilon(x), x \in \mathbb{R},$$

where $\bar{d}_0^\epsilon(x)$ is a "regularization" of initial data by introducing small discontinuity at $r_0(0)$.

Then we have to pass to the limit with ϵ in the hope that in this way we will select a unique interfacial curve r_0. In my talk I will present the limiting process, which involves:
(a) Proving existence of the limit of the matching curves r_0^ϵ and the independence of the limit of the chosen regularization.
(b) Showing that $(d^\epsilon, L_0^\epsilon)$ the unique solution of the regularized problem (2) forms a viscosity solution $\bar{d}^\epsilon$ to a Hamilton-Jacobi equation (3) with a *discontinuous* Hamiltonian $\bar{H}^\epsilon$, where

$$\bar{H}^\epsilon(t, x, u, p) = \begin{cases} -\sigma(t, (r_0^\epsilon)^*(t), u)m(p), & \text{if } x < r_0^\epsilon(t) \\ -\sigma(t, x, u)m(p), & \text{if } x \geq r_0^\epsilon(t). \end{cases}$$

(c) Proving existence of $\bar{d}$ a uniform limit of $\bar{d}^\epsilon$;
(d) Finding a proper notion of convergence of Hamiltonians $\bar{H}^\epsilon$ to $\bar{H}$.
(e) Showing that $\bar{d}$ is a viscosity solution to the equation with Hamiltonian $\bar{H}$. This is achieved by a Stability Theorem for a proper notion of convergence discontinuous Hamiltonians.
(f) Showing that $\bar{d}$ is unique a viscosity solution to (2). In order to prove this, we have to show a Comparison Principle when the Hamiltonian $\bar{H}$ is discontinuous, but the super- and subsolutions are special.

Pattern formations in the Kolmogorov flow

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July 28, 2009

Abstract

We consider Kolmogorov's problem for the 2D Navier-Stokes equations. Stability of and bifurcation from the trivial solution are studied numerically and analytically. More specifically, we compute solutions with large Reynolds numbers with a prescribed family of external forces of increasing degree of oscillation. We found that, whatever the external force may be, a stable steady-state of simple geometric character exits for sufficiently large Reynolds numbers. We thus observe a kind of universal outlook of the solutions, which is independent of the external force. This observation is reinforced by an asymptotic analysis of a simple equation called the Proudman-Johnson equation.

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Bursting Dynamics of the 3D Euler Equations in Cylindrical Domains

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Abstract

A class of three-dimensional initial data characterized by uniformly large vorticity is considered for the 3D incompressible Euler equations in bounded cylindrical domains. The fast singular oscillating limits of the 3D Euler equations are investigated for parametrically resonant cylinders. Resonances of fast oscillating swirling Beltrami waves deplete the Euler nonlinearity. These waves are exact solutions of the 3D Euler equations. We construct the 3D resonant Euler systems; the latter are countable uncoupled and coupled $SO(3;\mathbf{C})$ and $SO(3;\mathbf{R})$ rigid body systems. They conserve both energy and helicity. The 3D resonant Euler systems are vested with bursting dynamics, where the ratio of the enstrophy at time $t = t^*$ to the enstrophy at $t = 0$ of some remarkable orbits becomes very large for very small times t^*; similarly for higher norms $\mathbf{H}^s$, $s \geq 2$. These orbits are topologically close to homoclinic cycles. For the time intervals where $\mathbf{H}^s$ norms, $s \geq 7/2$ of the limit resonant orbits do not blow up, we prove that the full 3D Euler equations possess smooth solutions close to the resonant orbits uniformly in strong norms.

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The second term of the semi-classical asymptotic expansion for Feynman path integrals with integrand of polynomial growth.

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July 18, 2009

This is a short summary of the joint work with Naoto Kumano-go published in The Journal of Math. Soc. Japan [11].

Feynman path integral was invented by Feynman to quantize the motion of a particle moving in a potential field. It gives the integral kernel of evolution operator, which is the operator describing time evolution of wave function of a particle moving in a potential field. Contrary to Schrödinger equation, it does not use Hamiltonian but uses Lagrangian function.

Let $V(t,x)$ be a time dependent potential on the configuration space $\mathbf{R}^d$. Then the Lagrangian is

$$L(t,\dot{x},x) = \frac{1}{2}|\dot{x}|^2 - V(t,x).$$

A path γ is a continuous or sufficiently smooth map from the time interval $[s,s']$ to $\mathbf{R}^d$. The action $S(\gamma)$ of a path γ is the functional

$$S(\gamma) = \int_s^{s'} L(t, \frac{d}{dt}\gamma(t), \gamma(t))dt. \tag{2}$$

In [4] Feynman introduced the notion of an integral over the path space Ω, which is called Feynman path integral and is often denoted by

$$\int_{\Omega} e^{i\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma], \tag{3}$$

where $\nu = 2\pi h^{-1}$ with Planck's constant h. It was expected that Feynman path integral could have been defined as a measure theoretic integral if a suitable complex measure on the path space had been defined. However, Cameron [2] proved that this is not the case. (cf. also Johnson & Lapidus [13].)

Feynman himself gave the meaning to (3) as the limit of integrals over finite dimensional spaces. We call this method time slicing approximation method. Before we explain it in more detail, we make some preparation.

We assume that $V(t,x)$ is continuous in t and smooth in x and that it satisfies the following estimate: For any non-negative integer m there exists a non-negative constant v_m such that

$$\max_{|\alpha|=m} \sup_{(t,x)\in[0,T]\times\mathbf{R}^d} |\partial_x^\alpha V(t,x)| \leq v_m(1+|x|)^{\max\{2-m,0\}}. \tag{4}$$

Our assumption is close to that of Pauli [3].

Let $[s,s']$ be an interval of time. A path γ is called classical if it is a solution to the Euler equation

$$\frac{d^2}{dt^2}\gamma(t) + (\nabla V)(t,\gamma(t)) = 0 \quad \text{for } s < t < s'. \tag{5}$$

Here and hereafter ∇ stands for the nabla operator in the space $\mathbf{R}^d$.

For arbitrary pair of points $x, y \in \mathbf{R}^d$ there exists one and only one classical path γ which satisfies the boundary condition

$$\gamma(s) = y, \quad \gamma(s') = x \tag{6}$$

if $|s'-s| \leq \mu$ with sufficiently small μ, say for instance,

$$\frac{\mu^2 d v_2}{8} < 1. \tag{7}$$

In this case the action $S(\gamma)$ of γ is a function of (s',s,x,y) and is denoted by $S(s',s,x,y)$, i.e.,

$$S(s',s,x,y) = \int_s^{s'} L(t, \frac{d}{dt}\gamma(t), \gamma(t))\, dt. \tag{8}$$

Now we explain time slicing approximation method. Let

$$\Delta : 0 = T_0 < T_1 < \cdots < T_J < T_{J+1} = T \tag{9}$$

be a division of the interval $[0, T]$. Then we set $t_j = T_j - T_{j-1}$ and define the mesh $|\Delta|$ of the division Δ by $|\Delta| = \max_j\{t_j\}$. We always assume that

$$|\Delta| \leq \mu. \tag{10}$$

Let

$$x_j \in \mathbf{R}^d, \quad j = 0, 1, \ldots, J, J+1, \tag{11}$$

be arbitrary $J+2$ points of the configuration space $\mathbf{R}^d$. The piecewise classical path γ_Δ with vertices $(x_{J+1}, x_J, \ldots, x_1, x_0) \in \mathbf{R}^{d(J+2)}$ is the broken path that satisfies the Euler equation

$$\frac{d^2}{dt^2}\gamma_\Delta(t) + (\nabla V)(t, \gamma_\Delta(t)) = 0 \tag{12}$$

for $T_{j-1} < t < T_j (j = 1, 2, \ldots, J+1)$ and boundary conditions

$$\gamma_\Delta(T_j) = x_j, \quad j = 0, 1, \ldots, J, J+1, \tag{13}$$

where $x = x_{J+1}$ and $y = x_0$. When we emphasize the fact that this path γ_Δ depends on $(x_{J+1}, x_J, \ldots, x_1, x_0)$, we denote it by $\gamma_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ or $\gamma_\Delta(t; x_{J+1}, x_J, \ldots, x_1, x_0)$, where t is the time variable.

Let $F(\gamma)$ be a functional defined for paths γ. Its value $F(\gamma_\Delta)$ at γ_Δ can be written as a function $F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ of $(x_{J+1}, x_J, \ldots, x_1, x_0)$. For example the action functional $S(\gamma_\Delta)$ of γ_Δ is given by

$$\begin{aligned} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) = S(\gamma_\Delta) &= \int_0^T L(t, \frac{d}{dt}\gamma_\Delta(t), \gamma_\Delta(t))dt \\ &= \sum_{j=1}^{J+1} S_j(x_j, x_{j-1}), \end{aligned} \tag{14}$$

where we used the abbreviation

$$S_j(x_j, x_{j-1}) = S(T_j, T_{j-1}, x_j, x_{j-1}) = \int_{T_{j-1}}^{T_j} L(t, \frac{d}{dt}\gamma_\Delta(t), \gamma_\Delta(t))\, dt. \tag{15}$$

A piecewise classical time slicing approximation to Feynman path integral (3) with the integrand $F(\gamma)$ is an oscillatory integral

$$
\begin{aligned}
& I[F_\Delta](\Delta; x, y) \qquad (16)\\
&= \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{i\nu S(\gamma_\Delta)} F(\gamma_\Delta) \prod_{j=1}^{J} dx_j \\
&= \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{i\nu S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)} F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) \prod_{j=1}^{J} dx_j,
\end{aligned}
$$

where $x_{J+1} = x$ and $x_0 = y$.

Feynman's definition of path integral (3) is

$$
\int_\Omega e^{i\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma] = \lim_{|\Delta| \to 0} I[F_\Delta](\Delta; x, y), \qquad (17)
$$

if the limit on the right hand side exists. See Feynman [4].

One can ask questions:

1. Does this limit exists ?
2. What does this limit looks like if it exists ?

1 Existance of the limit.

In the case $F(\gamma) \equiv 1$ existence of the limit in (17) was proved by [5], [6]. [7],[16] and the Feynman path integral is in fact the fundamental solution of Schrödinger equation as Feynman expected.

In the case $F(\gamma) \not\equiv$ *constant* we give here a sufficient condition for the limit in (17) to converge. To explain our assumptions we make some preparation. The set $\Gamma(\Delta)$ of all piecewise classical paths associated with the division Δ forms a differentiable manifold of dimension $d(J+2)$. For a pair of divisions Δ' and Δ we use symbol $\Delta \prec \Delta'$ if Δ' is a refinement of Δ. The set Γ of all piecewise classical paths is the inductive limit of $\{\Gamma(\Delta), \prec\}$,i.e., $\Gamma = \lim_{\rightarrow} \Gamma(\Delta)$. Γ is a dense subset of the Sobolev space $H^1([0,T]; \mathbf{R}^d)$ of order 1 with values in $\mathbf{R}^d$ and hence it is also dense in the space $C([0,T]; \mathbf{R}^d)$ of all continuous paths. Let $\gamma_\Delta \in \Gamma(\Delta)$. Then the tangent space $T_{\gamma_\Delta}\Gamma$ to Γ at γ_Δ is the inductive limit $\lim_{\rightarrow} T_{\gamma_\Delta}\Gamma(\Delta')$, which is a dense linear subspace of the Sobolev space $H^1([0,T]; \mathbf{R}^d)$.

Let $F(\gamma)$ be a functional defined on Γ. We denote its Fréchet differential at $\gamma \in \Gamma$ by DF_γ if it exists. And $DF_\gamma[\zeta]$ stands for its value at the tangent vector $\zeta \in T_\gamma\Gamma$. For any integer $n > 0$ and for $\zeta_j \in T_\gamma\Gamma \quad (j = 1, 2, \ldots, n)$, we denote by $D^n F_\gamma[\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n]$, the symmetric n-linear form on the tangent space arising from the n-th jet modulo $n-1$-th jet of F at γ.

Our assumptions are the followings:.

Assumption 1 *Let $m \geq 0$. For any non-negative integer K there exist positive constants A_K and X_K such that for any division Δ of the form (9) and any integer $n_j \quad (0 \leq j \leq J+1)$ with $0 \leq n_j \leq K$*

$$\left| D^{\sum_{j=0}^{J+1} n_j} F_{\gamma_\Delta}[\otimes_{j=0}^{J+1} \otimes_{k=1}^{n_j} \zeta_{j,k}] \right| \tag{18}$$
$$\leq A_K X_K^{J+2} (1 + \|\gamma_\Delta\| + |||\gamma_\Delta|||)^m \prod_{j=0}^{J+1} \prod_{k=1}^{n_j} \|\zeta_{j,k}\|,$$

as far as $\zeta_{j,k} \in T_{\gamma_\Delta}\Gamma$ satisfies

$$\mathrm{supp}\zeta_{j,k} \subset \begin{cases} [0, T_1] & \text{if} \quad j = 0 \\ [T_{j-1}, T_{j+1}] & \text{if} \quad 1 \leq j \leq J \\ [T_J, T_{J+1}] & \text{if} \quad j = J+1, \end{cases} \tag{19}$$

where $\|\zeta\| = \max_{0 \leq t \leq T} |\zeta(t)|$ and $|||\gamma_\Delta||| =$ total variation of γ_Δ.

Assumption 2 *[15] [9]. There exists a positive bounded Borel measure ρ on $[0, T]$ such that with the same A_K, X_K as above*

$$\left| D^{1+\sum_{j=0}^{J+1} n_j} F_{\gamma_\Delta}[\eta \otimes \otimes_{j=0}^{J+1} \otimes_{k=1}^{n_j} \zeta_{j,k}] \right| \tag{20}$$
$$\leq A_K X_K^{J+2} (1 + \|\gamma_\Delta\| + |||\gamma_\Delta|||)^m \int_{[0,T]} |\eta(t)| \rho(dt) \prod_{j=0}^{J+1} \prod_{k=1}^{n_j} \|\zeta_{j,k}\|,$$

for any division Δ, integer $n_j \leq K$ and $\zeta_{j,k}$ which are the same as in Assumption 1. And η is an arbitrary element of $T_{\gamma_\Delta}\Gamma$.

Example:
Let $m \geq 0$, $f(t, x)$ be a smooth function and ρ be a function of bounded variation. Assume that for any multi-index α there exists a constant a constant C_α such that

$$|\partial_x^\alpha f(t, x)| \leq C_\alpha (1 + |x|)^m. \tag{21}$$

Then the Stieltjes integral

$$F(\gamma) = \int_0^T f(t, \gamma(t))\, d\rho. \tag{22}$$

is an example satisfying our assumptions.

We have the following

Theorem 1 *Assume that the integrand $F(\gamma)$ satisfies Assumption 1 and Assumption 2 above and T is so small that $|T| \leq \mu$, Then the limit of the right hand side of (17) converges compact-uniformly with respect to $(x, y) \in \mathbf{R}^{2d}$.*

We remark that Feynman [4] used also piecewise linear paths in place of piecewise classical paths. N. Kumano-go [15] proved the limit in (17) exists in the case of more general class of functional F using piecewise linear paths in place of piecewise classical paths.

We shall make more precise statement. In what follows we always assume that $0 < T \leq \mu$. For any fixed $(x, y) \in \mathbf{R}^{2d}$ the action $S(\gamma)$ has a unique critical point γ^*, which is the unique classical path starting y at time 0 and reaching x at time T. The critical point is non-degenerate. Similarly, if $0 < T \leq \mu$, the function $S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ of $(x_J, \ldots, x_1)$ has only one critical point, which is non-degenerate. Regarding ν as a parameter satisfying $\nu \geq 1$, we can apply stationary phase method to (16). Stationary phase method says that $I[F_\Delta](\Delta; x, y)$ is of the following form:

$$\begin{aligned} &I[F_\Delta](\Delta; x, y) \\ &= \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(\Delta; x, y)^{-1/2} e^{i\nu S(\gamma^*)} \Big(F(\gamma^*) + \nu^{-1} R_\Delta[F_\Delta](\nu, x, y)\Big). \end{aligned} \tag{23}$$

Here we used the following symbol

$$D(\Delta; x, y) = \left(\frac{t_{J+1} t_J \ldots t_1}{T}\right)^d \det \mathrm{Hess} S(\gamma_\Delta), \tag{24}$$

where $\mathrm{Hess} S(\gamma_\Delta)$ denotes the Hessian of $S(\gamma_\Delta)$ with respect to $(x_J, x_{J-1}, \ldots, x_1)$ evaluated at the critical point.

We know (cf. [7]) that $D(T, x, y) = \lim_{|\Delta| \to 0} D(\Delta; x, y)$ exists.

The function $T^{-d} D(T, x, y)$ coincides with the famous Morette-VanVleck determinant (cf. [7]).

Theorem 2 *Under the same assumption as in Theorem 1 we can write the limit* $\lim_{|\Delta|\to 0} I[F_\Delta](\Delta; x, y)$ *in the following way:*

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma] = \lim_{|\Delta|\to 0} I[F_\Delta](\Delta; x, y) \tag{25}$$
$$= \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(T, x, y)^{-1/2} e^{i\nu S(\gamma^*)} \Big(F(\gamma^*) + \nu^{-1} R[F](\nu, x, y)\Big).$$

For any non-negative integer K *there exist a positive constant* C_K *and a non-negative integer* $M(K)$ *independent of* ν *such that*

$$|\partial_x^\alpha \partial_y^\beta R[F](\nu, x, y)| \le C_K A_{M(K)} T(T + \rho([0, T]))(1 + |x| + |y|)^m. \tag{26}$$

2 The second term of semi-classical asymptotic expansion

Although ν is a constant in Physics, it is offten treated as a large positive parameter. It is expected that the Newton's classical mechanics is the limit of $\nu \to \infty$ (semi-classical limit) of quantum mechanics. Feynman discussed the asymptotic behaviour of Feynman path integral (3) as $\nu \to \infty$. i.e., the semiclassical asymptotic behaviour of Feynman path integrals. And he explained that the asymptotic behaviour of (3) is a result of "stationary phase method on path space". This is very interesting and challenging idea. **Can one make it mathematically rigorous?** Here is our partial answer.

It is expected that the following semi-classical asymptotic expansion holds;

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma] \tag{27}$$
$$= \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(T, x, y)^{-1/2} e^{i\nu S(\gamma^*)} \Big(A_0 + \nu^{-1} A_1 + O(\nu^{-2})\Big)$$
$$\text{as } \nu \to \infty.$$

Theorem 2 implies $A_0 = F(\gamma^*)$. What is the next term A_1 ?

In the case $F(\gamma) \equiv 1$ assuming the existence of expansion, Birkhoff gave the answer [1]. In fact, he gave even higher order terms of asymptotic expansion. However, if $F(\gamma) \neq constant$, then his method does not apply.

We write down the second term A_1 of (27) for general $F(\gamma)$ and prove that the asymptotic expression (27) actually holds. For this purpose we introduce

a new piece-wise classical path. Let ϵ be an arbitrary small positive number. And $\Delta(t,\epsilon)$ be the division

$$\Delta(t,\epsilon) : 0 = T_0 < t < t+\epsilon < T. \tag{28}$$

Let z be an arbitrary point in $\mathbf{R}^d$. We abbreviate the piecewise classical path $\gamma_{\Delta(t,\epsilon)}(s; x, \gamma^*(t+\epsilon), z, y)$ associated with the division $\Delta(t,\epsilon)$ by $\gamma_{\{t,\epsilon\}}(s,z)$, i.e., $\gamma_{\{t,\epsilon\}}(s,z)$ is the piecewise classical path which satisfies conditions:

$$\gamma_{\{t,\epsilon\}}(0,z) = y, \quad \gamma_{\{t,\epsilon\}}(t,z) = z, \quad \gamma_{\{t,\epsilon\}}(t+\epsilon,z) = \gamma^*(t+\epsilon), \quad \gamma_{\{t,\epsilon\}}(T,z) = x. \tag{29}$$

It is clear that $\gamma_{\{t,\epsilon\}}(s,z)$ coincides with $\gamma^*(s)$ for $t+\epsilon \le s \le T$ independently of z. Therefore, $\partial_z \gamma_{\{t,\epsilon\}}(s,z) = 0$ for $t+\epsilon \le s \le T$.

Lemma 1 *Under Assumption 1 and Assumption 2 we have bounded convergence of the limit*

$$q(t) = \lim_{\epsilon \to +0} \left[\Delta_z (D(t,z,y)^{-1/2} F(\gamma_{\{t,\epsilon\}}(*,z)))\Big|_{z=\gamma^*(t)} \right], \tag{30}$$

where Δ_z stands for the Laplacian with respect to z.

Theorem 3 *In addition to Assumptions 1 and 2 we further assume that the function $q(t)$ of Lemma 1 is Riemannian integrable over $[0,T]$. Set*

$$A_1 = \frac{i}{2} \int_0^T D(t,\gamma^*(t),y)^{1/2} q(t) dt. \tag{31}$$

Then, there holds the asymptotic formula, as $\nu \to \infty$,

$$\begin{aligned} &\int_\Omega e^{i\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma] \\ &= \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(T,x,y)^{-1/2} e^{i\nu S(\gamma^*)} \Big(A_0 + \nu^{-1} A_1 + \nu^{-2} r(\nu,x,y) \Big), \end{aligned} \tag{32}$$

where for any α, β the remainder term $r(\nu,x,y)$ satisfies estimate

$$|\partial^\alpha \partial^\beta r(\nu,x,y)| \le C_{\alpha,\beta} T^2 (1+|x|+|y|)^m. \tag{33}$$

We can calculate $q(t)$ in more detail for simple functionals $F(\gamma)$ of example (22).

Since our method is based on "stationary phase method of oscillatory integrals over a space of large dimension [8] and [10] ", it is completely different from Birkhoff's method, it may be interesting to see that this formula coincides with Birkhoff's result in the case of $F(\gamma) \equiv 1$.

Remark 1 *In this note the Lagrangian has no vector potential. Kitada-Kumano-go[14], Yajima [17] and Tshuchida-Fujiwara [12] discussed the case of Lagrangian with non zero vector potential. They proved that the limit (17) exists and the limit is the fundamental solution of Schrödinger equation if* $F(\gamma) \equiv 1$. *However we do not know whether the limit (17) exists or not if* $F(\gamma) \neq$ *constant and Lagrangian has non-zero vector potential.*

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Asymptotic stability of N-solitons of the FPU lattices

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The FPU lattice equation describes motion of particles connected by nonlinear springs:

$$\ddot{q}(t,n) = V'(q(t,n+1) - q(t,n)) - V'(q(t,n) - q(t,n-1)). \tag{1}$$

Here $q(t,n)$ denotes the displacement of the n-th particle at time t.

The FPU lattice equation was first studied by Fermi, Pasta and Ulam [3] who found recurrence phenomena for (1) with $V(r) = r^2/2 + r^3/6$ (α-FPU lattice) and $V(r) = r^2/2 + r^4/24$ (β-FPU lattice). A similar phenomena was observed by Zabusky and Kruskal [15] for KdV which is considered to be one of the continuous limit of (1). They found *solitons* which consists of multiple solitary waves that are stable, collide each other elastically, can be back to the initial state after a certain time. In this talk, we will discuss stability of multi-solitary waves of FPU.

First, we recall that FPU is a Hamiltonian system. Let $r(t,n) = q(t,n+1) - q(t,n)$, $p(t,n) = \dot{q}(t,n)$, $u(t,n) = (r(t,n), p(t,n))$ and

$$H(u(t)) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2} p(t,n)^2 + V(r(t,n)) \right). \tag{2}$$

Then (1) can be translated into a Hamiltonian system

$$\frac{du}{dt} = J\nabla_u H(u), \tag{3}$$

where $J = \begin{pmatrix} 0 & e^{\partial} - 1 \\ 1 - e^{-\partial} & 0 \end{pmatrix}$ and $e^{\pm\partial}$ are shift operators such that $(e^{\pm\partial} f)(n) = f(n \pm 1)$. Every finite energy solution of (3) satisfies a conservation law

$$H(u(t)) = H(u(0)) \quad \text{for every } t \geq 0. \tag{4}$$

FPU has two-parameter family of solitary wave solutions

$$\{u_c(n - ct + \delta) \mid \pm c > 1,\ \delta \in \mathbb{R}\},$$

where $u_c = (r_c, p_c)$, $p_c = -c\partial_x (e^{\partial} - 1)^{-1} r_c$ and

$$c^2 \partial_x^2 r_c = (e^{\partial} - 2 + e^{\partial}) V'(r_c). \tag{5}$$

A solitary wave is a single hump and exponentially localized pattern which does not change its shape and speed for all the time and its shape becomes similar to that of a KdV 1-soliton in the continuous limit, that is as $c \downarrow 1$ (see [4]). Being different from KdV 1-solitons, solitary wave

solutions of FPU are not characterized as a critical point of conservation laws because FPU lacks continuous symmetry in the spatial variable although their existence can be shown by a variational method (see Friesecke and Wattis [5]). Therefore, an energy based argument (e.g. [6]) fails to explain stability of solitary waves of (1).

Friesecke and Pego [4] proved stability of a solitary wave solution by using propagation estimates. Their idea is to build up a theory analogous to [12] using the fact that a major solitary wave outruns from other waves caused by perturbation. In fact, small solitary waves moves inherit linear stability property of KdV solitons in an exponentially weighted space.

In this talk, I will show stability of multi-soliton solutions which are moving to the same direction.

Theorem 1. *Suppose* $V \in C^\infty(\mathbb{R};\mathbb{R})$, $V(0) = V'(0) = 0$, $V''(0) = 1$, $V'''(0) \neq 0$, $0 < k_1 < \cdots < k_N$ *and* $c_{i,0} = 1 + \frac{k_i^2\varepsilon^2}{6}$ $(1 \le i \le N)$. *There exist positive numbers* ε_0, γ_0, A_0, L_0 *and* δ_0 *satisfying the following: Suppose* $\varepsilon \in (0, \varepsilon_0)$, $L > L_0$ *and that* $u(t)$ *is a solution to* (3) *such that* $\|v_0\|_{l^2} < \delta_0\varepsilon^2$,

$$u(\cdot, 0) = \sum_{i=1}^{N} u_{c_{i,0}}(\cdot - x_{i,0}) + v_0, \tag{6}$$

$$\min_{2\le i\le N} \varepsilon(x_{i,0} - x_{i-1,0}) > L. \tag{7}$$

Then there exist C^1*-functions* $x_i(t)$ $(i = 1, \cdots, N)$ *such that*

$$\sup_{t\ge 0} \left\| u(\cdot, t) - \sum_{i=1}^{N} u_{c_{i,0}}(\cdot - x_i(t)) \right\|_{l^2} < A_0(\|v_0\|_{l^2} + \varepsilon^{\frac{3}{2}} e^{-\gamma_0 L}). \tag{8}$$

Furthermore, there exist $c_{N,+} > \cdots > c_{1,+} > 1$ *and* $c_* \in (1, (1 + c_{1,0})/2)$ *such that*

$$\lim_{t\to\infty} \left\| u(\cdot, t) - \sum_{i=1}^{N} u_{c_{i,+}}(\cdot - x_i(t)) \right\|_{l^2(n\ge c_* t)} = 0, \tag{9}$$

$$\lim_{t\to\infty} \dot{x}_i(t) = c_{i,+} \quad \textit{and} \quad |c_{i,+} - c_{i,0}| < A_0(\varepsilon^{-1}\|v_0\|_{l^2}^2 + \varepsilon^2 e^{-\gamma_0 L}) \quad \textit{for } 1 \le i \le N. \tag{10}$$

Remark 1. *Eq.* (8) *implies orbital stability of FPU co-propagating* N*-solitons since by (P4),*

$$\|u_{c_{i,0}}\|_{l^2}^2 = 2\int_{\mathbb{R}} r_{c_{i,0}}(x)^2 dx(1 + o(1)) = O(\varepsilon^3).$$

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On reconstruction of polygonal cavities in an elastic body with one measurement

Masaru Ikehata and Hiromichi Itou (presenter)

1 Introduction

Inverse problems have been received a great deal of attention in various fields of science and engineering. Typical examples of inverse problems are to extract information about discontinuity embedded in a medium, such as cracks, cavities, obstacles and inclusions, from observed data. This kind of problems arises in geophysics, medical imaging such as Computed Tomography (CT) and Magnetic Resonance Imaging (MRI), Nondestructive testing, etc.

In the mathematical model, the problems are often described as inverse boundary value problems of partial differential equations called the governing equations and the solution is to extract information about location, shape and size of unknown discontinuity from boundary data. However, inverse problems are typically nonlinear and ill posed which means the lack of at least one of three conditions; existence, uniqueness, stability of the solution or solutions. Therefore, one cannot expect to obtain the desired solution in general. Then it is quite important to impose reasonable a priori assumptions on the problems in a practical sense and to extract the useful information of unknown discontinuity from fewer boundary data. From their standpoints we consider the following problem [9].

Let Ω be a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary and represent a homogeneous isotropic linearized elastic plate in both states of plane stress and plane strain. Let D denote cavities such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. As a priori assumption for unknown D we require that

(A1) $D = D_1 \cup D_2 \cup \cdots \cup D_m$, $\overline{D_j} \cap \overline{D_k} = \emptyset$ for $j \neq k$, where each D_j is a simply connected open set and polygon.

Problem . *Reconstruct unknown D satisfying* (A1) *from a single set of a surface force and the corresponding displacement field on the boundary of* Ω.

In order to solve the Problem we employ the *enclosure method* introduced by Ikehata [3]. By means of this method we show that one can reconstruct the convex hull of D from a single set of boundary data, stated as Theorem 1 in Section 3.

2 The enclosure method

The enclosure method is a methodology in inverse problems for partial differential equations. The method yields a partial information about the location of unknown discontinuity which appears as discontinuity of the coefficients of a partial differential equation or a part of the boundary of the common domain of definition of solutions of the equation. Although the original enclosure method makes use of infinitely many observation data, the single measurement version has been introduced in [2] and can be divided into three parts.

1. Find a special solution of the formal adjoint of the governing equation for the background medium which is parameterized by a large parameter τ and divides the whole space into two parts: in one part the absolute value of the solution decays as $\tau \to \infty$; in another part the solution grows as $\tau \to \infty$.

2. Construct an indicator function of independent variable τ by multiplying the governing equation of the medium by the special solution constructed in 1, integrating over the domain of definition and extracting only the integral on the known boundary of the domain.

3. Study the asymptotic behaviour of the indicator function as $\tau \to \infty$.

There are several existing applications of the single measurement version of the enclosure method to inverse problems, for the detail see his expository papers [4, 5] and recent papers [6, 7, 8, 10].

3 Statement of the main result

Let $\boldsymbol{n}$ be the unit outward normal vector to $\partial(\Omega\backslash\overline{D})$. Let the displacement vector $\boldsymbol{u} = (u_1, u_2)^{\mathrm{T}} \in \{H^1(\Omega \setminus \overline{D})\}^2$ satisfy the Navier equation in the absence of any body forces

$$\frac{\tilde{E}}{2(1+\tilde{\nu})}\triangle \boldsymbol{u} + \frac{\tilde{E}}{2(1-\tilde{\nu})}\nabla(\nabla\cdot\boldsymbol{u}) = \boldsymbol{0} \quad \text{in} \quad \Omega\setminus\overline{D} \tag{3.1}$$

and the traction free boundary condition

$$T\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \partial D \tag{3.2}$$

where $T\boldsymbol{u}$ is the stress vector expressed by

$$T\boldsymbol{u} = \frac{\tilde{\nu}\tilde{E}}{1-\tilde{\nu}^2}(\nabla\cdot\boldsymbol{u})\boldsymbol{n} + \frac{\tilde{E}}{2(1+\tilde{\nu})}\{\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\}\boldsymbol{n}$$

and

$$\tilde{E} = \begin{cases} E & \text{(plane stress)}, \\ \dfrac{E}{1-\nu^2} & \text{(plane strain)}, \end{cases} \qquad \tilde{\nu} = \begin{cases} \nu & \text{(plane stress)}, \\ \dfrac{\nu}{1-\nu} & \text{(plain strain)}. \end{cases}$$

Here E and ν is Young's modulus and Poisson's ratio of the elastic medium, respectively. Since both the shear modulus and the bulk modulus are required to be positive, we suppose $E > 0$ and $-1 < \nu < \frac{1}{2}$.

We denote by h_D the support function of D:

$$h_D(\boldsymbol{\omega}) = \sup_{\boldsymbol{x}\in D} \boldsymbol{x}\cdot\boldsymbol{\omega} \quad \text{for} \quad \boldsymbol{\omega} \in S^1.$$

The function $h_D(\,\cdot\,)$ is called the support function of D and its values give the signed distances from the origin of coordinates to the support lines of D.

To state the main result, we introduce the following assumptions for $\boldsymbol{\omega}$.

(W1) $\boldsymbol{\omega}$ satisfies that the intersection of the line $\boldsymbol{x}\cdot\boldsymbol{\omega} = h_D(\boldsymbol{\omega})$ with ∂D consists of only one point,

(W2) $\boldsymbol{\omega}$ satisfies (W1) and that the interior angle bisector of D at the point $\{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} \cdot \boldsymbol{\omega} = h_D(\boldsymbol{\omega})\} \cap \partial D$ is not perpendicular to the line $\boldsymbol{x} \cdot \boldsymbol{\omega} = h_D(\boldsymbol{\omega})$.

As shown in Figure 1 in Section 4, (W2) is equivalent to a condition $p + q + \pi \neq 0$. Then our main result is the following formula:

Theorem 1 ([9]). *Let $\boldsymbol{u}$ be not a rigid displacement. Under the assumptions of* (A1) *and* (W2) *the formula*

$$h_D(\boldsymbol{\omega}) = \lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log \left| \int_{\partial\Omega} (T\boldsymbol{u} \cdot \boldsymbol{v} - T\boldsymbol{v} \cdot \boldsymbol{u}) \, \mathrm{d}S \right|,$$

is valid, where $\boldsymbol{v}(\boldsymbol{x}) = (\boldsymbol{\omega} + i\boldsymbol{\omega}^{\perp})e^{\tau \boldsymbol{x} \cdot (\boldsymbol{\omega} + i\boldsymbol{\omega}^{\perp})}$, $\tau > 0$; $\boldsymbol{\omega}^{\perp} \in S^1$ is perpendicular to $\boldsymbol{\omega}$ and satisfies $\det(\boldsymbol{\omega}^{\perp} \; \boldsymbol{\omega}) > 0$.

Some remarks on Theorem 1 are in order.

- For given D satisfying (A1) the set of all $\boldsymbol{\omega}$ that does not satisfy (W2) (or (W1)) is a finite set. Since the support function is continuous, it follows from Theorem 1 that a single set of the data $\boldsymbol{u}$, $T\boldsymbol{u}$ on $\partial\Omega$ uniquely determine $h_D(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in S^1$ and thus the convex hull of D via the formula

  $$\bigcap_{\boldsymbol{\omega} \in S^1} \{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} \cdot \boldsymbol{\omega} < h_D(\boldsymbol{\omega})\}.$$

  This is the origin of the name "the enclosure method" and simultaneously the solution of Problem.

- In contrast to the related results, we do not require any other a priori assumptions for D excepting (A1) (e.g. [2]) and any constraints on boundary data (e.g. [6, 7, 8]).

- If we assume (W1) instead of (W2), then the formula is valid by replacing $\lim\limits_{\tau \longrightarrow \infty}$ with $\limsup\limits_{\tau \longrightarrow \infty}$, for the detail see the end of Section 4.

- In two-dimensional case rigid displacements can be described in the form

  $$F(\boldsymbol{x})\boldsymbol{k} = (k_0 + k_2 x_2, k_1 - k_2 x_1)^{\mathrm{T}}$$

  with an arbitrary constant vector $\boldsymbol{k} = (k_0, k_1, k_2)^{\mathrm{T}}$.

- It follows from the compatibility condition on the data that

  $$\int_{\partial\Omega} T\boldsymbol{u} \cdot F(\boldsymbol{x})\boldsymbol{k} \, \mathrm{d}S = 0$$

  for arbitrary $\boldsymbol{k} \in \mathbb{R}^3$.

4 Outline of the proof of Theorem 1

The proof of Theorem 1 proceeds along the same lines as results [2, 6, 7, 8].

The first part of the enclosure method mentioned in Section 2 is to find the special solution for the Navier equation (3.1), which corresponds to $\boldsymbol{v}(\boldsymbol{x})$ in Theorem 1.

The indicator function in the second part is just

$$\tau \longmapsto \int_{\partial\Omega} (T\boldsymbol{u}\cdot\boldsymbol{v} - T\boldsymbol{v}\cdot\boldsymbol{u})\,\mathrm{d}S.$$

The third part of the enclosure method is to study the asymptotic behaviour of the indicator function as $\tau \longrightarrow \infty$. This part is the most important in this analysis, because if one can see that

$$I_{\boldsymbol{\omega}}(\tau) \equiv e^{-\tau h_D(\boldsymbol{\omega})} \int_{\partial\Omega} (T\boldsymbol{u}\cdot\boldsymbol{v} - T\boldsymbol{v}\cdot\boldsymbol{u})\,\mathrm{d}S$$

is truly algebraic decaying as $\tau \longrightarrow \infty$, then it is easy to see that

$$\lim_{\tau\longrightarrow\infty} \frac{1}{\tau} \log |I_{\boldsymbol{\omega}}(\tau)| = 0 \tag{4.1}$$

and then the formula in Theorem 1 automatically follows.

The assumptions (A1) and (W1) yield that there exists a unique point $\boldsymbol{Q}$ on ∂D such that $\boldsymbol{Q}\cdot\omega = h_D(\boldsymbol{\omega})$, that is, $\boldsymbol{Q}$ has to be a vertex of a connected component of D. Hereafter we always assume that $\boldsymbol{\omega}$ satisfies (W1). Using the special form of $\boldsymbol{v}$ and integration by parts, we see that $I_{\boldsymbol{\omega}}(\tau)$ coincides with the function

$$e^{-\tau h_D(\boldsymbol{\omega})} \int_{\partial D\cap B_R(\boldsymbol{Q})} T\boldsymbol{v}\cdot\boldsymbol{u}\,\mathrm{d}S, \quad B_R(\boldsymbol{Q}) = \{\boldsymbol{x}\in\mathbb{R}^2 \mid |\boldsymbol{x}-\boldsymbol{Q}| < R\} \quad R > 0,$$

modulo exponentially decaying as $\tau \longrightarrow \infty$. Thus it is quite important to know the behaviour of $\boldsymbol{u}$ in a neighbourhood of $\boldsymbol{Q}$.

4.1 A convergent series expansion near a corner

In order to derive the convergent series expansion of $\boldsymbol{u}$ around $\boldsymbol{Q}$, first let us fix some notation. For $R > 0$ we set $D_R = B_R(\{\boldsymbol{Q}) \cap (\Omega \setminus \overline{D})$. We see that if $R < 1$ is sufficiently small, then D_R becomes a sector with center angle $2\alpha > \pi$, see Figure 1. We now fix such R.

Next, we identify each point $\boldsymbol{x}$ in D_R with $\boldsymbol{X}$ in a local coordinate system $\boldsymbol{X} = (X_1, X_2)^{\mathrm{T}} = (r\cos\theta, r\sin\theta)^{\mathrm{T}}$ with respect to the origin $\boldsymbol{Q}$ as follows, see Figure 1.

$$\boldsymbol{x} = \boldsymbol{Q} + r(\cos\theta\,\boldsymbol{a} + \sin\theta\,\boldsymbol{a}^{\perp}) \quad 0 < r < R \quad -\alpha < \theta < \alpha,$$

where $\frac{\pi}{2} < \alpha < \pi$,

$$\begin{aligned} \boldsymbol{a} &= \sin(\alpha+p)\,\boldsymbol{\omega} + \cos(\alpha+p)\,\boldsymbol{\omega}^{\perp} = \sin(q-\alpha)\,\boldsymbol{\omega} + \cos(q-\alpha)\,\boldsymbol{\omega}^{\perp}, \\ \boldsymbol{a}^{\perp} &= \cos(\alpha+p)\,\boldsymbol{\omega} - \sin(\alpha+p)\,\boldsymbol{\omega}^{\perp} = \cos(q-\alpha)\,\boldsymbol{\omega} - \sin(q-\alpha)\,\boldsymbol{\omega}^{\perp}, \end{aligned}$$

$-\pi < q < p < 0$ and $p - q = 2\pi - 2\alpha$.

Then, we can derive the convergent series expansion of $\boldsymbol{u}$ near $\boldsymbol{Q}$ in D_ρ for $0 < \rho < R$.

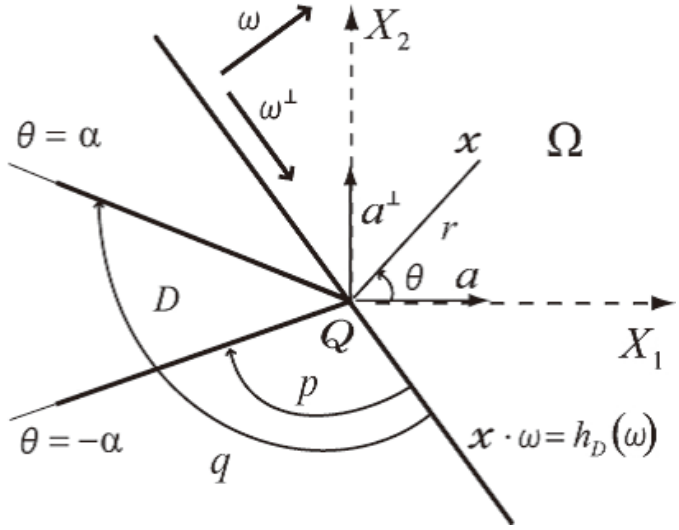

Figure 1: Polar coordinates

Proposition 1 ([9]). *If $2\alpha \neq \tan 2\alpha$, then there exist complex numbers $A_{j,m}$, $B_{j,m}(s)$ for $j = 1, 2,\ m = 1, 2, 3, \cdots$ and a constant vector $\tilde{\boldsymbol{k}}$ such that*

$$\boldsymbol{u}(r,\theta) = \sum_{j,m} A_{j,m} r^{-s_{j,m}-1} \boldsymbol{\Phi_j}(s_{j,m},\theta) + \sum_{j,m} \frac{\partial}{\partial s}\left(r^{-s-1} B_{j,m}(s) \boldsymbol{\Phi_j}(s,\theta)\right)\Big|_{s=\tilde{s}_{j,m}} + F(\boldsymbol{X})\tilde{\boldsymbol{k}}, \quad (4.2)$$

where $\boldsymbol{\Phi_j}$ is the following vector field :

$$\boldsymbol{\Phi_1}(s,\theta) = \begin{pmatrix} (1+\tilde{\nu})s\sin(s+2)\alpha\sin(s+1)\theta + (\tilde{\nu}-3)\sin s\alpha\sin(s+1)\theta \\ \quad -(1+\tilde{\nu})(s+1)\sin s\alpha\sin(s+3)\theta \\ -(1+\tilde{\nu})s\sin(s+2)\alpha\cos(s+1)\theta + (\tilde{\nu}-3)\sin s\alpha\cos(s+1)\theta \\ \quad +(1+\tilde{\nu})(s+1)\sin s\alpha\cos(s+3)\theta \end{pmatrix},$$

$$\boldsymbol{\Phi_2}(s,\theta) = \begin{pmatrix} (1+\tilde{\nu})s\cos(s+2)\alpha\cos(s+1)\theta + (\tilde{\nu}-3)\cos s\alpha\cos(s+1)\theta \\ \quad -(1+\tilde{\nu})(s+1)\cos s\alpha\cos(s+3)\theta \\ (1+\tilde{\nu})s\cos(s+2)\alpha\sin(s+1)\theta - (\tilde{\nu}-3)\cos s\alpha\sin(s+1)\theta \\ \quad -(1+\tilde{\nu})(s+1)\cos s\alpha\sin(s+3)\theta \end{pmatrix}$$

and $s_{j,m}$, $\tilde{s}_{j,m}$ which are not -2, are the simple and double roots of $h_j(\alpha, s) = 0$;

$$\begin{aligned} h_1(\alpha,s) &= (s+1)\sin 2\alpha - \sin 2(s+1)\alpha, \\ h_2(\alpha,s) &= (s+1)\sin 2\alpha + \sin 2(s+1)\alpha, \end{aligned}$$

respectively, following decreasing of $\mathbf{Re}s$ *order for $m = 1, 2, 3, \cdots$ in* $\mathbf{Re}s < -1$.
Moreover, the series (4.2) is convergent, absolutely in $H^1(D_\rho)$ and uniformly in D_ρ.

Remark 1. *The coefficients $A_{j,m}$, $B_{j,m}(s)$ in (4.2) have the following properties*:

1. *If* $s_{j,m} \in \mathbb{R}$, *then the corresponding* $A_{j,m} \in \mathbb{R}$.

2. *If* $s_{j,m} \in \mathbb{C} \setminus \mathbb{R}$ *such that* $s_{j,m+1} = \overline{s}_{j,m}$ *which denotes the complex conjugate of* $s_{j,m}$, *then it holds* $A_{j,m+1} = \overline{A}_{j,m}$.

3. $B_{j,m}(\tilde{s}_{j,m}) \in \mathbb{R}$ *and* $\dfrac{\partial}{\partial s} B_{j,m}(\tilde{s}_{j,m}) \in \mathbb{R}$.

Remark 2. *If* $2\alpha = \tan 2\alpha$, *then the following additional term with a real constant* A^* *should be taken into account to the formula* (4.2);

$$rA^* \begin{pmatrix} (1+\tilde{\nu})(1+\cos 2\alpha)\sin\theta + 2(1-\tilde{\nu})(\cos 2\alpha)\theta\cos\theta + 4\log r\cos 2\alpha\sin\theta \\ (1+\tilde{\nu})(1-\cos 2\alpha)\cos\theta + 2(1-\tilde{\nu})(\cos 2\alpha)\theta\sin\theta - 4\log r\cos 2\alpha\cos\theta \end{pmatrix}.$$

Sketch of the proof of Proposition 1

Here we give the essence of the proof of Proposition 1, for the detail see [9]. First we construct Airy's stress function $U \in H^2(D_R)$ by using Poincaré's lemma. Then, the problem (3.1)–(3.2) in D_R can be reduced to the Dirichlet boundary value problem for the biharmonic equation of U. Next, applying the Mellin transform to the problem for U, we have a boundary value problem for a fourth order ordinary differential equation in the transformed complex domain. For the problem we represent the solution by Green's function (e.g. [15]) and investigate the property. Since according to [1] and [11] it is well known that U multiplying by a cut-off function belongs to a weighted Sobolev space, we can apply the inverse Mellin transform to the solution for $r < \rho$. Furthermore, the residue theorem enable us to change the integration path. Here the residue of the integrand is calculated by the roots of the transcendental equations $h_j(\alpha, s) = 0$, which are not explicitly solvable and are complex. This is completely different from that of the case that the governing equation is the Laplace equation [2] and the case that D is a linear crack [6]. In the matter of the transcendental equations the following results are known, e.g. [1], [9] and [12].

Lemma 1.

1. *When* $\frac{\pi}{2} < \alpha < \alpha_0$, *it holds* $-2 < \mathbf{Re}s_{2,1} < -\frac{3}{2}$.
   When $\alpha_0 < \alpha < \pi$, *it holds* $-2 < \mathbf{Re}s_{1,1} < -\frac{\pi}{2\alpha} - 1 < \mathbf{Re}s_{2,1} < -\frac{3}{2}$.

2. *In* $\mathbf{Re}s < -1$ *the lines* $\mathbf{Re}s = \ell_{k+1} \equiv -k\frac{\pi}{2\alpha} - 1$ $(k = 1, 2, 3, \cdots)$ *contain no roots of each* $h_j(\alpha, s) = 0$.

3. *If* k *is an odd number, then the strip* $\ell_{k+2} < \mathbf{Re}s < \ell_{k+1}$ *contains only two roots of* $h_1(\alpha, s) = 0$ *(including complex conjugate and multiplicity) for* $\frac{\pi}{2} < \alpha < \pi$.

4. *If* k *is an even number, then the strip contains only two roots of* $h_2(\alpha, s) = 0$ *for* $\frac{\pi}{2} < \alpha < \pi$.

5. *The multiplicity of the roots of each* $h_j(\alpha, s) = 0$ *is not greater than two. Moreover, the double roots are all real and at most two.*

6. *For* $\frac{\pi}{2} < \alpha < \pi$ *the roots of each* $h_j(\alpha, s) = 0$ *in* $\mathbf{Re}s < -2$ *are not integers.*

7. *For* $\frac{\pi}{2} < \alpha < \pi$ *the roots of each* $h_j(\alpha, s) = 0$ *in* $s(s+2) \neq 0$ *satisfy* $\sin s\alpha \cos s\alpha \neq 0$.

Thus we obtain an asymptotic expansion of U with the remainder term expressed by the integral form.

Finally, by using the estimate of the Green's function we have the estimate of the remainder term and we can derive convergent series of U. Since $\boldsymbol{u}$ is uniquely determined by U up to any rigid displacements $F(\boldsymbol{X})\boldsymbol{k}$, we obtain Proposition 1.

4.2 The asymptotic behaviour of $I_{\boldsymbol{\omega}}(\tau)$ as $\tau \longrightarrow \infty$

Next, to show (4.1) we consider the asymptotic behaviour of $I_{\boldsymbol{\omega}}(\tau)$ by using Proposition 1. The arbitrary truncation of the expansion (4.2) together with a remainder estimate yields the complete asymptotic expansion of $I_{\boldsymbol{\omega}}(\tau)$.

Proposition 2. *Assume D contains no corner of angle satisfying $2\alpha = \tan 2\alpha$. For each $k \geq 1$, as $\tau \to \infty$,*

$$\begin{aligned} I_{\boldsymbol{\omega}}(\tau) &= \sum_{\mathbf{Re} s_{j,m} > \ell_{k+1}} \frac{C_j(s_{j,m}) A_{j,m}}{\tau^{-s_{j,m}-1}} + \sum_{\mathbf{Re} \tilde{s}_{j,m} > \ell_{k+1}} \frac{\partial}{\partial s} \left(\frac{C_j(s) B_{j,m}(s)}{\tau^{-s-1}} \right) \Bigg|_{s=\tilde{s}_{j,m}} \\ &\quad + O\left(\frac{1}{\tau^{\frac{k\pi}{2\alpha}}} \right), \end{aligned} \tag{4.3}$$

where

$$C_1(s) = 4 \frac{\tilde{E}}{1+\tilde{\nu}} \sin s\alpha \, e^{i\tau \boldsymbol{Q}\cdot\boldsymbol{\omega}^{\perp}} e^{-i\frac{\pi}{2}s} e^{-i(s+2)(p+\alpha)} \Gamma(-s) \left\{ 1 - e^{is2\pi} \right\},$$

$$C_2(s) = 4i \frac{\tilde{E}}{1+\tilde{\nu}} \cos s\alpha \, e^{i\tau \boldsymbol{Q}\cdot\boldsymbol{\omega}^{\perp}} e^{-i\frac{\pi}{2}s} e^{-i(s+2)(p+\alpha)} \Gamma(-s) \left\{ 1 - e^{is2\pi} \right\}.$$

Remark 3. *If D contains some corners of angle satisfying $2\alpha = \tan 2\alpha$, then* Proposition 2 *should be modified by adding the following term to* (4.3);

$$-A^* \left(8\pi i \frac{\tilde{E}}{1+\tilde{\nu}} \cos 2\alpha \, e^{i\tau \boldsymbol{Q}\cdot\boldsymbol{\omega}^{\perp}} \right) \frac{1}{\tau}. \tag{4.4}$$

However, since Proposition 2 only implies $I_{\boldsymbol{\omega}}(\tau)$ decays at most algebraically as $\tau \longrightarrow \infty$, this is not enough to prove (4.1) without any restrictions on boundary data in itself.

For this we need to show that there exist a positive constants τ_0, c independent of τ and a real number $\mu_0 < -1$ such that for all $\tau \geq \tau_0$

$$|I_{\boldsymbol{\omega}}(\tau)| \geq c\tau^{\mu_0}. \tag{4.5}$$

Now we set

$$P_1 \equiv \{ s_{j,m} \in \mathbb{C} \mid A_{j,m} \neq 0 \}, \quad P_2 \equiv \left\{ \tilde{s}_{j,m} \in \mathbb{R} \;\middle|\; B_{j,m}(\tilde{s}_{j,m}) \neq 0 \text{ or } \frac{\partial}{\partial s} B_{j,m}(\tilde{s}_{j,m}) \neq 0 \right\}.$$

Since $\boldsymbol{u}$ is not a rigid displacement from the hypothesis in Theorem 1, the unique continuation theorem ensures $P_1 \cup P_2 \neq \emptyset$. Note here that the convergence of (4.2) is indispensable in this part because $P_1 \cup P_2 = \emptyset$ implies $\boldsymbol{u}(r,\theta) = F(\boldsymbol{X})\tilde{\boldsymbol{k}}$.

Accordingly, we can choose $\lambda \in P_1 \cup P_2$ such that $\mathbf{Re}\lambda \geq \mathbf{Re} s$ for any $s \in P_1 \cup P_2$. Then the following three cases are considered.

Case 1. $\lambda = s_{j,m} \in \mathbb{R}$.
In this case, from *6* and *7* in Lemma 1 one sees that $C_j(s_{j,m})$ in Proposition 2 never vanishes. Namely, it holds that

$$I_{\boldsymbol{\omega}}(\tau) = C_j(s_{j,m}) A_{j,m} \tau^{s_{j,m}+1} + o(\tau^{s_{j,m}+1}),$$

which simultaneously means (4.5).

Case 2. $\lambda = \tilde{s}_{j,m} \in \mathbb{R}$.
Similarly to *Case 1*, one sees $C_j(\tilde{s}_{j,m}) \neq 0$. From *3* in Remark 1, it holds that

$$I_{\boldsymbol{\omega}}(\tau) = \frac{\partial}{\partial s}\left(C_j(s)B_{j,m}(s)\tau^{s+1}\right)\Big|_{s=\tilde{s}_{j,m}} + o(\tau^{\tilde{s}_{j,m}+1}),$$

which leads to (4.5).

Case 3. $\lambda = s_{j,m} \in \mathbb{C} \setminus \mathbb{R}$ such that $s_{j,m+1} = \overline{s}_{j,m}$. Namely,

$$I_{\boldsymbol{\omega}}(\tau) = C_j(s_{j,m})A_{j,m}\tau^{s_{j,m}+1} + C_j(s_{j,m+1})A_{j,m+1}\tau^{s_{j,m+1}+1} + o(\tau^{s_{j,m}+1}). \tag{4.6}$$

In this case, it follows from *6*, *7* in Lemma 1 and *2* in Remark 1 that $C_j(s_{j,m})A_{j,m} \neq 0$ and $C_j(s_{j,m+1})A_{j,m+1} \neq 0$. And one sees $|C_j(s_{j,m})A_{j,m}| = |C_j(s_{j,m+1})A_{j,m+1}|$ if and only if $p+q+\pi = 0$. Hence, when $\boldsymbol{\omega}$ satisfies (W2) which means $p+q+\pi \neq 0$, from (4.6) we obtain (4.5).

Summing up the above, (4.1) is verified in the case that D contains no corner of angle satisfying $\tan 2\alpha = 2\alpha$. If $\tan 2\alpha = 2\alpha$, then note that (4.4) in Remark 3 vanishes if and only if $A^* = 0$. In consequence, by the same argument as above, (4.1) is still valid even in the case that some angles of D satisfy $\tan 2\alpha = 2\alpha$ and the proof of Theorem 1 is completed.

Lastly, if we weaken the assumption (W2) to (W1), then the formula in Theorem 1 does not always hold. Now let consider the case $p+q+\pi = 0$ in *Case 3*. Then, it can be easily seen that there exists a sequence $\{\tau_N\}_{N=1}^{\infty}$ such that $\tau_N \to \infty$ as $N \to \infty$ and it holds

$$\lim_{N\to\infty} \tau_N^{-(\mathbf{Re}\, s_{j,m}+1)} \left| C_j(s_{j,m})A_{j,m}\tau_N^{s_{j,m}+1} + C_j(\overline{s}_{j,m})\overline{A}_{j,m}\tau_N^{\overline{s}_{j,m}+1} \right| > 0.$$

Namely, it follows from (4.6) that

$$\lim_{\tau_N \longrightarrow \infty} \frac{1}{\tau_N} \log |I_{\boldsymbol{\omega}}(\tau_N)| = 0.$$

Hence, one concludes that the formula

$$h_D(\boldsymbol{\omega}) = \limsup_{\tau \longrightarrow \infty} \frac{1}{\tau} \log \left| \int_{\partial\Omega} \left(T\boldsymbol{u}\cdot\boldsymbol{v} - T\boldsymbol{v}\cdot\boldsymbol{u}\right)\, \mathrm{d}S \right|$$

is valid even when $p+q+\pi = 0$, that is, under the assumption (W1) not (W2).

Acknowledgments

M. Ikehata was partially supported by Grant-in-Aid for Scientific Research (C)(No. 21540162) of Japan Society for the Promotion of Science(JSPS). H. Itou was partially supported by Grant-in-Aid for Young Scientists (B)(No. 21740091) of the Ministry of Education, Culture, Sports, Science and Technology(MEXT).

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Variational Problems for Anisotropic Surface Energies

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Abstract

We study surfaces which are in equilibrium for a constant coefficient parametric elliptic functional with a volume constraint. The Euler-Lagrange equation is a nonlinear elliptic PDE. We study the stability of solutions of certain free boundary problems.

1 Introduction

An anisotropic surface energy is one that depends on the direction of a surface at each point. They were introduced by Georg Wulff to model the equilibrium shape of a crystal [21]. Whereas the surface energy of a liquid drop is isotropic, the ordered arrangement of molecules in a crystal mean that its interfacial energy depends on the surface direction. Some time after the discovery of liquid crystals in 1888, anisotropic surface energies were applied to study their interfacial surface energy also. In this article, we will consider the simple case where the energy density γ is a function of the surface normal ν alone. Let $\gamma : \mathbf{R}^3 \to \mathbf{R}^+$ be a nonnegative function which satisfies the following properties:

(i) $\gamma(X) = 0$ if and only if $X = \mathbf{0}$;

(ii) C^∞ in $\mathbf{R}^3 - \{\mathbf{0}\}$;

(iii) positive homogeneity of degree one : $\forall r \geq 0$, $\gamma(rX) = r\gamma(X)$;

(iv) strict convexity :

$$\sum_{i,j} D_{x_i x_j}\gamma(X)\xi_i\xi_j \geq |X|^{-1}|\xi'|^2, \qquad \xi' = \xi - \left(\xi \cdot \frac{X}{|X|}\right)\frac{X}{|X|}$$

holds for all $X = (x_1, x_2, x_3) \in \mathbf{R}^3 - \{\mathbf{0}\}$ and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$.

These conditions are equivalent to saying that the Frank diagram Frank $\gamma = \{X \in \mathbf{R}^3 \,;\, \gamma(X) \leq 1\}$ has smooth boundary with positive principal curvatures.

For a smooth, oriented immersed surface (we will simply write surface) $X : \Sigma = \Sigma^2 \to \mathbf{R}^3$ whose Gauss map (unit normal) is $\nu = (\nu_1, \nu_2, \nu_3) : \Sigma \to S^2 = \{X \in \mathbf{R}^3 \,;\, |X| = 1\}$, we define the functional

$$\mathcal{F}(X) := \int_\Sigma \gamma(\nu)\, d\Sigma, \tag{1}$$

where $d\Sigma$ is the area element of X. It is known that, up to translation, there exists a unique absolute minimizer $W(V)$ of $\mathcal{F}$ among all closed surfaces in $\mathbf{R}^3$ enclosing the same

The author is partially supported by Grant-in-Aid for Scientific Research (C) No. 19540217 of the Japan Society for the Promotion of Science.

3-dimensional volume V, and it is a smooth convex surface ([16]). Thus $W(V)$ solves the isoperimetric problem for the functional $\mathcal{F}$.

The closed convex surface which is given by

$$(\gamma_{x_1}, \gamma_{x_2}, \gamma_{x_3})|_{|X|=1} \tag{2}$$

is called the *Wulff shape* (of γ). We denote this surface by W (or W_γ). The unique minimizer $W(V)$ mentioned above is a homothety of W.

If $\gamma(-X) = \gamma(X)$ holds, then γ defines a norm in $\mathbf{R}^3$, and the unit sphere

$$\{Y \in \mathbf{R}^3 \,;\, \gamma^*(Y) = 1\}$$

of the dual norm

$$\gamma^*(Y) = \sup\{Y \cdot Z \,;\, \gamma(Z) \leq 1\}$$

coincides with the Wulff shape W.

In the special case where $\gamma(X) \equiv |X|$, $\mathcal{F}(X) = \displaystyle\int_\Sigma d\Sigma$ and so $\mathcal{F}(X)$ is the usual area of the surface X, and W is the unit sphere with center at the origin.

The functional appearing in (1) is sometimes called a (constant coefficient) *parametric elliptic functional* ([4], [7]). It was extensively studied from the viewpoint of geometric measure theory and convex analysis. However, in differential geometry it had not been studied very much until rather recently. One notable geometric investigation is [15].

In this article, we consider natural variational problems of the functional $\mathcal{F}$ for volume-preserving variations. In order to see the Euler-Lagrange equation, first we consider a surface which is a graph of a C^∞ function $f : D(\subset \mathbf{R}^2) \to \mathbf{R}$ as follows:

$$X : D \to \mathbf{R}^3, \quad X(u_1, u_2) = (u_1, u_2, f(u_1, u_2)).$$

Then, the unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ to X is given by

$$\nu = \frac{(-f_1, -f_2, 1)}{(1 + |Df|^2)^{1/2}},$$

where

$$f_1 := f_{u_1}, \quad f_2 := f_{u_2}, \quad Df := (f_1, f_2).$$

The Euler-Lagrange equation for the functional $\mathcal{F}$ for compactly supported volume-preserving variations is

$$\sum_{i,j=1,2} \gamma_{x_i x_j}\Big|_{X=(-Df,1)} f_{u_i u_j} = \text{constant}. \tag{3}$$

In the special case where $\gamma(X) \equiv |X|$, the left hand side of (3) is

$$\frac{(1 + (f_2))^2 f_{11} - 2 f_1 f_2 f_{12} + (1 + (f_1)^2) f_{22}}{((f_1)^2 + (f_2)^2 + 1)^{3/2}},$$

and this is the twice of the mean curvature H of X. In view of (3), the *anisotropic mean curvature* Λ of X is defined as (cf. [15], [8])

$$\Lambda := \sum_{i,j=1,2} \gamma_{x_i x_j}\Big|_{X=(-Df,1)} f_{u_i u_j}.$$

If Λ is constant, X is called a *surface of constant anisotropic mean curvature* (*CAMC surface*). The anisotropic mean curvature of the Wulff shape is -2 with respect to the outward pointing

normal. In the case where $\gamma(X) \equiv |X|$, $\Lambda = 2H$ holds. Hence, the study of CAMC surfaces includes the study of CMC surfaces (surfaces of constant mean curvature) and minimal surfaces (surfaces of zero mean curvature).

The convexity condition implies that the Euler-Lagrange equation "Λ = constant" is absolutely elliptic.

There are many natural variational problems for the anisotropic surface energy with given boundary conditions. Since a CAMC surface is a critical point of the energy, it is natural to ask whether it attains a local minimum of the energy or not. A CAMC surface X is said to be *stable* if the second variation of $\mathcal{F}$ is nonnegative for all volume-preserving variations of X which satisfy the given boundary condition.

In this article, we mainly consider a variational problem for anisotropic surface energy for surfaces which lie between two parallel planes and have free boundaries on these planes (§2). Here a wetting energy term for the surface to plane interface is included. We will give results about existence, uniqueness, and geometric properties of solutions which attain the local minimum of the total energy (§4).

There is a criterion for the stability for CAMC surfaces which is expressed in terms of properties of eigenvalues and eigenfunctions of an eigenvalue problem which is associated with the second variation of the energy (§3). In §5 we will give ideas of a part of proofs of our main results.

In §6, we will give some related topics and future subjects.

In Appendix, we will give a classification and a representation formula for CAMC surfaces of revolution for rotationally symmetric Wulff shape (§7.1), and we will give examples of complete CAMC surfaces for more general Wulff shapes such that all of the horizontal slices are homothetic (§7.2).

We should remark about the assumption on the regularity. In our analysis, it is sufficient that critical points of our variational problems are of $C^{3+\alpha}$ $(0 < \alpha < 1)$. In this article, for simplicity, we assume that the energy density γ is of C^∞ and we treat only surfaces of C^∞.

We give three more remarks: Although we obtained some results which are new even for CMC surfaces, there is not sufficient space to point them out here. Some of the results in this article can be generalized to hypersurfaces in $\mathbf{R}^{n+1}$. In this article, we use notations from analysis rather than notations from differential geometry (cf. [8]-[13]).

2 A free boundary problem

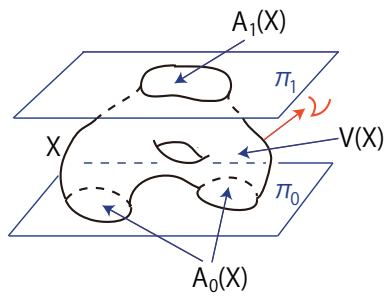


Figure 1: Surface between two parallel planes

Assume that γ is rotationally symmetric: $\gamma = \gamma(\nu_3)$. Set

$$\Pi_0 := \{x_3 = 0\}, \quad \Pi_1 := \{x_3 = h\}, \quad (h > 0), \qquad \Omega := \{0 \le x_3 \le h\}.$$

For an oriented embedded surface $X : (\Sigma^2, \partial\Sigma) \to (\Omega, \Pi_0 \cup \Pi_1)$, we define an energy:

$$\mathcal{E}[X] := \mathcal{F}[X] + \omega_0 \mathcal{A}_0[X] + \omega_1 \mathcal{A}_1[X],$$

where $\mathcal{A}_i[X]$ is the area in Π_i which is bounded by $C_i := X(\partial\Sigma) \cap \Pi_i$, and ω_i are constants. $\omega_i \mathcal{A}_i[X]$ is called a wetting energy.

We define the volume $V[X]$ of X as the usual volume of the three-dimensional domain enclosed by $X(\Sigma) \cup \Pi_1 \cup \Pi_2$. A variation $X_t : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ of X will be called an *admissible variation* if $V[X_t] = V[X]$ for all t.

For $p \in \Sigma$, there exists a uniquely determined point $\bar{G}(p)$ in the Wulff shape W such that the normal $\nu(p)$ of X coincides with the outward pointing unit normal to W at $\bar{G}(p)$. We call this map $\bar{G} : \Sigma \to W$ the anisotropic Gauss map of X.

If X is a critical point of $\mathcal{E}$ for all admissible variations, we call X a capillary surface.

Proposition 2.1 ([10]) *An embedding $X : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ is a capillary surface if and only if there holds: $\Lambda \equiv \Lambda_0$ in Σ for some constant Λ_0, and $\langle \overline{G}, E_3 \rangle \equiv -(-1)^i \omega_i$ on C_i $i = 0, 1$, that is, the contact angle of X with each Π_i is a constant.*

By using the maximum principle for solutions of absolutely elliptic partial differential equations, we can show the following:

Corollary 2.1 ([10]) *A capillary surface $X : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ for $\gamma = \gamma(\nu_3)$ is a CAMC surface of revolution with vertical rotation axis, and its genus is zero.*

In view of Corollary 2.1, a capillary surface $X : (\Sigma, \partial\Sigma) \to (\Omega, \Pi_0 \cup \Pi_1)$ is represented as

$$X(s, \theta) = (x(s)\cos\theta, x(s)\sin\theta, z(s)), \quad s_1 \le s \le s_2.$$

Let $X_t = X + (\xi + \psi\nu)t + \mathcal{O}(t^2)$ be an admissible variation of X, where ξ is the tangential component of the variation vector field. Then, the second variation of energy is

$$\delta^2\mathcal{E} := \left.\frac{d^2\mathcal{E}(X_t)}{dt^2}\right|_{t=0} = -\int_\Sigma \psi L[\psi]\, d\Sigma + \oint_{\partial\Sigma} \psi B[\psi]\, d\tilde{s} =: \mathcal{I}[\psi], \tag{4}$$

where L is the self-adjoint Jacobi operator

$$L[\psi] = x^{-1}\{(\mu_1^{-1} x \psi_s)_s + \mu_2^{-1} x^{-1} \psi_{\theta\theta}\} + \{\mu_1^{-1}(x''z' - x'z'')^2 + \mu_2^{-1}(x^{-1}z')^2\}\psi,$$

and

$$B[\psi] = \begin{cases} -(-1)^i \mu_1^{-1}\{\psi_s - (z''/z')\psi\}, & \sin\eta_i \ne 0, \\ \psi, & \sin\eta_i = 0 \end{cases}$$

on the boundary C_i. Here η_i is the contact angle, that is,

$$(x', z')|_{C_i} = (\cos\eta_i, \sin\eta_i), \qquad i = 0,\ 1,$$

and μ_1, μ_2 are the principal curvatures of W which are given by

$$\frac{1}{\mu_1} = (1 - \nu_3^2)\gamma_{\nu_3\nu_3} + \frac{1}{\mu_2} = -\frac{1 - \nu_3^2}{\nu_3}\left(\frac{1}{\mu_2}\right)_{\nu_3} + \frac{1}{\mu_2}, \qquad \frac{1}{\mu_2} = \gamma - \nu_3\gamma_{\nu_3}. \tag{5}$$

μ_1, μ_2 are positive functions which depend only on ν_3.

Definition 2.1 A capillary surface is said to be *stable* if $\delta^2\mathcal{E} \ge 0$ for all admissible variations, otherwise it is said to be unstable.

The analytic condition for stability is

Proposition 2.2 *A capillary surface $X : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ is stable if and only if $\mathcal{I}[\psi] \ge 0$ holds for all C^∞ functions ψ on Σ which satisfy*

(i) $\psi(w) = 0$ for $w \in \partial\Sigma$ where X is tangent to $\partial\Omega$, and (ii) $\int_\Sigma \psi\, d\Sigma = 0$.

3 Eigenvalue problem associated with the second variation

Let $X : (\Sigma, \partial\Sigma) \to (\Omega, \Pi_0 \cup \Pi_1)$ be a capillary surface. Consider the eigenvalue problem

$$L[\psi] = -\lambda\psi, \quad B[\psi]|_{\partial\Sigma} = 0, \qquad \psi \in H^1(\Sigma). \tag{6}$$

Denote by $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ the eigenvalues. Denote by E the eigenspace belonging to 0. The following criteria for the stability are known.

Lemma 3.1 (Maddocks, Vogel, Koiso) **(I)** *If $\lambda_1 \geq 0$, then X is stable.*
(II) *If $\lambda_1 < 0 < \lambda_2$, there exists a unique function $f \in C^\infty(\Sigma)$ satisfying $L[f] = 1$ and $B[f]|_{\partial\Sigma} = 0$, and the following (II-1) and (II-2) hold.*
(II-1) *If $\int_\Sigma f\, d\Sigma \geq 0$, then X is stable.*
(II-2) *If $\int_\Sigma f\, d\Sigma < 0$, then X is unstable.*
(III) *If $\lambda_1 < 0 = \lambda_2$, the following (III-1) and (III-2) hold.*
(III-A) *If there is an eigenfunction g belonging to $\lambda_2 = 0$ satisfying $\int_\Sigma g\, d\Sigma \neq 0$, then X is unstable.*
(III-B) *If $\int_\Sigma g\, d\Sigma = 0$ for all eigenfunctions g belonging to $\lambda_2 = 0$, then there exists a unique function $f \in E^\perp \cap C^\infty(\Sigma)$ satisfying $L[f] = 1$ and $B[f]|_{\partial\Sigma} = 0$, and the following (III-B1) and (III-B2) hold.*
(III-B1) *If $\int_\Sigma f\, d\Sigma \geq 0$, then X is stable.*
(III-B2) *If $\int_\Sigma f\, d\Sigma < 0$, then X is unstable.*
(IV) *If $\lambda_2 < 0$, then X is unstable.*

Lemma 3.2 *Assume that $\lambda_1 < 0 < \lambda_2$ holds. Assume also that there exists a one parameter family of capillary surfaces X_t ($X_0 = X$). Denote by $\Lambda(t)$, $V(t)$ the anisotropic mean curvature of X_t, the volume of X_t, respectively.*
(i) If $\Lambda'(0)V'(0) > 0$, then X is stable.
(ii) If $\Lambda'(0)V'(0) < 0$, then X is unstable.

Remark 3.1 By using geometric properties of the problem, we can show the following:

$$L[\nu_j] = 0, \quad j = 1, 2, 3, \qquad L[\langle X, \nu\rangle] = -\Lambda.$$

4 Main results

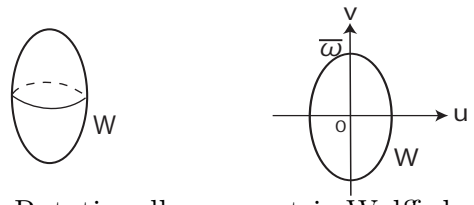
In this section, we assume that the Wulff shape W is rotationally invariant with vertical axis. Also, for simplicity, we assume that W is symmetric with respect to the horizontal plane $\{x_3 = 0\}$. We will consider existence and uniqueness for stable solutions, and the geometric properties of the solution. We should remark that, for CMC case, these subjects were studied by several authors ([2], [5], [17], [18], [19], [22], [23]).

By Proposition 2.1 and Corollary 2.1, capillary surfaces are CAMC surfaces of revolution with the same rotation axis as W. Such surfaces are called *anisotropic Delaunay surfaces.* They are classified into six classes: plane, Wulff shape (up to homothety), right circular cylinders, anisotropic catenoids, anisotropic unduloids and anisotropic nodoids (See §7.1).

First we remark that any part of W is stable. This result was essentially proved by Winterbottom [20]. We also obtained

Theorem 4.1 (Arroyo-Koiso-Palmer [1]) *Any convex anisotropic capillary surface is stable.*

Theorem 4.2 (Arroyo-Koiso-Palmer [1]) *If the generating curve of an anisotropic capillary surface contains two or more inflection points, then it is unstable.*



Rotationally symmetric Wulff shape (left),
generating curve of a Wulff shape which satisfies the condition $(*)$ (right)

Figure 2:

4.1 The case $\omega_0 = \omega_1 \geq 0$

With one boundary component, the only capillary surfaces are parts of W, which are energy minimizing by Winterbottom's theorem.

The simplest interesting problem is then the case of two boundary components and equal wetting constants: $\omega := \omega_0 = \omega_1$. In this subsection we consider the functional

$$\mathcal{E}[X] := \mathcal{F}[X] + \omega\mathcal{A}_0[X] + \omega\mathcal{A}_1[X], \qquad \omega \geq 0. \tag{7}$$

Now we pose a curvature condition for the Wulff shape W (see Figure 2):

$(*)$ The generating curve of W has non-decreasing curvature (with respect to the inward pointing normal) as a function of x_3 (the vertical coordinate) on $\{x_3 \geq 0\}$.

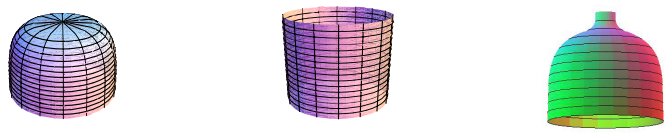
The following two results completely determine the stable equilibria when the condition $(*)$ holds. We let $\mu_i(0)$, $i = 1, 2$ denote the principal curvatures of W at the bulge $x_3 = 0$ with respect to the inward pointing normal. Here μ_1 denotes the curvature of the generating curve of W.

Theorem 4.3 (Koiso-Palmer [9], [10]) *Assume that the Wulff shape W is rotationally symmetric and satisfies the condition $(*)$. Let Σ be a capillary surface with free boundary on two horizontal planes for the functional $\mathcal{E}$ in (7).*

(i) If $\omega = 0$, then Σ is stable if and only if the surface is either homothetic to a half of the Wulff shape or a cylinder of height h and radius R, which is perpendicular to $\Pi_0 \cup \Pi_1$ which satisfies $h \leq \pi(\mu_1(0)/\mu_2(0))^{1/2}R$.

(ii) If $\omega > 0$, then Σ is stable if and only if Σ is a portion of an anisotropic Delaunay surface whose generating curve has no inflection points in its interior.

Remark 4.1 Recent numerical simulations [1] indicate that Theorem 4.3 (i) is no longer true without the condition $(*)$. It appears that for some functionals having energy density of the form $\gamma = 1 + e\nu_3^2$, with $e < 0$, stable parts of anisotropic unduloids appear when the volume is small, but not too small.



A half of Wulff shape (stable), a short cylinder (stable),
a half period of anisotropic unduloid (unstable)

Figure 3: No-wetting case ($\omega_0 = \omega_1 = 0$)

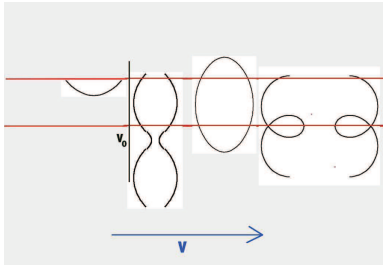


Figure 4: Stable capillary surfaces between two parallel planes for $\omega_0 = \omega_1 > 0$

Next we will give the existence and uniqueness of solutions. Denote by $(u(\sigma), v(\sigma))$ the profile curve of W. Set $\bar{\omega} := \max v$ (See Figure 2). We call a capillary surface *spanning* if its intersection with both supporting planes is a circle of positive radius.

Theorem 4.4 (Existence and uniqueness, Koiso-Palmer [11]) *Let W satisfy the same assumption as in Theorem 4.3. Assume $0 \leq \omega \leq \bar{\omega}$. Then, $\exists V_0 = V_0(h, \omega) > 0$ s.t.*
(i) For any $V \geq V_0$, there exists a unique stable spanning capillary surface $\Sigma(V) = \Sigma(V, h, \omega)$ which bounds volume V. And this is a convex surface.
(ii) For $V < V_0$, there is no stable panning capillary surface which bounds volume V.

Theorem 4.5 (Foliation, Koiso-Palmer [11]) *Assume that the Wulff shape satisfies the same condition as in Theorem 4.3. Assume $0 \leq \omega \leq \bar{\omega}$. Then, the surfaces $\{\Sigma(V) \mid V > V_0\}$ foliate the region of Ω which lies exterior to the surface $\Sigma(V_0)$.*

Remark 4.2 For $\omega \in [0, \overline{\omega}]$, for $V < V_0$, there is no stable spanning capillary surface, but there are unstable spanning capillary surfaces. If $\omega \notin [-\overline{\omega}, \overline{\omega}]$, there is no capillary surface.

We conjecture that below the volume V_0 there is a pitchfork bifurcation. As the volume is decreased, the family of stable symmetric surfaces with no inflection points on their boundary curves continue as a family of unstable symmetric surfaces with two interior inflection points on their boundary curves. This branch of surfaces is joined by two unstable branches, one of which is just the reflection through a horizontal plane of the other, whose generating curves have exactly one interior inflection point. It was stated by Vogel [19], that such a bifurcation occurs in the CMC case.

When the condition on the functional $(*)$ is dropped, numerical simulations [1], indicate that non convex, stable critica occur for comparatively small volumes while for large volumes, the stable equilibria are again convex.

4.2 The case $0 < \omega_0 \neq \omega_1 > 0$

In view of Theorems 4.1, 4.2, the remaining case is only the case where the capillary surface has exactly one inflection point. In this case, there are both cases: stable and unstable. We will discuss it in a forthcoming paper.

4.3 The case $\omega_0 = \omega_1 < 0$

In this case, we do not have such a beautiful geometric characterization of stable capillary surfaces as the case of nonnegative wetting which was given in §4.1. However, for sufficiently large volume, we have existence and uniqueness for the solution as follows.

Theorem 4.6 (Koiso-Palmer [11]) *Given $\omega_0 = \omega_1 =: \omega \in [-\overline{\omega}, 0)$ and h, there exists a constant $V^* = V^*(h, \omega) > 0$ such that, for any $V > V^*$, there exists a unique stable spanning capillary surface with volume V.*

Some geometric observations about this case are given in [10] and [1].

5 Idea of proof

By Proposition 2.1 and Corollary 2.1, capillary surfaces are parts of anisotropic Delaunay surfaces whose contact angles with Π_i are determined by ω_i. Let $(x(s), z(s))$ $(s_1 \leq s \leq s_2)$ be the generating curve of a capillary surface X. Then, we can show that, if $z'(s) \neq 0$ for $\forall s \in [s_1, s_2]$, then X is stable if and only if X is stable for rotationally symmetric variations. This means that the problem is reduced to be a one-dimensional problem.

Recall the criteria Lemmas 3.1, 3.2 for the stability.

In view of Remark 3.1,

$$L[\nu_j] = 0, \quad j = 1, 2, 3, \quad L[\langle X, \nu \rangle] = -\Lambda$$

holds for the unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ of X. However, in general, the functions ν_j, $\langle X, \nu \rangle$ do not satisfy the boundary condition. So, instead of the original eigenvalue problem, we consider suitably deformed eigenvalue problems to judge the stability of X. Here the representation formula of anisotropic Delaunay surfaces (Theorem 7.1) is essentially used.

If we know that $\lambda_1 < 0 < \lambda_2$ holds and we can find a one parameter family of capillary surfaces, then we may use Lemma 3.2. Here also the representation formula of anisotropic Delaunay surfaces (Theorem 7.1) is useful.

6 Related topics

[1] Variational problem of energy with line tension

We consider a similar problem of that in §2. However, here the total energy consists of the free energy $\mathcal{F}$, the wetting energy $\omega \cdot \mathcal{A} := \omega_0 \mathcal{A}_0 + \omega_1 \mathcal{A}_1$, and the *line tension* $\tau \cdot \mathcal{L} = \tau_0 \mathcal{L}_0 + \tau_1 \mathcal{L}_1$:

$$\tilde{\mathcal{E}} := \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}\,, \tag{8}$$

where τ_i are constants and $\mathcal{L}_i$ is the length of the boundary curve $C_i := \Sigma \cap \Pi_i$. Because particles on the boundary curves C_i are in contact with three phases, the curves should have their own energy cost. Line tension was introduced by Gibbs. It is known to only contribute

to the shape of drops of size in the range of a few microns. In [14], we studied stability of critical points for energy $\tilde{\mathcal{E}}$. When $\tau_i \geq 0$, we have many similar results to those for the case without line tension. When $\tau_i < 0$, we can prove that there exists no stable critical point, which contradicts some physical experiments. Maybe we need to construct new mathematical methods to treat such microscopic phenomena.

[2] A unified approach to surfaces of revolution which are critical points of more general energy functionals

Chen and Kamien ([3]) classified surfaces of revolution which are critical points of the functional

$$\hat{\mathcal{F}}(X) = \int_\Sigma \left(a + \frac{b}{r^2} \right) d\Sigma,$$

where r is the radius of the surface $X : \Sigma \to \mathbf{R}^3$. $\hat{\mathcal{F}}(X)$ can be applied to thin nematic liquid crystalline films. We pose a problem of classification of surfaces of revolution which are critical points of a general energy functional which includes both of $\mathcal{F}$ and $\hat{\mathcal{F}}$.

7 Appendix

7.1 Anisotropic Delaunay surfaces

In this section, we give classification of rotationally symmetric CAMC surfaces. We call them *anisotropic Delaunay surfaces*. They are not only basic examples of CAMC surfaces, but also important for research on general CAMC surfaces.

Assume that γ is rotationally invariant, say $\gamma = \gamma(\nu_3)$. Then the Wulff shape W is also rotationally invariant with respect to the third axis. In physics, certain nematic liquid crystals have this type of anisotropic surface energy. Denote by u_0 the maximum radius of the horizontal slices of W.

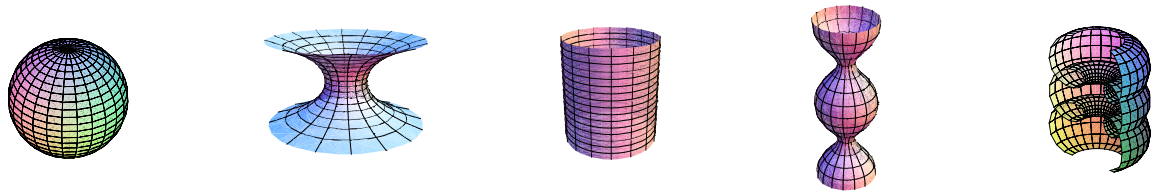
Consider an anisotropic Delaunay surface Σ parameterized by

$$X(s, \theta) = (x(s) \cos\theta, x(s) \sin\theta, z(s)) , \quad x(s) \geq 0,$$

where $(x(s), z(s))$ is the arc length parameterization of the generating curve. The orientation of X may be chosen so that $\Lambda \leq 0$ holds. Since the equation "$\Lambda =$ constant" is a second order ODE of $(x(s), z(s))$, we have two parameter family of solutions. Since one of the parameters gives vertical translation, we have essentially one parameter family of anisotropic Delaunay surfaces with anisotropic mean curvature Λ. By using a parameter c which is called a *flux parameter*, they are classified as follows ([8]):

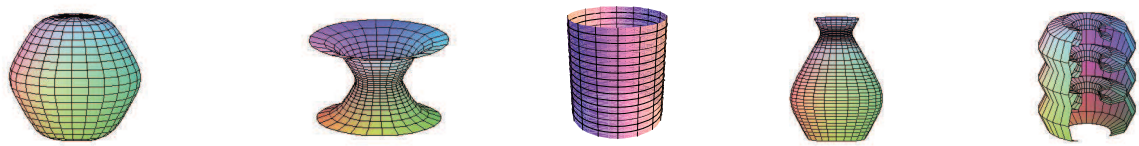
- (I-1) $\Lambda = 0$ and $c = 0$: *horizontal plane.*
- (I-2) $\Lambda = 0$ and $c \neq 0$: *anisotropic catenoid.*
- (II-1) $\Lambda < 0$ and $c = 0$: *Wulff shape (up to homothety).*
- (II-2) $\Lambda < 0$ and $c = c_0 := (u_0)^2 |\Lambda|^{-1}$: *cylinder of radius* $u_0|\Lambda|^{-1}$.
- (II-3) $\Lambda < 0$ and $0 < c < c_0$: *anisotropic unduloid.*
- (II-4) $\Lambda < 0$ and $c < 0$: *anisotropic nodoid.*

All surfaces above are complete. Anisotropic unduloid and nodoid are periodic surfaces, that is, each of them is invariant under a suitable parallel translation. An anisotropic unduloid dose not have self-intersection. An anisotropic nodoid has self-intersections.
The following formula is used essentially in our study on CAMC surfaces ([1], [9]-[14]).



Sphere, catenoid, cylinder, unduloid, and nodoid

Figure 5: Delaunay surfaces (CMC surfaces of revolution)



A Wulff shape, anisotropic catenoid, cylinder, anisotropic unduloid, and anisotropic nodoid

Figure 6: A Wulff shape and anisotropic Delauney surfaces

Theorem 7.1 ([8]) *Assume that the Wulff shape W is rotationally symmetric. Let $\sigma \mapsto (u(\sigma), v(\sigma))$ be the profile curve of W. Let $X(s,\theta) = (x(s)\cos\theta, x(s)\sin\theta, z(s))$ be a surface with constant anisotropic mean curvature $\Lambda \leq 0$, and let the Gauss map of X coincide with that of W at $s = s(\sigma)$. Then X is given as follows.*

(i) When X is an anisotropic catenoid, $x = c/(2u)$ for some nonzero constant c.

(ii) When X is an anisotropic unduloid, $x = \dfrac{u \pm \sqrt{u^2 + \Lambda c}}{-\Lambda}$ for some constants $c > 0$ and $\Lambda < 0$, where $x = x(u(\sigma))$ is defined in $\{\sigma | u \geq \sqrt{-\Lambda c}\}$.

(iii) When X is an anisotropic nodoid, $x = \dfrac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}$ for some constants $c < 0$ and $\Lambda < 0$, where $x = x(u(\sigma))$ is defined in $\{-\infty < \sigma < \infty\}$.

In all cases above, z is given by $z = \displaystyle\int^u v_u x_u \, du$.

7.2 Generalized anisotropic Delaunay surfaces

Here we consider CAMC surfaces for Wulff shape which may not be rotationally symmetric.

Let $\Omega_W : (u(\sigma), v(\sigma))$ be a closed convex curve parametrized by arc length σ which is symmetric with respect to the v-axis. Let $C : (\alpha(\tau), \beta(\tau))$ be a closed convex curve parameterized by arc length in the plane. We assume that the origin is inside the domain bounded by C. Consider the surface W given by $Y(\sigma,\tau) = (u(\sigma)\alpha(\tau), u(\sigma)\beta(\tau), v(\sigma))$. W is a convex surface such that all the curves obtained by intersecting W with planes $x_3 =$ constant are homothetic to each other. We say that such W is of *product type*.

If we define x and z as in Theorem 7.1, then $X(s,\tau) = (x(s)\alpha(\tau), x(s)\beta(\tau), z(s))$ defines a CAMC surface for the Wulff shape W given above ([12], [13]). We call these surfaces generalized anisotropic Delaunay surfaces (Figure 7). On the stability of them, a similar method to the case of CAMC surfaces of revolution can be applied ([14]).

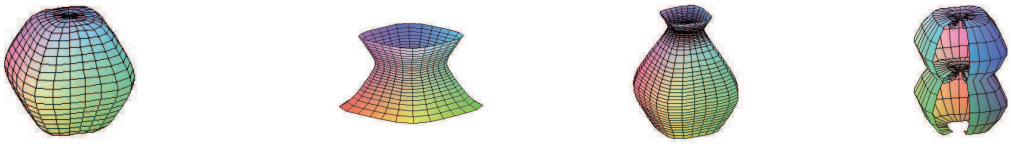


Figure 7: A Wulff shape (product type) and generalized anisotropic Delaunay surfaces

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The effect of boundary conditions to the dynamics of pulse solutions for reaction-diffusion systems

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Abstract.
We consider pulse-like localized solutions for reaction-diffusion systems on a half line and impose various boundary conditions at one end of the half line. It is shown that the movement of a pulse solution with the homogeneous Neumann boundary condition is completely opposite from that with the Dirichlet boundary condition. As general cases, Robin type bounary conditions are also considered. Introducing one parameter connecting the Neumann and the Dirichlet boundary conditons, we will show the existence of stationary solutions which have been known so far.

key words: Reaction-diffusion systems; Effect of boundary conditions; pulse solutions

Reaction diffusion systems have been widely treated to describe and study spatio-tempral patterns in dissipative systems. Among them, many reaction-diffusion systems which possess various types of localized solutions such as pulse-like localized solutions and front-like ones have been proposed while we omit the detail and merely refer to books ([6], [5]). To understand the dynamics of such solutions, reaction-difusion systems have been studied under various situations such as one or higher dimensional spaces, bounded or unbouded domains, and the Neumann boundary conditions or the Dirichlet ones according to considered problems. In fact, the dynamics solutions drastically change depending on the considered situations. As one example, let us consider the effect of boundary conditions for the Allen-Cahn equation:

$$u_t = \varepsilon^2 u_{xx} + f(u),\ t > 0,\ x \in I \subset \boldsymbol{R} \tag{1}$$

with the Neumann boundary conditions

$$u_x = 0,\ x \in \partial I, \tag{2}$$

or

$$u = \pm 1,\ x \in \partial I, \tag{3}$$

where $f(u) = \frac{1}{2}u(1-u^2)$ and $\varepsilon > 0$ is sufficiently small. For the simplicity, let $I = \boldsymbol{R}_+ := [0, \infty)$ and we impose the boundary conditions only at $x = 0$.

As known well, (1) on the whole line $\boldsymbol{R}$ has a stable standing front solution, say $\Phi(x) := \tanh x/2$ satisfying $\Phi(\pm\infty) = \pm 1$.

If we consider (1) on the half line $\boldsymbol{R}_+$ with the Neumann boundary condition (2) and the initial data is close to $\Phi(x - l_0)$ for $l_0 >> 1$, then it was shown by [1], [3] that the solution $u(t, x)$ remains close to $\Phi(x - l(t))$ and the movement is essentially governed by $\dot{l} = -12\varepsilon e^{-2l}$. That is, a front like localized solution approaches the boundary $x = 0$.

On the other hand, when we impose the Dirichlet boundary condition

$$u = -1, \ x = 0 \tag{4}$$

under the same situations as the previous one except the boundary condition, it is observed as in Fig 0.1 that the solution comes away from the boundary $x = 0$.

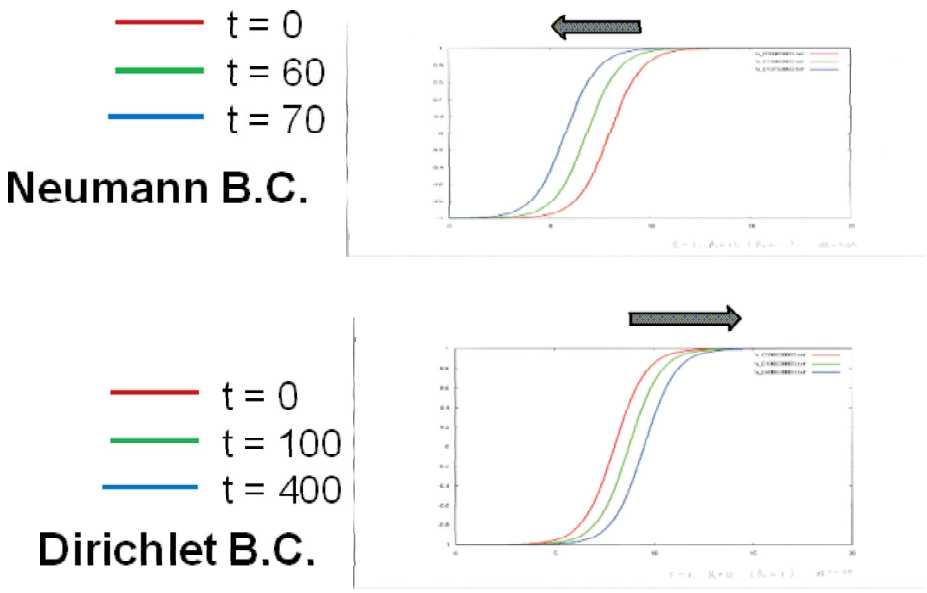

図 0.1: Movements of front solutions of (1) on $\boldsymbol{R}_+$.

In this talk, we consider farely general types of reaction-diffusion systems

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + F(\boldsymbol{u}), \ t > 0, \ x \in \boldsymbol{R}_+, \tag{5}$$

where $\boldsymbol{u} \in \boldsymbol{R}^N$, $D := diag\{d_1, \cdots, d_N)$ and $F : \boldsymbol{R}^N \to \boldsymbol{R}^N$ is a sufficiently smooth function.

First we consider the problem (5) on $\boldsymbol{R}$

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + F(\boldsymbol{u}), \ t > 0, \ x \in \boldsymbol{R}, \tag{6}$$

and assume several conditions for (6) as follows:

A1) There exists a stable symmetric stationary solution, say $S(x)$ satisfying $S(x) \to e^{-\alpha|x|}\boldsymbol{a}$ as $|x| \to \infty$ for $\alpha > 0$ and $\boldsymbol{a} \in \boldsymbol{R}^N$.

Let $L := \partial_{xx} + F'(S(x))$, the linearized operator of (6) with respect to $S(x)$.

A2) The spectral set $\sigma(L)$ of L is given $\sigma(L) = \sigma_0 \cup \sigma_1$, where $\sigma_0 := \{0\}$ and $\sigma_1 \subset \{Re\lambda < -\gamma_0\}$ for $\gamma_0 > 0$. Moreover, 0 is a simple eigenvalue of L.

Then there exists eigenfunction $\phi^*(x)$ of the adjoint operator L^* of L satisfying $L^*\phi^* = 0$ and $\phi^*(x) \to e^{-\alpha x}\boldsymbol{a}^*$ as $x \to +\infty$ for $\boldsymbol{a}^* \in \boldsymbol{R}^N$. Note that we can take $\phi^*(x)$ as an odd function and by the normalization $\langle S_x, \phi^* \rangle_{L^2} = 1$, $\phi^*(x)$ is uniquely determined.

Next coming back the original problem (5) on the half line $\boldsymbol{R}_+$. We impose the boundary condition

$$\boldsymbol{u}_x = \beta \boldsymbol{u}, \ x = 0. \tag{7}$$

Then we have

Theorem 1 *Assume A1) and A2). If the initial data $\boldsymbol{u}(0,x)$ is sufficiently close to $S(x-l_0)$ for $l_0 >> 1$, then the solution $\boldsymbol{u}(t,x)$ of (5) remains close to*

$$\boldsymbol{u}(t,x) = S(x - l(t)) + O(e^{-\alpha l(t)})$$

as long as $l(t) > l^*$ for $l^* >> 1$. $l(t)$ satisfies

$$\frac{dl}{dt} = \frac{2\alpha(\alpha-\beta)}{\alpha+\beta} M_0 e^{-2\alpha l}(1 + O(e^{-\alpha l})),$$

where $M_0 := \langle D\boldsymbol{a}, \boldsymbol{a}^* \rangle$.

In this talk, we will mention more precise analysis about the problems.

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