DIFFERENTIAL GEOMETRY OF SPACELIKE SUBMANIFOLDS IN DE SITTER SPACE

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Contents

1 Introduction

In this thesis, we investigate the differential geometry of spacelike submanifolds in de Sitter space as an application of the theory of Legendrian singularities.

Bleeker and Wilson [3] studied the singularities of the Gauss map of a surface in Euclidean 3-space. Gauss map is defined by the normal vector field of the surface. In their paper, the main theorem asserts that the generic singularities of the Gauss maps are folds or cusps. Banchoff *et al.* [2], Landis [15] and Platnova [17] studied geometric meanings of cusps of the Gauss map of the surface. The notion of Gauss-Kronecker curvature is defined by the determinant of the differential map of the Gauss map. Therefore, the set of singularities of the Gauss map coincides with the set of parabolic points, where the Gauss-Kronecker curvature vanishes. Bruce [4] and Romero-Fuster [18] have also independently studied the singularities of the Gauss map and the dual of a hypersurface in Euclidean space. The main tool of Bruce and Romero-Fuster in their study is the family of height functions on a hypersurface.

On the other hand, the differential geometry of Minkowski space, hyperbolic space and de Sitter space is also studied by several people. Izumiya, Pei and Sano [6] investigated extrinsic differential geometry of hypersurfaces in hyperbolic space as an application of Legendrian singularities. They observed the geometrical meanings of the singularities of lightcone Gauss indicatrice and lightcone Gauss maps of the spacelike hypersurface.

The case of spacelike submanifolds of codimension two are particularly important in the theory of general relativity. Izumiya, Kossowski, Pei and Romero Fuster [7] investigated lightlike hypersurfaces of spacelike surfaces in Minkowski four space, which is deeply related to the several kinds of horizons. Izumiya and Romero Fuster [8] investigated spacelike submanifolds of codimension two in general dimensional Minkowski space. They showed Gauss-Bonnet type formula in terms of the Gauss-Kronecker curvature with respect to the lightlike normals.

For general codimension case, the normal vector is not uniquely determined, however it is possible to construct hypersurfaces from normal unit vector fields of the spacelike submanifold. Izumiya, Pei, Romero Fuster and Takahashi [9] introduced the notion of canal hypersurfaces and horospherical hypersurfaces from the parallel normal frames of submanifolds in the hyperbolic space, and investigated submanifolds of higher codimension in the hyperbolic space.

It is known that de Sitter space is a Lorentzian space form with a positive curvature. In this thesis (cf. [11]) we investigate singularities of lightcone Gauss maps of spacelike hypersurfaces in de Sitter space, which is analogous to the case of hyperbolic space [6]. The singularities of the Gauss image coincide with the lightcone parabolic sets of spacelike hypersurface.

In the case of spacelike submanifold of codimension two, the normal direction of spacelike submanifold cannot be chosen uniquely. However, we can determine the lightcone normal frames. Fusho and Izumiya [5] investigated lightlike surfaces of spacelike curves in de Sitter 3-space by using the Frenet-Serret type formula and gave a classification of singularities of lightlike surfaces of generic spacelike curves, which are a cuspidal edge and a swallowtail. Here, we investigate singularities of the lightcone Gauss maps and lightlike hypersurfaces, as the generalization of the study [5]. We use analogous tools to those applied in [7, 8] to the study of spacelike submanifolds in Minkowski space. The singularities of lightlike hypersurfaces are described by the principal curvatures of spacelike submanifolds.

In the general codimension case, we investigate the differential geometry of spacelike submanifolds of codimension at least two in de Sitter space. In Euclidean space, the canal hypersurface of a submanifold is very useful and it has been classically known. Moreover, the analogous definition has been given in the hyperbolic space [9]. We introduced the notion of de Sitter horospherical hypersurfaces and spacelike canal hypersurfaces by using the parallel unit orthonormal section. The singular point of the de Sitter horospherical hypersurface corresponds to the parabolic point of spacelike canal hypersurface, which we call a de Sitter horospherical point.

Moreover, we can construct another kind of canal hypersurfaces. It is called as a timelike canal hypersurface. We study the geometrical property of the timelike canal hypersurfaces similarly as the case of spacelike canal hypersurface, and we find different properties from these of the spacelike canal hypersurface.

More precisely, in the first part of this thesis, we investigate the relationships between singularities of lightcone Gauss image and geometric properties of the spacelike hypersurface in de Sitter space.

In §3.1 we introduce the notion of the lightcone Gauss image and the lightcone Gauss-Kronecker curvature of the spacelike hypersurfaces. The lightcone Gauss-Kronecker curvature is an invariant under the Lorentzian transformation in de Sitter space. In §3.2 we introduce a family of functions that is called the lightcone height function on the spacelike hypersurface. The lightcone Gauss image is interpreted as the discriminant set of the family of height functions, and the singular set of the lightcone Gauss image is the lightcone parabolic set of the spacelike hypersurface. In §3.3 we discuss the contact of hypersurfaces with de Sitter hyper-horospheres. We apply the theory of Legendrian singularities for the study of lightcone Gauss images of generic hypersurfaces. In §3.5 we classify the singularities of lightcone Gauss images for generic spacelike surfaces in de Sitter 3-space. We have two types of singularities of the lightcone Gauss image in generic, which are cuspidal edges and swallowtails. In §3.4 and §3.6 we define the notion of spacelike Monge forms in de Sitter space. It makes us to give examples which are corresponding to generic singularities of the lightcone Gauss image.

In the second part of this thesis, we investigate the differential geometry of spacelike submanifolds of codimension two in de Sitter space.

In §4.1 we introduce the notion of the lightcone Gauss map, the normalized lightcone Gauss-Kronecker curvature and principal curvatures. The lightcone Gauss map does not depend on the choice of the future directed normal frame. In §4.2 we introduce the notion of the lightlike hypersurface and a family of functions that is called the Lorentzian distance squared function on the spacelike submanifold. The lightlike hypersurface is interpreted as the discriminant set of the family of the Lorentzian distance squared function, and the singular set of the lightlike hypersurface is described by the normalized lightcone principal curvatures of the spacelike submanifold. In §4.3 we discuss the contact of spacelike submanifold with lightcone in de Sitter space. We apply the theory of Legendrian singularities for the study of lightcone Gauss map of generic spacelike submanifolds. In §4.4 we introduce the notion of a family of functions that is called the lightcone height function. The lightcone Gauss map is interpreted as the discriminant set of the family of lightcone height function, and the singular set of the normalized lightcone Gauss map corresponds to the normalized lightcone parabolic set on the spacelike submanifold. In §4.5 We discuss the contact of spacelike submanifolds with lightlike cylinders in de Sitter space. In §4.6 we classify the singularities of lightlike hypersurfaces and lightcone Gauss maps of generic spacelike surfaces in de Sitter 4-space, and give some examples which have their singularities.

In the last part of this thesis, we investigate the differential geometry of spacelike submanifolds of codimension at least two in de Sitter space, and construct two types of canal hypersurfaces.

In §5 we consider geometric properties of the spacelike canal hypersurfaces. In §5.1 we define a timelike normal vector field of spacelike submanifolds in de Sitter space and introduce the notion of the de Sitter horospherical Gauss-Kronecker curvature and principal curvatures. In §5.2 and §5.3 we introduce a notion of de Sitter horospherical height function and de Sitter horospherical hypersurface. We also define a spacelike canal hypersurface, whose lightcone Gauss image is diffeomorphic to the de Sitter horospherical hypersurface. In §5.4 we naturally interpret the de Sitter horospherical hypersurfaces of the spacelike submanifold as a wave front set of de Sitter horospherical height functions in the theory of Legendrian singularities. In §5.5 we use the theory of contacts between the submanifolds due to Montaldi [16], and we discuss geometric properties of singularities of de Sitter horospherical hypersurfaces. We also consider generic properties of spacelike submanifolds.

In §6 we consider geometric properties of the spacelike canal hypersurfaces. In §6.1 we briefly review the differential geometry of timelike hypersurfaces in de Sitter space due to Izumiya [10]. In §6.2 we introduce the notion of the de Sitter horospherical Gauss-Kronecker curvature and principal curvatures with respect to the spacelike normal vector field. In §6.3 we construct the timelike hypersurface from the parallel pseudo orthonormal sections of the spacelike submanifold, and consider the geometric properties of the timelike canal hypersurface. We also use the notion of de Sitter Gauss image and de Sitter height function of the timelike hypersurfaces defined in [10]. The singularities of the timelike canal hypersurface and the de

Sitter Gauss image are related to the geometrical property of the spacelike submanifold. We also consider contacts between spacelike submanifolds and non-flat hyperbolic hyperquadrics. In §6.4 we finally consider the generic condition of the spacelike submanifolds.

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2 Preliminaries

2.1 Basic notations

In this section we review the basic notion to study the differential geometry of spacelike hypersurfaces and submanifolds in de Sitter space. Let $\mathbb{R}^{n+1} = \{\mathbf{x} = (x_0, \ldots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, \ldots, n)\}$ be an $(n+1)$-dimensional vector space. For any vectors $\mathbf{x} = (x_0, \ldots, x_n)$ and $\mathbf{y} = (y_0, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i$. We call $(\mathbb{R}^{n+1}, \langle, \rangle)$ a *Minkowski* $(n+1)$*-space* and write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle, \rangle)$. We say that a vector $\mathbf{x} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ is *spacelike*, *timelike* or *lightlike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}^{n+1}_1$ is defined by $||\mathbf{x}|| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$.

We respectively define *hyperbolic* n*-space* and *de Sitter* n*-space* by

$$\begin{aligned} H^n_{\pm}(-1) &= \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, sgn(x_0) = \pm 1\}, \\ S^n_1 &= \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}, \end{aligned}$$

and we write $H^n(-1) = H^n_+(-1) \cup H^n_-(-1)$. For any $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^{n+1}_1$, we define an pseudo-external product of vectors $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_n$ by

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_n = \det \begin{pmatrix} -\mathbf{e_0} & \mathbf{e_1} & \cdots & \mathbf{e_n} \\ x_0^1 & x_1^1 & \cdots & x_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{pmatrix},$$

where $\{\mathbf{e_1}, \ldots \mathbf{e_n}\}$ is a standard basis of $\mathbb{R}^{n+1}_1$ and $\mathbf{x_i} = (x_0^i, \ldots, x_n^i)$ for $i = 1, \ldots, n$. Since $\langle \mathbf{x}, \mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_n \rangle = \det(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n)$, so that $\mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_n$ is pseudo-orthogonal to any $\mathbf{x}_i$ for $i = 1, \ldots, n$. Let $\lambda \in \mathbb{R}^{n+1}_1$, we define a set

$$LC_\lambda = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x} - \lambda, \mathbf{x} - \lambda \rangle = 0\},$$

which is called a *closed lightcone* with vertex λ. We also define *future* (resp. *past*) *lightcone* at the origin by

$$\begin{aligned} LC^*_+ &= \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0,\ x_0 > 0\ \}, \\ LC^*_- &= \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0,\ x_0 < 0\ \}, \end{aligned}$$

and we write $LC^* = LC^*_+ \cap LC^*_-$.

For a vector $\mathbf{v} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ and a real number c, we define a *hyperplane with pseudo normal* $\mathbf{v}$ in the Minkowski space by

$$\mathrm{HP}(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}.$$

We say that a hyperplane $HP(\mathbf{v}, c)$ is spacelike, timelike or lightlike if the vector $\mathbf{v}$ is timelike, spacelike or lightlike. We now consider hyperquadrics in de Sitter space. We say that $HP(\mathbf{v}, c) \cap S^n_1$ is an *elliptic hyperquadric* or a *hyperbolic hyperquadric* if $HP(\mathbf{v}, c)$ is spacelike or timelike. Let $\mathbf{v}$ be lightlike, we say that $HP(\mathbf{v}, c) \cap S^n_1$ is a *de Sitter hyperhorosphere* and a *lightlike cylinder* if $c \neq 0$ and $c = 0$ respectively. We denote the de Sitter hyperhorosphere by

$$HS(\mathbf{v}, c) = HP(\mathbf{v}, c) \cap S^n_1.$$

Let $\mathbf{v}' = (1/c)\mathbf{v}$ then we have $HS(\mathbf{v}, c) = HS(\mathbf{v}', 1)$.

Let $\mathbf{v}$ be spacelike, we say that the hyperbolic hyperquadric $HP(\mathbf{v}, c) \cap S^n_1$ is a *flat timelike hyperquadric* and a *non-flat hyperbolic hyperquadric* if $c = 0$ and $c \neq 0$ respectively. If $c = ||\mathbf{v}||$ then the timelike hyperplane $HP(\mathbf{v}, c)$ is not transversal to de Sitter space at the point $(1/c)\mathbf{v}$. In this case the hyperbolic hyperquadric $HP(\mathbf{v}, c) \cap S^n_1$ is not a smooth manifold.

Part I Spacelike hypersurfaces in de Sitter space

3 Spacelike hypersurfaces

3.1 Spacelike hypersurfaces and lightcone Gauss images

In this section we study extrinsic differential geometry of spacelike hypersurfaces in de Sitter space. Let U be an open subset of $\mathbb{R}^{n-1}$ and $\mathbf{X} : U \longrightarrow S^n_1$ be an embedding map. We say that $\mathbf{X}$ is a *spacelike hypersurface* in de Sitter space S_1^n if every non zero vector generated by tangent vectors $\{\mathbf{X}_{u_i}(\mathbf{u})\}_{i=1}^{n-1}$ is always spacelike, where $\mathbf{u} = (u_1, \cdots, u_{n-1})$ is an element of U and $\mathbf{X}_{u_i}$ is a partial derivative of $\mathbf{X}$ with respect to u_i. We denote $M = \mathbf{X}(U)$ and identify M with U through the embedding $\mathbf{X}$. Since $\langle \mathbf{X}, \mathbf{X} \rangle \equiv 1$, we have $\langle \mathbf{X}_{u_i}, \mathbf{X} \rangle \equiv 0$ for $i = 1, \ldots, n-1$. It follows that a hyperplane spanned by $\{\mathbf{X}, \mathbf{X}_{u_1}, \cdots, \mathbf{X}_{u_{n-1}}\}$ is spacelike. We define a vector

$$\mathbf{e}(\mathbf{u}) = \frac{\mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}(\mathbf{u})}{||\mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}(\mathbf{u})||}.$$

Since $\langle \mathbf{e}, \mathbf{X}_{u_i} \rangle \equiv \langle \mathbf{e}, \mathbf{X} \rangle \equiv 0$ and $\langle \mathbf{e}, \mathbf{e} \rangle \equiv -1$ for $i = 1, \ldots, n-1$, so that $\mathbf{e}(\mathbf{u})$ is a timelike normal of the spacelike hypersurface M. By replacing coordinates on U, we may assume that $\mathbf{e}(\mathbf{u})$ is future directed. Therefore we have $\mathbf{X}(\mathbf{u}) \pm \mathbf{e}(\mathbf{u}) \in LC^*_{\pm}$. We define a map $\mathbb{L}^{\pm} : U \longrightarrow LC^*_{\pm}$ by

$$\mathbb{L}^{\pm}(\mathbf{u}) = \mathbf{X}(\mathbf{u}) \pm \mathbf{e}(\mathbf{u}),$$

which is called a *lightcone Gauss image* of $\mathbf{X}$.

We have the following proposition which is analogous to ([6], Proposition 2.2).

Proposition 3.1 ([11])**.** Let $\mathbf{X} : U \longrightarrow S_1^n$ be a spacelike hypersurface in S_1^n. The lightcone Gauss image $\mathbb{L}^{\pm}$ is constant if and only if the spacelike hypersurface $M = \mathbf{X}(U)$ is a part of a de Sitter hyperhorosphere.

Proof. Since $\mathbb{L}^{\pm}(\mathbf{u})$ is constant $\mathbb{L}^{\pm}$, so we have $\langle \mathbf{X}(\mathbf{u}), \mathbb{L}^{\pm} \rangle = \langle \mathbf{X}(\mathbf{u}), \mathbf{X}(\mathbf{u}) \pm \mathbf{e}(\mathbf{u}) \rangle = 1$ for any $\mathbf{u} \in U$. Therefore, we have $\mathbf{X}(U) \subset HP(\mathbb{L}^{\pm}, +1) \cap S_1^n$.

If $\mathbf{X}(U) \subset HP(\mathbf{v}, c) \cap S_1^n$ for some $\mathbf{v} \in LC^*$ and $c \neq 0$, then we have $\langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle = r$ and $\langle \mathbf{X}_{u_i}(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle = 0$. This means that $\mathbf{v} = r\mathbb{L}^{+}(\mathbf{u})$ or $\mathbf{v} = r\mathbb{L}^{-}(\mathbf{u})$. Therefore, $\mathbb{L}^{\pm}$ is a constant vector $(1/c)\mathbf{v}$. □

Let $\mathbf{u} \in U$ and $p = \mathbf{X}(\mathbf{u})$, under the identification of U and M, we can identify a derivative map $d\mathbf{X}(\mathbf{u})$ with an identity mapping id_{T_pM}. By the similar arguments as those of the proof of ([6], Lemma 2.1), $d\mathbf{e}(\mathbf{u})$ is a linear transformation on the tangent space T_pM. This means that a derivative of the lightcone Gauss image is a linear transformation on the tangent space T_pM given by $d\mathbb{L}^{\pm}(\mathbf{u}) = \mathrm{id}_{T_pM} \pm d\mathbf{e}(\mathbf{u})$. We respectively call $S_p^{\pm} = -d\mathbb{L}^{\pm}(\mathbf{u}) : T_pM \longrightarrow T_pM$ the *lightcone shape operator* of $M = \mathbf{X}(U)$ at $p = \mathbf{X}(\mathbf{u})$ and $A_p = -d\mathbf{e}(\mathbf{u}) : T_pM \longrightarrow T_pM$ the *shape operator* of M at p. We denote the eigenvalue of $S_p^{\pm}$ by $\bar{\kappa}_{p,i}^{\pm}$ and the eigenvalue of A_p by $\kappa_{p,i}$. We respectively call $\bar{\kappa}_{p,i}^{\pm}$ by *lightcone principal curvatures* of M at p and $\kappa_{p,i}^{\pm}$ by principal curvatures of M at p. By the relation $S_p^{\pm} = -\mathrm{id}_{T_pM} \pm A_p$, $S_p^{\pm}$ and A_p have the common eigenvectors and we have a relation $\bar{\kappa}_{p,i}^{\pm} = -1 \pm \kappa_{p,i}$. We also define a *lightcone Gauss-Kronecker curvature* of M at p by determinant of $S_p^{\pm}$ and denote it by $K_\ell^{\pm}(\mathbf{u})$. Since A_p is the shape operator with respect to the Riemannian metric on M induced from the Lorentzian metric on $\mathbb{R}_1^{n+1}$, we define the Gauss-Kronecker curvature of M at $p = \mathbf{X}(\mathbf{u})$ by $K(\mathbf{u}) = \det A_p$.

We say that a point $\mathbf{u} \in U$ or $p = \mathbf{X}(\mathbf{u})$ is a *lightcone parabolic point* if the Gauss-Kronecker curvature of M at p equals to zero. We also say that a point $\mathbf{u}$ or p is an *umbilic point* if all the eigenvalues of $S_p^{\pm}$ are equal. If all points on the spacelike hypersurface M are umbilic points, then M is called by *totally umbilic*. we say that a point p is a *lightcone flat point* if p is an umbilic point and a lightcone flat point. The following proposition is analogous to ([6], Proposition 2.3).

Proposition 3.2 ([11])**.** Suppose that $M = \mathbf{X}(U)$ is totally umbilic. Then $\bar{\kappa}_p^{\pm}, \kappa_p$ are constant functions $\bar{\kappa}^{\pm}, \kappa$. Under this condition, we have the following classification.

(1) If $0 \leq |\kappa| = |\bar{\kappa}^{\pm} + 1| < 1$, then M is a part of a hyperbolic hyperquadric.

(2) If $1 < |\kappa| = |\bar{\kappa}^{\pm} + 1|$, then M is a part of an elliptic hyperquadric.

(3) If $\bar{\kappa}^{\pm} = 0$, then M is a part of a de Sitter hyperhorosphere.

Proof. By definition, we have $-\mathbb{L}_{u_i}^{\pm}(\mathbf{u}) = \bar{\kappa}_p^{\pm}\mathbf{X}_{u_i}(\mathbf{u})$ (for $i = 1, \cdots, n-1$) for any $p = \mathbf{X}(\mathbf{u}) \in M$. Therefore, we have $\mathbb{L}_{u_iu_j}^{\pm}(\mathbf{u}) = \bar{\kappa}_{p,u_j}^{\pm}\mathbf{X}_{u_i}(\mathbf{u}) + \bar{\kappa}_p^{\pm}\mathbf{X}_{u_iu_j}(\mathbf{u})$. Since $\mathbb{L}_{u_iu_j}^{\pm} = \mathbb{L}_{u_ju_i}^{\pm}$ and $\mathbf{X}_{u_iu_j} = \mathbf{X}_{u_ju_i}$, we have $\bar{\kappa}_{p,u_j}^{\pm}\mathbf{X}_{u_i}(\mathbf{u}) - \bar{\kappa}_{p,u_i}^{\pm}\mathbf{X}_{u_j}(\mathbf{u}) = 0$. On the other hand, $\mathbf{X}_{u_i}$ for $i = 1, \cdots, n-1$ are linearly independent, so that $\bar{\kappa}_p^{\pm}$ is constant $\bar{\kappa}^{\pm}$. Since $\bar{\kappa}^{\pm} = \pm\kappa_p - 1$, this means that $\kappa_p^{\pm}$ is a constant function $\kappa^{\pm}$.

We now assume that $\bar{\kappa}^{\pm} \neq 0$. By the assumption, we have $-\mathbf{e}_{u_i} = \kappa\mathbf{X}_{u_i}$ (for $i = 1, \cdots, n-1$), so that there exists a constant vector $\mathbf{a} \in \mathbb{R}_1^{n+1}$ such that, $\mathbf{a} = \kappa\mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u})$ for any $\mathbf{u} \in U$. If

$|\kappa| = |\bar{\kappa}^{\pm} + 1| \neq 0$, then the vector $\mathbf{v} = (1/\kappa)\mathbf{a}$ satisfies $\langle \mathbf{v}, \mathbf{v} \rangle = 1 - 1/\kappa^2$ and $\langle \mathbf{X}, \mathbf{v} \rangle = +1$. If $\kappa = 0$, then $\mathbf{v} = \mathbf{a}$ satisfies $\langle \mathbf{v}, \mathbf{v} \rangle = -1$, $\langle \mathbf{X}, \mathbf{v} \rangle = 0$. so that the assertion (1), (2) follows.

Finally, we assume that $\bar{\kappa}^{\pm} = 0$. In this case, we have $\mathbb{L}^{\pm}_{u_i} = 0$ (for $i = 1, \cdots, n-1$), so that $\mathbb{L}^{\pm}$ is constant. Therefore we apply Proposition 3.1. This completes the proof. □

We now consider the Riemannian metric (the *first fundamental form*) $ds^2 = \sum_{i,j=1}^{n-1} g_{ij} du_i du_j$ on $M = \mathbf{X}(U)$, where $g_{ij}(\mathbf{u}) = \langle \mathbf{X}_{u_i}(\mathbf{u}), \mathbf{X}_{u_j}(\mathbf{u}) \rangle$ for any $\mathbf{u} \in U$. We also define a *positive* (or *negative*) *lightcone second fundamental form* $\bar{h}^{\pm}_{ij}(\mathbf{u}) = \langle -\mathbb{L}^{\pm}_{u_i}(\mathbf{u}), \mathbf{X}_{u_j}(\mathbf{u}) \rangle$ for any $\mathbf{u} \in U$. We have the following Weingarten-type formula which is analogous to ([6], Proposition 2.4).

$$\mathbb{L}^{\pm}_{u_i} = -\sum_{j=1}^{n-1} (\bar{h}^{\pm})_i^j \mathbf{X}_{u_j},$$

where $((\bar{h}^{\pm})_i^j) = (\bar{h}^{\pm}_{ik})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$.

Therefore, we have an explicit expression for the lightcone Gauss-Kronecker curvature by Riemannian metric and the lightcone second fundamental invariant.

$$K^{\pm}_{\ell} = \frac{\det(\bar{h}^{\pm}_{ij})}{\det(g_{\alpha\beta})}.$$

By the above formula, we have that a point p is a lightcone parabolic point (or, briefly an $L^{\pm}$-parabolic point) if and only if the second fundamental quantity matrix $(\bar{h}_{ij})(p)$ is not regular.

3.2 Lightcone height functions

In this section we introduce families of functions on a spacelike hypersurface in de Sitter space. Let $\mathbf{X} : U \longrightarrow S^n_1$ be a spacelike hypersurface in S^n_1. We define a family of functions $H : U \times LC^* \longrightarrow \mathbb{R}$ by

$$H(\mathbf{u}, \mathbf{v}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle - 1.$$

We call H a *lightcone height function* on $\mathbf{X} : U \longrightarrow S^n_1$.

We denote the Hessian matrix of the lightcone height function $h_{\mathbf{v}_0}(\mathbf{u}) = H(\mathbf{u}, \mathbf{v}_0)$ at $\mathbf{u}_0$ by $\mathrm{Hess}\,(h_{\mathbf{v}_0})(\mathbf{u}_0)$. We have the following lemma which is analogous to ([6], Propositions 3.1, 3.2).

Proposition 3.3 ([11])**.** Let $\mathbf{X} : U \longrightarrow S^n_1$ be a spacelike hypersurface in S^n_1, then $H(\mathbf{u},\mathbf{v}) = 0$ and $\partial H(\mathbf{u},\mathbf{v})/\partial u_i = 0$ $(i = 1, \cdots, n-1)$ if and only if $\mathbf{v} = \mathbb{L}^{\pm}(\mathbf{u})$. Under this condition, we have:

(1) $p_0 = \mathbf{X}(\mathbf{u}_0)$ is an $L^{\pm}$-parabolic point if and only if $\det \mathrm{Hess}\,(h^{\pm}_{\mathbf{v}_0})(\mathbf{u}_0) = 0$.

(2) $p_0 = \mathbf{X}(\mathbf{u}_0)$ is an $L^{\pm}$-flat point if and only if $\mathrm{rank}\,\mathrm{Hess}\,(h^{\pm}_{\mathbf{v}_0})(\mathbf{u}_0) = 0$.

Now we apply the arguments in Appendix A and naturally interpret the lightcone Gauss image of a spacelike hypersurface in S^n_1 as a wave front set in the theory of Legendrian singularities.

Proposition 3.4 ([11])**.** The lightcone height function $H : U \times LC^* \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces.

Proof. We denote $\mathbf{X}(\mathbf{u}) = (x_0(\mathbf{u}), \cdots, x_n(\mathbf{u}))$ and $\mathbf{X}_{u_i}(\mathbf{u}) = (x_{0,u_i}(\mathbf{u}), \cdots, x_{n,u_i}(\mathbf{u}))$. For any $\mathbf{v} = (v_0, \cdots, v_n) \in LC^*_{\pm}$, we have $v_0 \neq 0$. Without loss of generality, we assume that $v_0 = \sqrt{v_1^2 + \cdots + v_n^2} > 0$, so that we have $H(\mathbf{u},\mathbf{v}) = -1 - x_0(\mathbf{u})\sqrt{v_1^2 + \cdots + v_n^2} + \sum_{k=1}^n x_k(\mathbf{u})v_k$. We have to prove that the mapping $\Delta^* H : U \times LC^*_{\pm} \longrightarrow \mathbb{R}^n$ is non-singular on $(\Delta^* H)^{-1}(0)$. Therefore it is sufficient to show that the Jacobian matrix of $\Delta^* H$

$$J\Delta^* H(\mathbf{u},\mathbf{v}) = \begin{pmatrix} \left(-x_0\dfrac{v_j}{v_0} + x_j\right)_{j=1,\cdots,n} \\ \hline \left(-x_{0,u_i}\dfrac{v_j}{v_0} + x_{j,u_i}\right)_{\substack{j=1,\cdots,n \\ i=1,\cdots,n-1}} \end{pmatrix}$$

is regular on $\Sigma_*(H)$. We denote vectors $\bar{a}, \bar{b}_i$ $(i = 1, \cdots, n)$ by $\bar{a} = {}^t(x_0, x_{0,u_1}, \cdots, x_{0,u_{n-1}})$ and $\bar{b}_j = {}^t(x_j, x_{j,u_1}, \cdots, x_{j,u_{n-1}})$ for $j = 1, \cdots, n$. Then the determinant of $J\Delta^* H(\mathbf{u},\mathbf{v})$ is

$$\begin{aligned} \det J\Delta^* H(\mathbf{u},\mathbf{v}) &= \det\left(-\bar{a}\frac{v_1}{v_0} + \bar{b}_1, \cdots, -\bar{a}\frac{v_n}{v_0} + \bar{b}_n\right) \\ &= \det(\bar{b}_1, \cdots, \bar{b}_n) - \frac{v_1}{v_0}\det(\bar{a}, \bar{b}_2, \cdots, \bar{b}_n) - \cdots - \frac{v_n}{v_0}\det(\bar{b}_1, \cdots, \bar{b}_{n-1}, \bar{a}) \\ &= \left\langle \left(\frac{v_0}{v_0}, \cdots, \frac{v_n}{v_0}\right), \mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}(\mathbf{u}) \right\rangle = \frac{1}{v_0}\langle \mathbb{L}^{\pm}(\mathbf{u}), e(\mathbf{u})\rangle. \end{aligned}$$

Since $\mathbf{v} = \mathbb{L}^{\pm}(\mathbf{u})$, so that we have $\det J\Delta^* H(\mathbf{u},\mathbf{v}) = \mp 1/v_0 \neq 0$. This completes the proof. $\square$

By the arguments in appendix A, we obtain the following Legendrian immersion germs.

$$\mathcal{L}^{\pm} : (\Sigma_*^{\pm}(H), (\mathbf{u}_0, \mathbf{v}_0)) \longrightarrow PT^*(LC^*_{\pm}),$$

$$\mathcal{L}^{\pm}(\mathbf{u}, \mathbf{v}) = \left(\mathbf{v}, \left[\frac{\partial H}{\partial v_1}(\mathbf{u}, \mathbf{v}) : \cdots : \frac{\partial H}{\partial v_n}(\mathbf{u}, \mathbf{v})\right]\right),$$

where $\mathbf{v_0} = \mathbb{L}^{\pm}(\mathbf{u}_0)$ and $\Sigma_*^{\pm}(H)$ are singular sets of H

$$\Sigma_*^{\pm}(H) = \{(\mathbf{u}, \mathbf{v}) \in U \times LC^*_{\pm} \mid \mathbf{v} = \mathbb{L}^{\pm}(\mathbf{u})\}.$$

By Proposition 3.3, the lightcone Gauss images are the discriminant sets of the lightcone height function H. Therefore the wave front sets of the Legendrian immersion germs $\mathcal{L}^{\pm}$ are the lightcone Gauss image germs $\mathbb{L}^{\pm}$.

3.3 Contact with de Sitter hyperhorospheres

In this section we consider the contact between spacelike hypersurfaces and de Sitter hyperhorospheres. Let $\mathbf{v}_0 \in LC^*$, we define $\mathfrak{h}_{\mathbf{v}_0} : S^n_1 \longrightarrow \mathbb{R}$ by $\mathfrak{h}_{\mathbf{v}_0}(\mathbf{w}) = \langle \mathbf{w}, \mathbf{v}_0 \rangle - 1$. Then we have a de Sitter hyperhorosphere $\mathfrak{h}^{-1}_{\mathbf{v}_0}(0) = HS(\mathbf{v}_0, +1)$. For any $\mathbf{u}_0 \in U$, we consider the lightlike vector $\mathbf{v}^{\pm}_0 = \mathbb{L}^{\pm}(\mathbf{u}_0)$. Then we have

$$\begin{aligned}
(\mathfrak{h}_{\mathbf{v}^{\pm}_0} \circ \mathbf{X})(\mathbf{u}_0) &= H(\mathbf{u}_0, \mathbb{L}^{\pm}(\mathbf{u}_0)) = 0 \\
\frac{(\partial \mathfrak{h}_{\mathbf{v}^{\pm}_0} \circ \mathbf{X})}{\partial u_i}(\mathbf{u}_0) &= \langle \mathbf{X}_{u_i}(\mathbf{u}_0), \mathbb{L}^{\pm}(\mathbf{u}_0) \rangle = 0 \qquad \text{for } i = 1, \cdots, n-1.
\end{aligned}$$

This means that the de Sitter hyperhorosphere is tangent to the spacelike hypersurface $M = \mathbf{X}(U)$ at $p_0 = \mathbf{X}(\mathbf{u}_0)$. In this case, we call $HS(\mathbf{v}_0, +1)$ the *tangent de Sitter hyperhorosphere* of M at p_0 (or $\mathbf{u}_0$). Let $\mathbf{v}_1, \mathbf{v}_2 \in LC^*$, we say that de Sitter hyperhorospheres $HS(\mathbf{v}_i, +1)$ $(i = 1, 2)$ are *parallel* if $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. We have the following proposition which is analogous to ([6] Lemma 6.2).

Proposition 3.5 ([11])**.** Let $\mathbf{X} : U \longrightarrow S^n_1$ be a spacelike hypersurface. Consider two points $\mathbf{u}_1, \mathbf{u}_2 \in U$. Then $\mathbb{L}^{\pm}(\mathbf{u}_1) = \mathbb{L}^{\pm}(\mathbf{u}_2)$ if and only if $HS(\mathbf{X}, \mathbb{L}^{\pm}(\mathbf{u}_1)) = HS(\mathbf{X}, \mathbb{L}^{\pm}(\mathbf{u}_2))$.

Let $\mathbf{X}_i : (U, \mathbf{u}_i) \longrightarrow (S^n_1, p_i)$ be map germs of spacelike hypersurfaces and $\mathbb{L}^\pm_i$ be corresponding lightcone Gauss image germs. We denote germs of lightcone height functions with fixed parameters $\mathbf{v}^\pm_i = \mathbb{L}^\pm_i(\mathbf{u}_i)$ by $h_{i,\mathbf{v}^\pm_i} : (U, \mathbf{u}_i) \longrightarrow (\mathbb{R}, 0)$. Then we have $h_{i,\mathbf{v}^\pm_i}(\mathbf{u}) = (\mathfrak{h}_{\mathbf{v}^\pm_i}) \circ \mathbf{X}_i(\mathbf{u})$. By Theorem B.1, $K(\mathbf{X}_1(U), HS(\mathbf{X}, \mathbf{v}^\pm_1); \mathbf{u}_1) = K(\mathbf{X}_2(U), HS(\mathbf{X}, \mathbf{v}^\pm_2); \mathbf{u}_2)$ if and only if $h_{1,\mathbf{v}^\pm_1}$ and $h_{2,\mathbf{v}^\pm_2}$ are $\mathcal{K}$-equivalent.

Let $\mathbf{u}_0 \in U, \mathbf{v}^\pm_0 = \mathbb{L}^\pm(\mathbf{u}_0)$ and $C^\infty_{\mathbf{u}_0}(U)$ be the local ring of function germs at $\mathbf{u}_0$. We denote $Q^\pm(\mathbf{X}, \mathbf{u}_0)$ the local ring of the function germ $h_{\mathbf{v}^\pm_0} : (U, \mathbf{u}_0) \longrightarrow \mathbb{R}$ by

$$Q^\pm(\mathbf{X}, \mathbf{u}_0) = C^\infty_{\mathbf{u}_0}(U)/\langle h_{\mathbf{v}_0}\rangle_{C^\infty_{\mathbf{u}_0}(U)}.$$

By the above arguments and Appendix A, we have the following theorem which is analogous to ([6] Theorem 6.3).

Theorem 3.6 ([11])**.** Let $\mathbf{X}_i : (U, \mathbf{u}_i) \longrightarrow (S^n_1, p_i)$ (for $i = 1, 2$) be spacelike hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent:

(1) Lightcone Gauss image germs $\mathbb{L}^\pm_1$ and $\mathbb{L}^\pm_2$ are $\mathcal{A}$-equivalent.

(2) Legendrian immersion germs $\mathcal{L}_1$ and $\mathcal{L}_2$ are Legendrian equivalent.

(3) Lightcone height function germs H_1 and H_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(4) $h_{1,\mathbf{v}^\pm_1}$ and $h_{2,\mathbf{v}^\pm_2}$ are $\mathcal{K}$-equivalent.

(5) $K(\mathbf{X}_1(U), HS(\mathbf{X}_1, \mathbf{v}^\pm_1); \mathbf{u}_1) = K(\mathbf{X}_2(U), HS(\mathbf{X}_2, \mathbf{v}^\pm_2); \mathbf{u}_2)$

(6) $Q^\pm(\mathbf{X}_1, \mathbf{u}_1)$ and $Q^\pm(\mathbf{X}_2, \mathbf{u}_2)$ are isomorphic as $\mathbb{R}$-algebras.

Proof. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are Legendrian stable, regular sets of $\mathbb{L}_1$ and $\mathbb{L}_2$ are respectively dense. By Proposition A.2, the conditions (1) and (2) are equivalent. We apply Theorem A.3, the conditions (2) and (3) are equivalent. By the previous arguments from Theorem B.1, the conditions (4) and (5) are equivalent.

If we assume the condition (3), then the $\mathcal{P}$-$\mathcal{K}$-equivalence preserves the $\mathcal{K}$-equivalence, so that the condition (4) holds. Since the local ring $Q^{\pm}(\mathbf{X}_i, \mathbf{u}_i)$ is $\mathcal{K}$-invariant, this means that the condition (6) holds. By Proposition A.4, the condition (6) implies the condition (2). $\Box$

We now consider the generic property of spacelike hypersurfaces in de Sitter space. We consider the map space of spacelike embeddings Sp-Emb(U, S^n_1) with Whitney C^∞-topology. Applying the arguments in appendix C, we have the following proposition as a corollary of Theorem C.2.

Proposition 3.7. If $n \leq 6$, then assumption of the Theorem 3.6 is generic property. That is, there exists an open dense subset $\mathcal{O} \subset$ Sp-Emb(U, S^n_1) such that for any $\mathbf{X} \in \mathcal{O}$, corresponding Legendrian immersion germ $\mathcal{L}$ is Legendrian stable.

In general we have the following proposition.

Proposition 3.8 ([11])**.** Let $\mathbf{X}_i : (U, \mathbf{u}_i) \longrightarrow (S^n_1, p_i)$ (for $i = 1, 2$) be hypersurface germs such that their $L^{\pm}$-parabolic sets have no interior points as subspaces of U. If lightcone Gauss image germs $\mathbb{L}^{\pm}_1$ and $\mathbb{L}^{\pm}_2$ are $\mathcal{A}$-equivalent, then

$$K(\mathbf{X}_1(U), HS(\mathbf{X}_1, \mathbf{v}^{\pm}_1); \mathbf{u}_1) = K(\mathbf{X}_2(U), HS(\mathbf{X}_2, \mathbf{v}^{\pm}_2); \mathbf{u}_2),$$

where $\mathbf{v}_i = \mathbb{L}^{\pm}(\mathbf{u}_i)$ for $i = 1, 2$. In this case, $(\mathbf{X}_i^{-1}(HS(\mathbb{L}^{\pm}_i(\mathbf{u}_i), +1)), u_i)$ $(i = 1, 2)$ are diffeomorphic as set germs.

For a hypersurface germ $\mathbf{X}$, we call $(\mathbf{X}^{-1}(HS(\mathbb{L}^{\pm}(\mathbf{u}_0), +1)), \mathbf{u}_0)$ the *tangent de Sitter hyperhorospherical indicatrix germ* of $\mathbf{X}$. By Proposition 3.8, the diffeomorphic type of the tangent de Sitter horospherical indicatrix germ is an invariant under the $\mathcal{A}$-equivalence among lightcone Gauss image germs.

3.4 Spacelike Monge forms

In this section we introduce the notion of spacelike Monge forms in de Sitter space which is analogous to [6]. We now consider the function $f(u_1, \cdots, u_{n-1})$ with $f(0) = 0$ and $f_{u_i}(0) = 0$

for $i=1,\dots,n-1$. Then we have a spacelike hypersurface in S^n_1 by

$$\mathbf{X}_f(\mathbf{u})=\left(-f(\mathbf{u}),-\sqrt{1+f^2(\mathbf{u})-u_1^2-\cdots-u_{n-1}^2},u_1,\cdots,u_{n-1}\right).$$

where U is some sufficiently small open neighborhood at the origin. Since $\mathbf{e}(\mathbf{0})=(1,0,\cdots,0)$, so that we have $\mathbb{L}^{\pm}(\mathbf{0})=(\pm 1,-1,0,\cdots,0)$. We call $\mathbf{X}_f$ a *spacelike Monge form* in de Sitter space. Then we have the following proposition.

Proposition 3.9 ([11])**.** Any spacelike surface in S^n_1 is locally given by some spacelike Monge form.

Proof. Let $\mathbf{X}:U\longrightarrow S^n_1$ be a spacelike hypersurface. Since the Lorentzian group operates transitively S^n_1, we may assume that $p=\mathbf{X}(\mathbf{0})=(0,-1,0,\cdots,0)$. We have a basis $\{\mathbf{X}(\mathbf{0}),\mathbf{e}(\mathbf{0}),\mathbf{X}_{u_1}(\mathbf{0}),\cdots,\ \mathbf{X}_{u_{n-1}}(\mathbf{0})\}$ of $T_p\mathbb{R}^{n+1}$. Applying the Gram-Schmidt procedure on $\{\mathbf{X}_{u_1}(\mathbf{0}),\cdots,\ \mathbf{X}_{u_{n-1}}(\mathbf{0})\}$, we have a pseudo orthonormal basis $\{\mathbf{X}(\mathbf{0}),\mathbf{e}(\mathbf{0}),\ \mathbf{e}_1,\cdots,\mathbf{e}_{n-1}\}$ of $\mathbb{R}^{n+1}_1$ such that $T_pM=\langle\mathbf{e}_1,\cdots,\mathbf{e}_{n-1}\rangle_{\mathbb{R}}$. In particular, $\{\mathbf{e}_1,\cdots,\mathbf{e}_{n-1}\}$ is an orthonormal basis of spacelike subspace T_pM, so that T_pM is considered to be a subspace of $\mathbb{R}^n_0=\{(0,x_1,\cdots,x_n)\mid x_i\in\mathbb{R}\}$. By a rotation of the space $\mathbb{R}^n_0$, we assume that $T_pM=\{(0,0,x_2,\cdots,x_n)\mid x_i\in\mathbb{R}\}$. We remark that this rotation can be considered to be a Lorentzian motion of $\mathbb{R}^{n+1}_1$.

Therefore, the hypersurface germ (M,p) is written in the form

$$\mathbf{X}(\mathbf{u})=(-f(\mathbf{u}),-g(\mathbf{u}),u_1,\cdots,u_{n-1}).$$

with function germs $f(\mathbf{u}),g(\mathbf{u})$. Since $M\subset S^n_1$, we have a relation $g^2(\mathbf{u})=1+f^2(\mathbf{u})-u_1^2-\cdots-u_{n-1}^2$. By a rotation of the space $\mathbb{R}^n_0$, we can assume $g(\mathbf{u})\geq 0$, so that we have

$$g(\mathbf{u})=\sqrt{1+f^2(\mathbf{u})-u_1^2-\cdots-u_{n-1}^2}.$$

Since $T_pM=\{(0,0,x_2,\cdots,x_n)\mid x_i\in\mathbb{R}\}$, the conditions $f(\mathbf{0})=f_{u_i}(\mathbf{0})=0$ $(i=1,\cdots,n-1)$ are automatically satisfied. This completes the proof. □

For the lightlike vector $\mathbf{v}_0^{\pm}=(\pm 1,-1,0,\cdots,0)$, we consider the de Sitter hyperhorosphere $HS(\mathbf{v}_0^{\pm},+1)$. Then we have the spacelike Monge form of $HS(\mathbf{v}_0^{\pm},+1)$.

$$\mathbf{X}_{HS^{\pm}}(\mathbf{u})=\left(\mp\frac{1}{2}(u_1^2+\cdots+u_{n-1}^2),-1+\frac{1}{2}(u_1^2+\cdots+u_{n-1}^2),u_1,\cdots,u_{n-1}\right).$$

Here we can check the relation $\langle \mathbf{v}_0^{\pm}, \mathbf{X}_{HS^{\pm}}(\mathbf{u})\rangle = 1$. On the other hand, we have $\mathbf{X}_{HS^{\pm}}(\mathbf{0}) = (0, -1, 0, \cdots, 0)$ and $\mathbf{X}_{HS^{\pm},u_i}(\mathbf{0})$ is the x_{i+1}-axis for $i = 1, \cdots, n-1$. This means that $T_pM = T_p(\mathbf{X}_{HS^{\pm}}(U))$. Therefore $\mathbf{X}_{HS^{\pm}}(U) \subset HS(\mathbf{v}_0^{\pm}, +1)$ is the tangent de Sitter hyperhorosphere of $M = \mathbf{X}(U)$ at $p = \mathbf{X}(\mathbf{0})$. It follows from this fact that the tangent de Sitter hyperhorospherical indicatrix germ of the spacelike Monge form $\mathbf{X}_f$ is given as follows:

$$\mathbf{X}_f^{-1}(HS(\mathbf{v}_0^{\pm}, +1)) = \{(\mathbf{u}_1, \cdots, u_{n-1}) \mid \pm 2f(\mathbf{u}) = u_1^2 + \cdots + u_{n-1}^2\}.$$

Since the lightcone height function of $\mathbf{X}_f$ at $\mathbf{v}_0^{\pm}$ is

$$h_{\mathbf{v}_0^{\pm}}(\mathbf{u}) = \pm f(\mathbf{u}) + \sqrt{1 + f^2(\mathbf{u}) - u_1^2 - \cdots - u_{n-1}^2} - 1,$$

we can calculate the Hessian matrix, so that we have $\mathrm{Hess}\, h_{\mathbf{v}_0^{\pm}}(\mathbf{0}) = \pm\mathrm{Hess}\,(f(\mathbf{0})) - \mathrm{I}_{n-1}$, where I_{n-1} is an identity matrix.

On the other hand, since $f(\mathbf{0}) = f_{u_i}(\mathbf{0}) = 0$, we may write

$$f(\mathbf{u}) = \frac{1}{2}\kappa_1 u_1^2 + \cdots \frac{1}{2}\kappa_{n-1}u_{n-1}^2 + g(\mathbf{u}),$$

where $g \in \mathfrak{M}_{n-1}^3$ and $\kappa_1, \cdots, \kappa_{n-1}$ are eigenvalues of $\mathrm{Hess}\,(f(\mathbf{0}))$. Under this representation, we can easily calculate $(\mathbf{X}_f)_{u_i,u_j}(\mathbf{0}) = (-f_{u_i,u_j}(\mathbf{0}), \delta_{ij}, 0, \cdots, 0)$. It follows from that

$$\bar{h}_{ij}^{\pm}(\mathbf{0}) = \pm f_{u_i,u_j}(\mathbf{0}) - \delta_{ij} = \delta_{ij}(\pm\kappa_i - 1),$$

and $g_{ij}(\mathbf{0}) = \delta_{ij}$. Therefore, we have $\bar{\kappa}_i^{\pm}(\mathbf{0}) = -1 \pm \kappa_i$ and

$$K_\ell^{\pm}(\mathbf{0}) = \prod_{i=1}^{n-1} \bar{\kappa}_i^{\pm}(\mathbf{0}) = \prod_{i=1}^{n-1}(-1 \pm \kappa_i).$$

The tangent de Sitter hyperhorospherical indicatrix germ is given by

$$\mathbf{X}_f^{-1}(HS(\mathbf{v}_0^{\pm}, +1)) = \left\{ (u_1, \cdots, u_{n-1}) \;\middle|\; \sum_{i=1}^{n-1} \kappa_i^{\pm}(\mathbf{0})u_i^2 \pm 2g(\mathbf{u}) = 0 \right\}.$$

3.5 Spacelike surfaces in de Sitter 3-space

In this section we restrict the dimension $n = 3$ and observe the geometrical properties corresponding to the singularities of lightcone Gauss images. We consider the space of spacelike embeddings $\mathrm{Sp\text{-}Emb}(U, S_1^n)$ with Whitney C^∞-topology.

Let $\mathbf{u}_0 \in U \subset \mathbb{R}^3$ and $\mathbf{X} : U \longrightarrow S^3_1$ be a spacelike hypersurface in de Sitter three-space. We define the $\mathcal{K}$-codimension (or Tyurina number) of the lightcone height function germ $(h_{\mathbf{v}_0^\pm}, \mathbf{u}_0)$ by

$$\text{H-ord}^\pm(\mathbf{X}, \mathbf{u}_0) = \dim C^\infty_{\mathbf{u}_0} / \langle h_{\mathbf{v}_0^\pm}(\mathbf{u}_0), \partial h_{\mathbf{v}_0^\pm}(\mathbf{u}_0)/\partial u_i \rangle_{C^\infty_{\mathbf{u}_0}},$$

We also have the notion of corank of the function germ:

$$\text{H-corank}^\pm(\mathbf{X}, \mathbf{u}_0) = (n-1) - \operatorname{rank} \operatorname{Hess}(h_{\mathbf{v}_0^\pm}(\mathbf{u}_0)).$$

By Proposition 3.3, $p = \mathbf{X}(\mathbf{u}_0)$ is an $L^\pm$-parabolic point if and only if H-corank$^\pm$ $(\mathbf{X}, \mathbf{u}_0) \geq 1$. Moreover p is an $L^\pm$-flat point if and only if H-corank$^\pm(\mathbf{X}, \mathbf{u}_0) = 2$. (H-corank$^\pm(\mathbf{X}, \mathbf{u}_0) = n-1$ in general dimension n.)

We say that a function germ $f : (\mathbb{R}^{n-1}, \mathbf{a}) \longrightarrow \mathbb{R}$ has the $\mathcal{A}_k$-type singularity at $\mathbf{a}$ if f is $\mathcal{K}$-equivalent to the germ $g(u_1, \cdots, u_{n-1}) = \pm u_1 \pm \cdots \pm u_{n-2} + u_{n-1}^{k+1}$.

By the classification of stable Legendrian singularities for $n = 3$ [1, 20] and the transversal theorem of [19], we have the following theorem.

Theorem 3.10 ([11])**.** There exists an open dense subset $\mathcal{O} \subset \text{Sp-Emb}(U, S^n_1)$ such that for any $\mathbf{X} \in \mathcal{O}$, the following conditions holds.

(1) The $L^\pm$-parabolic set $K_\ell^{-1}(0)$ is a regular curve. We call such a curve the $L^\pm$*-parabolic curve.*

(2) The lightcone Gauss image $\mathbb{L}^\pm$ along the $L^\pm$-parabolic curve is a cuspidal edge except at isolated points. At this point $\mathbb{L}^\pm$ is swallowtail.

Here, a map germ $f : (\mathbb{R}^2, \mathbf{a}) \longrightarrow (\mathbb{R}^3, \mathbf{b})$ is called the *cuspidal edge* if it is $\mathcal{A}$-equivalent to the map germ $g_1 : (\mathbb{R}^2, \mathbf{0}) \longrightarrow (\mathbb{R}^3, \mathbf{0})$ with $g_1(\mathbf{u}) = (u_1, u_2^2, u_2^3)$ and the *swallowtail* if it is $\mathcal{A}$-equivalent to the map germ $g_2 : (\mathbb{R}^2, \mathbf{0}) \longrightarrow (\mathbb{R}^3, \mathbf{0})$ with $g_2(\mathbf{u}) = (3u_1^4 + u_1^2 u_2, 4u_1^3 + 2u_1 u_2, u_2)$. (c.f. Figure 1 and 2) The swallowtail point is an isolated singular point.

The assertion of Theorem 3.10 can be interpreted as saying that the Legendrian lift $\mathcal{L}^\pm$ of the lightcone Gauss image $\mathbb{L}^\pm$ is Legendrian stable at each point. In this case, the lightcone Gauss image $\mathbb{L}^\pm$ has only cuspidal edges and swallowtails as singularities.

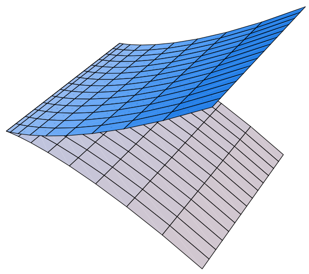

Figure 1: Cuspidal edge

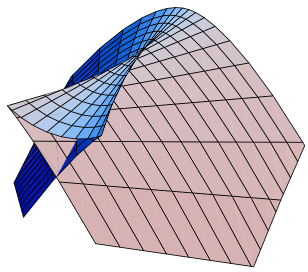

Figure 2: Swallowtail

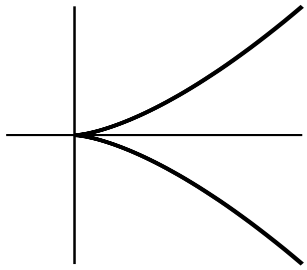

Figure 3: ordinary cusp

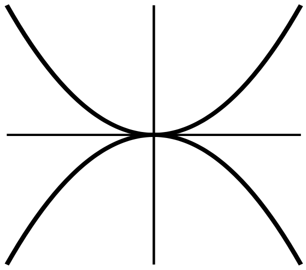

Figure 4: tachnodal

Corollary 3.11 ([11]). Let $\mathcal{O} \subset \text{Sp-Emb}(U, S^n_1)$ be the same open dense subset as in Theorem 3.10. Let $\mathbf{X} \in \mathcal{O}$, $\mathbf{v}_0^\pm = \mathbb{L}^\pm(\mathbf{u}_0)$ and $h_{\mathbf{v}_0^\pm} : (U, \mathbf{u}_0) \longrightarrow \mathbb{R}$ be the lightcone height function germ at $\mathbf{u}_0$. Then we have the following.

(1) The point $\mathbf{u}_0$ is an $L^\pm$-parabolic point of $\mathbf{X}$ if and only if $\text{H-corank}^\pm(\mathbf{X}, \mathbf{u}_0) = 1$ (that is, $\mathbf{u}_0$ is not an $L^\pm$-flat point). In this case, $h_{\mathbf{v}_0^\pm}$ has the $\mathcal{A}_k$-type singularity for $k = 2, 3$.

(2) Suppose that $\mathbf{u}_0$ is an $L^\pm$-parabolic point of $\mathbf{X}$. Then the following conditions are equivalent:

  (a) $\mathbb{L}^\pm$ has the cuspidal edge at $\mathbf{u}_0$;

  (b) $h_{\mathbf{v}_0^\pm}$ has the $\mathcal{A}_2$-type singularity;

  (c) $\text{H-ord}^\pm(\mathbf{X}, \mathbf{u}_0) = 2$;

  (d) The tangent de Sitter horospherical indicatrix germ is an ordinary cusp, where a curve $C \subset \mathbb{R}^2$ is called an ordinary cusp if it is diffeomorphic to the curve given by

$\{(u_1, u_2) \mid u_1^2 - u_2^3 = 0\}$;

(3) Suppose that $\mathbf{u}_0$ is an $L^\pm$-parabolic point of $\mathbf{X}$. Then the following conditions are equivalent:

(a) $\mathbb{L}^\pm$ has the swallowtail at $\mathbf{u}_0$;

(b) $h_{\mathbf{v}_0^\pm}$ has the $\mathcal{A}_3$-type singularity;

(c) H-ord$^\pm(\mathbf{X}, \mathbf{u}_0) = 3$;

(d) The tangent de Sitter horospherical indicatrix germ is a point or a tachnodal, where a curve $C \subset \mathbb{R}^2$ is called an tachnodal if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^4 = 0\}$;

(e) For each $\varepsilon > 0$, there exist $L^\pm$-non-parabolic points $u_1, u_2 \in U$ such that $||u_0 - u_i|| < \varepsilon$ for $i = 1, 2$, and the tangent de Sitter horospheres to $M = \mathbf{X}(U)$ at u_1 and u_2 are equal.

Proof. Since $n = 3$, $\mathbf{u}_0$ is an $L^\pm$-parabolic point if and only if H-corank$^\pm(\mathbf{X}, \mathbf{u}_0) \geq 1$. By the classification of singularities of function germs, $h_{\mathbf{v}_0^\pm}$ has only the $\mathcal{A}_2$ or $\mathcal{A}_3$-type singularities. We can avoid the case when $\mathbf{u}_0$ is an $L^\pm$-flat point, so that H-corank$^\pm(\mathbf{X}, \mathbf{u}_0) = 1$.

By Theorem 3.6, the conditions of (2) are equivalent. Similarly, the all conditions (a),(b),(c) and (d) of (3) are also equivalent. Suppose that corresponding Gauss image has swallowtail at $\mathbf{u}_0$. We can observe that there is a self-intersection curve approaching $\mathbf{u}_0$. (cf. Figure 2.) On this curve, there are two distinct points u_1 and u_2 such that $\mathbb{L}^\pm(\mathbf{u}_1) = \mathbb{L}^\pm(\mathbf{u}_2)$. By Lemma 3.5, this means that tangent de Sitter horospheres to $M = \mathbf{X}(U)$ at u_1 and u_2 are the same. On the other hand, if the Gauss image has cuspidal edge at $\mathbf{u}_0$, there are no self-intersection on $\mathbb{L}^\pm$. (cf. Figure 1.) This means that (3)(a) is equivalent to (3)(e). This completes the proof. □

3.6 Examples of spacelike surfaces in S_1^3

In this section we give some examples using the spacelike Monge form introduced in 3.4 in de Sitter space.

Example 3.12. If $f(u_1,u_2)=\frac{1}{3}u_1^3+\frac{1}{2}u_1^2$, then

$$\mathbf{X}_f(u_1,u_2)=\left(-\frac{1}{3}u_1^3-\frac{1}{2}u_1^2,-\sqrt{1+\left(\frac{1}{3}u_1^3+\frac{1}{2}u_1^2\right)^2-u_1^2-u_2^2},u_1,u_2\right),$$

and $\kappa_1=1,\kappa_2=0$. Then we have $\bar{\kappa}_1^+(\mathbf{0})=0$, $\bar{\kappa}_2^+(\mathbf{0})=-1$, $\bar{\kappa}_1^-(\mathbf{0})=-2$ and $\bar{\kappa}_2^-(\mathbf{0})=-1$. So the origin is not an L^--parabolic point but an L^+-parabolic point. The positive tangent de Sitter horospherical indicatrix germ is the ordinary cusp $\{(u_1,u_2)\mid 2u_1^3=3u_2^2\}$. Therefore, the lightcone Gauss image $\mathbb{L}^-$ is non-singular at the origin and $\mathbb{L}^+$ is a cuspidal edge at the origin.

Example 3.13. If $f(u_1,u_2)=\frac{1}{2}u_1^4+\frac{1}{2}u_1^2$, then

$$\mathbf{X}_f(u_1,u_2)=\left(-\frac{1}{2}u_1^4-\frac{1}{2}u_1^2,-\sqrt{1+\left(\frac{1}{2}u_1^4+\frac{1}{2}u_1^2\right)^2-u_1^2-u_2^2},u_1,u_2\right),$$

and $\kappa_1=1,\kappa_2=0$. By the same reason as in the previous example, the origin is not an L^--parabolic point but an L^+-parabolic point. The positive tangent de Sitter horospherical indicatrix germ is the tachnodal $\{(u_1,u_2)\mid u_1^4=u_2^2\}$. Therefore, the lightcone Gauss image $\mathbb{L}^-$ is non-singular at the origin and $\mathbb{L}^+$ is a swallowtail at the origin.

Part II Spacelike submanifolds of codimension two in de Sitter space

4 Spacelike submanifolds of codim two

4.1 Spacelike submanifolds and lightcone normal frames

Let U be an open subset of $\mathbb{R}^{n-2}$ and $\mathbf{X}:U\longrightarrow S_1^n$ be an embedding, we say that $\mathbf{X}$ is *spacelike* in S_1^n if every non zero vector generated by tangent vectors $\{\mathbf{X}_{u_i}(\mathbf{u})\}_{i=1}^{n-2}$ is always spacelike, where $\mathbf{u}\in U$ and $\mathbf{X}_{u_i}=\partial\mathbf{X}/\partial u_i$. We identify $M=\mathbf{X}(U)$ with U through the embedding $\mathbf{X}$ and call $\mathbf{X}$ and M a *spacelike submanifold of codimension two* in S_1^n.

Since $\langle \mathbf{X}, \mathbf{X} \rangle \equiv 1$, we have $\langle \mathbf{X}_{u_i}, \mathbf{X} \rangle \equiv 0$ for $i = 1, \cdots, n-2$. For any $p = \mathbf{X}(\mathbf{u})$, the pseudo-normal space $N_pM := T_pM^{\perp} \subset \mathbb{R}^{n+1}_1$ is a timelike 3-plane. we can choose a *future directed unit normal section* $\mathbf{n}^T(\mathbf{u}) \in N_pM \cap H^n_+(-1)$ satisfying $\langle \mathbf{n}^T(\mathbf{u}), \mathbf{X}(\mathbf{u}) \rangle = 0$. Therefore we can construct a spacelike unit normal section $\mathbf{n}^S(\mathbf{u}) \in N_pM \cap S^n_1$ by

$$\mathbf{n}^S(\mathbf{u}) = \frac{\mathbf{n}^T(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(\mathbf{u})}{||\mathbf{n}^T(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(\mathbf{u})||},$$

and we have $\langle \mathbf{n}^T(\mathbf{u}), \mathbf{n}^T(\mathbf{u}) \rangle = -1$, $\langle \mathbf{n}^T(\mathbf{u}), \mathbf{n}^S(\mathbf{u}) \rangle = 0$, $\langle \mathbf{n}^S(\mathbf{u}), \mathbf{n}^S(\mathbf{u}) \rangle = 1$. Therefore vectors $\mathbf{n}^T(\mathbf{u}) \pm \mathbf{n}^S(\mathbf{u})$ are lightlike. We call $(\mathbf{n}^T, \mathbf{n}^S)$ a *future directed normal frame along* $M = \mathbf{X}(U)$. The system $\{\mathbf{X}(\mathbf{u}), \mathbf{n}^T(\mathbf{u}), \mathbf{n}^S(\mathbf{u}), \mathbf{X}_{u_1}(\mathbf{u}), \cdots, \mathbf{X}_{u_{n-2}}(\mathbf{u})\}$ is a basis of $T_p\mathbb{R}^{n+1}_1$. We have the following lemma which is analogous to Lemma 3.1 in [8].

Lemma 4.1. ([12]) Let $\mathbf{n}^T(\mathbf{u})$ and $\bar{\mathbf{n}}^T(\mathbf{u}) \in N_pM$ be the future directed unit timelike normal sections of M then the corresponding lightlike normal sections $\mathbf{n}^T(\mathbf{u}) \pm \mathbf{n}^S(\mathbf{u})$ and $\bar{\mathbf{n}}^T(\mathbf{u}) \pm \bar{\mathbf{n}}^S(\mathbf{u})$ are parallel.

Proof. First of all, we show that $\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u})$ is parallel to either $\mathbf{n}^T(\mathbf{u}) + \mathbf{n}^S(\mathbf{u})$ or $\mathbf{n}^T(\mathbf{u}) - \mathbf{n}^S(\mathbf{u})$. By assumption, $\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u})$ is a lightlike normal at p on M, then there exist real numbers λ and μ such that

$$\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u}) = \lambda \mathbf{n}^T(\mathbf{u}) + \mu \mathbf{n}^S(\mathbf{u}).$$

Since $\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u})$ is lightlike, it follows that $|\lambda| = |\mu|$. Therefore we have

$$\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u}) = \lambda(\mathbf{n}^T(\mathbf{u}) + \sigma \mathbf{n}^S(\mathbf{u})).$$

for some $\sigma \in \{\pm 1\}$. It follows that both lightlike vectors $\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u})$ and $\mathbf{n}^T(\mathbf{u}) + \sigma \mathbf{n}^S(\mathbf{u})$ are future directed, therefore we have $\lambda > 0$.

Next, we show that $\sigma = +1$. By definition of $\mathbf{n}^S(\mathbf{u})$, there exist a positive number $\alpha > 0$ such that

$$\mathbf{n}^S(\mathbf{u}) = \alpha\, (\mathbf{n}^T(\mathbf{u}) \wedge \mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(\mathbf{u})\,).$$

On the other hand, there are some real number $\beta \in \mathbb{R}$ such that

$$\mathbf{n}^T(\mathbf{u}) = \beta\,(\mathbf{n}^S(\mathbf{u}) \wedge \mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(\mathbf{u})\,).$$

In this case we have

$$\langle \mathbf{n}^T(\mathbf{u}), \mathbf{n}^S(\mathbf{u}) \wedge \mathbf{X}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(\mathbf{u})\rangle \;\; = \;\; -\langle \mathbf{n}^S(\mathbf{u}), \mathbf{n}^T(\mathbf{u}) \wedge \mathbf{X}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(\mathbf{u})\rangle.$$

So that we have $-\alpha = \langle \mathbf{n}^T(\mathbf{u}), \alpha\mathbf{n}^T(\mathbf{u})\rangle = -\langle \mathbf{n}^S(\mathbf{u}), \beta\mathbf{n}^S(\mathbf{u})\rangle = -\beta$ and $\alpha = \beta$. Therefore,

$$\mathbf{n}^T(\mathbf{u}) + \mathbf{n}^S(\mathbf{u}) = \alpha\left(\mathbf{n}^T(\mathbf{u}) + \mathbf{n}^S(\mathbf{u})\right) \wedge \mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(\mathbf{u}).$$

On the other hand, we obtain a real number $\alpha' > 0$ such that

$$\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u}) = \alpha'\left(\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u})\right) \wedge \mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(\mathbf{u}).$$

Then we have,

$$\begin{aligned} \bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u}) &= \alpha'(\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u})) \wedge \mathbf{X}(\mathbf{u}) \wedge \cdots \\ &= \alpha'\lambda(\mathbf{n}^T(\mathbf{u}) + \sigma\mathbf{n}^S(\mathbf{u})) \wedge \mathbf{X}(\mathbf{u}) \wedge \cdots \\ &= \alpha^{-1}\alpha'\lambda(\mathbf{n}^S(\mathbf{u}) + \sigma\mathbf{n}^T(\mathbf{u})) \qquad (\alpha^{-1}\alpha'\lambda > 0). \end{aligned}$$

Since the vector $\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u})$ is future directed, so that we have $\sigma = +1$. Therefore

$$\bar{\mathbf{n}}^T(\mathbf{u}) + \bar{\mathbf{n}}^S(\mathbf{u}) = \lambda'(\mathbf{n}^T(\mathbf{u}) + \mathbf{n}^S(\mathbf{u})),$$

where $\lambda' = \alpha^{-1}\alpha'\lambda > 0$. We can show that the lightlike vectors $\bar{\mathbf{n}}^T(\mathbf{u}) - \bar{\mathbf{n}}^S(\mathbf{u})$ and $\mathbf{n}^T(\mathbf{u}) - \mathbf{n}^S(\mathbf{u})$ are parallel in similar way. In this case, we have

$$\bar{\mathbf{n}}^T(\mathbf{u}) - \bar{\mathbf{n}}^S(\mathbf{u}) = \frac{1}{\lambda'}\left(\mathbf{n}^T(\mathbf{u}) - \mathbf{n}^S(\mathbf{u})\right).$$

□

We remark that the matrix

$$\begin{pmatrix} \lambda' & 0 \\ 0 & 1/\lambda' \end{pmatrix} \qquad (\lambda' > 0)$$

is the matrix of a Lorentzian transformation on the 2-plane spanned by the lightlike normal frames, with respect to the basis $\{\mathbf{n}^T(\mathbf{u}) \pm \mathbf{n}^S(\mathbf{u})\}$ and $\{\bar{\mathbf{n}}^T(\mathbf{u}) \pm \bar{\mathbf{n}}^S(\mathbf{u})\}$.

Under the identification of M and U through $\mathbf{X}$, we have the linear mapping

$$d_p(\mathbf{n}^T \pm \mathbf{n}^S) : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_pM.$$

We consider a orthonormal projection $\pi^t : T_p\mathbb{R}^{n+1}_1 \longrightarrow T_pM$ and define a linear transformation $d_p(\mathbf{n}^T \pm \mathbf{n}^S)^t$ by

$$d_p(\mathbf{n}^T \pm \mathbf{n}^S)^t = \pi^t \circ d_p(\mathbf{n}^T \pm \mathbf{n}^S).$$

We call the linear transformation $S^{\pm}_p(\mathbf{n}^T, \mathbf{n}^S) = -d_p(\mathbf{n}^T \pm \mathbf{n}^S)^t$ an $(\mathbf{n}^T, \mathbf{n}^S)$-*shape operator of* $M = \mathbf{X}(U)$ at $p = \mathbf{X}(\mathbf{u})$.

The eigenvalues of $S^{\pm}_p(\mathbf{n}^T, \mathbf{n}^S)$ denoted by $\{\kappa^{\pm}_i(\mathbf{n}^T, \mathbf{n}^S)(p)\}_{i=1}^{n-2}$ are called the *lightcone principal curvatures* of M with respect to $(\mathbf{n}^T, \mathbf{n}^S)$ at p. We also define a *lightcone Gauss-Kronecker curvature* of M with respect to $(\mathbf{n}^T, \mathbf{n}^S)$ at p by

$$K^{\pm}_{\ell}(\mathbf{n}^T, \mathbf{n}^S)(p) = \det S^{\pm}_p(\mathbf{n}^T, \mathbf{n}^S).$$

We say that a point p is an $(\mathbf{n}^T, \mathbf{n}^S)$-*umbilic point* if all the principal curvatures coincide at p. In this case we have $S^{\pm}_p(\mathbf{n}^T, \mathbf{n}^S) = \kappa^{\pm}\, \mathrm{id}_{T_pM}$ for some $\kappa^{\pm} \in \mathbb{R}$. We say that M is $(\mathbf{n}^T, \mathbf{n}^S)$-*totally umbilic* if all points on M are $(\mathbf{n}^T, \mathbf{n}^S)$-umbilic.

Since $\mathbf{X}$ is spacelike submanifolds, we may define a *Riemannian metric* (or the *first fundamental form*) on M by $ds^2 = \sum_{i,j=1}^{n-2} g_{ij} du_i du_j$, where $g_{ij}(\mathbf{u}) = \langle \mathbf{X}_{u_i}(\mathbf{u}), \mathbf{X}_{u_j}(\mathbf{u})\rangle$ for any $\mathbf{u} \in U$. We also define a *lightcone second fundamental form* (or the *lightcone second fundamental invariant*) with respect to the normal vector field $(\mathbf{n}^T, \mathbf{n}^S)$ defined by $h^{\pm}_{ij}(\mathbf{u}) = -\langle (\mathbf{n}^T \pm \mathbf{n}^S)_{u_i}(\mathbf{u}), \mathbf{X}_{u_j}(\mathbf{u})\rangle$ for any $u \in U$. We have the following Weingarten-type formula.

Lemma 4.2 ([12])**.** Let $(\mathbf{n}^T, \mathbf{n}^S)$ be a lightlike normal frame on the spacelike submanifold M of codimension two in de Sitter space, then we have:

$$(\mathbf{n}^T \pm \mathbf{n}^S)_{u_i} = \pm\langle \mathbf{n}^S, \mathbf{n}^T_{u_i}\rangle(\mathbf{n}^T \pm \mathbf{n}^S) - \sum_{j=1}^{n-2} h^{\pm j}_i(\mathbf{n}^T, \mathbf{n}^S)\mathbf{X}_{u_j},$$

where $(h_i^{j\pm}(\mathbf{n}^T,\mathbf{n}^S))_{ij} = \left(h_{ik}^{\pm}(\mathbf{n}^T,\mathbf{n}^S)\right)_{ik}(g^{kj})_{kj}$ and $(g^{kj})_{kj} = (g_{kj})^{-1}$. Therefore we have

$$\pi^t \circ (\mathbf{n}^T \pm \mathbf{n}^S)_{u_i} = -\sum_{j=1}^{n-2} h_i^{j\pm}(\mathbf{n}^T,\mathbf{n}^S)\mathbf{X}_{u_j}.$$

Therefore we have an explicit expression of the lightcone Gauss-Kronecker curvature in terms of the lightcone first and second fundamental forms.

$$K_\ell^{\pm}(\mathbf{n}^T,\mathbf{n}^S)(p) = \frac{\det(h_{ij}^{\pm}(\mathbf{n}^T,\mathbf{n}^S)(\mathbf{u}))}{\det(g_{\alpha\beta}(\mathbf{u}))}$$

We say that a point p is an $(\mathbf{n}^T,\mathbf{n}^S)$-*parabolic point* if $K_\ell^{\pm}(\mathbf{n}^T,\mathbf{n}^S)(p) = 0$, and M is an $(\mathbf{n}^T,\mathbf{n}^S)$-*flat point* if p is $(\mathbf{n}^T,\mathbf{n}^S)$-umbilic and $K_\ell^{\pm}(\mathbf{n}^T,\mathbf{n}^S)(p) = 0$.

For a lightlike vector $\mathbf{v} = (v_0, v_1, \cdots, v_n)$ we define $\widetilde{\mathbf{v}} = (1, v_1/v_0, \cdots, v_n/v_0)$. By Lemma 4.1, if we choose another future directed unit timelike normal section $\bar{\mathbf{n}}^T(\mathbf{u})$, then we have $\mathbf{n}^T\widetilde{(\mathbf{u}) \pm \mathbf{n}}^S(\mathbf{u}) = \bar{\mathbf{n}}^T\widetilde{(\mathbf{u}) \pm \bar{\mathbf{n}}}^S(\mathbf{u}) \in S_+^{n-1}$. Therefore we define a *lightcone Gauss map* $\widetilde{\mathbb{L}}^{\pm} : U \longrightarrow S_+^{n-1}$ of $M = \mathbf{X}(U)$ by

$$\widetilde{\mathbb{L}}^{\pm}(\mathbf{u}) = \mathbf{n}^T\widetilde{(\mathbf{u}) \pm \mathbf{n}}^S(\mathbf{u}).$$

The lightcone Gauss map is analogous to the Minkowski space which is studied in [8]. Under the identification of U and M we define a linear mapping $d\widetilde{\mathbb{L}}^{\pm} : T_pM \longrightarrow T_p\mathbb{R}_1^{n+1}$, where $p = \mathbf{X}(\mathbf{u})$. By Lemma 4.2, we have the following normalized lightcone Weingarten formula.

$$\pi^t \circ \widetilde{\mathbb{L}}_{u_i}^{\pm} = \frac{1}{\ell_0^{\pm}}(\pi^t \circ \mathbb{L}_{u_i}^{\pm}) = -\sum_{j=1}^{n-2} \frac{1}{\ell_0^{\pm}} h_i^{\pm j}(\mathbf{n}^T,\mathbf{n}^S)\mathbf{X}_{u_j},$$

where $\mathbb{L}^{\pm}(\mathbf{u}) = (\ell_0^{\pm}(\mathbf{u}), \cdots, \ell_n^{\pm}(\mathbf{u}))$. We call linear transformation $S_p^{\pm} = -\pi^t \circ d\widetilde{\mathbb{L}_p^{\pm}} : T_pM \longrightarrow T_pM$ by a *normalized lightcone shape operator* of M at p. The eigenvalues $\{\widetilde{\kappa}_i^{\pm}(p)\}_{i=1}^{n-2}$ of $\widetilde{S}_p^{\pm}$ are called *normalized lightcone principal curvatures*. By the above proposition, we have a relation

$$\widetilde{\kappa}_i^{\pm}(p) = \frac{1}{\ell_0^{\pm}(\mathbf{u})}\kappa_i^{\pm}(\mathbf{n}^T,\mathbf{n}^S)(p).$$

The *normalized lightcone Gauss-Kronecker curvature* of M at p is defined to be $\widetilde{K}_\ell^{\pm}(\mathbf{u}) = \det \widetilde{S}_p^{\pm}$. Therefore we have the following relation between the normalized lightcone Gauss-Kronecker curvature and the lightcone Gauss-Kronecker curvature:

$$\widetilde{K}_\ell^{\pm}(\mathbf{u}) = \left(\frac{1}{\ell_0^{\pm}(\mathbf{u})}\right)^{n-2} K_\ell^{\pm}(\mathbf{n}^T,\mathbf{n}^S)(\mathbf{u}).$$

It is clear from the corresponding definitions that the lightcone Gauss map, the normalized lightcone principal curvatures and the normalized lightcone Gauss-Kronecker curvature are independent on the choice of the normal frame $(\mathbf{n}^T, \mathbf{n}^S)$ of M.

We say that a point $\mathbf{u} \in U$ or $p = \mathbf{X}(\mathbf{u})$ is a *lightlike umbilic point* if $\widetilde{S}_p^{\pm} = \widetilde{\kappa}_p^{\pm}(p)\mathrm{id}_{T_pM}$. By the above proposition, p is a lightlike umbilic point if and only if p is a $(\mathbf{n}^T, \mathbf{n}^S)$-umbilic point for any $(\mathbf{n}^T, \mathbf{n}^S)$. We say that M is *totally lightlike umbilic* if all points on M are lightlike umbilic. We also say that p is a *lightlike parabolic point* if $\widetilde{K}_\ell^{\pm}(\mathbf{u}) = 0$. Moreover, p is called a *lightlike flat point* if p is both lightlike umbilic and lightlike parabolic. The spacelike submanifold M in S_1^n is called *totally lightlike flat* if every point in M is lightlike flat.

4.2 Construction of lightlike hypersurfaces

In this section we define a lightlike hypersurface from the spacelike submanifold of codimension two in de Sitter space, and introduce the Lorentzian distance squared function in order to study the singularities of lightlike hypersurfaces.

Let $M = \mathbf{X}(U)$ be a spacelike submanifold of codimension two in de Sitter space. We define a hypersurface $LH_M^{\pm} : U \times \mathbb{R} \longrightarrow S_1^n$ by

$$LH_M^{\pm}(\mathbf{u}, \mu) = \mathbf{X}(\mathbf{u}) + \mu\widetilde{\mathbb{L}}^{\pm}(\mathbf{u}).$$

We call $LH_M^{\pm}$ the *lightlike hypersurface along M*. It is analogous to the Minkowski four space which is studied in [7]. It has been introduced by Izumiya and Fusho [5]. We define a family of functions $G : U \times S_1^n \longrightarrow \mathbb{R}$ on a spacelike submanifold M by

$$G(\mathbf{u}, \lambda) = \langle \mathbf{X}(\mathbf{u}) - \lambda, \mathbf{X}(\mathbf{u}) - \lambda \rangle,$$

where $p = \mathbf{X}(\mathbf{u})$. We call G by a *Lorentzian distance squared function* on the spacelike submanifold M. For any fixed point $\lambda_0 \in S_1^n$, we write $g_{\lambda_0}(\mathbf{u}) = G(\mathbf{u}, \lambda_0)$. We have following proposition.

Proposition 4.3 ([12]). Let M be a spacelike submanifold of codimension two in de Sitter space and $G : U \times S^n_1 \longrightarrow \mathbb{R}$ the Lorentzian distance squared function on M. Suppose that $p_0 = \mathbf{X}(\mathbf{u}_0) \neq \lambda_0$, then we have the following.

(1) $g_{\lambda_0}(\mathbf{u}_0) = \partial g_{\lambda_0}(\mathbf{u}_0)/\partial u_i = 0$ $(i = 1, \cdots, n-2)$ if and only if $\lambda_0 = LH^{\pm}_M(\mathbf{u}_0, \mu)$ for some $\mu \in \mathbb{R} \setminus \{0\}$.

(2) $g_{\lambda_0}(\mathbf{u}_0) = \partial g_{\lambda_0}(\mathbf{u}_0)/\partial u_i = 0$ $(i = 1, \cdots, n-2)$ and $\det \mathrm{Hess}\,(g_{\lambda_0})(\mathbf{u}_0) = 0$ if and only if $\lambda_0 = LH^{\pm}_M(\mathbf{u}_0, \mu_0)$ for some $\mu_0 \in \mathbb{R} \setminus \{0\}$ and $-1/\mu_0$ is one of the non-zero normalized lightcone principal curvatures $\widetilde{\kappa}^{\pm}_i(p_0)$.

Now we apply the arguments in Appendix A.

Proposition 4.4. Let G be the Lorentzian distance squared function on M. For any point $(\mathbf{u}, \lambda) \in \Delta^* G^{-1}(\mathbf{0})$, G is a Morse family of hypersurfaces around $(\mathbf{u}, \lambda)$.

Proof. For $\lambda = (\lambda_0, \cdots, \lambda_n) \in S^n_1$, $\lambda_k \neq 0$ for some k. Without loss of generality, we assume that $\lambda_n > 0$ and local coordinates around λ in de Sitter space S^n_1 is given by $\lambda = (\lambda_0, \cdots, \lambda_k, \cdots, \lambda_{n-1})$, where $\lambda_n = \sqrt{1 + \lambda_0^2 - \lambda_1^2 - \cdots - \lambda_{n-1}}$. Jacobian of $\Delta^* G$ is given by

$$B(\mathbf{u}, \lambda) = \left(\frac{\left(-X_j(\mathbf{u}) + \dfrac{X_n(\mathbf{u})}{\lambda_n} \lambda_j \right)_{j=0,\cdots,n-1}}{\left(X_{j,u_i}(\mathbf{u}) - \dfrac{X_{n,u_i}(\mathbf{u})}{\lambda_n} \lambda_j \right)_{\substack{j=0,\cdots,n-1 \\ i=1,\cdots,n-2}}} \right)$$

where $\mathbf{X}(\mathbf{u}) = (X_0(\mathbf{u}), \cdots, X_n(\mathbf{u})), \mathbf{X}_{u_i} = (X_{0,u_i}(\mathbf{u}), \cdots, X_{n,u_i}(\mathbf{u}))$ for $(i = 1, \cdots, n-1)$. On the other hand, $\lambda, \mathbf{X}(\mathbf{u}), \mathbf{X}_{u_1}(\mathbf{u}), \cdots, \mathbf{X}_{u_{n-2}}$ are linearly independent on $(\mathbf{u}, \lambda) \in \Delta^* G^{-1}(0)$, so that rank of $n \times (n-1)$ matrix

$$\begin{pmatrix} \lambda_0 & -\lambda_1 & \cdots & -\lambda_{n-1} & -\lambda_n \\ X_0(\mathbf{u}) & -X_1(\mathbf{u}) & \cdots & -X_{n-1}(\mathbf{u}) & -X_n(\mathbf{u}) \\ X_{0,u_1}(\mathbf{u}) & -X_{1,u_1}(\mathbf{u}) & \cdots & -X_{n-1,u_1}(\mathbf{u}) & -X_{n,u_1}(\mathbf{u}) \\ \vdots & \vdots & & \vdots & \vdots \\ X_{0,u_{n-2}}(\mathbf{u}) & -X_{1,u_{n-2}}(\mathbf{u}) & \cdots & -X_{n-1,u_{n-2}}(\mathbf{u}) & -X_{n,u_{n-2}}(\mathbf{u}) \end{pmatrix}$$

is n. We subtract the first row multiplied by $\mathbf{X}_n(\mathbf{u})/\lambda_n$ from the second row, and then subtract the first row multiplied by $\mathbf{X}_{n,u_k}(\mathbf{u})/\lambda_n$ from the $(2+k)$-th row for $k = 1, \cdots, n-2$. We have

$$\left(\begin{array}{ccc|c} \lambda_0 & -\lambda_1 \ \cdots & -\lambda_{n-1} & -\lambda_n \\ \hline & & & 0 \\ & \mathrm{B}(\mathbf{u}, \lambda) & & \vdots \\ & & & 0 \end{array}\right).$$

Therefore rank $B(\mathbf{u}, \lambda) = n - 1$. This completes the proof. □

We consider a point $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n) \in S_1^n$, then we have the relation

$$\lambda_k = \pm\sqrt{\lambda_0^2 - \cdots - \lambda_{i-1}^2 - \lambda_{i+1}^2 - \cdots - \lambda_n^2 + 1} \neq 0$$

for some k. So we adopt the coordinate system by $(\lambda_1, \cdots, \widehat{\lambda_k}, \cdots, \lambda_n)$ on $V^{\pm} := \{\lambda \mid \mathrm{sgn}(\lambda_k) = \pm 1\}$. We consider the local trivialization $PT^*(V_k^{\pm}) \equiv V_k^{\pm} \times P\mathbb{R}^{n-1}$ of $PT^*(S_1^n)$. Since G is a Morse family of hypersurfaces, we have the Legendrian immersion germs $\mathcal{L}_G^{\pm} : (\Sigma_*(G), (\mathbf{u}_0, \lambda_0)) \longrightarrow PT^*(S_1^n)$ by

$$\mathcal{L}_G^{\pm}(\mathbf{u}, \lambda) = \left(\lambda, \left[\frac{\partial G}{\partial \lambda^0}(\mathbf{u}, \lambda) : \cdots : \widehat{\frac{\partial G}{\partial \lambda^k}}(\mathbf{u}, \lambda) : \cdots : \frac{\partial G}{\partial \lambda^n}(\mathbf{u}, \lambda)\right]\right)$$

where $\lambda = (\lambda^0, \cdots, \lambda^n)$ is a local coordinate $V_k^{\pm}$ around $\lambda_0 \in S_1^n$. and $\Sigma_*(G) = (\Delta^* G)^{-1}(0) = \{(\mathbf{u}, \lambda) \in U \times S_1^n \mid \lambda = LH_M^{\pm}(\mathbf{u}, \mu),\ \mu \in \mathbb{R}\}$. The Lorentzian distance squared function G is a generating family of the Legendrian immersion $\mathcal{L}_G^{\pm}$ whose wave front set is the image of $LH_M^{\pm}$.

4.3 Contact with lightcones and spacelike submanifolds

In this section we use the theory of contacts between submanifolds due to Montaldi [16]. Let $\lambda_0 \in S_1^n$, we call a set

$$LC_{\lambda_0} \cap S_1^n = \{x \in S_1^n \mid \langle \mathbf{x} - \lambda_0, x - \lambda_0 \rangle = 0\}$$

by a *de Sitter lightcone* with vertex λ_0. The following proposition is generalization of Proposition 4.1 in [7].

Proposition 4.5 ([12]). Let $\lambda_0 \in S^n_1$ and M be a spacelike submanifold of codimension two without umbilic points satisfying $\widetilde{K_\ell} \neq 0$. Then $M \subset LC_{\lambda_0} \cap S^n_1$ if and only if λ_0 is an isolated singular value of the lightlike hypersurface $LH^\pm_M$ and $LH^\pm_M(U \times \mathbb{R}) \subset LC_{\lambda_0} \cap S^n_1$.

Proof. We assume that $M \subset LC(S^n_1)_{\lambda_0}$. By Proposition 4.3, there exists a smooth function $\mu : U \longrightarrow \mathbb{R}$ such that $\mathbf{X}(\mathbf{u}) = \lambda_0 + \mu(\mathbf{u}) \cdot (\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})(\mathbf{u})$. Therefore, $LH^\pm_M(U \times \mathbb{R}) \subset LC(S^n_1)_{\lambda_0}$.

We now show that λ_0 is isolated singularity. It follows that

$$
\begin{aligned}
\frac{\partial LH^\pm_M}{\partial t}(\mathbf{u}, t) &= (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(\mathbf{u}) \\
\frac{\partial LH^\pm_M}{\partial u_i}(\mathbf{u}, t) &= \mu_{u_i}(\mathbf{u})(\widetilde{\mathbf{n}^T + \mathbf{n}^S})(\mathbf{u}) + (t + \mu(\mathbf{u}))(\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_i}(\mathbf{u}) \quad (i = 1, \cdots, n-2).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
P(\mathbf{u}) &:= \mathbf{X}(\mathbf{u}) \wedge \frac{\partial LH^\pm_M}{\partial t}(\mathbf{u}, t) \wedge \frac{\partial LH^\pm_M}{\partial u_1}(\mathbf{u}, t) \wedge \cdots \wedge \frac{\partial LH^\pm_M}{\partial u_{n-2}}(\mathbf{u}, t) \\
&= (t + \mu(\mathbf{u}))^{n-2} \cdot \mathbf{X}(\mathbf{u}) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(\mathbf{u}) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_1}(\mathbf{u}) \wedge \cdots \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_{n-2}}(\mathbf{u}).
\end{aligned}
$$

On the other hand, $\mathbf{X}(\mathbf{u}) - \lambda_0 = \mu(\mathbf{u}) \cdot (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(\mathbf{u}) \neq 0$ is a lightlike vector and T_pM are spacelike, so that $\mathbf{X}(\mathbf{u}), \mathbf{X}(\mathbf{u}) - \lambda_0, \mathbf{X}_{u_1}(\mathbf{u}), \cdots, \mathbf{X}_{u_{n-2}}(\mathbf{u})$ are linearly independent. Therefore we have

$$
\begin{aligned}
\mathbf{0} &\neq \mathbf{X}(\mathbf{u}) \wedge (\mathbf{X}(\mathbf{u}) - \lambda_0) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \cdots \wedge \mathbf{X}_{u_{n-2}}(\mathbf{u}) \\
&= \mu(\mathbf{u})^{n-1} \cdot \mathbf{X}(\mathbf{u}) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(\mathbf{u}) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_1}(\mathbf{u}) \wedge \cdots \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_{n-2}}(\mathbf{u})
\end{aligned}
$$

so that $\mathbf{X}(\mathbf{u}) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})(\mathbf{u}) \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_1}(\mathbf{u}) \wedge \cdots \wedge (\widetilde{\mathbf{n}^T + \mathbf{n}^S})_{u_{n-2}}(\mathbf{u}) \neq \mathbf{0}$. Therefore $P(\mathbf{u}) = 0$ if and only if $t + \mu(\mathbf{u}) = 0$. This means that λ_0 is an isolated singular value of $LH^\pm_M$. The converse is trivial. □

We now consider the contact of spacelike submanifolds of codimension two with de Sitter lightcones due to Montaldi's result [16]. We define a function $\mathcal{G} : S^n_1 \times S^n_1 \longrightarrow \mathbb{R}$ by $\mathcal{G}(\mathbf{x}, \lambda) = \langle \mathbf{x} - \lambda, \mathbf{x} - \lambda \rangle$. For a given $\lambda_o \in S^n_1$, we denote $\mathfrak{g}_{\lambda_0}(\mathbf{x}) = \mathcal{G}(\mathbf{x}, \lambda_0)$, then we have $\mathfrak{g}^{-1}_{\lambda_0}(0) = LC(S^n_1)_{\lambda_0}$. For any $\mathbf{u}_0 \in U$, we take the point $\lambda^\pm_0 = \mathbf{X}(\mathbf{u}_0) + \mu_0 \widetilde{L^\pm}(\mathbf{u}_0)$ and have

$$
(\mathfrak{g}_{\lambda^\pm_0} \circ \mathbf{X})(\mathbf{u}_0) = \mathcal{G} \circ (\mathbf{X} \times \mathrm{id}_{S^n_1})(\mathbf{u}_0, \lambda^\pm_0) = G(\mathbf{u}_0, \lambda^\pm_0) = 0,
$$

where $p_0 = \mathbf{X}(\mathbf{u}_0)$ and $\mu_0 = -1/\widetilde{\kappa}_i^{\pm}(\mathbf{u}_0),\ (i = 1, \cdots, n-1)$. We also have

$$\frac{\partial(\mathfrak{g}_{\lambda_0^{\pm}} \circ \mathbf{X})}{\partial u_i}(\mathbf{u}_0) = \frac{\partial G}{\partial u_i}(\mathbf{u}_0, \lambda_0^{\pm}) = 0.$$

It follows that the de Sitter lightcone $\mathfrak{g}_{\lambda_0^{\pm}}^{-1}(0) = LC(S_1^n)_{\lambda_0} \cap S_1^n$ is tangent to M at $p_0 = \mathbf{X}(\mathbf{u}_0)$. In this case, we call each $LC_{\lambda_0^{\pm}}$ a *de Sitter tangent lightcone* of M at p_0.

We denote $g_{i,\lambda_i^{\pm}} : (U, \mathbf{u}_i) \longrightarrow (\mathbb{R}, \mathbf{0})$ by $g_{i,\lambda_i^{\pm}}(\mathbf{u}) = G_i(\mathbf{u}, \lambda_i^{\pm})$. Then we have $g_{i,\lambda_i^{\pm}}(\mathbf{u}) = (\mathfrak{g}_{i,\lambda_i^{\pm}} \circ \mathbf{X}_i)(\mathbf{u})$. By Theorem B.1,

$$K(\mathbf{X_1}(U), LC_{\lambda_1^{\pm}}; \lambda_1^{\pm}) = K(\mathbf{X_2}(U), LC_{\lambda_2^{\pm}}; \lambda_2^{\pm})$$

if and only if $g_{1,\lambda_1^{\pm}}$ and $g_{2,\lambda_2^{\pm}}$ are $\mathcal{K}$-equivalent.

Let $Q^{\pm}(\mathbf{X}, \mathbf{u}_0)$ be the local ring of the function germ $g_{\lambda_0^{\pm}} : (U, u_0) \longrightarrow \mathbb{R}$ defined by

$$Q^{\pm}(\mathbf{X}, \mathbf{u}_0) = C^{\infty}_{\mathbf{u}_0}(U)/\langle g_{\lambda_0^{\pm}} \rangle_{C^{\infty}_{\mathbf{u}_0}(U)},$$

where $\lambda_0 = LH_M^{\pm}(\mathbf{u}_0, \mu_0)$ and $C^{\infty}_{\mathbf{u}_0}(U)$ is the local ring of function germs at u_0.

By the arguments in Appendix A, we have following theorem.

Theorem 4.6 ([12])**.** Let $\mathbf{X}_i : (U, \mathbf{u}_i) \longrightarrow (S_1^n, p_i)$ $(i = 1, 2)$ be spacelike submanifold germs of codimension two in de Sitter space such that the corresponding Legendrian immersion germs $\mathcal{L}_i^{\pm}$ are Legendrian stable. Then the following conditions are equivalent:

(1) Lightlike hypersurface germs $LH^{\pm}_{M,1}$ and $LH^{\pm}_{M,2}$ are $\mathcal{A}$-equivalent.

(2) Legendrian immersion germs $\mathcal{L}_1^{\pm}$ and $\mathcal{L}_2^{\pm}$ are Legendrian equivalent.

(3) Lorentzian distance squared function germs G_1 and G_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(4) $g^{\pm}_{1,\lambda_1}$ and $g^{\pm}_{2,\lambda_2}$ are $\mathcal{K}$-equivalent.

(5) $K(\mathbf{X_1}(U), LC_{\lambda_1^{\pm}}; p_1) = K(\mathbf{X_2}(U), LC_{\lambda_2^{\pm}}; p_2)$

(6) Local rings $Q^{\pm}(\mathbf{X}_1, \mathbf{u}_1)$ and $Q^{\pm}(\mathbf{X}_2, \mathbf{u}_2)$ are isomorphic as $\mathbb{R}$-algebras.

Proof. Since $LH^{\pm}_{M,1}$ and $LH^{\pm}_{M,2}$ are Legendrian stable, regular sets of $LH^{\pm}_{M,1}$ and $LH^{\pm}_{M,2}$ are respectively dense, by Proposition A.2, the conditions (1) and (2) are equivalent. Applying Theorem A.3, the conditions (2) and (3) are equivalent. By the previous arguments from Theorem B.1, the conditions (4) and (5) are equivalent. If we assume the condition (3), then $\mathcal{P}$-$\mathcal{K}$-equivalence preserves the $\mathcal{K}$-equivalence, so that the condition (4) holds. Since the local ring $Q^{\pm}(\mathbf{X}_i, u_i)$ is $\mathcal{K}$-invariant, this means that the condition (6) holds. By Proposition A.4, the condition (6) implies the condition (2). $\square$

By Appendix C, the assumption of the above theorem is a generic property in the case when $n \leq 6$. In general we have the following proposition.

Proposition 4.7 ([12])**.** Let $\mathbf{X}_i : (U, \mathbf{u}_i) \longrightarrow (S^n_1, p_i)$ (for $i = 1, 2$) be spacelike submanifold germs of codimension two in de Sitter space and regular sets of their lightlike surfaces $LH^{\pm}_{M,i}$ are dense in U. If lightlike hypersurface germs $LH^{\pm}_{M,1}$ and $LH^{\pm}_{M,2}$ are $\mathcal{A}$-equivalent, then

$$K(\mathbf{X}_1(U), LC_{\lambda^{\pm}_1}; p_1) = K(\mathbf{X}_2(U), LC_{\lambda^{\pm}_2}; p_2).$$

In this case, $(\mathbf{X}_1^{-1}(LC_{\lambda^{\pm}_1}), \mathbf{u}_1)$ and $(\mathbf{X}_2^{-1}(LC_{\lambda^{\pm}_2}), \mathbf{u}_2)$ are diffeomorphic as set germs.

Proof. By Proposition A.2, if $LH^{\pm}_{M,1}$ and $LH^{\pm}_{M,1}$ are $\mathcal{A}$-equivalent, then $\mathcal{L}^{\pm}_1$ and $\mathcal{L}^{\pm}_2$ are Legendrian equivalent. By Theorem A.3, G_1 and G_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent, so that $g_{1,\lambda^{\pm}_1}$ and $g_{2,\lambda^{\pm}_2}$ are $\mathcal{K}$-equivalent. Applying Theorem B.1, the first assertion holds. On the other hand, $g^{-1}_{i,\lambda^{\pm}_i}(0) = (\mathbf{X}_i^{-1}(LC_{\lambda^{\pm}_i}), \mathbf{u}_i)$ and $\mathcal{K}$-equivalence preserves the zero level sets, so that $(\mathbf{X}_1^{-1}(LC_{\lambda^{\pm}_1}), \mathbf{u}_1)$ and $(\mathbf{X}_2^{-1}(LC_{\lambda^{\pm}_2}), \mathbf{u}_2)$ are diffeomorphic as set germs. $\square$

4.4 Lightcone Gauss maps and lightcone height functions

In this section, we define the lightcone height function whose wave front set is the image of the lightcone Gauss map. We also describe contacts of submanifolds with lightlike cylinders by applying Montaldi's theory.

We define a *lightcone height function* $H : U \times S^{n-1}_+ \longrightarrow \mathbb{R}$ by $H(\mathbf{u}, \mathbf{v}) = \langle X(\mathbf{u}), \mathbf{v}\rangle$. For $\mathbf{v}_0 \in S^{n-1}_+$, we write $h_{\mathbf{v}_0}(\mathbf{u}) = H(\mathbf{u}, \mathbf{v}_0)$ and have following proposition.

Proposition 4.8 ([12])**.** Let H be the lightcone height function of spacelike submanifold $\mathbf{X}$ of codimension two in de Sitter space S_1^n, then we have the following:

(1) $H(\mathbf{u}_0, \mathbf{v}_0) = H_{u_i}(\mathbf{u}_0, \mathbf{v}_0) = 0$ for $i = 1, \cdots, n-2$ if and only if $\mathbf{v}_0 = \widetilde{\mathbb{L}}^{\pm}(\mathbf{u}_0)$.

(2) $H(\mathbf{u}_0, \mathbf{v}_0) = H_{u_i}(\mathbf{u}_0, \mathbf{v}_0) = 0$ for $i = 1, \cdots, n-2$ and $\det \mathrm{Hess}\,(h_{\mathbf{v}_0})(\mathbf{u}_0)$ if and only if $\mathbf{v}_0 = \widetilde{\mathbb{L}}^{\pm}(\mathbf{u}_0)$ and $\widetilde{K}_{\ell}^{\pm}(\mathbf{u}_0) = 0$.

Proof. Let $\mathbf{v}_0 = \lambda \mathbf{X}(\mathbf{u}_0) + \eta^T \mathbf{n}^T(\mathbf{u}_0) + \eta^S \mathbf{n}^S(\mathbf{u}_0) + \sum_{j=1}^{n-2} \xi_j \mathbf{X}_j(\mathbf{u}_0)$ for some $\lambda, \eta^T, \eta^S, \xi_j \in \mathbb{R}$. By the assumption, we have $\lambda = 0$, $|\eta^T| = |\eta^S|$ and $\bar{\mathbf{H}}'(\mathbf{u}_0, \mathbf{v}_0) = (g_{ij}(\mathbf{u}_0))\,\bar{\xi}$, where $\bar{\mathbf{H}}' = {}^t(H_{u_1}, \cdots, H_{u_{n-2}})$, $\bar{\xi} = {}^t(\xi_1, \cdots, \xi_{n-2})$ and (g_{ij}) is the first fundamental form on M. Since $(g_{ij}(\mathbf{u}_0))$ is regular, $\bar{\mathbf{H}}'(\mathbf{u}_0, \mathbf{v}_0) = \mathbf{0}$ if and only if $\bar{\xi} = \mathbf{0}$. Therefore we have $\mathbf{v}_0 = \widetilde{\mathbb{L}}^{\pm}(\mathbf{u}_0)$. The converse of (1) is trivial. By the calculation,

$$\left(\frac{\partial^2 H}{\partial u_i \partial u_j}(\mathbf{u}_0, \mathbf{v}_0) \right)_{ij} = \left(\langle \mathbf{X}_{u_i u_j}(\mathbf{u}_0), \widetilde{\mathbb{L}}^{\pm}(\mathbf{u}_0) \rangle \right)_{ij} = \frac{1}{\ell_0^{\pm}(\mathbf{u}_0)} \left(h_{ij}^{\pm}(\mathbf{u}_0) \right),$$

where $\ell_0^{\pm}(\mathbf{u}_0)$ is the first component of $\widetilde{\mathbb{L}}^{\pm}(\mathbf{u}_0)$ and $(h_{ij}^{\pm}(\mathbf{u}_0))$ is the lightcone second fundamental form with respect to the lightcone normal frame $(\mathbf{n}^T, \mathbf{n}^S)$. Therefore $\mathrm{Hess}\, H(\mathbf{u}_0, \mathbf{v}_0)$ is degenerate if and only if u_0 is a lightcone parabolic point. This completes the proof. $\square$

By the above proposition, the discriminant set of the lightcone height function is given by

$$D_H = \left\{ \mathbf{v} \in S_+^{n-1} \,\middle|\, \mathbf{v} = \widetilde{\mathbb{L}}^{\pm}(\mathbf{u}),\ \mathbf{u} \in U \right\}.$$

Therefore D_H is the image of the lightcone Gauss map of M and the singular set of the lightcone Gauss map is the normalized lightcone parabolic set of M.

Proposition 4.9 ([12])**.** Let H is the lightcone height function on the spacelike submanifold M of codimension two in de Sitter space. Then H is a Morse family of hypersurfaces around $(\mathbf{u}, \mathbf{v}) \in \Delta^* H^{-1}(0)$.

Proof. We denote that $\mathbf{X} = (X_0, \cdots, X_n)$, $\mathbf{X}_{u_i} = (X_{0,u_i}, \cdots, X_{n,u_i})$ and $\mathbf{v} = (v_0, \cdots, v_n)$. Without the loss of generality, we assume that $v_n > 0$. Therefore we denote a matrix B and C by

$$\mathrm{B} = \left(\frac{\left(X_j(\mathbf{u}) - \dfrac{v_j}{v_n} X_n(\mathbf{u}) \right)_{j=1,\cdots,n-1}}{\left(X_{j,u_i}(\mathbf{u}) - \dfrac{v_j}{v_n} X_{n,u_i}(\mathbf{u}) \right)_{\substack{j=1,\cdots,n-1 \\ i=1,\cdots,n-2}}} \right), \mathrm{C} = \begin{pmatrix} 1 \quad 0 \quad \cdots \quad\quad 0 \\ \widetilde{\mathbb{L}}^{\pm}(\mathbf{u}) \\ \mathbf{X}(\mathbf{u}) \\ \mathbf{X}_{u_1}(\mathbf{u}) \\ \vdots \\ \mathbf{X}_{u_{n-2}}(\mathbf{u}) \end{pmatrix}.$$

Then we have $J(\Delta^* H) = (\,*\,|B)$ and $\det B = (-1)^{n-2} \det C / v_n$.

On the other hand, determinant of a matrix

$$\mathrm{C} \begin{pmatrix} -1 & 0 & & \\ 0 & 1 & & \mathrm{O} \\ & & \ddots & 0 \\ \mathrm{O} & & 0 & 1 \end{pmatrix} {}^t\mathrm{C} = \left(\begin{array}{cc|ccc} -1 & -1 & * & \cdots & * \\ -1 & 0 & 0 & \cdots & 0 \\ \hline * & 0 & 1 & \mathrm{O} & \\ \vdots & \vdots & & & \\ * & 0 & \mathrm{O} & (g_{ij}) & \end{array} \right)$$

equals to $-\det(g_{ij}) \neq 0$, where (g_{ij}) is the first fundamental form on M. This implies that both B and C are regular, therefore $\operatorname{rank} J(\Delta^* H) = n-1$. This completes the proof. $\square$

By the above proposition and Proposition A.1, we have the Legendrian immersion $\mathcal{L}_H^{\pm} : \Sigma_*(H) \longrightarrow PT^*(S_+^{n-1})$ defined by

$$\mathcal{L}_H^{\pm}(\mathbf{u}, \mathbf{v}) = \left(\lambda, \left[\frac{\partial H}{\partial v_1}(\mathbf{u}, \mathbf{v}) : \cdots \widehat{\frac{\partial H}{\partial v_k}}(\mathbf{u}, \mathbf{v}) : \cdots \frac{\partial H}{\partial v_n}(\mathbf{u}, \mathbf{v}) \right] \right)$$

where $\mathbf{v} = (v_0, v_1, \cdots, v_n) \in S_+^{n+1}$ and $\Sigma_*(H) = \{(\mathbf{u}, \mathbf{v}) \in U \mid \mathbf{v} = \widetilde{\mathbb{L}}^{\pm}(\mathbf{u}),\ \widetilde{K}_{\ell}^{\pm}(\mathbf{u}_0) = 0\}$. The lightcone height function H is the generating family of the Legendrian immersion $\mathcal{L}_H^{\pm}$ whose wave front set is the image of lightcone Gauss map $\widetilde{\mathbb{L}}^{\pm}$.

4.5 Contact with lightlike cylinders

In this section we describe contacts of submanifolds with lightlike cylinders by applying Montaldi's theory.

For any $\mathbf{v} \in S_+^{n-1}$, we define a *lightlike cylinder* along $\mathbf{v}$ by $HP(\mathbf{v},0) \cap S_1^n$. It is an $(n-1)$-dimensional submanifold in S_1^n which is isomorphic to $S^{n-2} \times \mathbb{R}$. We observe that its tangent space at each point has lightlike directions.

Proposition 4.10 ([12])**.** Let $\widetilde{\mathbb{L}}^{\pm}$ be a lightcone Gauss map of $\mathbf{X}$. Then $\widetilde{\mathbb{L}}^{\pm}$ is a constant map if and only if M is a part of lightlike cylinder $HP(\mathbf{v},0) \cap S_1^n$ for some $\mathbf{v} \in S_+^{n-1}$.

Proof. Necessity is trivial, so we prove sufficient condition. If $M \subset HP(\mathbf{v},0) \cap S_1^n$, then $\mathbf{v} = \alpha(\mathbf{u})\mathbf{n}^T(\mathbf{u}) + \beta(\mathbf{u})\mathbf{n}^S(\mathbf{u})$ for some functions $\alpha, \beta : U \longrightarrow \mathbb{R}$. Since $\mathbf{v}$ is lightlike, we have $\alpha = |\beta| > 0$. Therefore $\mathbf{v} = \widetilde{\mathbb{L}}^{\pm}(\mathbf{u})$ for all $\mathbf{u} \in U$. This completes the proof. □

We now consider the function $\mathcal{H} : S_1^n \times S_+^{n-1} \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\mathbf{x},\mathbf{v}) = \langle \mathbf{x},\mathbf{v}\rangle$. Given $\mathbf{v}_0 \in S_+^{n-1}$, we denote $\mathfrak{h}_{\mathbf{v}_0}(\mathbf{x}) = \mathcal{H}(\mathbf{x},\mathbf{v}_0)$, so that we have $\mathfrak{h}_{\mathbf{v}_0}^{-1}(0) = HP(\mathbf{v}_0,0) \cap S_1^n$. For any $\mathbf{u}_0 \in U$, we take the point $\mathbf{v}_0^{\pm} = \widetilde{\mathbb{L}}^{\pm}(\mathbf{u}_0)$ and have

$$(\mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{X})(\mathbf{u}_0) = \mathcal{H} \circ (\mathbf{X} \times \mathrm{id}_{S_+^{n-1}})(\mathbf{u}_0, \mathbf{v}_0^{\pm}) = H(\mathbf{u}_0, \mathbf{v}_0^{\pm}) = 0,$$

where $p_0 = \mathbf{X}(\mathbf{u}_0)$. We also have

$$\frac{\partial(\mathfrak{h}_{\mathbf{v}_0^{\pm}} \circ \mathbf{X})}{\partial u_i}(\mathbf{u}_0) = \frac{\partial H}{\partial u_i}(\mathbf{u}_0, \mathbf{v}_0^{\pm}) = 0.$$

It follows that the lightcone $\mathfrak{h}_{\mathbf{v}_0^{\pm}}^{-1}(0) = LC_{\mathbf{v}_0}$ is tangent to M at $p_0 = \mathbf{X}(\mathbf{u}_0)$. In this case, we call $LC_{\mathbf{v}_0^{\pm}}$ a *tangent lightlike cylinder* of M at p_0.

From the arguments in Appendix A and 4.5, we have following theorem.

Theorem 4.11 ([12])**.** $\mathbf{X}_i : (U, \mathbf{u}_i) \longrightarrow (S_1^n, p_i)$ $(i = 1, 2)$ be spacelike submanifold germs and $\mathbf{v}_i = \widetilde{\mathbb{L}}_i^{\pm}(u_i)$. If the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent:

(1) Lightcone Gauss map germs $\widetilde{\mathbb{L}}_1^{\pm}$ and $\widetilde{\mathbb{L}}_2^{\pm}$ are $\mathcal{A}$-equivalent.

(2) Legendrian immersion germs $\mathcal{L}_1^{\pm}$ and $\mathcal{L}_2^{\pm}$ are Legendrian equivalent.

(3) Lightcone height function germs H_1 and H_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(4) $h^{\pm}_{1,\mathbf{v}_1}$ and $h^{\pm}_{2,\mathbf{v}_2}$ are $\mathcal{K}$-equivalent.

(5) $K(\mathbf{X_1}(U), HP(\mathbf{v}_1, 0)\cap S^n_1; p_1) = K(\mathbf{X_2}(U), HP(\mathbf{v}_2, 0)\cap S^n_1; p_2)$

Proof. This proof is similar to the proof of Theorem 3.6. □

As in the case with the section 4.5, if $n \le 6$ there exists an open subset $\mathcal{O} \subset \text{Sp-Emb}(U, S^n_1)$ (with the Whitney C^∞-topology) such that for any $\mathbf{X} \in \mathcal{O}$ suffices the assumption of the above theorem.

Proposition 4.12 ([12])**.** Let $\mathbf{X}_i$ (for $i = 1, 2$) be spacelike submanifold germs and regular sets of their lightcone Gauss maps $\widetilde{\mathbb{L}}^{\pm}_i$ are dense in U. If lightcone Gauss map germs $\widetilde{\mathbb{L}}^{\pm}_1$ and $\widetilde{\mathbb{L}}^{\pm}_2$ are $\mathcal{A}$-equivalent, then we have

$$K(\mathbf{X_1}(U), HP(\mathbf{v}^{\pm}_1, 0)\cap S^n_1; p_1) = K(\mathbf{X_2}(U), HP(\mathbf{v}^{\pm}_2, 0)\cap S^n_1; p_2)$$

In this case, $({\mathbf{X_1}}^{-1}(HP(\mathbf{v}^{\pm}_1, 0)\cap S^n_1), \mathbf{u}_1)$ and $({\mathbf{X_2}}^{-1}(HP(\mathbf{v}^{\pm}_2, 0)\cap S^n_1), \mathbf{u}_2)$ are diffeomorphic as set germs.

Therefore we call $({\mathbf{X_i}}^{-1}(HP(\mathbf{v}^{\pm}_i, 0)\cap S^n_1), \mathbf{u}_i)$ a *tangent lightlike cylindrical indicatrix germ* of M_i at p_0.

4.6 Spacelike surfaces in de Sitter 4-space

In this section we consider the case of $n = 4$ and classify singularities of lightlike hypersurfaces and lightcone Gauss maps. We now define $\mathcal{K}$-invariants of spacelike surfaces in de Sitter 4-space. For open subset $U \subset \mathbb{R}^2$ and spacelike submanifold $X : U \longrightarrow S^4_1$, we define the $\mathcal{K}$-codimension (or Tyurina number) of the function germs $h_{\mathbf{v}^{\pm}_0}, g_{\lambda^{\pm}_0}$ and corank of $h_{\mathbf{v}^{\pm}_0}, g_{\lambda^{\pm}_0}$ by

$$\begin{aligned}
\text{H-ord}^{\pm}(\mathbf{X}, \mathbf{u}_0) &= \dim C^\infty_{\mathbf{u_0}}(U)/\langle h_{\mathbf{v}^{\pm}_0}(\mathbf{u}_0), \partial h_{\mathbf{v}^{\pm}_0}(\mathbf{u}_0)/\partial u_i\rangle_{C^\infty_{\mathbf{u_0}}(U)}, \\
\text{H-corank}^{\pm}(\mathbf{X}, \mathbf{u}_0) &= 2 - \operatorname{rank}\operatorname{Hess}(h_{\mathbf{v}^{\pm}_0}(\mathbf{u}_0)), \\
\text{G-ord}^{\pm}(\mathbf{X}, \mathbf{u}_0) &= \dim C^\infty_{\mathbf{u_0}}(U)/\langle g_{\lambda^{\pm}_0}(\mathbf{u}_0), \partial g_{\lambda^{\pm}_0}(\mathbf{u}_0)/\partial u_i\rangle_{C^\infty_{\mathbf{u_0}}(U)}, \\
\text{G-corank}^{\pm}(\mathbf{X}, \mathbf{u}_0) &= 2 - \operatorname{rank}\operatorname{Hess}(g_{\lambda^{\pm}_0}(\mathbf{u}_0)),
\end{aligned}$$

where $\mathbf{v}^{\pm}_0 = \widetilde{\mathbb{L}}^{\pm}(\mathbf{u}_0)$ and $\lambda^{\pm}_0 = \mathbf{X}(\mathbf{u}_0) + t_0\widetilde{\mathbb{L}}^{\pm}(\mathbf{u}_0)$.

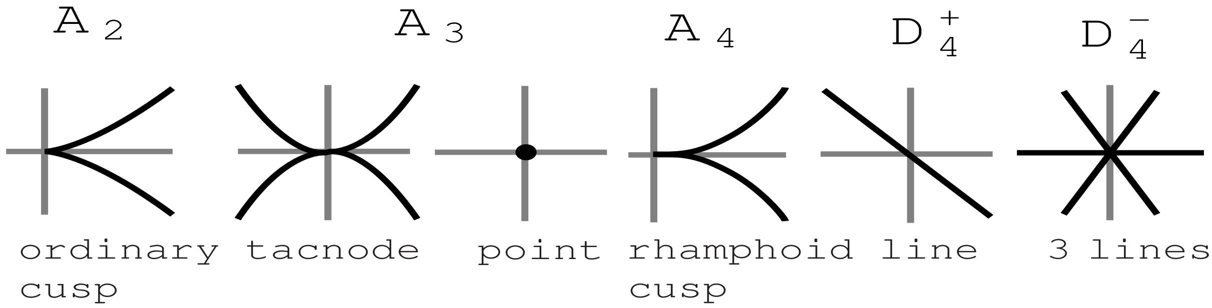

Figure 5: The list of tangent lightcone indicatrix germs

Theorem 4.13 ([12])**.** Let Sp-Emb(U, S^n_1) be the set of spacelike submanifolds. We have open dense subset $\mathcal{O} \subset$ Sp-Emb(U, S^n_1) such that for $\mathbf{X} \in \mathcal{O}$ and $\lambda_0^\pm = LH^\pm_M(\mathbf{u}_0, t_0)$, $\lambda_0^\pm$ is a singular value of $LH^\pm_M$ if and only if G-corank$^\pm(\mathbf{X}, \mathbf{u}_0) = 1$ or 2.

(1) If G-corank$^\pm(\mathbf{X}, \mathbf{u}_0) = 1$ then there are distinct normalized lightcone principal curvatures $\widetilde{\kappa}_1^\pm, \widetilde{\kappa}_2^\pm$ such that $\widetilde{\kappa}_1^\pm \neq 0$ and $t_0 = -1/\widetilde{\kappa}_1^\pm$. In this case we have G-ord$^\pm(\mathbf{X}, \mathbf{u}_0) = k$ ($k = 2, 3$ or 4) and the lightlike hypersurface $LH^\pm_M$ has $\mathcal{A}_k$ type singularity:

$$(\mathcal{A}_2) \quad f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_1, u_2)$$

$$(\mathcal{A}_3) \quad f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3)$$

$$(\mathcal{A}_4) \quad f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_2, u_3).$$

(2) If G-corank$^\pm(\mathbf{X}, \mathbf{u}_0) = 2$ then $\mathbf{u}_0$ is a non-flat umbilic point and $t_0 = -1/\widetilde{\kappa}_1^\pm$. In this case we have G-ord$^\pm(\mathbf{X}, \mathbf{u}_0) = 4$ and $LH^\pm_M$ has $\mathcal{D}_4^+$ or $\mathcal{D}_4^-$ type singularity:

$$(\mathcal{D}_4^+) \quad f(u_1, u_2, u_3) = (2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3)$$

$$(\mathcal{D}_4^-) \quad f(u_1, u_2, u_3) = (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3).$$

Theorem 4.14 ([12])**.** There exists an open dense subset $\mathcal{O}' \subset$ Sp-Emb(U, S^n_1) such that for any $\mathbf{X} \in \mathcal{O}'$, $\mathbf{u}_0 \in U$ is an $\widetilde{L}^\pm$-parabolic point if and only if H-corank$^\pm(\mathbf{X}, \mathbf{u}_0) = 1$. That is, M has non flat point and $\widetilde{K}_\ell^{-1}(0)$ is a regular curve.

(1) If H-ord$^{\pm}(\mathbf{X}, \mathbf{u}_0) = 2$, then $\widetilde{\mathbb{L}}^{\pm}$ has the cuspidal edge point at $\mathbf{u}_0$, and the tangent lightlike cylindrical indicatrix germ is an ordinary cusp.

(2) If H-ord$^{\pm}(\mathbf{X}, \mathbf{u}_0) = 3$, then $\widetilde{\mathbb{L}}^{\pm}$ has the swallowtail point at $\mathbf{u}_0$, and the tangent lightlike cylindrical indicatrix germ is a tacnode or a point.

Part III Spacelike submanifolds in de Sitter space

5 Spacelike canal hypersurfaces

We consider the differential geometry of spacelike submanifolds of codimension at least two in de Sitter space, which is analogous to [9]. We construct the spacelike canal hypersurfaces from the spacelike submanifold and observe their geometrical properties.

5.1 Spacelike submanifolds and timelike unit normal vector fields

Let $r \geq 2$ be an integer and $\mathbf{X} : U \longrightarrow S_1^n$ be an embedding from an open set $U \subset \mathbb{R}^{n-r}$. We say that $\mathbf{X}$ is *spacelike* in S_1^n if every non zero vector generated by $\{\mathbf{X}_{u_i}(\mathbf{u})\}_{i=1}^{n-r}$ is spacelike, where $\mathbf{u} \in U$ and $\mathbf{X}_{u_i} = \partial \mathbf{X}/\partial u_i$. We identify $M = \mathbf{X}(U)$ with U through the embedding $\mathbf{X}$ and call M a *spacelike submanifold of codimension* r in de Sitter space. Since $\langle \mathbf{X}, \mathbf{X} \rangle \equiv 1$, so that $\langle \mathbf{X}_{u_i}, \mathbf{X} \rangle \equiv 0$ for $i = 1, \ldots, n-r$. The tangent space of M at $p = \mathbf{X}(\mathbf{u})$ is spanned by the vectors $\mathbf{X}_{u_i}(\mathbf{u})$ for $i = 1, \ldots, n-r$.

Let N_pM be the normal space of M at p in $\mathbb{R}_1^{n+1}$ and we define $N_p^*(M) = N_pM \cap T_pS_1^n$. Let $\mathbf{n} : U \longrightarrow N_p^*(M)$ be a timelike unit normal vector field on M. Since $\langle \mathbf{n}, \mathbf{n} \rangle \equiv -1$ and $\langle \mathbf{X}, \mathbf{n} \rangle \equiv 0$, $\mathbf{n}_{u_i}$ is pseudo orthogonal to both of $\mathbf{X}$ and $\mathbf{n}$ for $i = 1, \ldots, n-r$. Therefore we have $\mathbf{n}_{u_i}(\mathbf{u}) \in T_pM \oplus N_p^*(M)$. Consider two pseudo orthonormal projections

$$\pi_p^t : T_p\mathbb{R}_1^{n+1} \longrightarrow T_pM, \quad \pi_p^n : T_p\mathbb{R}_1^{n+1} \longrightarrow N_pM.$$

Let $d_{\mathbf{u}}\mathbf{n}$ be the derivative of $\mathbf{n}$ at $\mathbf{u}$, under the identification of M and U through $\mathbf{X}$, we have the linear transformations on T_pM

$$d_p\mathbf{n}^T = \pi_p^t \circ d_{\mathbf{u}}\mathbf{n}, \quad d_p\mathbf{n}^N = \pi_p^n \circ d_{\mathbf{u}}\mathbf{n}.$$

We respectively call the linear transformation $A_p(\mathbf{n}) = -d_p\mathbf{n}^T$ and $S_p(\mathbf{n}) = -(\mathrm{id}_{T_pM} + d_p\mathbf{n}^T)$ an $\mathbf{n}$-*shape operator* and a *de Sitter horospherical* $\mathbf{n}$-*shape operator* of M at $p = \mathbf{X}(\mathbf{u})$. We also call the linear map $d_{\mathbf{u}}\mathbf{n}^N$ a *normal connection* with respect to the timelike normal $\mathbf{n}$ of M.

We denote eigenvalues of $A_p(\mathbf{n})$ and $S_p(\mathbf{n})$ by $\kappa_p(\mathbf{n})$ and $\bar{\kappa}_p(\mathbf{n})$, which we respectively call an $\mathbf{n}$-*principal curvature* and a *de Sitter horospherical* $\mathbf{n}$-*principal curvature*. The *de Sitter horospherical Gauss-Kronecker curvature* with respect to $\mathbf{n}$ at $p = \mathbf{X}(\mathbf{u})$ is defined to be

$$K_h(\mathbf{n})(\mathbf{u}) = \det S_p(\mathbf{n}).$$

We say that a point $p_0 = \mathbf{X}(\mathbf{u}_0)$ is an $\mathbf{n}$-*umbilic point* if $S_{p_0}(\mathbf{n}) = \bar{\kappa}_{p_0}(\mathbf{n})\mathrm{id}_{T_{p_0}M}$. Since the eigenvectors of $S_{p_0}(\mathbf{n})$ and $A_{p_0}(\mathbf{n})$ are the same, the above condition is equivalent to $A_{p_0}(\mathbf{n}) = \kappa_{p_0}(\mathbf{n})\mathrm{id}_{T_{p_0}M}$. We say that the spacelike submanifold M is *totally* $\mathbf{n}$-*umbilic* if every point on M is $\mathbf{n}$-umbilic. We also say that the timelike unit normal vector field $\mathbf{n}$ is *parallel at* p_0 if $d_{p_0}\mathbf{n}^N = 0_{T_{p_0}M}$. The timelike unit normal field $\mathbf{n}$ is *parallel* if $\mathbf{n}$ is parallel at any points on M. Then we have the following result which is analogous to ([9], Proposition 3.1).

Proposition 5.1 ([14])**.** Let $\mathbf{X}$ be a spacelike submanifold of codimension $r \geq 2$ in de Sitter space. Suppose that $\mathbf{n}$ is a timelike unit normal parallel vector field and $M = \mathbf{X}(U)$ is totally $\mathbf{n}$-umbilic. Then the principal curvatures $\kappa_p(\mathbf{n})$ and $\bar{\kappa}_p(\mathbf{n})$ are constant function $\kappa(\mathbf{n})$ and $\bar{\kappa}(\mathbf{n})$, and there exists a vector $\mathbf{v} \in \mathbb{R}^{n+1}_1$ and real number c such that M is a part of some hyperquadric $HP(\mathbf{v}, c) \cap S^n_1$ in de Sitter space. In this case, we have following classification.

(1) If $1 < |\bar{\kappa}(\mathbf{n}) + 1| = |\kappa(\mathbf{n})|$ then M is a part of a hyperbolic hyperquadric $HP(\mathbf{v}, +1)$.

(2) If $0 < |\bar{\kappa}(\mathbf{n}) + 1| = |\kappa(\mathbf{n})| < 1$ then M is a part of an elliptic hyperquadric $HP(\mathbf{v}, +1)$.

(3) If $\bar{\kappa}(\mathbf{n}) + 1 = \kappa(\mathbf{n}) = 0$ then M is a part of an elliptic hyperquadric $HP(\mathbf{v}, 0)$.

(4) If $\kappa(\mathbf{n}) = 1$ (namely $\bar{\kappa}(\mathbf{n}) = 0$) then M is a part of a de Sitter hyperhorosphere $HP(\mathbf{v}, +1)$.

Proof. By the assumption, we have $A_p(\mathbf{n}) \equiv \kappa_p \mathrm{id}_{T_pM}$. This means that $\pi_p^T \circ \mathbf{n}_{u_i}(\mathbf{u}) \equiv \kappa_p \mathbf{X}_{u_i}(\mathbf{u})$. Since $\mathbf{n}$ is parallel, we have $\mathbf{n}_{u_i}(\mathbf{u}) = \kappa_p \mathbf{X}_{u_i}(\mathbf{u})$. So that $\mathbf{n}_{u_i u_j}(\mathbf{u}) = \kappa_{u_j,p}\mathbf{X}_{u_i}(\mathbf{u}) + \kappa_p \mathbf{X}_{u_i u_j}(\mathbf{u})$ and $\mathbf{n}_{u_j u_i}(\mathbf{u}) = \kappa_{u_i,p}\mathbf{X}_{u_j}(\mathbf{u}) + \kappa_p \mathbf{X}_{u_j u_i}(\mathbf{u})$. It follows that $\mathbf{X}_{u_i u_j} \equiv \mathbf{X}_{u_j u_i}$ and $\mathbf{n}_{u_i u_j} \equiv \mathbf{n}_{u_j u_i}$, then we have $\kappa_{u_j,p}\mathbf{X}_{u_i}(\mathbf{u}) = \kappa_{u_i,p}\mathbf{X}_{u_j}(\mathbf{u})$. Since $\mathbf{X}_i(\mathbf{u})$ and $\mathbf{X}_j(\mathbf{u})$ are linearly independent, $\kappa_{u_i,p} = \kappa_{u_j,p} = 0$. This means that κ_p and $\bar{\kappa}_p$ are constant κ and $\bar{\kappa}$.

We now assume that $\bar{\kappa} + 1 = \kappa \neq 0$. By the assumption, we have $\mathbf{n}_{u_i}(\mathbf{u}) = -\kappa \mathbf{X}_{u_i}(\mathbf{u})$, so that there exists a constant vector $\mathbf{v}$ such that $\mathbf{X}(\mathbf{u}) = \mathbf{v} - (1/\kappa)\mathbf{n}(\mathbf{u})$. Then the vector $\mathbf{v}$ satisfies $\langle \mathbf{v}, \mathbf{v} \rangle = 1 - 1/\kappa^2$ and $\langle \mathbf{X}(\mathbf{u}) - \mathbf{v}, \mathbf{X}(\mathbf{u}) - \mathbf{v} \rangle = -1/\kappa^2$, so that $\langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle = 1$ for any $\mathbf{u} \in U$. This means that M is a part of a hyperquadric in de Sitter space $HP(\mathbf{v}, +1)$. Therefore we have (1), (2) and (4).

On the other hand, if $\bar{\kappa} + 1 = \kappa = 0$ then there exists a constant timelike vector $\mathbf{v}$ such that $\mathbf{n}(\mathbf{u}) = \mathbf{v}$ for any $\mathbf{u} \in U$. So that $\langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{X}(\mathbf{u}), \mathbf{n}(\mathbf{u}) \rangle = 0$ for any $\mathbf{u} \in U$. This means that $M \subset HP(\mathbf{v}, 0)$ Therefore (3) holds. This completes the proof. □

We now consider the following Weingarten type formula. Since $\mathbf{X}_{u_1}, \dots, \mathbf{X}_{u_{n-r}}$spans a spacelike vector subspace, we may induce a Riemannian metric by $ds^2 = \sum_{i,j=1}^{n-r} g_{ij} du_i du_j$ on $M = \mathbf{X}(U)$, where $g_{ij} = \langle \mathbf{X}_{u_i}, \mathbf{X}_{u_j} \rangle$. We respectively define the *second fundamental invariant* and *de Sitter horospherical second fundamental invariant* with respect to the timelike unit normal vector field $\mathbf{n}$ by $h_{ij}(\mathbf{n}) = -\langle \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle$ and $\bar{h}_{ij}(\mathbf{n}) = -\langle \mathbf{X}_{u_i} + \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle$. We have the relation

$$\bar{h}_{ij}(\mathbf{n}) = -g_{ij} + h_{ij}(\mathbf{n}),$$

for $i, j = 1, \dots, n-r$. We also have the following Weingarten-type formula with respect to $\mathbf{n}$, which is analogous to Proposition 3.2 in [9].

Proposition 5.2 ([14])**.** Let $\mathbf{X}$ be a spacelike submanifold of codimension $r \geq 2$ and $\mathbf{n}$ a timelike unit normal vector field on $M = \mathbf{X}(U)$. Then we have:

$$\pi^T \circ (\mathbf{X} + \mathbf{n})_{u_i} = -\sum_{k=1}^{n-r} \bar{h}_i^j(\mathbf{n}) \mathbf{X}_{u_j},$$

where $(\bar{h}_i^j(\mathbf{n}))_{ij} = (\bar{h}_{ik}(\mathbf{n}))_{ik}(g^{kj})_{kj}$ and $(g^{kj}) = (g_{kj})^{-1}$.

Therefore, the Gauss-Kronecker curvature with respect to $\mathbf{n}$ is given by

$$K_h(\mathbf{n}) = \frac{\det(\bar{h}_{ik}(\mathbf{n}))}{\det(g_{kj})}.$$

Since $\langle \mathbf{X} + \mathbf{n}, \mathbf{X}_{u_j} \rangle \equiv 0$, the coefficients of the second fundamental invariant with respect to the timelike parallel unit normal vector field $\mathbf{n}$ are expressed by

$$\begin{aligned} \bar{h}_{ij}(\mathbf{n}) &= -\langle \mathbf{X}_{u_i} + \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle = -\partial \langle \mathbf{X} + \mathbf{n}, \mathbf{X}_{u_j} \rangle / \partial u_i + \langle \mathbf{X} + \mathbf{n}, \mathbf{X}_{u_i u_j} \rangle \\ &= \langle \mathbf{X} + \mathbf{n}, \mathbf{X}_{u_i u_j} \rangle. \end{aligned}$$

So that the de Sitter horospherical second fundamental invariant at a point $p_0 = \mathbf{X}(\mathbf{u}_0)$ depends only on the timelike normal direction $\mathbf{n}_0 = \mathbf{n}(\mathbf{u}_0)$ at p_0. Therefore it is independent of the choice of timelike parallel unit normal vector field $\mathbf{n}$ with $\mathbf{n}_0 = \mathbf{n}(\mathbf{u}_0)$.

Let $\mathbf{n}_0$ be a timelike unit normal vector. We say that a point $p_0 = \mathbf{X}(\mathbf{u}_0)$ is an $\mathbf{n}_0$-*parabolic point* (resp. $\mathbf{n}_0$-*umbilic point*) of M if $K_h(\mathbf{n})(\mathbf{u}_0) = 0$ ($S_{p_0}(\mathbf{n}) = \bar{\kappa}_{p_0}(\mathbf{n})\mathrm{id}_{T_{p_0}M}$) for some timelike parallel unit normal vector field $\mathbf{n}$ with $\mathbf{n}(\mathbf{u}_0) = \mathbf{n}_0$. We also say that p_0 is an $\mathbf{n}_0$-*de Sitter horospherical point* if it is an $\mathbf{n}_0$-parabolic point and an $\mathbf{n}_0$-umbilic point.

5.2 Spacelike canal hypersurfaces

In this section we introduce the notion of spacelike canal hypersurfaces from the spacelike submanifolds. Let $\mathbf{X} : U \longrightarrow S^n_1$ be a spacelike submanifolds of codimension $r \geq 2$ in de Sitter space and $p = \mathbf{X}(\mathbf{u})$. We choose unit orthonormal sections

$$N_p(M) = \langle \mathbf{X}(\mathbf{u}), \mathbf{n}_0(\mathbf{u}), \mathbf{n}_1(\mathbf{u}), \ldots, \mathbf{n}_{r-1}(\mathbf{u}) \rangle_{\mathbb{R}},$$

where $\mathbf{n}_0(\mathbf{u})$ is a timelike unit normal vector and $\mathbf{n}_i(\mathbf{u})$ for $i = 1, \ldots, r-1$ are spacelike unit normal vectors. We define a map $\mathbf{e} : U \times H^{r-1}(-1) \longrightarrow H^{n-1}(-1)$ by

$$\mathbf{e}(\mathbf{u}, \bar{\mu}) = \mu_0 \mathbf{n}_0(\mathbf{u}) + \sum_{i=1}^{r-1} \mu_i \mathbf{n}_i(\mathbf{u}),$$

where $\bar{\mu} = (\mu_0, \ldots, \mu_{r-1})$. Let θ be a fixed real number, we also define a map $\bar{\mathbf{X}}_\theta : U \times H^{r-1}(-1) \longrightarrow S^n_1$ by

$$\bar{\mathbf{X}}_\theta(\mathbf{u}, \bar{\mu}) = \cosh\theta \mathbf{X}(\mathbf{u}) + \sinh\theta \mathbf{e}(\mathbf{u}, \bar{\mu}).$$

We remark that for any spacelike submanifold $\mathbf{X}$ and point $(\mathbf{u}_0, \bar{\mu}_0) \in U \times H^{r-1}(-1)$, there are a real number $\theta \neq 0$ and an open neighborhood V of $(\mathbf{u}_0, \bar{\mu}_0)$ such that $\bar{\mathbf{X}}_\theta$ is spacelike embedding on V. We assume that for any $(\mathbf{u}, \bar{\mu}) \in V$ then $(\mathbf{u}, -\bar{\mu}) \in V$. We write CM as an image $\bar{\mathbf{X}}_\theta(V)$ and call it a *spacelike canal hypersurface* of $M = \mathbf{X}(U)$. Izumiya, Pei, Romero Fuster and Takahashi [9] introduced the notion of canal surfaces of submanifolds in the hyperbolic space.

5.3 Horospherical hypersurfaces and height functions

We now introduce the notion of de Sitter horospherical hypersurface and de Sitter horospherical height function on a spacelike submanifold. For a spacelike submanifolds $\mathbf{X}$ of codimension r, we define the family of functions

$$H : U \times LC^* \longrightarrow \mathbb{R}$$

by $H(\mathbf{u}, \mathbf{v}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle - 1$, and we call H a *de Sitter horospherical height function on* M. For $\mathbf{v}_0 \in LC^*$ we denote $h_{\mathbf{v}_0}(\mathbf{u}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v}_0 \rangle - 1$. We have the following proposition which is analogous to ([9], Proposition 3.4).

Proposition 5.3. ([14]) Let $H : U \times LC^* \longrightarrow \mathbb{R}$ be a de Sitter horospherical height function of a spacelike submanifold $X : U \longrightarrow S^n_1$ of codimension r. Then $H(\mathbf{u}, \mathbf{v}) = \partial H(\mathbf{u}, \mathbf{v})/\partial u_i = 0$ for $i = 1, \dots, n-r$ if and only if $\mathbf{v} = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})$ for some $\bar{\mu} \in H^{r-1}(-1)$.

The proof of the above proposition is similar to that of Proposition 3.4 in [9], so it is omitted. The discriminant set of the de Sitter horospherical height function H is

$$D_H = \left\{ \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu}) \mid (\mathbf{u}, \bar{\mu}) \in U \times H^{r-1}(-1) \right\}.$$

We define a map $HS_{\mathbf{X}} : U \times H^{r-1}(-1) \longrightarrow LC^*$ by

$$HS_{\mathbf{X}}(\mathbf{u}, \bar{\mu}) = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu}),$$

which we call a *de Sitter horospherical hypersurface of* M. We remark that $HS_{\mathbf{X}}$ depends on the choice of the orthonormal frames of $N(M)$. Let $\{\mathbf{X}, \mathbf{n}_0, \dots, \mathbf{n}_{r-1}\}$ and $\{\mathbf{X}, \mathbf{n}'_0, \dots, \mathbf{n}'_{r-1}\}$

be two orthonormal frames of $N(M)$ with $\mathbf{n}_0, \mathbf{n}'_0 \in H^{n-1}_+(-1)$. Then we have $\mathbf{n}_i = \sum_{j=0}^{r-1} \lambda_i^j \mathbf{n}'_j$, where

$$\lambda_i^j(\mathbf{u}) = \begin{cases} -\langle \mathbf{n}_i, \mathbf{n}'_j \rangle & \text{if } j = 0 \\ \langle \mathbf{n}_i, \mathbf{n}'_j \rangle & \text{if } j = 1, \dots, r-1. \end{cases}$$

Then we have a diffeomorphism $\Phi : U \times H^{r-1}(-1) \longrightarrow U \times H^{r-1}(-1)$ defined by

$$\Phi(\mathbf{u}, \bar{\mu}) = \left(\mathbf{u}, \left(\sum_{i=0}^{r-1} \lambda_i^0(\mathbf{u})\mu_i, \dots, \sum_{i=0}^{r-1} \lambda_i^{r-1}(\mathbf{u})\mu_i \right) \right).$$

We also define $\mathbf{e}'(\mathbf{u}, \bar{\mu}) = \sum_{i=0}^{r-1} \mu_i \mathbf{n}'_i(\mathbf{u})$. It follows from the above that $\mathbf{e}(\mathbf{u}, \bar{\mu}) = \mathbf{e}' \circ \Phi(\mathbf{u}, \bar{\mu})$. Therefore we have

$$HS_{\mathbf{X}}(\mathbf{u}, \bar{\mu}) = HS'_{\mathbf{X}} \circ \Phi(\mathbf{u}, \bar{\mu}),$$

where $HS'_{\mathbf{X}} = \mathbf{X}(\mathbf{u}) + \mathbf{e}'(\mathbf{u}, \bar{\mu})$. This means that $HS_{\mathbf{X}}$ is independent to the choice of orthonormal frames of $N(M)$ up to the diffeomorphic parametrization. We have a following proposition which is analogous to ([9], Proposition 3.5).

Proposition 5.4 ([14])**.** Let $\mathbf{X} : U \longrightarrow S^n_1$ be a spacelike hypersurface of codimension $r \geq 2$ in de Sitter space, then $HS_{\mathbf{X}}(\mathbf{u}, \bar{\mu}) = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})$ is constant map for some smooth map $\bar{\mu} : U \longrightarrow H^{r-1}(-1)$ if and only if M is a part of de Sitter hyperhorosphere $HP(\mathbf{v}, 1) \cap S^n_1$. By Proposition 5.1, If M is totally $\mathbf{e}(\mathbf{u}, \bar{\mu}(\mathbf{u}))$-umbilic for some parallel normal vector field $\mathbf{e}(\mathbf{u}, \bar{\mu}(\mathbf{u}))$ and $K_h(\mathbf{e}(\mathbf{u}, \bar{\mu}(\mathbf{u})))(\mathbf{u}) = 0$, then the above assertion holds.

Proof. Suppose that $\mathbf{v}_0 = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})$ is a constant vector. Since $\mathbf{e}(\mathbf{u}, \bar{\mu})$ is pseudo orthogonal to $\mathbf{X}(\mathbf{u})$, then we have $\langle \mathbf{X}(\mathbf{u}), \mathbf{v}_0 \rangle = +1$ for any $\mathbf{u} \in U$. This means that M is a part of a de Sitter hyperhorosphere $HP(\mathbf{v}_0, 1) \cap S^n_1$. On the other hand, if $M \subset HP(\mathbf{v}_0, 1) \cap S^n_1$ for some lightlike vector, then $\langle \mathbf{v}_0 - \mathbf{X}(\mathbf{u}), \mathbf{X}(\mathbf{u}) \rangle = 0$ for any $\mathbf{u} \in U$. Since $\mathbf{X}(\mathbf{u})$ is pseudo orthogonal to $\mathbf{X}_{u_i}(\mathbf{u})$, it follows that $\langle \mathbf{v}_0 - \mathbf{X}(\mathbf{u}), \mathbf{X}_{u_i}(\mathbf{u}) \rangle = 0$. This means that $\mathbf{X}(\mathbf{u}) - \mathbf{v}_0$ is a normal vector of M at $p = \mathbf{X}(\mathbf{u})$. We define a function $\bar{\mu}(\mathbf{u})$ by

$$\bar{\mu}(\mathbf{u}) = -\langle \mathbf{X}(\mathbf{u}) - \mathbf{v}_0, \mathbf{n}_0(\mathbf{u}) \rangle \mathbf{n}_0(\mathbf{u}) + \sum_{i=1}^{r-1} \langle \mathbf{X}(\mathbf{u}) - \mathbf{v}_0, \mathbf{n}_i(\mathbf{u}) \rangle \mathbf{n}_i(\mathbf{u}).$$

Then we have $\mathbf{v}_0 - \mathbf{X}(\mathbf{u}) = \mathbf{e}(\mathbf{u}, \bar{\mu})$. This completes the proof. □

Since the image of $HS_{\mathbf{X}}$ is the discriminant set of the de Sitter horospherical height function H on M, the singular set of $HS_{\mathbf{X}}$ corresponds to the null set of the Hessian matrix of the de Sitter horospherical height function with the fixed parameter $\mathbf{v}$ at each point. Therefore we have the following proposition which is analogous to ([9], Proposition 3.6).

Proposition 5.5 ([14])**.** The singular set of $HS_{\mathbf{X}}$ is given by

$$\Sigma(HS_{\mathbf{X}}) = \left\{(\mathbf{u}, \bar{\mu}) \in U \times H^{r-1}(-1) \mid K_h(\mathbf{e}(\mathbf{u}, \bar{\mu}))(\mathbf{u}) = 0\right\}.$$

Proof. Let $h_{\mathbf{v}}(\mathbf{u})$ be a de Sitter horospherical height function with $\mathbf{v} \in LC^*$, then we have $\operatorname{Hess} h_{\mathbf{v}}(\mathbf{u}) = \langle \mathbf{X}_{u_i u_j}(\mathbf{u}), \mathbf{v}\rangle$. Suppose that $(\mathbf{u}, \mathbf{v}) \in \Sigma_*(H)$, then $\mathbf{v} = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})$ for some $\bar{\mu} \in H^{r-1}(-1)$. We recall that $\bar{h}_{ij}(\mathbf{v})(\mathbf{u}) = \langle \mathbf{X}_{u_i u_j}(\mathbf{u}), \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})\rangle$, where $(\bar{h}_{ij}(\mathbf{v})(\mathbf{u}))$ is the de Sitter horospherical second fundamental invariant with respect to the timelike direction $\mathbf{e}(\mathbf{u}, \bar{\mu})$. The de Sitter horospherical Gauss-Kronecker curvature is $K_h(\mathbf{e}(\mathbf{u}, \bar{\mu}))(\mathbf{u}) = \det(\langle \mathbf{X}_{u_i u_j}(\mathbf{u}), \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u})\rangle)/\det(g_{ij}(\mathbf{u})) = \det \operatorname{Hess} h_{\mathbf{v}}(\mathbf{u})/\det(g_{ij}(\mathbf{u}))$, where $(g_{ij}(\mathbf{u}))$ is the first fundamental invariant of M. Therefore $\operatorname{Hess} h_{\mathbf{v}}(\mathbf{u}) = 0$ if and only if $K_h(\mathbf{e}(\mathbf{u}, \bar{\mu}))(\mathbf{u}) = 0$. This completes the proof. □

The singular set of $HS_{\mathbf{X}}$ corresponds to the parabolic set of M with respect to some timelike parallel normal vector field $\mathbf{e}(\mathbf{u}, \bar{\mu})$. By the proof of above proposition, we have $\operatorname{rank} \operatorname{Hess} h_{\mathbf{v}_0}(\mathbf{u}_0) = \operatorname{rank}(\bar{h}_{ij}(\mathbf{v}_0)(\mathbf{u}_0))_{ij}$. Therefore we also have the following proposition which is analogous to ([9], Proposition 3.7).

Proposition 5.6 ([14])**.** For any spacelike submanifold $\mathbf{X}$ of codimension $r \geq 2$ and lightlike vector $\mathbf{v}_0 = \mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$, we have the following assertions.

(1) A point $p_0 = \mathbf{X}(\mathbf{u}_0)$ is an $\mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$-parabolic point if and only if $\det \operatorname{Hess} h_{\mathbf{v}_0}(\mathbf{u}_0) = 0$.

(2) A point p_0 is an $\mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$-de Sitter horospherical point if and only if $\operatorname{rank} \operatorname{Hess} h_{\mathbf{v}_0}(\mathbf{u}_0) = 0$.

Here $\operatorname{Hess} h_{\mathbf{v}_0}(\mathbf{u}_0)$ is a Hessian matrix of $h_{\mathbf{v}_0}(\mathbf{u})$ at $\mathbf{u} = \mathbf{u}_0$.

We now consider the lightcone height function and the lightcone Gauss image of spacelike canal hypersurface $\bar{\mathbf{X}}_\theta : V \longrightarrow S^n_1$ with $V \subset U \times H^{r-1}(-1)$. The lightcone height function $\bar{H} : V \times LC^* \longrightarrow \mathbb{R}$ of the spacelike hypersurface $\bar{\mathbf{X}}_\theta$ is

$$\bar{H}((\mathbf{u}, \bar{\mu}), \mathbf{v}) = \langle \bar{\mathbf{X}}_\theta(\mathbf{u}, \bar{\mu}), \mathbf{v} \rangle - 1.$$

We denote $\bar{h}_{\mathbf{v}}(\mathbf{u}) = \bar{H}((\mathbf{u}, \bar{\mu}), \mathbf{v})$ for any $\mathbf{v} \in LC^*$. Now we define a map $\bar{\mathbf{e}} : V \longrightarrow H^{n-1}(-1)$ by $\bar{\mathbf{e}}(\mathbf{u}, \bar{\mu}) = \sinh\theta \mathbf{X}(\mathbf{u}) + \cosh\theta \mathbf{e}(\mathbf{u}, \bar{\mu})$. Then we have $\langle \bar{\mathbf{e}}(\mathbf{u}, \bar{\mu}), \bar{\mathbf{X}}_\theta(\mathbf{u}) \rangle = \langle \bar{\mathbf{e}}(\mathbf{u}, \bar{\mu}), \bar{\mathbf{X}}_{\theta,u_i}(\mathbf{u}) \rangle = 0$ for any $(\mathbf{u}, \bar{\mu}) \in V$ and $i = 1, \ldots, n-r$. Therefore $\bar{\mathbf{e}}$ is a timelike normal of CM. The positive lightcone Gauss image $\mathbb{L}_{CM} : V \longrightarrow LC^*$ is defined by

$$\mathbb{L}_{CM}(\mathbf{u}, \bar{\mu}) = \bar{\mathbf{X}}_\theta(\mathbf{u}) + \bar{\mathbf{e}}(\mathbf{u}, \bar{\mu}) = (\cosh\theta + \sinh\theta)(\mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})).$$

By Proposition 3.3, $\bar{H}((\mathbf{u}, \bar{\mu}), \mathbf{v}) = \bar{H}_{u_i}((\mathbf{u}, \bar{\mu}), \mathbf{v}) = \bar{H}_{\mu_j}((\mathbf{u}, \bar{\mu}), \mathbf{v}) = 0$ for $i = 1, \ldots, n-r$ and $j = 0, \ldots, r-1$ if and only if $\mathbf{v} = \bar{\mathbf{X}}_\theta(\mathbf{u}) \pm \bar{\mathbf{e}}(\mathbf{u}, \bar{\mu}) = e^{\pm\theta}(\mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \pm\bar{\mu}))$. By assumption, $(\mathbf{u}, -\bar{\mu})$ is also an element of V. Therefore the discriminant set of the lightcone height function $\bar{H}$ is

$$D(\bar{H}) = \{e^{\pm\theta}(\mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})) \mid (\mathbf{u}, \bar{\mu}) \in V\}.$$

We now define a diffeomorphism

$$\mathcal{M}_c : LC^* \longrightarrow LC^*$$

given by $M_c(\mathbf{v}) = c\mathbf{v}$ for a fixed positive real number c. Then we have the following lemma, which is analogous to ([9], Proposition 3.9).

Lemma 5.7 ([14])**.** Under the above notations, we have

$$\mathcal{M}_c \circ HS_{\mathbf{X}}(\mathbf{u}, \bar{\mu}) = \mathbb{L}_{CM}(\mathbf{u}, \bar{\mu})$$

on $V \subset U \times H^{r-1}(-1)$, where $c = e^{\pm\theta}$.

By the above lemma, the de Sitter horospherical hypersurface $HS_{\mathbf{X}}$ is locally diffeomorphic to the lightcone Gauss image of the spacelike canal hypersurface $\bar{\mathbf{X}}_\theta$.

5.4 Horospherical hypersurfaces as wave fronts

In this section we naturally interpret the de Sitter horospherical hypersurfaces of M as a wave front set of the de Sitter horospherical height functions in the theory of Legendrian singularities.

By proceeding arguments in §2, the de Sitter horospherical hypersurface $HS_{\mathbf{X}}$ is the discriminant set of the de Sitter horospherical height function H, and the singular point set of the de Sitter horospherical hypersurface is the de Sitter horospherical point set. We have the following proposition which is analogous to ([9], Proposition 4.1).

Proposition 5.8 ([14])**.** Let $\mathbf{X} : U \longrightarrow S_1^n$ be a spacelike submanifold of codimension $r \geq 2$ and $H : U \times LC^* \longrightarrow \mathbb{R}$ be a de Sitter horospherical height function of M. Then H is a Morse family.

Proof. We denote $\mathbf{X}(\mathbf{u}) = (X_0(\mathbf{u}), \dots, X_n(\mathbf{u}))$ and $\mathbf{X}_{u_i}(\mathbf{u}) = (X_{0,u_i}(\mathbf{u}), \dots, X_{n,u_i}(\mathbf{u}))$. For any $\mathbf{v} = (v_0, \dots, v_n) \in LC^*$, we have $v_0 \neq 0$. Without loss of generality, we assume that $v_0 = \sqrt{v_1^2 + \cdots + v_n^2} > 0$, so that we have

$$H(\mathbf{u}, \mathbf{v}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle - 1 = -X_0 v_0 + \sum_{i=1}^{n} X_n v_n - 1.$$

We now prove a map

$$\Delta^* H = \left(H, \frac{\partial H}{\partial u_1}, \dots, \frac{\partial H}{\partial u_{n-r}} \right)$$

is non singular at any $(\mathbf{u}, \mathbf{v}) \in \Sigma_*(H)$. The Jacobian matrix of $\Delta^* H$ is

$$J\Delta^* H(\mathbf{u}, \mathbf{v}) = \left(\begin{array}{c|c} * & \dfrac{\partial H}{\partial v_j}(\mathbf{u}, \mathbf{v})_{j=1,\dots,n} \\ \hline * & \left(\dfrac{\partial^2 H}{\partial u_i \partial v_j}(\mathbf{u}, \mathbf{v}) \right)_{\substack{j=1,\dots,n \\ i=1,\dots,n-r}} \end{array} \right).$$

We denote an $(n - r + 1) \times n$ matrix B by $J\Delta^* H = (* \mid B)$. It is sufficient to show that rank $B = n - r + 1$ at $(\mathbf{u}, \mathbf{v}) \in \Sigma_*(H)$. We also denote an $(n-r+3) \times (n+1)$ matrix C by

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ v_0 & v_1 & \cdots & v_n \\ X_0 & X_1 & \cdots & X_n \\ X_{0,u_1} & X_{1,u_1} & \cdots & X_{n,u_1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{0,u_{n-r}} & X_{1,u_{n-r}} & \cdots & X_{n,u_{n-r}} \end{pmatrix}.$$

We now show that the rank of the matrix C is equal to $n - r + 3$. Since $\mathbf{v}, \mathbf{X}(\mathbf{u})$ and $\mathbf{X}_{u_i}(\mathbf{u})$ are linearly independent for all $(\mathbf{u}, \mathbf{v}) \in \Sigma_*(H)$, it is sufficient to show that timelike unit vector $\mathbf{e} = (1, 0, \ldots, 0)$ can not be written by a linear combination of $\mathbf{v}, \mathbf{X}(\mathbf{u})$ and $\mathbf{X}_{u_i}(\mathbf{u})$. If that is not so, there exists some real numbers η, μ, ξ_i such that $\mathbf{e} = \eta\mathbf{v} + \mu\mathbf{X}(\mathbf{u}) + \mathbf{w}$ and $\mathbf{w} = \sum_{i=1}^{n-r} \xi_i \mathbf{X}_{u_i}(\mathbf{u})$. Then we have $\langle \mathbf{e}, \mathbf{e} \rangle = \mu^2 + \langle \mathbf{w}, \mathbf{w} \rangle$. However, $\mathbf{w}$ is a spacelike vector, so that $\langle \mathbf{e}, \mathbf{e} \rangle$ would not be negative, which contradicts our assumption. This means that $\mathbf{e}, \mathbf{v}, \mathbf{X}(\mathbf{u})$ and $\mathbf{X}_{u_i}(\mathbf{u})$ are linearly independent, therefore we have rank $C = n - r + 3$.

We now show rank $B =$ rank $C' - 2$. We subtract the second row multiplied by X_0/v_0 from the third row of the matrix C, and add the second row multiplied by $X_{0,u_k}(\mathbf{u})/v_0$ from the $(3+k)$-th row for $k = 1, \ldots, n - r$. Then we have a matrix

$$C' = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ v_0 & v_1 & \cdots & v_n \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right).$$

Therefore we have rank $B =$ rank $C' - 2 = n - r + 1$. This completes the proof. □

Since H is a Morse family of hypersurfaces, we have the Legendrian immersion germ $\mathcal{L}_H : (\Sigma_*(H), (\mathbf{u}_0, \mathbf{v}_0)) \longrightarrow PT^*(LC^*)$ defined by

$$\mathcal{L}_H(\mathbf{u}, \mathbf{v}) = \left(\mathbf{v}, \left[\frac{\partial H}{\partial u_1}(\mathbf{u}, \mathbf{v}) : \ldots : \frac{\partial H}{\partial u_{n-r}}(\mathbf{u}, \mathbf{v})\right]\right).$$

We remark that the wave front set of the Legendrian immersion germ $\mathcal{L}_H$ is the de Sitter horospherical hypersurfaces $HS_{\mathbf{X}}$ of M. On the other hand, we define a contact diffeomorphism $\tilde{\mathcal{M}}_c : PT^*(LC^*) \longrightarrow PT^*(LC^*)$ by

$$\tilde{\mathcal{M}}_c(\mathbf{v}, [\xi]) = (c\mathbf{v}, [\xi]),$$

where c is a fixed real parameter with $c \neq 0$. By definition, we have the following theorem.

Theorem 5.9 ([14])**.** For a spacelike submanifold $\mathbf{X} : U \longrightarrow S^n_1$, we have

$$\tilde{\mathcal{M}}_c \circ \mathcal{L}_H = \mathcal{L}_{\bar{H}},$$

where $c = e^{\pm\theta}$ and $\mathcal{L}_{\bar{H}}$ is a Legendrian lift of the lightcone Gauss image $\mathbb{L}_{CM}$ of the spacelike canal hypersurface of M.

By the above theorem, the Legendrian lift of the lightcone Gauss image $\mathbb{L}_{CM}$ is $\mathcal{A}$-equivalent to the Legendrian lift of the de Sitter horospherical hypersurface $HS_{\mathbf{X}}$ of M.

5.5 Contact with de Sitter hyperhorospheres

In this section we use the theory of contacts between the spacelike submanifolds and the de Sitter hyperhorospheres, We consider the function $\mathcal{H} : S^n_1 \times LC^* \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\mathbf{x}, \mathbf{v}) = \langle \mathbf{x}, \mathbf{v} \rangle - 1$. Given $\mathbf{v}_0 \in LC^*$, we denote $\mathfrak{h}_{\mathbf{v}_0}(\mathbf{x}) = \mathcal{H}(\mathbf{x}, \mathbf{v}_0)$, so that we have $\mathfrak{h}^{-1}_{\mathbf{v}_0}(0) = HP(\mathbf{v}_0, +1) \cap S^n_1$. Let $\mathbf{X} : U \longrightarrow \mathbb{R}^n_1$ be a spacelike submanifold of codimension $r \geq 2$. For any $\mathbf{u}_0 \in U$ and $\bar{\mu}_0 \in H^{r-1}(-1)$, we take a point $\mathbf{v}_0 = \mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$. By Proposition 5.3, we have

$$(\mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{X})(\mathbf{u}_0) = \mathcal{H} \circ (\mathbf{X} \times \mathrm{id}_{LC^*})(\mathbf{u}_0, \mathbf{v}_0) = H(\mathbf{u}_0, \mathbf{v}_0) = 0.$$
$$\frac{\partial(\mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{X})}{\partial u_i}(\mathbf{u}_0) = \frac{\partial H}{\partial u_i}(\mathbf{u}_0, \mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)) = 0.$$

It follows that the de Sitter hyperhorosphere $\mathfrak{h}^{-1}_{\mathbf{v}_0}(0) = HP(\mathbf{v}_0, +1) \cap S^n_1$ is tangent to M at $p_0 = \mathbf{X}(\mathbf{u}_0)$. In this case we call $HP(\mathbf{v}_0, +1) \cap S^n_1$ a *tangent de Sitter hyperhorosphere* with respect to $\mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$. In section 3, we have considered the contacts of the spacelike canal surface $CM = \bar{\mathbf{X}}(V)$ and the de Sitter horospheres.

We denote $h_{i,\mathbf{v}_i} : (U, \mathbf{u}_i) \longrightarrow (\mathbb{R}, \mathbf{0})$ $(i = 1, 2)$ by $h_{i,\mathbf{v}_i}(\mathbf{u}) = H_i(\mathbf{u}, \mathbf{v}_i)$. Then we have $h_{i,\mathbf{v}_i}(\mathbf{u}) = (\mathfrak{h}_{i,\mathbf{v}_i} \circ \mathbf{X}_i)(\mathbf{u})$. By Theorem B.1,

$$K(\mathbf{X_1}(U), HP(\mathbf{v}_1, 1) \cap S^n_1; p_1) = K(\mathbf{X_2}(U), HP(\mathbf{v}_2, 1) \cap S^n_1; p_2)$$

if and only if $h_{1,\mathbf{v}_1}$ and $h_{2,\mathbf{v}_2}$ are $\mathcal{K}$-equivalent.

Let $Q(\mathbf{X}, \mathbf{u}_0)$ be the local ring of the de Sitter horospherical height function germ $h_{\mathbf{v}_0} : (U, \mathbf{u}_0) \longrightarrow \mathbb{R}$ defined by

$$Q(\mathbf{X}, \mathbf{u}_0; \bar{\mu}_0) = C^\infty_{\mathbf{u}_0}(U)/\langle h_{\mathbf{v}_0}\rangle_{C^\infty_{\mathbf{u}_0}(U)},$$

where $\mathbf{v}_0 = \mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$, $\bar{\mu}_0 \in H^{r-1}(-1)$ and $C^\infty_{\mathbf{u}_0}(U)$ is the local ring of function germs at $\mathbf{u}_0$ with the unique maximal ideal $\mathfrak{M}$. We also denote $Q(\bar{\mathbf{X}}_\theta, (\mathbf{u}_0, \bar{\mu}_0))$ as the local ring of the lightcone height function germ $\bar{h}_{\mathbf{v}'_0} : (U \times H^{r-1}(-1), (\mathbf{u}_0, \bar{\mu}_0)) \longrightarrow (\mathbb{R}, 0)$ of the canal hypersurface $\bar{\mathbf{X}}_\theta$, where $\mathbf{v}'_0 = \mathbb{L}_{CM}(\mathbf{u}_0, \bar{\mu}_0)$.

By the arguments in Appendix A, we have following theorem.

Theorem 5.10 ([14])**.** Let $\mathbf{X}_i : (U_i, \mathbf{u}_i) \longrightarrow (S^n_1, p_i)$ $(i = 1, 2)$ be spacelike submanifold germs of codimension at least two in de Sitter space. For $\bar{\mu}_i \in H^{r-1}(-1)$ $(i = 1, 2)$, we denote $\mathbf{v}_i = HS_i(\mathbf{u}_i, \bar{\mu}_i)$, $\mathbf{v}'_i = \mathbb{L}_{CM_i}(\mathbf{u}_i, \bar{\mu}_i)$, $h_{i,\mathbf{v}_i} = H_i\,|_{U\times\{\mathbf{v}_i\}}$, $\bar{h}_{i,\mathbf{v}'_i} = \bar{H}_i\,|_{U\times\{\mathbf{v}'_i\}}$ and $p'_i = \bar{\mathbf{X}}_{i,\theta_i}(\mathbf{u}_i, \bar{\mu}_i)$. If the corresponding Legendrian immersion germs $\mathcal{L}_{H_i}$ are Legendrian stable, then the following conditions are equivalent:

(1) Horospherical hypersurface germs $HS_{\mathbf{X}_1}$ and $HS_{\mathbf{X}_2}$ are $\mathcal{A}$-equivalent.

(2) Legendrian immersion germs $\mathcal{L}_{H_1}$ and $\mathcal{L}_{H_2}$ are Legendrian equivalent.

(3) Horospherical height function germs H_1 and H_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(4) $h_{1,\mathbf{v}_1}$ and $h_{2,\mathbf{v}_2}$ are $\mathcal{K}$-equivalent.

(5) $K(\mathbf{X_1}(U), HP(\mathbf{v}_1, 1) \cap S^n_1; p_1) = K(\mathbf{X_2}(U), HP(\mathbf{v}_2, 1) \cap S^n_1; p_2)$.

(6) $Q(\mathbf{X}_1, \mathbf{u_1})$ and $Q(\mathbf{X}_2, \mathbf{u_2})$ are isomorphic as $\mathbb{R}$-algebras.

(7) Lightcone Gauss image germs $\mathbb{L}_{CM_1}$ and $\mathbb{L}_{CM_2}$ are $\mathcal{A}$-equivalent.

(8) Legendrian immersion germs $\mathcal{L}_{\bar{H}_1}$ and $\mathcal{L}_{\bar{H}_2}$ are Legendrian equivalent.

(9) Lightcone height function germs $\bar{H}_1$ and $\bar{H}_2$ are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(10) $\bar{h}_{1,\mathbf{v}'_1}$ and $\bar{h}_{2,\mathbf{v}'_2}$ are $\mathcal{K}$-equivalent.

(11) $K(CM_1, HP(\mathbf{v}'_1, +1) \cap S^n_1; p'_1) = K(CM_2, HP(\mathbf{v}'_2, +1) \cap S^n_1; p'_2)$,

(12) $Q(\bar{\mathbf{X}}_{\theta_1}, (\mathbf{u_1}, \bar{\mu}_1))$ and $Q(\bar{\mathbf{X}}_{\theta_2}, (\mathbf{u_2}, \bar{\mu}_2))$ are isomorphic as $\mathbb{R}$-algebras.

In this case $({\mathbf{X}_1}^{-1}(HP(\mathbf{v}_1, 1) \cap S^n_1), \mathbf{u_1})$ and $({\mathbf{X}_2}^{-1}(HP(\mathbf{v}_2, 1) \cap S^n_1), \mathbf{u_2})$ are diffeomorphic as set germs.

Proof. Since $\mathcal{L}_{H_1}$ and $\mathcal{L}_{H_2}$ are Legendrian stable, regular sets of $HS_{\mathbf{X}_1}$ and $HS_{\mathbf{X}_2}$ are respectively dense, by applying Proposition A.2, the conditions (1) and (2) are equivalent. By Theorem A.3, the conditions (2) and (3) are equivalent. By the arguments in Theorem B.1, the conditions (4) and (5) are equivalent. If we assume the condition (3), then the $\mathcal{P}$-$\mathcal{K}$-equivalence of H_i $(i = 1, 2)$ preserves the $\mathcal{K}$-equivalence of $h_{i,\mathbf{v}_i}$, so that the condition (4) holds. Since the local rings $Q(\mathbf{X}_i, \mathbf{u}_i)$ are $\mathcal{K}$-invariant, this means that the condition (6) holds. By Proposition A.4, the condition (6) implies the condition (2). Therefore the statements from (1) to (6) are equivalent.

By Theorem 5.9, (2) and (8) are equivalent. Since $\mathcal{L}_{H_i}$ are Legendrian stable, $\mathcal{L}_{\bar{H}_i}$ are also Legendrian stable. So that we may similarly show the equivalence of the conditions from (7) to (12). On the other hand, $h^{-1}_{i,\mathbf{v}_i}(0) = ({\mathbf{X}_i}^{-1}(HP(\mathbf{v}_i, 1) \cap S^n_1), \mathbf{u}_i)$ and $\mathcal{K}$-equivalence preserves the zero level sets, so that $({\mathbf{X}_i}^{-1}(HP(\mathbf{v}_i, 1) \cap S^n_1), \mathbf{u}_i)$ $(i = 1, 2)$ are diffeomorphic as set germs. This completes the proof. □

We consider generic properties of spacelike submanifolds of codimension $r \geq 2$ in S^n_1. Let U be an open subset of $\mathbb{R}^{n-r}$. We consider the space of spacelike embeddings Sp-Emb(U, S^n_1) with Whitney C^∞-topology. By Appendix C, the assumption of Theorem 5.10 is generic conditions if $n \leq 6$.

6 Timelike canal hypersurfaces

We consider timelike canal hypersurfaces of spacelike submanifolds of codimension at least two in de Sitter space. In §5 we considered geometrical relations between spacelike canal hypersurfaces and spacelike submanifolds. We can construct the timelike hypersurfaces from the spacelike submanifolds and find out some geometrical relations between spacelike submanifolds and timelike hypersurfaces. In [10] the differential geometry of timelike hypersurfaces in de Sitter space was studied from the viewpoint of singularity theory. Singularities of de Sitter Gauss image are related to the de Sitter parabolic points, where de Sitter Gauss-Kronecker curvature is equal to zero.

6.1 Review of timelike hypersurfaces

We briefly review the differential geometry of timelike hypersurfaces in de Sitter space studied in [10]. Let U be an open subset of Euclidean $(n-1)$-space and $\mathbf{u} = (u_1, \cdots, u_{n-1}) \in U$. We say that an embedding map $\mathbf{X} : U \longrightarrow S^n_1$ is a *timelike hypersurface* if tangent space of $\mathbf{X}$ is timelike hyperplane at any point, and we denote $M = \mathbf{X}(U)$.

We define a *de Sitter Gauss image* of the timelike hypersurface by

$$\mathbf{x}^d : U \longrightarrow S^n_1,\ \mathbf{x}^d(\mathbf{u}) = \frac{\mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \ldots \wedge \mathbf{X}_{u_{n-1}}(\mathbf{u})}{||\mathbf{X}(\mathbf{u}) \wedge \ldots\ldots \wedge \mathbf{X}_{u_{n-1}}(\mathbf{u})||}.$$

We also define a *de Sitter shape operator* by $A^d_p := -d_{\mathbf{u}}\mathbf{x}^d : T_pM \longrightarrow T_pM$, a *de Sitter Gauss-Kronecker curvature* $K_d(\mathbf{u})$ by determinant of the de Sitter shape operator and *de Sitter principal curvatures* $\{\kappa_{d,i}(\mathbf{u})\}_i$ by eigenvalues of the de Sitter shape operator. We call a point with $K_d(\mathbf{u}) = 0$ by a *de Sitter parabolic point* and a point with $A^d_p = \kappa_d \cdot \mathrm{id}_{T_pM}$ by a *de Sitter umbilic point.*

Proposition 6.1. ([10]) Suppose that the timelike hypersurface M is totally umbilic, then the de Sitter principal curvatures $\kappa_{d,i}(\mathbf{u})$ are constant κ_d and M is a part of some hyperbolic hyperquadric $HP(\mathbf{v}, c) \cap S^n_1$.

(1) If $\kappa_d \neq 0$ then M is a part of some non-flat hyperbolic hyperquadric with

$$\mathbf{v} = \frac{1}{\sqrt{\kappa_d^2+1}}(\kappa_d \mathbf{X}(\mathbf{u}) + \mathbf{x}^d(\mathbf{u})) \text{ and } c = \frac{\kappa_d}{\sqrt{\kappa_d^2+1}} < ||\mathbf{v}||.$$

(2) If $\kappa_d = 0$ then M is a part of some flat timelike hyperquadric. In this case, we have $\mathbf{v} = \mathbf{x}^d(\mathbf{u})$ and $c = 0$.

We now consider Weingarten-type formula of the timelike hypersurfaces. We define a *de Sitter first fundamental form* by $ds^2 = \sum_{i,j=1}^{n-1} g_{ij} du_i du_j$, where $g_{ij} = \langle \mathbf{X}_{u_i}, \mathbf{X}_{u_j} \rangle$. Since the tangent space of M is always timelike, so ds^2 is a Lorentzian metric of index one. We also define coefficients of *de Sitter second fundamental form* by $h_{ij} = -\langle \mathbf{X}_{u_i}, x^d_{u_j} \rangle$. We have the following Weingarten-type formula.

Proposition 6.2 ([10])**.** Let $\mathbf{X}$ be a timelike hypersurface, the differential map of the de Sitter Gauss image $\mathbf{x}^d$ is given by

$$\mathbf{x}^d_{u_i} = \sum_{j=1}^{n-1} (h_i^j) \mathbf{x}^d_{u_j},$$

where $(h_i^j) = (h_i k)(g_k j)^{-1}$.

We define a *de Sitter height function* $H^d : U \times S_1^n \longrightarrow \mathbb{R}$ by

$$H^d(\mathbf{u}, \mathbf{v}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle,$$

and denote $h^d_{\mathbf{v}_0}(\mathbf{u}) = H^d(\mathbf{u}, \mathbf{v}_0)$. The discriminant set of the de Sitter height function is the image of de Sitter Gauss image of the timelike hypersurfaces in de Sitter space, and the singular point of de Sitter Gauss image corresponds to the de Sitter parabolic point on the timelike hypersurface.

Proposition 6.3 ([10])**.** The de Sitter height function H^d is a Morse families of hypersurfaces.

Therefore a Legendrian immersion germ $\mathcal{L}_{H^d} : (\Sigma_*(H^d), (\mathbf{u}_0, \mathbf{v}_0)) \longrightarrow PT^*(S_1^n)$ is defined by

$$\mathcal{L}_{H^d}(\mathbf{u}, \mathbf{v}) = \left(\mathbf{v}, \left[\frac{\partial H^d}{\partial v_1}(\mathbf{u}, \mathbf{v}) : \ldots : \frac{\partial H^d}{\partial v_n}(\mathbf{u}, \mathbf{v}) \right] \right),$$

The wave front set of the Legendrian immersion germ $\mathcal{L}_{H^d}$ is the de Sitter Gauss image germ $\mathbf{x}^d$.

We now consider the contacts of timelike hypersurfaces and flat timelike hyperquadrics. Let $\mathbf{v}_0 = \mathbf{x}^d(\mathbf{u}_0)$. Since $\mathbf{X}(\mathbf{u})$ is immersion and $h_{\mathbf{v}_0}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_0 \rangle$ is submersion, a flat timelike hyperquadric $h_{\mathbf{v}_0}^{-1}(0) = HP(\mathbf{v}_0, 0) \cap S_1^n$ is tangent to the timelike hypersurface $M = \mathbf{X}(U)$ at $p_0 = \mathbf{X}(\mathbf{u}_0)$. By Montaldi's theory, the contact type of submanifolds $K(M, HP(\mathbf{v}_0, 0) \cap S_1^n; p_0)$ corresponds to the $\mathcal{K}$-equivalence class of the de Sitter height function germs $h_{\mathbf{v}_0}^d$ at $p_0 = \mathbf{X}(\mathbf{u}_0)$. We call a germ $((h_{\mathbf{v}_0}^d)^{-1}(0), \mathbf{u}_0)$ a *tangent flat timelike hyperquadric indicatrix germ.*

Proposition 6.4 ([10])**.** Let $\mathbf{X}_i$ $(i = 1, 2)$ be timelike hypersurfaces germs at $u_i \in U$ such that their de Sitter parabolic sets have no interior points as subspaces of U. Then $\mathbf{x}_1^d$ and $\mathbf{x}_2^d$ are $\mathcal{A}$-equivalent if and only if the contact types $K(M, HP(\mathbf{v}_{i,0}, +1) \cap S_1^n; p_i)$ $(i = 1, 2)$ are equivalent. Under this condition, the tangent flat timelike hyperquadric indicatrix germ $h_{\mathbf{v}_{i,0}^d}^{-1}(0)$ are diffeomorphic as the set germs.

6.2 Spacelike submanifolds and spacelike unit normal vector fields

We now back to the spacelike submanifolds. Let $\mathbf{X} : U \longrightarrow S_1^n$ be a spacelike submanifold of codimension $r \geq 2$ in de Sitter space. We take a *spacelike unit normal vector field* $\mathbf{n} : U \longrightarrow S_1^n$ with $\mathbf{n}(\mathbf{u}) \perp \mathbf{X}(\mathbf{u})$ and $\mathbf{X}_{u_i}(\mathbf{u})$ for $i = 1, \ldots, n-r$ and $\mathbf{u} \in U$. Let $\pi_p^t : T_p\mathbb{R}_1^{n+1} \longrightarrow T_pM$ be an orthonormal projection to the tangent space. We define a *spacelike shape operator* with respect to $\mathbf{n}$ by $S_p(\mathbf{n}) = -\pi_p^t(d_p\mathbf{n})$. We also define *spacelike principal curvatures* $\kappa_i(\mathbf{n})(\mathbf{u})$ for $i = 1, \ldots, n-r$ and a *spacelike Gauss-Kronecker curvature* $K(\mathbf{n})(\mathbf{u})$ with respect to $\mathbf{n}$.

We now consider the totally umbilic case of spacelike submanifolds. We say that the spacelike unit normal vector field $\mathbf{n}$ is *parallel* if $\mathbf{n}_{u_i}(\mathbf{u}) \in T_{\mathbf{X}(\mathbf{u})}M$ for any $\mathbf{u} \in U$ and $i = 1, \ldots, n-r$. We have following proposition.

Proposition 6.5. Suppose that the spacelike unit normal vector field $\mathbf{n}$ is parallel and the spacelike submanifold M is totally umbilic with respect to $\mathbf{n}$, then the principal curvatures $\kappa(\mathbf{n})(\mathbf{u})$ are constant value $\kappa(\mathbf{n})$ and M is a part of some hyperbolic hyperquadric $HP(\mathbf{v}, c) \cap$

$S_1^n = \{\mathbf{x} \in S_1^n \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}$. Moreover we have the following cases:

(1) If $\kappa(\mathbf{n}) \neq 0$ then $c = \frac{\kappa(\mathbf{n})}{\sqrt{\kappa^2(\mathbf{n})+1}}$ and $\mathbf{v} = \frac{1}{\sqrt{\kappa^2(\mathbf{n})+1}}(\kappa(\mathbf{n})\mathbf{X}(\mathbf{u}) + \mathbf{n}(\mathbf{u}))$,

(2) If $\kappa(\mathbf{n}) = 0$ then $\mathbf{v} = \mathbf{n}(\mathbf{u})$ and $c = 0$.

Proof. We will show that the principal curvatures are constant. By assumption, we have $d_p\mathbf{n} = -\kappa(\mathbf{n})(\mathbf{u})\,\mathrm{id}_{T_pM}$. This means that $\mathbf{n}_{u_i}(\mathbf{u}) = -\kappa(\mathbf{n})(\mathbf{u})\mathbf{X}_{u_i}(\mathbf{u})$ for $i = 1, \ldots, n-r$, so that we have

$$\mathbf{n}_{u_iu_j}(\mathbf{u}) = -\kappa(\mathbf{n})_{u_j}(\mathbf{u})\mathbf{X}_{u_i}(\mathbf{u}) - \kappa(\mathbf{n})(\mathbf{u})\mathbf{X}_{u_iu_j}(\mathbf{u}).$$

It follows that $\kappa(\mathbf{n})_{u_j}(\mathbf{u})\mathbf{X}_{u_i}(\mathbf{u}) = \kappa(\mathbf{n})_{u_i}(\mathbf{u})\mathbf{X}_{u_j}(\mathbf{u})$. Since $\mathbf{X}_{u_i}(\mathbf{u})$ and $\mathbf{X}_{u_j}(\mathbf{u})$ are linearly independent, $\kappa(\mathbf{n})_{u_i}(\mathbf{u}) \equiv 0$ for all i, j and $\mathbf{u} \in U$. Therefore $\kappa(\mathbf{n})(\mathbf{u}) \equiv \kappa(\mathbf{n})$ for some real constant value $\kappa(\mathbf{n})$.

We now assume that $\kappa(\mathbf{n}) \neq 0$. We have $\mathbf{n}_{u_i}(\mathbf{u}) = -\kappa(\mathbf{n})\mathbf{X}_{u_i}(\mathbf{u})$, so that there exists a constant vector $\mathbf{v}$ such that $\mathbf{X}(\mathbf{u}) = \mathbf{v} - (1/\kappa(\mathbf{n}))\mathbf{n}(\mathbf{u})$. Then $\langle \mathbf{v}, \mathbf{v} \rangle = 1 + (1/\kappa(\mathbf{n}))^2$ and $\langle \mathbf{X}(\mathbf{u}) - \mathbf{v}, \mathbf{X}(\mathbf{u}) - \mathbf{v} \rangle = (1/\kappa(\mathbf{n}))^2$. So that $\langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle = 1$. Therefore the spacelike submanifold M is a part of a hyperbolic hyperquadric $HP(\mathbf{v}, +1) \cap S_1^n$ and $\mathbf{v}$ is a spacelike vector.

If $\kappa(\mathbf{n}) = 0$, then we have $\mathbf{n}_{u_i}(\mathbf{u}) \equiv 0$. There exists a constant vector $\mathbf{v}$ such that $\mathbf{n}(\mathbf{u}) \equiv \mathbf{v}$. So that $\langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle = 0$ for any $\mathbf{u} \in U$. This means that M is a part of a hyperbolic hyperquadric $HP(\mathbf{v}, 0) \cap S_1^n$ and $\mathbf{v}$ is spacelike. This completes the proof. □

Let $g_{ij} = \langle \mathbf{X}_{u_i}, \mathbf{X}_{u_j} \rangle$ be a coefficient of first fundamental quantity and $h_{ij}(\mathbf{n}) = -\langle \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle$ be that of second fundamental quantity. We have a Weingarten-type formula with respect to the spacelike normal vector field $\mathbf{n}$ of spacelike submanifolds.

$$\pi_p^t \circ \mathbf{n}_{u_i}(\mathbf{u}) = -\sum_{k=1}^{n-r}(h_{ij}(\mathbf{n})(\mathbf{u}))(g^{jk}(\mathbf{u}))\mathbf{X}_{u_k}(\mathbf{u}),$$

where π_p^t is an orthogonal projection $\pi_p^t : T_p\mathbb{R}_1^{n+1} \longrightarrow T_pM$. Therefore we have the following expression of the spacelike Gauss-Kronecker curvature.

$$K(\mathbf{n})(\mathbf{u}) = \det(h_{ij}(\mathbf{n})(\mathbf{u}))/\det(g_{ij}(\mathbf{u})).$$

If $\mathbf{n}$ and $\mathbf{n}'$ are parallel spacelike unit normal vector fields with $\mathbf{n}(\mathbf{u}_0) = \mathbf{n}'(\mathbf{u}_0) = \mathbf{n}_0$, then the corresponding second fundamental forms are coincide. This means that the spacelike Gauss-Kronecker curvature and spacelike principal curvatures are determined by the spacelike normal direction $\mathbf{n}_0$ at each point on the spacelike submanifolds M, so we may write $K_{p_0}(\mathbf{n}_0)$ and $\kappa_i(\mathbf{n}_0)$.

6.3 Timelike canal hypersurfaces

We now consider timelike canal hypersurfaces. We assume that there exists an unit parallel orthonormal frame of the spacelike submanifold M denoted by

$$\{\mathbf{X}(\mathbf{u}), \mathbf{n}_0(\mathbf{u}), \mathbf{n}_1(\mathbf{u}), \dots, \mathbf{n}_{r-1}(\mathbf{u})\},$$

where $\mathbf{n}_0(\mathbf{u})$ is a timelike unit normal section and others are spacelike unit normal section. Let $\theta \notin (\pi/2)\mathbb{Z}$ be a fixed real number, we define a map $\bar{\mathbf{X}}^t_\theta : U \times S^{r-1}_1 \longrightarrow S^n_1$ by

$$\bar{\mathbf{X}}^t_\theta(\mathbf{u}, \bar{\mu}) = \cos\theta \mathbf{X}(\mathbf{u}) + \sin\theta \mathbf{e}(\mathbf{u}, \bar{\mu}),$$

where $\mathbf{e}(\mathbf{u}, \bar{\mu}) = \sum_{i=0}^{r-1} \mu_i \mathbf{n}_i(\mathbf{u})$ is a spacelike unit normal. We call $\bar{\mathbf{X}}^t_\theta$ a *timelike canal hypersurface* of the spacelike submanifold M and denote its image by CM^t_θ. We remark that $\bar{\mathbf{X}}^t_\theta$ is not always an embedding map.

Proposition 6.6. An element $(\mathbf{u}, \bar{\mu}) \in U \times S^{r-1}_1$ is a singular point of the map $\bar{\mathbf{X}}^t_\theta$ if and only if a value $\cot\theta$ is one of the corresponding spacelike principal curvatures of spacelike submanifold M with respect to the spacelike unit normal direction $\mathbf{e}(\mathbf{u}, \bar{\mu})$.

Proof. Let $(\mathbf{u}, \bar{\mu}) = (u_1, \dots, u_{n-r}, \mu_0, \dots, \mu_{r-1}) \in U \times S^{r-1}_1$, without loss of generality, we assume that $\mu_{r-1} > 0$ and write $\mu_{r-1} = \sqrt{1 + \mu_0^2 - \sum_{k=1}^{r-2} \mu_k^2}$, then $(u_1, \dots, u_{n-r}, \mu_0, \dots, \mu_{r-2})$ is a local coordinate on $U \times S^{r-1}_1$. Partial derivatives of $\bar{\mathbf{X}}^t_\theta$ are

$$\begin{aligned}(\bar{\mathbf{X}}^t_\theta)_{u_i}(\mathbf{u}, \bar{\mu}) &= \cos\theta \mathbf{X}_{u_i}(\mathbf{u}) + \sin\theta \mathbf{e}_{u_i}(\mathbf{u}, \bar{\mu}) \in T_pM, \\ (\bar{\mathbf{X}}^t_\theta)_{\mu_j}(\mathbf{u}, \bar{\mu}) &= \sin\theta \mathbf{e}_{\mu_j}(\mathbf{u}, \bar{\mu}) \in N_pM,\end{aligned}$$

where N_pM is an orthonormal space of M at $p = \mathbf{X}(\mathbf{u})$. Since $\mathbf{n}_0, \cdots, \mathbf{n}_{r-1}$ are linearly independent and $\sin\theta \neq 0$, so that the vectors

$$\mathbf{e}_{\mu_j}(\mathbf{u}, \bar{\mu}) = \mathbf{n}_j(\mathbf{u}) + \frac{\mu_j}{\mu_{r-1}}\mathbf{n}_{r-1}(\mathbf{u}) \quad (j = 0, \ldots, r-2)$$

are also linearly independent for any $(\mathbf{u}, \bar{\mu})$. By the Weingarten-type formula for the spacelike submanifolds, we have

$$\begin{pmatrix} (\bar{\mathbf{X}}_\theta^t)_{u_1} \\ \vdots \\ (\bar{\mathbf{X}}_\theta^t)_{u_{n-r}} \end{pmatrix} = \left(\cos\theta E_n - \sin\theta I\!I_{\mathbf{e}(\mathbf{u},\bar{\mu})} I^{-1}\right) \begin{pmatrix} \mathbf{X}_{u_1} \\ \vdots \\ \mathbf{X}_{u_{n-r}} \end{pmatrix},$$

where E_n is an identity matrix and $I, I\!I_{\mathbf{e}(\mathbf{u},\bar{\mu})}$ are the first and second fundamental quantity matrices $(g_{ij})(\mathbf{u})$ and $(h_{ij}(\mathbf{e}(\mathbf{u}, \bar{\mu})))(\mathbf{u})$ of the spacelike submanifold M. Since $\mathbf{X}$ is an embedding map, the partial derivatives $(\bar{\mathbf{X}}_\theta^t)_{u_i}$ for $i = 1, \ldots, n-r$ are linearly dependent if and only if $\cot\theta \neq 0$ is one of the eigenvalues of the matrix $I\!I_{\mathbf{e}(\mathbf{u},\bar{\mu})} I^{-1}$, this means that the value $\cot\theta$ is a principal curvature of the spacelike submanifold M with respect to the spacelike unit normal direction $\mathbf{e}(\mathbf{u}, \bar{\mu})$. This completes the proof. □

We use the notion of de Sitter height functions and de Sitter Gauss images introduced in [10]. Let $V \subset U \times S_1^{r-1}$ be a regular part of timelike canal hypersurface $\bar{\mathbf{X}}_\theta^t$, and we denote $CM_\theta^t := \bar{\mathbf{X}}_\theta^t(V)$. The de Sitter height function $\bar{H}_\theta^d : V \times S_1^n \longrightarrow \mathbb{R}$ of timelike canal hypersurface is given by

$$\bar{H}_\theta^d((\mathbf{u}, \bar{\mu}), \mathbf{v}) = \langle \bar{\mathbf{X}}_\theta^t(\mathbf{u}, \bar{\mu}), \mathbf{v} \rangle.$$

For any $\mathbf{v}_0 \in S_1^n$, we denote $\bar{h}_{\mathbf{v}_0}^d(\mathbf{u}, \bar{\mu}) = \bar{H}_\theta^d((\mathbf{u}, \bar{\mu}), \mathbf{v}_0)$.

By calculation, the de Sitter Gauss image $\bar{\mathbf{x}}_{CM_\theta^t}^d : V \longrightarrow S_1^n$ of timelike canal hypersurface is given by

$$\bar{\mathbf{x}}_{CM_\theta^t}^d(\mathbf{u}, \bar{\mu}) = \sin\theta \mathbf{X}(\mathbf{u}) - \cos\theta \mathbf{e}(\mathbf{u}, \bar{\mu}).$$

We denote the de Sitter Gauss-Kronecker curvature by $\bar{K}_d(\mathbf{u}, \bar{\mu})$ and de Sitter principal curvatures by $\bar{\kappa}_{d,i}(\mathbf{u}, \bar{\mu})$.

We also observe the shape of the spacelike canal surface. Following proposition gives a reason that $\bar{\mathbf{x}}^d_{CM^t_\theta}$ is analogous notion of the canal surfaces introduced in Euclidean space.

Proposition 6.7. The timelike canal hypersurface $\bar{\mathbf{x}}^d_{CM^t_\theta}$ has $r-1$ principal directions $\partial/\partial\mu_j$ of the same principal curvature $\cot\theta$ at each point $\bar{p}$.

Proof. Let $(u_1,\dots,u_{n-r},\mu_0,\dots,\mu_{r-2})$ be a local coordinate introduced in the proof of the Proposition 6.6. The partial derivatives of $\bar{\mathbf{x}}^d_{CM^t_\theta}$ are

$$\begin{aligned}(\bar{\mathbf{x}}^d_{CM^t_\theta})_{u_i}(\mathbf{u},\bar{\mu}) &= \sin\theta\mathbf{X}_{u_i}(\mathbf{u})-\cos\theta\mathbf{e}_{u_i}(\mathbf{u},\bar{\mu}),\\ (\bar{\mathbf{x}}^d_{CM^t_\theta})_{\mu_j}(\mathbf{u},\bar{\mu}) &= -\cos\theta\mathbf{e}_{\mu_j}(\mathbf{u},\bar{\mu}).\end{aligned}$$

The differential map of the de Sitter Gauss image $\bar{\mathbf{x}}^d_{CM^t_\theta}$ is the de Sitter shape operator A^d_p of the spacelike canal hypersurface $\bar{\mathbf{X}}^t_\theta$. By Proposition 6.2, a representation matrix of A^d_p with respect to the basis $\{\partial/\partial u_i\}$ is given by $(h_{ij})(g_{ij})^{-1}$, where (g_{ij}) and (h_{ij}) are the de Sitter first and second fundamental forms. Since $(\bar{\mathbf{x}}^d_{CM^t_\theta})_{u_i},(\bar{\mathbf{X}}^t_\theta)_{u_i},(\bar{\mathbf{X}}^t_\theta)_{\mu_j}\in T_pM$ and $(\bar{\mathbf{x}}^d_{CM^t_\theta})_{\mu_j}\in N_pM$ for any $i=1,\dots,n-r$ and $j=0,\dots,r-2$, so that we have

$$\begin{aligned}\mathrm{I}=(g_{ij}) &= \left(\begin{array}{c|c}\langle(\bar{\mathbf{X}}^t_\theta)_{u_i},(\bar{\mathbf{X}}^t_\theta)_{u_j}\rangle & \langle(\bar{\mathbf{X}}^t_\theta)_{u_i},(\bar{\mathbf{X}}^t_\theta)_{\mu_j}\rangle\\ \hline \langle(\bar{\mathbf{X}}^t_\theta)_{\mu_i},(\bar{\mathbf{X}}^t_\theta)_{u_j}\rangle & \langle(\bar{\mathbf{X}}^t_\theta)_{\mu_i},(\bar{\mathbf{X}}^t_\theta)_{\mu_j}\rangle\end{array}\right)=\left(\begin{array}{c|c}* & O\\ \hline O & (\sin^2\theta\langle\mathbf{e}_{\mu_i},\mathbf{e}_{\mu_j}\rangle)_{ij}\end{array}\right),\\ \mathrm{II}_{\mathbf{e}(\mathbf{u},\bar{\mu})}=(h_{ij}) &= \left(\begin{array}{c|c}* & O\\ \hline O & (\sin\theta\cos\theta\langle\mathbf{e}_{\mu_i},\mathbf{e}_{\mu_j}\rangle)_{ij}\end{array}\right).\end{aligned}$$

Therefore we have

$$A_p=\left(\begin{array}{c|c}* & O\\ \hline O & \cot\theta E_{r-1}\end{array}\right),$$

where E_{r-1} is an identity matrix of $r-1$ raws and columns. Therefore the de Sitter shape operator A_p has an eigenvalue $\cot\theta$ with multiplicity at least $r-1$, and each vector $\partial/\partial\mu_j$ for $j=0,\dots,r-2$ is the principal direction. This completes the proof. □

On the other hand, the other $n-r$ principal curvatures are not always constant. We also have a following proposition which implies the relations between the spacelike submanifolds and the corresponding timelike canal hypersurfaces.

Proposition 6.8. Let $\bar{\mathbf{x}}^d_{CM^t_\theta}$ be the de Sitter Gauss image of timelike canal hypersurface $\bar{\mathbf{X}}^t_\theta$ and $\kappa_i(\mathbf{v})$ be principal curvatures of spacelike submanifold M with respect to the spacelike normal direction $\mathbf{v}$ at $\mathbf{X}(\mathbf{u})$, then we have

$$\operatorname{rank} d_{(\mathbf{u},\bar{\mu})}\bar{\mathbf{x}}^d_{CM^t_\theta} = (n-1) - \sharp\{i \mid \kappa_i(\mathbf{v})(\mathbf{u}) = \tan\theta\} \geq r-1.$$

Proof. Let $(u_1, \dots, u_{n-r}, \mu_0, \dots, \mu_{r-2})$ be a local coordinate of the regular part $V \subset U \times S^{r-1}_1$. Since $(\bar{\mathbf{x}}^d_{CM^t_\theta})_{\mu_j}(\mathbf{u}, \bar{\mu})$ are linearly independent and orthogonal to $(\bar{\mathbf{x}}^d_{CM^t_\theta})_{u_i}$ for all $i = 1, \dots, n-r$ and $j = 0, \dots, r-2$, so that the de Sitter Gauss image is degenerate if and only if $(\bar{\mathbf{x}}^d_{CM^t_\theta})_{u_i}$ for $i = 1, \dots, n-r$ are linearly dependent. Using a similar argument in the Proposition 6.6, this condition is equivalent to that the value $\tan\theta$ is a principal curvature of the spacelike submanifold M with respect to the spacelike unit normal direction $\mathbf{e}(\mathbf{u}, \bar{\mu})$. This completes the proof. □

Corollary 6.9. Let $\bar{p} = \bar{\mathbf{X}}^t_\theta(\mathbf{u}, \bar{\mu})$ and $p = \mathbf{X}(\mathbf{u})$, then the following conditions are equivalent:

(1) $\bar{p}$ is a singular point of the de Sitter Gauss image $\bar{\mathbf{x}}^d_{CM^t_\theta}$.

(2) $\bar{p}$ is a de Sitter parabolic point of $\bar{\mathbf{x}}^d_{CM^t_\theta}(V)$, that is $\bar{K}_d(\mathbf{u}, \bar{\mu}) = 0$.

(3) $\tan\theta \neq 0$ is one of the spacelike principal curvatures of spacelike submanifold M at p with respect to the spacelike normal direction $\bar{\mathbf{x}}^d_{CM^t_\theta}(\mathbf{u}, \bar{\mu})$.

Therefore each singularity of the de Sitter Gauss image relates to some timelike principal curvature of the spacelike submanifold, which corresponds to the de Sitter parabolic points of the timelike canal hypersurface.

We now back to the spacelike submanifolds and define de Sitter θ-height functions and θ-hypersurfaces. Let $\theta \notin (\pi/2)\mathbb{Z}$ be a real number, we define a map $H_\theta : U \times S^n_1 \longrightarrow \mathbb{R}$ by

$$H_\theta(\mathbf{u}, \mathbf{v}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle - \sin\theta$$

and $h_{\theta,\mathbf{v}}(\mathbf{u}) = H_\theta(\mathbf{u}, \mathbf{v})$. We call H_θ by a *de Sitter θ-height function* of M. We also define a *de Sitter θ-hypersurface* $\mathbf{e}^d_\theta : U \times S^{r-1}_1 \longrightarrow S^n_1$ of spacelike submanifold by

$$\mathbf{e}^d_\theta(\mathbf{u}, \bar{\mu}) = \sin\theta \mathbf{X}(\mathbf{u}) - \cos\theta \mathbf{e}(\mathbf{u}, \bar{\mu}).$$

We remark that $\bar{\mathbf{x}}^d_{CM^t_\theta}$ is a restriction map of $\mathbf{e}^d_\theta$ on V.

Proposition 6.10. Let $\mathbf{X}$ be a spacelike submanifold in de Sitter space. The discriminant set of the de Sitter height function H_θ is the image of de Sitter θ-hypersurface $\mathbf{e}^d_\theta(\mathbf{u}, \bar{\mu})$.

Proof. The discriminant set is given by $D_{H_\theta} = \{\mathbf{v} \in S^n_1 \mid H_\theta(\mathbf{u}, \mathbf{v}) = H_{\theta,u_i}(\mathbf{u}, \mathbf{v}) = 0 \ \forall i\}$. Let $\mathbf{v}$ is an element of D_{H_θ}. Since $\mathbf{X}(\mathbf{u}), \mathbf{X}_{u_i}(\mathbf{u}), \mathbf{n}_j(\mathbf{u})$ are basis of $\mathbb{R}^{n+1}_1$,

$$\mathbf{v} = \alpha\mathbf{X}(\mathbf{u}) + \sum_{i=0}^{n-r} \beta_i \mathbf{X}_i(\mathbf{u}) + \sum_{j=0}^{r-1} \gamma_j \mathbf{n}_j(\mathbf{u}),$$

for some $\alpha, \beta_i, \gamma_j \in \mathbb{R}$. By calculation, $\mathbf{v}$ is an element of D_{H_θ} if and only if $\alpha = \sin\theta$ and $\beta_i = 0$ for all i. The condition $\langle \mathbf{v}, \mathbf{v} \rangle = 1$ implies $-\gamma_0^2 + \sum_{j=1}^{r-1} \gamma_j^2 = \cos^2\theta$. Let $\bar{\mu} = (\mu_0, \dots, \mu_{r-1})$ be $\mu_j = \gamma_j / \sin\theta$ for $j = 0, \dots, r-1$, then we have $\bar{\mu} \in S^{r-1}_1$ and $\mathbf{v} = \sin\theta\mathbf{X}(\mathbf{u}) + \cos\theta\mathbf{e}(\mathbf{u}, \bar{\mu})$. Conversely, let $\mathbf{v} = \sin\theta\mathbf{X}(\mathbf{u}) + \cos\theta\mathbf{e}(\mathbf{u}, \bar{\mu})$ for some μ, then we can check that $\mathbf{v}$ suffices the condition to be an element of the discriminant set of the de Sitter θ-height function. This completes the proof. □

Let $(\mathbf{u}_0, \bar{\mu}_0) \in V$ and $\mathbf{v}_0 = \mathbf{e}^d_\theta(\mathbf{u}_0, \bar{\mu}_0)$, then

$$\operatorname{Hess} h_{\theta,\mathbf{v}_0}(\mathbf{u}_0) = \cos\theta I(\mathbf{u}_0)(\tan\theta \operatorname{id}_{T_pM} - I\!I_{\mathbf{v}_0}(\mathbf{u}_0) I^{-1}(\mathbf{u}_0)),$$

where $I(\mathbf{u}_0)$ is the first fundamental quantity matrix of M at $\mathbf{u}_0 \in U$ and $I\!I_{\mathbf{v}_0}(\mathbf{u}_0)$ is the second fundamental quantity matrix of M with respect to the spacelike normal direction $\mathbf{v}_0$.

We now review the theory of Montaldi to describe the contacts between the spacelike submanifolds and the non-flat hyperbolic hyperquadrics. Let $\mathbf{v}_0 = \mathbf{e}^d_\theta(\mathbf{u}_0, \bar{\mu}_0)$ be a spacelike vector, we consider an immersion germ $g : (U, \mathbf{u}_0) \longrightarrow (S^n_1, p_0)$ and a submersion germ $f_{\theta,\mathbf{v}_0} : (S^n_1, p_0) \longrightarrow (\mathbb{R}, 0)$ defined by $g(\mathbf{u}) = \mathbf{X}(\mathbf{u})$ and $f_{\theta,\mathbf{v}_0}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_0 \rangle - \sin\theta$. We have $f_{\theta,\mathbf{v}_0} \circ g(\mathbf{u}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v}_0 \rangle - \sin\theta = h_{\theta,\mathbf{v}_0}(\mathbf{u})$ and $h^{-1}_{\theta,\mathbf{v}_0}(0) = HP(\mathbf{v}_0, \sin\theta) \cap S^n_1$ is a non-flat hyperbolic hyperquadric. Moreover

$$h_{\theta,\mathbf{v}_0}(\mathbf{u}_0) = \frac{\partial h_{\theta,\mathbf{v}_0}}{\partial u_i}(\mathbf{u}_0) = 0 \qquad \text{for } i = 1, \dots, n-r,$$

so that the non-flat hyperbolic hyperquadric $HP(\mathbf{v}_0, \sin\theta) \cap S_1^n$ is tangent to the spacelike submanifold M at $p_0 = \mathbf{X}(\mathbf{u}_0)$. Therefore we call $HP(\mathbf{v}_0, \sin\theta) \cap S_1^n$ by a *tangent non-flat hyperbolic hyperquadric* of M at p_0.

By the theory of Montaldi, the contact type between the spacelike submanifold M and the tangent non-flat hyperbolic hyperquadric $HP(\mathbf{v}_0, \sin\theta) \cap S_1^n$ is correspond to the $\mathcal{K}$-equivalent class of the de Sitter θ-height function germs $h_{\theta,\mathbf{v}_0}$. Moreover this $\mathcal{K}$-equivalent class preserves the diffeomorphic type class of the zero level set germs $h_{\theta,\mathbf{v}_0}^{-1}(0)$, so we call $h_{\mathbf{v}_{i,0}}^{-1}(0)$ by *tangent non-flat hyperbolic hyperquadric indicatrix germ.*

We also have following proposition.

Proposition 6.11. Let $\mathbf{X}$ be a spacelike submanifolds in de Sitter space, then the de Sitter θ-height function H_θ is a Morse family of hypersurface.

Proof. Let $\mathbf{v}$ be an element of discriminant set of H_θ, it is sufficient to show that $\Delta^* H_\theta = (H_\theta, H_{\theta u_1}, \ldots, H_{\theta u_{n-r}})$ is regular at $(\mathbf{u}, \mathbf{v})$. Let $\mathbf{X}(\mathbf{u}) = (X_0(\mathbf{u}), \ldots, X_n(\mathbf{u}))$ and $(x_0, \ldots, x_n) \in S_1^n$, without loss of generality, we may assume that $v_n > 0$, then $v_n = \sqrt{1 + v_0^2 - \sum_{k=1}^{n-1} v_k^2}$. The Jacobian of $\Delta^* H_\theta$ is given by

$$J\Delta^* H_\theta = \left(\begin{array}{c|c} * & \dfrac{\partial H_\theta}{\partial v_j}(\mathbf{u}, \mathbf{v})_{\, j=0,\ldots,n-1} \\ \hline * & \left(\dfrac{\partial^2 H_\theta}{\partial u_i \partial v_j}(\mathbf{u}, \mathbf{v}) \right)_{\substack{j=0,\ldots,n-1 \\ i=1,\ldots,n-r}} \end{array} \right).$$

By calculation, we have

$$\begin{aligned} \frac{\partial H_\theta}{\partial v_j}(\mathbf{u}, \mathbf{v}) &= \begin{cases} -X_0(\mathbf{u}) + X_n(\mathbf{u}) \dfrac{v_0}{v_n} & \text{if } j = 0 \\ X_j(\mathbf{u}) - X_n(\mathbf{u}) \dfrac{v_j}{v_n} & \text{if } j = 1, \ldots, n-1 \end{cases} \\ \frac{\partial H_\theta}{\partial u_i \partial v_j}(\mathbf{u}, \mathbf{v}) &= \begin{cases} -X_{0,u_i}(\mathbf{u}) + X_{n,u_i}(\mathbf{u}) \dfrac{v_0}{v_n} & \text{if } j = 0 \\ X_{j,u_i}(\mathbf{u}) - X_{n,u_i}(\mathbf{u}) \dfrac{v_j}{v_n} & \text{if } j = 1, \ldots, n-1. \end{cases} \end{aligned}$$

We denote an $(n - r + 1) \times n$ matrix B by $J\Delta^* H_\theta = (* \mid B)$.

On the other hand, $\mathbf{v}, \mathbf{X}(\mathbf{u}), \mathbf{X}_{u_1}(\mathbf{u}), \ldots, \mathbf{X}_{u_{n-r}}(\mathbf{u})$ are linearly independent and a transformation

$$T : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}, \qquad T(x_0, x_1, \ldots, x_n) \longmapsto T(-x_0, x_1, \ldots, x_n)$$

is regular, so that $T\mathbf{v}, T\mathbf{X}(\mathbf{u}), T\mathbf{X}_{u_1}(\mathbf{u}), \ldots, T\mathbf{X}_{u_{n-r}}(\mathbf{u})$ are also linearly independent. Therefore the rank of a matrix

$$C = \begin{pmatrix} -v_0 & v_1 & \cdots & v_n \\ -X_0 & X_1 & \cdots & X_n \\ -X_{0,u_1} & X_{1,u_1} & \cdots & X_{n,u_1} \\ \vdots & \vdots & \ddots & \vdots \\ -X_{0,u_{n-r}} & X_{1,u_{n-r}} & \cdots & X_{n,u_{n-r}} \end{pmatrix}$$

is equal to $n - r + 2$. We now show that we obtain the matrix B from C by operating raws of the matrices. We subtract the first row multiplied by $\mathbf{X}_n(\mathbf{u})/\lambda_n$ from the second row, and then subtract the first row multiplied by $\mathbf{X}_{n,u_k}(\mathbf{u})/\lambda_n$ from the $(2 + k)$-th row for $k = 1, \cdots, n - 2$, then we have

$$\left(\begin{array}{c|c} -v_0 \ v_1 \ \cdots \ v_{n-1} & v_n \\ \hline \multirow{2}{*}{B} & 0 \\ & \vdots \end{array}\right).$$

Since rank of this matrix is $n - r + 2$, so that rank of the matrix B is $n - r + 1$. Therefore rank $J\Delta^* H_\theta$ is $n - r + 1$ at $(\mathbf{u}, \mathbf{v})$. This completes the proof. $\square$

Therefore we may construct a Legendrian immersion germ

$$\mathcal{L}_{H_\theta}(\mathbf{u}, \bar{\mu}, \mathbf{v}) = \left(\mathbf{v}, \left[\frac{\partial H_\theta}{\partial v_1}(\mathbf{u}, \bar{\mu}, \mathbf{v}) : \ldots : \frac{\partial H_\theta}{\partial v_n}(\mathbf{u}, \bar{\mu}, \mathbf{v})\right]\right),$$

where $\Sigma_*(H_\theta) = \{(\mathbf{u}, \bar{\mu}, \mathbf{v}) \mid \mathbf{v} = \mathbf{e}^d_\theta(\mathbf{u}, \bar{\mu}) \text{ for some } \bar{\mu} \in S^{r-1}_1\}$. The generating family of $\mathcal{L}_{H_\theta}$ is H_θ and the wave front set of the Legendrian immersion germ $\mathcal{L}_{H_\theta}$ is the de Sitter θ-hypersurface germ $\mathbf{e}^d_\theta$.

On the other hand, we already considered another Legendrian immersion germ $\mathcal{L}_H$ related to a timelike canal hypersurfaces. The generating family of $\mathcal{L}_{\bar{H}^d}$ is a de Sitter height function

$\bar{H}^d$ of the timelike canal hypersurface. Since the de Sitter Gauss image of CM_θ and de Sitter θ-hypersurface coincide as map germs, we have a following relation with the Legendrian immersion germs.

Proposition 6.12. Let $(\mathbf{u}, \bar{\mu}) \in V$ be a regular point of $\bar{\mathbf{X}}_\theta$ and $\mathbf{v} = \bar{\mathbf{x}}^d_{CM_\theta}(\mathbf{u}, \bar{\mu}) = \mathbf{e}^d_\theta(\mathbf{u}, \bar{\mu})$. Then the Legendrian immersion germs coincide $\mathcal{L}_{\bar{H}^d} = \mathcal{L}_{H_\theta}$ as map germs at $(\mathbf{u}, \bar{\mu}, \mathbf{v})$.

Proof. Let $\mathbf{X} = (\mathbf{X}_0, \ldots, \mathbf{X}_n)$ and $\mathbf{e} = (\mathbf{e}_0, \ldots, \mathbf{e}_n)$, without loss of generality, we assume that $v_n > 0$ for $\mathbf{v} = (v_0, \ldots, v_n) \in S^n_1$. Since $(\mathbf{u}, \bar{\mu})$ is a parameter of the timelike canal hypersurface, so that $\mathcal{L}_{\bar{H}^d}$ is given by

$$\mathcal{L}_{\bar{H}^d}(\mathbf{u}, \bar{\mu}, \mathbf{v}) = \left(\mathbf{v}, \left[\frac{\partial \bar{H}^d}{\partial v_0}(\mathbf{u}, \bar{\mu}, \mathbf{v}) : \ldots : \frac{\partial \bar{H}^d}{\partial v_{n-1}}(\mathbf{u}, \bar{\mu}, \mathbf{v})\right]\right),$$

where $\mathbf{v} = \bar{\mathbf{x}}^d_{CM_\theta}(\mathbf{u}, \bar{\mu})$ coincides $\mathbf{e}^d_\theta(\mathbf{u}, \bar{\mu})$. By calculation, we have

$$\begin{aligned} \frac{\partial \bar{H}^d}{\partial v_0}(\mathbf{u}, \bar{\mu}, \mathbf{v}) &= -\mathbf{X}_n(\mathbf{u})\mathbf{e}_0(\mathbf{u}, \bar{\mu}) + \mathbf{X}_0(\mathbf{u})\mathbf{e}_n(\mathbf{u}, \bar{\mu}), \\ \frac{\partial \bar{H}^d}{\partial v_j}(\mathbf{u}, \bar{\mu}, \mathbf{v}) &= \mathbf{X}_n(\mathbf{u})\mathbf{e}_j(\mathbf{u}, \bar{\mu}) - \mathbf{X}_j(\mathbf{u})\mathbf{e}_n(\mathbf{u}, \bar{\mu}). \qquad (j = 1, \ldots, n-1) \end{aligned}$$

On the other hand, fiber components of $\mathcal{L}_{H_\theta}$ at $\mathbf{v} \in S^n_1$ is given by

$$\begin{aligned} \frac{\partial H_\theta}{\partial v_0}(\mathbf{u}, \bar{\mu}, \mathbf{v}) &= \cos\theta(-\mathbf{X}_n(\mathbf{u})\mathbf{e}_0(\mathbf{u}, \bar{\mu}) + \mathbf{X}_0(\mathbf{u})\mathbf{e}_n(\mathbf{u}, \bar{\mu})), \\ \frac{\partial H_\theta}{\partial v_j}(\mathbf{u}, \bar{\mu}, \mathbf{v}) &= \cos\theta(\mathbf{X}_n(\mathbf{u})\mathbf{e}_j(\mathbf{u}, \bar{\mu}) - \mathbf{X}_j(\mathbf{u})\mathbf{e}_n(\mathbf{u}, \bar{\mu})), \qquad (j = 1, \ldots, n-1) \end{aligned}$$

so that $[\partial \bar{H}^d/\partial v_0, \ldots, \partial \bar{H}^d/\partial v_{n-1}] = [\partial H_\theta/\partial v_0, \ldots, \partial H_\theta/\partial v_{n-1}]$. Therefore $\mathcal{L}_{\bar{H}^d}(\mathbf{u}, \bar{\mu}, \mathbf{v}) = \mathcal{L}_{H_\theta}(\mathbf{u}, \bar{\mu}, \mathbf{v})$. This completes the proof. □

Therefore, for $(\mathbf{u}, \bar{\mu}) \in V$ the Legendrian immersion germ $\mathcal{L}_{\bar{H}^d}$ is Legendre stable if and only if $\mathcal{L}_{H_\theta}$ is Legendre stable.

6.4 Generic properties for the spacelike submanifolds

We now consider the generic condition of the spacelike submanifolds. Let $U \subset \mathbb{R}^{n-r}$ be open, We define the space of spacelike embeddings Sp-Emb(U, S^n_1) with Whitney C^∞-topology. By the arguments in Appendix A, we have the following theorem.

Theorem 6.13. Let $i = 1, 2$ and $\theta_i \notin (\pi/2)\mathbb{Z}$ be a real number. $(\mathbf{X}_i, \mathbf{u}_i)$, $(\bar{\mathbf{X}}^t_{\theta_i}, (\mathbf{u}_i, \bar{\mu}_i))$ be spacelike submanifold germs and corresponding regular timelike canal hypersurface germs for $i = 1, 2$. Suppose that the corresponding Legendrian immersion germs $\mathcal{L}_{H_{\theta_i}}$ are Legendrian stable, then the following conditions are equivalent.

(1) Legendrian immersion germs $\mathcal{L}_{H_{\theta_i}}$ are Legendrian equivalent.

(2) de Sitter θ_i-hypersurfaces $\mathbf{e}^d_{\theta,i}$ are $\mathcal{A}$-equivalent.

(3) Contact types $K(\mathbf{X}_i(U), DH(v_i, sin\theta_i); p_i)$ are equivalent.

(4) de Sitter θ_i-height functions H_{i,θ_i} are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(5) $h_{i,\theta_i,\mathbf{v}_i}$ are $\mathcal{K}$-equivalent.

(6) $Q(h_{i,\theta_i,\mathbf{v}_i})$ are isomorphic as $\mathbb{R}$-algebras.

(7) Legendrian immersion germs $\mathcal{L}_{\bar{H}^d_i}$ are Legendrian equivalent.

(8) de Sitter Gauss images $\bar{\mathbf{x}}^d_{CM_{\theta_i}}$ are $\mathcal{A}$-equivalent.

(9) Contact types $K(\bar{\mathbf{X}}^t_{\theta_i,i}(V), HP(\bar{v}_i, 0) \cap S^n_1; \bar{p}_i)$ are equivalent.

(10) de Sitter height functions $\bar{H}^d_i$ are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(11) $\bar{h}^d_{i,\bar{\mathbf{v}}_i}$ are $\mathcal{K}$-equivalent.

(12) $Q(\bar{h}^d_{i,\bar{\mathbf{v}}_i})$ are isomorphic as $\mathbb{R}$-algebras.

Where $p_i = \mathbf{u}_i$, $\mathbf{v}_i = \mathbf{e}^d_{\theta,i}(\mathbf{u}_i)$ and $\bar{\mathbf{v}}_i = \bar{\mathbf{x}}^d_{CM_{\theta_i}}(\mathbf{u}_i, \bar{\mu}_i)$. In this case, the corresponding tangent flat timelike hyperquadric indicatrix germs $((h^d_{i,\mathbf{v}_i})^{-1}(0), \mathbf{u}_i)$ are diffeomorphic to each other, and the corresponding tangent non-flat hyperbolic hyperquadric indicatrix germs $((h_{i,\theta_i,\mathbf{v}_i})^{-1}(0), \mathbf{u}_i)$ are also diffeomorphic to each other.

Proof. Suppose that $\mathcal{L}_{H_{\theta_i}}$ are Legendrian stable, by similar arguments in the proof of the Theorem 5.10, conditions (1) to (6) are equivalent. By Proposition 6.12, the corresponding

Legendrian immersion germs coincide $\mathcal{L}_{\bar{H}_i^d} = \mathcal{L}_{H_{\theta_i}}$ for $i = 1, 2$ and $\mathcal{L}_{\bar{H}_i^d}$ are also Legendrian stables, so that (1) and (7) are equivalent. Moreover, conditions (7) to (12) are also equivalent. This completes the proof. $\Box$

Part IV Appendix

A Theory of Legendrian singularities

In this appendix, we review a theory of Legendrian singularities. Let k be an integer $n-1$ or n and N be an k-dimensional smooth manifold. We consider N to be one of the sets S_1^n, $LC_{\pm}^*$ or S_+^{n-1}. Let $\pi : PT^*N \longrightarrow N$ be the projective tangent bundles with the canonical structures. Consider the tangent bundle $\tau : T(PT^*N) \longrightarrow PT^*N$ and the differential map $d\pi : T(PT^*N) \longrightarrow TN$ of π. For any $X \in T(PT^*N)$, there exists an element $\alpha \in T^*N$ such that $\tau(X) = [\alpha] \in PT^*N$. For any $x \in N$ and $V \in T_xN$, the property $\alpha(V) = 0$ does not depend on the choice of representative element α. So that we can define a canonical contact structure on PT^*N by

$$K = \{X \in T(PT^*N) \mid \tau(X)(d\pi(X)) = 0\}.$$

On the other hand, for a local coordinate $(U, \mathbf{v})$ on N, where $\mathbf{v} = (v_1, \ldots, v_k)$. We consider the local trivialization $PT^*(U) \equiv U \times P(\mathbb{R}^k)$ and we call $(v_1, \ldots, v_k, [\xi_1, \cdots, \xi_k])$ by homogeneous coordinates, where $[\xi_1 : \cdots : \xi_k]$ are the homogeneous coordinates of the dual projective space $P\mathbb{R}^k$. It is easy to show that $X_\bullet \in K_\bullet^\pm$ if and only if $\sum_{i=1}^k \mu_i \xi_i = 0$, where $\bullet = (\mathbf{v}, [\xi])$ and $d\pi_\bullet^\pm(X_\bullet) = \sum_{i=1}^k \mu_i \partial/\partial v_i \in T_\bullet N$. An immersion $i : L \longrightarrow PT^*(N)$ is said to be a *Legendrian immersion* if $\dim L = k-1$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called the *Legendrian map* and the image $W(i) = \mathrm{Im}(\pi \circ i)$, the *wave front* of i. Moreover, i (or the image of i) is called the *Legendrian lift* of $W(i)$.

Let $F : (\mathbb{R}^{n-1} \times \mathbb{R}^k, (\mathbf{u}_0, \mathbf{v}_0)) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* of hypersurfaces if the map germ $\Delta^*F : (\mathbb{R}^{n-1} \times \mathbb{R}^k, (\mathbf{u}_0, \mathbf{v}_0)) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ defined by

$\Delta^*F = (F, \partial F/\partial u_1, \cdots, \partial F/\partial u_{n-1})$ is non singular. In this case, we have a smooth $(k-1)$-dimensional smooth submanifold, $\Sigma_*(F) = \{(\mathbf{u},\mathbf{v}) \in (\mathbb{R}^{n-1}\times\mathbb{R}^k, (\mathbf{u}_0,\mathbf{v}_0)) \mid \Delta^*F(\mathbf{u},\mathbf{v}) = 0\}$, and the Legendrian immersion germ $\mathcal{L}_F : (\Sigma_*(F), (\mathbf{u}_0,\mathbf{v}_0)) \longrightarrow PT^*\mathbb{R}^k$ defined by

$$\mathcal{L}_F(\mathbf{u},\mathbf{v}) = \left(\mathbf{v}, \left[\frac{\partial F}{\partial v_1}(\mathbf{u},\mathbf{v}) : \cdots : \frac{\partial F}{\partial v_k}(\mathbf{u},\mathbf{v})\right]\right).$$

Then we have the following fundamental theorem of Arnold and Zakalyukin [1, 20].

Proposition A.1 ([1, 20])**.** All Legendrian submanifold germs in $PT^*\mathbb{R}^k$ are constructed by the above method.

We call F a generating family of $\mathcal{L}_F(\Sigma_*(F))$. The wave front is

$$W(\mathcal{L}_F) = \left\{\mathbf{v}\in\mathbb{R}^k \;\middle|\; \exists\mathbf{u}\in\mathbb{R}^{n-1} \text{ such that } F(\mathbf{u},\mathbf{v}) = \frac{\partial F}{\partial u_1}(\mathbf{u},\mathbf{v}) = \cdots = \frac{\partial F}{\partial u_{n-1}}(\mathbf{u},\mathbf{v}) = 0\right\}.$$

We call it the *discriminant set* of F.

We say that Legendrian immersion germs $\mathcal{L}_i : (U_i, \mathbf{u}_i) \longrightarrow (PT^*\mathbb{R}^k, p_i)$ $(i = 1, 2)$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^k, p_1) \longrightarrow (PT^*\mathbb{R}^k, p_2)$ such that H preserves fibers of π and $H(U_1) = U_2$. A Legendrian immersion germ at a point is said to be *Legendrian stable* if for every map with the given germ there are a neighborhood in the space of Legendrian immersions with the Whitney C^∞-topology and a neighborhood of the original point such that each Legendrian map belonging to the first neighborhood has a point in the second neighborhood, at which its germ is Legendrian equivalent to the original germ.

Proposition A.2 (Zakalyukin [21])**.** Let $\mathcal{L}_1, \mathcal{L}_2$ be Legendrian immersion germs such that regular sets of $\pi\circ\mathcal{L}_1$ and $\pi\circ\mathcal{L}_2$ are respectively dense. Then $\mathcal{L}_1, \mathcal{L}_2$ are Legendrian equivalent if and only if corresponding wave front sets $W(\mathcal{L}_1)$ and $W(\mathcal{L}_2)$ are diffeomorphic as set germs.

Let $F_i : (\mathbb{R}^n\times\mathbb{R}^k, (\mathbf{a}_i,\mathbf{b}_i)) \longrightarrow (\mathbb{R}, 0)$ $(i = 1, 2)$ be k-parameter unfoldings of function germs $f_i : (\mathbb{R}^n, \mathbf{a}_i) \longrightarrow (\mathbb{R}, 0)$. We say F_1 and F_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n\times\mathbb{R}^k, (\mathbf{a}_1,\mathbf{b}_1)) \longrightarrow (\mathbb{R}^n\times\mathbb{R}^k, (\mathbf{a}_2,\mathbf{b}_2))$ of the form $\Phi(\mathbf{a},\mathbf{b}) = (\phi_1(\mathbf{a},\mathbf{b}), \phi_2(\mathbf{b}))$ for $(\mathbf{a},\mathbf{b}) \in \mathbb{R}^n\times\mathbb{R}^k$ and a function germ $\lambda : (\mathbb{R}^n\times\mathbb{R}^k, (\mathbf{a}_1,\mathbf{b}_1)) \longrightarrow \mathbb{R}$ such that $\lambda(\mathbf{a}_1,\mathbf{b}_1) \neq 0$ and $F_1(\mathbf{a},\mathbf{b}) = \lambda(\mathbf{a},\mathbf{b})\cdot(F_2\circ\Phi)(\mathbf{a},\mathbf{b})$.

Theorem A.3 (Arnold, Zakalyukin [1, 20])**.** Let $F, G : (\mathbb{R}^n \times \mathbb{R}^k, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be Morse families and denote their Legendrian immersion by $\mathcal{L}_F, \mathcal{L}_G$. Then

(1) $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent if and only if F and G are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(2) $\mathcal{L}_F$ is Legendrian stable if and only if F is $\mathcal{K}$-versal deformation of $f = F\mid_{\mathbb{R}^n \times \{\mathbf{0}\}}$.

Let $g_i^{\pm} : (\mathbb{R}^n, \mathbf{a}_i) \longrightarrow (\mathbb{R}^k, \mathbf{b}_i)$ (for $i = 1, 2$) be map germs. We say g_1 and g_2 are $\mathcal{A}$-equivalent if and only if there exist diffeomorphism germs $\phi : (\mathbb{R}^n, \mathbf{a}_1) \longrightarrow (\mathbb{R}^n, \mathbf{a}_2)$ and $\Phi : (\mathbb{R}^k, \mathbf{b}_1) \longrightarrow (\mathbb{R}^k, \mathbf{b}_2)$ such that $\Phi \circ g_1 = g_2 \circ \phi$.

Let $Q(f)$ be the local ring of a function germ $f : (\mathbb{R}^n, \mathbf{u}_0) \longrightarrow (\mathbb{R}, 0)$ defined by

$$Q(f) = C^\infty_{\mathbf{u}_0}(\mathbb{R}^n)/\langle f \rangle_{C^\infty_{\mathbf{u}_0}(\mathbb{R}^n)},$$

where $C^\infty_{\mathbf{u}_0}(\mathbb{R}^n)$ is the set of local ring of function germs at $\mathbf{u}_0 \in \mathbb{R}^n$.

Proposition A.4. Let $F, G : (\mathbb{R}^n \times \mathbb{R}^k, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be Morse families. Suppose that Legendrian immersion germs $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian stable, then the following conditions are equivalent:

(1) $(W(\mathcal{L}_F), \lambda)$ and $(W(\mathcal{L}_G), \lambda')$ are diffeomorphic as set germs.

(2) $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent.

(3) $Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F\mid_{\mathbb{R}^n \times \{\mathbf{0}\}}$ and $g = G\mid_{\mathbb{R}^n \times \{\mathbf{0}\}}$.

B Contacts of submanifolds

In this appendix we review the theory of contact due to Montaldi [16] to study the contacts of submanifolds.

Let X_i and Y_i (for $i = 1, 2$) be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the contact of X_1 and Y_1 at $\mathbf{y}_1$ is the same type as the contact of X_2 and Y_2 at $\mathbf{y}_2$ if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, \mathbf{y}_1) \longrightarrow (\mathbb{R}^n, \mathbf{y}_2)$ such that $\Phi(X_1) =$

X_2 and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; \mathbf{y}_1) = K(X_2, Y_2; \mathbf{y}_2)$. Function germs $g_1, g_2 : (\mathbb{R}^n, \mathbf{a}_i) \longrightarrow (\mathbb{R}, 0)$ for $i = 1, 2$ are $\mathcal{K}$-*equivalent* if there are a diffeomorphism germ $\Phi : (\mathbb{R}^n, \mathbf{a}_1) \longrightarrow (\mathbb{R}^n, \mathbf{a}_2)$ and a function germ $\lambda : (\mathbb{R}^n, \mathbf{a}_1) \longrightarrow \mathbb{R}$ with $\lambda(\mathbf{a}_1) \neq 0$ such that $f_1 = \lambda \cdot (g_2 \circ \Phi)$. In [16] Montaldi has shown the following theorem.

Theorem B.1 (Montaldi [16])**.** Let X_i and Y_i (for $i = 1, 2$) be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_1, \mathbf{x}_1) \longrightarrow (\mathbb{R}^n, \mathbf{y}_i)$ be immersion germs and $f_i : (\mathbb{R}^n, \mathbf{y}_i) \longrightarrow (\mathbb{R}^p, \mathbf{0})$ be submersion germs with $(Y_i, \mathbf{y}_i) = (f_i^{-1}(\mathbf{0}), \mathbf{y}_i)$. Then $K(X_1, Y_1; \mathbf{y}_1) = K(X_2, Y_2; \mathbf{y}_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

We only consider the case of $p = 1$ and apply this theorem to the contacts submanifolds in de Sitter space.

C Generic properties

In this appendix, we review the generic properties of spacelike hypersurfaces and spacelike submanifolds in de Sitter space. Let n, r be integers with $n - r \geq 1$ and U be a open subset of $\mathbb{R}^{n-r}$. We consider the space of spacelike embeddings Sp-Emb(U, S_1^n) with Whitney C^∞-topology. We define a function $\mathcal{F} : S_1^n \times S_1^n \longrightarrow \mathbb{R}$ and denote $\mathfrak{f}_{\mathbf{v}}(\mathbf{x}) = \mathcal{F}(\mathbf{x}, \mathbf{v})$. We assume that $\mathfrak{f}_{\mathbf{v}}$ is a submersion. For any spacelike submanifolds $\mathbf{X} \in$ Sp-Emb(U, S_1^n), we have $F = \mathcal{F} \circ (\mathbf{X} \times \mathrm{id}_{S_1^n})$. We also have the ℓ-jet extension $j_1^\ell F : U \times S_1^n \longrightarrow J^\ell(U, \mathbb{R})$ defined by $j_1^\ell F(\mathbf{x}, \mathbf{v}) = j^\ell f_{\mathbf{v}}(u)$. We consider the trivialization $J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^\ell(n-1, 1)$. For any submanifold $Q \subset J^\ell(n-1, 1)$, we denote $\widetilde{Q} = U \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 due to Wassermann [19].

Proposition C.1. Let Q be a submanifold of $J^\ell(n-1, 1)$. Then the set

$$T_Q = \{\mathbf{X} \in \text{Sp-Emb}(U, S_1^n) \mid j_1^\ell G \text{ is transversal to } \widetilde{Q}\}$$

is a residual subset of Sp-Emb(U, S_1^n). If Q is a closed subset, then T_Q is open.

We remark that if the function $F_{\mathbf{v}_0}$ is ℓ-determined relative to $\mathcal{K}$, then F is a $\mathcal{K}$-versal deformation if and only if $j_1^\ell F$ is transversal to $\widetilde{\mathcal{K}}^\ell_{f,\mathbf{v}_0}$, where $\widetilde{\mathcal{K}}^\ell_{f,\mathbf{v}_0}$ is the $\mathcal{K}$-orbit through $j^\ell f_{\mathbf{v}_0}(\mathbf{0}) \in J^\ell(n-1,1)$. Applying Theorem A.3, this condition is equivalent to the condition that the corresponding Legendrian immersion germ is Legendrian stable. Moreover, we have the following theorem. (See [1] and Appendix of [7].)

Theorem C.2. if $n \leq 6$, there exists an open subset $\mathcal{O} \subset \text{Sp-Emb}(U, S^n_1)$ such that for any $\mathbf{X} \in \mathcal{O}$, the corresponding Legendrian immersion germ $\mathcal{L}_F$ is Legendrian stable.

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