A remark on the almost global existence theorems of Keel, Smith and Sogge

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Abstract

We shall give a new proof of temporally global existence of small solutions for systems of semi-linear wave equations. Our proof uses the Klainerman-Sideris inequality and a space-time L^2-estimate. We shall also discuss whether the scale-invariant version of the space-time L^2-estimates can hold, and obtain some related estimates. Among other things, we prove that the Keel-Smith-Sogge estimate actually holds in all space dimensions.

Key words: semi-linear wave equation, null condition, global existence, space-time L^2-estimate.
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1 Introduction

We consider the Cauchy problem for a system of semi-linear wave equations

$$\Box u = Q(\partial u) \ \text{ in } \mathbf{R}^{1+3}_{+} \tag{1.1}$$

subject to the smooth, compactly supported initial data

$$u(0) = f, \ \partial_t u(0) = g. \tag{1.2}$$

Here and in the rest of this paper we mean by ∂u the set of all the first derivatives of components of vector-valued functions $u : \mathbf{R}_+^{1+3} \to \mathbf{R}^m$, $m \geq 2$.

In the paper [10] Keel, Smith, and Sogge have devised an intriguing method of the analysis of the Cauchy problem for quadratic, semi-linear wave equations with small data in three space dimensions. The feature of their analysis lies in an efficient use of a temporal integrability estimate of the local energy, and they have given an essentially new proof of the almost global existence theorems of John and Klainerman [9], Klainerman [12], and Klainerman and Sideris [16] in the semi-linear case by using only the invariance of the d'Alembertian under translations and spatial rotations. The purpose of the present paper is to develop the method found in [10] further. We shall give a new proof of temporally global existence of small solutions for systems of semi-linear wave equations by working mainly with the generators of translations and spatial rotations, when the h-th component of the vector function Q is of the form

$$Q^h(\partial u) = \sum_{1\leq i<j\leq m} \sum_{1\leq k<l\leq 3} C_{ij}^{h,kl} Q_{kl}^{ij}(u,u), \quad h = 1, 2, \ldots, m$$

for real constants $C_{ij}^{h,kl}$. Here

$$Q_{kl}^{ij}(u,v) = \partial_k u^i \partial_l v^j - \partial_k v^j \partial_l u^i, \quad 1 \leq k < l \leq 3, \ 1 \leq i < j \leq m. \tag{1.3}$$

When we consider the problem of global existence, some of quadratic nonlinear terms cause the blow-up of small solutions [8], [22]. It is the null condition of Christodoulou [2] and Klainerman [13] that saves small, smooth solutions from blowing up in finite time. Among others, the quadratic term Q_{kl}^{ij} enjoys the feature that its improved time decay can been seen in the exterior region $|x| > t/2$ of the light cone with the help of a decomposition of the spatial derivatives into angular and radial derivatives (see (6.4) and (6.5) below). This is the reason why we consider the class of nonlinear terms as in (1.3).

It is noted here that, in the energy method using the generators of translations and spatial rotations, it is difficult to derive the asymptotic behaviors of solutions over the inner region $|x| < t/2$ of the light cone. To overcome this difficulty and prove global existence, we are hence obliged to make use of the generator of dilation $S = t\partial_t + x \cdot \nabla$. The use of the operator S allows the Klainerman-Sideris inequality (3.1) to come into play in our analysis, and this inequality together with some observations in [3, 4] provides some additional time decay in the inner region $|x| < t/2$.

Though we use the generators of dilation S, we shall work mainly with the generators of translations and spatial rotations. For clear explanations of some technical points it is suitable to give notation used in this paper now. Points in $\mathbf{R}^{1+3}_+ = (0,\infty) \times \mathbf{R}^3$ are denoted by $(x^0, x^1, x^2, x^3) = (t, x)$. In addition to the usual partial differential operators $\partial_\alpha = \partial/\partial x^\alpha$ $(\alpha = 0, \dots, 3)$ with the abbreviation $\partial = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla)$, we shall use the generators of Euclidean rotations $\Omega = (\Omega_{23}, \Omega_{31}, \Omega_{12})$, $\Omega_{jk} = x^j\partial_k - x^k\partial_j$, and the generator of dilation $S = t\partial_t + x \cdot \nabla$. The set of these 8 vector fields is denoted by $\Gamma = (\Gamma_0, \Gamma_1, \dots, \Gamma_7) = (\partial, \Omega, S)$, and we also denote $\Gamma \setminus \{S\}$ by $\bar{\Gamma} = (\Gamma_0, \Gamma_1, \dots, \Gamma_6) = (\partial, \Omega)$. For multi-indices $a = (a_0, \dots, a_7)$ and $b = (b_0, \dots, b_6)$, we denote

$$\Gamma^a = \Gamma_0^{a_0} \cdots \Gamma_7^{a_7}, \quad \bar{\Gamma}^b = \Gamma_0^{b_0} \cdots \Gamma_6^{b_6}. \tag{1.4}$$

The d'Alembertian, which acts on scalar functions $u : \mathbf{R}^{1+3}_+ \to \mathbf{R}$, is denoted by $\Box = \partial_t^2 - \Delta$. If there is no confusion, we also denote by

$$\Box = \mathrm{diag}(\underbrace{\Box, \dots, \Box}_{m})$$

the d'Alembertian acting on vector valued functions $u : \mathbf{R}^{1+3}_+ \to \mathbf{R}^m$. Associated with this operator, the energy is defined by

$$E_1(u(t)) = \frac{1}{2} \sum_{k=1}^{m} \int_{\mathbf{R}^3} (|\partial_t u^k(t,x)|^2 + |\nabla u^k(t,x)|^2) dx. \tag{1.5}$$

We also introduce two types of generalized energy such as

$$E_l(u(t)) = \sum_{\substack{|a|\le l-1\\ a_7\le 1}} E_1(\Gamma^a u(t)), \quad \bar{E}_l(u(t)) = \sum_{|a|\le l-1} E_1(\bar{\Gamma}^a u(t)) \tag{1.6}$$

for $l = 2, 3, \dots$.

The auxiliary norm

$$M_l(u(t)) = \sum_{k=1}^{m} \sum_{|a|=2} \sum_{|b|\le l-2} ||\langle t-r\rangle \partial^a \bar{\Gamma}^b u^k(t)||_{L^2(\mathbf{R}^3)}, \tag{1.7}$$

which appears in Lemma 3.1 as well as the Sobolev-type inequality (2.2), will play an intermediate role in the energy integral argument below. Here and in what follows we use the notation $\langle A\rangle = \sqrt{1+|A|^2}$ for a scalar or vector A. We shall simply denote the $L^p(\mathbf{R}^3)$-norm by $||\cdot||_{L^p}$. Finally, for $a>0$ we mean by $[a]$ the largest integer of all which are less than or equal to a.

The main theorem is as follows.

Theorem 1.1. *Let l be an integer satisfying*

$$\left[\frac{l}{2}\right] + 3 \le l-2. \tag{1.8}$$

There exists a small constant $\varepsilon > 0$ such that if the initial data (1.2) *satisfy*

$$E_l^{1/2}(u(0)) < \varepsilon \tag{1.9}$$

then the problem (1.1)–(1.3) *admits a unique solution u satisfying*

$$E_{l-2}^{1/2}(u(t)) < 2\varepsilon \text{ and } E_l^{1/2}(u(t)) \le E_l^{1/2}(u(0))(1+t)^{C\varepsilon}, \quad 0<t<\infty \tag{1.10}$$

with a constant $C>0$ independent of ε. The solution also satisfies

$$\begin{aligned} &\sum_{\substack{|a|\le l-3\\ a_7\le 1}} ||\langle x\rangle^{-1/2}\partial\Gamma^a u||_{L^2((0,T)\times\mathbf{R}^3)} \\ &\le C\varepsilon(1+\varepsilon\log(2+T))\sqrt{\log(2+T)}, \quad 0<T<\infty \end{aligned} \tag{1.11}$$

with a constant C independent of ε.

Remark. The quantity $E_l^{1/2}(u(0))$ depends on the size of the initial data (f, g). Indeed, for given data (f, g) at $t = 0$ we can calculate the derivatives of the solution u at $t = 0$ up to the l-th order by using the equation (1.1). In this way we can determine $E_l^{1/2}(u(0))$ explicitly.

It should be mentioned that the number of vector fields S is severely limited to at most one in the definition (1.6) of the generalized energy $E_l(u(t))$ which is employed in the a priori estimate of local solutions. This is in accordance with the thought in another paper of Keel, Smith and Sogge. In [11] the method of vector fields is shown effective in the proof of almost global existence of small solutions to initial-boundary value problems for quasi-linear wave equations in a domain exterior to a star-shaped obstacle with a compact, smooth boundary if the number of vector fields S is limited to at most one. In this paper we shall show the global existence of small solutions to the Cauchy problem for the semi-linear wave equation having the nonlinear term (1.3) by the energy method. The main point of this paper is that the operators involved in our proof are somewhat restricted. Namely, we mainly use the generators of translations and spatial rotations (∂, Ω) and we use the generator of dilation S with only a single power. We hope that our present analysis based on the vector fields approach will offer some insight into the study to prove global existence of small solutions to nonlinear wave equations in an exterior domain.

Finally we remark that our argument, which is on the basis of the Klainerman-Sideris inequality and the integrability estimate of the local energy, is general enough to prove global existence of small solutions to the Cauchy problem for quadratic, quasi-linear wave equations in space dimensions $n \geq 4$ [6].

This paper is organized as follows. In the next section some Sobolev-type inequalities as well as space-time L^2-estimates are presented. Section 3 is devoted to the weighted L^2-estimate of local solutions. In Section 4 we shall carry out the energy integral argument for the higher-order energy, and essentially the same computations are applied to the proof of space-time L^2-estimates in Section 5. In the final section we shall obtain a temporally uniform estimate of the lower-order energy and hence complete the proof of the main theorem. In Appendix we shall carry out two investigations concerning a scale-invariant form of Lemma 2.2 and a simple proof of that lemma.

2 Preliminaries

In what follows the commutation relations

$$
\begin{aligned}
&[\partial_\alpha, \Box] = 0,\ \ [\Omega_{ij}, \Box] = 0,\ \ [S, \Box] = -2\Box, \\
&[\Omega_{ij}, \partial_k] = -\delta_{ki}\partial_j + \delta_{kj}\partial_i,\ \ [S, \partial_\alpha] = -\partial_\alpha,\ \ 0 \le \alpha \le 3,\ 1 \le i, j, k \le 3
\end{aligned}
\tag{2.1}
$$

($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise) will be frequently used. We shall need the following Sobolev-type inequalities.

Lemma 2.1. *The following inequalities hold for sufficiently smooth functions $u : \mathbf{R}^{1+3}_+ \to \mathbf{R}^m$.*

$$
\langle r\rangle^{1/2}\langle t-r\rangle|\partial u(t,x)| \le C\bar{E}_2^{1/2}(u(t)) + CM_3(u(t)), \tag{2.2}
$$

$$
\langle r\rangle|\partial u(t,x)| \le C\bar{E}_3^{1/2}(u(t)). \tag{2.3}
$$

The inequality

$$
\langle r\rangle^{1/2}|u(t,x)| \le C\bar{E}_2^{1/2}(u(t)) \tag{2.4}
$$

also holds for sufficiently smooth functions $u : \mathbf{R}^{1+3}_+ \to \mathbf{R}^m$ decaying at the spatial infinity.

Proof. The inequalities (2.3) and (2.4) were shown by Sideris and Tu (see Lemma 6.1 of [24]). The proof of (2.2) starts with the inequality

$$r^{1/2}\Big(\int_{S^2}|v(r\zeta)|^4 dS\Big)^{1/4} \le C||\nabla v||_{L^2} \tag{2.5}$$

for any smooth function v decaying at the spatial infinity (see, e.g., (3.16) of Sideris [23]). Using the Sobolev embedding on the sphere S^2 and taking the commutation

$$[\Omega_{ij},\langle t-r\rangle\partial_k]=\langle t-r\rangle[\Omega_{ij},\partial_k]=\langle t-r\rangle(-\delta_{ki}\partial_j+\delta_{kj}\partial_i)$$

(by $\Omega_{ij}\langle t-r\rangle=0$ and (2.1)) into account, we obtain for all $r>0$

$$\begin{aligned}
&r^{1/2}|\langle t-r\rangle\partial_\alpha u(t,x)|\\
&\le C\sum_{|b|\le 1} r^{1/2}\Big(\int_{S^2}|\Omega^b(\langle t-r\rangle\partial_\alpha u(t,x))|^4 dS\Big)^{1/4}\\
&\le C\sum_{|b|\le 1} r^{1/2}\Big(\int_{S^2}|\langle t-r\rangle\partial_\alpha\Omega^b u(t,x)|^4 dS\Big)^{1/4}\\
&\quad +C\sum_{j=1}^{3} r^{1/2}\Big(\int_{S^2}|\langle t-r\rangle\partial_j u(t,x)|^4 dS\Big)^{1/4}\\
&\le C\sum_{|b|\le 1}||\nabla(\langle t-r\rangle\partial_\alpha\Omega^b u(t))||_{L^2}+C\sum_{j=1}^{3}||\nabla(\langle t-r\rangle\partial_j u(t))||_{L^2}\\
&\le C\sum_{\beta=0}^{3}\sum_{|b|\le 1}||\partial_\beta\Omega^b u(t)||_{L^2}\\
&+C\sum_{|b|\le 1}\sum_{j=1}^{3}||\langle t-r\rangle\partial_\alpha\partial_j\Omega^b u(t)||_{L^2}+\sum_{i,j=1}^{3}||\langle t-r\rangle\partial_i\partial_j u(t)||_{L^2}\\
&\le C\bar{E}_2^{1/2}(u(t))+CM_3(u(t)),
\end{aligned} \tag{2.6}$$

where $\Omega^b=\Omega_{23}^{b_1}\Omega_{31}^{b_2}\Omega_{12}^{b_3}$ for $b=(b_1,b_2,b_3)$. Let us introduce a cut-off function $\Phi\in C_0^\infty(\mathbf{R}^3)$, $\Phi=1$ for $|x|\le 1$, $\Phi=0$ for $|x|\ge 2$. When $|x|\le 1$, we assume without loss of generality

$t \geq 3$. We see

$$
\begin{aligned}
(2.7) \qquad & |\langle t-r\rangle \partial_\alpha u(t,x)| \leq (1+t)|\Phi(x)\partial_\alpha u(t,x)| \leq C(1+t)||\Phi\partial_\alpha u(t)||_{H^2} \\
& \leq C(1+t)\sum_{|b|=1}^{2} ||\partial_x^b(\Phi\partial_\alpha u(t))||_{L^2} \\
& \leq C(1+t)\sum_{|b|=1}^{2} ||\partial_x^b\partial_\alpha u(t)||_{L^2(\{x\in\mathbf{R}^3:|x|<2\})} + C(1+t)\sup_{1<|x|<2}|\partial_\alpha u(t,x)| \\
& \leq C\sum_{|b|=1}^{2} ||\langle t-r\rangle\partial_x^b\partial_\alpha u(t)||_{L^2} + \sup_{1<|x|<2}|\langle t-r\rangle\partial_\alpha u(t,x)| \\
& \leq C\bar{E}_2^{1/2}(u(t)) + CM_3(u(t)),
\end{aligned}
$$

where we have employed the Poincaré inequality at the third inequality and (2.6) at the last inequality. The inequality (2.2) follows from (2.6) and (2.7). □

It should be noted here that the boost operators $L_j := x_j\partial_t + t\partial_j$ are missing from the set Γ of vector fields. This is the main reason why we must refrain from employing the well-known inequality of Klainerman [13]. Part of this difficulty is overcome by the use of the weighted $L^2(\mathbf{R}^3)$-norm $M_l(u(t))$ because time decay can be shown by the Sobolev-type inequality (2.2). Since we are limiting the power of dilation operator S to at most one in the definition of the generalized energy, the time decay shown by (2.2) is not enough to prove the global existence. The key estimate to our result is the space-time L^2-estimate due to Keel, Smith and Sogge [10].

Lemma 2.2. *Suppose that* $u : (0,T)\times\mathbf{R}^3 \to \mathbf{R}$ *solves the wave equation* $\Box u = G$, $u(0) = f$, $\partial_t u(0) = g$. *Then the estimate*

$$
\begin{aligned}
(2.8) \qquad & (\log(2+T))^{-1/2}||\langle x\rangle^{-1/2}\partial u||_{L^2((0,T)\times\mathbf{R}^3)} \\
& \leq C(||\nabla f||_{L^2} + ||g||_{L^2}) + C||G||_{L^1(0,T;L^2(\mathbf{R}^3))}
\end{aligned}
$$

holds.

Remark. We explain here that Lemma 2.2 emerges as the critical case of the estimate of global (in time) integrability of the local energy. Let $\delta > 0$. We draw attention to the fact that, essentially in line with §27 of Mochizuki [20], one can prove by the multiplier method that the solution $u : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ of $\Box u = 0$ with data (f, g) at $t = 0$ satisfies

$$||\langle x \rangle^{-(1/2)-\delta} \partial u||_{L^2(\mathbf{R}\times\mathbf{R}^n)} \leq C(||\nabla f||_{L^2} + ||g||_{L^2}) \tag{2.9}$$

if $n = 1$ or $n \geq 3$. It is also possible by the Duhamel principle to show that the solution $v : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ of $\Box v = G$ with zero data at $t = 0$ satisfies

$$||\langle x \rangle^{-(1/2)-\delta} \partial v||_{L^2(\mathbf{R}\times\mathbf{R}^n)} \leq C||G||_{L^1(\mathbf{R};L^2(\mathbf{R}^n))} \tag{2.10}$$

if $n = 1$ or $n \geq 3$. Actually, as far as 1-D case is concerned, this type of estimate follows from the classical formula

$$u(t, x) = \frac{f(x + t) + f(x - t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy.$$

Since $2\partial_t u = f'(x + t) - f'(x - t) + g(x + t) + g(x - t)$, we have

$$||\partial_t u(\cdot, x)||_{L^2(\mathbf{R})} \leq C(||\nabla f||_{L^2} + ||g||_{L^2}),$$

which yields

$$||\langle x \rangle^{-(1/2)-\delta} \partial_t u||_{L^2(\mathbf{R}\times\mathbf{R})} = ||\langle x \rangle^{-(1/2)-\delta} ||\partial_t u(\cdot, x)||_{L^2_t(\mathbf{R})}||_{L^2_x(\mathbf{R})} \leq C(||\nabla f||_{L^2} + ||g||_{L^2})$$

as desired. The proof for $\partial_x u$ is the same.

In Proposition 2.8 of [18] Metcalfe has discussed global (in time) integrability of the local energy by making use of the space-time Fourier transform, and he has proved

$$||\langle x \rangle^{-(n-1)/4} \partial u||_{L^2(\mathbf{R}\times\mathbf{R}^n)} \leq C(||\nabla f||_{L^2} + ||g||_{L^2})$$

for $n \geq 4$. The present authors have verified that, with slight modifications, the argument of Metcalfe is actually valid for the proof of (2.9) for all $n \geq 1$. Therefore the estimate (2.10) is also true for all $n \geq 1$ by the Duhamel principle. Slightly modifying some arguments of Metcalfe [18]–[19], we shall show in Appendix that not only (2.8) but also a stronger estimate hold for all $n \geq 1$ (see Proposition B.1 in Appendix B below).

3 Weighted L^2-estimates

Since the weighted L^2-norm $M_l(u(t))$ appears on the right-hand side of the Sobolev-type inequality (2.2), it is necessary to bound the weighted norms $M_l(u(t))$ of local solutions by the generalized energies $E_l(u(t))$ in order to complete the energy integral argument. The next inequality is due to Klainerman and Sideris, and it plays a crucial role in our proof.

Lemma 3.1. *Let $\kappa \geq 2$ be an integer. Then for any smooth function $u : \mathbf{R}_+^{1+3} \to \mathbf{R}^m$ the inequality*

$$M_\kappa(u(t)) \leq C E_\kappa^{1/2}(u(t)) + C \sum_{|a| \leq \kappa-2} ||(t+r)\Box\bar{\Gamma}^a u(t)||_{L^2} \tag{3.1}$$

holds when the right-hand side is finite.

Proof. See Lemma 3.1 of Klainerman and Sideris [16] and Lemma 7.1 of Sideris and Tu [24]. □

Lemma 3.2. *Let $\kappa \geq 2$ be an integer and u a smooth local solution of* (1.1)–(1.3)*. Set $\kappa' = [(\kappa-1)/2] + 3$. Then for all $|a| \leq \kappa - 2$*

$$||(t+r)\Box\bar{\Gamma}^a u(t)||_{L^2} \leq C\bar{E}_{\kappa'}^{1/2}(u(t))\bar{E}_{\kappa-1}^{1/2}(u(t)) + CM_{\kappa'}(u(t))\bar{E}_{\kappa-1}^{1/2}(u(t)). \tag{3.2}$$

Proof. The proof of Lemma 3.2 is contained in [4]. We shall give the proof for the sake of completeness. We first recall that all the vector fields of $\bar{\Gamma}$ commute with $\Box$. In view of the commutation between Ω_{ij} and ∂_k (see (2.1)), we have for $h = 1, 2, \dots, m$ and $|a| \le \kappa - 2$

$$
\begin{aligned}
(3.3)\ ||(t+r)\Box\bar\Gamma^a u^h(t)||_{L^2} &\le \sum_{1\le i<j\le m}\sum_{1\le k<l\le 3} ||(t+r)\bar\Gamma^a(C_{ij}^{h,kl}Q_{kl}^{ij}(u,u))||_{L^2} \\
&\le \sum_{1\le i<j\le m}\sum_{1\le k<l\le 3}\Big(||(t+r)\bar\Gamma^a(C_{ij}^{h,kl}\partial_k u^i\cdot\partial_l u^j)||_{L^2} + ||(t+r)\bar\Gamma^a(C_{ij}^{h,kl}\partial_k u^j\cdot\partial_l u^i)||_{L^2}\Big) \\
&\le C\sum_{1\le i<j\le m}\sum_{1\le p,q\le 3}\sum_{|b|+|c|\le|a|} ||(t+r)\partial_p\bar\Gamma^b u^i\cdot\partial_q\bar\Gamma^c u^j||_{L^2}.
\end{aligned}
$$

Assuming $|b| \le [(\kappa-1)/2]$ without loss of generality and using an inequality $t + r \le |t-r| + 2r \le \langle t-r\rangle + 2r$, we have by (2.2) and (2.3)

$$
\begin{aligned}
(3.4)\qquad & ||(t+r)\partial_p\bar\Gamma^b u^i\cdot\partial_q\bar\Gamma^c u^j||_{L^2} \\
&\le ||\langle t-r\rangle\partial_p\bar\Gamma^b u^i\cdot\partial_q\bar\Gamma^c u^j||_{L^2} + 2||r\partial_p\bar\Gamma^b u^i\cdot\partial_q\bar\Gamma^c u^j||_{L^2} \\
&\le C\Big(||\langle t-r\rangle\partial_p\bar\Gamma^b u^i||_{L^\infty} + ||r\partial_p\bar\Gamma^b u^i||_{L^\infty}\Big)||\partial_q\bar\Gamma^c u^j||_{L^2} \\
&\le C\Big(\bar E^{1/2}_{|b|+3}(u(t)) + M_{|b|+3}(u(t))\Big)\bar E^{1/2}_{|c|+1}(u(t)) \\
&\le C\Big(\bar E^{1/2}_{\kappa'}(u(t)) + M_{\kappa'}(u(t))\Big)\bar E^{1/2}_{\kappa-1}(u(t))
\end{aligned}
$$

The proof of Lemma 3.2 has been completed. □

Lemma 3.3. *Suppose that an integer l satisfies* (1.8)*. Set $\mu = l - 2$. Suppose that, for a local smooth solution u of* (1.1)–(1.3)*, the supremum of $E_\mu^{1/2}(u(t))$ on an interval $(0,T)$ is sufficiently small. Then*

$$(3.5)\qquad M_\mu(u(t)) \le CE_\mu^{1/2}(u(t)),\ \ 0 \le t < T$$

and

$$(3.6)\qquad M_l(u(t)) \le CE_l^{1/2}(u(t)),\ \ 0 \le t < T$$

hold.

Remark. This lemma will turn out to be valid if an integer l satisfies $[(l-1)/2] + 3 \le l - 2$. We have assumed (1.8) for the later use.

Proof. Set

$$\varepsilon_0 := \sup_{0\le t<T} E_\mu^{1/2}(u(t)) \tag{3.7}$$

and

$$\mu' = \Big[\frac{\mu-1}{2}\Big]+3,\ \ l' = \Big[\frac{l-1}{2}\Big]+3. \tag{3.8}$$

Recalling (1.8), we see simply but crucially $\mu' \le l' \le \mu \le l$. We first employ (3.1) and (3.2), setting $\kappa \equiv \mu$ in the notation of Lemmas 3.1 and 3.2

$$\begin{aligned} M_\mu(u(t)) &\le CE_\mu^{1/2}(u(t)) + C\sum_{|a|\le\mu-2} ||(t+r)\Box\bar\Gamma^a u(t)||_{L^2} \\ &\le CE_\mu^{1/2}(u(t)) + C\bar E_{\mu'}^{1/2}(u(t))\bar E_{\mu-1}^{1/2}(u(t)) + CM_{\mu'}(u(t))\bar E_{\mu-1}^{1/2}(u(t)) \\ &\le CE_\mu^{1/2}(u(t)) + C\varepsilon_0\bar E_{\mu-1}^{1/2}(u(t)) + C\varepsilon_0 M_{\mu'}(u(t)), \end{aligned} \tag{3.9}$$

which yields (3.5). Returning to (3.1) and (3.2) and recalling $l' \le \mu$, we find

$$\begin{aligned} M_l(u(t)) &\le CE_l^{1/2}(u(t)) + C\sum_{|a|\le l-2} ||(t+r)\Box\bar\Gamma^a u(t)||_{L^2} \\ &\le CE_l^{1/2}(u(t)) + C\bar E_{l'}^{1/2}(u(t))\bar E_{l-1}^{1/2}(u(t)) + CM_{l'}(u(t))\bar E_{l-1}^{1/2}(u(t)) \\ &\le CE_l^{1/2}(u(t)) + C\varepsilon_0\bar E_{l-1}^{1/2}(u(t)), \end{aligned} \tag{3.10}$$

which leads to (3.6). The proof has been completed. □

4 Energy estimates. Higher-order energy

Carrying out estimates of the nonlinear terms carefully, we shall find that the generalized energy in (1.6), which contains at most one dilation operator S, indeed works well in our energy integral argument of global existence. In the course of proof, allowing the higher-order energy to grow polynomially in time but bounding the lower-order energy uniformly in time, we shall accomplish our energy integral argument. In the present paper two kinds of weighted norms $M_l(u(t))$ and $||\langle x\rangle^{-1/2}\partial\Gamma^\alpha u||_{L^2((0,T)\times\mathbf{R}^3)}$ will play intermediate roles.

We shall carry out the energy integral argument for the higher-order energy in this section and send the estimate of the lower-order energy, which requires more careful treatment, to Section 8.

Higher-order energy. Suppose that an integer l satisfies (1.8). For components of the data (f,g) we assume $(f^k,g^k)\in C_0^\infty(\mathbf{R}^3)\times C_0^\infty(\mathbf{R}^3)$ $(k=1,2,\dots,m)$. By the standard local existence theorem a unique smooth solution exists locally in time. We aim at obtaining an estimate of the higher-order energy $E_l(u(t))$. More precisely, the purpose of this section is to prove

Proposition 4.1. *Suppose that l satisfies* (1.8) *and set $\mu=l-2$ as before. Assume $E_l^{1/2}(u(0))<\varepsilon$ for a sufficiently small ε such that $2\varepsilon\le\varepsilon_0$ (see* (3.7) *for ε_0). Suppose that, for the unique solution u to* (1.1)–(1.3)*, T_0 is the supremum of all $T>0$ for which*

$$E_\mu^{1/2}(u(t))<2\varepsilon,\ \ 0\le t<T. \tag{4.1}$$

Then the solution u satisfies

$$E_l(u(t))\le E_l(u(0))(1+t)^{C\varepsilon},\ \ 0<t<T_0 \tag{4.2}$$

for a constant C independent of ε.

Remark. Suppose $T_0<\infty$. By the continuity of $E_\mu^{1/2}(u(t))$ on $[0,T_0]$ as well as the definition of T_0 we see that the maximum of $E_\mu^{1/2}(u(t))$ on the closed interval $[0,T_0]$ is 2ε. In Section 6 it will be shown that $E_\mu^{1/2}(u(t))<2\varepsilon$ on the interval $0\le t\le T_0$, which is the contradiction and hence means that the local solution actually exists for any length of time.

Proof. In what follows, we assume $0\le t<T_0$. By the standard argument we get

$$\frac{d}{dt}E_l(u(t))\le C\sum_{1\le p,q\le 3}\sum_{i,j,k}\sum_{\substack{|a|\le l-1\\ a_7\le 1}}\sum_{\substack{|b|+|c|\le|a|\\ b_7+c_7\le 1}}||\partial_p\Gamma^b u^i\cdot\partial_q\Gamma^c u^j||_{L^2}||\partial_t\Gamma^a u^k||_{L^2}. \tag{4.3}$$

When estimating the right-hand side of (4.3), we may assume $|b| \le [l/2]$ without loss of generality. Here attention is drawn to the fact that the generator of dilation S is included in Γ^a with only a single power. If Γ^b contains the operator S, then we have, noting $|c| \le l-2$ in this case,

$$
\begin{aligned}
(4.4)\qquad & \|\partial_p\Gamma^b u^i \cdot \partial_q\bar{\Gamma}^c u^j\|_{L^2} \\
&\le C\langle t\rangle^{-1}\Big(\|(r\partial_p\Gamma^b u^i)(r^{-1}\langle t-r\rangle\partial_q\bar{\Gamma}^c u^j)\|_{L^2} + \|\langle r\rangle\partial_p\Gamma^b u^i \cdot \partial_q\bar{\Gamma}^c u^j\|_{L^2}\Big) \\
&\le C\langle t\rangle^{-1}\|\langle r\rangle\partial_p\Gamma^b u^i\|_{L^\infty}\Big(\|\frac{1}{r}\langle t-r\rangle\partial_q\bar{\Gamma}^c u^j\|_{L^2} + \|\partial_q\bar{\Gamma}^c u^j\|_{L^2}\Big) \\
&\le C\langle t\rangle^{-1}E^{1/2}_{|b|+3}(u(t))\Big(E^{1/2}_{|c|+1}(u(t)) + M_{|c|+2}(u(t))\Big) \\
&\le C\langle t\rangle^{-1}E^{1/2}_{[l/2]+3}(u(t))\Big(E^{1/2}_{l-1}(u(t)) + M_l(u(t))\Big) \le C\langle t\rangle^{-1}E^{1/2}_{\mu}(u(t))E^{1/2}_l(u(t)).
\end{aligned}
$$

At the third inequality we have used the Hardy inequality. If Γ^b does not contain S, then we get, using (2.2)–(2.3)

$$
\begin{aligned}
(4.5)\qquad & \|\partial_p\bar{\Gamma}^b u^i \cdot \partial_q\Gamma^c u^j\|_{L^2} \\
&\le C\langle t\rangle^{-1}\Big(\|\langle t-r\rangle\partial_p\bar{\Gamma}^b u^i \cdot \partial_q\Gamma^c u^j\|_{L^2} + \|\langle r\rangle\partial_p\bar{\Gamma}^b u^i \cdot \partial_q\Gamma^c u^j\|_{L^2}\Big) \\
&\le C\langle t\rangle^{-1}\Big(\|\langle t-r\rangle\partial_p\bar{\Gamma}^b u^i\|_{L^\infty} + \|\langle r\rangle\partial_p\bar{\Gamma}^b u^i\|_{L^\infty}\Big)\|\partial_q\Gamma^c u^j\|_{L^2} \\
&\le C\langle t\rangle^{-1}\Big(E^{1/2}_{|b|+3}(u(t)) + M_{|b|+3}(u(t))\Big)E^{1/2}_{|c|+1}(u(t)) \\
&\le C\langle t\rangle^{-1}E^{1/2}_{[l/2]+3}(u(t))E^{1/2}_l(u(t)) \le C\langle t\rangle^{-1}E^{1/2}_{\mu}(u(t))E^{1/2}_l(u(t)).
\end{aligned}
$$

We therefore have from (4.1)

$$\frac{d}{dt}E_l(u(t)) \le C\varepsilon\langle t\rangle^{-1}E_l(u(t)),\ \ 0 \le t < T_0,$$

which yields

$$E_l(u(t)) \le E_l(u(0))(1+t)^{C\varepsilon},\ \ 0 \le t < T_0. \tag{4.6}$$

The proof has been completed. □

5 Space-time L^2 estimates

Before proceeding to the estimate of the lower-order energy, we note in this section that the local solution satisfies the following space-time L^2-estimate.

Proposition 5.1. *Under the same assumptions as in Proposition* 4.1 *the local solution satisfies*

$$
\begin{aligned}
&(\log(2+T))^{-1/2} \sum_{\substack{|a|\le\mu-1\\ a_7\le 1}} \|\langle x\rangle^{-1/2}\partial\Gamma^a u\|_{L^2((0,T)\times\mathbf{R}^3)} \\
&\le C\Big(E_\mu^{1/2}(u(0))+\varepsilon^2\log(2+T)\Big),\ T<T_0
\end{aligned}
\tag{5.1}
$$

with a constant $C>0$ independent of ε.

Proof. Taking the commutation relation (2.1) into account, we see by Lemma 2.2 that it suffices to estimate

$$
\sum_{1\le p<q\le 3}\sum_{i,j}\sum_{\substack{|b|+|c|\le\mu-1\\ b_7+c_7\le 1}} \|\partial_p\Gamma^b u^i\cdot\partial_q\Gamma^c u^j\|_{L^1(0,T;L^2(\mathbf{R}^3))}.
\tag{5.2}
$$

Using the two inequalities $E_\mu^{1/2}(u(t))<2\varepsilon$, $M_\mu(u(t))\le CE_\mu^{1/2}(u(t))$, $0<t<T_0$ (see (4.1), (3.5)) and repeating essentially the same computations as in (4.4)–(4.5), we can readily obtain (5.1). We may safely omit the detail. □

6 Lower-order energy

This final section is devoted to the proof of the first inequality in (1.10) for $0\le t\le T_0$, which completes the proof of Theorem 1.1 (see the remark below Proposition 4.1). The starting point is the standard inequality

$$
E_\mu^{1/2}(u(t))\le E_\mu^{1/2}(u(0))+C\sum_{\substack{|a|\le\mu-1\\ a_7\le 1}}\|\Gamma^a Q(\partial u)\|_{L^1(0,t;L^2(\mathbf{R}^3))}.
\tag{6.1}
$$

We divide the second term on the right-hand side into three pieces. It follows from (2.2), (3.6) and (4.2) that for any a with $|a| \le \mu - 1$, $a_7 \le 1$

$$
\begin{aligned}
(6.2)\quad & ||\Gamma^a Q(\partial u(\tau))||_{L^2(\{x\in\mathbf{R}^3: r<\tau/2\})} \\
&\le C \sum_{\substack{1\le p<q\le 3\\ 1\le i,j\le m}} \sum_{\substack{|b|+|c|\le \mu-1\\ b_7\le 1}} \langle\tau\rangle^{-1} ||\langle r\rangle^{-1/2}\partial_p \Gamma^b u^i(\tau)\langle r\rangle^{1/2}\langle \tau - r\rangle \partial_q \bar{\Gamma}^c u^j(\tau)||_{L^2(\{x\in\mathbf{R}^3: r<\tau/2\})} \\
&\le C \sum_{\substack{1\le p<q\le 3\\ 1\le i,j\le m}} \sum_{\substack{|b|+|c|\le \mu-1\\ b_7\le 1}} \langle\tau\rangle^{-1} ||\langle r\rangle^{-1/2}\partial_p \Gamma^b u^i(\tau)||_{L^2} ||\langle r\rangle^{1/2}\langle \tau - r\rangle \partial_q \bar{\Gamma}^c u^j(\tau)||_{L^\infty} \\
&\le C \sum_{\substack{1\le p\le 3\\ 1\le i\le m}} \sum_{\substack{|b|\le \mu-1\\ b_7\le 1}} \langle\tau\rangle^{-1} ||\langle r\rangle^{-1/2}\partial_p \Gamma^b u^i(\tau)||_{L^2} E_l^{1/2}(u(\tau)) \\
&\le C E_l^{1/2}(u(0)) \sum_{\substack{1\le p\le 3\\ 1\le i\le m}} \sum_{\substack{|b|\le \mu-1\\ b_7\le 1}} \langle\tau\rangle^{-1+C\varepsilon} ||\langle r\rangle^{-1/2}\partial_p \Gamma^b u^i(\tau)||_{L^2}.
\end{aligned}
$$

We therefore obtain, using the dyadic decomposition of the interval $(1,t)$ as in Sogge [25]

$$
\begin{aligned}
(6.3)\quad & \sum_{\substack{|a|\le \mu-1\\ a_7\le 1}} \int_1^t ||\Gamma^a Q(\partial u(\tau))||_{L^2(\{x\in\mathbf{R}^3: r<\tau/2\})} d\tau \\
&\le C \sum_{\substack{|a|\le \mu-1\\ a_7\le 1}} E_l^{1/2}(u(0)) \Big(\sum_{i,p} \sum_{j=0}^{N} \int_{2^j}^{2^{j+1}} \langle\tau\rangle^{-1+C\varepsilon} ||\langle r\rangle^{-1/2}\partial_p \Gamma^a u^i(\tau)||_{L^2} d\tau \Big) \\
&\le C\varepsilon \Big(\sum_{j=0}^{\infty} (2^j)^{(-1/2)+C\varepsilon} (\varepsilon + \varepsilon^2 \log(2+2^{j+1}))(\log(2+2^{j+1}))^{1/2} \Big) \\
&\le C\varepsilon^2.
\end{aligned}
$$

Here we have abused the notation to mean $2^{N+1} = T_0$.

On the other hand, it is the estimate of $||\Gamma^a Q(\partial u(\tau))||_{L^2(\{x\in\mathbf{R}^3: r>\tau/2\})}$ where the exact form of Q^{ij}_{kl} comes into play. Employing the decomposition of spatial derivatives

$$
(6.4)\qquad \nabla = \frac{x}{r}\partial_r - \frac{x}{r^2}\wedge\Omega,\ \ (\Omega = (\Omega_{23}, \Omega_{31}, \Omega_{12}))
$$

as in Klainerman and Sideris [16], Sideris and Tu [24], we observe that

$$
(6.5)\quad |Q^{ij}_{kl}(u(\tau,x), u(\tau,x))| \le \frac{C}{\langle\tau\rangle}(|\nabla u^i(\tau,x)||\Omega u^j(\tau,x)| + |\nabla u^j(\tau,x)||\Omega u^i(\tau,x)|)
$$

for $|x|>\tau/2$ with $\tau>1$, where it is worthwhile mentioning that the gain of some additional decay is seen without the generator of dilation S. Making use of the commutation relations

$$[\Omega_{ij},\partial_k]=-\delta_{ki}\partial_j+\delta_{kj}\partial_i \ \text{ and } \ [S,\partial_k]=-\partial_k \tag{6.6}$$

(recall (2.1)), we have for any a with $|a|\le\mu-1$, $a_7\le 1$

$$\begin{aligned}
&||\Gamma^a Q(\partial u(\tau))||_{L^2(\{x\in\mathbf{R}^3:r>\tau/2\})}\\
&\le C\sum_{\substack{1\le p\le 3\\ 1\le i,j\le m}}\ \sum_{\substack{|b|+|c|\le\mu-1\\ b_7\le 1}}\ \sum_{|d|=1}\langle\tau\rangle^{-1}\Big(||\partial_p\Gamma^b u^i(\tau)\Omega^d\bar\Gamma^c u^j(\tau)||_{L^2(\{x\in\mathbf{R}^3:r>\tau/2\})}\\
&\qquad\qquad\qquad\qquad\qquad +||\Omega^d\Gamma^b u^i(\tau)\partial_p\bar\Gamma^c u^j(\tau)||_{L^2(\{x\in\mathbf{R}^3:r>\tau/2\})}\Big)\\
&\le C\sum_{|b|+|c|\le\mu-1}\langle\tau\rangle^{-3/2}\Big(E^{1/2}_{|b|+1}(u(\tau))\bar E^{1/2}_{|c|+3}(u(\tau))+E^{1/2}_{|b|+3}(u(\tau))\bar E^{1/2}_{|c|+1}(u(\tau))\Big)\\
&\le C\langle\tau\rangle^{-3/2}E_l^{1/2}(u(\tau))E_\mu^{1/2}(u(\tau)),\ \ \tau>1.
\end{aligned}\tag{6.7}$$

At the second inequality we have used (2.4). Recalling (4.6), we then obtain

$$\begin{aligned}
&\sum_{\substack{|a|\le\mu-1\\ a_7\le 1}}\int_1^t ||\Gamma^a Q(\partial u(\tau))||_{L^2(\{x\in\mathbf{R}^3:r>\tau/2\})}d\tau\\
&\le CE_l^{1/2}(u(0))\int_1^t\langle\tau\rangle^{-(3/2)+C\varepsilon}E_\mu^{1/2}(u(\tau))d\tau\\
&\le CE_l^{1/2}(u(0))\sup_{t<T_0}E_\mu^{1/2}(u(t))\le C\varepsilon^2,\ \ t<T_0.
\end{aligned}\tag{6.8}$$

Finally, a simple observation yields

$$\sum_{\substack{|a|\le\mu-1\\ a_7\le 1}}\int_0^1 ||\Gamma^a Q(\partial u(\tau))||_{L^2}d\tau\le C\varepsilon^2. \tag{6.9}$$

Combining (6.3), (6.8)–(6.9) with (6.1), we have obtained

$$E_\mu^{1/2}(u(t))\le\varepsilon+C\varepsilon^2,\ \ 0<t<T_0 \tag{6.10}$$

for a constant $C > 0$ independent of ε. Since ε is sufficiently small, we can conclude $E_\mu^{1/2}(u(t)) \le 3\varepsilon/2$ for $t \in [0, T_0]$. The proof of the first inequality of (1.10) has been finished.

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7 Appendix A

The authors have received a number of helpful suggestions from the referee. Among them we are grateful for the suggestion that a scale-invariant form of Lemma 2.2 should be devised and a simple proof of that lemma should be given. The appendix goes into detail on these problems. In this section we discuss the problem on the scale-invariant version of Lemma 2.2.

Let us consider the Cauchy problem $\Box u = 0$ with data (f, g) at $t = 0$ in three space dimensions. We start with the two examples which show that the scale-invariant form of Lemma 2.2 such as

$$\| \, |x|^{-1/2} \partial u \, \|_{L^2((0,\infty)\times\mathbf{R}^3)} \le C(\|\nabla f\|_{L^2(\mathbf{R}^3)} + \|g\|_{L^2(\mathbf{R}^3)}) \tag{7.1}$$

is false in general.

Proposition A.1. (i) *Assume that a non-negative and spherically symmetric function $g \in C^2(\mathbf{R}^3)$ satisfies* $\operatorname{supp} g \subset \{\, x \in \mathbf{R}^3 \,:\, |x| \le 1 \,\}$ *and* $||g||_{L^2(\mathbf{R}^3)} = 1$. *Define* $g_\varepsilon(x) := \varepsilon^{-3/2} g(\varepsilon^{-1}x)$ *so that* $||g_\varepsilon||_{L^2(\mathbf{R}^3)} = 1$ *for all* $\varepsilon > 0$. *Let* u_ε *be the solution to* $\Box u_\varepsilon = 0$ *with* $(u_\varepsilon(0,x), \partial_t u_\varepsilon(0,x)) = (0, g_\varepsilon(x))$. *The solution* u_ε *satisfies*

$$|| \, |x|^{-1/2} \partial_t u_\varepsilon \, ||_{L^2((0,1)\times B)} \ge \frac{1}{2} \sqrt{\log\Big(1 + \frac{1}{\varepsilon}\Big)} \tag{7.2}$$

for all $0 < \varepsilon < 1$, *where* $B := \{\, x \in \mathbf{R}^3 \,:\, |x| < 2 \,\}$.

(ii) *Assume that a non-negative and spherically symmetric* $g \in C^2(\mathbf{R}^3)$ *satisfies* $\operatorname{supp} g \subset \{\, x \in \mathbf{R}^3 \,:\, |x| \le R \,\}$ *for some* $R > 0$. *Let* u *be the solution to* $\Box u = 0$ *with* $(u(0,x), \partial_t u(0,x)) = (0, g(x))$. *The solution* u *satisfies*

$$|| \, |x|^{-1/2} \partial_t u \, ||_{L^2((0,T)\times D)} \ge \frac{1}{2} ||g||_{L^2(\mathbf{R}^3)} \sqrt{\log\Big(\frac{R+T}{R+2}\Big)} \tag{7.3}$$

for all $T \ge 2$, *where* $D := \{\, x \in \mathbf{R}^3 \,:\, |x| > 2 \,\}$.

Proof. Let us start with the proof of (i). Since g is spherically symmetric, we may write $g(x)$ as $\psi(r)$ ($r := |x|$). We also denote $\psi_\varepsilon(r) := \varepsilon^{-3/2}\psi(r/\varepsilon)$.

It is known that u_ε is represented as

$$u_\varepsilon(t,x) = \frac{1}{2r} \int_{|r-t|}^{r+t} \rho \psi_\varepsilon(\rho) d\rho.$$

We easily see for $r > t$

$$\partial_t u_\varepsilon(t,x) = \frac{1}{2r}\Big((r+t)\psi_\varepsilon(r+t) + (r-t)\psi_\varepsilon(r-t)\Big) \ge \frac{1}{2r}(r-t)\psi_\varepsilon(r-t) \ge 0,$$

which yields

$$\begin{aligned} &|| \, |x|^{-1/2} \partial_t u_\varepsilon \, ||^2_{L^2((0,1)\times B)} \\ &= \int_0^1 dt \int_B |x|^{-1} |\partial_t u_\varepsilon|^2 dx = \int_0^1 dt \int_0^2 r^{-1} |\partial_t u_\varepsilon|^2 r^2 dr \int_{S^2} dS \end{aligned} \tag{7.4}$$

$$
\begin{aligned}
&\geq \int_0^1 dt \int_t^2 r^{-1}|\partial_t u_\varepsilon|^2 r^2 dr \int_{S^2} dS \\
&\geq \int_0^1 dt \int_t^2 r^{-1}\frac{1}{2^2 r^2}(r-t)^2\psi_\varepsilon^2(r-t) r^2 dr \int_{S^2} dS \\
&= \int_0^1 dt \int_0^{2-t} \frac{1}{4(\rho+t)}\rho^2\psi_\varepsilon^2(\rho)d\rho \int_{S^2} dS = \int_0^1 dt \int_0^{\varepsilon} \frac{1}{4(\rho+t)}\rho^2\psi_\varepsilon^2(\rho)d\rho \int_{S^2} dS \\
&\geq \int_0^1 \frac{dt}{4(\varepsilon+t)} \int_0^{\varepsilon} \rho^2\psi_\varepsilon^2(\rho)d\rho \int_{S^2} dS = \frac{1}{4}(\log(1+\varepsilon^{-1}))||g_\varepsilon||_{L^2}^2 = \frac{1}{4}\log(1+\varepsilon^{-1})
\end{aligned}
$$

for all $0 < \varepsilon < 1$ as desired.

Let us turn to the proof of (ii). We again denote $\psi(r) := g(x)$. Using the inequality

$$
\partial_t u(t,x) \geq \frac{1}{2r}(r-t)\psi(r-t) \geq 0 \qquad (r > t),
$$

we obtain

$$
\begin{aligned}
(7.5)\qquad & ||\,|x|^{-1/2}\partial_t u\,||_{L^2((0,T)\times D)}^2 \\
&= \int_0^T dt \int_D |x|^{-1}|\partial_t u|^2 dx = \int_0^T dt \int_2^\infty r^{-1}|\partial_t u|^2 r^2 dr \int_{S^2} dS \\
&\geq \int_2^T dt \int_t^\infty r^{-1}|\partial_t u|^2 r^2 dr \int_{S^2} dS \\
&\geq \int_2^T dt \int_t^\infty r^{-1}\frac{1}{2^2 r^2}(r-t)^2\psi^2(r-t) r^2 dr \int_{S^2} dS \\
&= \int_2^T dt \int_0^\infty \frac{1}{4(\rho+t)}\rho^2\psi^2(\rho)d\rho \int_{S^2} dS = \int_2^T dt \int_0^R \frac{1}{4(\rho+t)}\rho^2\psi^2(\rho)d\rho \int_{S^2} dS \\
&\geq \int_2^T \frac{dt}{4(R+t)} \int_0^R \rho^2\psi^2(\rho)d\rho \int_{S^2} dS = \frac{1}{4}||g||_{L^2(\mathbf{R}^3)}^2 \log\Big(\frac{R+T}{R+2}\Big)
\end{aligned}
$$

as desired. The proof has been completed. $\square$

We next intend to give another proof of Lemma 2.2 for spherically symmetric solutions. This insight will be useful later on. Here and in what follows we mean by $\dot{H}_2^s = \dot{H}_2^s(\mathbf{R}^3)$ the homogeneous Sobolev space of order s (see [1] on page 146 for the definition of homogeneous Sobolev spaces).

Proposition A.2. *Suppose that f and g are spherically symmetric functions with $f \in \dot{H}^1_2(\mathbf{R}^3)$, $g \in L^2(\mathbf{R}^3)$. Let u be the solution to $\Box u = 0$ with the initial data $(u(0), \partial_t u(0)) = (f, g)$ at $t = 0$. Let ε be an arbitrary positive number. For a positive constant C_ε ($C_\varepsilon \to \infty$ as $\varepsilon \to 0$) the estimate*

$$\begin{aligned}(7.6)\qquad & ||\langle x\rangle^{-\varepsilon}|x|^{-(3/2)+\varepsilon}u||_{L^2((0,T)\times\mathbf{R}^3)} + \sum_{\alpha=0}^{3} ||\langle x\rangle^{-\varepsilon}|x|^{-(1/2)+\varepsilon}\partial_\alpha u||_{L^2((0,T)\times\mathbf{R}^3)} \\ & \leq C_\varepsilon A(T)(||\nabla f||_{L^2(\mathbf{R}^3)} + ||g||_{L^2(\mathbf{R}^3)})\end{aligned}$$

holds, where

$$A(T) = \begin{cases} T^\varepsilon & \textit{for} \quad T \leq 1, \\ \\ \big(\log(2+T)\big)^{1/2} & \textit{for} \quad T > 1. \end{cases}$$

Remark. If u is a spherically symmetric solution to the inhomogeneous equation $\Box u = F$ with the radial data (f, g) at $t = 0$ and the forcing term $F(t, x)$ which is also spherically symmetric in the spatial variables, then the estimate

$$\begin{aligned}(7.7)\qquad & ||\langle x\rangle^{-\varepsilon}|x|^{-(3/2)+\varepsilon}u||_{L^2((0,T)\times\mathbf{R}^3)} + \sum_{\alpha=0}^{3} ||\langle x\rangle^{-\varepsilon}|x|^{-(1/2)+\varepsilon}\partial_\alpha u||_{L^2((0,T)\times\mathbf{R}^3)} \\ & \leq C_\varepsilon A(T)(||\nabla f||_{L^2(\mathbf{R}^3)} + ||g||_{L^2(\mathbf{R}^3)} + ||F||_{L^1(0,T;L^2(\mathbf{R}^3))})\end{aligned}$$

holds. For the proof we consider the problem $\Box u_\tau(t, x) = 0$ with data $(u_\tau(\tau, x), \partial_t u_\tau(\tau, x)) = (0, F(\tau, x))$ at $t = \tau$. By the Duhamel principle the solution u is written as

$$u(t, x) = u^0(t, x) + \int_0^t u_\tau(t, x)d\tau,$$

where u^0 satisfies $\Box u^0 = 0$ with data (f, g) at $t = 0$. Since Proposition A.2 gives the estimate of u^0, we may focus our attention on the integral term. Using the Minkowski inequality and noting the fact

$$(7.8)\qquad \partial_t \int_0^t u_\tau(t, x)d\tau = \int_0^t \partial_t u_\tau(t, x)d\tau,$$

we have for $0 < t < T$

$$
\begin{aligned}
(7.9) \qquad & ||\langle x\rangle^{-\varepsilon}|x|^{-(3/2)+\varepsilon}\int_0^t u_\tau(t,x)d\tau||_{L^2((0,T)\times\mathbf{R}^3)} \\
&\quad + \sum_{\alpha=0}^{3}||\langle x\rangle^{-\varepsilon}|x|^{-(1/2)+\varepsilon}\partial_\alpha\int_0^t u_\tau(t,x)d\tau||_{L^2((0,T)\times\mathbf{R}^3)} \\
&\le \int_0^T ||\langle x\rangle^{-\varepsilon}|x|^{-(3/2)+\varepsilon}u_\tau||_{L^2((0,T)\times\mathbf{R}^3)}d\tau \\
&\quad + \sum_{\alpha=0}^{3}\int_0^T ||\langle x\rangle^{-\varepsilon}|x|^{-(1/2)+\varepsilon}\partial_\alpha u_\tau||_{L^2((0,T)\times\mathbf{R}^3)}d\tau \\
&\le C_\varepsilon A(T)\int_0^T ||F(\tau)||_{L^2(\mathbf{R}^3)}d\tau.
\end{aligned}
$$

We have employed Proposition A.2 at the last inequality.

Proof of Proposition A.2. Without loss of generality we may assume $f,\ g \in \mathcal{S}(\mathbf{R}^3)$. Since f and g are smooth and spherically symmetric, we may suppose $f(x) = \varphi(r)$, $g(x) = \psi(r)$ $(r = |x|)$ for even functions $\varphi,\ \psi \in C^\infty(\mathbf{R})$. It is known that the solution $u(t,x)$, which will be denoted by $v(t,r)$ in what follows, is written as

$$
\begin{aligned}
(7.10) \qquad v(t,r) &= \frac{1}{2r}\Big((r+t)\varphi(r+t) + (r-t)\varphi(r-t)\Big) + \frac{1}{2r}\int_{r-t}^{r+t}\rho\psi(\rho)d\rho \\
&= \frac{1}{2r}\int_{t-r}^{t+r}(\rho\varphi(\rho))'d\rho + \frac{1}{2r}\int_{t-r}^{t+r}\rho\psi(\rho)d\rho.
\end{aligned}
$$

Inspired by Klainerman and Machedon [15] and Lindblad and Sogge [17], we shall make use of the boundedness of the Hardy-Littlewood maximal operator (see, e.g., Stein and Weiss [26] on page 58). Observe first

$$
\begin{aligned}
(7.11) \qquad & ||\langle r\rangle^{-\varepsilon}r^{-(3/2)+\varepsilon}\frac{1}{r}\int_{t-r}^{t+r}(\rho\varphi(\rho))'d\rho||^2_{L^2((0,T)\times\mathbf{R}^3)} \\
&= C\int_0^T\int_0^t \langle r\rangle^{-2\varepsilon}r^{-3+2\varepsilon}\left|\frac{1}{r}\int_{t-r}^{t+r}(\rho\varphi(\rho))'d\rho\right|^2 r^2drdt \\
&+ C\int_0^T\int_t^\infty \langle r\rangle^{-2\varepsilon}r^{-3+2\varepsilon}|(t+r)\varphi(t+r)-(t-r)\varphi(t-r)|^2drdt = I_1 + I_2.
\end{aligned}
$$

In what follows a simple inequality $|a+b|^2 \le 2(|a|^2+|b|^2)$ will be sometimes used. We continue the estimate of I_2 by defining $\lambda_\pm = t \pm r$. We have

$$
\begin{aligned}
I_2 &\le C \sum_{\star\in\{+,-\}} \int_0^T \int_t^\infty \langle r\rangle^{-2\varepsilon} r^{-3+2\varepsilon} |\lambda_\star \varphi(\lambda_\star)|^2 dr dt \\
&\le C \sum_{\star\in\{+,-\}} \int_0^T \langle t\rangle^{-2\varepsilon} t^{-1+2\varepsilon} \int_t^\infty |\frac{\lambda_\star}{r}\varphi(\lambda_\star)|^2 dr dt \\
&\le C_\varepsilon A^2(T) ||\varphi||^2_{L^2(\mathbf{R})} \le C_\varepsilon A^2(T) ||\frac{1}{|x|} f||^2_{L^2(\mathbf{R}^3)} \le C_\varepsilon A^2(T) ||\nabla f||^2_{L^2(\mathbf{R}^3)}.
\end{aligned} \tag{7.12}
$$

Here we have employed a simple inequality $|(t\pm r)/r| \le 2$ for $r > t$ and the Hardy inequality.

On the other hand, we shall use the Hardy-Littlewood maximal operator to estimate I_1 of (7.11). We get

$$
\begin{aligned}
I_1 &\le \int_0^T \int_0^t \langle r\rangle^{-2\varepsilon} r^{-1+2\varepsilon} \Big(\frac{1}{r}\int_{t-r}^{t+r} |(\rho\varphi(\rho))'| d\rho\Big)^2 dr dt \\
&\le C \int_0^T \langle r\rangle^{-2\varepsilon} r^{-1+2\varepsilon} \int_r^T \mathcal{M}((\rho\varphi)')^2(t) dt dr \\
&\le C_\varepsilon A^2(T) ||\mathcal{M}((\rho\varphi)')||^2_{L^2(\mathbf{R})} \le C_\varepsilon A^2(T) ||(\rho\varphi)'||^2_{L^2(\mathbf{R})} \\
&\le C_\varepsilon A^2(T) \Big(||\frac{1}{|x|} f||_{L^2(\mathbf{R}^3)} + ||\nabla f||_{L^2(\mathbf{R}^3)}\Big)^2 \le C_\varepsilon A^2(T) ||\nabla f||^2_{L^2(\mathbf{R}^3)}.
\end{aligned} \tag{7.13}
$$

Here, and in what follows, we denote the Hardy-Littlewood maximal operator as

$$
\mathcal{M}(\varphi)(t) = \sup_{\lambda>0} \frac{1}{2\lambda} \int_{t-\lambda}^{t+\lambda} |\varphi(s)| ds
$$

for $\varphi \in L^2(\mathbf{R})$.

As for the integral term involving ψ in (7.10) we proceed, noting that $\rho\psi(\rho)$ is odd in ρ, as

$$
\int_0^T \int_0^\infty \langle r\rangle^{-2\varepsilon} r^{-3+2\varepsilon} \left| \frac{1}{r} \int_{t-r}^{t+r} \rho\psi(\rho) d\rho \right|^2 r^2 dr dt \tag{7.14}
$$

$$
\begin{aligned}
&= \int_0^T \int_0^t \langle r\rangle^{-2\varepsilon} r^{-1+2\varepsilon} \left| \frac{1}{r} \int_{t-r}^{t+r} \rho\psi(\rho)d\rho \right|^2 drdt \\
&\quad + \int_0^T \int_t^\infty \langle r\rangle^{-2\varepsilon} r^{-1+2\varepsilon} \left| \frac{1}{r} \int_{r-t}^{r+t} \rho\psi(\rho)d\rho \right|^2 drdt \\
&\le C \int_0^T \langle r\rangle^{-2\varepsilon} r^{-1+2\varepsilon} \int_r^T \mathcal{M}(\rho\psi)^2(t)dtdr + \int_0^T \langle t\rangle^{-2\varepsilon} t^{-1+2\varepsilon} \int_t^\infty \mathcal{M}(\rho\psi)^2(r)drdt \\
&\le C_\varepsilon A^2(T)||\rho\psi||^2_{L^2(\mathbf{R})} \le C_\varepsilon A^2(T)||g||^2_{L^2(\mathbf{R}^3)}.
\end{aligned}
$$

Thus we have completed the required estimate of $\langle x\rangle^{-\varepsilon}|x|^{-(3/2)+\varepsilon}u$ in the space-time L^2-norm.

Let us turn to the estimate of the first derivatives. The simple identity

$$
\partial_r v = \frac{1}{r}\partial_r(rv) - \frac{1}{r}v \tag{7.15}
$$

is convenient to estimate $\partial_r v$. Since the estimate of $\langle r\rangle^{-\varepsilon} r^{-(1/2)+\varepsilon}(r^{-1}v)$ has been finished, it is enough to estimate $\langle r\rangle^{-\varepsilon} r^{-(1/2)+\varepsilon}(r^{-1}\partial_r(rv))$. Observe first

$$
\begin{aligned}
\partial_r(rv) &= \frac{1}{2}\Big(\varphi(r+t) + (r+t)\varphi'(r+t) + \varphi(r-t) + (r-t)\varphi'(r-t)\Big) \\
&\quad + \frac{1}{2}\Big((r+t)\psi(r+t) - (r-t)\psi(r-t)\Big).
\end{aligned} \tag{7.16}
$$

We get

$$
\begin{aligned}
&\int_0^T \int_0^\infty \langle r\rangle^{-2\varepsilon} r^{-3+2\varepsilon} |\varphi(r\pm t)|^2 r^2 drdt \\
&\le \int_0^T \int_0^t \langle r\rangle^{-2\varepsilon} r^{-1+2\varepsilon} |\varphi(r\pm t)|^2 drdt + \int_0^T \int_t^\infty \langle r\rangle^{-2\varepsilon} r^{-1+2\varepsilon} |\varphi(r\pm t)|^2 drdt \\
&\le \int_0^T \langle r\rangle^{-2\varepsilon} r^{-1+2\varepsilon} \Big(\int_r^T |\varphi(r\pm t)|^2 dt\Big) dr + \int_0^T \langle t\rangle^{-2\varepsilon} t^{-1+2\varepsilon} \Big(\int_t^\infty |\varphi(r\pm t)|^2 dr\Big) dt \\
&\le C_\varepsilon A^2(T)||\varphi||^2_{L^2(\mathbf{R})} \le C_\varepsilon A^2(T)||\nabla f||^2_{L^2(\mathbf{R}^3)}.
\end{aligned} \tag{7.17}
$$

Similarly, we have for $\lambda_\pm = r \pm t$

$$
\sum_{\star\in\{+,-\}} \int_0^T \int_0^\infty \langle r\rangle^{-2\varepsilon} r^{-3+2\varepsilon} \lambda_\star^2 |\varphi'(\lambda_\star)|^2 r^2 drdt \tag{7.18}
$$

$$
\begin{aligned}
&\leq \sum_{\star\in\{+,-\}} \int_0^T \langle r\rangle^{-2\varepsilon} r^{-1+2\varepsilon} \Big(\int_r^T \lambda_\star^2 |\varphi'(\lambda_\star)|^2 dt \Big) dr \\
&+ \sum_{\star\in\{+,-\}} \int_0^T \langle t\rangle^{-2\varepsilon} t^{-1+2\varepsilon} \Big(\int_t^\infty \lambda_\star^2 |\varphi'(\lambda_\star)|^2 dr \Big) dt \\
&\leq C_\varepsilon A^2(T) ||r\varphi'||^2_{L^2(\mathbf{R})} \leq C_\varepsilon A^2(T) ||\nabla f||^2_{L^2(\mathbf{R}^3)}.
\end{aligned}
$$

The estimate of $(r+t)\psi(r+t)$ and $(r-t)\psi(r-t)$ can be done in the same way as above.

The estimate of $\partial_t v$ remains to be done. But, making use of the formula

$$
\begin{aligned}
\partial_t v(t,r) = &\frac{1}{2r}\Big(\varphi(r+t) + (r+t)\varphi'(r+t) - \varphi(r-t) - (r-t)\varphi'(r-t)\Big) \\
&+ \frac{1}{2r}\Big((r+t)\psi(r+t) + (r-t)\psi(r-t)\Big),
\end{aligned}
\tag{7.19}
$$

we can complete the estimate of $\partial_t v$ in the same way as we have done for $r^{-1}\partial_r(rv)$ above. Therefore the proof of Proposition A.2 has been finished. $\square$

Though the scale-invariant version of Lemma 2.2 does not hold as it is, we can nevertheless obtain the following by devising estimates in the weak L^2-space $L^{2,\infty}(\mathbf{R}^3)$ which is strictly larger than $L^2(\mathbf{R}^3)$ (see [1] on page 7 for the definition of weak L^p-spaces).

Proposition A.3. *Suppose that f and g are spherically symmetric functions with $f \in \dot{H}^1_2(\mathbf{R}^3)$, $g \in L^2(\mathbf{R}^3)$. Suppose that $F \in L^1(\mathbf{R}; L^2(\mathbf{R}^3))$ and that F is spherically symmetric in the spatial variables. Let u be the solution to $\Box u = F$ with the initial data $(u(0), \partial_t u(0)) = (f, g)$ at $t = 0$. The solution satisfies*

$$
\begin{aligned}
&||\, |x|^{-3/2} u \,||_{L^{2,\infty}_x(\mathbf{R}^3; L^2_t(\mathbf{R}))} + \sum_{\alpha=0}^{3} ||\, |x|^{-1/2} \partial_\alpha u \,||_{L^{2,\infty}_x(\mathbf{R}^3; L^2_t(\mathbf{R}))} \\
&\leq C(||\nabla f||_{L^2(\mathbf{R}^3)} + ||g||_{L^2(\mathbf{R}^3)}) + C||F||_{L^1_t(\mathbf{R}; L^2_x(\mathbf{R}^3))}.
\end{aligned}
\tag{7.20}
$$

Remark. Let $F = 0$. The inequality

$$
||\, |x|^{-3/2} u \,||_{L^{2,\infty}_x(\mathbf{R}^3; L^2_t(\mathbf{R}))} \leq C(||\nabla f||_{L^2(\mathbf{R}^3)} + ||g||_{L^2(\mathbf{R}^3)})
$$

is actually true even if the solution is not spherically symmetric. See (8.6) below.

The inequality (7.20) is obviously scale-invariant. We shall use the following proposition in the proof of Proposition A.3.

Proposition A.4. *The spherically symmetric solution u described in Proposition A.2 satisfies*

$$(7.21)\ ||u||_{L_t^2(\mathbf{R};L_x^\infty(\mathbf{R}^3))} + \sum_{\alpha=0}^{3} \sup_{x\in\mathbf{R}^3} |x|||\partial_\alpha u(\cdot,x)||_{L_t^2(\mathbf{R})} \le C(||\nabla f||_{L^2(\mathbf{R}^3)} + ||g||_{L^2(\mathbf{R}^3)}).$$

Proof of Proposition A.4. The estimate of u in the $L_t^2 L_x^\infty$-norm is exactly what is called the end-point Strichartz estimate due to Klainerman and Machedon for spherically symmetric solutions [15]. For readers' convenience we give the proof here. Let us use the formula (7.10) to see

$$(7.22)\qquad \int_{\mathbf{R}} \Big(\sup_{r>0} \frac{1}{2r} \Big|\int_{t-r}^{t+r} (\rho\varphi(\rho))' d\rho\Big|\Big)^2 dt \le \int_{\mathbf{R}} \mathcal{M}((\rho\varphi(\rho)')^2(t)dt$$
$$\le C||(\rho\varphi)'||^2_{L^2(\mathbf{R})} \le C\Big(||\frac{1}{|x|}f||_{L^2(\mathbf{R}^3)} + ||\nabla f||_{L^2(\mathbf{R}^3)}\Big)^2 \le C||\nabla f||^2_{L^2(\mathbf{R}^3)},$$

where we have employed the Hardy inequality again. The estimate of the second term on the right-hand side of (7.10) can be done in the same way.

Let us turn to the estimate of the first derivatives in (7.21), which is an analogue of the elementary inequality

$$||\partial_t u||_{L_x^\infty(\mathbf{R};L_t^2(\mathbf{R}))} + ||\partial_x u||_{L_x^\infty(\mathbf{R};L_t^2(\mathbf{R}))} \le C(||\partial_x u(0)||_{L_x^2(\mathbf{R})} + ||\partial_t u(0)||_{L_x^2(\mathbf{R})})$$

for the one-dimensional wave equation. We start with the estimate of $\partial_r v$. Recalling the identity (7.15) and taking the fact $L_t^2(\mathbf{R}; L_x^\infty(\mathbf{R}^3)) \subset L_x^\infty(\mathbf{R}^3; L_t^2(\mathbf{R}))$ into account, we may obviously focus our effort on the estimate of $\partial_r(rv)$ in the $L_x^\infty(\mathbf{R}^3; L_t^2(\mathbf{R}))$-norm. The formula (7.16) is useful again. We see

$$(7.23)\qquad \int_{\mathbf{R}} |\varphi(r\pm t)|^2 dt = ||\varphi||^2_{L^2(\mathbf{R})} = C||\frac{1}{|x|}f||^2_{L^2(\mathbf{R}^3)} \le C||\nabla f||^2_{L^2(\mathbf{R}^3)}$$

and

$$\sum_{\star\in\{+,-\}}\Big(\int_{\mathbf{R}}\lambda_\star^2|\varphi'(\lambda_\star)|^2dt+\int_{\mathbf{R}}\lambda_\star^2|\psi(\lambda_\star)|^2dt\Big)$$
$$=||r\varphi'||^2_{L^2(\mathbf{R})}+||r\psi||^2_{L^2(\mathbf{R})}\le C(||\nabla f||^2_{L^2(\mathbf{R})}+||g||^2_{L^2(\mathbf{R})}). \tag{7.24}$$

Thus we have obtained the required estimate of $\partial_r v$.

The estimate of $\partial_t v$ remains to be shown. Using the formula (7.19), we can show the required estimate of $\partial_t v$ in the same way as above. Therefore the proof of Proposition A.4 has been completed. □

Proof of Proposition A.3. The proof is based on the Duhamel principle. We temporarily assume $F=0$. In this case Proposition A.3 is a direct consequence of Proposition A.4 and the fact $|x|^{-3/2}\in L^{2,\infty}(\mathbf{R}^3)$.

In the rest of the proof we may therefore suppose that $f=g=0$ without loss of generality. Let u_τ satisfy $\Box u_\tau=0$ with $(u_\tau(\tau,x),\partial_t u_\tau(\tau,x))=(0,F(\tau,x))$. The solution is written as

$$u(t,x)=\int_0^t u_\tau(t,x)d\tau.$$

Let $w=w(t)\in L^2(\mathbf{R})$. We have

$$\Big|\int_{\mathbf{R}}\int_0^t u_\tau(t,x)d\tau\bar{w}(t)dt\Big|$$
$$\le\int_0^\infty\int_\tau^\infty|u_\tau(t,x)\bar{w}(t)|dtd\tau+\int_{-\infty}^0\int_{-\infty}^\tau|u_\tau(t,x)\bar{w}(t)|dtd\tau$$
$$\le C\int_{\mathbf{R}}||u_\tau(\cdot,x)||_{L^2_t(\mathbf{R})}d\tau||\bar{w}||_{L^2_t(\mathbf{R})}, \tag{7.25}$$

and hence

$$||u(\cdot,x)||_{L^2_t(\mathbf{R})}\le C\int_{\mathbf{R}}||\sup_{x\in\mathbf{R}^3}|u_\tau(\cdot,x)|\,||_{L^2_t(\mathbf{R})}d\tau\le C\int_{\mathbf{R}}||F(\tau,\cdot)||_{L^2(\mathbf{R}^3)}d\tau. \tag{7.26}$$

Here we have employed the $L_t^2 L_x^\infty$-Strichartz estimate. Because of the fact $|x|^{-3/2} \in L^{2,\infty}(\mathbf{R}^3)$ the required estimate of $|x|^{-3/2}u$ in the $L^{2,\infty}(\mathbf{R}^3; L^2(\mathbf{R}))$-norm is a direct consequence of (7.26).

Let us turn to the estimate of the first derivatives. Since $u_\tau(t,x)$ is spherically symmetric in the spatial variables, we may denote $u_\tau(t,x)$ by $v_\tau(t,r)$. We get

$$
\begin{aligned}
(7.27) \qquad & ||\partial_r \int_0^t v_\tau(t,r)d\tau||_{L_t^2(\mathbf{R})} \\
& \le Cr^{-1} \int_{\mathbf{R}} r||\partial_r v_\tau(\cdot,r)||_{L_t^2(\mathbf{R})} d\tau \le Cr^{-1} \int_{\mathbf{R}} \sup_{r>0} r||\partial_r v_\tau(\cdot,r)||_{L_t^2(\mathbf{R}^3)} d\tau \\
& \le Cr^{-1} \int_{\mathbf{R}} ||F(\tau,\cdot)||_{L^2(\mathbf{R}^3)} d\tau
\end{aligned}
$$

by using Proposition A.4 at the last inequality. The required estimate of $\partial_r v$ is an immediate consequence of (7.27) and the fact $|x|^{-3/2} \in L^{2,\infty}(\mathbf{R}^3)$.

The estimate of $\partial_t u$ remains to be proven. Since

$$
\partial_t u(t,x) = \int_0^t \partial_t v_\tau(t,r) d\tau
$$

as in (7.8), the required estimate of $\partial_t u$ can be obtained in the same way as we have done for $\partial_r u$. Thus the proof has been finished. $\square$

Limiting our interest to radial solutions, we have so far discussed the problem how to obtain scale-invariant version of Lemma 2.2 by using the classical formula (7.10) and the boundedness of the Hardy-Littlewood maximal operator. One of the key points was to replace $L^2(\mathbf{R}^3)$ with a strictly larger space $L^{2,\infty}(\mathbf{R}^3)$. We next discuss the same problem mainly by working with the space-time Fourier transform. In what follows the spatial Fourier transform of $v(x)$ is denoted by $\hat{v}(\xi)$ and the space-time Fourier transform of $v(t,x)$ by $\tilde{v}(\tau,\xi)$. The spaces of functions we shall employ are defined as follows. We know that there exists a function $\Phi \in \mathcal{S}(\mathbf{R})$ such that $\operatorname{supp}\Phi \subset \{\,\tau \,:\, 1/2 \le |\tau| \le 2\,\}$,

$\Phi(\tau) > 0$ for $1/2 < |\tau| < 2$ and

$$\sum_{k=-\infty}^{\infty} \Phi(\tau/2^k) = 1 \qquad (\tau \neq 0)$$

(see, e.g., [1] on pages 135–136). For such a function Φ we denote $\Phi(\tau/2^k)$ by $\Phi_k(\tau)$. We recall the definition of norms of the homogeneous Besov spaces $\dot{B}^1_{2,p}(\mathbf{R}^3)$:

$$||v||_{\dot{B}^1_{2,\infty}(\mathbf{R}^3)} = \sup_{k\in\mathbf{Z}} \Big(\int_{\mathbf{R}^3} \Phi_k^2(|\xi|)|\xi|^2|\hat{v}(\xi)|^2 d\xi \Big)^{1/2}$$
$$||v||_{\dot{B}^1_{2,p}(\mathbf{R}^3)} = \Big(\sum_{k=-\infty}^{\infty} \Big(\int_{\mathbf{R}^3} \Phi_k^2(|\xi|)|\xi|^2|\hat{v}(\xi)|^2 d\xi \Big)^{p/2} \Big)^{1/p} \quad (1 \leq p < \infty).$$

Making use of this dyadic decomposition, we also define $\dot{\mathcal{B}}_p := \{\, v \in \mathcal{S}'(\mathbf{R}^{1+3}) : ||v||_{\dot{\mathcal{B}}_p} < \infty \,\}$ $(1 \leq p \leq \infty)$ where

$$||v||_{\dot{\mathcal{B}}_\infty} = \sup_{j\in\mathbf{Z}} \Big(\int_{D_j} \Big(\sup_{k\in\mathbf{Z}} (\int_{\mathbf{R}} \Phi_k^2(\tau)|\tilde{v}(\tau,\xi)|^2 d\tau)^{1/2} \Big)^2 d\xi \Big)^{1/2} \tag{7.28}$$

$$||v||_{\dot{\mathcal{B}}_p} = \sup_{j\in\mathbf{Z}} \Big(\int_{D_j} \Big(\sum_{k=-\infty}^{\infty} (\int_{\mathbf{R}} \Phi_k^2(\tau)|\tilde{v}(\tau,\xi)|^2 d\tau)^{p/2} \Big)^{2/p} d\xi \Big)^{1/2} \quad (1 \leq p < \infty), \tag{7.29}$$

$D_j = \{\, \xi \in \mathbf{R}^3 : 2^{j-1} < |\xi| < 2^{j+1} \,\}$, $j = 0, \pm 1, \pm 2, \dots$. We see that $\dot{\mathcal{B}}_{p_1} \subset \dot{\mathcal{B}}_{p_2}$ for $1 \leq p_1 < p_2 \leq \infty$ because $l^{p_1} \subset l^{p_2}$.

Using the space-time Fourier transform, we shall show

Proposition A.5. *Suppose* $1 \leq p \leq \infty$. *For any spherically symmetric* $f \in \dot{B}^1_{2,p}(\mathbf{R}^3)$ *the estimate*

$$||\, |x|^{-3/2} e^{\pm it\sqrt{-\Delta}} f \,||_{\dot{\mathcal{B}}_p} \leq C ||f||_{\dot{B}^1_{2,p}(\mathbf{R}^3)} \tag{7.30}$$

holds.

Remark. We note that $\dot{B}^1_{2,2}(\mathbf{R}^3) = \dot{H}^1_2(\mathbf{R}^3)$ and $\dot{B}^1_{2,p_1}(\mathbf{R}^3) \subset \dot{B}^1_{2,p_2}(\mathbf{R}^3)$ $(1 \leq p_1 < p_2 \leq \infty)$. Because $|x|^{-1/2} \in \dot{B}^1_{2,\infty}(\mathbf{R}^3)$, the estimate (7.30) is valid for a class of free, self-similar radial solutions.

Proof of Proposition A.5. It suffices to deal with the operator $e^{it\sqrt{-\Delta}}$ only. We denote $|x|^{-3/2}$ by $w(x)$. By direct computation we see for $\tau > 0$

$$
\begin{aligned}
(7.31)\qquad & (we^{it\sqrt{-\Delta}}f)\tilde{}\,(\tau,\xi) \\
&= C\int_{\mathbf{R}^3} \hat{w}(\xi-\eta)\delta(\tau-|\eta|)\hat{f}(\eta)d\eta = C\int_{S^2} |\xi-\tau\zeta|^{-3/2}\hat{f}(\tau\zeta)\tau^2 dS \\
&= C\int_{S^2} |\xi-\tau\zeta|^{-3/2} dS F(|\tau|)\tau^2 \quad (dS = dS_\zeta,\ \zeta\in S^2)
\end{aligned}
$$

for a suitable F which depends only on $|\tau|$, because $\hat{f}$ is also spherically symmetric. Note that the last integral over S^2 depends only on τ and $|\xi|$. Using the polar coordinate $\zeta = (\sin\varphi\sin\theta, \sin\varphi\cos\theta, \cos\varphi)$ $(0 < \theta < 2\pi,\ 0 < \varphi < \pi)$ and taking $\xi = (0,0,|\xi|)$ without loss of generality, we see for $\tau > 0$

$$
\begin{aligned}
(7.32)\qquad & \int_{S^2} |\xi-\tau\zeta|^{-3/2} dS \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} (|\xi|^2 - 2|\xi|\tau\cos\varphi + \tau^2)^{-3/4}\sin\varphi d\varphi \\
&= C\int_{-1}^{1} (|\xi|^2 - 2|\xi|\tau s + \tau^2)^{-3/4} ds = \frac{C}{|\xi|\tau}\Big(\sqrt{|\xi|+\tau} - \sqrt{||\xi|-\tau|}\Big) \\
&\le \frac{C(|\xi|+\tau - ||\xi|-\tau|)}{|\xi|\tau\Big(\sqrt{|\xi|+\tau} + \sqrt{||\xi|-\tau|}\Big)} \le \frac{C}{|\xi|\sqrt{|\xi|}}.
\end{aligned}
$$

Note that $(we^{it\sqrt{-\Delta}}f)\tilde{} = 0$ if $\tau < 0$. Hence we get by (7.31) and (7.32)

$$
\begin{aligned}
(7.33)\quad & \Big(\int_{\mathbf{R}} \Phi_k^2(\tau)|(we^{it\sqrt{-\Delta}}f)\tilde{}\,(\tau,\xi)|^2 d\tau\Big)^{1/2} \\
&\le C|\xi|^{-3/2}\Big(\int_{\mathbf{R}} \Phi_k^2(\tau)|F(|\tau|)\tau^2|^2 d\tau\Big)^{1/2} = C|\xi|^{-3/2}\Big(\int_{\mathbf{R}^3} \Phi_k^2(|\eta|)|\eta|^2|\hat{f}(\eta)|^2 d\eta\Big)^{1/2}.
\end{aligned}
$$

Taking the l^p-norm of both the sides, we obtain

$$
(7.34)\qquad \sup_{k\in\mathbf{Z}}\Big(\int_{\mathbf{R}} \Phi_k^2(\tau)|(we^{it\sqrt{-\Delta}}f)\tilde{}\,(\tau,\xi)|^2 d\tau\Big)^{1/2} \le C|\xi|^{-3/2}||f||_{\dot{B}^1_{2,\infty}(\mathbf{R}^3)}
$$

and

$$\left(\sum_{k=-\infty}^{\infty}\Big(\int_{\mathbf{R}}\Phi_k^2(\tau)|(we^{it\sqrt{-\Delta}}f)\widetilde{\ }(\tau,\xi)|^2d\tau\Big)^{p/2}\right)^{1/p}\le C|\xi|^{-3/2}||f||_{\dot{B}^1_{2,p}(\mathbf{R}^3)}. \tag{7.35}$$

Since

$$\int_{D_j}(|\xi|^{-3/2})^2d\xi=C\int_{2^{j-1}}^{2^{j+1}}\frac{dr}{r}=C\log 2$$

for a constant independent of j, the inequality (7.30) is a direct consequence of (7.34) and (7.35). $\square$

8 Appendix B

It has been suggested by the referee that a simple proof of Lemma 2.2 should be found. The authors regret to say that they could not find any proof which is essentially simpler than that due to Keel, Smith and Sogge. But they can instead strengthen and sharpen Lemma 2.2. Inspired by a series of papers of Metcalfe [18]–[19] as well as Proposition A.2, we prove the following.

Proposition B.1. *Let $n\ge 1$, $(f,g)\in\mathcal{S}(\mathbf{R}^n)\times\mathcal{S}(\mathbf{R}^n)$ and $\varepsilon>0$. Suppose that u solves the Cauchy problem $\Box u=0$ with the data $(u(0),\partial_t u(0))=(f,g)$. Then this solution satisfies*

$$\sum_{\alpha=0}^{n}||\langle x\rangle^{-\varepsilon}|x|^{-(1/2)+\varepsilon}\partial_\alpha u||_{L^2((0,T)\times\mathbf{R}^n)}\le C_\varepsilon A(T)(||\nabla f||_{L^2}+||g||_{L^2}), \tag{8.1}$$

where $A(T)$ is the same as in Proposition A.2 and the positive constant C_ε grows to infinity as $\varepsilon\to 0$. If $n\ge 3$ in addition, the estimate

$$||\langle x\rangle^{-\varepsilon}|x|^{-(3/2)+\varepsilon}u||_{L^2((0,T)\times\mathbf{R}^n)}\le C_\varepsilon A(T)(||\nabla f||_{L^2}+||g||_{L^2}), \tag{8.2}$$

also holds.

Remarks. (1) Let $F \in L^1((0,T);L^2(\mathbf{R}^n))$. Suppose that u solves the Cauchy problem $\Box u = F$ with zero data at $t=0$. Because the estimate (8.1) obviously remains true for the data $(0,g)$ with $g \in L^2(\mathbf{R}^n)$, we can obtain by the Duhamel principle

$$\sum_{\alpha=0}^{n} ||\langle x\rangle^{-\varepsilon}|x|^{-(1/2)+\varepsilon}\partial_\alpha u||_{L^2((0,T)\times\mathbf{R}^n)} \le C_\varepsilon A(T)\int_0^T ||F(\tau,\cdot)||_{L^2(\mathbf{R}^n)}d\tau$$

(see the remark below Proposition A.2).

(2) It follows from the radial identity of Morawetz [21] (see also [27] on page 11) that

$$|| \, |x|^{-3/2}u \, ||_{L^2((0,\infty)\times\mathbf{R}^n)} \le C(||\nabla f||_{L^2} + ||g||_{L^2}) \tag{8.3}$$

when $n \ge 4$. For $n=3$ that identity yields

$$\int_0^\infty u^2(t,x)dt \le C(||\nabla f||_{L^2}^2 + ||g||_{L^2}^2) \tag{8.4}$$

(see LEMMA of [21] on page 292), which immediately leads to

$$\begin{aligned} ||\langle x\rangle^{-(\varepsilon+\varepsilon')}|x|^{-(3/2)+\varepsilon}u||_{L^2((0,\infty)\times\mathbf{R}^3)} &= ||\langle x\rangle^{-(\varepsilon+\varepsilon')}|x|^{-(3/2)+\varepsilon}||u(\cdot,x)||_{L^2(0,\infty)}||_{L^2(\mathbf{R}^3)} \\ &\le C||\langle x\rangle^{-(\varepsilon+\varepsilon')}|x|^{-(3/2)+\varepsilon}||_{L^2(\mathbf{R}^3)}(||\nabla f||_{L^2}+||g||_{L^2}) \le C(||\nabla f||_{L^2}+||g||_{L^2}) \end{aligned} \tag{8.5}$$

for ε, $\varepsilon' > 0$ and $C = C(\varepsilon,\varepsilon')$ ($C \to \infty$ as $\varepsilon \to 0$ or $\varepsilon' \to 0$), and

$$|| \, |x|^{-3/2}u \, ||_{L^{2,\infty}(\mathbf{R}^3;L^2(0,\infty))} \le C(||\nabla f||_{L^2} + ||g||_{L^2}). \tag{8.6}$$

Our estimate (8.2) for $n=3$ is therefore regarded as the critical (i.e., $\varepsilon'=0$) case of (8.5). Our estimate (8.2) for $n \ge 4$ is weaker than (8.3) because the latter is scale-invariant and global in time. We finally note that our proof of (8.2) remains valid for that of (8.5) without any essential modification.

Our sharp estimates (8.1)–(8.2) have good application to some nonlinear problems (see [5], [7]). Let us return to the proof of Proposition B.1. For the proof the result due to Metcalfe is essential (see Corollary 2.7 of [18]).

Proposition B.2. *Let $R > 0$. The solution u described in Proposition B.1 satisfies*

$$\| \, |x|^{-3/2} u \, \|_{L^2(\{(t,x): 0<t<\infty, \, R<|x|<2R\})} \leq C(\|\nabla f\|_{L^2} + \|g\|_{L^2}) \tag{8.7}$$

for $n \geq 3$ and

$$\sum_{\alpha=0}^{n} \| \, |x|^{-1/2} \partial_\alpha u \, \|_{L^2(\{(t,x): 0<t<\infty, \, R<|x|<2R\})} \leq C(\|\nabla f\|_{L^2} + \|g\|_{L^2}) \tag{8.8}$$

for $n \geq 1$, where $C > 0$ is a constant independent of R.

Admitting Proposition B.2 temporarily, we enter into the proof of Proposition B.1.

Proof of Proposition B.1. For any $T > 0$ we divide $\mathbf{R}^n$ into $\{ x \, : \, |x| < T \}$ and $\{ x \, : \, |x| > T \}$. For the former we separate two cases $T < 1$ or $T > 1$. If $T < 1$, we use (8.8) with $R = 2^{j-1}$ to obtain

$$\begin{aligned} &\|\langle x \rangle^{-\varepsilon} |x|^{-(1/2)+\varepsilon} \partial_\alpha u\|^2_{L^2(\{(t,x):0<t<T, \, 0<|x|<T\})} \\ &\leq C \sum_{j=-\infty}^{-\left[|(\log T)/(\log 2)|\right]} (2^\varepsilon)^{2j} \| \, |x|^{-1/2} \partial_\alpha u \, \|^2_{L^2(\{(t,x):0<t<T, \, 2^{j-1}<|x|<2^j\})} \\ &\leq C_\varepsilon T^{2\varepsilon} (\|\nabla f\|_{L^2} + \|g\|_{L^2})^2 \end{aligned} \tag{8.9}$$

for a constant $C_\varepsilon > 0$ ($C_\varepsilon \to \infty$ as $\varepsilon \to 0$) independent of T. Similarly, we obtain for $n \geq 3$

$$\|\langle x \rangle^{-\varepsilon} |x|^{-(3/2)+\varepsilon} u\|^2_{L^2(\{(t,x):0<t<T, \, 0<|x|<T\})} \leq C_\varepsilon T^{2\varepsilon} (\|\nabla f\|_{L^2} + \|g\|_{L^2})^2. \tag{8.10}$$

On the other hand, if $T > 1$, then we first note that

$$\begin{aligned} &\|\langle x \rangle^{-\varepsilon} |x|^{-(1/2)+\varepsilon} \partial_\alpha u\|^2_{L^2(\{(t,x):0<t<T, \, 0<|x|<T\})} \\ &\leq \|\langle x \rangle^{-\varepsilon} |x|^{-(1/2)+\varepsilon} \partial_\alpha u\|^2_{L^2(\{(t,x):0<t<T, \, 0<|x|<1\})} \\ &+ \|\langle x \rangle^{-\varepsilon} |x|^{-(1/2)+\varepsilon} \partial_\alpha u\|^2_{L^2(\{(t,x):0<t<T, \, 1<|x|<T\})} \equiv I_1 + I_2. \end{aligned} \tag{8.11}$$

We get $I_1 \le C_\varepsilon(||\nabla f||_{L^2} + ||g||_{L^2})^2$ in the same way as in (8.9). For I_2 we use (8.8) with $R = 2^{j-1}$ to get

$$
\begin{aligned}
(8.12) \qquad I_2 &\le \sum_{j=1}^{\left[(\log T)/(\log 2)\right]+1} || \, |x|^{-1/2}\partial_\alpha u \, ||^2_{L^2(\{(t,x):0<t<T, \, 2^{j-1}<|x|<2^j\})} \\
&\le C(\log(2+T))(||\nabla f||_{L^2} + ||g||_{L^2})^2.
\end{aligned}
$$

The estimate of

$$
||\langle x\rangle^{-\varepsilon}|x|^{-(3/2)+\varepsilon}u||_{L^2(\{(t,x):0<t<T, \, 0<|x|<T\})} \qquad (n \ge 3)
$$

can be carried out in the same way as in (8.11)–(8.12).

We turn to the set $\{ \, x \, : \, |x| > T \, \}$. We easily obtain

$$
\begin{aligned}
(8.13) \qquad & ||\langle x\rangle^{-\varepsilon}|x|^{-(1/2)+\varepsilon}\partial_\alpha u||_{L^2(\{(t,x):0<t<T, \, |x|>T\})} \\
&\le T^{-1/2}||\partial_\alpha u||_{L^2(\{(t,x):0<t<T, \, x\in\mathbf{R}^n\})} \\
&\le \sup_{0<t<T}||\partial_\alpha u(t,\cdot)||_{L^2} \le C(||\nabla f||_{L^2} + ||g||_{L^2}).
\end{aligned}
$$

For the estimate of u itself over $\{ \, x \, : \, |x| > T \, \}$, we may use the Hardy inequality by the assumption $n \ge 3$. We obtain

$$
\begin{aligned}
(8.14) \qquad & ||\langle x\rangle^{-\varepsilon}|x|^{-(3/2)+\varepsilon}u||_{L^2(\{(t,x):0<t<T, \, |x|>T\})} \\
&\le T^{-1/2}||\frac{1}{|x|}u||_{L^2(\{(t,x):0<t<T, \, x\in\mathbf{R}^n\})} \\
&\le CT^{-1/2}||\nabla u||_{L^2(\{(t,x):0<t<T, \, x\in\mathbf{R}^n\})} \le C(||\nabla f||_{L^2} + ||g||_{L^2}).
\end{aligned}
$$

The proof of Proposition B.1 has been finished. $\square$

Proposition B.2 remains to be proven. It should be noted that this proposition has been essentially shown by Metcalfe [18]. For clarity of the exposition we give its complete proof here. We first prove the following two propositions.

Proposition B.3. *Let $n \geq 3$ and $\beta \in \mathcal{S}(\mathbf{R}^n)$. The inequality*

$$\sup_{\xi \in \mathbf{R}^n} |\xi|^2 \int_{\mathbf{R}^n_\eta} |\hat{\beta}(\xi - \eta)| \delta(\tau - |\eta|) d\eta \leq C\tau^2 \tag{8.15}$$

holds for all $\tau \geq 0$.

Proposition B.4. *Let $n \geq 1$ and $\beta \in \mathcal{S}(\mathbf{R}^n)$. The inequality*

$$\sup_{\xi \in \mathbf{R}^n} \int_{\mathbf{R}^n_\eta} |\hat{\beta}(\xi - \eta)| \delta(\tau - |\eta|) d\eta \leq C \tag{8.16}$$

holds for all $\tau \geq 0$.

Proof of Proposition B.3. We closely follow Metcalfe [19]. Using the polar coordinates, we see

$$\begin{aligned} &\int_{\mathbf{R}^n_\eta} |\hat{\beta}(\xi - \eta)| \delta(\tau - |\eta|) d\eta \\ &= \int_{S^{n-1}_\zeta} \int_0^\infty |\hat{\beta}(\xi - r\zeta)| \delta(\tau - r) r^{n-1} dr dS = \tau^{n-1} \int_{S^{n-1}_\zeta} |\hat{\beta}(\xi - \tau\zeta)| dS. \end{aligned} \tag{8.17}$$

We separate two cases: $|\xi| > 2\tau$ or $|\xi| < 2\tau$. Let us start with the former. Since $\beta \in \mathcal{S}(\mathbf{R}^n)$, we may suppose

$$|\hat{\beta}(\xi - \tau\zeta)| \leq \frac{C}{(1 + |\xi - \tau\zeta|)^{n-1}} \leq \frac{C}{(1 + |\xi|)^{n-1}}, \quad |\xi| > 2\tau. \tag{8.18}$$

We hence find

$$\begin{aligned} &\sup_{|\xi| > 2\tau} |\xi|^2 \int_{\mathbf{R}^n_\eta} |\hat{\beta}(\xi - \eta)| \delta(\tau - |\eta|) d\eta \\ &\leq C\tau^{n-1} \sup_{|\xi| > 2\tau} \frac{|\xi|^2}{(1 + |\xi|)^{n-1}} \leq C\tau^2 \sup_{|\xi| > 2\tau} \frac{\tau^{n-3}|\xi|^2}{(1 + |\xi|)^{n-1}} \leq C\tau^2, \end{aligned} \tag{8.19}$$

as desired. Note that the assumption $n \geq 3$ has played a role here.

We turn our attention to the case $|\xi| < 2\tau$. We may suppose

$$|\hat{\beta}(\xi - \tau\zeta)| \leq \frac{C}{\left(\sqrt{1 + |\xi - \tau\zeta|^2}\right)^{n+1}}. \tag{8.20}$$

Using the trace theorem, we get

$$\tau^{n-1}\int_{S^{n-1}_\zeta}|\hat\beta(\xi-\tau\zeta)|dS \le C\int_{S^{n-1}_\zeta}\frac{\tau^{n-1}}{\left(\sqrt{1+|\xi-\tau\zeta|^2}\right)^{n+1}}dS \tag{8.21}$$
$$\le C\|\left(\sqrt{1+|\xi-\cdot|^2}\right)^{-(n+1)}\|_{W^{1,1}(B_\tau)} \le C\|\left(\sqrt{1+|\cdot|^2}\right)^{-(n+1)}\|_{L^2(\mathbf{R}^n)} \le C,$$

where $B_\tau := \{\, x\in\mathbf{R}^n \,:\, |x|<\tau \,\}$ and we mean by $W^{1,1}(B_\tau)$ the standard Sobolev space over B_τ whose members and their weak derivatives of the first order are lying in $L^1(B_\tau)$. It hence follows from (8.17) that

$$\sup_{|\xi|<2\tau}|\xi|^2\int_{\mathbf{R}^n_\eta}|\hat\beta(\xi-\eta)|\delta(\tau-|\eta|)d\eta \le C\sup_{|\xi|<2\tau}|\xi|^2 = C\tau^2, \tag{8.22}$$

as desired. The proof has been finished. □

Proof of Proposition B.4. We have only to modify the argument in the proof of Proposition B.3 properly. We should mention that it is possible to get the inequality (8.16) for all $n\ge 1$. We may omit the details. □

Proposition B.5. *Let $\beta\in\mathcal{S}(\mathbf{R}^n)$.* (1) *If $n\ge 3$, then the inequality*

$$\int_{\mathbf{R}}\|\beta(\cdot)e^{\pm it\sqrt{-\Delta}}h(\cdot)\|^2_{\dot H^1_2(\mathbf{R}^n)}dt \le C\|h\|^2_{\dot H^1_2(\mathbf{R}^n)} \tag{8.23}$$

holds. (2) *Suppose that $n\ge 1$. The inequality*

$$\int_{\mathbf{R}}\|\beta(\cdot)e^{\pm it\sqrt{-\Delta}}h(\cdot)\|^2_{L^2(\mathbf{R}^n)}dt \le C\|h\|^2_{L^2(\mathbf{R}^n)} \tag{8.24}$$

holds.

Proof. Obviously, we have only to deal with $\beta e^{it\sqrt{-\Delta}}h$. Let us start with the proof of (8.23). We follow Metcalfe [18]. Note that $(\beta e^{it\sqrt{-\Delta}}h)\tilde{}(\tau,\xi)=0$ if $\tau<0$. Using the space-time Fourier transform, we have by the Plancherel theorem

$$\int_{\mathbf{R}}\|\beta(\cdot)e^{it\sqrt{-\Delta}}h(\cdot)\|^2_{\dot H^1_2(\mathbf{R}^n)}dt \tag{8.25}$$
$$= C\int_0^\infty\int_{\mathbf{R}^n_\xi}|\xi|^2\left|\int_{\mathbf{R}^n_\eta}\hat\beta(\xi-\eta)\delta(\tau-|\eta|)\hat h(\eta)d\eta\right|^2 d\xi d\tau.$$

Using the Schwarz inequality in η and then applying Proposition B.3, we estimate the right-hand side of (8.25) as

$$\begin{aligned}(8.26)\qquad \cdots \le C\int_0^\infty \int_{\mathbf{R}^n_\xi} |\xi|^2 \Big(\int_{\mathbf{R}^n_\eta} |\hat\beta(\xi-\eta)|\delta(\tau-|\eta|)d\eta\Big) \\ \times\Big(\int_{\mathbf{R}^n_\eta} |\hat\beta(\xi-\eta)|\delta(\tau-|\eta|)|\hat h(\eta)|^2 d\eta\Big)d\xi d\tau \\ \le C\int_{\mathbf{R}^n_\xi}\int_{\mathbf{R}^n_\eta} |\hat\beta(\xi-\eta)||\hat h(\eta)|^2|\eta|^2 d\eta d\xi = C||h||^2_{\dot H^1_2(\mathbf{R}^n)}.\end{aligned}$$

We have therefore finished the proof of (8.23).

For the proof of (8.24) we have only to apply Proposition B.4 instead of Proposition B.3. We may omit the details. □

We are ready to prove Proposition B.2. In what follows we assume $\beta \in C_0^\infty(\mathbf{R}^n)$. For $n \ge 3$ we have by applying the Poincaré inequality to (8.23)

$$(8.27)\qquad \int_{\mathbf{R}} ||\beta(\cdot)e^{\pm it\sqrt{-\Delta}}h(\cdot)||^2_{L^2(\mathbf{R}^n)}dt \le C||h||^2_{\dot H^1_2(\mathbf{R}^n)},$$

which yields

$$(8.28)\qquad ||\beta u||_{L^2(\mathbf{R}\times\mathbf{R}^n)} \le C(||\nabla f||_{L^2} + ||g||_{L^2}).$$

On the other hand, it immediately follows from (8.24) that for $n \ge 1$

$$(8.29)\qquad \sum_{a=0}^n ||\beta\partial_a u||_{L^2(\mathbf{R}\times\mathbf{R}^n)} \le (||\nabla f||_{L^2} + ||g||_{L^2}).$$

Choosing $\beta \in C_0^\infty(\mathbf{R}^n)$ so that $\beta = 1$ for $|x| \le 2$, we get from (8.28)–(8.29)

$$(8.30)\qquad ||u||_{L^2(\{(t,x):t\in\mathbf{R},\,1<|x|<2\})} \le C(||\nabla f||_{L^2} + ||g||_{L^2})$$

for $n \ge 3$ and

$$(8.31)\qquad \sum_{a=0}^n ||\partial_a u||_{L^2(\{(t,x):t\in\mathbf{R},\,1<|x|<2\})} \le C(||\nabla f||_{L^2} + ||g||_{L^2})$$

for $n \ge 1$. By scaling, the proof of (8.7) and that of (8.8) are reduced to (8.30) and (8.31), respectively. We have therefore finished the proof of Proposition B.2. □

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