



Title	On the theory of discrete KMO Langevin equations with reflection positivity (I)
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Citation	Hokkaido University Preprint Series in Mathematics, 1, 2-43
Issue Date	1987-05
DOI	10.14943/48745
Doc URL	<a href="http://eprints3.math.sci.hokudai.ac.jp/357/">http://eprints3.math.sci.hokudai.ac.jp/357/</a> ; <a href="http://hdl.handle.net/2115/45522">http://hdl.handle.net/2115/45522</a>
Type	bulletin (article)
File Information	pre1.pdf



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On the theory of discrete KMO-Langevin  
equations with reflection positivity (I)

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Series #1. May 1987

On the theory of discrete KMO-Langevin  
equations with reflection positivity (I)

by

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To the memory of Professor Gishiro Maruyama

\*) Partially supported by the Grant-in-Aid for Scientific  
Research of the Ministry of Education, Science and Culture  
of Japan.

## §1. Introduction

In a series of papers ([6]~[10]), the author has developed a theory of generalized Langevin equations for real continuous-time stationary Gaussian processes with reflection positivity. The time evolution of such a process  $X(t)$  can be described in terms of two kinds of Langevin equations with a notable difference in character of random forces ([9]) : One is the first KMO-Langevin equation having a white noise as a random force, and it has a root in his study ([7]) and [8]) of the  $[\alpha, \beta, \gamma]$ -Langevin equations. The other is the second KMO-Langevin equation where a colored noise named the Kubo noise is taken to be a random force.

With the linear response theory of R.Kubo([4]) in mind, we established in [10] the fluctuation-dissipation theorems based on these Langevin equations of the two types ; our discovery was that the classical Einstein relation for Ornstein-Uhlenbeck processes holds for the second type, but does not hold for the first one. In addition, we calculated the deviation from the classical Einstein relation. As a concrete example in physics, we discussed the Stokes-Boussinesque-Langevin equation with the Alder-Wainwright effect within our framework of the theory of KMO-Langevin equations possessing reflection positivity.

The purpose of the present and subsequent papers is to establish the discrete analogues of the results mentioned above for the first and second KMO-Langevin equations. <sup>Further</sup> development of these results will be discussed in the author's forthcoming third paper, with the same title. In contrast to the continuous-time

case, we will find that the Einstein relation for discrete-time series  $X(n)$  always deviates from the classical one in the Markovian case, not only for the first type (see §7 of the present (I)), but also for the second one (see §6 of (II)). In the third paper ([12]), we will discuss an entropy criterion and present an answer to the basic question in the problem of modelling such as : "Which noise, white or Kubo, should be taken to be a random force in the equation with given coefficients  $(\alpha, \beta, \gamma)$  called the KMO-Langevin data ?". It should be noted that a variety of covariance functions arising from the theory of ARMA processes ([1]) and of one-dimensional transformations ([5]) are all realized as those of real discrete-time stationary Gaussian processes with reflection positivity. And so they enter into the present framework of the theory of KMO-Langevin equations.

Now we will state the content of this paper. Let  $X = (X(n); n \in \mathbb{Z})$  be a real stationary Gaussian process with mean zero and covariance function  $R$ :

$$(1.1) \quad R(n) = E(X(n)X(0)) \quad (n \in \mathbb{Z}) .$$

In §2 we will briefly recall the spectral theory of  $X$ . In particular, for the Hardy spectral density  $\Delta$  of  $X$  such that  $\log \Delta \in L^1(-\pi, \pi)$ , we define the outer function  $h$  of  $X$  on  $U_1(0) \equiv \{z \in \mathbb{C}; |z| < 1\}$  by

$$(1.2) \quad h(z) = \exp\left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \Delta(\theta) d\theta\right) .$$

In a similar manner to the continuous-time case discussed in [6], we will in §3 define the reflection positivity of  $X$ , which

can be characterized by the condition : there exists a bounded Borel measure  $\sigma$  on  $[-1,1]$  such that

$$(1.3) \quad R(n) = \int_{[-1,1]} t^{|n|} \sigma(dt) \quad (n \in \mathbb{Z}) .$$

The following conditions are assumed in what follows:

$$(1.4) \quad \sigma((-1,1)) = 0$$

$$(1.5) \quad \int_{-1}^1 \left( \frac{1}{1+t} + \frac{1}{1-t} \right) \sigma(dt) < \infty .$$

From these conditions it follows that

$$(1.6) \quad R \in \mathcal{Q}^1(\mathbb{Z}) .$$

Section 4 is devoted to the study of the structure of the outer function  $h$  as well as of the canonical representation kernel  $E = \hat{h}$ , which will play an important role in this paper. By using the result for the continuous-time case obtained in [9], we have

Theorem 4.1. There exists a unique triple  $(\alpha_1, \beta_1, \rho_1)$  such that

- (i)  $\alpha_1 > 0$  and  $\beta_1 > 0$
- (ii)  $\rho_1$  is a bounded Borel measure on  $[-1,1]$  with  $\rho_1((-1,1)) = 0$
- (iii) for any  $z \in U_1(0)$

$$h(z) = \frac{\alpha_1}{\sqrt{2\pi}} \frac{1}{\beta_1(1+z)+1-z+(1-z^2) \int_{-1}^1 \frac{1}{1-tz} \rho_1(dt)} .$$

We call the triple  $(\alpha_1, \beta_1, \rho_1)$  in Theorem 4.1 the first KMO-Langevin data associated with  $\sigma$  (or  $R$ ) . Another triple

$(\alpha_1, \beta_1, \gamma_1)$  is also called by the same name, where  $\gamma_1$  is a function on  $Z$  defined by

$$(1.7) \quad \gamma_1 = \frac{1}{2\pi} \left( (1 - e^{2i \cdot}) \int_{-1}^1 \frac{1}{1 - te^{i \cdot}} \rho_1(dt) \right)^\wedge .$$

More explicitly,

$$(1.8) \quad \gamma_1(n) = \begin{cases} 0 & \text{for } n \in \{-1, -2, \dots\} \\ \int_{-1}^1 t^n \rho_1(dt) & \text{for } n \in \{0, 1\} \\ \int_{-1}^1 t^{n-2} (t^2 - 1) \rho_1(dt) & \text{for } n \in \{2, 3, \dots\} , \end{cases}$$

which implies that

$$(1.9) \quad \gamma_1 \in \mathcal{Q}^1(Z) .$$

It will be found in §5 that the correspondence between  $\sigma$  and  $(\alpha_1, \beta_1, \rho_1)$  is bijective (Theorem 5.1). Furthermore we will obtain an explicit formula of the triple  $(\alpha_1, \beta_1, \gamma_1)$  in terms of  $\sigma$  (Theorem 5.2).

By using Theorem 4.1, we will in §6 derive a stochastic difference equation with a white noise as its random force describing the time evolution of  $X$ .

Theorem 6.1.

$$(1.10) \quad X(n) - X(n-1) = -\beta_1(X(n)+X(n-1)) - (\gamma_1 * X)(n) + \alpha_1 \xi(n) \quad \text{a.s. } (n \in Z) ,$$

where  $\mathfrak{X} = (\xi(n); n \in Z)$  is a real Gaussian white noise.

This equation (1.10) is nothing but the first KMO-Langevin equation we are <sup>looking</sup> for in the discrete-time case. Conversely, we will show that the first KMO-Langevin equation can be uniquely solved for any given triple  $(\alpha_1, \beta_1, \rho_1)$  with conditions (i) and (ii) in Theorem 4.1 and a real Gaussian white noise  $\mathfrak{X}$  (Theorem 6.2).

Fundamental examples of  $X$  are given by the Markov processes  $X_p$ ,  $-1 < p < 1$ , with covariance functions  $R_p$ :

$$(1.11) \quad R_p(n) = p^{|n|} \quad (n \in \mathbb{Z}) .$$

It then follows that the outer function  $h_p$  of  $X_p$  becomes

$$h_p(z) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1-p^2}{1-pz}} \quad (z \in U_1(0)) .$$

By rewriting it in the form (iii) in Theorem 4.1, we have

$$(1.12) \quad h_p(z) = \frac{\alpha_p^{(1)}}{\sqrt{2\pi}} \frac{1}{\beta_p^{(1)}(1+z)+1-z} ,$$

where

$$(1.13) \quad \alpha_p^{(1)} = 2 \sqrt{\frac{1-p}{1+p}} \quad \text{and} \quad \beta_p^{(1)} = \frac{1-p}{1+p} .$$

Therefore, we find from Theorem 6.1 that the time evolution of  $X_p$  is governed by

$$(1.14) \quad X_p(n) - X_p(n-1) = -\beta_p^{(1)}(X_p(n)+X_p(n-1)) + \alpha_p^{(1)}X_0(n) \quad \text{a.s.} (n \in \mathbb{Z}) .$$

We note that  $X_0$  is a white noise.

Concerning a discrete analogue of the generalized fluctuation-dissipation theorem for the continuous-time case discussed in [10], we will in §7 show



Theorem 7.1.

(i) For any  $\theta \in (-\pi, \pi)$

$$\frac{1}{\mathcal{R}_1(1+e^{i\theta})+1-e^{i\theta}+2\pi\tilde{\gamma}_1(\theta)} = \frac{h(e^{i\theta})}{2 \lim_{\tau \downarrow -\pi} h(e^{i\tau})}$$

(ii)  $\frac{\alpha_1^2}{2} = R(0) C_{\mathcal{R}_1, \gamma_1},$

where

(1.15)  $C_{\mathcal{R}_1, \gamma_1} = \pi \left( \int_{-\pi}^{\pi} |\mathcal{R}_1(1+e^{i\theta}) + 1 - e^{i\theta} + 2\pi\tilde{\gamma}_1(\theta)|^{-2} d\theta \right)^{-1}$

(iii)  $D = \frac{\alpha_1^2}{2(2\mathcal{R}_1)^2},$

where

(1.16)  $D = \lim_{N \rightarrow \infty} \frac{1}{2N} E \left( \left( \sum_{n=0}^N X(n) \right)^2 \right).$

(iv)  $D = \frac{R(0)}{2\mathcal{R}_1} \frac{C_{\mathcal{R}_1, \gamma_1}}{2\mathcal{R}_1}$

(v)  $\frac{C_{\mathcal{R}_1, \gamma_1}}{2\mathcal{R}_1} - 1 = \frac{1}{R(0)} \int_{-1}^1 \int_{-1}^1 \frac{1+t}{1-tu} \sigma(dt) \rho_1(du).$

By taking account of the physical meaning of the generalized fluctuation-dissipation theorem for the continuous-time case given in [10], we call the relations (i), (ii) and (iv) in Theorem 7.1 the generalized first fluctuation-dissipation theorem, the generalized second fluctuation-dissipation theorem and the generalized Einstein relation, respectively. And we call the constants  $C_{\mathcal{R}_1, \gamma_1}$  in (1.15) and  $D$  in (1.16) the generalized

friction coefficient and the diffusion constant of  $X$ , respectively.

We note that this  $D$  is also expressed in terms of  $R$ :

$$(1.17) \quad D = \sum_{n=0}^{\infty} R(n) - \frac{R(0)}{2} .$$

In particular, the diffusion constant  $D_p$  of  $X_p$  is given by

$$(1.18) \quad D_p = \frac{R_p(0)}{2\mathcal{B}_p(1)} \quad \left( = \frac{1+p}{2(1-p)} \right) \quad (p \in (-1,1)) .$$

This expression is the classical Einstein relation. And so we conclude from the generalized Einstein relation (iv) in Theorem 7.1 that there occurs a deviation from the classical Einstein relation (1.18) in general non-Markovian cases. Such a deviation can be calculated explicitly by the formula (v) in Theorem 7.1.

The author would like to thank the referees for their valuable and constructive advices.

## §2. Preliminaries

Let  $X = (X(n); n \in \mathbb{Z})$  be a real stationary Gaussian process with discrete time on a probability space  $(\Omega, \mathcal{F}, P)$  and let  $R$  be its covariance function:

$$(2.1) \quad R(n) = E(X(n)X(0)) \quad (n \in \mathbb{Z}) .$$

We assume that the spectral measure of  $X$  has a spectral density  $\Delta = \Delta(\theta)$  such that

$$(2.2) \quad \log \Delta \in L^1((-\pi, \pi)) ,$$

$$(2.3) \quad R(n) = \int_{-\pi}^{\pi} e^{-in\theta} \Delta(\theta) d\theta .$$

Then we can define an outer function  $h$  of  $\Delta$  by

$$(2.4) \quad h(z) = \exp\left\{\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \Delta(\theta) d\theta\right\}$$

for  $z \in U_1(0) = \{z \in \mathbb{C}; |z| < 1\}$ . Such a function  $h$  has the following properties ([2]):

$$(2.5) \quad \sup_{0 < r < 1} \int_{-\pi}^{\pi} |h(re^{i\theta})|^2 d\theta < \infty;$$

$$(2.6) \quad h(e^{i\theta}) = \lim_{r \uparrow 1} h(re^{i\theta}) \text{ exists} \quad \text{a.e. } \theta \in (-\pi, \pi);$$

$$(2.7) \quad |h(e^{i\theta})|^2 = \Delta(\theta) \quad \text{a.e. } \theta \in (-\pi, \pi);$$

$$(2.8) \quad \text{l.i.m.}_{r \uparrow 1} h(re^{i\cdot}) = h(e^{i\cdot}) \quad \text{in } L^2((-\pi, \pi)) .$$

Next we define a function  $E$  on  $\mathbb{Z}$  by

$$(2.9) \quad E(n) = \hat{h}(n) = \int_{-\pi}^{\pi} e^{-in\theta} h(e^{i\theta}) d\theta .$$

It then follows from the above properties of  $h$  that

$$(2.10) \quad E \in \mathfrak{L}^2(\mathbb{Z}) ,$$

$$(2.11) \quad E(n) = 0 \quad (n=-1, -2, -3, \dots) ,$$

$$(2.12) \quad \lim_{n \rightarrow \infty} E(n) = 0 , \quad \text{and}$$

$$(2.13) \quad R(n) = \frac{1}{2\pi} \sum_{m=0}^{\infty} E(|n+m|)E(m) \quad (n \in \mathbb{Z}) .$$

As is well known ([3]), there exists a normalized Gaussian white noise  $\mathfrak{X} = (\xi(n); n \in \mathbb{Z})$  such that

$$(2.14) \quad X(n) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^n E(n-m)\xi(m) \quad \text{in } L^2(\Omega, \mathcal{F}, P) .$$

$$(2.15) \quad \sigma(X(m); m \leq n) = \sigma(\xi(m); m \leq n) \quad (n \in \mathbb{Z}) .$$

By taking account of what have been obtained so far, the function  $E$  is said to be a canonical representation kernel of  $X$  .

### §3. Reflection positivity

Let  $X = (X(n); n \in \mathbb{Z})$  be a real stationary Gaussian process on  $(\Omega, \mathcal{F}, P)$  with covariance function  $R$ . We denote by  $M$  (resp.  $M^+$ ) the closed linear hull of  $\{X(n); n \in \mathbb{Z}\}$  (resp.  $\{X(n); n \geq 0\}$ ) in  $L^2(\Omega, \mathcal{F}, P)$ , and by  $P_{M^+}$  the orthogonal projection on  $M^+$ . The time reflection operator  $T$  is a unitary and self-adjoint operator on  $M$  defined by

$$T(X(n)) = X(-n) \quad (n \in \mathbb{Z}) .$$

As in the continuous-time case ([6]), we say that  $X$  has reflection positivity (T-positivity) if and only if  $P_{M^+} T P_{M^+}$  is non-negative.

In the sequel, we assume that  $X$  has reflection positivity. By taking the same consideration as §2 in [6], we see that there exists a unique Borel measure  $\sigma$  on  $[-1, 1]$  such that

$$(3.1) \quad R(n) = \int_{[-1, 1]} t^{|n|} \sigma(dt) \quad (n \in \mathbb{Z}) .$$

Example 3.1. For each  $p \in (-1, 1)$  consider a non-negative definite function  $R_p$  corresponding to the case  $\sigma = \delta_{(p)}$ , i.e.,

$$(3.2) \quad R_p(n) = p^{|n|} .$$

Then the spectral density  $\Delta_p$  of  $R_p$  is given by

$$(3.3) \quad \Delta_p(\theta) = \frac{1}{2\pi} \frac{1-p^2}{|1-pe^{i\theta}|^2} \quad (\theta \in (-\pi, \pi)) .$$

We now impose the additional assumption on the measure  $\sigma$  in (3.1):

$$(3.4) \quad \sigma((-1,1)) = 0 .$$

Then it immediately follows that the spectral measure of  $R$  has the density  $\Delta$  of the form

$$(3.5) \quad \Delta(\theta) = \frac{1}{2\pi} \int_{-1}^1 \frac{1-t^2}{|1-te^{i\theta}|^2} \sigma(dt) \quad (\theta \in (-\pi, \pi)) .$$

By using a homeomorphism  $\mathcal{P}$  from  $(-1,1]$  onto  $[0, \infty)$ :

$$(3.6) \quad \mathcal{P}(t) = \frac{1-t}{1+t} ,$$

(we define) a bounded Borel measure  $\sigma_c$  on  $[0, \infty)$  by

$$(3.7) \quad \sigma_c = \mathcal{P}(\sigma) .$$

A direct calculation ~~implies~~ <sup>yields</sup> that

$$(3.8) \quad \Delta(2 \tan^{-1} \xi) = \frac{1+\xi^2}{2\pi} \int_0^\infty \frac{\lambda}{\xi^2 + \lambda^2} \sigma_c(d\lambda) \quad (\xi \in \mathbb{R}) ,$$

which leads us to consider

$$(3.9) \quad \Delta_c(\xi) = \frac{1}{\pi} \int_0^\infty \frac{\lambda}{\xi^2 + \lambda^2} \sigma_c(d\lambda) .$$

Since  $\sigma_c(\{0\}) = 0$ , it follows from Lemma 2.12 in [6] that  $\Delta_c$  is a Hardy weight, that is,

$$(3.10) \quad \frac{\log \Delta_c(\xi)}{1+\xi^2} \in L^1(\mathbb{R}) .$$

And so we can define an outer function  $h_c$  of  $\Delta_c$  on  $\mathbb{C}^+$  by

$$(3.11) \quad h_c(z) = \exp\left(\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1+\xi z}{\xi-z} \frac{\log \Delta_c(\xi)}{1+\xi^2} d\xi\right) .$$

Noting that

$$\int_{-\pi}^{\pi} |\log \Delta(\theta)| d\theta = 2 \int_{\mathbb{R}} \left| \frac{\log \Delta(2 \tan^{-1} \xi)}{1+\xi^2} \right| d\xi ,$$

we see from (3.8), (3.9) and (3.10) that the spectral density  $\Delta$  of  $R$  satisfies condition (2.2). By a direct calculation, the outer function  $h$  of  $\Delta$  can be rewritten into the form

$$(3.12) \quad h(z) = \exp\left(\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1+\lambda\zeta}{\lambda-\zeta} \frac{\log \Delta(2 \tan^{-1} \lambda)}{1+\lambda^2} d\lambda\right),$$

where  $z \in U_1(0)$  and  $\zeta = i \frac{1-z}{1+z} \in \mathbb{C}^+$ .

Since for any  $\zeta \in \mathbb{C}^+$

$$\exp\left(\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1+\xi\zeta}{\xi-\zeta} \frac{\log(2/1+\xi^2)}{1+\xi^2} d\xi\right) = \frac{\sqrt{2}}{1-i\zeta},$$

it follows from (3.8), (3.9), (3.11) and (3.12) that

Lemma 3.1.

$$(i) \quad h(z) = \frac{\sqrt{2}}{1+z} h_c\left(i \frac{1-z}{1+z}\right) \quad (z \in U_1(0))$$

$$(ii) \quad h_c(\zeta) = \frac{\sqrt{2}}{1-i\zeta} h\left(\frac{i-\zeta}{i+\zeta}\right) \quad (\zeta \in \mathbb{C}^+).$$

Furthermore, immediately from (3.7), we obtain

Lemma 3.2.

$$(i) \quad \int_{-1}^1 \frac{1+t}{1-t} \sigma(dt) = \int_0^{\infty} \lambda^{-1} \sigma_c(d\lambda)$$

$$(ii) \quad \int_{-1}^1 \frac{1-t}{1+t} \sigma(dt) = \int_0^{\infty} \lambda \sigma_c(d\lambda).$$

§4. Outer function h and canonical representation kernel E

Let  $X = (X(n); n \in \mathbb{Z})$  be a real stationary Gaussian process on  $(\Omega, \mathcal{F}, P)$  satisfying reflection positivity. It then follows that the covariance function  $R$  of  $X$  has such a representation as (3.1) with a bounded Borel measure  $\sigma$  on  $[-1, 1]$ . In the sequel, we assume the following conditions:

$$(4.1) \quad \sigma((-1, 1)) = 0$$

$$(4.2) \quad \int_{-1}^1 \left( \frac{1}{1+t} + \frac{1}{1-t} \right) \sigma(dt) < \infty .$$

At first we will show the following structure theorem for the outer function  $h$  of  $X$ , which will be the key to derive a description of the time evolution of  $X$  in §6.

Theorem 4.1. There exists a unique triple  $(\alpha_1, \beta_1, \rho_1)$  such that

$$(i) \quad \alpha_1 > 0 \quad \text{and} \quad \beta_1 > 0$$

$$(ii) \quad \rho_1 \text{ is a bounded Borel measure on } [-1, 1] \text{ with } \rho_1((-1, 1)) = 0$$

$$(iii) \quad \text{for any } z \in U_1(0)$$

$$h(z) = \frac{\alpha_1}{\sqrt{2\pi}} \frac{1}{\beta_1 (1+z) + 1-z + (1-z^2) \int_{-1}^1 \frac{1}{1-tz} \rho_1(dt)} .$$

Proof. Since it follows from Lemma 3.2 and condition (4.2) that  $\int_0^\infty (\lambda^{-1} + \lambda) \sigma_c(d\lambda) < \infty$ , we can apply Theorem 2.2 in [9] to

see that there exists a unique triple  $(\alpha_c, \beta_c, \rho_c)$  such that



$$(4.3) \quad \alpha_c > 0 \quad \text{and} \quad \beta_c > 0$$

(4.4)  $\rho_c$  is a Borel measure on  $[0, \infty)$  satisfying

$$\rho_c(\{0\}) = 0 \quad \text{and} \quad \int_0^\infty \frac{1}{1+\lambda} \rho_c(d\lambda) < \infty$$

$$(4.5) \quad h_c(z) = \frac{\alpha_c}{\sqrt{2\pi}} \frac{1}{\beta_c - iz - iz \int_0^\infty \frac{1}{\lambda - iz} \rho_c(d\lambda)} \quad \text{for any } z \in \mathbb{C}^+ .$$

Then we define a triple  $(\alpha_1, \beta_1, \rho_1)$  by

$$(4.6) \quad \alpha_1 = \sqrt{2}\alpha_c$$

$$(4.7) \quad \beta_1 = \beta_c$$

$$(4.8) \quad \rho_1(dt) = \frac{1+t}{2} (\varphi^{-1} \rho_c)(dt) .$$

In particular, we have

$$(4.9) \quad \rho_1([-1, 1]) = \int_0^\infty \frac{1}{\lambda+1} \rho_c(d\lambda) .$$

Therefore, we find that the triple  $(\alpha_1, \beta_1, \rho_1)$  satisfies (i) and (ii) in Theorem 4.1. Furthermore, it follows from Lemma 3.1(i) and (4.5) that for any  $z \in U_1(0)$

$$h(z) = \frac{\sqrt{2}}{1+z} \frac{\alpha_c}{\sqrt{2\pi}} \frac{1}{\beta_c + \frac{1-z}{1+z} + \frac{1-z}{1+z} \int_0^\infty (\lambda + \frac{1-z}{1+z})^{-1} \rho_c(d\lambda)} .$$

On the other hand, by the definition of  $\varphi$ ,

$$\begin{aligned} \int_0^\infty (\lambda + \frac{1-z}{1+z})^{-1} \rho_c(d\lambda) &= \int_{-1}^1 (\frac{1-t}{1+t} + \frac{1-z}{1+z})^{-1} (\varphi^{-1} \rho_c)(dt) \\ &= (1+z) \int_{-1}^1 \frac{1}{1-tz} \frac{1+t}{2} (\varphi^{-1} \rho_c)(dt) . \end{aligned}$$

Therefore, we find from (4.6), (4.7) and (4.8) that the triple  $(\alpha_1, \beta_1, \rho_1)$  satisfies relation (iii) in Theorem 4.1.

The uniqueness of such a triple  $(\alpha_1, \beta_1, \rho_1)$  can be proved as follows : since  $0 \leq \frac{(1+x)(1-x)}{1-tx} \leq 2$  for any  $t, x \in (-1, 1)$ , we see that

$$(4.10) \quad \lim_{x \downarrow -1} h(x) = \frac{\alpha_1}{2\sqrt{2\pi}}$$

and

$$(4.11) \quad \lim_{x \uparrow 1} h(x) = \frac{\alpha_1}{2\sqrt{2\pi}\beta_1}$$

which determine uniquely the pair  $(\alpha_1, \beta_1)$ . Furthermore, since

$$\int_{-1}^1 \frac{1}{1+xt} \rho_1(dt) = \frac{1}{x} \int_0^1 \frac{1}{\frac{1-x}{x} + s} \rho_1(d(s-1)) \quad \text{for any } x \in (0, 1),$$

it follows from the uniqueness of Stieltjes transform that the measure  $\rho_1$  is uniquely determined. (Q.E.D.)

Next we will show the following expression of  $h$ , closely related to the one established in Theorem 4.1.

Theorem 4.2. There exists a unique Borel measure  $\nu$  on  $[-1, 1]$  such that

$$(i) \quad \nu((-1, 1)) = 0 \quad \text{and} \quad \nu([-1, 1]) < \infty$$

$$(ii) \quad \int_{-1}^1 \left( \frac{1}{1-t} + \frac{1}{1+t} \right) \nu(dt) < \infty$$

$$(iii) \quad h(z) = \frac{1}{2\pi} \int_{-1}^1 \frac{1}{1-tz} \nu(dt) \quad \text{for any } z \in U_1(0).$$

Proof. By Theorem 2.1 and (2.19) in [9], we see that there exists a unique Borel measure  $\nu_c$  on  $[0, \infty)$  such that

$$(4.12) \quad \nu_c(\{0\}) = 0 \quad \text{and} \quad \nu_c([0, \infty]) < \infty$$

$$(4.13) \quad \int_0^{\infty} \lambda^{-1} \nu_c(d\lambda) < \infty$$

$$(4.14) \quad h_c(z) = \frac{1}{2\pi} \int_0^{\infty} \frac{1}{\lambda - iz} \nu_c(d\lambda) \quad \text{for any } z \in \mathbb{C}^+ .$$

Then we define a Borel measure  $\nu$  on  $[-1, 1]$  by

$$(4.15) \quad \nu(dt) = \frac{1+t}{\sqrt{2}} (\varphi^{-1} \nu_c)(dt) .$$

Since

$$\begin{aligned} \int_0^{\infty} \lambda^{-1} \nu_c(d\lambda) &= \int_{-1}^1 (\varphi(t))^{-1} (\varphi^{-1} \nu_c)(dt) \\ &= \sqrt{2} \int_{-1}^1 \frac{1}{1-t} \nu(dt) , \end{aligned}$$

it follows from (4.12), (4.13) and (4.15) that the measure  $\nu$  satisfies (i) and (ii) in Theorem 4.2. Furthermore, we see from Lemma 3.1(i) and (4.14) that for any  $z \in U_1(0)$

$$\begin{aligned} h(z) &= \frac{\sqrt{2}}{1+z} \frac{1}{2\pi} \int_0^{\infty} \left(\lambda + \frac{1-z}{1+z}\right)^{-1} \nu_c(d\lambda) \\ &= \frac{\sqrt{2}}{1+z} \frac{1}{2\pi} \int_{-1}^1 \left(\frac{1-t}{1+t} + \frac{1-z}{1+z}\right)^{-1} (\varphi^{-1} \nu_c)(dt) \\ &= \frac{1}{2\pi} \int_{-1}^1 \frac{1}{1-tz} \frac{1+t}{\sqrt{2}} (\varphi^{-1} \nu_c)(dt) , \end{aligned}$$

which, together with (4.15), yields (iii) in Theorem 4.2. The uniqueness of a Borel measure  $\nu$  satisfying (i), (ii) and (iii)

can be proved by using the uniqueness of Stieltjes transform.

(Q.E.D.)

By taking a boundary value of the outer function  $h$ , we can rephrase Theorems 4.1 and 4.2 as follows.

Corollary 4.1. For almost all  $\theta \in (-\pi, \pi)$ ,

$$(i) \quad h(e^{i\theta}) = \frac{\alpha_1}{\sqrt{2\pi}} \frac{1}{\beta_1(1+e^{i\theta})+1-e^{i\theta}+(1-e^{2i\theta})} \int_{-1}^1 \frac{1}{1-te^{i\theta}} \rho_1(dt)$$

$$(ii) \quad h(e^{i\theta}) = \frac{1}{2\pi} \int_{-1}^1 \frac{1}{1-te^{i\theta}} \nu(dt) .$$

As a consequence of the above expression (ii) of  $h$ , we can obtain an expression of the canonical representation kernel  $E$  in (2.9), which says that  $E(n)$ ,  $n \geq 0$ , is nothing but the moment sequence of the measure  $\nu$ .

Theorem 4.3.

$$E(n) = x_{[0, \infty)}(n) \int_{-1}^1 t^n \nu(dt) \quad \text{for any } n \in \mathbb{Z} .$$

Proof. By Theorem 4.2 (ii) and Corollary 4.1 (ii), the following series

$$h(e^{i\theta}) = \frac{1}{2\pi} \sum_{n=0}^{\infty} e^{i\theta n} \left( \int_{-1}^1 t^n \nu(dt) \right)$$

is absolutely convergent (a.e.). Therefore, by (2.9), we have the assertion. (Q.E.D.)

Concerning a relation between the measures  $\sigma$  in (3.1) and

$\nu$  in Theorem 4.2, we will show

Theorem 4.4.

$$(4.16) \quad \sigma(dt) = \frac{1}{2\pi} \left( \int_{-1}^1 \frac{1}{1-ts} \nu(ds) \right) \nu(dt) .$$

Proof. By Theorem 4.3, for any  $n \in \{0, 1, 2, \dots\}$ ,

$$\sum_{m=0}^{\infty} E(n+m)E(m) = \int_{-1}^1 t^n \left( \int_{-1}^1 \frac{1}{1-ts} \nu(ds) \right) \nu(dt) .$$

Therefore, by (2.14) and (3.1), we have Theorem 4.4. (Q.E.D.)

Remark 4.1. By Lemma 2.6 (i) in [9], the measure  $\sigma_c$  in (3.7) is related to the measure  $\nu_c$  in (4.12) as follows:

$$(4.17) \quad \sigma_c(d\lambda) = \frac{1}{2\pi} \left( \int_0^{\infty} \frac{1}{\lambda+\lambda'} \nu_c(d\lambda') \right) \nu_c(d\lambda) .$$

In view of (3.7) and (4.15), it turns out that this equality is equivalent to (4.16).

Example 4.1. For each fixed  $p \in (-1, 1)$ , consider the non-negative definite function  $R_p$  in (3.2). Then we see that the outer function  $h_p$  and the canonical representation kernel  $E_p$  become

$$(4.18) \quad h_p(z) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1-p^2}{1-pz}} \quad (z \in U_1(0))$$

$$(4.19) \quad E_p(n) = \chi_{[0, \infty)}(n) \sqrt{2\pi(1-p^2)} p^n \quad (n \in \mathbb{Z}) .$$

Furthermore we observe that

$$(4.20) \quad h_p(z) = \frac{1}{\sqrt{2\pi}} 2 \sqrt{\frac{1-p}{1+p}} \frac{1}{\frac{1-p}{1+p}(1+z)+1-z} \quad (z \in U_1(0)) ,$$

which implies that the triple  $(\alpha_p^{(1)}, \beta_p^{(1)}, \rho_p^{(1)})$  in Theorem 4.1 associated with  $R_p$  becomes

$$(4.21) \quad \alpha_p^{(1)} = 2 \sqrt{\frac{1-p}{1+p}} , \quad \beta_p^{(1)} = \frac{1-p}{1+p} \quad \text{and} \quad \rho_p^{(1)} = 0 .$$

Example 4.2. Let  $R$  be the non-negative definite function given by

$$(4.22) \quad R(n) = \sigma_1 p_1^{|n|} + \sigma_2 p_2^{|n|} \quad (n \in \mathbb{Z}) ,$$

where  $\sigma_1, \sigma_2 > 0$  and  $-1 < p_1 < p_2 < 1$ . Put

$$(4.23) \quad a_1 = \sigma_1(1-p_1^2)p_2 + \sigma_2(1-p_2^2)p_1$$

$$(4.24) \quad a_2 = \sigma_1(1-p_1^2)(1+p_2^2) + \sigma_2(1-p_2^2)(1+p_1^2) .$$

By a direct calculation, we can conclude that the outer function  $h$ , the triple  $(\alpha_1, \beta_1, \rho_1)$  and the measure  $\nu$  associated with this  $R$  are given as follows:

(i) The case  $a_1 = 0$  ( $-1 < p_1 < 0 < p_2 < 1$ ):

$$(4.25) \quad h(z) = \sqrt{\frac{a_2}{2\pi}} \frac{1}{(1-p_1z)(1-p_2z)} \quad (z \in (U_1(0)))$$

$$(4.26) \quad \left\{ \begin{array}{l} \alpha_1 = \frac{2\sqrt{a_2}}{(1+p_1)(1+p_2)} \\ \beta_1 = \frac{(1-p_1)(1-p_2)}{(1+p_1)(1+p_2)} \\ \rho_1(dt) = \frac{-2p_1p_2}{(1+p_1)(1+p_2)} \delta_{\{0\}}(dt) \end{array} \right.$$

$$(4.27) \quad v = \frac{\sqrt{2\pi a_2}}{p_2 - p_1} \{ (-p_1)\delta_{\{p_1\}} + p_2\delta_{\{p_2\}} \}$$

(ii) The case  $a_1 \neq 0$ :

$$(4.28) \quad h(z) = \sqrt{\frac{r_1}{2\pi}} \frac{1 - q_1 z}{(1 - p_1 z)(1 - p_2 z)} \quad (z \in (U_1(0)))$$

$$(4.29) \quad \left\{ \begin{array}{l} \alpha_1 = \frac{2\sqrt{r_1}(1+q_1)}{(1+p_1)(1+p_2)} \\ \beta_1 = \frac{(1+q_1)(1-p_1)(1-p_2)}{(1-q_1)(1+p_1)(1+p_2)} \\ \rho_1(dt) = \frac{2(q_1 - p_1)(p_2 - q_1)}{(1-q_1)(1+p_1)(1+p_2)} \delta_{\{q_1\}}(dt) \end{array} \right.$$

$$(4.30) \quad v = \frac{\sqrt{2\pi r_1}}{p_2 - p_1} \{ (q_1 - p_1)\delta_{\{p_1\}} + (p_2 - q_1)\delta_{\{p_2\}} \},$$

where

$$(4.31) \quad q_1 = \frac{1}{2} \left( \frac{a_2}{a_1} \pm \sqrt{\left( \frac{a_2}{a_1} \right)^2 - 4} \right) \quad \text{if } a_1 \begin{matrix} > 0 \\ < 0 \end{matrix}$$

$$(4.32) \quad r_1 = \frac{a_1}{q_1}.$$

We note that  $p_1 < q_1 < p_2$  and  $q_1 > 0$  if  $a_1 > 0$ .



§5. The first KMO-Langevin data

This section is devoted to the further study of the key expression of  $h$  in Theorem 4.1. In order to investigate the correspondence between  $\sigma$  in (3.1) and  $(\alpha_1, \beta_1, \rho_1)$  in Theorem 4.1, we introduce two classes  $\Sigma_1$  and  $\mathcal{L}_1$  by

(5.1)  $\Sigma_1 = \{\sigma; \sigma \text{ is a bounded Borel measure on } [-1, 1] \text{ such that}$

$$\sigma((-1, 1)) = 0 \quad \text{and} \quad \int_{-1}^1 \left( \frac{1}{1+t} + \frac{1}{1-t} \right) \sigma(dt) < \infty$$

and

(5.2)  $\mathcal{L}_1 = \{(\alpha, \beta, \rho); \alpha > 0, \beta > 0 \text{ and } \rho \text{ is a bounded Borel measure on } [-1, 1] \text{ such that } \rho((-1, 1)) = 0\}.$

For each  $\sigma \in \Sigma_1$ , we denote by  $R_\sigma$ ,  $\Delta_\sigma$  and  $h_\sigma$  the non-negative definite function, the spectral density and the outer function associated with  $\sigma$ , respectively:

$$(5.3) \quad \left\{ \begin{array}{l} R_\sigma(n) = \int_{-1}^1 t^{|n|} \sigma(dt) \quad (n \in \mathbb{Z}) \\ \Delta_\sigma(\theta) = \frac{1}{2\pi} \int_{-1}^1 \frac{1-t^2}{|1-te^{i\theta}|^2} \sigma(dt) \quad (\theta \in (-\pi, \pi)) \\ h_\sigma(z) = \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \Delta_\sigma(\theta) d\theta \right\} \quad (z \in U_1(0)) . \end{array} \right.$$

Theorem 5.1. There exists a bijective mapping  $L_1$  from  $\Sigma_1$  onto  $\mathcal{L}_1$  such that for any  $\sigma \in \Sigma_1$  and  $(\alpha_1, \beta_1, \rho_1) = L_1(\sigma) \in \mathcal{L}_1$

$$(5.4) \quad h_{\sigma}(z) = \frac{\alpha_1}{\sqrt{2\pi}} \frac{1}{\beta_1(1+z)+1-z+(1-z^2) \int_{-1}^1 \frac{1}{1-tz} \rho_1(dt)} \quad (z \in U_1(0)) .$$

Proof. By Theorem 4.1, we have an injective mapping  $L_1$  from  $\Sigma_1$  into  $\mathcal{L}_1$  satisfying relation (5.4) and so we have only to show that  $L_1$  is surjective. Let  $(\alpha_1, \beta_1, \rho_1)$  be any element of  $\mathcal{L}_1$ . We define a new triple  $(\alpha_c, \beta_c, \rho_c)$  by

$$(5.5) \quad \alpha_c = \frac{\alpha_1}{\sqrt{2}}, \quad \beta_c = \beta_1 \quad \text{and} \quad \rho_c = \mathcal{F}\left(\frac{2}{1+\cdot} \rho_1\right) .$$

By noting (4.9), we see from Theorem 3.1 in [9] that there exists a unique bounded Borel measure  $\sigma_c$  on  $[0, \infty)$  such that

$$(5.6) \quad \sigma_c(\{0\}) = 0 \quad \text{and} \quad \int_0^{\infty} (\lambda + \lambda^{-1}) \sigma_c(d\lambda) < \infty$$

$$(5.7) \quad h_c(z) = \frac{\alpha_c}{\sqrt{2\pi}} \frac{1}{\beta_c - iz - iz \int_0^{\infty} \frac{1}{\lambda - iz} \rho_c(d\lambda)} \quad (z \in \mathbb{C}^+) ,$$

where  $h_c$  is the outer function of the Hardy weight  $\Delta_c$  of the form

$$(5.8) \quad \Delta_c(\xi) = \frac{1}{\pi} \int_0^{\infty} \frac{\lambda}{\xi^2 + \lambda^2} \sigma_c(d\lambda) \quad (\xi \in \mathbb{R}) .$$

We then define a bounded Borel measure  $\sigma$  on  $[-1, 1]$  by

$$(5.9) \quad \sigma = \mathcal{F}^{-1}(\sigma_c) .$$

By Lemma 3.2, (5.6) and (5.9), we see that  $\sigma \in \Sigma_1$ . Furthermore, it follows from Lemma 3.1(i) and (3.12) that

$$h_{\sigma}(z) = \frac{\sqrt{2}}{1+z} h_c\left(i\frac{1-z}{1+z}\right) \quad (z \in U_1(0)) .$$

By combining this with (5.7) and then noting (5.5), we conclude that  $L_1(\sigma) = (\alpha_1, \beta_1, \rho_1)$  . (Q.E.D.)

For each bounded Borel measure  $\rho$  on  $[-1,1]$  with  $\rho((-1,1)) = 0$  , we define a function  $\gamma$  on  $Z$  by

$$(5.10) \quad \gamma = \frac{1}{2\pi} \left( (1 - e^{2i\cdot}) \int_{-1}^1 \frac{1}{1 - te^{i\cdot}} \rho(dt) \right)^\wedge .$$

We note that  $\gamma$  is well-defined, because

$$(5.11) \quad \left| \frac{(1 + e^{i\theta})(1 - e^{i\theta})}{1 - te^{i\theta}} \right| \leq 2 \quad \text{for any } t \in (-1,1) \text{ and } \theta \in (-\pi, \pi) .$$

Some properties of  $\gamma$  are listed as

Proposition 5.1.

$$(i) \quad \gamma(n) = \begin{cases} 0 & \text{for } n \in \{-1, -2, \dots\} \\ \int_{-1}^1 t^n \rho(dt) & \text{for } n \in \{0, 1\} \\ \int_{-1}^1 (t^n - t^{n-2}) \rho(dt) & \text{for } n \in \{2, 3, \dots\} \end{cases}$$

$$(ii) \quad \gamma \in \mathcal{L}^1(Z)$$

$$(iii) \quad \sum_{n=0}^{\infty} \gamma(n) = 0$$

$$(iv) \quad \sum_{n=0}^{\infty} (-1)^n \gamma(n) = 0 .$$

Proof. By the definition of  $\gamma$  , we have

$$\gamma(n) = \frac{1}{2\pi} \int_{-1}^1 \left( \int_{-\pi}^{\pi} e^{-in\theta} \frac{1 - e^{i2\theta}}{1 - te^{i\theta}} d\theta \right) \rho(dt) \quad (n \in Z) .$$

Hence (i) follows from the simple fact that for any  $t \in (-1,1)$

$$\int_{-\pi}^{\pi} e^{-in\theta} \frac{1-e^{i2\theta}}{1-te^{i\theta}} d\theta = \begin{cases} 0 & \text{for } n \in \{-1,-2,\dots\} \\ 2\pi t^n & \text{for } n \in \{0,1\} \\ 2\pi(t^n - t^{n-2}) & \text{for } n \in \{2,3,\dots\} . \end{cases}$$

Since

$$\sum_{n=2}^{\infty} |\gamma(n)| \leq \int_{-1}^1 \left( \sum_{n=2}^{\infty} |t|^{n-2} \right) (1-t^2) \rho(dt) = \int_{-1}^1 (1+|t|) \rho(dt) < \infty ,$$

we have (ii). By (ii), we can take the inverse Fourier transform of (5.10) to get

$$2\pi \tilde{\gamma}(\theta) = (1-e^{2i\theta}) \int_{-1}^1 \frac{1}{1-te^{i\theta}} \rho(dt) \quad \text{for any } \theta \in (-\pi, \pi) .$$

By taking  $\theta = 0$  and  $\theta = -\pi$  in the above expression, we have (iii) and (iv), respectively. (Q.E.D.)

Definition 5.2. For each  $\sigma \in \Sigma_1$ , we call a triple  $(\alpha_1, \beta_1, \rho_1) (=L_1(\sigma))$  or  $(\alpha_1, \beta_1, \gamma_1)$  the first KMO-Langevin data associated with  $\sigma$  or  $R_\sigma$ .

We will give a formula concerning the first KMO-Langevin data  $(\alpha_1, \beta_1, \gamma_1)$  associated with a fixed  $\sigma (\in \Sigma_1)$ .

Theorem 5.2.

$$(i) \quad \alpha_1 = 2 \sqrt{\int_{-1}^1 \frac{1-t}{1+t} \sigma(dt)}$$

$$(ii) \quad \beta_1 = \sqrt{\left( \int_{-1}^1 \frac{1-t}{1+t} \sigma(dt) \right) \left( \int_{-1}^1 \frac{1+t}{1-t} \sigma(dt) \right)^{-1}}$$

$$(iii) \quad \gamma_1(n) = \frac{\alpha_1}{(2\pi)^{3/2}} \left( \frac{1}{h_\sigma(e^{i\cdot})} \right)^{\wedge(n)} - (B_1+1)\delta_{n,0} - (B_1-1)\delta_{n,1}$$

(n ∈ Z) .

Proof. By noting (4.10), we see from (2.6), (2.7) and (3.5) that

$$\begin{aligned} \alpha_1^2 &= (2\sqrt{2\pi})^2 \lim_{r \downarrow -1} |h_\sigma(r)|^2 \\ &= (2\sqrt{2\pi})^2 \Delta(-\pi) \\ &= 4 \int_{-1}^1 \frac{1-t}{1+t} \sigma(dt) , \end{aligned}$$

which implies (i). Similarly, by noting (4.11), we have

$$\begin{aligned} B_1^2 &= \Delta(-\pi) (\Delta(0))^{-1} \\ &= \int_{-1}^1 \frac{1-t}{1+t} \sigma(dt) \left( \int_{-1}^1 \frac{1+t}{1-t} \sigma(dt) \right)^{-1} , \end{aligned}$$

which implies (ii). By Corollary 4.1 (i), for a.e.  $\theta \in (-\pi, \pi)$  ,

$$\begin{aligned} &(1-e^{2i\theta}) \int_{-1}^1 \frac{1}{1-te^{i\theta}} \rho_1(dt) \\ &= \frac{\alpha_1}{\sqrt{2\pi}} \frac{1}{h_\sigma(e^{i\theta})} - B_1(1+e^{i\theta}) - (1-e^{i\theta}) . \end{aligned}$$

By taking Fourier transform of both hand sides, we have (iii).

(Q.E.D.)

Remark 5.1. We see from (2.4), (3.5) and (4.2) that

$$(i) \quad \frac{1}{h_\sigma(z)} = \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\Delta_\sigma(\theta)^{-1}) d\theta \right\} \quad (z \in U_1(0))$$

(ii) there exist positive constants  $c_1$  and  $c_2$  such that

$$c_1 \leq \Delta_{\sigma}(\theta) \leq c_2 \quad \text{for any } \theta \in (-\pi, \pi)$$

(iii)  $\frac{1}{h_{\sigma}}$  is the outer function of the Hardy density  $\Lambda_{\sigma}^{-1}$ .

§6. The first KMO-Langevin equation

Returning to §4, we will derive a stochastic difference equation describing the time evolution of a real stationary Gaussian process  $X = (X(n); n \in \mathbb{Z})$  on a probability space  $(\Omega, \mathcal{F}, P)$ ; the covariance function  $R$  takes the form (3.1) with some bounded Borel measure  $\sigma \in \Sigma_1$ .

Let the triple  $(\alpha_1, \beta_1, \gamma_1)$  be the first KMO-Langevin data associated with  $\sigma$ . By using the normalized Gaussian white noise  $\xi = (\xi(n); n \in \mathbb{Z})$  in (2.14), we will show

Theorem 6.1.

$$(6.1) \quad X(n) - X(n-1) = -\beta_1(X(n)+X(n-1)) - (\gamma_1 * X)(n) + \alpha_1 \xi(n) \\ \text{a.s. } (n \in \mathbb{Z}).$$

Proof By Theorems 4.2 (ii) and 4.3, we have

$$(6.2) \quad E \in \mathcal{Q}^1(\mathbb{Z}).$$

Furthermore, by noting Proposition 5.1 (ii), we see from (2.14) that the following two random series are absolutely convergent (a.s.) : for any  $n \in \mathbb{Z}$

$$(6.3) \quad X(n) - X(n-1) + \beta_1(X(n)+X(n-1)) \\ = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \{E(n-m) - E(n-1-m) + \beta_1(E(n-m) + E(n-m-1))\} \xi(m)$$

$$(6.4) \quad (\gamma_1 * X)(n) = \sum_{\mathcal{Q}=-\infty}^{\infty} \gamma_1(n-\mathcal{Q})X(\mathcal{Q}) \\ = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \left( \sum_{\mathcal{Q}=-\infty}^{\infty} \gamma_1(n-\mathcal{Q})E(\mathcal{Q}-m) \right) \xi(m).$$

On the other hand, it follows from (2.9), (5.10), Proposition 5.1 (ii) and (6.2) that for any  $n, m \in \mathbb{Z}$

$$(6.5) \quad E(n-m) - E(n-1-m) + \beta_1(E(n-m) + E(n-m-1))$$

$$= \int_{-\pi}^{\pi} e^{-i(n-m)\theta} \{1 - e^{i\theta} + \beta_1(1 + e^{i\theta})\} h(e^{i\theta}) d\theta$$

$$(6.6) \quad \sum_{\varrho=-\infty}^{\infty} \gamma_1(n-\varrho) E(\varrho-m)$$

$$= \int_{-\pi}^{\pi} e^{-i(n-m)\theta} (1 - e^{2i\theta}) \int_{-1}^1 \frac{1}{1 - te^{i\theta}} \rho_1(dt) h(e^{i\theta}) d\theta .$$

Therefore, by substituting (6.5) and (6.6) into (6.3) and (6.4), respectively, we conclude from Corollary 4.1 (i) that for any  $n \in \mathbb{Z}$

$$X(n) - X(n-1) + \beta_1(X(n) + X(n-1)) + (\gamma_1 * X)(n)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \left( \int_{-\pi}^{\pi} e^{-i(n-m)\theta} \frac{\alpha_1}{\sqrt{2\pi}} d\theta \right) \xi(m)$$

$$= \alpha_1 \xi(n) \quad \text{a.s.},$$

which completes the proof of Theorem 6.1. (Q.E.D.)

Definition 6.1. We call the stochastic difference equation (6.1) the first KMO-Langevin equation associated with  $X$ .

As the converse of Theorem 6.1, we will show

Theorem 6.2. For each triple  $(\alpha_1, \beta_1, \rho_1) \in \mathcal{L}_1$  and each normalized Gaussian white noise  $\xi = (\xi(n); n \in \mathbb{Z})$  on a probability space  $(\Omega, \mathcal{F}, P)$ , there exists a unique real stationary Gaussian process  $X = (X(n); n \in \mathbb{Z})$  on  $(\Omega, \mathcal{F}, P)$  with reflection positivity such that  $X$  satisfies the first KMO-Langevin equation (6.1); the covariance function of this  $X$  coincides with  $R_{\sigma}$ ,  $\sigma = L_1^{-1}((\alpha_1, \beta_1, \rho_1))$ .



Proof By Theorem 5.1, we get a bounded Borel measure

$\sigma = L_1^{-1}((\alpha_1, \beta_1, \rho_1))$  in  $\Sigma_1$  such that the outer function  $h_\sigma$

associated with  $\sigma$  takes the form (5.4). Choose an  $\mathcal{L}^2(Z)$ -

function  $E = \hat{h}_\sigma$  as a canonical representation kernel to define a

real stationary Gaussian process  $X = (X(n); n \in \mathbb{Z})$ :

$$X(n) = (\sqrt{2\pi})^{-1} \sum_{m=-\infty}^{\infty} E(n-m)\xi(m) .$$

Then it follows from Theorem 6.1 that this  $X$  is our desired process.

To prove the uniqueness of such a process  $X$ , let  $Y = (Y(n); n \in \mathbb{Z})$  be another real stationary Gaussian process on  $(\Omega, \mathcal{F}, P)$  satisfying the same equation (6.1). Fix any  $m \in \mathbb{Z}$ . By multiplying both hand sides of equation (6.1) by  $(h_\sigma(\cdot)e^{-im\cdot})^\sim(n)$  and then summing up with respect to  $n$ , we can observe from (2.10), Corollary 4.1, (5.10), Proposition 5.1 (ii) and (6.2) that

$$Y(m) = (\sqrt{2\pi})^{-1} \sum_{n=-\infty}^m E(m-n)\xi(m) ,$$

which implies  $Y = X$ .

(Q.E.D.)

Example 6.1. Let  $X_p = (X_p(n); n \in \mathbb{Z})$  be a real stationary Gaussian process with the covariance function  $R_p$  given in (3.2) ( $p \in (-1, 1)$ ). We note that each  $X_p$  has the simple Markov property, and in particular  $X_0$  represents a normalized Gaussian white noise  $\mathfrak{F}$ . The first KMO-Langevin equation of  $X_p$  takes the simplest form

$$(6.13) \quad X_p(n) - X_p(n-1) = -\beta_p^{(1)}(X_p(n)+X_p(n-1)) + \alpha_p^{(1)}\xi(n) \\ \text{a.s.}(n \in Z) ,$$

where the pair  $(\alpha_p^{(1)}, \beta_p^{(1)})$  was given by (4.21). In case  $p = 0$ , the above form (6.13) for the white noise  $\xi$  becomes trivial:

$$(6.14) \quad \xi(n) - \xi(n-1) = -(\xi(n)+\xi(n-1)) + 2\xi(n) \quad \text{a.s.}(n \in Z) .$$

Example 6.2. Let  $X = (X(n); n \in Z)$  be a real stationary Gaussian process with the covariance function  $R$  of the form (4.22). It follows from Proposition 5.1 (i) that

$$(6.15) \quad \gamma_1(n) = \begin{cases} 0 & \text{for } n \in \{-1, -2, \dots\} \\ \rho_0 q_1^n & \text{for } n \in \{0, 1\} \\ \rho_0 q_1^{n-2} (q_1^2 - 1) & \text{for } n \in \{2, 3, \dots\} , \end{cases}$$

where

$$(6.16) \quad \rho_0 = \frac{2(q_1 - p_1)(p_2 - q_1)}{(1 - q_1)(1 + p_1)(1 + p_2)} .$$

We note that (i) in Example 4.2 is a special case  $q_1 = 0$  of (ii) in Example 4.2. Therefore, we see from Theorem 6.1 that  $X$  satisfies the following KMO-Langevin equation:

$$(6.17) \quad X(n) - X(n-1) = -\beta_1(X(n)+X(n-1)) - (\gamma_1 * X)(n) + \alpha_1 \xi(n) \\ \text{a.s.}(n \in Z) ,$$

where the pair  $(\alpha_1, \beta_1)$  was given by (4.26) or (4.29). It deserves mention that the second term on the right hand side of equation (6.17) depends upon the whole past of  $X$  in case  $q_1 \neq 0$ , i.e.  $a_1 \neq 0$ .

## §7. Generalized fluctuation-dissipation theorems

In this final section we will prove a couple of relations between our objects - the first KMO-Langevin data  $(\alpha_1, \beta_1, \gamma_1) \in \mathcal{L}_1$ , the outer function  $h$  of  $X$  and other important quantities. The physical meaning of these relations will be explained in detail later (see Remarks 7.2~7.4).

Let  $(\alpha_1, \beta_1, \rho_1)$  be any element of  $\mathcal{L}_1$  and  $\xi = (\xi(n); n \in \mathbb{Z})$  be a normalized Gaussian white noise. By Theorem 6.2, we obtain a real stationary Gaussian process  $X = (X(n); n \in \mathbb{Z})$  as the unique solution of the first KMO-Langevin equation:

$$(7.1) \quad X(n) - X(n-1) = -\beta_1(X(n)+X(n-1)) - (\gamma_1 * X)(n) + \alpha_1 \xi(n) \\ \text{a.s. } (n \in \mathbb{Z}),$$

where  $\gamma_1$  is given by (5.10). This process has the covariance function  $R = R_\sigma \in \mathcal{Q}^1(\mathbb{Z})$ , with  $\sigma = L_1^{-1}((\alpha_1, \beta_1, \rho_1))$ .

We will begin with

Lemma 7.1. The following limit exists:

$$(7.2) \quad D \equiv \lim_{N \rightarrow \infty} \frac{1}{2N} E\left(\left(\sum_{n=0}^N X(n)\right)^2\right) = \sum_{n=0}^{\infty} R(n) - \frac{R(0)}{2}.$$

Proof. For any  $N \in \mathbb{N}$ ,

$$E\left(\left(\sum_{n=0}^N X(n)\right)^2\right) = (N+1)R(0) + 2 \sum_{n=0}^{N-1} \left(\sum_{m=n+1}^N R(n-m)\right) \\ = (N+1)R(0) + 2 \sum_{n=1}^N \left(\sum_{\varrho=1}^n R(\varrho)\right).$$

Therefore, we have the assertion, noting that  $R \in \mathcal{Q}^1(Z)$  .

(Q.E.D.)

Definition 7.1. In view of the definition of diffusion constant for the continuous-time case (cf.(2.30) in [10]), the above limit  $D$  is called the diffusion constant of the process  $X$  .

As a discrete analogue of Theorem 2.1 in [10], we will show

Theorem 7.1.

(i) For any  $\theta \in (-\pi, \pi)$

$$\frac{1}{\beta_1(1+e^{i\theta}) + 1-e^{i\theta} + 2\pi\tilde{\gamma}_1(\theta)} = \frac{h(e^{i\theta})}{2 \lim_{\tau \downarrow -\pi} h(e^{i\tau})} .$$

(ii)  $\frac{\alpha_1^2}{2} = R(0)C_{\beta_1, \gamma_1}$  ,

where

(7.3)  $C_{\beta_1, \gamma_1} = \pi \left( \int_{-\pi}^{\pi} |\beta_1(1+e^{i\theta}) + 1-e^{i\theta} + 2\pi\tilde{\gamma}_1(\theta)|^{-2} d\theta \right)^{-1}$  .

(iii)  $D = \frac{\alpha_1^2}{2(2\beta_1)^2}$

(iv)  $D = \frac{R(0)}{2\beta_1} \frac{C_{\beta_1, \gamma_1}}{2\beta_1}$

(v)  $\frac{C_{\beta_1, \gamma_1}}{2\beta_1} - 1 = \frac{1}{R(0)} \int_{-1}^1 \int_{-1}^1 \frac{1+t}{1-tu} \sigma(dt) \rho_1(du)$  .

Proof. By noting (5.11), we see from Corollary 4.1 that

(7.4)  $2 \lim_{\theta \downarrow -\pi} h(e^{i\theta}) = \frac{\alpha_1}{\sqrt{2\pi}}$  .

Therefore, (i) follows from Corollary 4.1 (i), (5.10) and Proposition 5.1 (ii).

By using Corollary 4.1 (i), (5.10) and Proposition 5.1 (ii) again, we see from (2.7) that

$$\Delta(\theta) = \frac{\alpha_1^2}{2\pi} |\beta_1(1+e^{i\theta}) + 1-e^{i\theta} + 2\pi\tilde{\gamma}_1(\theta)|^{-2} \quad \text{a.e. } \theta \in (-\pi, \pi) .$$

By integrating both hand sides with respect to  $\theta$ , we have (ii).

We now compute the diffusion constant  $D$ . By (3.1) and (3.7), we have

$$(7.5) \quad \sum_{n=0}^{\infty} R(n) = \int_{-1}^1 \frac{1}{1-t} \sigma(dt)$$

$$(7.6) \quad \int_{-1}^1 \frac{1}{1-t} \sigma(dt) = \frac{1}{2} \int_0^{\infty} (1+\frac{1}{\lambda}) \sigma_c(d\lambda)$$

$$(7.7) \quad R(0) = \sigma_c([0, \infty)) .$$

Hence, appealing to the result in [10], Theorem 2.1(iii), we get

$$\begin{aligned} D &= \frac{1}{2} \left( \int_0^{\infty} (1+\frac{1}{\lambda}) \sigma_c(d\lambda) - \sigma_c([0, \infty]) \right) = \frac{1}{2} \int_0^{\infty} \frac{1}{\lambda} \sigma_c(d\lambda) \\ &= \frac{\alpha_c^2}{4\beta_c^2} = \frac{\alpha_1^2}{8\beta_1^2} , \end{aligned}$$

which completes the proof of (iii). (iv) is an immediate consequence of (ii) and (iii).

Now, we proceed to the proof of (v). We first claim

$$(7.8) \quad E(0) = \frac{\sqrt{2\pi}\alpha_1}{1+\beta_1+\gamma_1(0)}$$

$$(7.9) \quad E(n) = \frac{1-\beta_1}{1+\beta_1} E(0) - \frac{2\beta_1}{1+\beta_1} \sum_{m=1}^{n-1} E(m) - \frac{1}{1+\beta_1} \sum_{m=1}^n (\gamma_1 * E)(m) \quad (n \geq 1) .$$

By Theorems 4.2 and 4.3, we have

$$(7.10) \quad \frac{1}{2\pi} \sum_{n=0}^{\infty} E(n) z^n = h(z) \quad (z \in U_1(0)) .$$

In particular,

$$E(0) = 2\pi h(z) \Big|_{z=0} .$$

Therefore, by taking  $z = 0$  (resp.  $n=0$ ) in Theorem 4.1 (resp. Proposition 5.1(i)), we get (7.8). Furthermore, by using Theorem 4.1 again, we see from (7.10) that for any  $n \in \mathbb{N}$

$$E(n) - E(n-1) = -\beta_1 (E(n)+E(n-1)) - (\gamma_1 * E)(n)$$

and so

$$\begin{aligned} E(n) - E(0) &= \sum_{m=1}^n (E(m) - E(m-1)) \\ &= -\beta_1 (E(0)+E(n)+2 \sum_{m=1}^{n-1} E(m)) - \sum_{m=1}^n (\gamma_1 * E)(m) , \end{aligned}$$

which implies (7.9).

Next we prove the key formula

$$(7.11) \quad \begin{aligned} 2\pi R(0) &= \left(1 + \frac{\gamma_1(0)}{1+\beta_1}\right) E(0) \sum_{n=0}^{\infty} E(n) - \frac{4\pi\beta_1}{1+\beta_1} \sum_{n=1}^{\infty} R(n) \\ &\quad - \frac{2\pi}{1+\beta_1} \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} R(m+k) \gamma_1(k) \right) . \end{aligned}$$

By substituting (7.8) and (7.9) into (2.13), we have

$$\begin{aligned} 2\pi R(0) &= E(0)^2 + \frac{1-\beta_1}{1+\beta_1} E(0) \sum_{n=1}^{\infty} E(n) - \frac{2\beta_1}{1+\beta_1} \sum_{n=1}^{\infty} \left( \sum_{\varrho=1}^{n-1} E(\varrho) \right) E(n) \\ &\quad - \frac{1}{1+\beta_1} \sum_{n=1}^{\infty} \left( \sum_{\varrho=1}^n (\gamma_1 * E)(\varrho) \right) E(n) . \end{aligned}$$

On the other hand, we see from (2.11), (2.13) and (6.2) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \left( \sum_{\varrho=1}^{n-1} E(\varrho) \right) E(n) &= \sum_{\varrho=1}^{\infty} \left( \sum_{m=1}^{\infty} E(\varrho+m) \right) E(\varrho) \\
&= \sum_{m=1}^{\infty} \left( \sum_{\varrho=0}^{\infty} E(m+\varrho) E(\varrho) - E(0) E(m+1) \right) \\
&= 2\pi \sum_{m=1}^{\infty} R(m) - E(0) \sum_{m=1}^{\infty} E(m) ,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left( \sum_{\varrho=1}^n (\gamma_1 * E)(\varrho) \right) E(n) \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left( \sum_{\varrho=1}^{\infty} E(\varrho+m) E(\varrho-k) \right) \gamma_1(k) \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \gamma_1(k) \left( \sum_{\varrho=0}^{\infty} E(\varrho+m) E(\varrho-k) - E(m) E(-k) \right) \\
&= 2\pi \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} R(m+k) \gamma_1(k) \right) - \gamma_1(0) E(0) \sum_{m=0}^{\infty} E(m) .
\end{aligned}$$

And so

$$\begin{aligned}
2\pi R(0) &= E(0)^2 + \frac{1-\beta_1}{1+\beta_1} E(0) \sum_{n=1}^{\infty} E(n) \\
&\quad - \frac{2\beta_1}{1+\beta_1} \left( 2\pi \sum_{n=1}^{\infty} R(n) - E(0) \sum_{m=1}^{\infty} E(m) \right) \\
&\quad - \frac{1}{1+\beta_1} \left\{ 2\pi \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} R(m+k) \gamma_1(k) \right) - \gamma_1(0) E(0) \sum_{m=0}^{\infty} E(m) \right\} \\
&= E(0)^2 + E(0) \sum_{n=1}^{\infty} E(n) + \frac{\gamma_1(0)}{1+\beta_1} E(0) \sum_{m=0}^{\infty} E(m) \\
&\quad - \frac{4\pi\beta_1}{1+\beta_1} \sum_{n=1}^{\infty} R(n) - \frac{2\pi}{1+\beta_1} \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} R(m+k) \gamma_1(k) \right) ,
\end{aligned}$$

which implies (7.11).

We are now ready to show

$$(7.12) \quad \frac{C_{\beta_1, \gamma_1}}{2\beta_1} - 1 = \frac{1}{R(0)} \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} R(m+k) \gamma_1(k) \right),$$

which is proved in the following manner.

By (7.10),

$$\sum_{n=0}^{\infty} E(n) = 2\pi \lim_{x \uparrow 1} h(x).$$

And so by (4.11)

$$(7.13) \quad \sum_{n=0}^{\infty} E(n) = \frac{\sqrt{2\pi}\alpha_1}{2\beta_1}.$$

Combining this with (7.8), we have

$$\left(1 + \frac{\gamma_1(0)}{1+\beta_1}\right) E(0) \sum_{n=0}^{\infty} E(n) = \frac{\pi\alpha_1^2}{\beta_1(1+\beta_1)}.$$

On the other hand, by Lemma 7.1 and Theorem 7.1 (iii),

$$\sum_{n=1}^{\infty} R(n) = \frac{\alpha_1^2}{8\beta_1^2} - \frac{R(0)}{2}.$$

Therefore, by combining these with (7.11), we see that

$$\begin{aligned} R(0) &= \frac{\alpha_1^2}{2\beta_1(1+\beta_1)} - \frac{2\beta_1}{1+\beta_1} \left( \frac{\alpha_1^2}{8\beta_1^2} - \frac{R(0)}{2} \right) \\ &\quad - \frac{1}{1+\beta_1} \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} R(m+k) \gamma_1(k) \right) \end{aligned}$$

and so

$$R(0) = \frac{\alpha_1^2}{4\beta_1} - \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} R(m+k) \gamma_1(k) \right),$$

which, together with Theorem 7.1 (ii), implies (7.12).



Next we claim

$$(7.14) \quad \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} R(m+k) \gamma_1(k) \right) = \int_{-1}^1 \int_{-1}^1 \frac{1+t}{1-tu} \sigma(dt) \rho_1(du) .$$

By (3.1) and Proposition 5.1(1) ,

the left hand side of (7.14)

$$\begin{aligned} &= \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} R(m+k) \right) \gamma_1(k) \\ &= \sum_{k=0}^{\infty} \left( \int_{-1}^1 \frac{t^k}{1-t} \sigma(dt) \right) \gamma_1(k) \\ &= \int_{-1}^1 \left( \sum_{k=0}^{\infty} \gamma_1(k) t^k \right) \frac{\sigma(dt)}{1-t} \\ &= \int_{-1}^1 \left( (1-t^2) \int_{-1}^1 \frac{\rho_1(du)}{1-tu} \right) \frac{\sigma(dt)}{1-t} , \end{aligned}$$

which implies (7.14). Thus, we conclude from (7.12) and (7.14) that (v) holds. (Q.E.D.)

Before we go into the explanation of the physical meaning of Theorem 7.1, we will consider the simplest

Example 7.1. Let  $X_p$  be the same stochastic process as in Example 6.1, and  $D_p$  be the diffusion constant of  $X_p$ . By (3.2) and Lemma 7.1,

$$(7.15) \quad R_p(0) = 1$$

$$(7.16) \quad D_p = \frac{1+p}{2(1-p)} .$$

By combining these with (4.21), we see that

$$(7.19) \quad \frac{(\alpha_p^{(1)})^2}{2} = R_p(0) (2D_p^{(1)})$$

$$(7.20) \quad D_p = \frac{R_p^{(0)}}{2B_p^{(1)}},$$

which, together with Theorem 7.1 (ii) or Theorem 7.1 (iv), imply

$$(7.21) \quad C_{B_p^{(1)}, \gamma_p^{(1)}} = C_{B_p^{(1)}, 0} = 2B_p^{(1)}.$$

In addition, we see from (3.2), (4.19) and (4.21) that a remarkable relation between  $R_p$  and  $E_p$  holds:

$$(7.22) \quad R_p^{(n)} = \frac{1+B_p^{(1)}}{\sqrt{2\pi\alpha_p^{(1)}}} E_p^{(n)} \quad (n \geq 0).$$

We will return to the general case and give some characterization of the simple Markovian property. As a discrete analogue of Theorem 2.2 in [10], we can see from Theorem 7.1 that

Theorem 7.2.

$$(i) \quad \frac{C_{B_1, \gamma_1}}{2B_1} \geq 1$$

(ii) The following four statements are equivalent;

$$(a) \quad \frac{C_{B_1, \gamma_1}}{2B_1} = 1$$

$$(b) \quad \gamma_1 = 0$$

$$(c) \quad \rho_1 = 0$$

$$(d) \quad X = X_p \quad \text{with some } p \in (-1, 1).$$

Remark 7.1. As we have seen in Theorem 2.2 in [10], the relation (7.22) characterizes the simple Markovian property for the continuous-time processes. However, this is no longer true

for the present discrete-time processes. We will give such an example. Let  $X$  be a real stationary Gaussian process discussed in the case (i) of Example 6.2 such that

$$(7.23) \quad p_1 = -p_2 \quad \text{and} \quad \sigma_1 = \sigma_2 = \frac{1}{2(1-p_1^2)} .$$

It then follows from Theorem 4.3 and (4.27) the canonical representation kernel  $E$  of  $X$  becomes

$$(7.24) \quad E(n) = \chi_{[0, \infty)}(n) \sqrt{2\pi(1+p_1^2)} (p_1^n + (-p_1)^n) ,$$

which implies the desired relation

$$(7.25) \quad R(n) = \frac{1}{2(1-p_1^2)\sqrt{2(1-p_1^2)}} E(n) \quad (n \geq 0) .$$

Finally we will give three remarks concerning the physical meaning of Theorem 7.1 (cf. [10] for the continuous-time case).

Remark 7.2. In relation (i) in Theorem 7.1, the left hand side denotes a complex mobility of the system  $X$  described by equation (7.1), which represents the response of the system  $X$  to the external force  $\alpha_1 \mathfrak{F}$ . On the other hand, the right hand side is determined by the outer function of  $X$ , which represents the thermal fluctuation of the system in equilibrium without the external force. The relation (i) in Theorem 7.1 might be said to be the generalized first fluctuation-dissipation theorem.

Remark 7.3. We are now concerned with relation (ii) in Theorem 7.1. The fluctuation power of a random force  $\alpha_1 \mathfrak{F}$  in

equation (7.1) is  $\frac{\alpha_1^2}{2}$ . While, the positive constant  $C_{\beta_1, \gamma_1}$  is expressed in terms of the drift coefficient representing the systematic part of equation (7.1). And from the physical point of view we can regard  $R(0)$  as the absolute constant  $k T$ , where  $k$  and  $T$  denote the Boltzman constant and absolute temperature of the system in equilibrium, respectively. This leads us to think of  $C_{\beta_1, \gamma_1}$  as the generalized friction constant, and the relation (ii) itself might be said to be the generalized second fluctuation-dissipation theorem.

Remark 7.4. For the Markov process  $X_p$  in Example 7.1, we found that the diffusion constant  $D_p$  is inversely proportional to the friction constant  $\beta_p^{(1)}$ . This relation (7.20) is analogous to the classical Einstein relation valid for the Ornstein-Uhlenbeck Brownian motion with continuous time (see (2.29) in [10]). For this reason, we call relation (7.20) for  $X_p$  the Einstein relation. In a general system described by equation (7.1) with  $\gamma_1 \neq 0$ , however, we found a significant deviation (iv) in Theorem 7.1 from the Einstein relation (7.20) with  $\gamma_1 = 0$ , and obtained the formula (v) in Theorem 7.1 expressing the degree of such a deviation. In view of the analogous fact in the continuous-time case (Theorem 2.1 in [10]), we call the relation (iv) in Theorem 7.1 the generalized Einstein relation.

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