A factorization theorem for unfoldings of analytic functions

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Series #8. July 1987
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A factorization theorem for unfoldings of analytic functions

Tatsuo Suwa

Abstract

Let \( \tilde{f} \) and \( g \) be holomorphic function germs at 0 in \( \mathbb{C}^n \times \mathbb{C}^q \). If \( d_x g \wedge d_x \tilde{f} = 0 \) and if \( f(x) = \tilde{f}(x,0) \) is not a power or a unit, then there exists a germ \( \lambda \) at 0 in \( \mathbb{C} \times \mathbb{C}^q \) such that \( g(x,s) = \lambda(\tilde{f}(x,s),s) \). The result has the implication that the notion of an RL-morphism in the unfolding theory of foliation germs generalizes that of a right-left morphism in the function germ case.

The notion of an RL-morphism in the unfolding theory of foliation singularities was introduced in [5] to describe the determinacy results and in [6] the versality theorem for these morphisms is proved. This note, which should be considered as an appendix to [5] or [6], contains a factorization theorem implying that an RL-morphism is a generalization of a right-left morphism in the unfolding theory of function germs. It depends on the Mattei-Moussu factorization theorem ([1]) and is a generalization of a result of Moussu [2].

1980 Mathematics Subject Classification (1985 Revision).
Primary 32A10, 32G11 ; Secondary 58C27, 58F14.
A codim 1 foliation germ at 0 in $\mathbb{C}^n$ is a module $F = (\omega)$ over the ring of holomorphic function germs generated by a germ of an integrable 1-form $\omega$ (see Section 2). An unfolding of $F$ with parameter space $\mathbb{C}^m = \{t\}$ is a codim 1 foliation germ $\tilde{F} = (\tilde{\omega})$ at 0 in $\mathbb{C}^n \times \mathbb{C}^m$ with a generator $\tilde{\omega}$ whose restriction to $\mathbb{C}^n \times \{0\}$ is $\omega$. We let $F_t$ be the foliation germ generated by the restriction $\omega_t$ of $\tilde{\omega}$ to $\mathbb{C}^n \times \{t\}$. Let $F'$ be another unfolding of $F$ with parameter space $\mathbb{C}^q = \{s\}$. A morphism from $F'$ to $\tilde{F}$ is a holomorphic map germ $\phi: (\mathbb{C}^n \times \mathbb{C}^q, 0) \to (\mathbb{C}^n \times \mathbb{C}^m, 0)$ such that (a) $\phi(x, s) = (\phi(x, s), \psi(s))$ for some holomorphic map germs $\phi: (\mathbb{C}^n \times \mathbb{C}^q, 0) \to (\mathbb{C}^q, 0)$ and $\psi: (\mathbb{C}^q, 0) \to (\mathbb{C}^m, 0)$, (b) $\phi(x, 0) = x$ and (c) the pull back $\phi^*\tilde{\omega}$ of $\tilde{\omega}$ by $\phi$ generates $F'$. Thus, if we set $\phi_s(x) = \phi(x, s)$, we may think of $(\phi_s)$ as a family of local coordinate changes of $(\mathbb{C}^n, 0)$. For an RL-morphism, in place of (c), we only require that $\phi_s^*\omega_s \psi(s)$ generates $F'_s$ for each $s$ (see (2.1) Definition). Our previous result shows that if $F$ has a generator of the form $df$ for some holomorphic function germ $f$ (strong first integral for $F$), then every unfolding of $F$ admits a generator of the form $\tilde{\omega}$ with $\tilde{\omega}$ an unfolding of $f$. In the unfolding theory of function germs, there are notions of a right morphism and a right-left morphism. The former involves coordinate changes in the source space $(\mathbb{C}^n, 0)$, whereas the latter involves coordinate changes in the target space $\mathbb{C}$ as well. It is not difficult to see that our morphism generalizes a right morphism in the sense that when $F$ admits a strong first integral $f$, then it becomes a (strict) right morphism in the unfolding theory of $f$. For a foliation
without first integrals, it may not seem relevant to talk about right-left morphisms. However, as stated above, our factorization theorem shows that an RL-morphism is a natural generalization of a right-left morphism, since when $F = (df)$, an RL-morphism is exactly a right-left morphism in the unfolding theory of $f$. We also note that RL-morphisms are closely related to integrating factors of the foliation ((2.2) Remark 2).
1. The factorization theorem.

We denote by $O_n$ the ring of germs of holomorphic functions at the origin 0 in $\mathbb{C}^n = \{(x_1, \cdots, x_n)\}$. A germ $f$ in $O_n$ is said to be a power if $f = f_0^m$ for some positive integer $m$ and a non-unit $f_0$ in $O_n$. If we denote the critical set of $f$ by $C(f)$, then $\text{codim} \ C(f) \geq 2$ implies that $f$ is not a power. We quote the following factorization theorem of Mattei and Moussu.

(1.1) Theorem ([1]). Let $f$ be a germ in $O_n$ which is not a power or a unit. If $g$ is a germ in $O_n$ with $dg \wedge df = 0$, then there exists a germ $\lambda$ in $O_1$ such that $g = \lambda \cdot f$.

The theorem is proved using the reduction theory of singularities of holomorphic 1-forms due to Seidenberg and Van den Essen. The proof is rather simple if we assume $\text{codim} \ C(f) \geq 2$ (see Moussu-Tougeron[3]). If $\tilde{f}$ is a germ in $O_{n+\mathbb{A}^1}$, we may think of $\tilde{f}$ as an unfolding of $f(x) = \tilde{f}(x,0)$ with parameter space $\mathbb{C}^\mathbb{A} = \{(s_1, \cdots, s_{\mathbb{A}})\}$. We denote by $d_x f$ the exterior derivation with respect to $x$; $d_x \tilde{f} = \sum_{i=1}^{\mathbb{A}} \frac{\partial f}{\partial x_i}(x,s)dx_i$.

(1.2) Theorem. Let $\tilde{f}$ be a germ in $O_{n+\mathbb{A}^1}$ such that $f(x) = \tilde{f}(x,0)$ is not a power or a unit in $O_n$. If $g$ is a germ in $O_{n+\mathbb{A}^1}$ with $d_x g \wedge d_x \tilde{f} = 0$, then there exists a germ $\lambda$ in $O_{1+\mathbb{A}^1}$ such that $g(x,s) = \lambda(\tilde{f}(x,s),s)$.

Proof. First we show the existence of $\lambda$ as a formal power series in $s$. Thus we express $\lambda$ as
\[ \lambda(y,s) = \sum_{|\nu| \geq 0} \lambda^{(\nu)}(y) s^\nu, \quad \lambda^{(\nu)} \in O_1, \]

where \( \nu \) denote an \( n \)-tuple \( (\nu_1, \ldots, \nu_n) \) of non-negative integers, \( |\nu| = \nu_1 + \cdots + \nu_n \) and \( s^\nu = s_1^{\nu_1} \cdots s_n^{\nu_n} \). In general, if \( \sigma = \sum_{|\nu| \geq 0} \sigma^{(\nu)} s^\nu \) is a series in \( s \) with \( \sigma^{(\nu)} \in O_n^r \) for some \( r \), we set

\[
[\sigma]_p = \sum_{|\nu| = p} \sigma^{(\nu)} s^\nu \quad \text{and} \quad \sigma|_p = \sum_{|\nu| = 0} \sigma^{(\nu)} s^\nu
\]

for a non-negative integer \( p \).

We look for \( \lambda \) satisfying the congruence

\[ (1.3)_p \quad g(x,s) = \lambda|_p(f(x),s),s) \]

for \( p \geq 0 \), where \( = \) denotes the equality mod \( s^\nu \), \( |\nu| = p + 1 \).

First, \( (1.3)_0 \) is equivalent to

\[ g(x,0) = \lambda^{(0)}(f(x)) \, . \]

From the condition of the theorem, we have \( d(g(x,0)) \wedge df = 0 \).

Hence by (1.1), there exists a germ \( \lambda^{(0)} \) in \( O_1 \) satisfying the above. Now we suppose that we have \( \lambda|_p \) satisfying \( (1.3)_p \) and look for \( [\lambda]_{p+1} \). The congruence \( (1.3)_{p+1} \) reads

\[ g(x,s) = \sum_{|\nu| = p+1} \lambda^{(\nu)}(f(x)) s^\nu + \lambda|_p(f(x),s),s) \, . \]

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Hence, for our purpose, it suffices to show that

\[(1.4) \quad d_x [g(x,s) - \lambda \|p(f(x,s),s)]_{p+1} \wedge df = 0 .\]

By \((1.3)_p\), we have \((1.4)\) if we show that

\[d_x (g(x,s) - \lambda \|p(f(x,s),s)) \wedge d_x f = 0 .\]

But this follows from the condition of the theorem and

\[d_x (g(x,s) - \lambda \|p(f(x,s),s)) \wedge d_x f = 0 .\]

Thus we have a formal power series

\[\lambda(y,s) = \sum_{|\nu| \geq 0} \lambda^{(\nu)}(y)s^\nu , \quad \lambda^{(\nu)} \in O_1\]

in \(s\) such that \(g(x,s) = \lambda(\tilde{f}(x,s),s)\) as power series in \((x,s)\).

Since \(\tilde{f}\) and \(g\) are both convergent, \(\lambda\) must be also convergent.

\((1.5)\) Remarks.

1. The germ \(\lambda\) is determined uniquely by \(g\) (and \(\tilde{f}\)). If we assume that \(g(x,0) = f(x)\), then \(\lambda(y,0) = y\).

2. The above theorem generalizes Corollaire 1 in [2] Ch.II,1 in the case \(X = H\).
2. Some types of morphisms in the unfolding theory of foliation germs.

We denote by $\mathcal{Q}_n$ the $\mathcal{Q}_n$-modules of germs of holomorphic 1-forms at $0$ in $\mathbb{C}^n$. We recall (Cf. [4], [5]) that a codim 1 foliation germ at $0$ in $\mathbb{C}^n$ is a rank 1 free sub-$\mathcal{Q}_n$-module $F = (\omega)$ of $\mathcal{Q}_n$ with a generator satisfying the integrability condition $d\omega \wedge \omega = 0$. The singular set $\mathcal{S}(F)$ of $F$ is defined to be the singular set $\{x \mid \omega(x) = 0\}$ of $\omega$. We always assume that codim $S(F) \geq 2$. An unfolding of $F = (\omega)$ is a codim 1 foliation germ $\mathcal{F} = (\tilde{\omega})$ at $0$ in $\mathbb{C}^n \times \mathbb{C}^m = \{(x, t)\}$, for some $m$, with a generator $\tilde{\omega}$ satisfying $\iota^* \tilde{\omega} = \omega$, where $\iota$ denotes the embedding of $\mathbb{C}^n$ into $\mathbb{C}^n \times \mathbb{C}^m$ given by $\iota(x) = (x, 0)$. We call $\mathbb{C}^m$ the parameter space of $\mathcal{F}$. We recall the following definition ([5](2.1), [6](1.1)).

(2.1) Definition. Let $\mathcal{F}$ and $\mathcal{F}'$ be two unfoldings of $F$ with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^\alpha = ((s_1, \cdots, s_\alpha))$, respectively.

(I) An RL-morphism from $\mathcal{F}'$ to $\mathcal{F}$ is a pair $(\Phi, \Psi)$ satisfying the following conditions:

(a) $\Phi$ and $\Psi$ are holomorphic map germs making the diagram

$$
\begin{array}{ccc}
(C^n \times C^\alpha, 0) & \xrightarrow{\Phi} & (C^n \times C^m, 0) \\
\downarrow & & \downarrow \\
(C^\alpha, 0) & \xrightarrow{\Psi} & (C^m, 0)
\end{array}
$$

commutative, where the vertical maps are the projections.
(b) \( \Phi(x,0) = (x,0) \).

(c) For any generator \( \tilde{\omega} \) of \( \tilde{\mathcal{F}} \), there is a germ \( \alpha = (\alpha_1, \ldots, \alpha_\lambda) \) in \( O_{n+\lambda}^\mathbb{Q} \) such that the germ

\[
\Phi^*\tilde{\omega} + \sum_{k=1}^\lambda \alpha_k ds_k
\]

generates \( \tilde{\mathcal{F}}' \).

(II) A morphism from \( \mathcal{F}' \) to \( \mathcal{F} \) is an RL-morphism such that for any generator \( \tilde{\omega} \) of \( \tilde{\mathcal{F}} \), we may choose \( \alpha = 0 \) in (c).

(2.2) Remarks.

1. In the both cases, we may replace "any" by "some".

2. From the integrability condition we see that, for \( \alpha \) in (c), each \( \alpha_k(x,0) \) is an integrating factor of \( \omega = \tilde{\omega}^* \), i.e.,

\[
\alpha_k(x,0)d\omega = d(\alpha_k(x,0)) \wedge \omega.
\]

3. We have a "versality theorem" for each type of morphisms ([4],[6]).

If a germ \( \tilde{f} \) in \( O_{n+m}^\mathbb{Q} \) is an unfolding of \( f \), i.e., if \( \tilde{f} = f \), then \( \tilde{\mathcal{F}} = (df) \) is an unfolding of \( \mathcal{F} = (df) \) with parameter space \( \mathbb{C}^m \) and conversely, any unfolding of \( \mathcal{F} = (df) \) has a generator of the form \( df \) with \( \tilde{f} \) an unfolding of \( f \) ([4] p.47).

We recall the following definition (cf.[7] Definition 3.2).

(2.4) Definition. Let \( \tilde{f} \) and \( g \) be two unfoldings of \( f \) with parameter spaces \( \mathbb{C}^m \) and \( \mathbb{C}^\mathbb{Q} = ((s_1, \ldots, s_\lambda)) \), respectively.

(I) A right-left morphism from \( g \) to \( \tilde{f} \) is a pair \( (\Phi, \Psi) \) satisfying (I)(a) and (b) in (2.1) Definition and
(c) $g(x,s) = \lambda(\Phi^* f(x,s), s)$

for some $\lambda$ in $O_{1+\varnothing}$ with $\lambda(y,0) = y$.

(II) A strict right morphism from $g$ to $\tilde{f}$ is a right-left morphism such that $\lambda(y,s) = y$ in (c).

The following is a direct consequence of (1.2) Theorem.

(2.4) Proposition. Let $\tilde{f}$ and $g$ be unfoldings of $f$. A pair $(\Phi, \psi)$ is, respectively, a right-left morphism or a strict right morphism from $g$ to $\tilde{f}$ if and only if it is an RL-morphism or a morphism from $\tilde{F}' = (dg)$ to $\tilde{F} = (df)$.
References


