<table>
<thead>
<tr>
<th>Title</th>
<th>A factorization theorem for unfoldings of analytic functions</th>
</tr>
</thead>
<tbody>
<tr>
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A factorization theorem for unfoldings
of analytic functions

Tatsuo Suwa

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<table>
<thead>
<tr>
<th>#</th>
<th>Author</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Y. Okabe</td>
<td>On the theory of discrete KMO-Langevin equations with reflection positivity (I)</td>
</tr>
<tr>
<td>2.</td>
<td>Y. Giga and T. Kambe</td>
<td>Large time behavior of the vorticity of two-dimensional flow and its application to vortex formation</td>
</tr>
<tr>
<td>3.</td>
<td>A. Arai</td>
<td>Path Integral Representation of the Index of Kahler-Dirac Operators on an Infinite Dimensional Manifold</td>
</tr>
<tr>
<td>4.</td>
<td>I. Nakamura</td>
<td>Threefolds Homeomorphic to a Hyperquadric in $\mathbb{P}^4$</td>
</tr>
<tr>
<td>5.</td>
<td>T. Nakazi</td>
<td>Notes on Interpolation by Bounded Analytic Functions</td>
</tr>
<tr>
<td>6.</td>
<td>T. Nakazi</td>
<td>A Spectral Dilation of Some Non-Dirichlet Algebra</td>
</tr>
<tr>
<td>7.</td>
<td>H. Hida</td>
<td>A $p$-adic measure attached to the zeta functions associated with two elliptic modular forms II</td>
</tr>
</tbody>
</table>
A factorization theorem for unfoldings of analytic functions

Tatsuo Suwa

Abstract

Let \( \tilde{f} \) and \( g \) be holomorphic function germs at 0 in \( \mathbb{C}^n \times \mathbb{C}^q = \{(x,s)\} \). If \( d_x g \wedge d_x \tilde{f} = 0 \) and if \( f(x) = \tilde{f}(x,0) \) is not a power or a unit, then there exists a germ \( \lambda \) at 0 in \( \mathbb{C} \times \mathbb{C}^q \) such that \( g(x,s) = \lambda(f(x,s),s) \). The result has the implication that the notion of an RL-morphism in the unfolding theory of foliation germs generalizes that of a right-left morphism in the function germ case.

The notion of an RL-morphism in the unfolding theory of foliation singularities was introduced in [5] to describe the determinacy results and in [6] the versality theorem for these morphisms is proved. This note, which should be considered as an appendix to [5] or [6], contains a factorization theorem implying that an RL-morphism is a generalization of a right-left morphism in the unfolding theory of function germs. It depends on the Mattei-Moussu factorization theorem ([1]) and is a generalization of a result of Moussu [2].

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Primary 32A10, 32G11 ; Secondary 58C27, 58F14.
A codim 1 foliation germ at 0 in $\mathbb{C}^n$ is a module $F = (\omega)$ over the ring of holomorphic function germs generated by a germ of an integrable 1-form $\omega$ (see Section 2). An unfolding of $F$ with parameter space $\mathbb{C}^m = \{t\}$ is a codim 1 foliation germ $\widetilde{F} = (\tilde{\omega})$ at 0 in $\mathbb{C}^n \times \mathbb{C}^m$ with a generator $\tilde{\omega}$ whose restriction to $\mathbb{C}^n \times \{0\}$ is $\omega$. We let $F_t$ be the foliation germ generated by the restriction $\omega_t$ of $\tilde{\omega}$ to $\mathbb{C}^n \times \{t\}$. Let $F'$ be another unfolding of $F$ with parameter space $\mathbb{C}^q = \{s\}$. A morphism from $F'$ to $\tilde{F}$ is a holomorphic map germ $\phi : (\mathbb{C}^n \times \mathbb{C}^q, 0) \to (\mathbb{C}^n \times \mathbb{C}^m, 0)$ such that (a) $\phi(x,s) = (\phi(x,s), \psi(s))$ for some holomorphic map germs $\phi : (\mathbb{C}^n \times \mathbb{C}^q, 0) \to (\mathbb{C}^n, 0)$ and $\psi : (\mathbb{C}^q, 0) \to (\mathbb{C}^m, 0)$, (b) $\phi(x,0) = x$ and (c) the pull back $\phi^* \tilde{\omega}$ of $\tilde{\omega}$ by $\phi$ generates $F'$. Thus, if we set $\phi_s(x) = \phi(x,s)$, we may think of $(\phi_s)$ as a family of local coordinate changes of $(\mathbb{C}^n, 0)$. For an RL-morphism, in place of (c), we only require that $\phi_s^* \omega$ generates $F'_s$ for each $s$ (see (2.1) Definition). Our previous result shows that if $F$ has a generator of the form $df$ for some holomorphic function germ $f$ (strong first integral for $F$), then every unfolding of $F$ admits a generator of the form $\tilde{f}$ with $\tilde{f}$ an unfolding of $f$. In the unfolding theory of function germs, there are notions of a right morphism and a right-left morphism. The former involves coordinate changes in the source space $(\mathbb{C}^n, 0)$, whereas the latter involves coordinate changes in the target space $\mathbb{C}$ as well. It is not difficult to see that our morphism generalizes a right morphism in the sense that when $F$ admits a strong first integral $f$, then it becomes a (strict) right morphism in the unfolding theory of $f$. For a foliation
without first integrals, it may not seem relevant to talk about right-left morphisms. However, as stated above, our factorization theorem shows that an RL-morphism is a natural generalization of a right-left morphism, since when $F = (df)$, an RL-morphism is exactly a right-left morphism in the unfolding theory of $f$. We also note that RL-morphisms are closely related to integrating factors of the foliation ((2.2) Remark 2).
1. The factorization theorem.

We denote by \( O_n \) the ring of germs of holomorphic functions at the origin 0 in \( \mathbb{C}^n = \{(x_1, \ldots, x_n)\} \). A germ \( f \) in \( O_n \) is said to be a power if \( f = f_0^m \) for some positive integer \( m \) and a non-unit \( f_0 \) in \( O_n \). If we denote the critical set of \( f \) by \( C(f) \), then \( \text{codim} \ C(f) \geq 2 \) implies that \( f \) is not a power. We quote the following factorization theorem of Mattei and Moussu.

(1.1) Theorem ([1]). Let \( f \) be a germ in \( O_n \) which is not a power or a unit. If \( g \) is a germ in \( O_n \) with \( dg \wedge df = 0 \), then there exists a germ \( \lambda \) in \( O_1 \) such that \( g = \lambda \cdot f \).

The theorem is proved using the reduction theory of singularities of holomorphic 1-forms due to Seidenberg and Van den Essen. The proof is rather simple if we assume \( \text{codim} \ C(f) \geq 2 \) (see Moussu-Tougeron[3]). If \( \tilde{f} \) is a germ in \( O_{n+\lambda} \), we may think of \( \tilde{f} \) as an unfolding of \( f(x) = \tilde{f}(x,0) \) with parameter space \( \mathbb{C}^\lambda = \{(s_1, \ldots, s_\lambda)\} \). We denote by \( d_x \) the exterior derivation with respect to \( x \); \( d_x = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x,s)dx_i \).

(1.2) Theorem. Let \( \tilde{f} \) be a germ in \( O_{n+\lambda} \) such that \( f(x) = \tilde{f}(x,0) \) is not a power or a unit in \( O_n \). If \( g \) is a germ in \( O_{n+\lambda} \) with \( d_x g \wedge d_x \tilde{f} = 0 \), then there exists a germ \( \lambda \) in \( O_1+\lambda \) such that \( g(x,s) = \lambda(\tilde{f}(x,s),s) \).

Proof. First we show the existence of \( \lambda \) as a formal power series in \( s \). Thus we express \( \lambda \) as
\[ \lambda(y,s) = \sum_{|\nu| \geq 0} \lambda^{(\nu)}(y)s^\nu, \quad \lambda^{(\nu)} \in \mathcal{O}_1, \]

where \( \nu \) denote an \( \mathbb{Q} \)-tuple \((\nu_1, \ldots, \nu_\mathbb{Q})\) of non-negative integers, \( |\nu| = \nu_1 + \cdots + \nu_\mathbb{Q} \) and \( s^\nu = s_1^{\nu_1} \cdots s_\mathbb{Q}^{\nu_\mathbb{Q}} \). In general, if \( \sigma = \sum_{|\nu| \geq 0} \sigma^{(\nu)} s^\nu \) is a series in \( s \) with \( \sigma^{(\nu)} \in \mathcal{O}_n^r \) for some \( r \), we set

\[ [\sigma]_p = \sum_{|\nu| = p} \sigma^{(\nu)} s^\nu \quad \text{and} \quad [\sigma|_p = \sum_{|\nu| = 0}^{p} \sigma^{(\nu)} s^\nu \]

for a non-negative integer \( p \).

We look for \( \lambda \) satisfying the congruence

\[ (1.3)_p \quad g(x,s) = \lambda|_p (f(x,s),s) \]

for \( p \geq 0 \), where \( = \) denotes the equality mod \( s^\nu \), \( |\nu| = p + 1 \).

First, \((1.3)_0\) is equivalent to

\[ g(x,0) = \lambda^{(0)}(f(x)) \, . \]

From the condition of the theorem, we have \( d(g(x,0)) \wedge df = 0 \). Hence by \((1.1)\), there exists a germ \( \lambda^{(0)} \) in \( \mathcal{O}_1 \) satisfying the above. Now we suppose that we have \( \lambda|_p \) satisfying \((1.3)_p\) and look for \([\lambda]|_{p+1} \). The congruence \((1.3)_{p+1}\) reads

\[ g(x,s) = \sum_{|\nu| = p+1} \lambda^{(\nu)}(f(x))s^\nu + \lambda|_p (f(x,s),s) \, . \]
Hence, for our purpose, it suffices to show that

\[(1.4) \quad d_x [g(x,s) - \lambda |p(f(x,s),s)] p+1 \wedge df = 0.\]

By (1.3)p, we have (1.4) if we show that

\[d_x (g(x,s) - \lambda |p(f(x,s),s)) \wedge d_x f = 0.\]

But this follows from the condition of the theorem and

\[d_x \lambda |p(f(x,s),s) = \frac{\partial \lambda |p(f(x,s),s)}{\partial y} d_x f.\]

Thus we have a formal power series

\[\lambda(y,s) = \sum_{|\nu| \geq 0} \lambda^{(\nu)}(y)s^\nu, \quad \lambda^{(\nu)} \in O_1\]

in s such that \(g(x,s) = \lambda(\tilde{f}(x,s),s)\) as power series in \((x,s)\).

Since \(\tilde{f}\) and \(g\) are both convergent, \(\lambda\) must be also convergent.

\((1.5)\) Remarks.

1. The germ \(\lambda\) is determined uniquely by \(g\) (and \(\tilde{f}\)). If we assume that \(g(x,0) = f(x)\), then \(\lambda(y,0) = y\).
2. The above theorem generalizes Corollaire 1 in [2] Ch.II,1 in the case \(X = H\).
2. Some types of morphisms in the unfolding theory of foliation germs.

We denote by $Q_n$ the $O_n$-modules of germs of holomorphic 1-forms at 0 in $C^n$. We recall (Cf. [4], [5]) that a codim 1 foliation germ at 0 in $C^n$ is a rank 1 free sub-$O_n$-module $F = (\omega)$ of $Q_n$ with a generator satisfying the integrability condition $d\omega \wedge \omega = 0$. The singular set $S(F)$ of $F$ is defined to be the singular set \{x|\omega(x) = 0\} of $\omega$. We always assume that codim $S(F) \geq 2$. An unfolding of $F = (\omega)$ is a codim 1 foliation germ $\tilde{F} = (\tilde{\omega})$ at 0 in $C^n \times C^m = ((x,t))$, for some $m$, with a generator $\tilde{\omega}$ satisfying $i^*\tilde{\omega} = \omega$, where $i$ denotes the embedding of $C^n$ into $C^n \times C^m$ given by $i(x) = (x,0)$. We call $C^m$ the parameter space of $\tilde{F}$. We recall the following definition ([5](2.1), [6](1.1))

(2.1) Definition. Let $\tilde{T}$ and $\tilde{T}'$ be two unfoldings of $F$ with parameter spaces $C^m$ and $C^\alpha = ((s_1, \ldots, s_\alpha))$, respectively. (I) An RL-morphism from $\tilde{T}'$ to $\tilde{T}$ is a pair $(\Phi, \Psi)$ satisfying the following conditions:
(a) $\Phi$ and $\Psi$ are holomorphic map germs making the diagram

\[
\begin{array}{ccc}
(C^n \times C^\alpha, 0) & \xrightarrow{\Phi} & (C^n \times C^m, 0) \\
\downarrow & & \downarrow \\
(C^\alpha, 0) & \xrightarrow{\Psi} & (C^m, 0)
\end{array}
\]

commutative, where the vertical maps are the projections.
(b) $\Phi(x,0) = (x,0)$.

(c) For any generator $\tilde{\omega}$ of $\mathcal{F}$, there is a germ $\alpha = (\alpha_1, \ldots, \alpha_q)$ in $O_n^q$ such that the germ

$$\Phi^*\tilde{\omega} + \sum_{k=1}^q \alpha_k d s_k$$

generates $\mathcal{F}'$.

(II) A morphism from $\mathcal{F}'$ to $\mathcal{F}$ is an RL-morphism such that for any generator $\tilde{\omega}$ of $\mathcal{F}$, we may choose $\alpha = 0$ in (c).

(2.2) Remarks.

1. In the both cases, we may replace "any" by "some".

2. From the integrability condition we see that, for $\alpha$ in (c), each $\alpha_k(x,0)$ is an integrating factor of $\omega = \tilde{\omega}$, i.e.,

$$\alpha_k(x,0) d\omega = d(\alpha_k(x,0)) \wedge \omega.$$  

3. We have a "versality theorem" for each type of morphisms ([4],[6]).

If a germ $\tilde{f}$ in $O_n^m$ is an unfolding of $f$, i.e., $\tilde{f} = f$, then $\mathcal{F} = (df)$ is an unfolding of $F = (df)$ with parameter space $C^m$ and conversely, any unfolding of $F = (df)$ has a generator of the form $df$ with $\tilde{f}$ an unfolding of $f$([4] p.47). We recall the following definition (cf.[7] Definition 3.2).

(2.4) Definition. Let $\tilde{f}$ and $g$ be two unfoldings of $f$ with parameter spaces $C^m$ and $C^q = (s_1, \ldots, s_q)$, respectively.

(I) A right-left morphism from $g$ to $\tilde{f}$ is a pair $(\Phi, \Psi)$ satisfying (I)(a) and (b) in (2.1) Definition and
(c) \( g(x,s) = \lambda(\Phi^* f(x,s),s) \)

for some \( \lambda \) in \( O_{1+Q} \) with \( \lambda(y,0) = y \).

(II) A strict right morphism from \( g \) to \( \bar{f} \) is a right-left morphism such that \( \lambda(y,s) = y \) in (c).

The following is a direct consequence of (1.2) Theorem.

(2.4) Proposition. Let \( \bar{f} \) and \( g \) be unfoldings of \( f \). A pair \((\Phi,\psi)\) is, respectively, a right-left morphism or a strict right morphism from \( g \) to \( \bar{f} \) if and only if it is an RL-morphism or a morphism from \( \bar{f}' = (dg) \) to \( \bar{f} = (df) \).
References


