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A factorization theorem for unfoldings
of analytic functions

Tatsuo Suwa

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# Author Title

2. Y. Giga and T. Kambe, Large time behavior of the vorticity of two-dimensional flow and its application to vortex formation
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4. I. Nakamura, Threefolds Homeomorphic to a Hyperquadric in $P^4$
5. T. Nakazi, Notes on Interpolation by Bounded Analytic Functions
6. T. Nakazi, A Spectral Dilation of Some Non-Dirichlet Algebra
7. H. Hida, A $p$-adic measure attached to the zeta functions associated with two elliptic modular forms II
A factorization theorem for unfoldings of analytic functions

Tatsuo Suwa

Abstract

Let \( \tilde{f} \) and \( g \) be holomorphic function germs at 0 in \( \mathbb{C}^n \times \mathbb{C}^\lambda \). If \( d_x g \wedge d_x \tilde{f} = 0 \) and if \( f(x) = \tilde{f}(x,0) \) is not a power or a unit, then there exists a germ \( \lambda \) at 0 in \( \mathbb{C} \times \mathbb{C}^\lambda \) such that \( g(x,s) = \lambda(f(x,s),s) \). The result has the implication that the notion of an RL-morphism in the unfolding theory of foliation germs generalizes that of a right-left morphism in the function germ case.

The notion of an RL-morphism in the unfolding theory of foliation singularities was introduced in [5] to describe the determinacy results and in [6] the versality theorem for these morphisms is proved. This note, which should be considered as an appendix to [5] or [6], contains a factorization theorem implying that an RL-morphism is a generalization of a right-left morphism in the unfolding theory of function germs. It depends on the Mattei-Moussu factorization theorem ([1]) and is a generalization of a result of Moussu [2].

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A codim 1 foliation germ at 0 in \( \mathbb{C}^n \) is a module \( F = (\omega) \) over the ring of holomorphic function germs generated by a germ of an integrable 1-form \( \omega \) (see Section 2). An unfolding of \( F \) with parameter space \( \mathbb{C}^m = \{t\} \) is a codim 1 foliation germ \( \mathcal{F} = (\tilde{\omega}) \) at 0 in \( \mathbb{C}^n \times \mathbb{C}^m \) with a generator \( \tilde{\omega} \) whose restriction to \( \mathbb{C}^n \times \{0\} \) is \( \omega \). We let \( F_t \) be the foliation germ generated by the restriction \( \omega_t \) of \( \tilde{\omega} \) to \( \mathbb{C}^n \times \{t\} \). Let \( \mathcal{F}' \) be another unfolding of \( F \) with parameter space \( \mathbb{C}^q = \{s\} \). A morphism from \( \mathcal{F}' \) to \( \mathcal{F} \) is a holomorphic map germ \( \phi : (\mathbb{C}^n \times \mathbb{C}^q, 0) \rightarrow (\mathbb{C}^n \times \mathbb{C}^m, 0) \) such that (a) \( \phi(x, s) = (\phi(x, s), \psi(s)) \) for some holomorphic map germs \( \phi : (\mathbb{C}^n \times \mathbb{C}^q, 0) \rightarrow (\mathbb{C}^n, 0) \) and \( \psi : (\mathbb{C}^q, 0) \rightarrow (\mathbb{C}^m, 0) \), (b) \( \phi(x, 0) = x \) and (c) the pull back \( \phi^* \tilde{\omega} \) of \( \tilde{\omega} \) by \( \phi \) generates \( \mathcal{F}' \). Thus, if we set \( \phi_s(x) = \phi(x, s) \), we may think of \( \phi_s \) as a family of local coordinate changes of \( (\mathbb{C}^n, 0) \). For an RL-morphism, in place of (c), we only require that \( \phi_s^* \omega \psi(s) \) generates \( F'_s \) for each \( s \) (see (2.1) Definition). Our previous result shows that if \( F \) has a generator of the form \( df \) for some holomorphic function germ \( f \) (strong first integral for \( F \)), then every unfolding of \( F \) admits a generator of the form \( \tilde{\omega} \) with \( \tilde{\omega} \) an unfolding of \( f \). In the unfolding theory of function germs, there are notions of a right morphism and a right-left morphism. The former involves coordinate changes in the source space \( (\mathbb{C}^n, 0) \), whereas the latter involves coordinate changes in the target space \( \mathbb{C} \) as well. It is not difficult to see that our morphism generalizes a right morphism in the sense that when \( F \) admits a strong first integral \( f \), then it becomes a (strict) right morphism in the unfolding theory of \( f \). For a foliation
without first integrals, it may not seem relevant to talk about right-left morphisms. However, as stated above, our factorization theorem shows that an RL-morphism is a natural generalization of a right-left morphism, since when $F = (df)$, an RL-morphism is exactly a right-left morphism in the unfolding theory of $f$. We also note that RL-morphisms are closely related to integrating factors of the foliation ((2.2) Remark 2).
1. The factorization theorem.

We denote by $O_n$ the ring of germs of holomorphic functions at the origin 0 in $\mathbb{C}^n = \{(x_1, \ldots, x_n)\}$. A germ $f$ in $O_n$ is said to be a power if $f = f_0^m$ for some positive integer $m$ and a non-unit $f_0$ in $O_n$. If we denote the critical set of $f$ by $\mathcal{C}(f)$, then $\text{codim} \mathcal{C}(f) \geq 2$ implies that $f$ is not a power. We quote the following factorization theorem of Mattei and Moussu.

(1.1) Theorem (1)). Let $f$ be a germ in $O_n$ which is not a power or a unit. If $g$ is a germ in $O_n$ with $dg \wedge df = 0$, then there exists a germ $\lambda$ in $O_1$ such that $g = \lambda \cdot f$.

The theorem is proved using the reduction theory of singularities of holomorphic 1-forms due to Seidenberg and Van den Essen. The proof is rather simple if we assume $\text{codim} \mathcal{C}(f) \geq 2$ (see Moussu-Tougeron[3]). If $\tilde{f}$ is a germ in $O_{n+\mathbb{R}}$, we may think of $\tilde{f}$ as an unfolding of $f(x) = \tilde{f}(x,0)$ with parameter space $\mathbb{C}^\mathbb{R} = \{ (s_1, \ldots, s_n) \}$. We denote by $d_x^\infty$ the exterior derivation with respect to $x$; $d_x^\infty f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (x,s) dx_i$.

(1.2) Theorem. Let $\tilde{f}$ be a germ in $O_{n+\mathbb{R}}$ such that $f(x) = \tilde{f}(x,0)$ is not a power or a unit in $O_n$. If $g$ is a germ in $O_{n+\mathbb{R}}$ with $d_x g \wedge d_x \tilde{f} = 0$, then there exists a germ $\lambda$ in $O_{1+\mathbb{R}}$ such that $g(x,s) = \lambda(\tilde{f}(x,s),s)$.

Proof. First we show the existence of $\lambda$ as a formal power series in $s$. Thus we express $\lambda$ as
\[ \lambda(y,s) = \sum_{|\nu| \geq 0} \lambda^{(\nu)}(y)s^\nu, \quad \lambda^{(\nu)} \in O_1, \]

where \( \nu \) denote an \( \ell \)-tuple \((\nu_1, \ldots, \nu_\ell)\) of non-negative integers, \(|\nu| = \nu_1 + \cdots + \nu_\ell\) and \( s^\nu = s_1^{\nu_1} \cdots s_\ell^{\nu_\ell} \). In general, if \( \sigma = \sum_{|\nu| \geq 0} \sigma^{(\nu)}s^\nu \) is a series in \( s \) with \( \sigma^{(\nu)} \in O^n_r \) for some \( r \), we set

\[
[\sigma]_p = \sum_{|\nu| = p} \sigma^{(\nu)}s^\nu \quad \text{and} \quad [\sigma]_0 = \sum_{|\nu| = 0} \sigma^{(\nu)}s^\nu
\]

for a non-negative integer \( p \).

We look for \( \lambda \) satisfying the congruence

\[
(1.3)_p \quad g(x,s) = \lambda^{[p]}(f(x),s)
\]

for \( p \geq 0 \), where \( = \) denotes the equality mod \( s^\nu \), \(|\nu| = p + 1 \).

First, \((1.3)_0\) is equivalent to

\[
g(x,0) = \lambda^{(0)}(f(x)) \,.
\]

From the condition of the theorem, we have \( d(g(x,0)) \land df = 0 \). Hence by \((1.1)\), there exists a germ \( \lambda^{(0)} \) in \( O_1 \) satisfying the above. Now we suppose that we have \( \lambda^{[p]} \) satisfying \((1.3)_p\) and look for \([\lambda]_{p+1} \). The congruence \((1.3)_{p+1}\) reads

\[
g(x,s) = \sum_{|\nu| = p+1} \lambda^{(\nu)}(f(x))s^\nu + \lambda^{[p]}(f(x),s),s) \,.
\]
Hence, for our purpose, it suffices to show that

\[(1.4)\quad d_x [p(x,s) - \lambda |p(f(x,s), s)]_{p+1} \wedge df = 0.\]

By (1.3), we have (1.4) if we show that

\[d_x (g(x,s) - \lambda |p(f(x,s), s)) \wedge d_x f = 0.\]

But this follows from the condition of the theorem and

\[d_x \lambda |p(f(x,s), s) = \frac{\partial \lambda |p}{\partial y}(f(x,s), s) d_x f.\]

Thus we have a formal power series

\[\lambda(y,s) = \sum_{|\nu| \geq 0} \lambda^{(\nu)}(y)s^\nu, \quad \lambda^{(\nu)} \in O_1\]

in \(s\) such that \(g(x,s) = \lambda(f(x,s), s)\) as power series in \((x,s)\).

Since \(\approx f\) and \(g\) are both convergent, \(\lambda\) must be also convergent.

(1.5) Remarks.

1. The germ \(\lambda\) is determined uniquely by \(g\) (and \(\approx f\)). If we assume that \(g(x,0) = f(x)\), then \(\lambda(y,0) = y\).

2. The above theorem generalizes Corollaire 1 in [2] Ch.II,1 in the case \(X = H\).
2. Some types of morphisms in the unfolding theory of foliation germs.

We denote by $Q_n$ the $O_n$-modules of germs of holomorphic 1-forms at 0 in $\mathbb{C}^n$. We recall (Cf.[4],[5]) that a codim 1 foliation germ at 0 in $\mathbb{C}^n$ is a rank 1 free sub-$O_n$-module $F = (\omega)$ of $Q_n$ with a generator satisfying the integrability condition $d\omega \wedge \omega = 0$. The singular set $\mathcal{S}(F)$ of $F$ is defined to be the singular set $\{x|\omega(x) = 0\}$ of $\omega$. We always assume that $\text{codim } S(F) \geq 2$. An unfolding of $F = (\omega)$ is a codim 1 foliation germ $\mathcal{F} = (\tilde{\omega})$ at 0 in $\mathbb{C}^n \times \mathbb{C}^m = \{(x,t)\}$, for some $m$, with a generator $\tilde{\omega}$ satisfying $\iota^* \tilde{\omega} = \omega$, where $\iota$ denotes the embedding of $\mathbb{C}^n$ into $\mathbb{C}^n \times \mathbb{C}^m$ given by $\iota(x) = (x,0)$. We call $\mathbb{C}^m$ the parameter space of $\mathcal{F}$. We recall the following definition ([5](2.1),[6](1.1)).

(2.1) Definition. Let $\mathcal{F}$ and $\mathcal{F}'$ be two unfoldings of $F$ with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^\alpha = \{(s_1,\ldots,s_\alpha)\}$, respectively.

(1) An RL-morphism from $\mathcal{F}'$ to $\mathcal{F}$ is a pair $(\Phi,\Psi)$ satisfying the following conditions:

(a) $\Phi$ and $\Psi$ are holomorphic map germs making the diagram

$$(\mathbb{C}^n \times \mathbb{C}^\alpha,0) \xrightarrow{\Phi} (\mathbb{C}^n \times \mathbb{C}^m,0)$$

$$\downarrow \quad \downarrow$$

$$(\mathbb{C}^\alpha,0) \xrightarrow{\Psi} (\mathbb{C}^m,0)$$

commutative, where the vertical maps are the projections.
(b) \( \Phi(x,0) = (x,0) \).

(c) For any generator \( \tilde{\omega} \) of \( \mathcal{F} \), there is a germ \( \alpha = (\alpha_1, \ldots, \alpha_\lambda) \) in \( O_{n+\lambda}^\lambda \) such that the germ

\[
\Phi^* \tilde{\omega} + \sum_{k=1}^{\lambda} \alpha_k d s_k
\]

generates \( \mathcal{F}' \).

(II) A morphism from \( \mathcal{F}' \) to \( \mathcal{F} \) is an RL-morphism such that for any generator \( \tilde{\omega} \) of \( \mathcal{F} \), we may choose \( \alpha = 0 \) in (c).

(2.2) Remarks.

1. In the both cases, we may replace "any" by "some".

2. From the integrability condition we see that, for \( \alpha \) in (c), each \( \alpha_k(x,0) \) is an integrating factor of \( \omega = \tilde{\omega} \), i.e.,

\[
\alpha_k(x,0) d\omega = d(\alpha_k(x,0)) \wedge \omega.
\]

3. We have a "versality theorem" for each type of morphisms ([4],[6]).

If a germ \( \tilde{f} \) in \( O_{n+m}^m \) is an unfolding of \( f \), i.e., if \( \tilde{f} \) = \( f \), then \( \mathcal{F} = (df) \) is an unfolding of \( F = (df) \) with parameter space \( C^m \) and conversely, any unfolding of \( F = (df) \) has a generator of the form \( df \) with \( \tilde{f} \) an unfolding of \( f \) ([4] p.47). We recall the following definition (cf.[7] Definition 3.2).

(2.4) Definition. Let \( \tilde{f} \) and \( g \) be two unfoldings of \( f \) with parameter spaces \( C^m \) and \( C^{\lambda} = ((s_1, \ldots, s_\lambda)) \), respectively.

(I) A right-left morphism from \( g \) to \( \tilde{f} \) is a pair \( (\Phi, \psi) \) satisfying (I)(a) and (b) in (2.1) Definition and
(c) \( g(x,s) = \lambda(\Phi \Phi^* f(x,s),s) \)
for some \( \lambda \) in \( \mathcal{O}_{1+\mathcal{S}} \) with \( \lambda(y,0) = y \).

(II) A strict right morphism from \( g \) to \( f \) is a right-left morphism such that \( \lambda(y,s) = y \) in (c).

The following is a direct consequence of (1.2) Theorem.

(2.4) Proposition. Let \( f \) and \( g \) be unfoldings of \( f \). A pair \( (\Phi, \psi) \) is, respectively, a right-left morphism or a strict right morphism from \( g \) to \( f \) if and only if it is an RL-morphism or a morphism from \( \tilde{F}' = (dg) \) to \( \tilde{F} = (df) \).
References


