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On the spaces of Eisenstein series of  
Hilbert modular groups

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Introduction

In [ 3 ], Shimura investigated automorphic eigen forms and Eisenstein series of Hilbert modular groups with parameter in great detail. There he proved that in most cases the orthogonal complements  $N(\sigma, \lambda, \Delta)$  of the spaces  $S(\sigma, \lambda, \Delta)$  of cusp forms in the spaces  $A(\sigma, \lambda, \Delta)$  of automorphic eigen forms are spanned by special values of Eisenstein series (and some other forms derived from Eisenstein series in certain special cases). There he excluded the case when  $\lambda$  is multiple, but he suggested that this fact would be true even in that case. The purpose of this paper is to prove it following his idea with some modification.

1. Automorphic eigen forms

Let  $F$  be a totally real number field and  $\mathfrak{a}$  the set of all archimedean primes of  $F$ . For each set  $X$ , we denote by  $X^{\mathfrak{a}}$  the product of a copies of  $X$  or the set  $\{(x_v)_v \mid v \in \mathfrak{a}\}$ . For each element  $x$  of  $X^{\mathfrak{a}}$ , we denote by  $x_v$  the  $v$ -component of  $x$ . For two elements  $c$  and  $x$  of  $C^{\mathfrak{a}}$ , we put

$$c^x = \prod_v c_v^{x_v}$$

whenever each factor is well defined.

Let  $H$  be the upper half plane  $\{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ . For each  $g \in \text{SL}_2(\mathbb{R})$  and  $z \in H$ , we put

$$g(z) = \frac{az+b}{cz+d}, \quad j(g, z) = cz+d \quad \text{if } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By  $z \rightarrow g(z)$ ,  $\text{SL}_2(\mathbb{R})$  acts on  $H$  and therefore  $\text{SL}_2(\mathbb{R})^{\mathfrak{a}}$  acts on  $H^{\mathfrak{a}}$ .

For  $g \in \text{SL}_2(\mathbb{R})^{\mathfrak{a}}$ ,  $z = (z_v) \in H^{\mathfrak{a}}$  and  $\sigma = (\sigma_v) \in Z^{\mathfrak{a}}$ , we put

$$j_g(z)^{\sigma} = j(g, z)^{\sigma} = \prod_v j(g_v, z_v)^{\sigma_v}.$$

We shall define automorphic forms of integral and half integral weight. Put  $u=(1,1,\dots,1) \in \mathbb{R}^a$ . A weight will be either an element of  $Z^a$  (Case I, integral weight) or an element of  $(1/2)u+Z^a$  (Case II, half integral weight). For each weight  $\sigma$ , we denote by  $G_\sigma$  the set of all pairs  $(g, l(z))$  with  $g \in SL_2(F)$  and  $l(z)$  a holomorphic function on  $H^a$  such that

$$l(z)^2 = t j(g, z)^\sigma, \quad t \in \mathbb{C}, \quad |t| = 1.$$

The set  $G_\sigma$  is a group by the group law defined by

$$(g, l)(g', l') = (gg', l(g'(z))l'(z)).$$

We denote the projection of  $G_\sigma$  to  $SL_2(F)$  by  $pr$ , or  $pr((g, l)) = g$ .

For  $\alpha = (g, l(z)) \in G_\sigma$ , we denote  $l(z)$  by  $l_\alpha(z)$  and put  $\alpha(z) = g(z)$  for

$z \in H^a$  and  $j_\alpha^\sigma = j_g^\sigma$ . For a function  $f$  on  $H^a$  and  $\alpha \in G_\sigma$ , we define the function  $f \parallel \alpha$  by

$$(f \parallel \alpha)(z) = l_\alpha(z)^{-1} f(\alpha(z)), \quad z \in H^a.$$

For  $z = (z_v) \in H^a$ , we put  $y_v = \text{Im}(z_v)$  and  $y = (y_v)$ , and consider  $y$  as an  $\mathbb{R}^a$ -valued function on  $H^a$ . Then

$$(1.1) \quad y^p \parallel \alpha = l_\alpha^{-1} |j_\alpha|^{-2p} y^p \quad (p \in \mathbb{R}^a, \alpha \in G_\sigma)$$

For  $v \in a$  and  $\sigma \in \mathbb{R}^a$ , we define differential operators  $\varepsilon_v, \delta_v^\sigma$  and  $L_v^\sigma$  operating on  $C^\infty$ -functions  $f$  on  $H^a$  by

$$(1.2) \quad \varepsilon_v f = -y_v^2 \partial f / \partial \bar{z}_v$$

$$(1.3) \quad \delta_v^\sigma = -y_v^\sigma \partial (y_v^\sigma f) / \partial z_v$$

$$(1.4) \quad L_v^\sigma = 4 \delta_v^{\sigma'}, \quad \varepsilon_v, \quad \sigma_v' = \sigma_v - 2$$

Let  $W$  be the subset of  $SL_2(F)$  defined by [ 3 , (1.10) ] and  $h_g$  the holomorphic function on  $H^a$  for each  $g \in W$  given in [ 3, Prop.3.2 ]. For each weight  $\sigma$ , let  $\Lambda_\sigma$  be the injection

$$\begin{aligned} \Lambda_\sigma &: SL_2(F) \longrightarrow G_\sigma && \text{(Case I) ,} \\ \Lambda_\sigma &: W \longrightarrow G_\sigma && \text{(Case II).} \end{aligned}$$

given by

$$\Lambda_\sigma(g) = \begin{cases} (g, j_g^\sigma) , & g \in SL_2(F) & \text{(Case I),} \\ (g, h_g j_g^{\sigma-u/2}) , & g \in W & \text{(Case II).} \end{cases}$$

Let  $\mathfrak{o}$  be the maximal order of  $F$ ,  $\mathfrak{o}^\times$  the unit group of  $\mathfrak{o}$  and  $d$  the different of  $F$ . For each integral ideal  $c$  of  $F$ , we put

$$\Gamma(c) = \{ \alpha \in SL_2(F) \cap M_2(\mathfrak{o}) \mid \alpha^{-1} \in cM_2(\mathfrak{o}) \}$$

and call it a congruence subgroup of  $SL_2(F)$ . Further we call a subgroup  $\Delta$  of  $G_\sigma$ , a congruence subgroup if it satisfies the following two conditions:

( 1.5 )  $\Delta$  is isomorphic to a subgroup of  $SL_2(F)$  by pr,

( 1.6 )  $\Delta$  contains  $\Lambda_\sigma(\Gamma(c))$  as a subgroup of finite index for some  $c$  ( $c \subset 8d^{-1}$  in Case II).

A real analytic function  $f$  on  $H^a$  is called an automorphic eigen form with respect to a congruence subgroup  $\Delta$  of  $G_\sigma$ , if it satisfies the following three conditions:

( 1.7 )  $f \parallel \alpha = f$  for every  $\alpha \in \Delta$ ;

( 1.8 )  $L_v^\sigma f = \lambda_v f$  with  $\lambda_v \in \mathbb{C}$  for every  $v \in \mathfrak{a}$ ;

( 1.9 ) for every  $\alpha \in G_\sigma$ , there exist positive numbers  $A$ ,  $B$  and  $C$  (depending on  $f$  and  $\alpha$ ) such that

$$y^{\sigma/2} | (f \parallel \alpha)(x+iy) | \leq Ay^{cu} \quad \text{if } y^u > B.$$

For  $\lambda = (\lambda_v) \in C^a$ , we denote by  $A(\sigma, \lambda, \Delta)$  the set of all such  $f$  and by  $A(\sigma, \lambda)$  the union of  $A(\sigma, \lambda, \Delta)$  for all congruence subgroups  $\Delta$  of  $G_\sigma$ . We know  $A(\sigma, \lambda, \Delta)$  is finite dimensional and  $A(\sigma, \lambda)$  is stable under the action of  $\alpha \in G_\sigma$ . If  $f \in A(\sigma, \lambda, \Delta)$ , then it has a Fourier expansion of the form

$$f(x+iy) = b(y) + \sum_{0 \neq h \in m} b_h W(hy; \sigma, \lambda) e(hx)$$

with  $m$  a lattice of  $F$ ,  $b_h \in C$ ,  $b(y)$  a function on  $R^a$ ,  $e(z) = \exp(2\pi i \sum z_v)$

for  $z \in C^a$ , and  $W$  the Whittaker function defined by [3, (2.19) and (2.20)]. We call  $b(y)$  the constant term of  $f$  and call  $f$  a cusp form if the constant term of  $f \parallel \alpha$  vanishes for every  $\alpha \in G_\sigma$ . We denote the set of all cusp forms in  $A(\sigma, \lambda)$  by  $S(\sigma, \lambda)$  and put  $S(\sigma, \lambda, \Delta) = A(\sigma, \lambda, \Delta) \cap S(\sigma, \lambda)$ .

For two continuous functions  $f$  and  $g$  satisfying (1.7), we put

$$\langle f, g \rangle = \mu(\Delta \setminus H^a)^{-1} \int_{\Delta \setminus H^a} \bar{f} g y^\sigma d\mu(z)$$

where  $d\mu(z) = y^{-2u} \prod_{v \in a} dx_v dy_v$ . This does not depend on the choice of  $\Delta$ .

We define subspaces  $N(\sigma, \lambda)$  and  $N(\sigma, \lambda, \Delta)$  of  $A(\sigma, \lambda)$  by

$$\begin{aligned} N(\sigma, \lambda, \Delta) &= \{g \in A(\sigma, \lambda, \Delta) \mid \langle f, g \rangle = 0 \text{ for all } f \in S(\sigma, \lambda, \Delta)\} \\ N(\sigma, \lambda) &= \{g \in A(\sigma, \lambda) \mid \langle f, g \rangle = 0 \text{ for all } f \in S(\sigma, \lambda)\} \end{aligned}$$

Then  $N(\sigma, \lambda, \Delta) = N(\sigma, \lambda) \cap A(\sigma, \lambda, \Delta)$ .

Let  $U$  be a subgroup of  $g^X$  of finite index. We call  $\tau = (\tau_v) \in R^a$  U-admissible if

$$(1.10) \quad \sum \tau_v = 0 \quad \text{and} \quad |a|^i \tau = 1 \text{ for all } a \in U.$$

We call  $\tau \in R^a$  admissible if it is  $U$ -admissible for some  $U$  and denote by  $T_U$  the set of all  $U$ -admissible  $\tau$ .

Hereafter we fix a weight  $\sigma$  and write  $G = G_\sigma$  and  $\Lambda = \Lambda_\sigma$ . We call

$\lambda \in C^a$  critical if  $4\lambda_v = (1 - \sigma_v)^2$  for all  $v \in a$ , and call  $\lambda$  non-critical if it is not critical.

Proposition 1.1 ( [ 3 , Prop. 3.1 ] )

The constant term  $b(y)$  of an element  $f \in A(\sigma, \lambda)$  has one of the following forms.

(1) If  $\lambda$  is critical then

$$b(y) = a_1 y^q + a_2 y^q \log y^u$$

where  $q = (q_v)$  and  $q_v$  is the multiple root of  $X^2 - (1 - \sigma_v)X + \lambda_v = 0$ .

(2) If  $\lambda$  is non-critical then  $b(y)$  is a linear combination of  $y^p$  with  $p = (p_v) \in C^a$  satisfying

$$(1.11) \quad p_v \text{ is a root of } \psi_v(x) = X^2 - (1 - \sigma_v)X + \lambda_v.$$

$$(1.12) \quad p = su - (\sigma - i\tau)/2 \text{ with } s \in \mathbb{C} \text{ and an admissible } \tau (\in \mathbb{R}^a).$$

When  $\lambda$  is non-critical, an element  $p \in C^a$  is called an exponent attached to  $\lambda$  if it satisfies (1.11) and (1.12). We denote by  $C(\sigma, \lambda)$  the set of all exponents attached to  $\lambda$ . For  $p = (p_v) \in C^a$ , we put  $\bar{p} = (\bar{p}_v) \in C^a$ . Then  $C(\sigma, \bar{\lambda}) = \{\bar{p} \mid p \in C(\sigma, \lambda)\}$ . We note if  $C(\sigma, \lambda) = \emptyset$ , then  $A(\sigma, \lambda) = S(\sigma, \lambda)$ . We call  $\lambda$  simple either if  $\lambda$  is critical or if  $\lambda$  is non-critical and  $C(\sigma, \lambda)$  consists of exactly two elements. We also call  $\lambda$  multiple if  $C(\sigma, \lambda)$  has more than two elements.

Lemma 1.2 Assume  $\lambda$  is non-critical. For  $p \in C(\sigma, \lambda)$ , put  $p' = u - \sigma - p$ . Then  $p' \in C(\sigma, \lambda)$  and  $p' \neq p$ . Furthermore  $\bar{\lambda}$  is non-critical and  $\overline{p'} = \bar{p}' \in C(\sigma, \bar{\lambda})$ .

Proof. Since  $p_v$  is a root of  $\psi_v(X) = 0$ ,  $1 - \sigma_v - p_v$  is also a root of  $\psi_v(X) = 0$ . As  $\psi_v(X) = 0$  has simple roots for at least one  $v$ , we have  $p' \neq p$ . Further since  $p' = (1-s)u - (\sigma + i\tau)/2$ ,  $p'$  satisfies also (1.12). The last statement is obvious. (Q.E.D.)



2. A bilinear relation of coefficients of constant terms.

The purpose of this section is to generalize [ 3 , Theorem 6.1 ] to multiple  $\lambda$ . Hereafter we assume  $\lambda$  is non-critical. Let  $\Delta$  be a congruence subgroup of  $G$  and put  $\Gamma = \text{pr}(\Delta)$ . Put

$$P = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(F) \mid c=0 \right\}, \quad P = \left\{ \alpha \in G \mid \text{pr}(\alpha) \in P \right\}.$$

Then  $P \backslash G / \Delta$  is a finite set. We call classes of  $P \backslash G / \Delta$  cusp classes. Take a complete set of representatives  $X$  for  $P \backslash G / \Delta$ . For each  $\xi \in X$ , we put  $Q_\xi = P \cap \text{pr}(\xi \Delta \xi^{-1})$ . Let  $\Theta$  be a subgroup of  $P$  of the form

$$\Theta = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mid a \in U_1, b \in \mathfrak{m} \right\},$$

with a fractional ideal  $\mathfrak{m}$  of  $F$  and a subgroup  $U$  of  $\mathfrak{g}^X$  of finite index. Take  $U_1$  and  $\mathfrak{m}$  so that  $\Lambda(\Theta) \subset \xi \Delta \xi^{-1}$  for all  $\xi \in X$  and  $a \gg 0$  for every  $a \in U_1$ . For  $0 < r \in \mathbb{R}$ , put

$$T_r = \{ z \in H^a \mid y^u > r \}, \quad M_r = \{ z \in H^a \mid y^u = r \}.$$

Put  $U = \{ a^2 \mid a \in U_1 \}$ . Then  $\Theta \backslash M_r$  is isomorphic to the product of  $\mathbb{R}^a / \mathfrak{m}$  and  $\{ y \in \mathbb{R}^a \mid y^u = r, y \gg 0 \} / U$  up to the difference of orientations of  $(-1)^{n(n-1)/2}$ . For a fixed  $v \in \mathfrak{a}$ , we put

$$\omega = y^{-2u} \prod_{v \in \mathfrak{a}} dx_v \wedge dy_v, \quad \xi_v = (i/2) y_v^{-2u} d\bar{z}_v \prod_{w \neq v} dx_w \wedge dy_w.$$

Since [ 3 , Lemma 6.2 ] holds only when  $\tau = 0$ , its proper statement should be the following

Lemma 2.1. For  $s \in \mathbb{C}$  and  $\tau \in T_U$ , we have

$$\int_{\Theta \backslash M_r} y^{\text{su} + \tau + v} \xi_v = \begin{cases} (-i/2) \mu(\mathbb{R}^a / \mathfrak{m}) R_U r^{s-1} & (\tau = 0) \\ 0 & (\tau \neq 0), \end{cases}$$

where  $R_U = R_F [ \mathfrak{g}^X : U \cdot \{ \pm 1 \} ]$  with the regulator  $R_F$  of  $F$  and  $\mu(\mathbb{R}^a / \mathfrak{m})$  is the volume of  $\mathbb{R}^a / \mathfrak{m}$ .

Let  $f \in A(\sigma, \lambda, \Delta)$  and  $g \in A(\sigma, \bar{\lambda}, \Delta)$ . For each  $\xi \in X$ , we write

$$f \parallel \xi^{-1} = \sum_{p, \xi} a_{p, \xi} y^p + (\text{non-constant terms}),$$

and

$$g \parallel \xi^{-1} = \sum_{p, \xi} b_{p, \xi} y^{\bar{p}} + (\text{non-constant terms}).$$

where  $p$  is taken over the elements of  $C(\sigma, \lambda)$ . Let  $C_1 = C_1(\sigma, \lambda)$  be a subset of  $C(\sigma, \lambda)$  such that

$$C(\sigma, \lambda) = C_1(\sigma, \lambda) \cup \{p' \mid p \in C_1(\sigma, \lambda)\} \quad (\text{disjoint}).$$

Then we can generalize [ 3 , Theorem 6.1 ] to the following

**Theorem 2.2.** We have

$$\sum_{p \in C_1} (p_v - p'_v) \sum_{\xi \in X} \nu_\xi (a_{p, \xi} \bar{b}_{p', \xi} - a_{p', \xi} \bar{b}_{p, \xi}) = 0 ,$$

where  $\nu_\xi = [ Q_\xi \{ \pm 1 \} : \Theta \{ \pm 1 \} ]^{-1}$ .

Proof. Take a positive number  $r$  so that any two sets  $\xi^{-1} Q_\xi \setminus \xi^{-1}(T_r)$  ( $\subset \Gamma \setminus H^a$ ,  $\xi \in X$ ) have no intersection points. Let  $J$  be a union of small neighbourhoods of elliptic points on  $\Gamma \setminus H^a$ , which are compact manifolds with boundary. Inducing a natural orientation into each set, we see that

$$\partial K = \sum_{\xi \in X} \xi^{-1} Q_\xi \setminus \xi^{-1}(T_r) - \partial J.$$

Therefore for a  $\Gamma$ -invariant  $C^\infty$ -form  $\phi$  on  $H^a$  of codegree 1, we have

$$\int_K d\phi = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} \phi \cdot \xi^{-1} - \int_{\partial J} \phi ,$$

where  $B_\xi = \xi^{-1} \Theta \setminus \xi^{-1}(M_r)$ . We put  $\phi = \bar{f}(\xi_v g) y^\sigma \xi_v$  and  $\phi' = \bar{g}(\xi_v f) y^\sigma \xi_v$ .

Then

$$d\phi = \frac{1}{4} \bar{f} L_v^\sigma g y^\sigma \omega - (\overline{\xi_v f})(\xi_v g) y^{\sigma'} \omega \quad (\sigma' = \sigma - 2v)$$

and

$$\begin{aligned} d(\bar{\phi} - \phi') &= \frac{1}{4} (f \overline{L_v^\sigma g} - L_v^\sigma f \bar{g}) y^\sigma \omega \\ &= \frac{1}{4} (\lambda_v - \lambda_v) f \bar{g} y^\sigma \omega = 0 . \end{aligned}$$

This implies that

$$\int_{\partial J} (\bar{\phi} - \phi') = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} (\bar{\phi} - \phi') \cdot \xi^{-1}.$$

Now we have the expansions

$$\bar{\phi} \cdot \xi^{-1} = \frac{i}{2} \sum_{p,q} q_v a_{p, \xi} \bar{b}_{q, \xi} y^{\bar{p}+\bar{q}+v+\sigma} \xi_v + \dots, \quad ,$$

and

$$\phi' \cdot \xi^{-1} = -\frac{i}{2} \sum_{p,q} p_v a_{p, \xi} \bar{b}_{q, \xi} y^{\bar{p}+\bar{q}+v+\sigma} \xi_v + \dots, \quad ,$$

where  $p$  and  $q$  are taken over  $C(\sigma, \lambda)$ . Though the unwritten terms also contain terms which do not contain  $e(hx)$ , they tend to 0 in our later process of  $r \rightarrow \infty$  by [ 3 , Prop.2.1(2) ]. Applying Lemma 2.1, we obtain

$$\int_{\partial J} (\bar{\phi} - \phi') = \mu(R^a/m)R_U \sum_{p \in C_1} (p_v - p'_v) \sum_{\xi \in X} v_\xi (a_{p, \xi} \bar{b}_{p', \xi} - a_{p', \xi} \bar{b}_{p, \xi}) + \dots,$$

where  $C_1 = C_1(\sigma, \lambda)$ . Since the unwritten terms tend to 0 when  $r \rightarrow \infty$

as we mentioned above and  $\int_{\partial J} (\bar{\phi} - \phi') \rightarrow 0$  when  $J \rightarrow \phi$ , we have the

assertion. (Q.E.D.)

### 3. Eisenstein series

Let  $\rho$  be an element of  $C^a$  such that

$$\rho = (\sigma - \tau)/2$$

with an admissible  $\tau$ . A cusp class  $P \xi \Delta$  ( $\xi \in X$ ) is called  $\rho$ -regular if

$$y^{-\rho} \parallel \alpha = y^{-\rho} \quad \text{for every } \alpha \in P \cap \xi \Delta \xi^{-1}.$$

We denote by  $Y(\rho)$  the subset of  $X$  that represents all  $\rho$ -regular cusps.

We also denote by  $\kappa(\rho)$  the number of elements of  $Y(\rho)$ . For each congruence subgroup  $\Delta$ , we define its Eisenstein series by

$$E(z, s; \rho, \Delta) = \begin{cases} \sum_{\alpha \in P \cap \Delta} y^{su-\rho} \parallel \alpha & \text{if } P \Delta \text{ is } \rho\text{-regular,} \\ 0 & \text{otherwise.} \end{cases}$$

The series is convergent for  $\text{Re}(s) > 1$  and can be continued as a meromorphic function in  $s$  to the whole  $s$ -plane. If  $\Delta \supset \Delta'$ , we see that

$$(3.1) \quad [P \cap \Delta : P \cap \Delta'] E(z, s; \rho, \Delta) \\ = \sum_{\gamma \in \Delta \setminus \Delta'} E(z, s; \rho, \Delta') \parallel \gamma .$$

For each cusp class  $P \xi \Delta$ , we put

$$E_{\xi}(z, s; \rho, \Delta) = E(z, s; \rho, \xi \Delta \xi^{-1}) \parallel \xi .$$

Then we see

$$(3.2) \quad E_{\xi}(z, s; \rho, \Delta) \parallel \alpha = E_{\xi}(z, s; \rho, \Delta) \quad \text{for every } \alpha \in \Delta .$$

We denote by  $E[\rho, \Delta]$  the complex vector space generated by the functions  $E_{\xi}(z, s; \rho, \Delta)$  for all  $\xi \in X$ . Using (3.1), we can prove that

if  $\Delta \supset \Delta'$ , then  $E[\rho, \Delta] \subset E[\rho, \Delta']$  and

$$(3.3) \quad E[\rho, \Delta] \\ = \{ g(z, s) \in E[\rho, \Delta'] \mid g \parallel \alpha = g \text{ for every } \alpha \in \Delta \} .$$

Now by [3, Prop.5.2], we have, for  $\xi, \eta \in Y(\rho)$ ,

$$(3.4) \quad E_{\xi} \parallel \eta^{-1} = \delta_{\xi \eta} y^{su-\rho} + f_{\xi \eta} y^{u-su-\bar{\rho}} \\ + \sum_{0 \neq h \in n} g_{\xi \eta}(h, s, y) e(hx) .$$

where  $f_{\xi \eta}$  and  $g_{\xi \eta}$  are meromorphic functions in  $s$ ,  $\delta_{\xi \eta}$  is the Kronecker's delta and  $n$  is a lattice in  $F$ .

For  $s_0 \in C$ , we denote by  $E[s_0, \rho, \Delta]$  the subspace of  $E[\rho, \Delta]$  consisting of all functions  $g(z, s)$  that are holomorphic at  $s_0$ . We put

$$E(s_0, \rho, \Delta) = \{ g(z, s_0) \mid g \in E[s_0, \rho, \Delta] \} .$$

Then by [3, Prop.7.1],

$$E(s_0, \rho, \Delta) \subset A(\sigma, \lambda, \Delta)$$

with  $\lambda = (\lambda_v)$ ,  $\lambda_v = (s_0 - \rho_v)(1 - s_0 - \bar{\rho}_v)$ .

The following lemma is stated in [3, Prop.7.2] under the assumption that  $\lambda$  is simple, but the assertion holds also for multiple  $\lambda$  without any changes of the proof.

Lemma 3.1. (1)  $\dim E[\rho, \Delta] = \kappa(\rho)$ .

(2) The map  $g(z, s) \rightarrow g(z, s_0)$  gives an isomorphism of  $E[s_0, \rho, \Delta]$  onto  $E(s_0, \rho, \Delta)$ .

Conversely for a fixed non-critical  $\lambda$ , we express  $p \in C(\sigma, \lambda)$  as

$$p = s_p u - (\sigma - i \tau_p)/2$$

with  $s_p \in \mathbb{C}$  and an admissible  $\tau_p$ . We put  $\rho_p = (\sigma - i \tau_p)/2$  and also set  $Y(p) = Y(\rho_p)$  and  $\kappa(p) = \kappa(\rho_p)$ . We note

$$p' = (1 - s_p)u - \rho_p \quad \text{and} \quad \kappa(p') = \kappa(p)$$

by [ 3 , Prop.7.5 ] .

**Theorem 3.2.** Suppose  $\lambda$  is non-critical and  $E[\rho_p, \Delta] = E[s_p, \rho_p, \Delta]$  and  $E[\bar{\rho}_p, \Delta] = E[\bar{s}_p, \bar{\rho}_p, \Delta]$  for any  $p \in C_1(\sigma, \lambda)$ . Then

$$N(\sigma, \lambda, \Delta) = \bigoplus_{p \in C_1} E_p(s_p, \rho_p, \Delta) \quad (C_1 = C_1(\sigma, \lambda)).$$

**Proof.** It is easy to see that the right-hand side is a direct sum and is contained in  $N(\sigma, \lambda, \Delta)$  by [ 3 , Prop.7.1 ] . Therefore we have  $\dim(N(\sigma, \lambda, \Delta)) \geq \sum_{p \in C_1} \kappa(p)$ . For each  $p \in C_1$ , let  $Y'(p)$  be the set of

all  $\xi \in X$  such that  $\prod_{m=1}^m \xi \Delta$  is  $\bar{\rho}_p$ -regular. Then the number of elements of  $Y'(p)$  is  $\kappa(p)$  by [ 3 , Prop.7.5 ] . For  $f \in A(\sigma, \lambda, \Delta)$  and  $\xi \in X$ , write

$$f \parallel \xi^{-1} = \sum_{p \in C_1} (a_{p, \xi} y^p + a_{p', \xi} y^{p'}) + (\text{non-constant terms}).$$

Then the map  $\Psi: f \rightarrow ((a_{p, \xi})_{p \in C_1, \xi \in Y(p)}, (a_{p', \xi})_{p \in C_1, \xi \in Y'(p)})$

gives an injection of  $A(\sigma, \lambda, \Delta) / S(\sigma, \lambda, \Delta)$  into  $C^{2\mu}$  ( $\mu = \sum_{p \in C_1} \kappa(p)$ ).

For each  $p \in C_1$ , take  $v \in a$  so that  $p_v \neq p'_v$ . Let  $g \in E(\bar{s}_p, \bar{\rho}_p, \Delta)$  and

$\xi \in Y'(p)$ . Denote the Fourier expansion of  $g \parallel \xi^{-1}$  be

$$g \parallel \xi^{-1} = b_{p, \xi}^- y^{\bar{p}} + b_{p', \xi}^- y^{\bar{p}'} + (\text{non-constant terms}).$$

Then using Theorem 2.2, we have a linear relation

$$\sum_{\xi \in Y'(p)} \nu_{\xi} (a_{p, \xi} \bar{b}_{p', \xi}^- - a_{p', \xi} \bar{b}_{p, \xi}^-) = 0$$

among  $(a_{p, \xi}, a_{p', \xi})$  for each  $g$ . Since these linear relations are independent if  $p$ 's are different, we have at least  $\mu$  independent linear relations. This implies the dimension of the image of  $\Psi$  is at most  $\mu$  and therefore is equal to  $\mu$ . (Q.E.D.)

If  $\lambda$  is non-critical, by [ 3 , Remark 7.4 (1),(2) ] , we can take the set  $C_1(\sigma, \lambda)$  and  $s_p$  for each  $p \in C_1(\sigma, \lambda)$  so that they satisfy the conditions of Theorem 3.2 , except for the case when  $\lambda=0, \sigma=0$  ( $p=u, p'=0, s_p=1$ ) in Case I and the case when  $\sigma_v-1/2$  is either an even non-negative integers or an odd negative integer for every  $v \in a$  ( $p=3/4-(1/2)\sigma, p'=1/4-(1/2)\sigma, s_p=3/4$ ) in Case II. To discuss these cases, we denote by  $E^*[s_0, \rho, \Delta]$  the set of elements of  $E[\rho, \Delta]$  that have at most a simple pole at  $s_0$  , and by  $E^*(s_0, \rho, \Delta)$  the set of residues of all elements of  $E^*[s_0, \rho, \Delta]$  . The following theorem is a generalization of [ 3 , Theorem 7.9 ] , which can be proved similarly as [ 3 , Theorem 7.9 ] together with the modification used in Theorem 3.2.

Theorem 3.3. Suppose  $\lambda$  is real and non-critical. Suppose also for all  $p \in C_1(\sigma, \lambda)$ ,  $E[\rho_p, \Delta] = E^*[s_p, \rho_p, \Delta]$  and a cusp class of  $\Delta$  is  $\rho_p$ -regular if and only if it is  $\bar{\rho}_p$ -regular. Then  $N(\sigma, \lambda, \Delta)$  has dimension  $\sum_{p \in C_1} \kappa(p)$  and is the direct sum

$$\bigoplus_{p \in C_1} (E(s_p, \rho_p, \Delta) \oplus E^*(s_p, \rho_p, \Delta)) , \quad (C_1 = C_1(\sigma, \lambda)) .$$

Using Theorem 3.2 and Theorem 3.3 , we obtain the following

Theorem 3.4. If  $\lambda$  is non-critical, then  $N(\sigma, \lambda, \Delta)$  is generated by special values and the residues of Eisenstein series and has dimension  $\sum_{p \in C_1} \kappa(p)$  (  $C_1=C_1(\sigma, \lambda)$  ).

Proof. This is a direct result of Theorem 3.2 and Theorem 3.3 together with [ 3 , Remark 7.10 ] . The only thing we would like to mention here is that in Theorem 3.3, we have assumed a cusp class of  $\Delta$  is  $\rho_p$ -regular if and only if it is  $\bar{\rho}_p$ -regular for all  $p \in C_1$ .

To avoid this restriction, take a subgroup  $\Delta'$  of  $\Delta$  of finite index so small that any cusp class of  $\Delta'$  is  $\rho_p$ -regular and also  $\bar{\rho}_p$ -regular for all  $p \in C_1$ . Then by ( 3.3 ), we have the result. (Q.E.D.)

#### 4. Remarks on multiple $\lambda$

In [ 3 , Remark 5.5 ] , Shimura gave an example of multiple  $\lambda$  , when the field  $F$  is a quadratic field. We explain it in a slightly more general situation, because it seems the only case when multiple  $\lambda$  appears. Let  $F$  be a totally real number field of degree  $2n$  containing a quadratic field  $L$  as a subfield. Denote by  $\{ v, w \}$  the set of archimedean primes of  $L$  and by  $a = \{ v_1, \dots, v_n, w_1, \dots, w_n \}$  the set of archimedean primes of  $F$  of which  $v_1, \dots, v_n$  are lying over  $v$  and  $w_1, \dots, w_n$  are over  $w$ . Let  $U_1$  be a subgroup of the unit group of  $L$  of finite index and take  $\theta$  so that  $T_{U_1} = Z\theta$ . Let  $U$  be a subgroup of the unit group of  $F$  of finite index such that  $\{ N_{F/L}(\varepsilon) \mid \varepsilon \in U \} \subset U_1$ . Let  $\tau$  be an element of  $R^a$  such that

$$\tau_{v_i} = \theta_v, \quad \tau_{w_i} = \theta_w \quad \text{for any } i (1 \leq i \leq n).$$

Then  $|\varepsilon|^i \tau = |N_{F/L}(\varepsilon)|^i \theta = 1$  for any  $\varepsilon \in U$ . Put for integers  $m, n$  ( $m \neq n$ ),

$$p = (1 + in \theta_v)u - (\sigma - 2mi \tau) / 2, \\ q = (1 + im \theta_v)u - (\sigma - 2ni \tau) / 2.$$

Then  $p, \bar{p}, q, \bar{q}$  are all distinct and are exponents belonging to  $C(\sigma, \lambda)$  with  $\lambda = (\lambda_{v_1}, \dots, \lambda_{v_n}, \lambda_{w_1}, \dots, \lambda_{w_n})$  given by

$$4 \lambda_{v_i} = (1 - \sigma_{v_i})^2 + (m+n)^2 \theta_v^2, \quad 4 \lambda_{w_i} = (1 - \sigma_{w_i})^2 + (m-n)^2 \theta_w^2.$$

Therefore  $\lambda$  is multiple.

Now we return to the general situation and assume  $\lambda$  is multiple. Then by [ 3, Prop. 3.2 ],  $\lambda$  is real and  $X^2 - (1 - \sigma_v)X + \lambda_v = 0$  has either a multiple root or two simple roots which are complex conjugate.

Therefore if  $p$  is an exponent attached to  $\lambda$ , then  $p' = \bar{p}$  and for any other exponent  $q \in C(\sigma, \lambda)$ , we see that  $q_v$  is either  $p_v$  or  $\bar{p}_v$  for all  $v \in a$ . The following proposition suggests that even for multiple  $\lambda$ ,  $C(\sigma, \lambda)$  cannot contain so many exponents.

Proposition 4.1. Let  $F$  be a totally real number field of degree  $n$ . Assume  $\lambda$  is multiple. Let  $p = (p_1, \dots, p_n)$  ( $p_i = p_{v_i}$  for  $v_i \in a$ ) be an

exponent attached to  $\lambda$ . Assume  $\bar{p}_i \neq p_i$  and put  $q = (p_1, \dots, \bar{p}_i, \dots, p_n)$ .

Then  $q$  is not an exponent attached to  $\lambda$ .

Proof. We may assume  $i=1$  by changing the indices. Assume  $q$  is also an exponent attached to  $\lambda$  and put

$$p = su - (\sigma - i\tau)/2, \quad p' = s'u - (\sigma - i\tau')/2$$

with admissible  $\tau$  and  $\tau'$ . Since  $\text{Re}(s) = \text{Re}(s') = 1/2$ , we can write

$$s = 1/2 + it, \quad s' = 1/2 + it'$$

with  $t, t' \in \mathbb{R}$ . Then we see

$$t' + \tau_1' = -(t + \tau_1), \quad t' + \tau_j' = t + \tau_j \quad (2 \leq j \leq n).$$

Therefore we see  $\tau_j - \tau_j' = t' - t$  for all  $j$  ( $2 \leq j \leq n$ ). Putting  $a = t - t'$ , we obtain

$$\tau_1 - \tau_1' = -\left(\sum_{j=2}^n \tau - \sum_{j=2}^n \tau'\right) = (n-1)a.$$

This implies

$$\tau - \tau' = -au + (na, 0, \dots, 0).$$

Take a subgroup of the unit group of  $F$  of finite index such that  $\tau, \tau'$  are  $U$ -admissible. Then we see for any  $\varepsilon \in U$ ,

$$1 = |\varepsilon|^{i(\tau - \tau')} = |\varepsilon_i|^{ina}.$$

Since the rank of  $U$  is  $n-1$ , we see  $a=0$  if  $n \geq 3$ . This implies  $t=t', \tau = \tau'$  and therefore  $p_1 = \bar{p}_1$ , which is a contradiction. (Q.E.D.)

It is an interesting problem to determine whether all multiple  $\lambda$  can be obtained as Shimura's example mentioned in the beginning of this section or not. It seems a problem solely on the structure of the unit group of number fields.



Corollary 4.2. If  $[F:Q] = 3$ , then there exists no multiple  $\lambda$  for any weight  $\sigma$ .

Proof. Assume  $\lambda$  is a multiple eigenvalue and let  $p$  be an exponent attached to  $\lambda$ . Then  $\bar{p}$  is also an exponent. If  $q$  is an exponent attached to  $\lambda$ , then  $q$  is obtained by changing  $p$  or  $\bar{p}$  at only one prime  $v$ . But this is not allowed by Prop. 4.1. (Q.E.D.)

#### References

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