Note on the double centralizers in an H-separable extension

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NOTE ON THE DOUBLE CENTRALIZERS
IN AN H-SEPARABLE EXTENSION
By Kazuo Niita

Throughout this note, $A$ denotes an associative ring with an identity $1$ and $B$ a subring of $A$ with the common identity. And we let $E = \{x \in A : x \otimes 1 = 1 \otimes x \in A \otimes_B A\}$, $V_A(S) = \{x \in A : xy = yx\text{ for all } y \in A\}$ where $S$ is any subset of $A$. These $E$ and $V_A(S)$ are both subrings of $A$ and it is easily shown that the inclusions $B \subseteq E \subseteq V_A(V_A(B))$ always hold.

The aim of this paper is to give necessary and sufficient conditions for any ring extension $A$ of $B$ to hold the equality $B = E$, that is, the equivalence of the followings (Theorem 1.);

(a) $B = E$
(b) The canonical map $A/B \longrightarrow A \otimes_B (A/B)$ is a monomorphism.
(c) For every $f \in \text{Hom}_B(M, BA/B)$ where $M$ is a left $B$-module,

$0 = 1 \otimes f : A \otimes_B M \longrightarrow A \otimes_B (A/B)$ implies $f = 0$.

In case $A$ is a ring extension of $B$ satisfying $E = V_A(V_A(B))$, these conditions are also equivalent to that $B = V_A(V_A(B))$. And this result can be applied especially to H-separable extensions (Proposition and Theorem 2.)

$A$ is to be called an H-separable extension of $B$ if $A \otimes_B A$ is isomorphic to a direct summand of a finite direct sum of copies of $A$ as $(A,A)$-bimodules. This special type of separable ring extensions was first introduced by K. Hirata in [1] to be a
generalization of central separable algebras, and studied exten­sively by him and K. Sugano. In this case, as Y. Kurata and S. Morimoto showed recently in [3], the equality $E = V_A(V_A(B))$ holds. For some other fundamental properties of this extension, see [2], [5] and [6].

Before proceeding to the subject, here is another remark on notations. For any subring $B'$ of $A$, we denote by $A/B'$ a factor module of $A$ modulo $B'$ as an additive $(B',B')$-bimodule. Furthermore, for an element $x \in A$, $\overline{x}$ denotes a coset element of $A/B'$ which contains $x$.

**Lemma** Let $B'$ be a subring of $A$ such that $B \subseteq B' \subseteq E$. Then

$$\xi_{B'} : A \otimes_B (A/B') \rightarrow A \otimes_B A$$

defined by $\xi_{B'}(x \otimes y) = xy \otimes 1 - x \otimes y \cdot (x,y \in A)$ is a split mono­morphism of $(A,B')$-bimodules.

**Proof.** Define $\delta : A \rightarrow A \otimes_B A$ by $\delta(x) = x \otimes 1 - 1 \otimes x$ ($x \in A$).

Clearly $\delta$ is a homomorphism of $(E,E)$-bimodules and $\text{Ker} \, \delta = E$.

Then, since $B' \subseteq E$, $\delta$ induces

$$A/B' \rightarrow A \otimes_B A, \overline{x} \mapsto x \otimes 1 - 1 \otimes x \quad (x \in A),$$

and this leads us to a homomorphism of $(A,B')$-bimodules

$$A \otimes_B (A/B') \rightarrow A \otimes_B A \otimes_B A, x \otimes \overline{y} \mapsto x \otimes y \otimes 1 - x \otimes 1 \otimes y \quad (x,y \in A).$$

Now the requested map $\xi_{B'}$ is given by composing this homomorphism and the multiplication of the first and second tensorial factors of $A \otimes_B A \otimes_B A$. It is easily shown that this is a split monomor-
Phism by taking a map $A \otimes_B A \rightarrow A \otimes_B (A/B')$ which is yielded by a natural epimorphism $A \rightarrow A/B'$, since $0 = x \otimes 1 \in A \otimes_B (A/B')$ for any $x \in A$.

Note that we could also show, similarly to this lemma, that a map $(A/B') \otimes_B A \rightarrow A \otimes_B A$ defined by $x \otimes y \mapsto x \otimes y' - 1 \otimes xy$ is a split monomorphism of $(B',A)$-bimodules. And this "the other side version" of tensor products is valid for the following corollaries, too.

**Corollary 1.** The map $A \otimes_B (A/B) \rightarrow A \otimes_B (A/E)$ given by $x \otimes \bar{y} \mapsto x \otimes \bar{y} (x,y \in A)$ is an isomorphism of $(A,B)$-bimodules.

**Proof.** Since $A/B \rightarrow A/E$ is an epimorphism, it is sufficient to show that the given map is injective. But this is an easy consequence by the commutativity of the following diagram where $\xi_B$ and $\xi_E$ are monomorphisms defined in the last lemma.

**Corollary 2.** The sequence of $(A,B)$-bimodules

$$0 \rightarrow A \otimes_B (A/B) \xrightarrow{\xi_B} A \otimes_B A \xrightarrow{\mu} A \rightarrow 0$$

where $\mu$ is the multiplication, is split exact.

**Proof.** This is immediate by the lemma and the well-known fact that $\text{Ker } \mu = \Sigma\{x(y \otimes 1 - 1 \otimes y) : x,y \in A\}$. 

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Theorem 1. The followings are equivalent for any ring extension $A$ of $B$.

(a) $B = E$

(b) $0 = 1 \otimes x \in A \otimes_B (A/B)$ implies $x \in B$.

(b') $0 = x \otimes 1 \in (A/B) \otimes_B A$ implies $x \in B$.

(c) For every $f \in \text{Hom}_{B}(M, BA/B)$ where $M$ is a left $B$-module,

$$0 = 1 \otimes f : A \otimes_B M \rightarrow A \otimes_B (A/B)$$

implies $f = 0$.

(c') For every $f \in \text{Hom}(M_B, A_B/B)$ where $M$ is a right $B$-module,

$$0 = f \otimes 1 : M \otimes_B A \rightarrow (A/B) \otimes_B M$$

implies $f = 0$.

Proof. We will only show the equivalence of (a), (b) and (c).

(a) $\Rightarrow$ (b) If $0 = 1 \otimes x \in A \otimes_B (A/B)$, then, by the lemma, $0 = e_B(1 \otimes x) = x \otimes 1 - 1 \otimes x \in A \otimes_B A$, that is, $x \otimes 1 = 1 \otimes x$. Thus, assuming (a), we have $x \in E = B$.

(b) $\Rightarrow$ (c) This is clear.

(c) $\Rightarrow$ (a) Let $f : E \rightarrow A/B$ and $g : E \rightarrow A/E$ be the restrictions to $E$ of natural epimorphisms $A \rightarrow A/B$ and $A \rightarrow A/E$ respectively. Clearly, $g$ is a zero map. Consider the following commutative diagram where the vertical map is an isomorphism of Corollary 1.

$$\begin{array}{ccc} A \otimes_B (A/B) & \xrightarrow{\otimes g} & A \otimes_B (A/E) \\ \downarrow \odot f & & \downarrow \odot g \\ A \otimes_B E & \xrightarrow{1 \otimes g} & A \otimes_B (A/E) \end{array}$$

Since $1 \otimes g = 0$, it follows easily that $1 \otimes f = 0$. Then, by the assumption, we have $f = 0$, that is, $E \subseteq B$. This implies (a).
In the case A is an H-separable extension of B, we know that \( E = V_A(V_A(B)) \). So, by this theorem, we have the next proposition.

**Proposition** Let A be an H-separable extension of B. Then the followings are equivalent.

(a) \( B = V_A(V_A(B)) \)

(b) \( A/B \longrightarrow A \otimes_B (A/B) \) given by \( \bar{x} \longrightarrow 1 \otimes \bar{x} \) is a monomorphism.

(c) \( A/B \longrightarrow (A/B) \otimes_B A \) given by \( \bar{x} \longrightarrow \bar{x} \otimes 1 \) is a monomorphism.

It should be noted here that this proposition is essentially to be contained in the work of Kurata and Morimoto (See Theorem 3.12. of [3].).

We are now led to the following theorem.

**Theorem 2.** If A is an H-separable extension of B such that \( A/B \) is flat as a left or right B-module, then \( B = V_A(V_A(B)) \).

If B is a regular ring, every B-module is flat. Thus the next corollary, which has shown by Sugano in [7], is immediate by this theorem.

**Corollary 3.** If A is an H-separable extension of a regular ring B, then \( B = V_A(V_A(B)) \).
REFERENCES


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