Note on the double centralizers in an H-separable extension

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Series #11. September 1987

HOKKAIDO UNIVERSITY PREPRINT SERIES IN MATHEMATICS

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NOTE ON THE DOUBLE CENTRALIZERS
IN AN H-SEPARABLE EXTENSION

By Kazuo Niita

Throughout this note, A denotes an associative ring with an identity 1 and B a subring of A with the common identity. And we let $E = \{x \in A : x \otimes 1 = 1 \otimes x \in A \otimes_B A\}$, $V_A(S) = \{x \in A : xy = yx$ for all $y \in A\}$ where S is any subset of A. These E and $V_A(S)$ are both subrings of A and it is easily shown that the inclusions $B \subseteq E \subseteq V_A(V_A(B))$ always hold.

The aim of this paper is to give necessary and sufficient conditions for any ring extension A of B to hold the equality $B = E$, that is, the equivalence of the followings (Theorem 1.);

(a) $B = E$

(b) The canonical map $A/B \longrightarrow A \otimes_B (A/B)$ is a monomorphism.

(c) For every $f \in \mathrm{Hom}({}_B M, {}_B A/B)$ where M is a left B-module,

$0 = 1 \otimes f : A \otimes_B M \longrightarrow A \otimes_B (A/B)$ implies $f = 0$.

In case A is a ring extension of B satisfying $E = V_A(V_A(B))$, these conditions are also equivalent to that $B = V_A(V_A(B))$. And this result can be applied especially to H-separable extensions (Proposition and Theorem 2.)

A is to be called an H-separable extension of B if $A \otimes_B A$ is isomorphic to a direct summand of a finite direct sum of copies of A as (A,A)-bimodules. This special type of separable ring extensions was first introduced by K. Hirata in [1] to be a generalization of central separable algebras, and studied extensively by him and K. Sugano. In this case, as Y. Kurata and S. Morimoto showed recently in [3], the equality $E = V_A(V_A(B))$ holds. For some other fundamental properties of this extension, see [2], [5] and [6].

Before proceeding to the subject, here is another remark on notations. For any subring B' of A, we denote by A/B' a factor module of A modulo B' as an additive (B',B')-bimodule. Furthermore, for an element $x \in A$, $\bar{x}$ denotes a coset element of A/B' which contains x.

<u>Lemma</u> Let B' be a subring of A such that $B \subseteq B' \subseteq E$. Then

$$\varepsilon_{B'} : A \otimes_B (A/B') \longrightarrow A \otimes_B A$$

defined by $\varepsilon_{B'}(x \otimes \bar{y}) = xy \otimes 1 - x \otimes y$ $(x,y \in A)$ is a split monomorphism of (A,B')-bimodules.

<u>Proof.</u> Define $\delta : A \longrightarrow A \otimes_B A$ by $\delta(x) = x \otimes 1 - 1 \otimes x$ $(x \in A)$. Clearly δ is a homomorphism of (E,E)-bimodules and $\mathrm{Ker}\,\delta = E$. Then, since $B' \subseteq E$, δ induces

$$A/B' \longrightarrow A \otimes_B A,\ \bar{x} \longmapsto x \otimes 1 - 1 \otimes x \quad (x \in A),$$

and this leads us to a homomorphism of (A,B')-bimodules

$$A \otimes_B (A/B') \longrightarrow A \otimes_B A \otimes_B A,\ x \otimes \bar{y} \longmapsto x \otimes y \otimes 1 - x \otimes 1 \otimes y$$
$$(x,y \in A).$$

Now the requested map $\varepsilon_{B'}$ is given by composing this homomorphism and the multiplication of the first and second tensorial factors of $A \otimes_B A \otimes_B A$. It is easily shown that this is a split monomorphism by taking a map $A \otimes_B A \longrightarrow A \otimes_B (A/B')$ which is yielded by a natural epimorphism $A \longrightarrow A/B'$, since $0 = x \otimes 1 \in A \otimes_B (A/B')$ for any $x \in A$.

Note that we could also show, similarly to this lemma, that a map $(A/B') \otimes_B A \longrightarrow A \otimes_B A$ defined by $\bar{x} \otimes y \longmapsto x \otimes y - 1 \otimes xy$ is a split monomorphism of (B',A)-bimodules. And this "the other side version" of tensor products is valid for the following corollaries, too.

<u>Corollary 1.</u> The map $A \otimes_B (A/B) \longrightarrow A \otimes_B (A/E)$ given by $x \otimes \bar{y} \longmapsto x \otimes \bar{y}$ $(x,y \in A)$ is an isomorphism of (A,B)-bimodules.

<u>Proof.</u> Since $A/B \longrightarrow A/E$ is an epimorphism, it is sufficient to show that the given map is injective. But this is an easy consequence by the commutativity of the following diagram where ε_B and ε_E are monomorphisms defined in the last lemma.

$$\begin{tikzcd} A \otimes_B (A/B) \arrow[rr] \arrow[dr, "\varepsilon_B"'] & & A \otimes_B (A/E) \arrow[dl, "\varepsilon_E"] \\ & A \otimes_B A & \end{tikzcd}$$

<u>Corollary 2.</u> The sequence of (A,B)-bimodules

$$0 \longrightarrow A \otimes_B (A/B) \xrightarrow{\varepsilon_B} A \otimes_B A \xrightarrow{\mu} A \longrightarrow 0$$

where μ is the multiplication, is split exact.

<u>Proof.</u> This is immediate by the lemma and the well-known fact that $\mathrm{Ker}\,\mu = \sum\{x(y \otimes 1 - 1 \otimes y) : x,y \in A\}$.

<u>Theorem 1.</u> The followings are equivalent for any ring extension A of B.

(a) $B = E$

(b) $0 = 1 \otimes \bar{x} \in A \otimes_B (A/B)$ implies $x \in B$.

(b') $0 = \bar{x} \otimes 1 \in (A/B) \otimes_B A$ implies $x \in B$.

(c) For every $f \in \mathrm{Hom}({}_BM, {}_BA/B)$ where M is a left B-module,

$0 = 1 \otimes f : A \otimes_B M \longrightarrow A \otimes_B (A/B)$ implies $f = 0$.

(c') For every $f \in \mathrm{Hom}(M_B, A/B_B)$ where M is a right B-module,

$0 = f \otimes 1 : M \otimes_B A \longrightarrow (A/B) \otimes_B M$ implies $f = 0$.

<u>Proof.</u> We will only show the equivalence of (a),(b) and (c).

(a)$\Longrightarrow$(b) If $0 = 1 \otimes \bar{x} \in A \otimes_B (A/B)$, then, by the lemma, $0 = \varepsilon_B(1 \otimes \bar{x}) = x \otimes 1 - 1 \otimes x \in A \otimes_B A$, that is, $x \otimes 1 = 1 \otimes x$. Thus, assuming (a), we have $x \in E = B$.

(b)$\Longrightarrow$(c) This is clear.

(c)$\Longrightarrow$(a) Let $f : E \longrightarrow A/B$ and $g : E \longrightarrow A/E$ be the restrictions to E of natural epimorphisms $A \longrightarrow A/B$ and $A \longrightarrow A/E$ respectively. Clearly, g is a zero map. Consider the following commutative diagram where the vertical map is an isomorphism of Corollary 1.

$$\begin{tikzcd} & A \otimes_B (A/B) \arrow[d, "\wr"] \\ A\ x_B E \arrow[ur, "1\otimes f"] \arrow[r, "1\otimes g"'] & A \otimes_B (A/E) \end{tikzcd}$$

Since $1 \otimes g = 0$, it follows easily that $1 \otimes f = 0$. Then, by the assumption, we have $f = 0$, that is, $E \subseteq B$. This implies (a).

In the case A is an H-separable extension of B, we know that $E = V_A(V_A(B))$. So, by this theorem, we have the next proposition.

<u>Proposition</u> Let A be an H-separable extension of B. Then the followings are equivalent.

(a) $B = V_A(V_A(B))$

(b) $A/B \longrightarrow A \otimes_B (A/B)$ given by $\bar{x} \longmapsto 1 \otimes \bar{x}$ is a monomorphism.

(c) $A/B \longrightarrow (A/B) \otimes_B A$ given by $\bar{x} \longmapsto \bar{x} \otimes 1$ is a monomorphism.

It should be noted here that this proposition is essentially to be contained in the work of Kurata and Morimoto (See Theorem 3.12. of [3].).

We are now led to the following theorem.

<u>Theorem 2.</u> If A is an H-separable extension of B such that A/B is flat as a left or right B-module, then $B = V_A(V_A(B))$.

If B is a regular ring, every B-module is flat. Thus the next corollary, which has shown by Sugano in [7], is immediate by this theorem.

<u>Corollary 3.</u> If A is an H-separable extension of a regular ring B, then $B = V_A(V_A(B))$.

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