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Citation
Hokkaido University Preprint Series in Mathematics, 11, 1-6

Issue Date
1987-09

DOI
10.14943/49131

Doc URL
http://eprints3.math.sci.hokudai.ac.jp/907/; http://hdl.handle.net/2115/45529

Type
bulletin (article)

File Information
pre11.pdf

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NOTE ON THE DOUBLE CENTRALIZERS
IN AN H-SEPARABLE EXTENSION

By Kazuo Niita

Throughout this note, A denotes an associative ring with an identity 1 and B a subring of A with the common identity. And we let $E = \{ x \in A : x \otimes 1 = 1 \otimes x \in A \otimes_B A \}$, $V_A(S) = \{ x \in A : xy = yx \text{ for all } y \in A \}$ where $S$ is any subset of $A$. These $E$ and $V_A(S)$ are both subrings of $A$ and it is easily shown that the inclusions $B \subseteq E \subseteq V_A(V_A(B))$ always hold.

The aim of this paper is to give necessary and sufficient conditions for any ring extension $A$ of $B$ to hold the equality $B = E$, that is, the equivalence of the followings (Theorem 1.);

(a) $B = E$

(b) The canonical map $A/B \rightarrow A \otimes_B (A/B)$ is a monomorphism.

(c) For every $f \in \text{Hom}(B^M, B^{A/B})$ where $M$ is a left $B$-module,

$$0 = 1 \otimes f : A \otimes_B M \rightarrow A \otimes_B (A/B)$$

implies $f = 0$.

In case $A$ is a ring extension of $B$ satisfying $E = V_A(V_A(B))$, these conditions are also equivalent to that $B = V_A(V_A(B))$. And this result can be applied especially to $H$-separable extensions (Proposition and Theorem 2.)

$A$ is to be called an $H$-separable extension of $B$ if $A \otimes_B A$ is isomorphic to a direct summand of a finite direct sum of copies of $A$ as $(A,A)$-bimodules. This special type of separable ring extensions was first introduced by K. Hirata in [1] to be a
generalization of central separable algebras, and studied extensively by him and K. Sugano. In this case, as Y. Kurata and S. Morimoto showed recently in [3], the equality \( E = V_A(V_A(B)) \) holds.

For some other fundamental properties of this extension, see [2], [5] and [6].

Before proceeding to the subject, here is another remark on notations. For any subring \( B' \) of \( A \), we denote by \( A/B' \) a factor module of \( A \) modulo \( B' \) as an additive \((B',B')\)-bimodule. Furthermore, for an element \( x \in A \), \( \overline{x} \) denotes a coset element of \( A/B' \) which contains \( x \).

**Lemma.** Let \( B' \) be a subring of \( A \) such that \( B \subseteq B' \subseteq E \). Then

\[
E_{B'} : A \otimes_B (A/B') \rightarrow A \otimes_B A
\]

defined by \( E_{B'}(x \otimes y) = xy \otimes 1 - x \otimes y \cdot (x,y \in A) \) is a split monomorphism of \((A,B')\)-bimodules.

**Proof.** Define \( \delta : A \rightarrow A \otimes_B A \) by \( \delta(x) = x \otimes 1 - 1 \otimes x \cdot (x \in A) \). Clearly \( \delta \) is a homomorphism of \((E,E)\)-bimodules and \( \text{Ker} \delta = E \). Then, since \( B' \subseteq E \), \( \delta \) induces

\[
A/B' \rightarrow A \otimes_B A, \quad \overline{x} \mapsto x \otimes 1 - 1 \otimes x \quad (x \in A),
\]

and this leads us to a homomorphism of \((A,B')\)-bimodules

\[
A \otimes_B (A/B') \rightarrow A \otimes_B A \otimes_B A, \quad x \otimes y \mapsto x \otimes y \otimes 1 - x \otimes 1 \otimes y \quad (x,y \in A).
\]

Now the requested map \( E_{B'} \) is given by composing this homomorphism and the multiplication of the first and second tensorial factors of \( A \otimes_B A \otimes_B A \). It is easily shown that this is a split monomor-
phism by taking a map \( A \otimes_B A \rightarrow A \otimes_B (A/B') \) which is yielded by a natural epimorphism \( A \rightarrow A/B' \), since \( 0 = x \otimes 1 \in A \otimes_B (A/B') \) for any \( x \in A \).

Note that we could also show, similarly to this lemma, that a map \( (A/B') \otimes_B A \rightarrow A \otimes_B A \) defined by \( x \otimes y \mapsto x \otimes y' - l \otimes xy \) is a split monomorphism of \((B', A)\)-bimodules. And this "the other side version" of tensor products is valid for the following corollaries, too.

**Corollary 1.** The map \( A \otimes_B (A/B) \rightarrow A \otimes_B (A/E) \) given by \( x \otimes \overline{y} \mapsto x \otimes \overline{y} \) \((x, y \in A)\) is an isomorphism of \((A, B)\)-bimodules.

**Proof.** Since \( A/B \rightarrow A/E \) is an epimorphism, it is sufficient to show that the given map is injective. But this is an easy consequence by the commutativity of the following diagram where \( \varepsilon_B \) and \( \varepsilon_E \) are monomorphisms defined in the last lemma.

\[
\begin{array}{ccc}
A \otimes_B (A/B) & \rightarrow & A \otimes_B (A/E) \\
\varepsilon_B & & \varepsilon_E \\
A \otimes_B A & \rightleftharpoons & A \otimes_B A
\end{array}
\]

**Corollary 2.** The sequence of \((A, B)\)-bimodules

\[0 \rightarrow A \otimes_B (A/B) \xrightarrow{\varepsilon_B} A \otimes_B A \xrightarrow{\mu} A \rightarrow 0\]

where \( \mu \) is the multiplication, is split exact.

**Proof.** This is immediate by the lemma and the well-known fact that \( \text{Ker} \mu = \sum \{x(y \otimes 1 - 1 \otimes y) : x, y \in A\} \).
Theorem 1. The followings are equivalent for any ring extension $A$ of $B$.

(a) $B = E$

(b) $0 = 1 \otimes x \in A \otimes_B (A/B)$ implies $x \in B$.

(b') $0 = x \otimes 1 \in (A/B) \otimes_B A$ implies $x \in B$.

(c) For every $f \in \text{Hom}(E, A)$ where $M$ is a left $B$-module,

$$0 = 1 \otimes f : A \otimes_B M \rightarrow A \otimes_B (A/B)$$

implies $f = 0$.

(c') For every $f \in \text{Hom}(M, A)$ where $M$ is a right $B$-module,

$$0 = f \otimes 1 : M \otimes_B A \rightarrow (A/B) \otimes_B M$$

implies $f = 0$.

Proof. We will only show the equivalence of (a), (b) and (c).

(a)$\Rightarrow$(b) If $0 = l \otimes x \in A \otimes_B (A/B)$, then, by the lemma, $0 = \varepsilon_B(l \otimes x) = x \otimes 1 - 1 \otimes x \in A \otimes_B A$, that is, $x \otimes 1 = 1 \otimes x$. Thus, assuming (a), we have $x \in E = B$.

(b)$\Rightarrow$(c) This is clear.

(c)$\Rightarrow$(a) Let $f : E \rightarrow A/B$ and $g : E \rightarrow A/E$ be the restrictions to $E$ of natural epimorphisms $A \rightarrow A/B$ and $A \rightarrow A/E$ respectively. Clearly, $g$ is a zero map. Consider the following commutative diagram where the vertical map is an isomorphism of Corollary 1.

\[
\begin{array}{ccc}
A & \otimes_B (A/B) & \\
\downarrow & & \\
A \otimes_B (A/E) & \end{array}
\]

Since $1 \otimes g = 0$, it follows easily that $1 \otimes f = 0$. Then, by the assumption, we have $f = 0$, that is, $E \subseteq B$. This implies (a).
In the case $A$ is an $H$-separable extension of $B$, we know that $E = V_A(V_A(B))$. So, by this theorem, we have the next proposition.

**Proposition** Let $A$ be an $H$-separable extension of $B$. Then the followings are equivalent.

(a) $B = V_A(V_A(B))$

(b) $A/B \rightarrow A \otimes_B (A/B)$ given by $x \mapsto 1 \otimes x$ is a monomorphism.

(c) $A/B \rightarrow (A/B) \otimes_B A$ given by $x \mapsto x \otimes 1$ is a monomorphism.

It should be noted here that this proposition is essentially to be contained in the work of Kurata and Morimoto (See Theorem 3.12. of [3].).

We are now led to the following theorem.

**Theorem 2.** If $A$ is an $H$-separable extension of $B$ such that $A/B$ is flat as a left or right $B$-module, then $B = V_A(V_A(B))$.

If $B$ is a regular ring, every $B$-module is flat. Thus the next corollary, which has shown by Sugano in [7], is immediate by this theorem.

**Corollary 3.** If $A$ is an $H$-separable extension of a regular ring $B$, then $B = V_A(V_A(B))$. 

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