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Note on the double centralizers in an H-separable extension

Kazuo Nitta

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NOTE ON THE DOUBLE CENTRALIZERS
IN AN H-SEPARABLE EXTENSION

By Kazuo Niita

Throughout this note, \( A \) denotes an associative ring with an identity \( 1 \) and \( B \) a subring of \( A \) with the common identity. And we let \( E = \{ x \in A : x \otimes 1 = 1 \otimes x \in A \otimes_B A \} \), \( V_A(S) = \{ x \in A : xy = yx \) for all \( y \in A \} \) where \( S \) is any subset of \( A \). These \( E \) and \( V_A(S) \) are both subrings of \( A \) and it is easily shown that the inclusions \( B \subseteq E \subseteq V_A(V_A(B)) \) always hold.

The aim of this paper is to give necessary and sufficient conditions for any ring extension \( A \) of \( B \) to hold the equality \( B = E \), that is, the equivalence of the followings (Theorem 1.);

(a) \( B = E \)

(b) The canonical map \( A/B \to A \otimes_B (A/B) \) is a monomorphism.

(c) For every \( f \in \text{Hom}_{B-M, B}(A/B) \) where \( M \) is a left \( B \)-module,

\[ 0 = 1 \otimes f : A \otimes_B M \to A \otimes_B (A/B) \] implies \( f = 0 \).

In case \( A \) is a ring extension of \( B \) satisfying \( E = V_A(V_A(B)) \), these conditions are also equivalent to that \( B = V_A(V_A(B)) \). And this result can be applied especially to \( H \)-separable extensions (Proposition and Theorem 2.)

\( A \) is to be called an \( H \)-separable extension of \( B \) if \( A \otimes_B A \) is isomorphic to a direct summand of a finite direct sum of copies of \( A \) as \( (A,A) \)-bimodules. This special type of separable ring extensions was first introduced by K. Hirata in [1] to be a
generalization of central separable algebras, and studied extensively by him and K. Sugano. In this case, as Y. Kurata and S. Morimoto showed recently in [3], the equality \( E = \text{VA}(V_A(B)) \) holds. For some other fundamental properties of this extension, see [2], [5] and [6].

Before proceeding to the subject, here is another remark on notations. For any subring \( B' \) of \( A \), we denote by \( A/B' \) a factor module of \( A \) modulo \( B' \) as an additive \((B',B')\)-bimodule. Furthermore, for an element \( x \in A \), \( \overline{x} \) denotes a coset element of \( A/B' \) which contains \( x \).

**Lemma** Let \( B' \) be a subring of \( A \) such that \( B \subseteq B' \subseteq E \). Then

\[ \mathcal{E}_{B'} : A \otimes_B (A/B') \rightarrow A \otimes_B A \]

defined by \( \mathcal{E}_{B'}(x \otimes y) = xy \otimes 1 - x \otimes y \cdot (x,y \in A) \) is a split monomorphism of \((A,B')\)-bimodules.

**Proof.** Define \( \delta : A \rightarrow A \otimes_B A \) by \( \delta(x) = x \otimes 1 - l \otimes x \ (x \in A) \). Clearly \( \delta \) is a homomorphism of \((E,E)\)-bimodules and \( \text{Ker} \delta = E \). Then, since \( B' \subseteq E \), \( \delta \) induces

\[ A/B' \rightarrow A \otimes_B A, \overline{x} \mapsto x \otimes 1 - l \otimes x \ (x \in A), \]

and this leads us to a homomorphism of \((A,B')\)-bimodules

\[ A \otimes_B (A/B') \rightarrow A \otimes_B A \otimes_B A, \ x \otimes y \mapsto x \otimes y \otimes 1 - x \otimes l \otimes y \]

\( (x,y \in A) \).

Now the requested map \( \mathcal{E}_{B'} \), is given by composing this homomorphism and the multiplication of the first and second tensorial factors of \( A \otimes_B A \otimes_B A \). It is easily shown that this is a split monomor-
phism by taking a map $A \otimes B A \to A \otimes_B (A/B')$ which is yielded by a natural epimorphism $A \to A/B'$, since $0 = x \otimes 1 \in A \otimes_B (A/B')$ for any $x \in A$.

Note that we could also show, similarly to this lemma, that a map $(A/B') \otimes_B A \to A \otimes_B (A/B)$ defined by $x \otimes y \mapsto x \otimes y' - 1 \otimes xy$ is a split monomorphism of $(B',A)$-bimodules. And this "the other side version" of tensor products is valid for the following corollaries, too.

Corollary 1. The map $A \otimes_B (A/B) \to A \otimes_B (A/E)$ given by $x \otimes \bar{y} \mapsto x \otimes y$ $(x,y \in A)$ is an isomorphism of $(A,B)$-bimodules.

Proof. Since $A/B \to A/E$ is an epimorphism, it is sufficient to show that the given map is injective. But this is an easy consequence by the commutativity of the following diagram where $\xi_B$ and $\xi_E$ are monomorphisms defined in the last lemma.

$$
\begin{array}{c}
A \otimes_B (A/B) \to A \otimes_B (A/E) \\
\downarrow \xi_B \\
A \otimes_B A
\end{array}
$$

Corollary 2. The sequence of $(A,B)$-bimodules

$\begin{array}{c}
0 \to A \otimes_B (A/B) \xrightarrow{\xi_B} A \otimes_B A \xrightarrow{\mu} A \to 0
\end{array}$

where $\mu$ is the multiplication, is split exact.

Proof. This is immediate by the lemma and the well-known fact that $\text{Ker } \mu = \sum \{x(y \otimes 1 - 1 \otimes y) : x,y \in A\}$. 

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Theorem 1. The followings are equivalent for any ring extension $A$ of $B$.

(a) $B = E$

(b) $0 = 1 \otimes \bar{x} \in A \otimes_B(A/B)$ implies $x \in B$.

(b') $0 = \bar{x} \otimes 1 \in (A/B) \otimes_A A$ implies $x \in B$.

(c) For every $f \in \text{Hom}(BM, BA/B)$ where $M$ is a left $B$-module,

$$0 = 1 \otimes f : A \otimes_B M \rightarrow A \otimes_B (A/B)$$

implies $f = 0$.

(c') For every $f \in \text{Hom}(M_B, A/B_B)$ where $M$ is a right $B$-module,

$$0 = f \otimes 1 : M \otimes_A A \rightarrow (A/B) \otimes_A M$$

implies $f = 0$.

Proof. We will only show the equivalence of (a), (b) and (c).

(a)$\Rightarrow$(b) If $0 = 1 \otimes \bar{x} \in A \otimes_B(A/B)$, then, by the lemma,

$$0 = \mathcal{E}_B(1 \otimes \bar{x}) = x \otimes 1 - 1 \otimes x \in A \otimes_B A,$$

that is, $x \otimes 1 = 1 \otimes x$. Thus, assuming (a), we have $x \in E = B$.

(b)$\Rightarrow$(c) This is clear.

(c)$\Rightarrow$(a) Let $f : E \rightarrow A/B$ and $g : E \rightarrow A/E$ be the restrictions to $E$ of natural epimorphisms $A \rightarrow A/B$ and $A \rightarrow A/E$ respectively. Clearly, $g$ is a zero map. Consider the following commutative diagram where the vertical map is an isomorphism of Corollary 1.

$$\begin{array}{ccc}
A \otimes_B(A/B) & \rightarrow & (A/B) \otimes_B A \\
\downarrow & & \downarrow \\
A \otimes_B(A/E) & \rightarrow & E
\end{array}$$

Since $1 \otimes g = 0$, it follows easily that $1 \otimes f = 0$. Then, by the assumption, we have $f = 0$, that is, $E \subseteq B$. This implies (a).
In the case $A$ is an $H$-separable extension of $B$, we know that $E = V_A(V_A(B))$. So, by this theorem, we have the next proposition.

**Proposition** Let $A$ be an $H$-separable extension of $B$. Then the followings are equivalent.

(a) $B = V_A(V_A(B))$

(b) $A/B \to A \otimes_B (A/B)$ given by $\bar{x} \mapsto 1 \otimes \bar{x}$ is a monomorphism.

(c) $A/B \to (A/B) \otimes_B A$ given by $\bar{x} \mapsto \bar{x} \otimes 1$ is a monomorphism.

It should be noted here that this proposition is essentially to be contained in the work of Kurata and Morimoto (See Theorem 3.12. of [3]).

We are now led to the following theorem.

**Theorem 2.** If $A$ is an $H$-separable extension of $B$ such that $A/B$ is flat as a left or right $B$-module, then $B = V_A(V_A(B))$.

If $B$ is a regular ring, every $B$-module is flat. Thus the next corollary, which has shown by Sugano in [7], is immediate by this theorem.

**Corollary 3.** If $A$ is an $H$-separable extension of a regular ring $B$, then $B = V_A(V_A(B))$. 

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REFERENCES


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