Note on the double centralizers
in an H-separable extension

Kazuo Nitta

Series #11. September 1987
<table>
<thead>
<tr>
<th>#</th>
<th>Author</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Y. Okabe</td>
<td>On the theory of discrete KMO-Langevin equations with reflection positivity (I)</td>
</tr>
<tr>
<td>2.</td>
<td>Y. Giga and T. Kambe</td>
<td>Large time behavior of the vorticity of two-dimensional flow and its application to vortex formation</td>
</tr>
<tr>
<td>3.</td>
<td>A. Arai</td>
<td>Path Integral Representation of the Index of Kahler-Dirac Operators on an Infinite Dimensional Manifold</td>
</tr>
<tr>
<td>4.</td>
<td>I. Nakamura</td>
<td>Threefolds Homeomorphic to a Hyperquadric in $\mathbb{P}^4$</td>
</tr>
<tr>
<td>5.</td>
<td>T. Nakazi</td>
<td>Notes on Interpolation by Bounded Analytic Functions</td>
</tr>
<tr>
<td>6.</td>
<td>T. Nakazi</td>
<td>A Spectral Dilation of Some Non-Dirichlet Algebra</td>
</tr>
<tr>
<td>7.</td>
<td>H. Hida</td>
<td>A $p$-adic measure attached to the zeta functions associated with two elliptic modular forms II</td>
</tr>
<tr>
<td>8.</td>
<td>T. Suwa</td>
<td>A factorization theorem for unfoldings of analytic functions</td>
</tr>
<tr>
<td>9.</td>
<td>T. Nakazi</td>
<td>Weighted norm inequalities and uniform algebras</td>
</tr>
<tr>
<td>10.</td>
<td>T. Miyake</td>
<td>On the spaces of Eisenstein series of Hilbert modular groups</td>
</tr>
</tbody>
</table>
Throughout this note, \( A \) denotes an associative ring with an identity \( 1 \) and \( B \) a subring of \( A \) with the common identity. And we let \( E = \{ x \in A : x \otimes 1 = 1 \otimes x \in A \otimes_B A \} \), \( V_A(S) = \{ x \in A : xy = yx \) for all \( y \in A \} \) where \( S \) is any subset of \( A \). These \( E \) and \( V_A(S) \) are both subrings of \( A \) and it is easily shown that the inclusions \( B \subseteq E \subseteq V_A(V_A(B)) \) always hold.

The aim of this paper is to give necessary and sufficient conditions for any ring extension \( A \) of \( B \) to hold the equality \( B = E \), that is, the equivalence of the followings (Theorem 1.);

(a) \( B = E \)

(b) The canonical map \( A/B \longrightarrow A \otimes_B (A/B) \) is a monomorphism.

(c) For every \( f \in \text{Hom}_{B,M,A/B} \) where \( M \) is a left \( B \)-module, 
\[
0 = 1 \otimes f : A \otimes_B M \longrightarrow A \otimes_B (A/B) \implies f = 0.
\]

In case \( A \) is a ring extension of \( B \) satisfying \( E = V_A(V_A(B)) \), these conditions are also equivalent to that \( B = V_A(V_A(B)) \). And this result can be applied especially to \( H \)-separable extensions (Proposition and Theorem 2.)

\( A \) is to be called an \( H \)-separable extension of \( B \) if \( A \otimes_B A \) is isomorphic to a direct summand of a finite direct sum of copies of \( A \) as \( (A,A) \)-bimodules. This special type of separable ring extensions was first introduced by K. Hirata in [1] to be a
generalization of central separable algebras, and studied exten-
sively by him and K. Sugano. In this case, as Y. Kurata and S.
Morimoto showed recently in [3], the equality $E = V_A(V_A(B))$ holds.
For some other fundamental properties of this extension, see [2],
[5] and [6].

Before proceeding to the subject, here is another remark on
notations. For any subring $B'$ of $A$, we denote by $A/B'$ a factor
module of $A$ modulo $B'$ as an additive $(B',B')$-bimodule. Further-
more, for an element $x \in A$, $\bar{x}$ denotes a coset element of $A/B'$
which contains $x$.

**Lemma.** Let $B'$ be a subring of $A$ such that $B \subseteq B' \subseteq E$. Then

$$\mathcal{E}_{B'} : A \otimes_B (A/B') \rightarrow A \otimes_B A$$

defined by $\mathcal{E}_{B'}(x \otimes y) = xy \otimes 1 - x \otimes y \cdot (x,y \in A)$ is a split mono-
morphism of $(A,B')$-bimodules.

**Proof.** Define $\delta : A \rightarrow A \otimes_B A$ by $\delta(x) = x \otimes 1 - 1 \otimes x \cdot (x \in A)$. Clearly $\delta$ is a homomorphism of $(E,E)$-bimodules and $\ker \delta = E$.

Then, since $B' \subseteq E$, $\delta$ induces

$$A/B' \rightarrow A \otimes_B A, \quad x \mapsto x \otimes 1 - 1 \otimes x \cdot (x \in A),$$

and this leads us to a homomorphism of $(A,B')$-bimodules

$$A \otimes_B (A/B') \rightarrow A \otimes_B A \otimes_B A, \quad x \otimes y \mapsto x \otimes y \otimes 1 - x \otimes 1 \otimes y \cdot (x,y \in A).$$

Now the requested map $\mathcal{E}_{B'}$ is given by composing this homomorphism
and the multiplication of the first and second tensorial factors
of $A \otimes_B A \otimes_B A$. It is easily shown that this is a split monomor-
phism by taking a map \( A \otimes_B A \longrightarrow A \otimes_B (A/B') \) which is yielded by a natural epimorphism \( A \longrightarrow A/B' \), since \( 0 = x \otimes 1 \in A \otimes_B (A/B') \) for any \( x \in A \).

Note that we could also show, similarly to this lemma, that a map \((A/B') \otimes_B A \longrightarrow A \otimes_B (A/E)\) defined by \( x \otimes y \longmapsto x \otimes y' - 1 \otimes xy \) is a split monomorphism of \((B',A)\)-bimodules. And this "the other side version" of tensor products is valid for the following corollaries, too.

**Corollary 1.** The map \( \mu(B) \longrightarrow A \otimes_B (A/E) \) given by \( x \otimes y \longmapsto x \otimes y \) \((x,y \in A)\) is an isomorphism of \((A,B)\)-bimodules.

**Proof.** Since \( A/B \longrightarrow A/E \) is an epimorphism, it is sufficient to show that the given map is injective. But this is an easy consequence by the commutativity of the following diagram where \( \xi_B \) and \( \xi_E \) are monomorphisms defined in the last lemma.

\[
\begin{array}{ccc}
A \otimes_B (A/B) & \longrightarrow & A \otimes_B (A/E) \\
\xi_B & & \xi_E \\
A \otimes_B A & \ni & A \otimes_B A
\end{array}
\]

**Corollary 2.** The sequence of \((A,B)\)-bimodules

\[
0 \longrightarrow A \otimes_B (A/B) \xrightarrow{\xi_B} A \otimes_B A \xrightarrow{\mu} A \longrightarrow 0
\]

where \( \mu \) is the multiplication, is split exact.

**Proof.** This is immediate by the lemma and the well-known fact that \( \ker \mu = \sum \{ x(y \otimes 1 - 1 \otimes y) : x,y \in A \} \).
Theorem 1. The followings are equivalent for any ring extension $A$ of $B$.

(a) $B = E$

(b) $0 = l \otimes x \in A \otimes_B (A/B)$ implies $x \in B$.

(b') $0 = x \otimes l \in (A/B) \otimes_B A$ implies $x \in B$.

(c) For every $f \in \text{Hom}(B, A/B)$ where $M$ is a left $B$-module,

$$0 = l \otimes f : A \otimes_B M \to A \otimes_B (A/B)$$
implies $f = 0$.

(c') For every $f \in \text{Hom}(M, A/B)$ where $M$ is a right $B$-module,

$$0 = f \otimes l : M \otimes_B A \to (A/B) \otimes_B M$$
implies $f = 0$.

Proof. We will only show the equivalence of (a), (b) and (c).

(a)$\Rightarrow$ (b) If $0 = l \otimes x \in A \otimes_B (A/B)$, then, by the lemma, $0 = \xi_B(l \otimes x) = x \otimes l - l \otimes x \in A \otimes_B A$, that is, $x \otimes l = l \otimes x$. Thus, assuming (a), we have $x \in E = B$.

(b)$\Rightarrow$ (c) This is clear.

(c)$\Rightarrow$ (a) Let $f : E \to A/B$ and $g : E \to A/E$ be the restrictions to $E$ of natural epimorphisms $A \to A/B$ and $A \to A/E$ respectively. Clearly, $g$ is a zero map. Consider the following commutative diagram where the vertical map is an isomorphism of Corollary 1.

\[
\begin{array}{ccc}
A & \xrightarrow{\otimes f} & A \otimes_B (A/B) \\
\downarrow \quad & & \downarrow \\
\otimes g : A \otimes_B (A/E) & & \\
\end{array}
\]

Since $1 \otimes g = 0$, it follows easily that $l \otimes f = 0$. Then, by the assumption, we have $f = 0$, that is, $E \subseteq B$. This implies (a).
In the case $A$ is an $H$-separable extension of $B$, we know that $E = V_A(V_A(B))$. So, by this theorem, we have the next proposition.

**Proposition** Let $A$ be an $H$-separable extension of $B$. Then the followings are equivalent.

(a) $B = V_A(V_A(B))$

(b) $A/B \twoheadrightarrow A \otimes_B (A/B)$ given by $\overline{x} \mapsto 1 \otimes \overline{x}$ is a monomorphism.

(c) $A/B \twoheadrightarrow (A/B) \otimes_B A$ given by $\overline{x} \mapsto \overline{x} \otimes 1$ is a monomorphism.

It should be noted here that this proposition is essentially to be contained in the work of Kurata and Morimoto (See Theorem 3.12. of [3].).

We are now led to the following theorem.

**Theorem 2.** If $A$ is an $H$-separable extension of $B$ such that $A/B$ is flat as a left or right $B$-module, then $B = V_A(V_A(B))$.

If $B$ is a regular ring, every $B$-module is flat. Thus the next corollary, which has shown by Sugano in [7], is immediate by this theorem.

**Corollary 3.** If $A$ is an $H$-separable extension of a regular ring $B$, then $B = V_A(V_A(B))$. 
REFERENCES


