Microlocal parametrices and propagation of singularities near gliding points for hyperbolic mixed problems II

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Microlocal parametrices and propagation of singularities near gliding points for hyperbolic mixed problems II

By Kôji KUBOTA

§1. Introduction

This work is a continuation of [11] in which we have shown the existence of a parametrix and studied the propagation of singularities near a gliding point for a mixed (initial boundary value) problem for a second order hyperbolic equation. In the present paper we are concerned with first order symmetric hyperbolic systems. We do not repeat here the introductions of [10], [11], where the reader can find references and short histrical surveys of this problem.

In the sequel the hypothesis (H_2) of [10] will be removed and the assumption that the boundary condition is maximally dissipative will be relaxed so that (1.3) below holds. Besides, using the methods of this paper, one can simplify fairly the equation (4.14) of [11] on the boundary and, as a result, remove the hypothesis that the subprincipal symbol of P is pure imaginary. (See Remark 5.1 below).

Let $P(x,D)$ be a symmetric hyperbolic system in R^{n+1} $(n \geqq 2)$ of the form

$$P(x,D) = \Sigma_{k=0}^{n} A_k(x) D_k + C(x), \quad D_k = -i\partial/\partial x_k,$$

where $x = (x_0, x_1, \cdots, x_n)$, A_k are hermitian $m \times m$ matrices and

A_0 is positive definite. Let us consider the following mixed problem in a closed half space $X = X' \times [0,\infty) = \{x = (x', x_n);$ $x_n \geqq 0,\ x' = (x_0, x_1, \cdots, x_{n-1}) \in X' = R^n\}$ with boundary ∂X:

$$
\begin{aligned}
&P(x,D)u = 0 \quad \text{in } X, \\
(1.1)\qquad &B(x)u = f \quad \text{on } \partial X, \\
&u = 0 \quad \text{in } X \cap \{x_0 << 0\}.
\end{aligned}
$$

We assume P is of constant multiplicity. Then, denoting by $P_1(x,\xi)$ the principal symbol of P with $\xi = (\xi', \xi_n)$ the covariable of $x = (x', x_n)$, one can write

$$(1.2)\qquad \det P_1(x,\xi) = Q_1(x,\xi)^{m_1} \cdots Q_r(x,\xi)^{m_r} \tilde{Q}(x,\xi'),$$

where $Q_1, \cdots, Q_r$ and $\tilde{Q}$ are homogeneous polynomials in ξ which have no common zero in ξ_0, $Q_1, \cdots, Q_r$ are strictly hyperbolic with respect to ξ_0, and $\tilde{Q}$ is independent of ξ_n. (See Matsuura [14]). Setting

$$Q(x,\xi) = \Pi_{j=1}^{r} Q_j(x,\xi),$$

we assume that the boundary ∂X is noncharacteristic for Q and that, for each $x \in \partial X$ and $\xi' \in R^n \setminus 0$, the multiplicity of real roots ξ_n of the equation $Q(x,\xi',\xi_n) = 0$ is at most double and there is at most one double real root. Let d^+ be the number of the positive eigenvalues of $A_n(x)$, which is independent of $x \in \partial X$. We also suppose that $B(x)$ is a $d^+ \times m$ matrix of maximal rank and that $A_n(x)$, $C(x)$ and $B(x)$ are smooth (i.e. C^∞), constant for $|x|$ large enough. Moreover we assume

$$(1.3)\qquad \ker A_n(x) \subset \ker B(x) \quad \text{for } x \in \partial X.$$

(See e.g. [12] for what this condition means).

Now let $(\bar{x}',\bar{\xi}') \in T^*X'\setminus 0$ be a (fixed) gliding point, by definition, a point such that for some $1 \leqq j \leqq r$, say $j = 1$, the equation $Q_1(\bar{x}',0,\bar{\xi}',\xi_n) = 0$ has a real double root $\bar{\xi}_n$ and

$$(1.4) \qquad \{Q_1, \partial Q_1/\partial\xi_n\}(\bar{x},\bar{\xi}) < 0,$$

where $\bar{x} = (\bar{x}',0) \in \partial X$, $\bar{\xi} = (\bar{\xi}',\bar{\xi}_n)$ and $\{,\}$ denotes the Poisson bracket on T^*X. Then, since $Q_1(\bar{x},\bar{\xi}) = (\partial Q_1/\partial\xi_n)(\bar{x},\bar{\xi}) = 0$ and $(\partial^2 Q_1/\partial\xi_n^2)(\bar{x},\bar{\xi}) \neq 0$, one can write

$$(1.5) \qquad \begin{aligned} Q_1(x,\xi) &= Q_0(x,\xi)Q_1'(x,\xi) \quad \text{with } Q_1'(\bar{x},\bar{\xi}) \neq 0, \\ Q_0(x,\xi) &= (\xi_n - \lambda(x,\xi'))^2 - \mu(x,\xi') \end{aligned}$$

in a conic neighborhood of $(\bar{x},\bar{\xi})$. Here $\lambda(x,\xi')$, $\mu(x,\xi')$ are real valued smooth functions, homogeneous in ξ' of degree 1, 2 , respectively, such that $\mu(\bar{x},\bar{\xi}') = 0$, $\lambda(\bar{x},\bar{\xi}') = \bar{\xi}_n$ and (1.4) is equivalent to

$$(1.4)' \qquad \{\xi_n - \lambda, \mu\}(\bar{x},\bar{\xi}) < 0.$$

Since $Q_1(x,\xi)$ is strictly hyperbolic with respect to ξ_0, we have $(\partial\mu/\partial\xi_0) \neq 0$. From now on we suppose, for definiteness,

$$(1.6) \qquad (\partial\mu/\partial\xi_0)(\bar{x},\bar{\xi}') > 0.$$

Note that, near $(\bar{x},\bar{\xi})$, $Q_1(x,\xi) = 0$ is equivalent to $Q_0(x,\xi) = 0$ and if $\xi_0 > \bar{\xi}_0$ then $\mu(\bar{x},\xi_0,\bar{\xi}'') > 0$ hence the equation $Q_0(\bar{x},\xi_0,\bar{\xi}'',\xi_n) = 0$ with respect to ξ_n has two simple real roots. Here we have set $\xi' = (\xi_0,\xi'')$ and $\xi'' = (\xi_1,\cdots,\xi_{n-1})$. Besides, for

$\xi_0 >> 1$, all roots ξ_n of $Q_1(\bar{x},\xi_0,\bar{\xi}'',\xi_n) = 0$ are simple real. Bearing these in mind we make the following assumption on the polynomial Q_1.

(1.7) The two roots of the equation $Q_0(\bar{x},\xi_0,\bar{\xi}'',\xi_n) = 0$ with respect to ξ_n, regarded as single-valued continuous functions of $\xi_0 > \bar{\xi}_0$, are continued up to $\bar{\xi}_0 < \xi_0 < \infty$ as simple real roots of $Q_1(\bar{x},\xi_0,\bar{\xi}'',\xi_n) = 0$.

This condition will be used only to essure (2.4) and hence the solvability of the transport equation (3.23).

We also make certain assumptions on the boundary condition. To state these let $W(x',\xi',\xi_n)$ be a smooth $m\times m_1$ matrix of maximal rank, analytic in ξ_n, whose columns form a basis of $\ker P_1(x',0,\xi',\xi_n)$ when $Q_0(x',0,\xi',\xi_n) = 0$, where m_1 is the multiplicity of Q_1 in (1.2). Besides, let $W_h(x',\xi')$ or $W_e(x',\xi')$ be, respectively, a smooth basis of the root subspace of $P_1(x',0,\xi',\xi_n)$, corresponding to the simple real roots ξ_n of $(Q/Q_0)(x',0,\xi',\xi_n) = 0$ such that $\partial\xi_n/\partial\xi_0 < 0$ or to the nonreal roots with positive imaginary parts. Noting that (W,W_h,W_e) is an $m\times d^+$ matrix, we set

$$(1.8)\qquad \begin{aligned} \tilde{R}(x',\xi',\xi_n) &= B(x',0)(W(x',\xi',\xi_n),W_h(x',\xi'),W_e(x',\xi')),\\ R(x',\xi',\xi_n) &= \det \tilde{R}(x',\xi',\xi_n). \end{aligned}$$

Moreover let $\xi_n^+(x',\xi')$ be the root of $Q_0(x',0,\xi',\xi_n) = 0$ such that $\partial\xi_n^+/\partial\xi_0 < 0$ for $\mu > 0$, $\operatorname{Im} \xi_n^+ > 0$ for $\mu < 0$. Then

$$R(x',\xi',\xi_n^+(x',\xi'))$$

is called a Lopatinski determinant. Note that $\xi_n^+(x',\xi') = \lambda(x',0,\xi') - \mu(x',0,\xi')^{1/2}$ for $\mu \geqq 0$ under (1.6). We also say that the strong Lopatinski condition is satisfied at $(\bar{x}',\bar{\xi}')$ if

$$(1.9) \qquad R(\bar{x}',\bar{\xi}',\bar{\xi}_n) \neq 0 \quad \text{with } \bar{\xi}_n = \lambda(\bar{x}',0,\bar{\xi}').$$

To the contrary, suppose in what follows

$$(1.10) \qquad R(\bar{x}',\bar{\xi}',\bar{\xi}_n) = 0.$$

Then we assume

$$(1.11) \qquad (\partial R/\partial\xi_n)(\bar{x}',\bar{\xi}',\bar{\xi}_n) \neq 0.$$

Set for convenience

$$\mu_0(x',\xi') = \mu(x',0,\xi'),\ \lambda_0(x',\xi') = \lambda(x',0,\xi'),$$

$$N_0 = \{(x',\xi') \in T^*X'\setminus 0;\ \mu_0(x',\xi') = 0\},$$

$$N_\pm = \{(x',\xi') \in T^*X'\setminus 0;\ \mu_0(x',\xi') \gtrless 0\}$$

and

$$(1.12) \qquad R_\lambda(x',\xi') = (R/(\partial R/\partial\xi_n))(x',\xi',\lambda_0(x',\xi')).$$

Our main assumption on the boundary condition is concerned with the range of the normalized Lopantinski determinant R_λ, restricted to the glancing surface N_0.

(H_1) There are a positive number $\delta_0 \leqq \pi/2$ and a conic neighborhood Σ_0 of $(\bar{x}',\bar{\xi}')$ such that, for $(x',\xi') \in N_0 \cap \Sigma_0$,

$$\arg R_\lambda(x',\xi') \subset [\pi/2,(3/2)\pi - \delta_0] \quad \text{if } m_1 = 1,$$

$$\arg R_\lambda(x',\xi') \subset [\pi/2 + \delta_0,(3/2)\pi - \delta_0] \quad \text{if } m_1 \geqq 2,$$

where m_1 is the multiplicity of Q_1 in (1.2).

It is desirable for (H_1) to be relaxed, e.g., as

$(H_1)_0$ $\quad \operatorname{Re} R_\lambda(x',\xi') \leqq 0$ for $(x',\xi') \in N_0 \cap \Sigma_0$.

Unfortunately, when $m_1 \geqq 2$, in the set N_+ of hyperbolic points we must make an additional assumption on R_λ or on reflection coefficients c_{jk} with $j,k = 1,2,\cdots,m_1$. Here, for $(x',\xi') \in N_+$, the $c_{jk}(x',\xi')$ is defined to be the (j,k) entry of the following matrix

$$(1.13) \qquad \tilde{R}(x',\xi',\xi_n^+(x',\xi'))^{-1}\tilde{R}(x',\xi',\xi_n^-(x',\xi')),$$

where $\xi_n^-(x',\xi') = \lambda_0(x',\xi') + \mu_0(x',\xi')^{1/2}$ is another root of $Q_0(x',0,\xi',\xi_n) = 0$. (See [8],§6 for an interpretation of c_{jk}). Condition $(H_1)_0$ implies that, near $(\bar{x}',\bar{\xi}')$,

$$(1.14) \qquad |R(x',\xi',\xi_n^+(x',\xi'))| \geqq C\mu_0(x',\xi')^{1/2}$$

for $(x',\xi') \in N_+$ with $|\xi'| = 1$, where C is a positive constant, hence $c_{jk}(x',\xi')$ are well defined for $(x',\xi') \in N_+$. To see this we note that

$$(1.15) \qquad R(x',\xi',\xi_n^+) = (\partial R/\partial\xi_n)(x',\xi',\lambda_0)(R_\lambda - \sqrt{\mu_0} + O(\mu_0))$$

for $|\xi'| = 1$. Thus (1.14) follows from $(H_1)_0$. Moreover

$c_{jk}(x',\xi')$ are bounded in N_+ near $(\bar{x}',\bar{\xi}')$, because $\xi_n^- = \xi_n^+ + 2\sqrt{\mu_0}$ and hence

$$(1.16) \qquad c_{jk} = \delta_{jk} + O(\sqrt{\mu_0}/\ R(x',\xi',\xi_n^+)) \quad \text{for } |\xi'| = 1.$$

Besides, it follows from (1.8), (1.10) and (1.11) that, for some $1 \leqq j \leqq m_1$, the j-th column of $\tilde{R}(\bar{x}',\bar{\xi}',\bar{\xi}_n)$ is a linear combination of the others. One can assume without loss of generality that $j = 1$. Bearing these in mind we finally impose the following condition.

(H_2) Let $m_1 \geqq 2$. Suppose the first column of $\tilde{R}(\bar{x}',\bar{\xi}',\bar{\xi}_n)$ is a linear combination of the last $d^+ - 1$ columns. Then there is a conic neighborhood Σ_1 of $(\bar{x}',\bar{\xi}')$ such that, in $N_+ \cap \Sigma_1$, either

$$(1.17) \qquad R(x',\xi',\xi_n^+(x',\xi')) = O(\sqrt{\mu_0}) \quad \text{for } |\xi'| = 1$$

or

$$(1.18) \qquad c_{1k}(x',\xi') = O(\sqrt{\mu_0}) \quad \text{for } k = 2,\cdots,m_1 \text{ and } |\xi'| = 1.$$

Note that, according to (1.15), (1.17) is equivalent to

$$(1.19) \qquad R_\lambda(x',\xi') = O(\mu_0) \text{ for } |\xi'| = 1 \text{ in } N_+ \cap \Sigma_1,$$

because $\partial\mu_0/\partial\xi_0 \neq 0$ hence R_λ can be regarded as a smooth function of (x',ξ'',μ_0). For many classical boundary conditions, either (1.19) is satisfied or $c_{1k} = O(\ \mu_0/R(x',\xi',\xi_n^+))$ for $k = 2,\cdots,m_1$ and $|\xi'| = 1$, which implies (1.18).

We are now in a position to state our main results, analogous to [11],Theorems 1.1, 1.2 and 1.3. We shall keep using the notations in the preceding paper, where $P_2(x,\xi)$ is replaced by the symbol $Q_0(x,\xi)$ in (1.5), unless stated otherwise. Since the boundary ∂X may be now characteristic for P, we also need the same function space $H^{\infty,-\infty}_{loc}(V)$ as in [10], where V is a relative open set in X. For a nonnegative integer k and a real number s we mean by $H^{k,s}(X)$ the set of extensible distribution $u \in \mathcal{D}'(X)$ such that $(1 - \Delta_{x'})^{(k-j+s)/2} D_n^j u \in L^2(X)$ for $j = 0,1,\cdots,k$. We then denote by $H^{\infty,-\infty}_{loc}(V)$ the union of $\cap_{k=0}^{\infty} H^{k,s_k}_{loc}(V)$ for all decreasing sequence $\{s_k\}_{k=0}^{\infty}$ of real numbers, where $u \in H^{k,s}_{loc}(V)$ means that $\phi u \in H^{k,s}(X)$ for all $\phi \in C_0^{\infty}(R^{n+1})$ with $(\text{supp } \phi) \cap X \subset V$. By $\Gamma(\bar{x}',\bar{\xi}')$ we also denote the gliding ray (i.e., null bicharacteristics of $\mu_0(x',\xi')$) through $(\bar{x}',\bar{\xi}')$. Suppose for simplicity of description that $\bar{x}_0 = 0$ with $\bar{x}' = (\bar{x}_0,\bar{x}_1,\cdots,\bar{x}_{n-1})$.

<u>Theorem 1.1</u> (main theorem). Let $f \in \mathcal{E}'(X')$ be a distribution with compact support such that WF(f) is contained in a small conic neighborhood of $\Gamma(\bar{x}',\bar{\xi}') \cap \{x_0 \geqq 0\}$. Assume conditions (H_1) and (H_2) hold in the case of (1.10). Then there exist a parametrix E(f) for (1.1) and a positive number T such that

(1.20) $\quad E(f) \in H^{\infty,-\infty}_{loc}(X_T)$, where $X_T = X \cap \{x_0 < T\}$.

(1.21) $\quad E(f) \in C^\infty(X \cap \{x_0 << 0\})$,

(1.22) $\quad PE(f) \in C^\infty(X_T)$,

(1.23) $\quad BE(f)|_{x_n=0} - f \in C^\infty(X'_T)$, where $X'_T = X' \cap \{x_0 < T\}$.

Moreover $E(f)$ is smooth up to the boundary at each point $(x',\xi') \in T^*X'\backslash 0$, with $x_0 < T$, which does not belong to the set

(1.24) $\quad WF(f) \cup M_0^+(f) \cup (\cup_{k=1}^\infty \phi_+^k(WF(f) \cap N_+))$.

Such results have been obtained by Petkov [16], [17] in the case of (1.9) and [18] in the case of (1.19). (See also Melrose and Taylor [15], [21] when $m_1 = 1$). Note that from the above theorem one can derive an outer estimate for the wave front set of $E(f)$ in the interior of X_T, using results on propagation of singularities in the free space (see e.g. Taylor [20], pp.153-155).

Next we shall describe the propagation of singularities of solutions to (1.1). Noting that $A_n(x)$ is of constant rank, we set

$$\text{rank } A_n(x) = d \quad \text{for } x \in X.$$

Then, after a change of the unknown, one can assume without loss of generality that $A_0(x) = I_m$,

(1.25) $$A_n(x) = \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix},$$

where $A(x)$ is a nonsingular matrix of order d, and

$$(1.26) \qquad A(x) = \begin{bmatrix} A^+(x) & 0 \\ 0 & A^-(x) \end{bmatrix},$$

where $A^+(x)$, $A^-(x)$ are square matrices of order d^+, d^-, respectively, with $d^+ + d^- = d$, and $A^+(x)$, $-A^-(x)$ are positive definite. Hereafter I_k stands for the unit matrix of order k. Note that, under (1.25), condition (1.3) yields

$$(1.27) \qquad B(x) = (B_1(x), 0) \quad \text{for } x \in \partial X.$$

where B_1 is a $d^+ \times d$ matrix of maximal rank. We shall assume as usual that, under (1.26), the left $d^+ \times d^+$ block of B_1 is nonsingular and hence

$$(1.28) \qquad B_1(x) = (I_{d^+}, S(x)),$$

where S is a $d^+ \times d^-$ matrix. This hypothesis is satisfied for a wide class of boundary conditions (see [12],§2, in particular, Lemmas 2.9 and 2.10).

<u>Theorem 1.2</u>. Assume (1.28) as well as the hypotheses of Theoerm 1.1 holds and let T be such a positive number as before. Let $u \in H^{0,\bar{s}}_{loc}(X_T)$ with some $\bar{s} \in R^1$. Suppose that $u \in C^\infty(X \cap \{x_0 << 0\})$, $Pu \in C^\infty(X_T)$, $Bu|_{x_n=0} - f \in C^\infty(X'_T)$ and $WF(A_n u|_{x_n=0})$ is contained in a small conic neighborhood of $\Gamma(\bar{x}', \bar{\xi}') \cap \{x_0 \geqq 0\}$. Then u is smooth up to the boundary at each such point $(x', \xi') \in T^*X' \setminus 0$ as described in the preceding theorem.

<u>Theorem 1.3</u>. Assume (H_1) and (H_2) are satisfied in the case of (1.10). Besides, suppose (1.28) holds. Let $u \in H^{\infty,-\infty}_{loc}(V)$, where V is a neighborhood of $\bar{x}$ in X. Assume that $Pu \in C^\infty(V)$,

$$WF(Bu|_{x_n=0}) \cap \Gamma(\bar{x}',\bar{\xi}') \cap \{-\delta < x_0 \leqq 0\} = \phi$$

with some $\delta > 0$, and that $WF(u|_{V\setminus\partial X})$ intersects no incoming null bicharacteristics of $(Q/Q_0)(x,\xi)$ which arrive at $\iota^{*-1}(\bar{x}',\bar{\xi}')$. Here ι^* stands for the pullback of $T^*X|_{\partial X}$ into T^*X' induced by the natural injection ι of X' into X such that $\iota(X') = \partial X$. Then

$$(\bar{x}',\bar{\xi}') \in \cup_{k=0}^{\infty} WF(D_n^k u|_{x_n=0})$$

implies

$$\Gamma(\bar{x}',\bar{\xi}') \cap \{-\delta < x_0 \leqq 0\} \subset WF(u|_{x_n=0})$$

provided δ is small.

<u>Remark</u>. When (1.1) has an appropriate regularity property, Theorem 1.2 is a direct consequence of Theorem 1.1. Besides, a global version of Theorem 1.3 is also valid. Indeed, assume (H_1) and (H_2) as well as (1.28) are satisfied at each point $(\hat{x}',\hat{\xi}') \in \Gamma(\bar{x}',\bar{\xi}') \cap \{x_0 \leqq 0\}$. Suppose $u \in H^{\infty,-\infty}_{loc}(X)$, $Pu \in C^\infty(X)$ and

$$WF(Bu|_{x_n=0}) \cap \Gamma(\bar{x}',\bar{\xi}') \cap \{x_0 \leqq 0\} = \phi.$$

Then

$$(\bar{x}',\bar{\xi}') \in \cup_{k=0}^{\infty} WF(D_n^k u|_{x_n=0})$$

implies

$$\Gamma(\bar{x}',\bar{\xi}') \cap \{x_0 \leqq 0\} \subset WF(u|_{x_n=0}).$$

In the preceding paper [11], we have obtained the analogous results in the case where P is a scalar differential operator of the second order for which the boundary ∂X is noncharacteristic. We have also shown in [8] the existence of a parametrix near a diffractive point for such a hyperbolic system as in the present article. For the purpose of proving Theorem 1.1 we will combine the methods of [11] with those of [8], although some devices are required.

The plan of this paper is as follows. In §2 we first refine the construction in [8] of bases of ker $P_1(x,\xi)$ with $Q_0(x,\xi) = 0$, so that the hyperbolicity of the transport equation (3.23) is clear. Next we give an extension of Andersson and Melrose [1], Proposition 4.16 to the present case. Finally we show that (1.1) has a regularity property provided the boundary condition is maximally dissipative. In §3 we refine the construction in [8] of asymptotic solutions to $Pu = 0$. In §4 we give a summary of [11], §3 which is a collection of properties of Airy operators appearing in the boundary values of the parametrix $E(f)$. §§5 through 7 are devoted to study the equation $BE(f)|_{\partial X} = f$ on the boundary. In §5 we choose appropriately the initial data for the transport equation (3.23) and reduce $BE(f)|_{\partial X} = f$ to (5.10), (5.17) or (5.22). Basic a priori estimates for solutions of the reduced equation are derived in §6, and the singularities of the solutions are examined in §7. The proof of Theorem 1.1 is completed in §8. Finally, Theorem 1.2 an 1.3 are proved in §9.

§2. Preliminaries

2.1. We first refine the basis of ker $P_1(x,\xi)$ with $Q_0(x,\xi) = 0$ which is obtained in [8]. In the case that ∂X is characteristic for P, namely, $d < m$, we write according as (1.25)

$$(2.1)\qquad P_1(x,\xi) = \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}\xi_n + \begin{bmatrix} A_{11}(x,\xi') & A_{12}(x,\xi') \\ A_{21}(x,\xi') & A_{22}(x,\xi') \end{bmatrix},$$

where A_{11}, A_{22} are square matrices of order d, m-d, respectively. Note that $A_{22}(x,\xi')$ is nonsingular if and only if $\tilde{Q}(x,\xi') \neq 0$, where $\tilde{Q}$ is the polynomial in (1.2). In fact, for each (x,ξ'),

$$\xi_n^{-d}\det P_1(x,\xi',\xi_n) = (\det A)\det A_{22} + O(\xi_n^{-1}) \quad \text{as } \xi_n \to \infty,$$

while (1.2) yields

$$\xi_n^{-d}\det P_1(x,\xi',\xi_n) = Q_1(x,0,1)^{m_1}\cdots Q_r(x,0,1)^{m_r}\tilde{Q}(x,\xi') + O(\xi_n^{-1}) \quad \text{as } \xi_n \to \infty.$$

Therefore we have, modulo a nonzero factor,

$$(2.2)\qquad \det A_{22}(x,\xi') = \tilde{Q}(x,\xi').$$

In particular, $A_{22}(\bar{x},\bar{\xi}')$ is nonsingular, because Q_1 and $\tilde{Q}$ have no common zero in ξ_0. For (x,ξ') with $\tilde{Q}(x,\xi') \neq 0$ we set

$$M(x,\xi') = -A^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})$$

and rewrite P_1 as

$$(2.3)\qquad P_1(x,\xi) = \begin{bmatrix} A(\xi_n - M) & A_{12}A_{22}^{-1} \\ 0 & I_{m-d} \end{bmatrix}\begin{bmatrix} I_d & 0 \\ A_{21} & A_{22} \end{bmatrix}.$$

In the case where ∂X is noncharacteristic for P, namely, $d = m$, we set $M(x,\xi') = -A^{-1}A_{11} = \xi_n I_m - A_n(x)^{-1}P_1(x,\xi)$. Such modification will be often required throughout this paper. The following lemma refines [8], Lemma 2.3.

<u>Lemma 2.1</u>. There is a nonsingular smooth matrix $S(x,\xi')$ of order d defined on a conic neighborhood of $(\bar{x},\bar{\xi}')$, analytic and homogeneous of degree zero in ξ', such that $MS = S\tilde{M}$,

$$\tilde{M} = \begin{bmatrix} M_0 & & & & 0 \\ & M_h^+ & & & \\ & & M_h^- & & \\ & & & M_e^+ & \\ 0 & & & & M_e^- \end{bmatrix}, \quad M_0 = \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_{m_1} \end{bmatrix}$$

and

$$M_j(x,\xi') = \begin{bmatrix} \lambda(x,\xi') & 1 \\ \mu(x,\xi') & \lambda(x,\xi') \end{bmatrix} \quad \text{for } |\xi'| = 1,\ j = 1,\cdots,m_1.$$

Here M_h^+ or M_h^- is a diagonal matrix whose eigenvalues are simple real roots ξ_n of $(Q/Q_0)(x,\xi',\xi_n) = 0$ such that $\partial\xi_n/\partial\xi_0$ are negative or positive, respectively; the imaginary parts of the eigenvalues of M_e^+ or M_e^- are positive or negative, respectively. Moreover

(2.4) $\quad S_2^* A(\lambda - M)S_2(\bar{x},\bar{\xi}')$ is positive definite,

where $S_2(x,\xi') = (s_2, s_4, \cdots, s_{2m_1})$ with s_j the j-th column of S. The construction of the last $m-2m_1$ columns of S is as usual. To choose the first $2m_1$ columns $s_1, \cdots, s_{2m_1}$ so that (2.4) also is satisfied, we use the following lemma. For (x,ξ') with $\mu(x,\xi') \geqq 0$ we denote by $\Pi(x,\xi')$ the eigenprojection for eigenvalues $\xi_n^+(x,\xi')$ and $\xi_n^-(x,\xi')$ of $M(x,\xi')$, where

$$\xi_n^{\pm}(x,\xi') = \lambda(x,\xi') \mp \mu(x,\xi')^{1/2}.$$

Then we have

<u>Lemma 2.2</u>. $\Pi(\bar{x},\bar{\xi}')$ is an orhtogonal projection whose range is an invariant subspace of the hermitian matrix

$$A(\bar{x})(\xi_n^+(\bar{x},\bar{\xi}') - M(\bar{x},\bar{\xi}')).$$

Moreover the restriction of the matrix to the range of $\Pi(\bar{x},\bar{\xi}')$ is of rank m_1 and all nonzero eigenvalues are positive.

<u>Proof</u>. For ξ_0 close to $\bar{\xi}_0$ we set

$$\Phi(\xi_0) = A(\bar{x})(\xi_n^+ - M)\Pi(\bar{x},\xi_0,\bar{\xi}").$$

Since (1.7) implies that $\xi_n^{\pm}(\bar{x},\xi_0,\bar{\xi}")$ are continued analytically up to $\xi_0 > \bar{\xi}_0$, so is $\Phi(\xi_0)$. Besides, we have

$$\lim_{\xi_0\to\infty} \xi_n^{\pm}(\bar{x},\xi_0,\bar{\xi}")/\xi_0 = \alpha_{\pm},$$

where $\alpha_{\pm}$ are roots of the equation $Q_1(\bar{x},1,0,\xi_n) = 0$ such that $\alpha_+ < 0$, $\alpha_- > 0$. Noting that $-1/\alpha_{\pm}$ are eigenvalues of $A(\bar{x})$ because $M(x,1,0) = -A(x)^{-1}$, one can assume without loss of generality that

$$A(\bar{x}) = \begin{bmatrix} -(1/\alpha_+)I_{m_1} & 0 & 0 \\ 0 & -(1/\alpha_-)I_{m_1} & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

where β is a nonsingular matrix of order $d-2m_1$ whose eigenvalues are different from $-1/\alpha_{\pm}$. Hence it follows from (1.2) and (2.3) that, for $\xi_0 >> 1$, $A(\bar{x})(\xi_n^+ - M)(\bar{x},\xi_0,\bar{\xi}")$ has zero eigenvalue of multiplicity m_1 and m_1 positive eigenvalues close to $(1 - \alpha_+/\alpha_-)\xi_0$, while the others are far away from those.

Now let $\tilde{\Pi}(\xi_0)$ with $\xi_0 >> 1$ be the orthogonal projection for the zero eigenvalue and those close to $(1 - \alpha_+/\alpha_-)\xi_0$ of $A(\bar{x})(\xi_n^+ - M)(\bar{x},\xi_0,\bar{\xi}")$. Since the matrix is hermitian and analytic for $\xi_0 > \bar{\xi}_0$, $\tilde{\Pi}(\xi_0)$ can be continued analytically up to $\xi_0 > \bar{\xi}_0$. (See Kato [7], p. 120). Moreover $\Pi(\bar{x}, \xi_0,\bar{\xi}")$ coincides with $\tilde{\Pi}(\xi_0)$ for $\xi_0 >> 1$ hence for $\xi_0 > \bar{\xi}_0$ by analiticity. Thus $\Pi(\bar{x},\bar{\xi}_0,\bar{\xi}") = \tilde{\Pi}(\bar{\xi}_0)$ by continuity.

It is now clear that the range of $\Pi(\bar{x},\bar{\xi}') = \tilde{\Pi}(\bar{\xi}_0)$ is an invariant subspace of $A(\bar{x})(\xi_n^+ - M)(\bar{x},\bar{\xi}')$ and $\Phi(\bar{\xi}_0)$ has m_1 positive eigenvalues. Moreover we have rank $\Phi(\bar{\xi}_0) = m_1$ according to [8],Lemma 2.3. Thus we complete the proof.

Proof of Lemma 2.1. By virtue of Lemma 2.2, the restriction of $A(\bar{x})(\lambda - M)(\bar{x},\bar{\xi}')$ to the range of $\Pi(\bar{x},\bar{\xi}')$ has m_1 positive eigenvalues, say, $\alpha_1, \cdots, \alpha_{m_1}$. Let $h_1, \cdots, h_{m_1}$ be an orthonormal system of eigenvectors of $A(\bar{x})(\lambda - M)(\bar{x},\bar{\xi}')$ belonging to $\alpha_1, \cdots, \alpha_{m_1}$, respectively, such that $h_j = \Pi(\bar{x},\bar{\xi}')h_j$. For $j = 1, \cdots, m_1$ and (x,ξ') near $(\bar{x},\bar{\xi}')$, set

$$s_{2j}(x,\xi') = \Pi(x,\xi')h_j,$$
$$s_{2j-1}(x,\xi') = (M(x,\xi') - \lambda(x,\xi'))s_{2j}(x,\xi').$$

Then we shall show that $2m_1$ vectors $s_1, \cdots, s_{2m_1}$ have the required properties. To this end we need only to prove that (2.4) and

$$(M(x,\xi') - \lambda(x,\xi'))s_{2j-1}(x,\xi') = \mu(x,\xi')s_{2j}(x,\xi') \tag{2.5}$$

hold and that $s_1, s_3, \cdots, s_{2m_1-1}$ are linearly independent.

By [8], Lemma 2.3 we have

$$(M - \lambda)^2\Pi(x,\xi') = \mu\Pi(x,\xi'),$$

which implies (2.5). Moreover, since

$$A(\bar{x})(\lambda - M)(\bar{x},\bar{\xi}')h_j = \alpha_j h_j,$$

we have

$$S_2^*A(\lambda - M)S_2(\bar{x},\bar{\xi}') = (h_1, \cdots, h_{m_1})^*(\alpha_1 h_1, \cdots, \alpha_{m_1} h_{m_1})$$

$$= \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_{m_1} \end{bmatrix},$$

which yields (2.4). It is now clear that $s_1, s_3, \cdots, s_{2m_1-1}$ are linearly independent, because $A(\bar{x})s_{2j-1}(\bar{x},\bar{\xi}') = -\alpha_j h_j$. The proof is complete.

Using the S in Lemma 2.1, one can construct a basis $W(x,\xi)$ of ker $P_1(x,\xi)$ with $Q_0(x,\xi) = 0$ which is very convenient in the following analysis. Indeed we define, as in [8],

$$(2.6)\qquad \begin{aligned} W(x,\xi',\xi_n) &= \begin{bmatrix} I_d \\ -A_{22}^{-1}A_{21}(x,\xi') \end{bmatrix} S_0(x,\xi',\xi_n), \\ S_0(x,\xi',\xi_n) &= S_1(x,\xi') + (\xi_n - \lambda(x,\xi'))|\xi'|^{-1}S_2(x,\xi'), \end{aligned}$$

where $S_1 = (s_1, s_3, \cdots, s_{2m_1-1})$, $S_2 = (s_2, s_4, \cdots, s_{2m_1})$ and s_j is the j-th column of S. Then (2.3) yields

$$(2.7)\qquad \begin{aligned} P_1(x,\xi',\xi_n)W(x,\xi',\xi_n) &= \begin{bmatrix} I_d \\ 0 \end{bmatrix} A(\xi_n - M)S_0 \\ &= Q_0(x,\xi',\xi_n)|\xi'|^{-1}\begin{bmatrix} I_d \\ 0 \end{bmatrix} A(x)S_2(x,\xi'), \end{aligned}$$

because $MS_1 = \lambda S_1 + \mu S_2$, $MS_2 = S_1 + \lambda S_2$ for $|\xi'| = 1$.

Similarly, one can construct bases $W_h^+(x,\xi')$, $W_e^+(x,\xi')$ of the root subspaces of $P_1(x,\xi)$ for the eigenvalues of $M_h^+(x,\xi')$, $M_e^+(x,\xi')$. Denoting by S_h^+, S_e^+ the blocks of S coresponding to M_h^+, M_e^+, respectively, we set

$$(2.8)\qquad (W_h^+, W_e^+)(x,\xi') = \begin{bmatrix} I_d \\ -A_{22}^{-1}A_{21} \end{bmatrix} (S_h^+, S_e^+).$$

Note that one can take the restrictions of W, W_h^+ and W_e^+ to $x_n = 0$ as W, W_h, W_e in (1.8).

2.2. To prove the last statement of Theorem 1.1 we need an extension of [1], Proposition 4.16 which gives a connection between the regularity of boundary values of extensible distribution and the smoothness up to the boundary. When ∂X is characteristic for P we also need the following lemma.

Lemma 2.3. Let $(\hat{x},\hat{\xi}) = (\hat{x}',\hat{x}_n,\hat{\xi}',\hat{\xi}_n) \in T^*R^{n+1}\backslash 0$ be a point such that $\hat{\xi}' = 0$ and $\hat{\xi}_n = 1$. Let $\psi_1(x',D') \in OPS^0_{1,0}(R^n)$ be a pseudodifferential operator such that $\tilde{Q}(x',\hat{x}_n,\xi') \neq 0$ on supp $\psi_1(x',\xi')$. Let $u \in H^{\infty,-\infty}_{loc}(V)$, where V is a neighborhood of $\hat{x}$ in R^{n+1}. Suppose $(\hat{x},\hat{\xi}) \notin WF(\psi_1 Pu)$. Then $(\hat{x},\hat{\xi}) \notin WF(\psi_2 u)$ for any pseudodifferential operator $\psi_2(x',D') \in OPS^0_{1,0}(R^n)$ such that $\psi_1(x',\xi') = 1$ on a conic neighborhood of supp $\psi_2(x',\xi')$.

Proof. By the assumption that $(\hat{x},\hat{\xi}) \notin WF(\psi_1 Pu)$ there is a pseudodifferential operator $\chi_1(x,D) \in OPS^0_{1,0}(R^{n+1})$ such that $\chi_1\psi_1 Pu \in C^\infty(R^{n+1})$ and $\chi_1(x,\xi) = 1$ on a conic neighborhood of $(\hat{x},\hat{\xi})$. In view of (2.1) we write

$$(2.9) \qquad P(x,D) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } Pu = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where P_{11} is the upper left d×d block of P and u_1, f_1 are the upper d×1 blocks of u, Pu, respectively. Then, since (2.2) implies that $P_{22}(x',\hat{x}_n,D') \in OPS^1_{1,0}(R^n)$ is elliptic on supp $\psi_1(x',\xi')$, there is a pseudodifferential operator $Q_{22}(x',x_n,D') \in OPS^{-1}_{1,0}(R^n)$, depending smoothly on the parameter x_n close to $\hat{x}_n$, such that

$$\psi_1 Q_{22} P_{22} = \psi_1, \quad \text{mod } OPS^{-\infty}_{1,0}(R^n).$$

Note that $\psi_1 Q_{22} P_{22} u_2 = \psi_1 u_2$, mod $C^\infty(R^{n+1})$. Similarly, there is a microlocal parametrix $Q_{11}(x,D) \in OPS^{-1}_{1,0}(R^{n+1})$ for $P_{11} - P_{12}Q_{22}P_{21}$ at $(\hat{x},\hat{\xi})$. We now define $Q(x,D) \in OPS^{-1}_{1,0}(R^{n+1})$ by

$$Q = \begin{bmatrix} Q_{11} & 0 \\ -Q_{22}P_{21}Q_{11} & \langle D_{x'}\rangle^{-1} I_{m-d} \end{bmatrix} \begin{bmatrix} I_d & -P_{12}Q_{22} \\ 0 & \langle D_{x'}\rangle Q_{22} \end{bmatrix},$$

where $\langle D_{x'}\rangle \in OPS^1_{1,0}(R^n)$ is the pseudodifferential operator with symbol $(1 + |\xi'|^2)^{1/2}$. Then there is a pseudodifferential operator $\chi_2(x,D) \in OPS^0_{1,0}(R^{n+1})$, elliptic at $(\hat{x},\hat{\xi})$, such that $\chi_1(x,\xi) = 1$ on supp $\chi_2(x,\xi)$ and $\chi_2\psi_2 QPu = \chi_2\psi_2 u$, mod $C^\infty(R^{n+1})$ for each $\psi_2(x',D') \in OPS^0_{1,0}(R^n)$ with the properties stated in the lemma. Since $\chi_2\psi_2 QPu = \chi_2\psi_2 Q\chi_1\psi_1 Pu$ mod $C^\infty(R^{n+1})$, we conclude that $\chi_2\psi_2 u \in C^\infty(R^{n+1})$ and hence $(\hat{x},\hat{\xi}) \notin WF(\psi_2 u)$. Thus we prove the lemma.

One can now prove the following extension of [1], (4.16) to systems.

<u>Proposition 2.4</u>. Let $(\hat{x}',\hat{\xi}') \in T^*X'\backslash 0$ be a point such that $\tilde{Q}(\hat{x}',0,\hat{\xi}') \neq 0$. Let $u \in H^{\infty,-\infty}_{loc}(V)$, where V is a neighborhood of $(\hat{x}',0)$ in X. Suppose $Pu \in C^\infty(V)$ and $(\hat{x}',\hat{\xi}') \notin WF(u|_{x_n=0})$. Moreover assume that each of null bicharacteristics of $Q(x,\xi)$ through $\iota^{*-1}(\hat{x}',\hat{\xi}')$ either immediately enters $x_n < 0$ in at least one direction or intersects a point $(x,\xi) \in T^*X\backslash 0$ such that $(x,\xi) \notin WF(u)$, $x_n > 0$ and $x \in V$. Then u is smooth up to the boumdary at $(\hat{x}',\hat{\xi}')$, namely, there is a pseudodifferential operator $\psi(x',\xi') \in OPS^0_{1,0}(R^n)$, elliptic at $(\hat{x}',\hat{\xi}')$, such that $\psi u \in C^\infty(X \cap \{0 \leqq x_n << 1\})$.

Proof. We shall keep using the notations in (2.9). Since $P_{22}(x',0,\xi')$ is elliptic at $(\hat{x}',\hat{\xi}')$ according to (2.2), one can find a small conic neighborhood Σ of $(\hat{x}',\hat{\xi}')$ such that

$$(2.10) \qquad WF(u|_{x_n=0}) \cap \Sigma = \phi$$

and $P_{22}(x',0,\xi')$ is elliptic on Σ. We shall show

$$(2.11) \qquad WF(D_n^j u|_{x_n=0}) \cap \Sigma = \phi \quad \text{for all } j \geqq 0.$$

Set

$$P'_{11}(x,D') = P_{11}(x,D) - A(x)D_n.$$

Then, since $f_1 \in C^\infty(V)$ and $D_n u_1 = A^{-1}(f_1 - P'_{11}u_1 - P_{12}u_2)$, it follows from (2.10) that

$$(2.12) \qquad WF(D_n u_1|_{x_n=0}) \cap \Sigma = \phi.$$

Next, applying D_n to $P_{22}u_2 = f_2 - P_{21}u_1$, we have

$$P_{22}(x,D')D_n u_2 = D_n f_2 - P_{21}D_n u_1 - [D_n, P_{22}]u_2 - [D_n, P_{21}]u_1.$$

Therefore (2.10) and (2.12) yield (2.11) for $j = 1$, because $P_{22}(x',0,D')$ is elliptic on Σ. Analogously we obtain (2.11) for $j \geqq 2$. Thus, as in [1],P.210, we can prove the proposition, by using (2.11), Lemma 2.3 and results on propagation of singularity in the free space.

2.3. To prove Theorem 1.2 we will use the following regularity of solutions to dissipative boundary value problems for P. Note that all roots $\tau_1(x,\xi'',\xi_n),\cdots,\tau_m(x,\xi'',\xi_n)$ of the equation $\det P_1(x,\xi_0,\xi'',\xi_n) = 0$ with respect to ξ_0 are bounded real-valued functions of $(x,\xi'',\xi_n) \in X\times\{(\xi'',\xi_n) \in R^n;\ |\xi''|^2 + \xi_n^2 = 1\}$. Let τ_0 be the least upper bound of $|\tau_k(x,\xi'',\xi_n)|$ for such (x,ξ'',ξ_n) and $k = 1,\cdots,m$. For a point $\bar{x} \in X$ we denote by $\Gamma_0(\bar{x})$ the interior of a backward cone with vertex $\bar{x}$, more precisely, we set

$$\Gamma_0(\bar{x}) = \{x \in R^{n+1};\ \tau_0(\bar{x}_0 - x_0) > (|x'' - \bar{x}''|^2 + |x_n - \bar{x}_n|^2)^{1/2}\}.$$

<u>Proposition 2.5</u>. Let $\bar{x} \in X$ and $u \in H^{0,s}_{loc}(\Gamma_0(\bar{x}) \cap X)$ for some $s \in R^1$. Suppose $Pu \in C^\infty(\Gamma_0(\bar{x}) \cap X)$, $Bu|_{x_n=0} \in C^\infty(\Gamma_0(\bar{x}) \cap \{x_n=0\})$ and $u \in C^\infty(\Gamma_0(\bar{x}) \cap X \cap \{x_0 << \bar{x}_0\})$. Moreover assume that boundary condition $Bu|_{x_n=0} = 0$ is maximally nonpositive for P. Then $u \in C^\infty(\Gamma_0(\bar{x}) \cap X)$.

Although this proposition seems to be well known in essence, we shall give a proof for the sake of completeness of description.

For an integer k and real numbers s, γ with $\gamma \neq 0$ we denote by $H^{k,s}_\gamma(X)$ the set of extensible distributions $u \in \mathcal{D}'(X\backslash\partial X)$ such that $e^{-\gamma x_0}u \in H^{k,s}(X)$. (See e.g. [4], p. 51 for the space $H^{k,s}(X)$ with $k < 0$).

<u>Lemma 2.6</u>. For each nonnegative integer s, there is a positive number γ_s such that if $\gamma \geqq \gamma_s$, $u \in H^{-1,0}_\gamma(X)$ and $P_{22}u \in H^{0,s}_\gamma(X)$ then $u \in H^{0,s}_\gamma(X)$. Here P_{22} is the differential operator in (2.9).

Proof. Since P_{22} is a symmetric hyperbolic system, there is a positive number γ_s' such that for each $\gamma \geqq \gamma_s'$ there exists a function $w \in H_\gamma^{0,s}(X)$ satisfying $P_{22}w = P_{22}u$ if $P_{22}u \in H_\gamma^{0,s}(X)$. Therefore it suffices to prove the uniqueness.

Let $u \in H_\gamma^{-1,0}(X)$ and $P_{22}u = 0$ in $H_\gamma^{0,s}(X)$. Then we need only to show that $u = 0$ in $\mathcal{D}'(X\setminus\partial X)$, because $H_\gamma^{-1,0}(X)$ is a subspace of $\mathcal{D}'(X\setminus\partial X)$. Let $g \in C_0^\infty(R^{n+1})$ be a test function, supported in $\{x_n > 0\}$. Then the equation $P_{22}^* v = g$ in R^{n+1} has a unique solution $v \in H_{-\gamma}^{2,0}(R^{n+1})$ for $\gamma \geqq \gamma_0$, where P_{22}^* is the formal adjoint of P_{22} and γ_0 a positive number. Moreover, since P_{22} does not contain D_n, we observe that supp $v(x) \subset \{x_n > 0\}$. Hence $(u,g) = (P_{22}u,v) = 0$. The proof is complete.

Lemma 2.7. Suppose the boundary condition $Bu|_{x_n=0} = 0$ is maximally nonpositive for P. Then for each positive integer k there is a positive number γ_k such that, for any $\gamma \geqq \gamma_k$ and $f \in H_\gamma^{2k,0}(X)$, the following boundary value problem

$$(2.13)\qquad \begin{aligned} Pu &= f \quad \text{in } X, \\ Bu &= 0 \quad \text{on } \partial X \end{aligned}$$

has a unique solution $u \in H_\gamma^{k,0}(X)$. Moreover

$$(2.14)\qquad u(x) = 0 \quad \text{in } \Gamma_0(\bar{x}) \cap X \text{ if so is } f$$

for each $\bar{x} \in X$.

Proof. By the hypothesis of the lemma one can assume the $B_1(x)$ in (1.27) is of the form (1.28). (See [12], Lemmas 2.9 and 2.10). Set

$$H(x) = \begin{bmatrix} I_{d^+} & -S(x) \\ S^*(x) & I_{d^-} \end{bmatrix},$$

which is a nonsingular matrix of order d. Besides, set $\tilde{B}_1 = B_1 H$ and $\tilde{u}_1 = H^{-1}u_1$ with the notations in (2.9). Then $\tilde{B}_1\tilde{u}_1 = B_1 u_1$ and $\tilde{B}_1 = (I_{d^+} + SS^*, 0)$, where $I_{d^+} + SS^*$ is nonsingular. Thus after a change of the unknown u_1, one can assume B_1 is of the form

$$B_1(x) = (I_{d^+}, 0), \tag{2.15}$$

although (1.26) may be not preserved. Moreover we define a $d^- \times d$ matrix $B_1'(x)$ by

$$B_1'(x) = (0, I_{d^-})A(x). \tag{2.16}$$

Then the adjoint boundary condition is given by $B_1' v_1|_{x_n=0} = 0$, because

$$(Pu,v)_{L^2(X)} - (u,P^*v)_{L^2(X)} = i(u_1,Av_1)_{L^2(\partial X)}$$

for $u, v \in C_0^\infty(X)$. Furthermore the hypothesis of the lemma implies that

$$A(x)v_1 \cdot v_1 \geqq 0 \quad \text{for } v_1 \in \ker B_1'(x) \text{ and } x \in \partial X, \tag{2.17}$$

since the boundary condition $B_1' v_1|_{x_n=0} = 0$ is maximally nonpositive for P* (see Lax and Phillips [13]). Therefore we see by a standard argument that for each positive integer k there is a positive number γ_k' such that (2.13) has a unique solution $u \in H_\gamma^{0,2k}(X)$ for any $\gamma \geqq \gamma_k'$ and $f \in H_\gamma^{0,2k}(X)$.

Now let $f \in H_\gamma^{1,2k-1}(X)$. We shall show there is another positive number $\gamma_1 \geqq \gamma_1'$ such that

$$(2.18) \qquad u \in H_\gamma^{1,2k-2}(X) \quad \text{for } \gamma \geqq \gamma_1.$$

We keep using the notations in (2.9). As in the proof of Proposition 2.4 we then have $D_n u_1 \in H_\gamma^{0,2k-1}(X)$ hence $u_1 \in H_\gamma^{1,2k-1}(X)$, because $u_1 \in H_\gamma^{0,2k}(X)$. We also see that $P_{22}D_n u_2 \in H_\gamma^{0,2k-2}(X)$. Since $D_n u_2 \in H_\gamma^{-1,0}(X)$, it follows from the preceding lemma that $D_n u_2 \in H_\gamma^{0,2k-2}(X)$ hence $u_2 \in H_\gamma^{1,2k-2}(X)$ if γ is large enough. Thus there is a positive number γ_1 such that (2.18) holds. Analogously one can find γ_k with the desired properties.

Finally we shall prove (2.14). Let $\nu(x) = (\nu_0(x), \nu_1(x), \cdots, \nu_n(x))$ be the outward unit normal to $\Gamma_0(\bar{x}) \cap X$ at a point x on the boundary $\partial(\Gamma_0(\bar{x}) \cap X)$ of $\Gamma_0(\bar{x}) \cap X$ and set

$$A^\nu(x) = \Sigma_{k=0}^n \nu_k(x) A_k(x).$$

For convenience we write

$$(e^{-\gamma x_0}u,\ e^{-\gamma x_0}v)_{L^2(\Gamma_0(\bar{x}) \cap X)} = (u,v)_\gamma,$$

$$(e^{-\gamma x_0}u,\ e^{-\gamma x_0}v)_{L^2(\partial(\Gamma_0(\bar{x}) \cap X))} = \langle u,v\rangle_\gamma$$

and $\|u\|_\gamma = ((u,u)_\gamma)^{1/2}$. Noting that

$$P(e^{-\gamma x_0}u) = e^{-\gamma x_0}(Pu + i\gamma A_0 u),$$

$$P^*v = Pv + (\Sigma_{k=0}^n D_k A_k + C^* - C)v,$$

and setting

$$\tilde{C}(x) = \Sigma_{k=0}^n \partial A_k(x)/\partial x_k + i(C^* - C)(x),$$

by Green's formula we have

$$-2\gamma\|u\|_\gamma^2 - 2\mathrm{Im}\ (f,u)_\gamma + (u,\tilde{C}u)_\gamma = \langle A^\nu u,u\rangle_\gamma.$$

Moreover we claim that

$$\langle A^\nu u,u\rangle_\gamma \geqq 0. \tag{2.19}$$

In fact, since $A^\nu(x) = -A_n(x)$ on ∂X, by assumption we have

$$A^\nu(x)u\cdot u \geqq 0 \quad \text{for } u \in \ker B(x) \text{ and } x \in \partial X.$$

Furthermore, on the boundary of $\Gamma_0(\bar{x})$ we have

$$\nu(x) = (1 + \tau_0^2)^{-1/2}(\tau_0,\xi'',\xi_n),$$

where $(\xi'',\xi_n) = (|x'' - \bar{x}''|^2 + |x_n - \bar{x}_n|^2)^{-1/2}(x'' - \bar{x}'',x_n - \bar{x}_n)$, so

$$A^\nu(x) = (1 + \tau_0^2)^{-1/2}(\tau_0 A_0(x) + \Sigma_{k=1}^n \xi_k A_k(x)),$$

which is positive semidefinite because of the definition of τ_0. Thus we obtain (2.19) and hence

$$2\gamma\|u\|_\gamma^2 + 2\mathrm{Im}\ (f,u)_\gamma - (u,\tilde{C}u)_\gamma \leqq 0.$$

Besides, since $\tilde{C}(x)$ is bounded in X, there is a constant C_0 such that $|(u,\tilde{C}u)_\gamma| \leqq C_0\|u\|_\gamma^2$. Now let $f = 0$ in $\Gamma_0(\bar{x}) \cap X$. Then

$$(2\gamma - C_0)\|u\|_\gamma^2 \leqq 0,$$

so $u = 0$ in $\Gamma_0(\bar{x}) \cap X$ for $\gamma > C_0/2$. Thus we prove the lemma.

Remark 2.8. From the proof we also observe the following. Suppose to the contrary that $Bu|_{x_n=0} = 0$ is maximally nonnegative for P. Then the conclusion of Lemma 2.7 is still valid provided $f \in H_\gamma^{2k,0}(X)$, $u \in H_\gamma^{k,0}(X)$ and $\Gamma_0(\bar{x})$ are replaced, respectively, by $f \in H_{-\gamma}^{2k,0}(X)$, $u \in H_{-\gamma}^{k,0}(X)$ and the forward cone

$$\{x \in R^{n+1};\ \tau_0(x_0 - \bar{x}_0) > (|x'' - \bar{x}''|^2 + |x_n - \bar{x}_n|^2)^{1/2}\}.$$

Corollary 2.9. Suppose the hypothesis of Lemma 2.7 is satisfied. Let $u \in H_{loc}^{0,s}(\Gamma_0(\bar{x}) \cap X)$ for some $s \in R^1$. Suppose $Pu = 0$ in $\Gamma_0(\bar{x}) \cap X$, $Bu = 0$ on $\Gamma_0(\bar{x}) \cap \partial X$ and $u = 0$ in $\Gamma_0(\bar{x}) \cap X \cap \{x_0 << \bar{x}_0\}$. Then $u = 0$ in $\mathcal{D}'(\Gamma_0(\bar{x}) \cap \{x_n > 0\})$.

Proof. In view of (2.16) and (2.17) we see from the preceding remark that, for each $\gamma \geqq \gamma_1$ and $g \in C_0^\infty(\Gamma_0(\bar{x}) \cap \{x_n > 0\})$, there is a function $v \in H_{-\gamma}^{1,|s|}(X)$ such that $P^*v = g$ in X, $B_1'v_1|_{x_n=0} = 0$ and $(\mathrm{supp}\ v) \cap X \subset \Gamma_0(\bar{x})$. Hence by Green's formula we have $(u,g)_{L^2(X)} = 0$, which proves the corollary.

<u>Proof of Proposition 2.5</u>. One can assume without loss of generality that $u = 0$ in $\Gamma_0(\bar{x}) \cap X \cap \{x_0 << \bar{x}_0\}$. Let $\hat{x} \in \Gamma_0(\bar{x}) \cap X$ and let k be an arbitrary positive integer such that $k > s$. Then it suffices to prove

$$(2.20) \qquad u \in H^{k,0}_{loc}(\Gamma_0(\hat{x}) \cap X).$$

In view of (2.15) one can also assume $Bu = 0$ on $\Gamma_0(x) \cap \{x_n = 0\}$. Let $\phi \in C^\infty(R^{n+1})$ be a cutoff function, suppoted in $\Gamma_0(\bar{x})$, such that $\phi(x) = 1$ on $\Gamma_0(\hat{x})$, and set $f = \phi Pu$. Then $f \in C_0^\infty(X)$, in particular, $f \in H^{2k,0}_\gamma(X)$ for all $\gamma > 0$. Hence by Lemma 2.7 there is a solution $v \in H^{k,0}_\gamma(X)$ of (2.13) such that $v = 0$ in $X \cap \{x_0 << x_0\}$. Set $w = u - v$. Then $w \in H^{0,s}_{loc}(\Gamma_0(\hat{x}) \cap X)$, $Pw = 0$ in $\Gamma_0(\hat{x}) \cap X$, $Bw = 0$ on $\Gamma_0(\hat{x}) \cap \{x_n = 0\}$ and $w = 0$ in $\Gamma_0(\hat{x}) \cap X \cap \{x_0 << \hat{x}_0\}$. Therefore by Corallary 2.9 we have $w = 0$ in $\mathcal{D}'(\Gamma_0(\hat{x}) \cap \{x_n > 0\})$ and hence (2.20) follows, because $H^{k,0}_{loc}(\Gamma_0(\hat{x}) \cap X)$ is a subspace of $\mathcal{D}'(\Gamma_0(\hat{x}) \cap \{x_n > 0\})$. Thus we prove the proposition.

§3. Construction of a parametrix

In order to construct the parametrix $E(f)$ we use the same phase functions $\theta(x,\eta')$, $\rho(x,\eta')$ and Airy functions $A_0(z)$, $A_\pm(z)$ as in [//]. Recall that $\theta(x,\eta')$, $\rho(x,\eta')$ with $\eta' = (\eta_0,\eta'') = (\eta_0,\eta_1,\cdots,\eta_{n-1})$ are real valued smooth functions defined on a conic neighborhood of $(\bar{x},\bar{\eta}')$ in $X\times(R^n\setminus 0)$, homogeneous in η' of degree 1, 2/3, respectively, where $\bar{\eta}_0 = 0$, $\bar{\eta}'' = \bar{\xi}'' \neq 0$. Moreover the functions $\phi^\pm = \theta \pm (2/3)\rho^{3/2}$ solve the eikonal equation $Q_0(x,\phi_x^\pm) = 0$ in the following sense, where $\phi_x^\pm = \partial\phi^\pm/\partial x$ and $Q_0(x,\xi)$ is the function in (1.5). Writing $\lambda(x, \theta_{x'} \pm \sqrt{\rho}\rho_{x'}) = \lambda_1 \pm \sqrt{\rho}\lambda_2$, $\mu(x, \theta_{x'} \pm \sqrt{\rho}\rho_{x'}) = \mu_1 \pm \sqrt{\rho}\mu_2$ for $\rho > 0$, where λ_1, λ_2, μ_1 and μ_2 are even functions of $\sqrt{\rho}$, we have for $\rho > 0$

$$(3.1)_+ \qquad \begin{aligned} &(\theta_{x_n} - \lambda_1)^2 + \rho(\rho_{x_n} - \lambda_2)^2 - \mu_1 = 0, \\ &2(\theta_{x_n} - \lambda_1)(\rho_{x_n} - \lambda_2) - \mu_2 = 0, \end{aligned}$$

and, for $\eta_0 < 0$ and $0 \leqq x_n << 1$,

$$(3.1)_- \qquad \begin{aligned} &(\theta_{x_n} - \lambda_1)^2 + \rho(\rho_{x_n} - \lambda_2)^2 - \mu_1 = O(x_n^\infty|\eta'|^2), \\ &2(\theta_{x_n} - \lambda_1)(\rho_{x_n} - \lambda_2) - \mu_2 = O(x_n|\eta'|^{5/3}). \end{aligned}$$

Furthermore, for $x_n = 0$,

$$(3.2) \qquad \det \theta_{x'\eta'} > 0, \text{ where } \theta_{x'\eta'} = \partial^2\theta/\partial x'\partial\eta',$$

$$(3.3) \qquad \theta_{x_0\eta_0} > 0,$$

$$(3.4) \qquad \rho_{x_n} < 0,$$

$$(3.5) \qquad \rho(x',0,\eta') = \eta_0|\eta''|^{-1/3} = \alpha|\eta''|^{2/3}, \text{ where } \alpha = \eta_0/|\eta''|.$$

We also have for $x_n = 0$

(3.6) $\theta_{x_n} = \lambda(x,\theta_{x'})$,

(3.7) $\mu(x,\theta_{x'}) = \alpha(\rho_{x_n})^2$ for $|\eta''| = 1$,

because $\lambda_2 = \mu_2 = 0$, $\lambda_1 = \lambda$ and $\mu_1 = \mu$ for $x_n = 0$.

From now on, for $x_n = 0$ we shall extend θ, ρ, θ_{x_n} and ρ_{x_n} to $R^n \times R^n$ in such a way that (3.2) through (3.7) are preserved for $|\alpha| < 1$ and that $\theta(x',0,\eta') = x'\eta'$ outside a conic neighborhood of $(\bar{x}',\bar{\eta}')$. Then $\theta(x',0,\eta')$ generates a canonical transformation $\phi_1(y',\eta') = (x',\xi')$ of $T^*R^n\backslash 0$ into $T^*X'\backslash 0$ defined by

(3.8) $\xi' = \theta_{x'}(x',0,\eta')$, $y' = \theta_{\eta'}(x',0,\eta')$.

Moreover, under ϕ_1^{-1}, the gliding ray $\Gamma(\bar{x}',\bar{\xi}')$ is mapped (locally) onto the straight line through $(\bar{y}',\bar{\eta}') = \phi_1^{-1}(\bar{x}',\bar{\xi}')$ which is parallel to the y_0 axis, where $y' = (y_0,y'') = (y_0,y_1,\cdots,y_{n-1})$, and on which y_0 increases as x_0 does.

Now we shall look for the parametrix $E(f)$ in the same form as (6) of [10], namely,

(3.9) $Gv = G_0v_0 + G_hv_h + G_ev_e$.

Here $v(y') = {}^t({}^tv_0, {}^tv_h, {}^tv_e)$ is a vector with d^+ components in $H^{-\infty}(R^n)$, and G_h, G_e are essentially the same operators as the $G^{(2)}$, $G^{(3)}$ in [8], respectively, while G_0 is the same one as the G_1 defined by (8) of [10], in other words, we take $q_1 = 1$ and $q_2 = 0$ in the (7). More precisely, we define

$$(3.10)\qquad (G_0v_0)(x) = \int e^{i\check{\theta}}(A_0(\check{\rho})\check{a} - iA_0'(\check{\rho})\check{b})(A_+(\zeta)^{-1}\chi_1 + A_0(\zeta)^{-1}(1-\chi_1))\hat{v}_0(\eta')d\eta',$$

where

$$\hat{v}_0(\eta') = \int e^{-iy'\eta'}v_0(y')dy'$$

and $v_0(y')$ is determined later so that (1.23) holds. Here A_0, A_+, ζ and χ_1 are the same functions as in [11],(2.15). (See also (4.1), (4.4) and (4.7) below). Besides, $\check{\theta}$, $\check{\rho}$, $\check{a}$ and $\check{b}$ are almost analytic continuations of θ, ρ, a and b, analogous to (2.17) of that paper (see also (3.30) below), while the amplitudes $a(x,\eta')$ and $b(x,\eta')$ are given in the following theorem.

<u>Theorem 3.1</u>. There exist smooth $m\times m_1$ matrices $a(x,\eta')$, $b(x,\eta')$ defined on a conic neighborhood of $(\bar{x},\bar{\eta}')$ in $X\times(R^n\setminus 0)$, which have asymptotic expansions

$$a \sim \Sigma_{k=0}^{\infty}a_k,\quad b \sim \Sigma_{k=0}^{\infty}b_k,$$

where $a_k(x,\eta')$, $b_k(x,\eta')$ are homogeneous in η' of degree $-k$, $-k-1/3$, respectively. Moreover if we write

$$(3.11)\qquad \begin{aligned} &e^{-i\theta}P(x,D)(e^{i\theta}(A_0(\rho)a - iA_0'(\rho)b)) = A_0(\rho)c - iA_0'(\rho)d,\\ &c \sim \Sigma_{k=-1}^{\infty}c_k,\quad d \sim \Sigma_{k=-1}^{\infty}d_k,\end{aligned}$$

where $c_k(x,\eta')$, $d_k(x,\eta')$ are homogeneous in η' of degree $-k$, $-k-1/3$, respectively, then

$$(3.12)\qquad c_k,\ d_k = \begin{cases} 0 & \text{for } \rho \geqq 0, \\ O(x_n^\infty) & \text{for } 0 \leqq x_n << 1,\ \eta_0 < 0 \text{ and } |\eta'| = 1. \end{cases}$$

This theorem has been essentially obtained in [8],§§3 and 4. Nevertheless, since the proof given there is somewhat inaccessible, we shall give another proof. (See also Petkov [7]).

<u>Proof of Theorem 3.1</u>. We first seek a_k, b_k for $\rho > 0$ and then extend them to $\rho \leqq 0$. Using the equation $A_0''(\rho) = -\rho A_0(\rho)$ and setting the coefficients of $A_0(\rho)$, $-iA_0'(\rho)$ on the left hand side of (3.11) equal to c, d, respectively, we have for $k = -1, 0, 1, 2, \cdots$,

$$c_k = P_1(x,\theta_x)a_{k+1} + \rho P_1(x,\rho_x)b_{k+1} + P(x,D)a_k,$$

$$d_k = P_1(x,\theta_x)b_{k+1} + P_1(x,\rho_x)a_{k+1} + P(x,D)b_k,$$

where $a_{-1} = 0$, $b_{-1} = 0$.

Let $\rho > 0$. Then (3.12) is equivalent to

$$(3.13)\qquad c_k \pm \sqrt{\rho}d_k = 0,$$

where

$$c_{-1} \pm \sqrt{\rho}d_{-1} = P_1(x,\ \theta_x \pm \sqrt{\rho}\rho_x)(a_0 \pm \sqrt{\rho}b_0).$$

Let $W(x,\xi)$ be the matrix defined by (2.6). Then it follows from (2.7) and the eikonal equation $(3.1)_+$ that

$$(3.14)\qquad P_1(x,\ \theta_x \pm \sqrt{\rho}\rho_x)W(x,\ \theta_x \pm \sqrt{\rho}\rho_x) = 0.$$

Thus, setting

$$a_0 \pm \sqrt{\rho} b_0 = W(x, \theta_x \pm \sqrt{\rho}\rho_x)(g_0 \pm \sqrt{\rho} h_0), \tag{3.15}$$

we obtain (3.13) for $k = -1$, where $g_0(x,\eta')$, $h_0(x,\eta')$ are arbitrary smooth $m_1 \times m_1$ matrices, homogeneous in η' of degree 0, $-1/3$, respectively. More preciesly, we define a_0, b_0 as follows. Setting

$$\begin{aligned} W_1(x,\eta') &= (W(x, \theta_x + \sqrt{\rho}\rho_x) + W(x, \theta_x - \sqrt{\rho}\rho_x))/2, \\ W_2(x,\eta') &= (W(x, \theta_x + \sqrt{\rho}\rho_x) - W(x, \theta_x - \sqrt{\rho}\rho_x))/(2\sqrt{\rho}) \end{aligned} \tag{3.16}$$

so that $W(x,\theta_x \pm \sqrt{\rho}\rho_x) = W_1 \pm \sqrt{\rho} W_2$, we define

$$\begin{aligned} a_0 &= W_1 g_0 + \rho W_2 h_0 \\ b_0 &= W_1 h_0 + W_2 g_0 \end{aligned} \tag{3.17}$$

Then (3.15) holds and $a_0(x,\eta')$, $b_0(x,\eta')$ are smooth near $(\bar{x},\bar{\eta}')$, homogeneous in η' of degree 0, $-1/3$, respectivly, because so are $W_1(x,\eta')$, $W_2(x,\eta')$.

Next let $k = 0$ and $|\eta'| = 1$. Then (3.13) is equivalent to

$$P_1(x,\theta_x \pm \sqrt{\rho}\rho_x)(a_1 \pm \sqrt{\rho} b_1) + P(x,D)a_0 \pm \sqrt{\rho} P(x,D) b_0 = 0, \tag{3.18}$$

so we look for a_1, b_1 in the form

$$\begin{aligned} a_1 &= W_1 g_1 + \rho W_2 h_1 + \tilde{a}_1, \\ b_1 &= W_1 h_1 + W_2 g_1 + \tilde{b}_1, \end{aligned} \tag{3.19}$$

where $\tilde{a}_1 \pm \sqrt{\rho}\tilde{b}_1$ are special solutions of (3.18) with a_0, b_0 given.

We also see, as above, that (3.18) with (3.19) becomes the following system of linear equations for $\tilde{a}_1 \pm \sqrt{\rho}\tilde{b}_1$:

$$(3.20) \qquad P_1(x,\theta_x \pm \sqrt{\rho}\rho_x)(\tilde{a}_1 \pm \sqrt{\rho}\tilde{b}_1) + P(x,D)a_0 \pm \sqrt{\rho}P(x,D)b_0 = 0,$$

which is solvable if and only if

$$(3.21) \qquad W^*(x,\theta_x \pm \sqrt{\rho}\rho_x)(P(x,D)a_0 \pm \sqrt{\rho}P(x,D)b_0) = 0.$$

Here W^* denotes the adjoint matrix of W, so the rows of $W^*(x,\theta_x \pm \sqrt{\rho}\rho_x)$ are right null vectors of $P_1(x,\theta_x \pm \sqrt{\rho}\rho_x)$. Notice that (3.21) with (3.17) is equivalent to the following transport equation for $g_0 \pm \sqrt{\rho}h_0$:

$$(3.22) \qquad \sum_{j=0}^{n} A_j^{\pm}(x,\eta',\sqrt{\rho})\frac{\partial}{\partial x_j}(g_0 \pm \sqrt{\rho}h_0) + C^{\pm}(x,\eta',\sqrt{\rho})(g_0 \pm \sqrt{\rho}h_0) = 0,$$

where

$$A_j^{\pm}(x,\eta',t) = W^*(x,\theta_x \pm t\rho_x)A_j(x)W(x,\theta_x \pm t\rho_x),$$

$$C^{\pm}(x,\eta',t) = iW^*(x,\theta_x \pm t\rho_x)P(x,D)W(x,\theta_x \pm t\rho_x).$$

In fact, (3.17) yields

$$\begin{aligned} P(x,D)a_0 \pm \sqrt{\rho}P(x,D)b_0 &= \Sigma_{j=0}^{n}A_j(x)(W_1 \pm \sqrt{\rho}W_2)D_j(g_0 \pm \sqrt{\rho}h_0) \\ &\quad + (P(x,D)W_1 \pm \sqrt{\rho}P(x,D)W_2)(g_0 \pm \sqrt{\rho}h_0) \\ &\quad \pm (i/(2\sqrt{\rho}))(P_1(x,\rho_x)W_1 \mp \sqrt{\rho}P_1(x,\rho_x)W_2)h_0. \end{aligned}$$

Moreover from (3.14) and (3.16) we have $P_1(x,\rho_x)W_1 = -P_1(x,\theta_x)W_2$. Therefore the left hand side of (3.21) coincides with $(-i)$ times that of (3.22), because $(W_1^* \pm \sqrt{\rho}W_2^*)P_1(x,\theta_x \pm \sqrt{\rho}\rho_x) = 0$. Furthermore, making changes of variables $(x',x_n) \to (x',\rho)$

and $(x',\rho) \to (x',t)$ with $t = \sqrt{\rho}$, and setting

$$a^{\pm}(x',t,\eta') = g_0(x,\eta') \pm th_0(x,\eta'),$$

we see that (3.22) is equivalent to

$$(3.23)_{\pm} \qquad C_n^{\pm}(x',t,\eta')\frac{\partial a^{\pm}}{\partial t} + \sum_{j=0}^{n-1} C_j^{\pm}(x',t,\eta')\frac{\partial a}{\partial x_j} + C_{n+1}^{\pm}(x',t,\eta')a^{\pm} = 0,$$

where

$$C_n^{\pm}(x',t,\eta') = (2t)^{-1}W^*(x,\theta_x \pm t\rho_x)P_1(x,\rho_x)W(x,\theta_x \pm t\rho_x),$$

$$C_j^{\pm}(x',t,\eta') = A_j^{\pm}(x,\eta',t),\ j = 0,\ 1,\cdots,n-1,$$

$$C_{n+1}^{\pm}(x',t,\eta') = C^{\pm}(x,\eta',t).$$

We shall show that $(3.23)_{\pm}$ are hyperbolic with respect to dt. Since (3.14) and (3.16) yield $P_1(x,\rho_x)W_1 = -P_1(x,\theta_x)W_2$ hence $W_1^*P_1(x,\rho_x) = -W_2^*P_1(x,\theta_x)$, we have

$$(W_1^* \pm tW_2^*)P_1(x,\rho_x)(W_1 \pm tW_2) = -W_2^*P_1(x,\theta_x)W_1 \mp 2tW_2^*P_1(x,\theta_x)W_2 + O(t^2).$$

Besides $P_1(x,\theta_x)W_1 = O(t^2)$. Therefore we see that $C_n^{\pm}(x',t,\eta')$ are smoothly extended to $t \leqq 0$ and

$$C_n^{\pm}(x',t,\eta') = \mp W_2^*P_1(x,\theta_x)W_2 + O(t).$$

(See [8],Lemma 4.1, although the factor 1/2 in (4.9) must be replaced by 1). Moreover, since (3.5) and (3.16) imply

$$W_2(x,\eta') = \rho_{x_n}W_{\xi_n}(x,\theta_x) + O(\rho) \quad \text{for } x_n = 0,$$

we have

$$C_n^{\pm}(\bar{x}',0,\bar{\eta}') = \mp(\rho_{x_n}(\bar{x},\bar{\eta}'))^2(W_{\xi_n}^* P_1 W_{\xi_n})(\bar{x},\theta_x(\bar{x},\bar{\eta}')).$$

Besides, (2.3) and (2.6) yield

$$W_{\xi_n}^* P_1 W_{\xi_n} = S_2^* A(\xi_n - M)S_2/|\xi'|^2.$$

Thus we see from (2.4), (3.4), (3.6) and (3.8) that $\mp C_n^{\pm}(x',t,\eta')$ are positive definite near $(\bar{x}',0,\bar{\eta}')$ hence $(3.23)_{\pm}$ are symmetric hyperbolic systems which are really ordinary differential equations along bicharacteristic curves of Q_0. Consequently $(3.23)_{\pm}$ have the unique smooth solutions for arbitrary smooth data prescribed on $t = 0$ such that $a^-(x',t,\eta') = a^+(x',-t,\eta')$. Furhtermore, if we set

$$g_0(x,\eta') = (a^+(x',\sqrt{\rho},\eta') + a^+(x',-\sqrt{\rho},\eta'))/2,$$

$$h_0(x,\eta') = (a^+(x',\sqrt{\rho},\eta') - a^+(x',-\sqrt{\rho},\eta'))/2\sqrt{\rho},$$

then g_0, h_0 are smooth up to $\rho \geqq 0$ and (3.21) also holds.

Now we shall extend the a_0, b_0 given by (3.17) to $\rho \leqq 0$. Since $W(x,\xi)$ is analytic in ξ according to (2.6), the W_1 and W_2 defined by (3.16) are even functions of $\sqrt{\rho}$ hence they can be extended to $\rho \leqq 0$ in a natural way. Let g_0, h_0 be arbitrary smooth extensions to $\rho < 0$. Then we have (3.12) for $k = -1$ and $\eta_0 < 0$, because $P_1(x,\theta_x)W_1 + \rho P_1(x,\rho_x)W_2$ and $P_1(x,\theta_x)W_2 + P_1(x,\rho_x)W_1$ are $O(x_n^{\infty})$ for $|\eta'| = 1$ according to (2.7) with $\xi = \theta_x \pm \sqrt{\rho}\rho_x$ and $(3.1)_-$.

Next we shall construct special solutions $\tilde{a}_1$, $\tilde{b}_1$ of (3.20) so that (3.12) holds for $k = 0$. As in [8],pp. 284-285, one can extend g_0, h_0 to $\rho \leqq 0$ so that (3.21) is satisfied to infinite order on $x_n = 0$ for $\alpha < 0$, more precisely, so that both $W_1^* P(x,D)a_0 + \rho W_2^* P(x,D)b_0$ and $W_1^* P(x,D)b_0 + W_2^* P(x,D)a_0$ are $O(x_n^\infty)$ for $\eta_0 < 0$ and $0 \leqq x_n << 1$. Therefore it suffices to construct $\tilde{a}_1$, $\tilde{b}_1$ so that if the left hand sides of (3.20) and (3.21) are written as $B^{(1)} \pm \sqrt{\rho}B^{(2)}$ and $B^{(3)} \pm \sqrt{\rho}B^{(4)}$, respectively, where $B^{(j)}(x,\eta')$ are smooth, then each element of $B^{(1)}$ and $B^{(2)}$ is, mod $O(x_n^\infty)$, a linear combination of those of $B^{(3)}$ and $B^{(4)}$ with smooth coefficients. The procedure below will refine the proof of [8],Proposition 3.1.

Introducing an extra variable z in place of $\pm\sqrt{\rho}$, we set $\xi = \theta_x(x,\eta') + z\rho_x(x,\eta')$ and

$$a(x,\eta',z) = \begin{bmatrix} a_I \\ a_{II} \end{bmatrix} = \tilde{a}_1(x,\eta') + z\tilde{b}_1(x,\eta'),$$

$$F(x,\eta',z) = \begin{bmatrix} F_I \\ F_{II} \end{bmatrix} = -P(x,D)a_0(x,\eta') - zP(x,D)b_0(x,\eta'),$$

where a_I, F_I are the upper $d\times m_1$ blocks of a, F, respectively. We shall dominate $P_1(x,\xi)a(x,\eta',z) - F(x,\eta',z)$ by $W^*(x,\xi)F(x,\eta',z) + O(Q_0)$, where a is constructed similarly to the a_{-1}^0 in [8], Proposition 3.1.

It follows (2.3) that

$$P_1 a = \begin{bmatrix} A(\xi_n - M)a_I + A_{12}A_{22}^{-1}(A_{21}a_I + a_{22}a_{II}) \\ A_{21}a_I + A_{22}a_{II} \end{bmatrix}.$$

Hence, setting

$$a_{II} = A_{22}^{-1}(F_{II} - A_{21}a_I),$$

we need to estimate

$$A(\xi_n - M)a_I - F_I + A_{12}A_{22}^{-1}F_{II}.$$

Moreover, putting

$$(AS)^{-1}(F_I - A_{12}A_{22}^{-1}F_{II}) = \begin{bmatrix} F_0 \\ F_h \\ F_e \end{bmatrix},$$

$$S^{-1}a_I = \begin{bmatrix} a_0 \\ a_h \\ a_e \end{bmatrix},$$

we see from Lemma 2.1 that

$$(AS)^{-1}(A(\xi_n - M)a_I - F_I + A_{12}A_{22}^{-1}F_{II}) = \begin{bmatrix} (\xi_n - M_0)a_0 - F_0 \\ (\xi_n - M_h)a_h - F_h \\ (\xi_n - M_e)a_e - F_e \end{bmatrix},$$

where

$$M_h = \begin{bmatrix} M_h^+ & 0 \\ 0 & M_h^- \end{bmatrix}, \quad M_e = \begin{bmatrix} M_e^+ & 0 \\ 0 & M_e^- \end{bmatrix}.$$

Since the eigenvalues of $M_h(\bar{x},\bar{\xi}')$, $M_e(\bar{x},\bar{\xi}')$ are different from $\bar{\xi}_n$, we define

$$a_h = (\xi_n - M_h)^{-1}F_h,$$

$$a_e = (\xi_n - M_e)^{-1}F_e.$$

Thus it suffices to construct a_0 so that elements of $(\xi_n - M_0)a_0 - F_0$ are linear combinations of those of W^*F, mod $O(Q_0)$.

Now, (2.6) yields

$$(3.24) \qquad W^*F = S_0^*(F_I - A_{12}A_{22}^{-1}F_{II}) = S_0^*AS\begin{bmatrix} F_0 \\ F_h \\ F_e \end{bmatrix}.$$

Noting that S_0^*AS is an $m_1 \times d$ matrix of maximal rank, we set

$$S_0^*AS = (T_1,\cdots,T_d)$$

and suppose $|\xi'| = 1$. Then we shall show that

$$(3.25) \qquad T_{2j-1} = (\xi_n - \lambda)T_{2j},\ (\xi_n - \lambda)T_{2j-1} - \mu T_{2j} = 0, \text{ mod } O(Q_0)$$

for $j = 1,\cdots,m_1$, and

$$(3.26) \qquad T_j = O(Q_0) \quad \text{for } 2m_1 < j \leqq d.$$

Since (2.7) implies $A(\xi_n - M)S_0 = O(Q_0)$, so is $S_0^*A(\xi_n - M)S = (A(\xi_n - M)S_0)^*S$. Therefore by Lemma 2.1 we have

$$S_0^*AS(\xi_n - \tilde{M}) = S_0^*A(\xi_n - M)S = O(Q_0),$$

which yields (3.26). Moreover, since $(T_{2j-1}, T_{2j})(\xi_n - M_j) = O(Q_0)$, we obtain (3.25).

Now, define

$$a_0 = -\begin{bmatrix} 0 \\ F_1 \\ 0 \\ F_3 \\ \vdots \\ 0 \\ F_{2m_1-1} \end{bmatrix} \quad \text{with } F_0 = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{2m_1} \end{bmatrix}.$$

Then we have

$$(\xi_n - M_0)a_0 = \begin{bmatrix} F_1 \\ 0 \\ F_3 \\ 0 \\ \vdots \\ F_{2m_1-1} \\ 0 \end{bmatrix} - (\xi_n - \lambda)\begin{bmatrix} 0 \\ F_1 \\ 0 \\ F_3 \\ \vdots \\ 0 \\ F_{2m_1-1} \end{bmatrix},$$

while (3.24), (3.25) and (3.26) imply that $T_2, T_4, \cdots, T_{2m_1}$ are linearly independent and

$$(T_2, T_4, \cdots, T_{2m_1})^{-1} W^*F = (\xi_n - \lambda)\begin{bmatrix} F_1 \\ F_3 \\ \vdots \\ F_{2m_1-1} \end{bmatrix} + \begin{bmatrix} F_2 \\ F_4 \\ \vdots \\ F_{2m_1} \end{bmatrix} + O(Q_0).$$

Consequently we see that elements of $(\xi_n - M_0)a_0 - F_0$ are linear combinatons of those of W^*F, mod $O(Q_0)$.

Finally we set

$$\tilde{a}_1(x,\eta') = (1/2)(a(x,\eta',z) + a(x,\eta',-z))\big|_{z^2=\rho},$$

$$\tilde{b}_1(x,\eta') = (1/(2z))(a(x,\eta',z) - a(x,\eta',-z))\big|_{z^2=\rho}.$$

Then, since $a(x,\eta',z)$ is an analytic function of z, $\tilde{a}_1(x,\eta')$ and $\tilde{b}_1(x,\eta')$ are smooth in (x,η') and have the desired properties. Analogously one can construct a_k, b_k for $k \geqq 2$ so that (3.12) holds. Thus we prove the theorem.

The construction of G_h, G_e in (3.9) is similar to that of $G^{(2)}$, $G^{(3)}$ in [8]. First we take a Fourier integral operator $\tilde{G}_h$ (on X') and a pseudodifferential operator $\tilde{G}_e$ on X', depending smoothly on parameter x_n, such that $P(x,D)G_h w_h \in C^\infty(X)$, $P(x,D)G_e w_e \in C^\infty(X)$ near $\bar{x}$ for all $w_h(x')$, $w_e(x') \in H^{-\infty}(X')$ with $WF(w_h)$, $WF(w_e)$ contained in a small conic neighborhood of $WF(f)$, whose boundary values are classical pseudodifferential operators of the form

$$(\tilde{G}_h w_h)(x',0) = (2\pi)^{-n}\int e^{ix'\xi'} W_h(x',\xi')\tilde{\chi}(\xi')\hat{w}_h(\xi')d\xi',$$

(3.27)

$$(\tilde{G}_e w_e)(x',0) = (2\pi)^{-n}\int e^{ix'\xi'} W_e(x',\xi')\tilde{\chi}(\xi')\hat{w}_e(\xi')d\xi'.$$

Here $W_h(x',\xi') = W_h^+(x',0,\xi')$, $W_e(x',\xi') = W_e^+(x',0,\xi')$ mod $S^{-1}_{1,0}$, where $W_h^+(x,\xi')$, $W_e^+(x,\xi')$ are the matrices defined by (2.8), and $\tilde{\chi}$ is a cutoff function such that $\tilde{\chi}(\xi') = 1$ for (x',ξ') in a conic neighborhood of $WF(f)$. The construction of such $\tilde{G}_h$, $\tilde{G}_e$ is well known (see for example [20], Chap. IX). Next

let Φ_1 be a Fourier integral operator, with the canonical transformation ϕ_1 defined by (3.8), whose amplitude is $e^{i(\check{\theta} - \theta)(x',0,\eta')} \in S^0_{1,0}$, namely, define

$$(3.28) \qquad (\Phi_1 v)(x') = \int e^{i\theta(x',0,\eta')} e^{i(\check{\theta} - \theta)(x',0,\eta')} \hat{v}(\eta') d\eta'$$

for $v(y') \in H^{-\infty}(R^n)$. Hereafter $\check{\theta}(x',0,\eta')$ is extended to $R^n \times (R^n \setminus 0)$ in such a way that $\check{\theta}(x',0,\eta') = \theta(x',0,\eta') = x'\eta'$ outside a conic neighborhood of $(\bar{x}',\bar{\eta}')$. Finally we define G_h, G_e as the compositions

$$(3.29) \qquad G_h = \tilde{G}_h \circ \Phi_1, \; G_e = \tilde{G}_e \circ \Phi_1.$$

To deduce (1.22) we also use the following lemma. Recall that the almost analytic continuation $\check{a}(x,\eta')$ of $a(x,\eta')$ is defined by

$$(3.30) \qquad \check{a}(x,\eta') = \sum_{k=0}^{\infty} \frac{\partial^k a(x,\eta')}{\partial \eta_0^k} \frac{(-i\tau)^k}{k!} \chi_0(N_k \tau |\eta''|^{-k}),$$

where τ is a positive number, $\chi_0(t) \in C_0^\infty(R^1)$ a cutoff function, supported in $|t| < 2$, such that $\chi_0(t) = 1$ for $|t| < 1$, and $\{N_k\}_{k=0}^\infty$ with $N_0 = 1$ is a sequence of positive numbers which increases fast enough. Applying $P(x,D)$ to each side of (3.10) and using $A_0''(\rho) = -\rho A_0(\rho)$, we have

$$(3.31) \qquad P(x,D)G_0 v_0 = \int e^{i\check{\theta}} (A_0(\check{\rho})\tilde{c} - iA_0'(\check{\rho})\tilde{d})(A_+(\zeta)^{-1}\chi_1 + A_0(\zeta)^{-1}(1 - \chi_1))\hat{v}_0(\eta') d\eta'.$$

Here, setting

$$c(x,\theta,\rho,a,b) = P_1(x,\theta_x)a + \rho P_1(x,\rho_x)b + P(x,D)a,$$

$$d(x,\theta,\rho,a,b) = P_1(x,\theta_x)b + P_1(x,\rho_x)a + P(x,D)b,$$

which coincide with $c(x,\eta')$ and $d(x,\eta')$ in (3.11), respectively, we have

$$\tilde{c}(x,\eta') = c(x,\check{\theta},\check{\rho},\check{a},\check{b}),$$

$$\tilde{d}(x,\eta') = d(x,\check{\theta},\check{\rho},\check{a},\check{b}).$$

Moreover we obtain

<u>Lemma 3.2</u>. Let k be a nonnegative integer and let $|\eta''| \geqq \tau N_k$. Then

$$\tilde{c}(x,\eta') = \begin{cases} O(|\eta'|^{-k}) & \text{for } \rho \geqq 0, \\ O((\rho|\eta'|^{-2/3})^{\infty}|\eta'|) + O(|\eta'|^{-k}) & \text{for } \rho < 0, \\ O(x_n^{\infty}|\eta'|) + O(|\eta'|^{-k}) & \text{for } \eta_0 < 0 \text{ and } 0 \leqq x_n << 1. \end{cases}$$

The analogous estimate for $\tilde{d}$ also holds with $|\eta'|$ and $|\eta'|^{-k}$ replaced by $|\eta'|^{2/3}$ and $|\eta'|^{-k-1/3}$, respectively.

<u>Proof</u>. Let $\check{c}(x,\eta')$ be the almost analytic continuation of $c(x,\theta,\rho,a,b)$. Then, since $\chi_0(N_j\tau|\eta''|^{-1}) = 1$ for $j \leqq k$, we have $\tilde{c} - \check{c} = O(|\eta'|^{-k})$ and

$$\check{c} = \sum_{j=0}^{k} \frac{\partial^j c}{\partial \eta_0^j} \frac{(-i\tau)^j}{j!} + O(|\eta'|^{-k}).$$

Hence the desired estimate follows immediately from Theorem 3.1.

§4. Airy operators

The purpose of this section is to give a summary of [11],§3. Let Ai(z) be the Airy function of the first kind, given by

$$\mathrm{Ai}(z) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{izt} e^{it^3/3} dt.$$

We define $A_\pm$, A_0 by

$$(4.1)\qquad \begin{aligned} A_\pm(z) &= e^{\mp i\pi/3}\mathrm{Ai}(e^{\mp i\pi/3}z), \\ A_0(z) &= A_+(z) + A_-(z) = \mathrm{Ai}(-z). \end{aligned}$$

Then

$$(4.2)\qquad A_\pm''(z) + zA_\pm(z) = A_0''(z) + zA_0(z) = 0$$

and $A_\pm$ have the asymptotic expansions for $|z| >> 1$ with $-\pi \pm \pi/3 < \arg z < \pi \pm \pi/3$

$$(4.3)\qquad \begin{aligned} A_\pm(z) &= z^{-1/4} e^{\pm i(2/3)z^{3/2}} \Psi_\pm(z), \\ \Psi_\pm(z) &\sim e^{\mp i\pi/4} \Sigma_{k=0}^{\infty} (\pm i)^k a_k z^{-(3/2)k}, \end{aligned}$$

where a_k are real and $a_0 = 1/(2\sqrt{\pi})$. Besides, $A_+(z) \neq 0$ for $\mathrm{Re}\, z \geqq 0$, $A_-(z) \neq 0$ for $\mathrm{Im}\, z \leqq 0$ and $\mathrm{Re}\, A_+(x) = \mathrm{Re}\, A_-(x) = A_0(x)/2$ for x real.

Throughout the present paper, all functions of (x',η') will be modified for $|\eta''| < 1$ so that they are smooth in $R^n \times R^n$. Noting that $\check{\rho}(x',0,\eta') = (\eta_0 - i\tau)|\eta''|^{-1/3}$ for $|\eta''| > \tau N_1$, where τ, N_1 are the positive numbers in (3.30), we set

$$(4.4) \qquad \zeta = (\eta_0 - i\tau)|\eta''|^{-1/3} = (\alpha - i\tau|\eta''|^{-1})|\eta''|^{2/3}.$$

The parameter τ will be fixed so large as all a priori estimats in Propositions 6.1, 6.2, 6.3, Corollaries 6.4 and 6.5 hold. Moreover we set

$$(4.5) \qquad \begin{aligned} K_\pm(\eta') &= -i|\eta''|^{-1/3}(A_\pm'/A_\pm)(\zeta), \\ K_0(\eta') &= -i|\eta''|^{-1/3}(A_0'/A_0)(\zeta), \end{aligned}$$

$$(4.6) \qquad \begin{aligned} L(\eta') &= (A_-/A_+)(\zeta), \\ \mathcal{L}(\eta') &= (K_+ + K_-L)\chi_1 + K_0(1 - \chi_1), \end{aligned}$$

$$(4.7) \qquad \chi_1(\eta') = \chi(\alpha|\eta''|^{2/3}/t_0) = \chi(\eta_0|\eta''|^{-1/3}/t_0),$$

$$(4.8) \qquad \chi_\varepsilon(\eta') = \chi(\alpha|\eta''|^{\varepsilon}) = \chi(\eta_0|\eta''|^{\varepsilon-1})$$

and

$$(4.9) \qquad \gamma = (\alpha^2 + |\eta''|^{-4/3})^{1/4} = (\eta_0^2 + |\eta''|^{2/3})^{1/4}|\eta''|^{-1/2}.$$

Here t_0 is a positive number with $t_0 < 1$ such that $A_0(t) > 0$ for $t \leqq 3t_0$ and $\operatorname{Im} A_+(t) < 0$ for $0 \leqq t \leqq 3t_0$, and $\chi(t) \in C^\infty(R^1)$ is a real valued function, supported in $t > 3/2$, such that $\chi(t) = 1$ for $t > 2$ and $\chi'(t) \geqq 0$. The ε is an arbitrary (fixed) positive number with $\varepsilon < 1/2$. Note that $\chi_1 \in S^0_{1/3,0}$, $\chi_\varepsilon \in S^0_{1-\varepsilon,0}$ for $|\alpha|$ bounded. We also denote the Fourier multiplires corresponding to (4.5) through (4.9) by the same letters. For example, L denotes the Fourier multiplier defined by

$$(Lv)(y') = (2\pi)^{-n}\int e^{iy'\eta'}L(\eta')\hat{v}(\eta')d\eta'.$$

From now on we suppose $|\alpha| < 1$ and $|\eta''|^{1/3} >> \tau > 1$. In addition, we denote by $L_0^2(R^n)$ or $H_0^s(R^n)$, respectively, the set of functions $v(y') \in L^2(R^n)$ or $v(y') \in H^s(R^n)$ such that supp $\hat{v}(\eta') \subset \{|\alpha| < 1\}$. Here $-\infty \leqq s \leqq \infty$. We also denote constants independent of τ by C, C', C_k and so on, while $O(|\eta''|^{-1})$ etc. may be depend on τ.

<u>Lemma 4.1</u>. Let q be a real number. Then

$$|\partial_{\eta_0}^k \partial_{\eta''}^\beta \gamma^q| \leqq C_{q,k,\beta}|\eta''|^{-k-|\beta|}\gamma^{q-2k} \quad \text{for } k,\ |\beta| \geqq 0.$$

In particular, γ^q belongs to $S^0_{1/3,0}$ if $q > 0$ and to $S^{-q/3}_{1/3,0}$ if $q < 0$.

<u>Lemma 4.2</u>. The functions $K_+\chi_1$, K_- and $K_0(1 - \chi_1)$ belong to $S^0_{1/3,0}$. More percisely,

$$|\partial_{\eta_0}^k \partial_{\eta''}^\beta K_-(\eta')| \leqq C_{k,\beta}|\eta''|^{-k-|\beta|}\gamma^{1-2k}(1 + O(|\eta''|^{-1}) \quad \text{for } k,\ |\beta| \geqq 0.$$

The analogous estimates also hold for K_+ and K_0 if $\alpha \geqq 0$ and $\alpha|\eta''|^{2/3} \leqq 3t_0$, respectively, where t_0 is the number in (4.7).

<u>Lemma 4.3</u>. The operators $K_+\chi_1$, K_- and $K_0(1 - x_1)$ are bounded on $L_0^2(R^n)$. More precisely,

$$\|K_-v\|^2 \leqq C\|\gamma v\|^2 + O(\|v\|^2_{-1/2}) \quad \text{for } v \in L_0^2(R^n).$$

The analogous estimates also hold for K_+ and K_0 if supp $\hat{v}(\eta') \subset \{\alpha \geqq 0\}$ and $\{\alpha|\eta''|^{2/3} \leqq 3t_0\}$, respectively.

For the commutators involving γ, $K_\pm$ and K_0 we have

<u>Lemma 4.4</u>. Let $a(y',\eta') \in S^0_{1/3,0}$ and q be a real number. Then

$$|(\gamma^{1-q}[\gamma^q, a]v, w)| \leqq C_q\|\gamma^{-1/2}v\|^2_{-1/2} + \|\gamma^{-1/2}w\|^2_{-1/2} + O(\|v\|^2_{-1/2} + \|w\|^2_{-1/2}),$$

$$|([K_-,a]v, w)| \leqq C\|\gamma^{-1/2}v\|^2_{-1/2} + \|\gamma^{-1/2}w\|^2_{-1/2} + O(\|v\|^2_{-1/2} + \|w\|^2_{-1/2})$$

and

$$|(K_-[\chi_1, a]v, w)| \leqq C'\|\gamma^{-1/2}v\|^2_{-1/2} + \|\gamma^{-1/2}w\|^2_{-1/2} + O(\|v\|^2_{-1/2} + \|w\|^2_{-1/2})$$

for $v, w \in L^2_0(R^n)$. The analogous estimates also hold for K_+ and K_0 if supp $\hat{v}(\eta')$ is as in the preceding lemma.

For the proofs see those of [11], Lemmas 3.1 through 3.4.

The following three lemmas will play basic roles in dealing with the operators L or $\mathcal{L}$.

<u>Lemma 4.5</u>. Let $x \geqq 0$ and $0 < y << (1 + x)^{-1/2}$. Then

$$|A_\pm(x - iy)|^2 = |A_\pm(x)|^2 \pm b_0 y + O(y^2(1 + x)^{1/2}),$$

$$|A'_\pm(x - iy)|^2 = |A'_\pm(x)|^2 \pm b_0 xy + O(y^2(1 + x)^{3/2}),$$

where $b_0 = -\sqrt{3}Ai(0)Ai'(0) > 0$. Moreover $|A_-(x)| = |A_+(x)|$ and , when $x >> 1$,

$$|A_+(x)|^2 = a_0^2 x^{-1/2}(1 + O(x^{-3})),$$

$$|A_\pm'(x)|^2 = a_0^2 x^{1/2}(1 + O(x^{-3})),$$

where a_0 is the positive number in (4.3)

<u>Lemma 4.6</u>. Let $\alpha \geqq 0$ and set

$$\ell(\eta') = L(\eta')e^{i(4/3)\alpha^{3/2}|\eta''|}.$$

Then

$$\ell(\eta') = ie^{-2\tau\sqrt{\alpha}}(1 + O(\zeta^{-3/2})) \quad \text{for } \alpha|\eta''|^{2/3} >> 1$$

and

$$|\partial_{\eta_0}^k \partial_{\eta''}^\beta L(\eta')| \leqq C_{k,\beta}\gamma^{k+3|\beta|}(1 + O(|\eta''|^{-1})) \quad \text{for } k, |\beta| \geqq 0.$$

In particular, $(L(1 - \chi_\varepsilon)\chi_1)(\eta') \in S^0_{\varepsilon/2,0}$ and $L(D_{y'})\chi_\varepsilon$ is a Fourier integral operator with singular phase function

$$\varphi(y',\eta') = y'\eta' - (4/3)\alpha^{3/2}|\eta''|,$$

with amplitude $\ell(\eta')\chi_\varepsilon(\eta') \in S^0_{1-\varepsilon,0}$, where χ_1 and χ_ε are the cutoff functios defined by (4.7) and (4.8), respectively. Moreover denote by ϕ_2 the canonical transformation associated with $L\chi_\varepsilon$ which is defined by $\phi_2^{-1}(y',\eta') = (\varphi_{\eta'}(y',\eta'),\eta')$. Then

$$\phi_2(y',\eta') = (y_0 + 2\sqrt{\alpha},\ y'' - (2/3)\alpha^{3/2}\eta''/|\eta''|,\ \eta') \quad \text{for } \alpha \geqq 0. \tag{4.10}$$

<u>Lemma 4.7</u>. Let $a(y',\eta')$, $b(y',\eta') \in S^m_{1-\varepsilon,0}$ be homogeneous in η'. Then there exist symbols $a^{(0)}$, $b^{(0)} \in S^m_{1-\varepsilon,0}$ and $a^{(1)}$, $b^{(1)} \in S^{m-1+\varepsilon}_{1-\varepsilon,0}$ such that, modulo smoothing operators,

$$(L\chi_1)(D_{y'})a(y',D_{y'})\chi_\varepsilon = (a^{(0)} + a^{(1)})(y',D_{y'})L(D_{y'})\chi_\varepsilon,$$

$$b(y',D_{y'})L(D_{y'})\chi_\varepsilon = (L\chi_1)(D_{y'})(b^{(0)} + b^{(1)})(y',D_{y'})\chi_\varepsilon.$$

Here $a^{(0)}(y',\eta') = a\circ\phi_2^{-1}(y',\eta')$, $b^{(0)}(y',\eta') = b\circ\phi_2(y',\eta')$, $a^{(1)}$ and $b^{(1)}$ are $O(|\eta''|^{m-1}\gamma^{-1})$, supp $a^{(1)} \subset \phi_2(\text{supp } \partial_{y'}a)$ and supp $b^{(1)} \subset \phi_2^{-1}(\text{supp } \partial_{y'}b)$. In particular, if $a(y',\eta') \geqq 0$ and

$$a\circ\phi_2^{-1}(y',\eta') \leqq a(y',\eta'),$$

then supp $a^{(j)} \subset$ supp a for $j = 0, 1$; if $b(y',\eta') \geqq 0$ and

$$b\circ\phi_2(y',\eta') \leqq b(y',\eta'),$$

then supp $b^{(j)} \subset$ supp b for $j = 0, 1$.

For the proofs see those of [//],Lemmas 3.9, 3.5 and 3.7. The following lemma is a direct consequence of Lemma 4.5 if we set $x = \alpha|\eta''|^{2/3}$ and $y = \tau|\eta''|^{-1/3}$.

<u>Lemma 4.8</u>. Let b_1 be the positive number in [//],Lemma 3.10. Then

$$1 - |L(\eta')| \geqq b_1\tau\gamma - O(|\eta''|^{-1}) \quad \text{if } 0 \leqq \alpha << \tau^{-2},$$

$$1 - |L(\eta')| \geqq \delta - O(|\eta''|^{-1}) \quad \text{if } \alpha \geqq \delta^2\tau^{-2} \text{ and } 0 < \delta < 1/2.$$

Moreover for $v \in L_0^2(R^n)$

$$\mathrm{Re}\ ((1 + L\chi_1)v,\ v) \geqq b_1\tau\|\gamma^{1/2}\chi_1 v\|^2 + \|(1 - \chi_1)v\|^2 - O(\|v\|^2_{-1/2})$$

if supp $\hat{v}(\eta') \subset \{\alpha << \tau^{-2}\}$, and

$$\mathrm{Re}\ ((1 + L\chi_1)v,\ v) \geqq \delta\|v\|^2 - O(\|v\|^2_{-1/2})$$

if supp $\hat{v}(\eta') \subset \{\alpha \geqq \delta^2\tau^{-2}\}$ and $0 < \delta < 1/2$. In particular,

$$\|Lv\|^2 \leqq \|v\|^2 + O(\|v\|^2_{-1/2}) \tag{4.11}$$

for $v \in L^2(R^n)$ with supp $\hat{v}(\eta') \subset \{0 \leqq \alpha < 1\}$.

Note that Lemma 4.3 and (4.11) yield

$$\|\mathcal{L}v\|^2 \leqq C\|\gamma v\|^2 + O(\|v\|^2_{-1/2}) \tag{4.12}$$

for $v \in L_0^2(R^n)$.

For the commutators involving $L\chi_1$ or $\mathcal{L}$ we have

<u>Lemma 4.9</u>. Let $a(y',\eta') \in S^0_{1-\varepsilon,0}$. Then

$$\|[L\chi_1,\ a]v\|^2 \leqq C\|\gamma v\|^2 + O(\|v\|^2_{-1/2})$$

and

$$|([\mathcal{L},\ a]v,\ w)| \leqq C'(\|\gamma v\|^2 + \|\gamma w\|^2) + O(\|v\|^2_{-/12} + \|w\|^2_{-1/2})$$

for $v,\ w \in L_0^2(R^n)$.

For the proof see that of [//],Lemma 3.8. The following a priori estimate for the operator $\mathcal{L}$ will play an essential role in deriving a basic estimate in this paper which will be given in Proposition 6.1.

<u>Lemma 4.10</u>. There is a positive number b_2 such that

$$\mathrm{Re}\ (\mathcal{L}v,\ (1 + L\chi_1)v) \geqq b_2\tau(\|\gamma\chi_1 v\|^2 + \|\gamma^{-1/2}(1 - \chi_1)v\|^2_{-1/2}) - O(\|v\|^2_{-1/2})$$

for $v \in L^2_0(R^n)$ with supp $\hat{v}(\eta') \subset \{\alpha << \tau^{-2}\}$.

For the proof see that of [//],Proposition 3.12. The following lemmas are supplements to the above estimate.

<u>Lemma 4.11</u>. Let $0 < \delta \leqq \pi/2$ and set

$$C_\delta = (\sin\delta)\inf_{x\leqq 0}\ (1 - x)^{-1/2}A_0'(x)/A_0(x).$$

Then c_δ is positive and

$$\mathrm{Re}\ (e^{i\delta}\gamma K_0 v,\ v) \geqq C_\delta\|\gamma v\|^2 - O(\|v\|^2_{-1/2})$$

for $v \in L^2(R^n)$ with supp $\hat{v}(\eta') \subset \{-1 < \alpha \leqq 0\}$.

<u>Lemma 4.12</u>. Let $\alpha|\eta''|^\varepsilon > 1$ and $0 < \varepsilon < 1/2$. Then

$$K_\pm(\eta') = \pm\sqrt{\alpha} + i(4\alpha|\eta''|)^{-1} + O(\alpha^{-1/2}|\eta''|^{-1}).$$

<u>Lemma 4.13</u>. Let $0 < \delta < 1/2$. Then

$$\mathrm{Re}\ (\mathcal{L}v,\ (1 + L\chi_1)v) \geqq \delta^2\tau^{-1}\|v\|^2 - O(\|v\|^2_{-1/2})$$

for $v \in L^2_0(R^n)$ with supp $\hat{v}(\eta') \subset \{\tau^2\alpha > \delta^2\}$.

For the proofs see those of [//],Lemmas 3.17, 3.18 and 3.19.

We will also use in §7 the family of pseudodifferential operators Λ_t with symbols $\Lambda_t(y',\eta') = \langle\eta''\rangle^{-ty_0}$. Here $t > 0$ is a parameter and

$$\langle\eta''\rangle = |\eta''|\chi(|\eta''|) + (1+|\eta''|^2)^{1/2}(1-\chi(|\eta''|))$$

with the χ in (4.7), so that $\langle\eta''\rangle = |\eta''|$ if $|\eta''| > 2$. For $\tau > 1$ and $t > 0$ we set, as in [//],

$$\tau_t = \tau + t\log\langle\eta''\rangle, \quad \zeta_t = (\eta_0 - i\tau_t)\langle\eta''\rangle^{-1/3},$$

$$\begin{aligned} &L_t(\eta') = (A_-/A_+)(\zeta_t), \\ (4.13)\qquad &K_{\pm,t}(\eta') = -i\langle\eta''\rangle^{-1/3}(A_\pm'/A_\pm)(\zeta_t), \\ &K_{0,t}(\eta') = -i\langle\eta''\rangle^{-1/3}(A_0'/A_0)(\zeta_t), \end{aligned}$$

$$(4.14)\qquad \chi_{1t}(\eta') = \sum_{k=0}^{\infty}\frac{\partial^k\chi_1(\eta')}{\partial\eta_0^k}\,\frac{(-it\log\langle\eta''\rangle)^k}{k!}\chi_0(N_k t\langle\eta''\rangle^{-1}\log\langle\eta''\rangle)$$

with such χ_0, N_k as in (3.30), and

$$(4.15)\qquad \mathcal{L}_t = (K_{+,t} + K_{-,t}L_t)\chi_{1t} + K_{0t}(1-\chi_{1t}).$$

Let $v(y') \in H^{-\infty}(R^n)$ be a function such that $v \in H^\infty(R^n\setminus K)$ for a compact set K and $\operatorname{supp}\hat{v}(\eta') \subset \{|\alpha| < c_0\langle\eta''\rangle^{-\delta}\}$ with some positive numbers c_0, δ. Then we have

$$(4.16)\qquad \Lambda_t L\chi_1 v = L_t\chi_{1t}\Lambda_t v, \quad \Lambda_t\mathcal{L}v = \mathcal{L}_t\Lambda_t v$$

mod $H^\infty(R^n \cap \{y_0 > T\})$ for any real number T. Moreover Lemmas 4.2, 4.3, 4.4, 4.9 and 4.11 are still valid even if τ and χ_1 are replaced by τ_t and χ_{1t}, respectively. Hereafter we suppose that $|\alpha| < c_0\langle\eta''\rangle^{-\delta}$ and v is as above. Then we obtain the following estimates.

<u>Lemma 4.14</u>. Let b_1 be the positive number in Lemma 4.8. Then

$$1 - |L_t(\eta')| \geqq b_1\tau_t\gamma - O(\langle\eta''\rangle^{-1}) \quad \text{if } \alpha \geqq 0$$

and

$$\mathrm{Re}\ ((1 + L_t\chi_{1t})v,\ v) \geqq b_1\|(\tau_t\gamma)^{1/2}\chi_{1t}v\|^2 + \|(1 - \chi_{1t})v\|^2 - O(\|\tau_t v\|^2_{-1/3}) \quad \text{if } v \in L^2(R^n).$$

<u>Lemma 4.15</u>. Let b_2 be the positive number in Lemma 4.10. Then

$$\mathrm{Re}\ (\mathcal{L}_t v,\ (1 + L_t\chi_{1t})v)$$
$$\geqq b_2(\|\tau_t^{1/2}\gamma\chi_{1t}v\|^2 + \|\tau_t^{1/2}\gamma^{-1/2}(1 - \chi_{1t})v\|^2_{-1/2}) - O(\|\tau_t v\|^2_{-1/2}) \quad \text{if } v \in L^2(R^n).$$

For the proofs see the end of [//], §3. We also see from Lemmas 4.14 and 4.3 that (4.11) and (4.12) hold for such v as described above even if L and $\mathcal{L}$ are replaced by L_t and $\mathcal{L}_t$, respectively.

§5. Equation on the boundary

Our next task is to solve, mod C^∞,

$$(5.1)\qquad BGv|_{x_n=0} = f,$$

where G is the operator defined by (3.9), (3.10) and (3.29). From now on we suppose that $x_n = 0$, $|\alpha| << 1$, $(x',\xi') = \phi_1(y',\xi')$, where ϕ_1 is the canonical transformation given by (3.8), and often abbreviate $(x',0) \in \partial X$ as x', so $\theta_x(x',\eta') = (\theta_{x'}(x',0,\eta'), \theta_{x_n}(x',0,\eta'))$ and so on.

Let Φ^{-1} be an elliptic Fourier integral operator with canonical transformation ϕ_1^{-1} such that

$$(5.2)\qquad \Phi_1\Phi_1^{-1} = \Phi_1^{-1}\Phi_1 = \text{the identity, mod } OPS_{1,0}^{-\infty},$$

where Φ_1 is the Fourier integral operator defined by (3.28). Then (5.1) is equivalent to

$$(5.3)\qquad \Phi_1^{-1}BG_0v_0 + \Phi_1^{-1}B(G_h,\ G_e)\begin{bmatrix} v_h \\ v_e \end{bmatrix} = \Phi_1^{-1}f,$$

where (3.27) and (3.29) imply that $\Phi_1^{-1}B(G_h,\ G_e)$ is a classical pseudodifferential operator with principal symbol $B(x')(W_h,\ W_e)(x',\xi')$. Moreover, since $\check{\rho}(x',\eta') = \zeta$ for $|\eta''| >> 1$, it follows from (4.1), (4.5) and (4.6) that, mod $C^\infty(X')$,

$$(BG_0v_0)(x') = \int e^{i\check{\theta}(x',\eta')}B(x')\check{a}(x',\eta')(1 + L\chi_1)(\eta')\hat{v}_0(\eta')d\eta'$$
$$+ \int e^{i\check{\theta}(x',\eta')}B(x')\check{b}(x',\eta')|\eta''|^{1/3}\mathcal{L}(\eta')\hat{v}_0(\eta')d\eta'.$$

Therefore, applying Φ_1^{-1} to each side, we have

$$(5.4)\qquad (\Phi_1^{-1}BG_0v_0)(y') = \tilde{c}(1 + L\chi_1)v_0 + \tilde{d}\mathcal{L}v_0,$$

where $\tilde{c}$, $\tilde{d} \in OPS^0_{1,0}$ and, mod $S^{-1}_{1,0}$,

$$\tilde{c}(y',\eta') = B(x')a_0(x',\eta'),$$

$$\tilde{d}(y',\eta') = B(x')b_0(x',\eta')|\eta''|^{1/3}$$

with a_0, b_0 the symbols given by (3.17). Note that (2.6), (3.5), (3.6) and (3.16) yield

$$(5.5)\qquad \begin{aligned} a_0(x',\eta') &= W(x',\xi',\ \lambda(x',\xi'))g_0(x',\eta') \\ &\quad + \alpha|\eta''|^{2/3}\rho_{x_n}(x',\eta')W_{\xi_n}(x',\xi')h_0(x',\eta') \\ b_0(x',\eta') &= W(x',\xi',\ \lambda(x',\xi'))h_0(x',\eta') \\ &\quad + \rho_{x_n}(x',\eta')W_{\xi_n}(x',\xi')g_0(x',\eta'), \end{aligned}$$

because W_{ξ_n} is independent of ξ_n.

We shall here specify the initial data on $t = 0$ for the transport equation $(3.23)_\pm$. First suppose (1.9) holds. Let $E(t',D_{y'}) \in OPS^0_{1,0}$ be an elliptic pseudodifferential operator whose symbol is the matrix $\tilde{R}(x',\xi',\xi_n)$ given by (1.8) with $\xi_n = \lambda(x',\xi')$. Applying a parametrix E^{-1} for E to each side of (5.3), one can write

$$(5.6)\qquad \begin{aligned} B_{11}v_0 + B_{12}\begin{bmatrix} v_h \\ v_e \end{bmatrix} &= F_1, \\ B_{21}v_0 + B_{22}\begin{bmatrix} v_h \\ v_e \end{bmatrix} &= F_2. \end{aligned}$$

Here B_{11}, B_{22} are square matrices of order m_1, $d_+ - m_1$, respectively; B_{12}, $B_{22} \in OPS^0_{1,0}$ and, mod $S^{-1}_{1,0}$, $B_{12}(y',\eta') = 0$, $B_{22}(y',\eta') = I_{d_+ - m_1}$. Hence, setting

$$\begin{bmatrix} v_h \\ v_e \end{bmatrix} = B_{22}^{-1}(F_2 - B_{21}v_0),$$

we see from (5.4) and (5.5) that (5.6) becomes

$$(5.7) \qquad (\tilde{c}_1 g_0 + \tilde{c}_2 h_0)(1 + L\chi_1)v_0 + (\tilde{d}_1 h_0 + \tilde{d}_2 g_0)\mathcal{L} v_0 = F_1 - B_{12}B_{22}^{-1}F_2,$$

where $\tilde{c}_1$, $\tilde{d}_2 \in OPS^0_{1,0}$, $\tilde{d}_1$, $\tilde{c}_2 \in OPS^{1/3}_{1,0}$ and $\tilde{c}_1(y',\eta') = I_{m_1} + O(\alpha)$ mod $S^{-1}_{1,0}$, $\tilde{c}_2(y',\eta') = O(\alpha|\eta''|^{1/3})$, $\tilde{d}_1(y',\eta') = (I_{m_1} + O(\alpha))|\eta''|^{1/3}$ mod $S^{-2/3}_{1,0}$.

We shall now take $g_0(x, \eta')$, $h_0(x,\eta')$ for $\rho = 0$ in such a way that g_0 is elliptic and, for $x_n = 0$,

$$(5.8) \qquad h_0|\eta''|^{1/3} + \tilde{d}_2 g_0 = O(\alpha), \text{ mod } S^{-1}_{1,0}.$$

In fact, setting $t = 0$ in $(3.23)_+$, we have

$$(5.9) \qquad \sum_{j=0}^{n-1} C_j^+(x',0,\eta')\frac{\partial g_0}{\partial x_j} + C_{n+1}^+(x',0,\eta')g_0 = -C_n^+(x',0,\eta')h_0.$$

Therefore, if we define $h_0 = -\tilde{d}_2 g_0|\eta''|^{-1/3}$ for $t = 0$, the above equation becomes a symmetric hyperbolic system for $g_0(x, \eta')|_{\rho=0}$, because $C_0^+(x',0,\eta')$ is positive definite. Thus (5.9) has a unique solution with initial data $g_0 = 1$ on $x_0 = 0$. Consequently, g_0 is elliptic and, by (3.5), (5.8) holds.

Finally, applying a parametrix for $\tilde{c}_1 g_0 + \tilde{c}_2 h_0$ to each side of (5.7), we arrive at

$$(5.10) \qquad (1 + L\chi_1)v_0 + b\mathcal{L}v_0 = f_0,$$

where $b(y',\eta') = O(\alpha) \bmod S^{-1}_{1,0}$, and

$$f_0 = (\tilde{c}_1 g_0 + \tilde{c}_2 h_0)^{-1}(F_1 - B_{12}B_{22}^{-1}F_2).$$

Next suppose (1.10) and (1.11) hold. For convenience we denote by $\tilde{R}_{jk}(x',\xi',\xi_n)$, $j, k = 1,\cdots,m_1$, the matrix $\tilde{R}(x',\xi',\xi_n)$ with the j-th column of $W(x',\xi',\xi_n)$ replaced by the k-th column of $W_{\xi_n}(x',\xi')|\eta''|$ and set $R_{jk} = \det \tilde{R}_{jk}$. Then (1.11) means

$$\Sigma_{k=1}^{m_1} R_{kk}(\bar{x}',\bar{\xi}',\bar{\xi}_n) \neq 0.$$

Hereafter we suppose for definiteness that $R_{11}(\bar{x}',\bar{\xi}',\bar{\xi}_n) \neq 0$. Let $E_1 \in OPS^0_{1,0}$ be an elliptic pseudoidfferential operator with symbol $\tilde{R}_{11}(x',\xi',\lambda(x',\xi'))$. Applying a parametrix E_1^{-1} for E_1 to each side of (5.3), we obtain an equation of the same form as (5.6), where B_{12}, B_{22} are as before. Set

$$(5.11) \qquad \begin{bmatrix} v_h \\ v_e \end{bmatrix} = B_{22}^{-1}(F_2 - B_{21}v_0)$$

with

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = E_1^{-1}\Phi_1^{-1}f.$$

Then (5.3) is equivalent to

$$(5.12)\qquad (B_{11} - B_{12}B_{22}^{-1}B_{21})v_0 = F_1 - B_{12}B_{22}^{-1}F_2,$$

the left hand side of which is of the same form as that of (5.7). Since $B_{12}B_{22}^{-1} \in OPS_{1,0}^{-1}$, it suffices to examine B_{11} only.

Write

$$\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \tilde{c}'(1 + L\chi_1) + \tilde{d}'\mathcal{L}.$$

Then, mod $S_{1,0}^{-1}$,

$$\tilde{c}'(y',\eta') = E_1^{-1}BWg_0 + O(\alpha),$$

$$\tilde{d}'(y',\eta') = (E_1^{-1}BWh_0 + \rho_{x_n}E_1^{-1}BW_{\xi_n}g_0)|\eta''|^{1/3},$$

where $W = W(x',\xi', \lambda(x',\xi'))$, $W_{\xi_n} = W_{\xi_n}(x',\xi')$. Therefore, setting

$$(5.13)\qquad E_1^{-1}(y',\eta')B(x')W(x',\xi', \lambda(x',\xi')) = \begin{bmatrix} e_{11} & \cdots & e_{1m_1} \\ \vdots & & \vdots \\ e_{d_+1} & \cdots & e_{d_+m_1} \end{bmatrix}(y',\eta'),$$

we have, mod $S_{1,0}^{-1}$,

$$e_{11} = (R/R_{11})(x',\xi', \lambda(x',\xi'))|\eta''|^{-1} + O(\alpha),$$

$$(5.14)\qquad e_{j1} = -(R_{j1}/R_{11})(x',\xi',\lambda(x',\xi')) + O(\alpha) \text{ for } 2 \leqq j \leqq m_1,$$

$$e_{jk} = \delta_{jk} + O(\alpha) \quad \text{for } 1 \leqq j \leqq m_1,\ 2 \leqq k \leqq m_1.$$

We shall now take the initial data $g_0(x,\eta')$ and $h_0(x,\eta')$ on $\rho = 0$ for $(3.23)_\pm$ as follows. In the case of $m_1 = 1$, we

define $h_0(x,\eta') = 0$ for $\rho = 0$ and then solve (5.9) with initial data $g_0 = 1$ on $x_0 = 0$, so that g_0 is elliptic and , by (3.5), we have

$$(5.15) \qquad h_0(x',\eta') = O(\alpha|\eta''|^{-1/3}).$$

Moreover (5.12) can be written as

$$\tilde{\tilde{c}}(1 + L\chi_1) + \tilde{\tilde{d}}\mathcal{L} = F_1 - B_{12}B_{22}^{-1}F_2,$$

where, mod $S_{1,0}^{-1}$,

$$\tilde{\tilde{c}}(y',\eta') = e_{11}g_0(y',\eta') + O(\alpha),$$

$$\tilde{\tilde{d}}(y',\eta') = g_0(x',\eta')\rho_{x_n}(x',\eta')|\eta''|^{-2/3} + O(\alpha).$$

Since $\tilde{\tilde{d}} \in S_{1,0}^0$ is elliptic, applying a parametrix $\tilde{\tilde{d}}^{-1}$ to each side, we arrive at

$$(5.16) \qquad \mathcal{B}v_0 = f_0$$

with $f_0 = \tilde{\tilde{d}}^{-1}(F_1 - B_{12}B_{22}^{-1}F_2)$ and

$$(5.17) \qquad \mathcal{B} = a(1 + L\chi_1) + \mathcal{L}.$$

Here $a \in OPS_{1,0}^0$ and, mod $S_{1,0}^{-1}$,

$$(5.18) \qquad a(y',\eta') = R_\lambda(x',\xi')/(\rho_{x_n}(x',\eta')|\eta''|^{1/3}) + O(\alpha),$$

so that condition (H_1) and (3.4) yield

$$\arg a(y',\eta') \subset [-\pi/2, \pi/2 - \delta_0] \quad \text{for } \alpha = 0.$$

In the case of $m_1 \geqq 2$, we take

$$(5.19) \qquad g_0(x,\eta')|_{\rho=0} = \begin{bmatrix} 1 & 0 \\ -e_{21} & \\ \vdots & I_{m_1-1} \\ -e_{m_1 1} & \end{bmatrix}$$

and define $h_0(x,\eta')|_{\rho=0}$ so that (5.9) holds. Then (5.12) becomes

$$(5.20) \qquad \tilde{a}'(1 + L\chi_1)v_0 + \tilde{b}'\mathcal{L}v_0 = F_1 - B_{12}B_{22}^{-1}F_2,$$

where $\tilde{a}'$, $\tilde{b}' \in OPS^0_{1,0}$ and, mod $S^{-1}_{1,0}$,

$$\tilde{a}'(y',\eta') = \begin{bmatrix} e_{11} & 0 \\ 0 & I_{m_1-1} \end{bmatrix} + O(\alpha).$$

Moreover, denoting by $\tilde{b}'_{1k}(y',\eta')$ the $(1,k)$ entry of $\tilde{b}'(y',\eta')$, we see from (5.13), (5.14) and (5.19) that, mod $S^{-1}_{1,0}$,

$$\tilde{b}'_{11}(y',\eta') = (\Sigma_{k=1}^{m_1} R_{kk}/R_{11})(x',\xi',\lambda(x',\xi'))\rho_{x_n}|\eta''|^{-2/3} + O(e_{11}) + O(\alpha)$$

and, for $2 \leqq k \leqq m_1$,

$$\tilde{b}'_{1k}(y',\eta') = (R_{1k}/R_{11})(x',\xi',\lambda(x',\xi'))\rho_{x_n}|\eta''|^{1/3} + O(e_{11}) + O(\alpha)$$

because the $(1,k)$ entry of $E_1^{-1}BW_{\xi_n}$ is equal to R_{1k}/R_{11}, mod $O(e_{11})$, for $k \geqq 2$. (See the proof of [8], (5.25)).

Now let E_2 be an elliptic pseudodifferential operator whose symbol is equal to I_{m_1} with the (1,1) entry replaced by $\tilde{b}_{11}(y',\eta')$. Applying a parametrix E_2^{-1} to each side of (5.20), we arrive at

$$(5.21) \qquad \mathcal{B} v_0 = f_0$$

with $f_0 = E_2^{-1}(F_1 - B_{12}B_{22}^{-1}F_2)$ and

$$(5.22) \qquad \mathcal{B} = a(1 + L\chi_1) + b\mathcal{L},$$

where $a, b \in OPS^0_{1,0}$. Moreover, setting

$$(5.23) \qquad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \; v_0 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \; f_0 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where a_{11}, b_{11}, v_1, f_1 are scalar, and a_{22}, b_{22} square matrices of order $m_1 - 1$, we have, mod $S^{-1}_{1,0}$,

$$(5.24) \qquad a_{11}(y',\eta') = \frac{R_\lambda(x',\xi')}{\rho_{x_n}(x',\eta')|\eta''|^{1/3}}\left(1 + O\left(\frac{R_\lambda}{\rho_{x_n}|\eta''|^{1/3}}\right)\right) + O(\alpha),$$

$$(5.25) \qquad a_{12}(y',\eta') = O(\alpha), \; a_{21}(y',\eta') = O(\alpha),$$

$$(5.26) \qquad a_{22}(y',\eta') = I_{m_1-1} + O(\alpha),$$

$$(5.27) \qquad b_{11}(y',\eta') = 1$$

and

$$(5.28) \qquad b_{12}(y',\eta') = R^{-1}_{\xi_n}(R_{12},\cdots,R_{1m_1})(x',\xi',\lambda(x',\xi')) + O(a_{11}) + O(\alpha).$$

Here R_{1k} are the functions described above (5.11).

Finally we modify $\mathcal{B}$ for $y_0 << \bar{y}_0$ as follows. Since WF(f) is contained in a small conic neighborhood of $\Gamma(\bar{x}',\bar{\xi}') \cap \{x_0 \geqq 0\}$, one can assume there is a positive number δ_1 such that

$$f_0 \in H^{\infty}(R^n \cap \{y_0 < \bar{y}_0 - \delta_1\}). \tag{5.29}$$

Let $q_1(y_0)$, $q_2(y_0) \in C^{\infty}(R^1)$ be cutoff functions such that $q_j'(y_0) \geqq 0$ and

$$\begin{aligned} &q_1(y_0) = 1 \quad \text{for } y_0 > \bar{y}_0 - 2\delta_1, \ \operatorname{supp} q_1 \subset (\bar{y}_0 - 3\delta_1, \infty), \\ &q_2(y_0) = 1 \quad \text{for } y_0 > \bar{y}_0 - 4\delta_1, \ \operatorname{supp} q_2 \subset (\bar{y}_0 - 5\delta_1, \infty). \end{aligned} \tag{5.30}$$

Then we set

$$\begin{aligned} &\tilde{a} = q_1 a + (1 - q_1) I_{m_1}, \quad \tilde{b} = q_2 b, \\ &\tilde{\mathcal{B}} = \tilde{a}(1 + L\chi_1) + \tilde{b}\mathcal{L}, \end{aligned} \tag{5.31}$$

so that

$$\tilde{\mathcal{B}} = \begin{cases} \mathcal{B} & \text{for } y_0 > \bar{y}_0 - 2\delta_1, \\ \tilde{a}(1 + L\chi_1) + b\mathcal{L} & \text{for } y_0 > \bar{y}_0 - 4\delta_1, \\ (1 + L\chi_1) I_{m_1} + \tilde{b}\mathcal{L} & \text{for } y_0 < \bar{y}_0 - 3\delta_1, \\ (1 + L\chi_1) I_{m_1} & \text{for } y_0 < \bar{y}_0 - 5\delta_1. \end{cases}$$

In the sequel we will find a solution $v_0 \in H^{-\infty}(R^n)$ of $\tilde{\mathcal{B}} v_0 = f_0$ such that $v_0 \in H^{\infty}(R^n \cap \{y_0 < \bar{y}_0 - \delta_1\})$ and hence $\mathcal{B} v_0 = f_0$, mod $H^{\infty}(R^n)$.

Remark 5.1. Adopting such a modification as (5.31), one can simplify fairly the procedure in [11]. In the rest of this remark we shall use the notations in the preceding paper except for q_1 and q_2, and restrict ourselves to the case of (1.7). First we replace (2.13) by $Gv = G_1 v$, so that (4.15) becomes (5.10). (The (2.12),(4.8) and (4.13) are unnecessary). Next we modify $\mathcal{B}_0$ for $y_0 << \bar{y}_0$ analogously to (5.31) of the present article. Denote the modified operator by $\mathcal{B}$. Then (5.5) is replaced by

$$S = q_3^2\{(1 + L\chi_1) + \sqrt{\tau}e^{-i\delta_0}\gamma\chi_{-1}^2\} + (1 - q_3)\gamma(1 - q_3),$$

where q_3 is a cutoff function such that $q_3 = 1$ on supp q_1 and $q_2 = 1$ on supp q_3. (The (5.2), (5.3), (5.4) and Lemma 5.8 are unnecessary). Besides, in the proof of Lemma 6.8 or 6.9 one can assume that $\mathcal{B} = 1 + L\chi_1$ or $\mathcal{B} = 1 + L\chi_1 + q_2\mathcal{L}$, respectively. (See the proof of Lemma 7.5 below).

Remark 5.2. It should be pointed out that, in the case where $(\bar{x}', \bar{\xi}')$ is a diffractive point, the equation (5.21) can be replaced by $(a + bK_-)v_0 = f_0$ with $\tau = 0$ in (4.5). Since $a_{22} + b_{22}K_- \in S^0_{1/3,0}$ is elliptic, the system of m_1 equations is reduced to a single equation for v_1 only, namely, to (5.28) of [8]. Therefore, using Theorem B.1 of Eskin [5] in the references of [11] (Comm. in P. D. E., Vol. 10 (1985), pp. 1117-1212), one can relax the hypothesis (iv) of [8] so that arg $R_\lambda(x', \xi')$ is contained in the closed interval $[\delta_0, (3/2)\pi - \delta_0]$ for $(x', \xi') \in N_0 \cap \Sigma_0$, where δ_0, N_0 and Σ_0 are the notations in (H_1).

§6. A priori estimates for the equation on the boundary

In the rest of this paper we deal with the more difficult case where (1.10) holds, unless stated otherwise. (For the case of (1.9) see Remarks 6.11 and 7.10 below).

The main purpose of the present section is to derive a priori estimates for solutions of $\tilde{\mathcal{B}} v_0 = f_0$ and $\mathcal{B} v_0 = f_0$ which will be stated in Propositions 6.1 and 6.3, respectively. Here $\mathcal{B}$ is the operator defined by (5.17) when $m_1 = 1$ and by (5.22) when $m_1 \geqq 2$, and $\tilde{\mathcal{B}}$ the modified operator given by (5.31), where $b = 1$ if $m_1 = 1$. From now on we assume the symbols $a(y',\eta')$, $b(y',\eta')$ are homogeneous in η' for $|\eta'| > 1$.

First suppose $m_1 = 1$. Then

$$\mathcal{B} = a(1 + L\chi_1) + \mathcal{L}$$

is a scalar operator, where L, $\mathcal{L}$ and χ_1 are the Fourier multipliers given by (4.6), (4.7). Moreover, by (5.18), the condition (H_1) implies that

$$\text{(6.1)} \qquad \arg a(y',\eta') \subset [-\pi/2,\ \pi/2 - \delta_0] \quad \text{for } \alpha = 0,$$

in particular,

$$(6.1)_0 \qquad \operatorname{Re} a(y',\eta') \geqq 0 \quad \text{for } \alpha = 0.$$

Next suppose $m_1 \geqq 2$. Then, according to (5.24) and (H_1), one can assume without loss of generality that

$$\text{(6.2)} \qquad \arg a_{11}(y',\eta') \subset [-\pi/2,\ \pi/2 - \delta_0] \quad \text{for } \alpha = 0,$$

in particular,

$$(6.2)_0 \qquad \operatorname{Re} a_{11}(y',\eta') \geqq 0 \quad \text{for } \alpha = 0.$$

In order to state a basic a priori estimate for $\widetilde{\mathcal{B}}$ we now introduce an auxiliary bounded operator S on $L^2(R^n)$, defined by

$$(6.3) \qquad Sv_0 = q^2\begin{bmatrix} S_1v_1 \\ S_2v_0 \end{bmatrix} + (1-q)\gamma(1-q)v_0$$

with

$$(6.4) \qquad S_1 = (1 + L\chi_1) + \delta_2\tau e^{-i\delta_0}\gamma\chi_{-1}^2,$$

$$(6.5) \qquad S_2v_0 = (1 + L\chi_1)v_2 + \gamma v_2 + (\tilde{b}_{21}\mathcal{L}v_1 + \tilde{b}_{22}\mathcal{L}v_2),$$

where if $m_1 = 1$ then $v_0 = v_1$ and $S_2v_0 = 0$. Here $\chi_{-1}(\eta') = \chi_1(-\eta')$, γ is the Fourier multiplier given by (4.7), τ the parameter in (4.4) and δ_2 a small positive number. Moreover $q = q(y_0) \in C^\infty(R^1)$ is a cutoff function such that $q(y_0) = 1$ for $y_0 > \bar{y}_0 - 3\delta_1$ and supp $q \subset (\bar{y}_0 - 4\delta_1, \infty)$. Note that (5.30) yields $qq_1 = q_1$ and $qq_2 = q$.

The following a priori estimate for $\widetilde{\mathcal{B}}$ will play a basic role in the following analysis.

<u>Proposition 6.1</u>. Assume (6.1) or (6.2) holds in the case of $m_1 = 1$ or $m_1 \geqq 2$, respectively. Then there are positive numbers τ_1, C_1 and δ_2 such that

$$(6.6) \qquad \mathrm{Re}\,(\widetilde{\mathcal{B}}v_0, Sv_0) \geqq C_1\tau\|\gamma v_0\|^2 - O(\|v_0\|_{-1/2}^2)$$

for $\tau \geqq \tau_1$ and $v_0 \in L^2(R^n)$ with supp $\hat{v}_0(\eta') \subset \{\gamma < \tau^{-1}$ and $\alpha << \tau^{-2}\}$.

Note that (6.6) yields

$$(6.7)\qquad \tau\|\gamma v_0\|_s^2 \leqq C_1'\|\gamma^{-1}\tilde{\mathcal{B}}v_0\|_s^2 + C_{\tau,s}\|\gamma^{-1}v_0\|_{s-1}^2$$

for any real number s and $\gamma^{-1}v_0 \in H^s(R^n)$ with supp $\hat{v}_0$ as above, in particular,

$$(6.8)\qquad \tau\|v_0\|_{s-1/3}^2 \leqq C_1'\|\tilde{\mathcal{B}}v_0\|_{s+1/3}^2 + C_{\tau,s}\|v_0\|_{s-2/3}^2$$

if $v_0 \in H^{s+1/3}(R^n)$, where C_1' is a constant independent of τ, s.

For the purpose of showing that there exists a solution of $\tilde{\mathcal{B}}v_0 = f_0$, we need also the following a priori estimate for S.

<u>Proposition 6.2</u>. There are positive numbers τ_2 and C_2, independent of δ_2, such that

$$(6.9)\qquad \mathrm{Re}\,(Sv_0, v_0) \geqq C_2(\tau\|\gamma^{1/2}qv_0\|^2 + \|\gamma^{1/2}(1-q)v_0\|^2) - O(\|\gamma^{-1/2}v_0\|_{-1/2}^2)$$

for $\tau \geqq \tau_2$ and such $v_0 \in L^2(R^n)$ as in the preceding proposition.

To study the propagation of singularities in the region $\alpha|\eta''|^{\varepsilon} >> 1$ we use the following a priori estimate for $\mathcal{B}$.

<u>Proposition 6.3</u>. Assume that $(6.1)_0$ holds in the case of $m_1 = 1$ and that (H_1), (H_2) hold in the case of $m_1 \geqq 2$. Then there are positive numbers τ_3, C_3 and δ_3 such that,

if $p(y',\eta') \in S^0_{1,0}$ is homogeneous in η', $0 \leqq p(y',\eta') \leqq 1$ and $p\circ\phi_2(y',\eta') \leqq p(y',\eta')$, ϕ_2 being the canonical transformation given by (4.10), then

$$(6.10) \qquad \tau\|\gamma^2 p v_0\|^2 \leqq C_3(\|p\mathcal{B}v_0\|^2 + \|pL\chi_\varepsilon \mathcal{B}v_0\|^2) + O(\|\gamma^2 v_0\|^2_{-\varepsilon_0})$$

for $\tau \geqq \tau_3$ and $v_0 \in L^2(R^n)$ with supp $\hat{v}_0(\eta') \subset \{2|\eta''|^{-\varepsilon} < \alpha < \delta_3\tau^{-2}\}$. Here $\varepsilon_0 = 1/2 - (3/4)\varepsilon$ with ε the number in (4.8), and τ_3, C_3 and δ_3 are independent of p.

We also need an analogue to (6.6) for $\mathcal{B}$. Denoting by S_0 the operator S defined by (6.3) with $q = 1$, we have

<u>Corollary 6.4</u>. Assume (6.1) or (6.2) holds in the case of $m_1 = 1$ or $m_1 \geqq 2$, respectively. Then there are positive numbers τ_4, C_4 and δ_2 such that

$$(6.11) \qquad \mathrm{Re}\ (\mathcal{B}v_0,\ S_0v_0) \geqq C_4\tau\|\gamma v_0\| - O(\|v_0\|^2_{-1/2})$$

for $\tau \geqq \tau_4$ and such $v_0 \in L^2(R^n)$ as in Proposition 6.1.

<u>Corollary 6.5</u>. There are positive numbers τ_5, C_5 such that

$$(6.12) \qquad \mathrm{Re}\ (S_0v_0,\ v_0) \geqq C_5\tau\|\gamma^{1/2}v_0\|^2 - O(\|\gamma^{-1/2}v_0\|^2_{-1/2})$$

for $\tau \geqq \tau_5$ and such $v_0 \in L^2(R^n)$ as above.

The rest of this section will be devoted to the proofs of the above estimates. From now on we suppose $v_0 \in L^2(R^n)$, $\operatorname{supp} \hat{v}_0(\eta') \subset \{\gamma < \tau^{-1}, -1 < \alpha << \tau^{-2}\}$, and denote constants independent of τ by C_k, C and so on.

<u>Proof of Proposition 6.1 in the case of $m_1 = 1$.</u> Write

$$\begin{aligned}(\tilde{\mathcal{B}} v_0, Sv_0) &= (q\tilde{\mathcal{B}} v_0, q(1 + L\chi_1)v_0) \\ &\quad + \delta_2\tau(e^{i\delta_0} q\tilde{\mathcal{B}} v_0, q\gamma\chi_{-1}^2 v_0) + ((1 - q)\tilde{\mathcal{B}} v_0, \gamma(1-q)v_0) \\ &= I_1 + \delta_2\tau I_2 + I_3.\end{aligned}$$

Then, since $q\tilde{\mathcal{B}} = q\tilde{a}(1 + L\chi_1) + q\mathcal{L}$ and $(6.1)_0$ implies that

$$\operatorname{Re} \tilde{a}(y',\eta') \geqq 0 \quad \text{for } \alpha = 0,$$

we obtain, analogously to [//], Lemma 5.5,

$$\begin{aligned}(6.13) \qquad \operatorname{Re} I_1 &\geqq b_2\tau(\|\gamma\chi_1 qv_0\|^2 + \|\gamma^{-1/2}(1 - \chi_1)qv_0\|^2_{-1/2}) \\ &\quad - C\|\gamma v_0\|^2 - O(\|v_0\|^2_{-1/2}),\end{aligned}$$

where b_2 is the positive number in Lemma 4.10. We also have, analogously to [//], Lemma 5.6,

$$\begin{aligned}(6.14) \qquad \operatorname{Re} I_2 &\geqq C_{\delta_0}\|\gamma\chi_{-1}qv_0\|^2 - C\|\gamma^{3/2}\chi_{-1}qv_0\|^2 \\ &\quad - C'\|\gamma^{-1/2}(1 - \chi_1)v_0\|^2_{-1/2} - O(\|v_0\|^2_{-1/2}),\end{aligned}$$

where C_{δ_0} is the positive number in Lemma 4.11 with $\delta = \delta_0$. Finally consider I_3. Since

$$(1 - q)\tilde{\mathcal{B}} = (1 - q)(1 + L\chi_1) + (1 - q)q_2\mathcal{L},$$

one can write

$$I_3 = ((1 + L\chi_1)(1 - q)v_0, \gamma(1 - q)v_0)$$
$$+ ([1 - q, L\chi_1]v_0, \gamma(1 - q)v_0) + ((1 - q)q_2\mathcal{L}v_0, \gamma(1 - q)v_0).$$

Therefore, by virtue of Lemmas 4.8, 4.9 and (4.12), we obtain

$$\text{(6.15)} \qquad \operatorname{Re} I_3 \geqq b_1\tau\|\gamma\chi_1(1 - q)v_0\|^2 + \|\gamma^{1/2}(1 - \chi_1)(1 - q)v_0\|^2 - C\|\gamma v_0\|^2 - O(\|v_0\|^2_{-1/2}),$$

where b_1 is the positive number in Lemma 4.8. Now (6.6) follows from (6.13), (6.14) and (6.15), if we take δ_2 small relatively to b_1, b_2 and 1. In fact, mod $O(\|v_0\|^2_{-1/2})$,

$$\|\gamma^{3/2}\chi_{-1}qv_0\|^2 \leqq \tau^{-1}\|\gamma\chi_{-1}qv_0\|^2,$$

$$\|\gamma^{-1/2}(1 - \chi_1)(1 - q)v_0\|^2_{-1/2} \leqq \tau^{-1}\|\gamma^{1/2}(1 - \chi_1)(1 - q)v_0\|^2,$$

$$\|\gamma^{1/2}(1 - \chi_1)(1 - q)v_0\|^2 \geqq \tau\|\gamma(1 - \chi_1)(1 - q)v_0\|^2$$

and

$$\|\gamma(1 - \chi_1 - \chi_{-1})qv_0\|^2 \leqq C\|\gamma^{-1/2}(1 - \chi_1)qv_0\|^2_{-1/2}.$$

Therefore we obtain (6.6) for $\tau >> 1$ and complete the proof.

To prove Proposition 6.1 in the case of $m_1 \geqq 2$ we represent the $\tilde{a}$, $\tilde{b}$ in (5.31) analogously to (5.23). Then, in view of (5.25) and (5.26), one can assume without loss of generality that

$$\tilde{a} = \begin{bmatrix} \tilde{a}_{11} & 0 \\ 0 & I_{m_1-1} \end{bmatrix}.$$

For convenience set

$$\tilde{\mathcal{B}} v_0 = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix},$$

so that

$$\tilde{f}_1 = \tilde{a}_{11}(1 + L\chi_1) + \tilde{b}_{11}\mathcal{L}v_1 + \tilde{b}_{12}\mathcal{L}v_2, \tag{6.16}$$

$$\tilde{f}_2 = (1 + L\chi_1)v_2 + \tilde{b}_{21}\mathcal{L}v_1 + \tilde{b}_{22}\mathcal{L}v_2. \tag{6.17}$$

We also write

$$\begin{aligned}(\tilde{\mathcal{B}} v_0, Sv_0) &= (q\tilde{f}_1, q(1 + L\chi_1)v_1) \\ &\quad + \delta_2\tau(e^{i\delta_0}q\tilde{f}_1, q\gamma\chi_{-1}^2 v_1) + (q\tilde{f}_2, qS_2v_0) \\ &\quad + ((1 - q)\tilde{\mathcal{B}} v_0, \gamma(1 - q)v_0).\end{aligned} \tag{6.18}$$

<u>Lemma 6.6.</u> Suppose $(6.2)_0$ holds. Then

$$\begin{aligned}&\mathrm{Re}\ (q\tilde{f}_1, q(1 + L\chi_1)v_1) \\ &\geqq b_2\tau(\|\gamma\chi_1 qv_1\|^2 + \|\gamma^{-1/2}(1 - \chi_1)qv_1\|_{-1/2}^2) \\ &\quad - (1/2)\|(1 + L\chi_1)qv_2\|^2 - C_6\|\gamma v_0\|^2 - O(\|v_0\|_{-1/2}^2).\end{aligned} \tag{6.19}$$

Here b_2 is the positive number in Lemma 4.10.

<u>Proof</u>. As we derived (6.13), it follows from $(6.2)_0$ and (5.27) that

$$\begin{aligned}
&\mathrm{Re}\ (q\tilde{f}_1,\ q(1 + L\chi_1)v_1) \\
&\geqq \mathrm{Re}\ (b_{12}\mathcal{L}qv_2,\ (1 + L\chi_1)qv_1) \\
&+ b_2\tau(\|\gamma\chi_1 qv_1\|^2 + \|\gamma^{-1/2}(1 - \chi_1)qv_1\|^2_{-1/2}) - C\|\gamma v_0\|^2 \\
&\qquad\qquad - O(\|v_0\|^2_{-1/2}).
\end{aligned}$$

Hence we need only to prove

$$\begin{aligned}
(6.20)\qquad &|(b_{12}\mathcal{L}qv_2,\ (1 + L\chi_1)qv_1)| \\
&\leqq (1/2)\|(1 + L\chi_1)qv_2\|^2 + C'\|\gamma v_0\|^2 + O(\|v_0\|^2_{-1/2}).
\end{aligned}$$

By (4.6) one can write

$$\mathcal{L} = K_-(1 + L\chi_1) + K,$$

where $K = K_+\chi_1 + K_0(1 - \chi_1) - K_-$, hence

$$\begin{aligned}
&(b_{12}\mathcal{L}qv_2,\ (1 + L\chi_1)qv_1) \\
&= ((1 + L\chi_1)qv_2,\ K_-^*b_{12}^*(1 + L\chi_1)qv_1) \\
&\quad + (qv_2,\ K^*b_{12}^*(1 + L\chi_1)qv_1) \\
&= I_1 + I_2.
\end{aligned}$$

Besides, Lemma 4.3 yields

$$|I_1| \leqq (1/4)\|(1 + L\chi_1)qv_2\|^2 + C\|\gamma v_1\|^2 + O(\|v_1\|^2_{-1/2}).$$

Moreover, writing

$$I_2 = ((1 + L\chi_1)^* qv_2,\ K^* b_{12}^* qv_1) + (qv_2,\ K^*[b_{12}^*,\ L\chi_1]qv_1),$$

we have

$$|I_2| \leqq (1/4)\|(1 + L\chi_1)^* qv_2\|^2 + C\|\gamma v_0\|^2 + O(\|v_1\|_{-1/2}^2).$$

Since $\|(1 + L\chi_1)^* qv_2\|^2 = \|(1 + L\chi_1)qv_2\|^2$, we thus obtain (6.20) and complete the proof.

<u>Lemma 6.7</u>. Suppose (6.2) holds. Then

$$\begin{aligned}(6.21)\qquad & \mathrm{Re}\ (e^{i\delta_0} q\tilde{f}_1,\ q\gamma\chi_{-1}^2 v_1) \\ & \geqq (1/2)C_{\delta_0}\|\gamma\chi_{-1}qv_1\|^2 - C_7\|\gamma\chi_{-1}qv_2\|^2 \\ & - C_7'\|\gamma^{-1/2}(1 - \chi_1)v_0\|_{-1/2}^2 - C_7''\|\gamma^{3/2}\chi_{-1}qv_1\|^2 - O(\|v_0\|_{-1/2}^2),\end{aligned}$$

where C_{δ_0} is the positive number in Lemma 4.11.

<u>Proof</u>. Since it follows from (6.16) and (4.6) that, mod $H^\infty(R^n)$,

$$\chi_{-1}q\tilde{f}_1 = \chi_{-1}q(\tilde{a}_{11}v_1 + b_{11}K_0v_1 + b_{12}K_0v_2),$$

we have, analogously to (6.14),

$$\begin{aligned}& \mathrm{Re}\ (e^{i\delta_0} q\tilde{f}_1,\ q\gamma\chi_{-1}^2 v_1) \\ & \geqq C_{\delta_0}\|\gamma\chi_{-1}qv_1\|^2 - C\|\gamma^{3/2}\chi_{-1}qv_1\|^2 - C'\|\gamma^{-1/2}(1 - \chi_1)v_0\|_{-1/2}^2 \\ & - O(\|v_0\|_{-1/2}^2) + \mathrm{Re}\ (b_{12}K_0\chi_{-1}qv_1,\ \gamma\chi_{-1}qv_2).\end{aligned}$$

Besides, by Lemma 4.3, the last term is estimate from below by

$$-(1/2)C_{\delta_0}\|\gamma\chi_{-1}qv_1\|^2 - C''\|\gamma\chi_{-1}qv_2\|^2 - O(\|v_0\|^2_{-1/2}).$$

Therefore we obtain (6.21).

<u>Lemma 6.8</u>. We have

$$\begin{aligned}(6.22)\qquad & \mathrm{Re}\ (q\tilde{f}_2,\ qS_2v_0)\\ &\geqq b_1\tau\|\gamma\chi_1qv_2\|^2 + \|\gamma^{1/2}(1-\chi_1)qv_2\|^2\\ &+ (3/4)\|(1+L\chi_1)qv_2\|^2 - C_8\|\gamma v_0\|^2 - C_8'\|\gamma^{-1/2}v_0\|^2_{-1/2}\\ &\qquad - O(\|v_0\|^2_{-1/2}).\end{aligned}$$

<u>Proof</u>. It follows from (6.5) and (6.17) that

$$\begin{aligned}&\mathrm{Re}\ (q\tilde{f}_2,\ qS_2v_0)\\ &\geqq \|q(1+L\chi_1)v_2\|^2 + \mathrm{Re}\ (q(1+L\chi_1)v_2,\ q\gamma v_2)\\ &+ 2\mathrm{Re}\ (q(1+L\chi_1)v_2,\ q(\tilde{b}_{22}\mathcal{L}v_2 + \tilde{b}_{21}\mathcal{L}v_1)) - C'\|\gamma v_0\|^2\\ &\qquad - O(\|v_0\|^2_{-1/2}).\end{aligned}$$

Applying Lemma 4.8 to the second term on the right hand side, from (4.12), Lemmas 4.4 and 4.9 we have therefore

$$\begin{aligned}&\mathrm{Re}\ (q\tilde{f}_2,\ qS_2v_0)\\ &\geqq b_1\tau\|\gamma\chi_1qv_2\|^2 + \|\gamma^{1/2}(1-\chi_1)qv_2\|^2 + (1-\delta)\|q(1+L\chi_1)v_2\|^2\\ &- C_\delta\|\gamma v_0\|^2 - C'\|\gamma^{-1/2}v_0\|^2_{-1/2} - O(\|v_0\|^2_{-1/2})\end{aligned}$$

for $\delta > 0$. Besides,

$$\|q(1 + L\chi_1)v_2\|^2 \geqq (1 - \delta)\|(1 + L\chi_1)qv_2\|^2 - C'_\delta\|\gamma v_2\|^2 - O(\|v_2\|^2_{-1/2}).$$

Hence we obtain (6.22)

<u>Proof of Proposition 6.1 in the case of $m_1 \geqq 2$.</u> Since

$$(1 - q)\widetilde{\mathcal{B}} = (1 - q)(1 + L\chi_1) + (1 - q)\tilde{b}\mathcal{L},$$

the last term of (6.18) is estimated similarly to (6.15). Therefore (6.6) follows from (6.19), (6.21) and (6.22), as in the case of $m_1 = 1$. Thus we prove the proposition.

<u>Proof of Proposition 6.2</u>. Write

$$
\begin{aligned}
(Sv_0, v_0) &= (q(1 + L\chi_1)v_0, qv_0) + (\gamma(1 - q)v_0, (1 - q)v_0) \\
&\quad + \delta_2\tau e^{-i\delta_0}(q\gamma\chi_{-1}^2 v_1, qv_1) \\
&\quad + (q(\gamma + \tilde{b}_{22}\mathcal{L})v_2 + q\tilde{b}_{21}\mathcal{L}v_1, qv_2) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

By virtue of Lemma 4.8 together with Lemmas 4.1 and 4.9 we have

$$
\begin{aligned}
\mathrm{Re}\ I_1 \geqq\ & b_1\tau\|\gamma^{1/2}\chi_1 qv_0\|^2 + \|(1 - \chi_1)qv_0\|^2 - \tau^{1/2}\|\gamma^{1/2}qv_0\|^2 \\
& - C\tau^{-1/2}\|\gamma^{1/2}v_0\|^2 - O(\|\gamma^{-1/2}v_0\|_{-1/2}^2).
\end{aligned}
$$

Clearly

$$
I_2 = \|\gamma^{1/2}(1 - q)v_0\|^2.
$$

It is also not hard to show that

$$
\mathrm{Re}\ I_3 = \delta_2\tau(\cos\ \delta_0)\|\gamma^{1/2}\chi_{-1}qv_1\|^2 + O(\|\gamma^{-1/2}v_1\|_{-1/2}^2).
$$

Besides,

$$
|I_4| \leqq C\|\gamma^{1/2}qv_0\|^2 + C'\|\gamma^{3/2}v_0\|^2 + O(\|\gamma^{-1/2}v_0\|_{-1/2}^2).
$$

Thus, noting that

$$
\|(1 - \chi_1)qv_0\|^2 \geqq \tau\|\gamma^{1/2}(1 - \chi_1)qv_0\|^2 + O(\|v_0\|_{-1/2}^2),
$$

we complete the proof.

We shall now proceed to the proof of Proposition 6.3.

Lemma 6.9. Let $m_1 \geqq 2$. Assume (H_1) and (1.18) hold. Then there is a positive constant C such that

$$(6.23) \qquad |b_{12}(y',\eta')| \leqq C(\mathrm{Re}\ a_{11}(y',\eta') + \sqrt{\alpha}) \quad \text{for } 0 < \alpha << 1,$$

where a_{11}, b_{12} are the symbols in (5.23).

Proof. In view of (5.28) it suffices to prove (6.23) with b_{12} replaced by $R_{1k}(x',\xi',\lambda_0(x',\xi'))$ for $2 \leqq k \leqq m_1$, where $(x',\xi') = \phi_1(y',\eta')$ and R_{1k} is the notation described above (5.11), because (H_1) and (5.24) together with (3.4) imply that $|a_{11}(y',\eta')| \leqq C\mathrm{Re}\ a_{11}(y',\eta')$ for $\alpha = 0$, mod $S^{-1}_{1,0}$. Hereafter we omit the variables except ξ_n, so $R(\xi_n)$ stands for $R(x',\xi',\xi_n)$ and so on.

We first show

$$(6.24) \qquad 2\sqrt{\mu_0}R_{1k}(\xi_n^+) = c_{1k}R(\xi_n^+) \quad \text{for } \mu_0 > 0 \text{ and } 2 \leqq k \leqq m_1.$$

Set

$$W(\xi_n) = (V_1(\xi_n),\cdots,V_{m_1}(\xi_n)),$$

$$W_{\xi_n} = (V_1',\cdots,V_{m_1}'),$$

so that

$$\tilde{R}(\xi_n) = (V_1(\xi_n),\cdots,V_{m_1}(\xi_n),\ W_h,\ W_e).$$

Then by (1.13) we have

$$\det B(V_k(\xi_n^-),V_2(\xi_n^+),\cdots,V_{m_1}(\xi_n^+),W_h,W_e) = c_{1k}R(\xi_n^+).$$

Since $\xi_n^- = \xi_n^+ + 2\sqrt{\mu_0}$ and V_k' is independet of ξ_n, we also see that

$$V_k(\xi_n^-) = V_k(\xi_n^+) + 2\sqrt{\mu_0}V_k'.$$

Therefore we obtain (6.24).

Now, since $\xi_n^+ = \lambda_0 - \sqrt{\mu_0}$ and, by (3.4) and (3.7), $-\sqrt{\mu_0} = \sqrt{\alpha}\rho_{x_n}|\eta''|^{1/3}$, we have from (1.12)

$$R(\xi_n^+) = R_{\xi_n}(\lambda_0)(R_\lambda + \sqrt{\alpha}\rho_{x_n}|\eta''|^{1/3}) + O(\alpha),$$

so (6.24) yields

$$2\sqrt{\mu_0}R_{1k}(\lambda_0) = c_{1k}R_{\xi_n}(\lambda_0)(R_\lambda + \sqrt{\alpha}\rho_{x_n}|\eta''|^{1/3}) + O(\alpha).$$

Consequently we deduce from (1.18), (5.24) and (H_1) that $R_{1k}(\lambda_0)/(\mathrm{Re}\ a_{11} + \sqrt{\alpha})$ is bounded. Thus we prove the lemma.

Now, by Lemma 4.8 there is a positive number δ_3 such that

$$(6.25) \qquad 1 - |L(\eta')|^2 \geqq b_1\tau\gamma - O(|\eta''|^{-1}) \text{ for } 0 < \alpha < \delta_3\tau^{-2}.$$

Suppose v_0 is as in (6.10) with the above δ_3. Then (4.6) and (4.7) imply that $L\chi_1 v_0 = Lv_0$ and $\mathcal{L}v_0 = (K_+ + K_-L)v_0$. Hence by Lemma 4.12 and (4.9) we have

$$(6.26) \qquad \mathcal{L}v_0 = \gamma(1 - L)v_0 + O(\|\gamma^{-2}v_0\|_{-1}).$$

We also see from Lemma 4.7 that there are pseudodifferential operators $p^{(0)}$, $p^{(1)}$ such that, modulo a smoothing operator,

$$pLv_0 = L\chi_\varepsilon p^{(0)}v_0 + L\chi_\varepsilon p^{(1)}v_0,$$

where $p^{(0)}(y',\eta') = p\circ\phi_2(y',\eta')$ with the canonical transformation ϕ_2 given by (4.10), and $p^{(1)}(y',\eta') \in S^{-1+\varepsilon}_{1-\varepsilon,0}$. Hereafter $L(\eta')$ is modified outside supp $\hat{v}_0$ in such a way that $L(\eta')(1 - \chi_\varepsilon(\eta')) \in S^0_{\varepsilon/2,0}$.

<u>Lemma 6.10</u>. Let $p \in OPS^0_{1-\varepsilon,0}$ be as in Proposition 6.3 and let q,s be real numbers. Then for $v \in H^s(R^n)$ with supp $\hat{v}(\eta') \subset \{|\eta''|^{-\varepsilon} < \alpha < 1\}$ we have

$$(6.27) \qquad \|\gamma^q p^{(0)}v\|_s^2 \leqq \|\gamma^q pv\|_s^2 + O(\|\gamma^q v\|_{s+(-1+\varepsilon)/2}^2),$$

$$(6.28) \qquad \|\gamma^q pLv\|_s^2 \leqq \|L\gamma^q pv\|_s^2 + O(\|\gamma^q v\|_{s+(-1+\varepsilon)/2}^2),$$

$$(6.29) \qquad \|\gamma^q pLv\|^2 \leqq \|\gamma^q pv\|_s^2 + O(\|\gamma^q v\|_{s+(-1+\varepsilon)/2}^2).$$

Proof. The first estimate can be proved analogously to [//], (5.16). To derive (6.28) one can assume $q = 0$, because the symbol of the commutator $[\gamma^q, p]$ is $O(\gamma^q|\eta''|^{-1+\varepsilon})$ on supp $\hat{v}$. Now we have

$$\|pLv\|^2 = \|Lp^{(0)}v\|^2 + O(\|v\|^2_{(-1+\varepsilon)/2}).$$

Since Lemma 4.6 implies that $|L(\eta')| \in S^0_{1-\varepsilon,0}$, we denote by $|L|$ the pseudodifferential operator with symbol $|L(\eta')|$. Then, noting that $\|Lp^{(0)}v\|^2 = \||L|p^{(0)}v\|^2$ and $[|L|, p^{(0)}] \in OPS^{-1+\varepsilon}_{1-\varepsilon,0}$, we get

$$\|Lp^{(0)}v\|^2 = \|p^{(0)}|L|v\|^2 + O(\|v\|^2_{(-1+\varepsilon)}).$$

Therefore by (6.27) we obtain (6.28), which yields (6.29), because of (4.11).

<u>Proof of Proposition 6.3</u>. Since the case of $m_1 = 1$ can be handled analogously to [//], Proposition 5.3, we suppose that $m_1 \geqq 2$ and v_0 is as in (6.10). In view of (6.26), we may also replace $\mathcal{B}$ by

$$\mathcal{B}_0 = a(1 + L) + b\gamma(1 - L).$$

Moreover by (5.25), (5.26) and (5.27) one can assume that

$$a(y',\eta') = \begin{bmatrix} a_{11} & 0 \\ 0 & I_{m_1-1} \end{bmatrix},$$

$$b(y',\eta') = \begin{bmatrix} 1 & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

and

$$a_{11}(y',\eta') = a_{11}(y',0,\eta'').$$

Now set

$$\mathcal{B}_0 v_0 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

so that

$$f_1 = a_{11}(1 + L)v_1 + \gamma(1 - L)v_1 + b_{12}\gamma(1 - L)v_2, \tag{6.30}$$

$$f_2 = (1 + L)v_2 + b_{21}\gamma(1 - L)v_1 + b_{22}\gamma(1 - L)v_2. \tag{6.31}$$

First suppose (1.19) holds. Then (5.24) implies $a_{11} = O(\alpha)$. Hence one can assume

$$f_1 = \gamma(1 - L)v_1 + b_{12}\gamma(1 - L)v_2. \tag{6.30$_0$}$$

We shall prove

$$(6.32)\qquad b_1\tau\|\gamma^2 pv_0\|^2 \leqq \|pf_1\|^2 + \|pLf_1\|^2 + C(\|\gamma pf_2\|^2 + \|\gamma pLf_2\|^2) + C'\|\gamma^2 pv_0\|^2 + O(\|\gamma^2 v_0\|^2_{-\varepsilon_0}),$$

where b_1 is the positive number in Lemma 4.8 and C, C' are independent of τ. Let us apply $1 + L$ to each side of $(6.30)_0$. Since Lemma 4.7 implies that $Lb_{12} = b_{12}L + b'_{12}L$, where $b'_{12} \in OPS^0_{1-\varepsilon,0}$ and $b'_{12}(y',\eta') = O(\gamma) \bmod S^{-1+\varepsilon}_{1-\varepsilon,0}$, we have

$$\gamma(1 - L^2)v_1 = (1 + L)f_1 - b_{12}\gamma(1 - L)(1 + L)v_2 - b'_{12}\gamma L(1 - L)v_2.$$

Moreover (6.31) yields

$$b_{12}\gamma(1 - L)(1 + L)v_2$$
$$= b_{12}\gamma(1 - L)f_2 - b_{12}\gamma(1 - L)\{b_{21}\gamma(1 - L)v_1 + b_{22}\gamma(1 - L)v_2\}.$$

Hence by (6.28) and (6.29) we get

$$2|(p\gamma(1 - L^2)v_1, \gamma^2 pv_1)|$$
$$\leqq \|pf_1\|^2 + \|pLf_1\|^2 + C(\|\gamma pf_2\|^2 + \|\gamma pLf_2\|^2) + C'\|\gamma^2 pv_0\|^2 + O(\|\gamma^2 v_0\|^2_{(-1+\varepsilon)/2}),$$

while

$$2\mathrm{Re}\ (\gamma p(1 - L^2)v_1, \gamma^2 pv_1) \geqq \|\gamma^{3/2} pv_1\|^2 - \|L\gamma^{3/2} pv_1\|^2 - O(\|\gamma^{3/2} v_1\|^2_{(-1+\varepsilon)/2}).$$

Therefore by (6.25) we obtain

$$b_1\tau\|\gamma^2 pv_1\|^2 \leqq \|pf_1\|^2 + \|pLf_1\|^2 + C(\|\gamma pf_2\|^2 + \|\gamma pLf_2\|^2)$$
$$+ C'\|\gamma^2 pv_0\|^2 + O(\|\gamma^{3/2}v_0\|^2_{(-1+\varepsilon)/2}).$$

Similarly, (6.31) yields

$$b_1\tau\|\gamma^2 pv_2\|^2 \leqq \|\gamma pf_2\|^2 + C\|\gamma^2 pv_0\|^2 + O(\|\gamma^{3/2}v_0\|^2_{(-1+\varepsilon)/2}). \tag{6.33}$$

Thus we obtain (6.32), because

$$\|\gamma^{3/2}v_0\|^2_{(-1+\varepsilon)/2} = O(\|\gamma^2 v_0\|^2_{-\varepsilon_0}).$$

Next suppose $(6.2)_0$ and (6.23) hold. Then we shall rewrite (6.30) as

$$(a_{11} + \gamma)v_1 + (a_{11} - \gamma)Lv_1 = f_1 - b_{12}\gamma(1 - L)v_2$$

and prove

$$b_1\tau\|\gamma^2 pv_0\|^2 \leqq \|pf_1\|^2 + \|\gamma pf_2\|^2 + C\|\gamma^2 pv_0\|^2 + O(\|\gamma^2 v_0\|^2_{-\varepsilon_0}), \tag{6.34}$$

where C is independent of τ. For convenience we modify γ outside supp $\hat{\hat{v}}_0$ as

$$\tilde{\gamma} = \gamma\chi_\varepsilon + (1 + |\eta''|^2)^{-\varepsilon/4}(1 - \chi_\varepsilon)$$

with χ_ε given by (4.8), so that $\gamma v_0 = \tilde{\gamma}v_0$ and $\tilde{\gamma} \in S^0_{1-\varepsilon,0}$. Since Re $a_{11}(y',\eta') \geqq 0$, there is a parametrix $\Phi \in S^{\varepsilon/2}_{1-\varepsilon,\varepsilon/2}$ for $a_{11} + \tilde{\gamma}$ such that $\Phi(y',\eta') = (a_{11}(y',\eta') + \tilde{\gamma})^{-1}$, mod $S^{-1+2\varepsilon}_{1-\varepsilon,\varepsilon/2}$. Applying Φp to each side of (6.30), we have

$$(6.35)\qquad pv_1 + \Phi(a_{11} - \tilde{\gamma})pLv_1 = \Phi pf_1 - \Phi b_{12}\gamma p(1 - L)v_2 + O(\|\gamma^{-2}v_0\|_{-1}).$$

Now, since $\Phi(a_{11} - \tilde{\gamma}) \in OPS^0_{1-\varepsilon,\varepsilon/2}$ and

$$|(a_{11}(y',\eta') + \tilde{\gamma})^{-1}(a_{11}(y',\eta') - \tilde{\gamma})| \leqq 1,$$

we have

$$\|\Phi(a_{11} - \tilde{\gamma})\gamma^{3/2}pLv_1\|^2$$
$$\leqq \|\gamma^{3/2}pLv_1\|^2 + O(\|\gamma^{3/2}pLv_1\|^2_{-1/2} + (3/4)\varepsilon)$$

and hence by (6.28) and (6.25) we obtain

$$2\mathrm{Re}\ (pv_1 + \Phi(a_{11} - \tilde{\gamma})pLv_1,\ \gamma^3 pv_1)$$
$$\geqq b_1\tau\|\gamma^2 pv_1\|^2 - C\|\gamma^2 pv_1\|^2_{-1/2+\varepsilon} - O(\|\gamma^{3/2}v_1\|^2_{(-1+\varepsilon)/2}).$$

On the other hand we see from (6.23) that $\Phi b_{12} \in OPS^0_{1-\varepsilon,\varepsilon/2}$. Therefore by (6.35) we have

$$2|(pv_1 + \Phi(a_{11} - \tilde{\gamma})pLv_1,\ \gamma^3 pv_1)|$$
$$\leqq \|pf_1\|^2 + C'\|\gamma^2 pv_0\|^2 + O(\|\gamma^2 v_0\|^2_{(-1+\varepsilon)/2}),$$

because $\Phi^*\gamma \in S^0_{1-\varepsilon,\varepsilon/2}$. Thus we get

$$b_1\tau\|\gamma^2 pv_1\|^2 \leqq \|pf_1\|^2 + C\|\gamma^2 pv_0\|^2 + O(\|\gamma^2 v_0\|^2_{-\varepsilon_0}).$$

By this and (6.33) we obtain (6.34) and hence (6.10) if $\tau >> 1$. The proof is complete.

<u>Remark 6.11</u>. Suppose (1.9) holds. Set $\mathcal{B} = (1 + L\chi_1) + b\mathcal{L}$ and $\widetilde{\mathcal{B}} = (1 + L\chi_1) + q_2 b\mathcal{L}$, where $b \in OPS^0_{1,0}$ is the pseudodifferential operator in (5.10) and q_2 the cutoff function in (5.31). Then, using Lemma 4.8 and (4.12), we obtain easily

$$\text{(6.36)} \qquad \operatorname{Re}\,(\widetilde{\mathcal{B}} v_0, \gamma v_0) \geqq C\tau \|\gamma v_0\|^2 - O(\|v_0\|^2_{-1/2})$$

for such τ and v_0 as in (6.6), in particular,

$$\text{(6.37)} \qquad \tau \|\gamma v_0\|^2_s \leqq C' \|\widetilde{\mathcal{B}} v_0\|^2_s + C_{\tau,s} \|\gamma^{-1} v_0\|^2_{s-1}$$

for $s \in R^1$ and $v_0 \in H^s(R^n)$ with supp $\hat{v}_0(\eta')$ as before. Here C, C' are positive constants independent of τ, s and v_0. Moreover, by virtue of (6.25), (6.26) and (6.28) we have

$$\text{(6.38)} \qquad \tau \|\gamma p v_0\|^2 \leqq C'' \|p\mathcal{B} v_0\|^2 + O(\|\gamma v_0\|^2_{-\varepsilon_0})$$

for such p, τ and v_0 as in (6.10). Here ε_0 is the same number as before and C'' a constant independent of p, τ and v_0.

§7. Singularities on the boundary

In this section we study the propagation of singularities of solutions to (5.16) for $m_1 = 1$ and (5.21) for $m_1 \geqq 2$. From now on we shall fix the parameter τ in (4.4) and choose a positive number δ_4 in such a way that (6.6), (6.9), (6.11) and (6.12) hold for $v_0 \in L^2(R^n)$ with supp $\hat{v}_0(\eta') \subset \{|\alpha| < 2\delta_4\}$ and that (6.10) holds for $v_0 \in L^2(R^n)$ with supp $\hat{v}_0(\eta') \subset \{2|\eta''|^{-\varepsilon} < \alpha < 2\delta_4\}$. For convenience we rewrite (6.7), (6.8) as

$$(7.1) \qquad \|\gamma v_0\|_s \leqq C_1\|\gamma^{-1}\widetilde{\mathcal{B}} v_0\|_s + C_s\|\gamma^{-1}v\|_{s-1}$$

for $s \in R^1$ and $\gamma^{-1}v_0 \in H^s(R^n)$ with supp $\hat{v}_0(\eta') \subset \{|\alpha| < 2\delta_4\}$, in particular,

$$(7.2) \qquad \|v_0\|_{s-1/3} \leqq C_1\|\widetilde{\mathcal{B}} v_0\|_{s+1/3} + C_s\|v_0\|_{s-2/3}$$

if $v_0 \in H^{s+1/3}(R^n)$. The analogous estimates for $\mathcal{B}$ also follows from (6.11). Moreover (6.10) implies that

$$(7.3) \qquad \|\gamma^2 p v_0\|_s \leqq C_0(\|p\mathcal{B} v_0\|_s + \|pL\chi_\varepsilon\mathcal{B} v_0\|_s) + C_{p,s}\|\gamma^2 v_0\|_{s-\varepsilon_0}$$

for such $p \in OPS^0_{1,0}$, $s \in R^1$ and $v_0 \in H^s(R^n)$ with supp $\hat{v}_0(\eta')$ $\subset \{2|\eta''|^{-\varepsilon} < \alpha < 2\delta_4\}$, where C_0 is independent of p, s. One can also assume without loss of generality that

$$(7.4) \qquad WF(f_0) \subset \{|\alpha| < \delta_4\}.$$

For the existence of solutions we have

Proposition 7.1. Assume (6.6) and (6.9) hold. Suppose $f_0 \in H^{s+1/3}(R^n)$ for some $s \in R^1$ and (7.4) holds. Then there is a solution $v_0 \in H^{s-1/3}(R^n)$ of $\tilde{\mathcal{B}} v_0 = f_0 \bmod H^\infty(R^n)$ such that $WF(v_0) \subset \{|\alpha| < \delta_4\}$, supp $\hat{v}_0(\eta') \subset \{|\alpha| < 2\delta_4\}$ and

$$\|\gamma v_0\|_s \leq C_1\|\gamma^{-1}f_0\|_s + C_s\|\gamma^{-1}v_0\|_{s-1},$$

where $\tilde{\mathcal{B}}$ is the modified operator, given by (5.31). Besides, the solution is unique mod $H^\infty(R^n)$ in the set $\{v_0 \in H^{-\infty}(R^n);$ supp $\hat{v}_0(\eta') \subset \{|\alpha| < 2\delta_4\}\}$. The analogous result with $\tilde{\mathcal{B}}$ replaced by $\mathcal{B}$ is also valid, provided (6.11) and (6.12) are satisfied instead of (6.6) and (6.9).

For the proof see that of [//], Proposition 6.1.

To examine the singularities of the solution we divide it into two parts, as in the preceding paper. Set

$$\tilde{\chi}_\varepsilon(\eta') = \chi(\alpha|\eta''|^\varepsilon - 1)$$

with the χ in (4.8), so that $\tilde{\chi}_\varepsilon \in S^0_{1-\varepsilon,0}$ and $\chi_\varepsilon = 1$ on supp $\tilde{\chi}_\varepsilon$.

Proposition 7.2. Let $v_{01} \in H^{-\infty}(R^n)$ be a solution of $\mathcal{B} v_{01} = \tilde{\chi}_\varepsilon f_0 \bmod H^\infty(R^n)$ such that supp $\hat{v}_{01}(\eta') \subset \{|\alpha| < 2\delta_4\}$. Suppose (7.1) with $\tilde{\mathcal{B}}$ replaced by $\mathcal{B}$ holds. Then

$$v_{01} = \chi_2 v_{01}, \bmod H^\infty(R^n), \tag{7.5}$$

for any cutoff function $\chi_2(\eta') \in S^0_{1-\varepsilon,0}$, like χ_ε, such that $\chi_2 = 1$ on supp $\tilde{\chi}_\varepsilon$. Furthermore assume (5.29) and (7.3) hold.

Then

$$(7.6) \qquad v_{01} \in H^{\infty}(R^n \cap \{y_0 < \bar{y}_0 - \delta_1\})$$

and

$$(7.7) \qquad WF(v_{01}) \cap \hat{N}_0 \subset \hat{M}_0^+(f_0),$$

where

$$\hat{N}_0 = \{(y',\eta') \in T^*R^n\backslash 0;\ \alpha = 0\},$$

$$\hat{M}_0^+(f_0) = \{(y_0 + t,\ y'',\ \eta') \in \hat{N}_0;\ (y',\eta') \in WF(f_0),\ t \geqq 0\}.$$

<u>Proof</u>. The proofs of (7.5) and (7.6) are analogous to those of [11], (6.12) and (6.14), respectively. To prove (7.7) let $(\tilde{y}',\tilde{\eta}')$ be a point in $\hat{N}_0$ such that $(\tilde{y}',\tilde{\eta}') \notin \hat{M}_0^+(\tilde{\chi}_\varepsilon f_0)$ and set $\tilde{y}' = (\tilde{y}_0,\ \tilde{y}'')$. Then it suffices to show that $(y_0,\ \tilde{y}'',\ \tilde{\eta}') \notin WF(v_{01})$ for $\bar{y}_0 - \delta_1 \leqq y_0 \leqq \tilde{y}_0$.

As in the proof of [11], Proposition 6.4, one can find a sequence of cutoff functions $p_k(y',\ \eta') \in S^0_{1,0}$ $(k = 1,2,\cdots)$ such that $p_k\tilde{\chi}_\varepsilon f_0 \in H^\infty(R^n)$, $p_k(y_0,\ \tilde{y}'',\ \tilde{\eta}'/|\tilde{\eta}'|) = 1$ for $y_0 \leqq \tilde{y}_0$, $p_{k+1}(y',\ \eta') = 1$ on supp p_k, $0 \leqq p_k \leqq 1$ and $p_k \circ \phi_2(y',\ \eta') \leqq p_k(y',\ \eta')$ for $\alpha > 0$. Moreover Lemma 4.7 yields that, mod $H^\infty(R^n)$, $p_k \mathcal{B} p_{k+1} v_{01} = p_k \mathcal{B} v_{01}$ and $p_k L\chi_\varepsilon \mathcal{B} p_{k+1} v_{01} = p_k L\chi_\varepsilon \mathcal{B} v_{01}$. Therefore by virtue of (7.3) we have $\gamma^2 p_1 v_{01} \in H^{s+N\varepsilon_0}(R^n)$ for each positive integer N provided $\gamma^2 v_{01} \in H^s(R^n)$, and hence we deduce that $p_1 v_{01} \in H^\infty(R^n)$. Thus we conclude that $(y_0,\ \tilde{y}'',\ \tilde{\eta}') \notin WF(v_{01})$ for $y_0 \leqq \tilde{y}_0$, which completes the proof.

<u>Proposition 7.3</u>. Let $v_{01} \in H^{-\infty}(R^n)$ be a solution of $\mathcal{B} v_{01} = \tilde{\chi}_\varepsilon f_0 \bmod H^\infty(R^n)$ satisfying (7.5) and (7.6). Suppose $(6.1)_0$ or $(6.2)_0$ holds in the case of $m_1 = 1$ or $m_1 \geqq 2$, respectively. Then

$$(7.8) \qquad WF(v_{01}) \cap \hat{N}_+ \subset \cup_{k=0}^{\infty} \phi_2^k (WF(f_0) \cap \hat{N}_+),$$

where ϕ_2 is the canonical transformation given by (4.10) and

$$\hat{N}_+ = \{(y', \eta') \in T^*R^n \backslash 0;\ \alpha > 0\}.$$

<u>Proof</u>. Since the case of $m_1 = 1$ can be handled analogously to [//], Proposition 6.5, we assume $m_1 \geqq 2$. Then (5.22), (4.6) and (7.5) imply

$$cv_{01} + dL\chi_\varepsilon v_{01} = \tilde{\chi}_\varepsilon f_0, \bmod H^\infty(R^n),$$

where

$$c = (a + K_+ I_{m_1})\chi_\varepsilon + (1 - \chi_\varepsilon) I_{m_1},$$

$$d = a + K_- I_{m_1}.$$

Therefore, in view of the proof of the proposition cited above, it suffices to show that there is a left parametrix $c^{-1} \in S^{\varepsilon/2}_{1-(3/2)\varepsilon, \varepsilon/2}$ for c.

Setting, as (5.23),

$$c = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

we see from (5.26) and Lemma 4.12 that $c_{22} \in S^0_{1-\varepsilon,0}$ is elliptic. Let $c_{22}^{-1} \in S^0_{1-\varepsilon,0}$ be a parametrix for c_{22}. Then we have

$$\begin{bmatrix} 1 & -c_{12}c_{22}^{-1} \\ 0 & c_{22}^{-1} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} p & 0 \\ c_{22}^{-1}c_{21} & I_{m_1-1} \end{bmatrix}, \text{ mod } OPS^{-\infty},$$

where

$$p = c_{11} - c_{12}c_{22}^{-1}c_{21}.$$

Moreover $(6.2)_0$ and (5.25) imply that there is a parametrix $p^{-1} \in S^{\varepsilon/2}_{1-(3/2)\varepsilon,\varepsilon/2}$ for p (see the proof of [//], Proposition 6.5). Therefore, setting

$$c^{-1} = \begin{bmatrix} p^{-1} & 0 \\ -c_{22}^{-1}c_{21}p^{-1} & I_{m_1-1} \end{bmatrix} \begin{bmatrix} 1 & -c_{12}c_{22}^{-1} \\ 0 & c_{22}^{-1} \end{bmatrix},$$

we have $c^{-1}c = I_{m_1}$ mod $OPS^{-\infty}$. The proof is complete.

For the purpose of studying the propagation of singularities of the solution v_0 to $\mathcal{B}\, v_0 = (1 - \tilde{\chi}_\varepsilon) f_0$, we first take a solution v_{02} of the modified equation $\widetilde{\mathcal{B}}\, v_{02} = (1 - \tilde{\chi}_\varepsilon) f_0$ and then show that $WF(v_{02}) \subset \{y_0 > \bar{y}_0 - 2\delta_1\}$ hence (5.31) yields $\widetilde{\mathcal{B}}\, v_{02} = \mathcal{B}\, v_{02} \bmod H^\infty(R^n)$.

<u>Proposition 7.4</u>. Let $v_{02} \in H^{-\infty}(R^n)$ be a solution of $\widetilde{\mathcal{B}}\, v_{02} = (1 - \tilde{\chi}_\varepsilon) f_0 \bmod H^\infty(R^n)$ such that $\operatorname{supp} \hat{v}_{02}(\eta') \subset \{|\alpha| < 2\delta_4\}$. Suppose (7.1) holds. Then

$$v_{02} = \chi_3 v_{02}, \bmod H^\infty(R^n), \tag{7.9}$$

for any cutoff function $\chi_3 \in S^0_{1-\varepsilon,0}$, like $1 - \chi_\varepsilon$, such that $\chi_3(\eta') = 1 - \chi(\alpha|\eta''|^\varepsilon - t)$ with $2 < t < 3$, so $\chi_3(\eta') = 0$ for $\alpha|\eta''|^\varepsilon > 2 + t$ and $\chi_3(\eta') = 1$ for $\alpha|\eta''|^\varepsilon < 1 + t$. In particular $WF(v_{02}) \cap \hat{N}_+ = \phi$. Moreover

$$WF(v_{02}) \cap \hat{N}_- \subset WF(f_0), \tag{7.10}$$

$$WF(v_{02}) \cap \hat{N}_0 \subset \hat{M}_0(f_0), \tag{7.11}$$

where

$$\hat{N}_- = \{(y', \eta') \in T^*R^n \backslash 0;\ \alpha < 0\},$$

$$\hat{M}_0(f_0) = \{(y_0 + t, y'', \eta') \in \hat{N}_0;\ (y', \eta') \in WF(f_0),\ -\infty < t < \infty\}.$$

<u>Proof</u>. The proof of (7.9) is analogous to that of (7.5). Besides, (7.10) is a direct consequence of (7.1). To prove (7.11) let $(\tilde{y}_0, \tilde{y}'', \tilde{\eta}') \in \hat{N}_0$ be a point such that $(\tilde{y}_0, \tilde{y}'', \tilde{\eta}') \notin \hat{M}_0(f_0)$. Then $(y_0, \tilde{y}'', \tilde{\eta}') \notin WF(f_0)$ for all $y_0 \in R^1$. Moreover one can

find a conic neighborhood Σ of $(\tilde{y}'', \tilde{\eta}')$ such that $(y_0, y'', \eta') \notin WF(f_0)$ for all $y_0 \in R^1$ and $(y'', \eta') \in \Sigma$. Let $p(y'', \eta') \in S^0_{1,0}$ be a symbol, independent of y_0, such that supp $p \subset \Sigma$. Then $p\tilde{\mathcal{B}} v_{02} \in H^\infty(R^n)$ and it follows from (4.7), Lemmas 4.1 and 4.2 that the commutators $[p, \chi_1]$, $[p, \chi_3]$ and $\gamma^{-1}[p, \gamma\ ]$ belong to $OPS^{-1}_{1/3,0}$, and the symbols of $[p, K_+\chi_1]$, $[p, K_-]$ and $[p, K_0(1 - \chi_1)]$ are $O(\gamma|\eta''|^{-1})$. Furthermore, by Lemma 4.6, the symbol of $[p, L\chi_1\chi_3]$ is $O(\gamma^2|\eta''|^{-\varepsilon/2})$. Therefore by (7.1) we deduce that $\gamma p v_{02} \in H^{s+\varepsilon/2}(R^n)$ if $\gamma v_{02} \in H^s(R^n)$. Thus we obtain (7.11).

<u>Lemma 7.5</u>. Let $v_{02} \in H^{-\infty}(R^n)$ be a solution of $\tilde{\mathcal{B}} v_{02} = (1 - \tilde{\chi}_\varepsilon) f_0$, mod $H^\infty(R^n)$, satisfying (7.9), (7.11) and supp $\hat{v}_{02}(\eta') \subset \{|\alpha| < 2\delta_4\}$. Suppose (5.29) and (7.1) hold. Then

$$v_{02} \in H^\infty(R^n \cap \{y_0 < \bar{y}_0 - 3\delta_1\}). \tag{7.12}$$

<u>Proof</u>. We first show that v_{02} is smooth for $y_0 << \bar{y}_0$, more precisely,

$$v_{02} \in H^\infty(R^n \cap \{y_0 < \bar{y}_0 - 5\delta_1\}). \tag{7.13}$$

Let $p(y_0) \in C^\infty(R^1)$ be a cutoff function such that $p(y_0) = 1$ for $y_0 < \bar{y}_0 - 5\delta_1 + \delta$ and $p(y_0) = 0$ for $y_0 > \bar{y}_0 - 5\delta_1 + 2\delta$ with small $\delta > 0$. Then from (5.30) and (5.31) we have $p\tilde{\mathcal{B}} = p(1 + L\chi_1)$. Hence we obtain (7.13) similarly to [//], Lemma 6.8.

Now, the proof of (7.12) is also similar to that of Lemma 6.9 in the paper cited above. Note that by virtue of (7.1) and (7.9) one can assume

$$\text{(7.14)} \qquad \operatorname{supp}\,(\hat{v}_{02})(\eta') \subset \{|\alpha| < 5|\eta''|^{-\varepsilon}\},$$

because $\widetilde{\mathcal{B}}\chi_{-2} \in OPS^{0}_{1-\varepsilon,0}$ and $[\chi_{-2}, \widetilde{\mathcal{B}}] \in OPS^{-1+\varepsilon}_{1-\varepsilon,0}$ if $\chi_{-2}(\eta') = \chi(-\alpha|\eta''|^{\varepsilon})$. Let $p_0(y_0) \in C_0^{\infty}(R^n)$ be a cutoff function, supported in a small neighborhood of $[\bar{y}_0 - 5\delta_1, \bar{y}_0 - 3\delta_1]$, such that $p = 1$ on a smaller one. Then we have

$$\widetilde{\mathcal{B}}\, p_0 v_{02} = g$$

with

$$g = p_0(1 - \tilde{\chi}_{\varepsilon}) f_0 - [p_0, \widetilde{\mathcal{B}}] v_{02},$$

so that $g \in H^{\infty}(R^n \cap \{y_0 < \bar{y}_0 - 3\delta_1\})$ and (5.31) together with (5.30) implies

$$\widetilde{\mathcal{B}}\, p_0 v_{02} = (1 + L\chi_1) p_0 v_{02} + \tilde{b}\mathcal{L} p_0 v_{02}, \bmod H^{\infty}(R^n).$$

Moreover (4.16) yields

$$p\Lambda_t \widetilde{\mathcal{B}}\, p_0 v_{02} = \widetilde{\mathcal{B}}_t \Lambda_t p_0 v_{02}, \bmod H^{\infty}(R^n),$$

for any cutoff function $p(y_0) \in C_0^{\infty}(R^1)$ such that $p = 1$ on $\operatorname{supp} p_0$. Here

$$\widetilde{\mathcal{B}}_t = (1 + L_t \chi_{1t}) + \tilde{b}_t \mathcal{L}_t,$$

where L_t, χ_{1t} and $\mathcal{L}_t$ are the Fourier multipliers defined by (4.13), (4.14) and (4.15), respectively, and $\tilde{b}_t = \tilde{b} \bmod OPS_{1,0}^{-1+\delta}$ for any $\delta > 0$. Therefore, applying $p\Lambda^s\Lambda_t$ to each side of $\widetilde{\mathcal{B}}\, p_0 v_{02} = g$ and setting $v_t = \Lambda_t p_0 v_{02}$, we arrive at

$$\widetilde{\mathcal{B}}_{t,s}\Lambda^s v_t = p\Lambda^s\Lambda_t g, \bmod H^\infty(R^n), \tag{7.15}$$

where Λ^s is the pseudodifferential operator with symbol $\langle\eta''\rangle^s$ and

$$\widetilde{\mathcal{B}}_{t,s} = (1 + L_t\chi_{1t}) + (\tilde{b}_t + [\Lambda^s, \tilde{b}_t]\Lambda^{-s})\mathcal{L}_t.$$

Furthermore by virtue of Lemma 4.14 and so on we obtain

$$\mathrm{Re}\ (\widetilde{\mathcal{B}}_{t,s}\Lambda^s v_t, \gamma\Lambda^s v_t) \geqq C_1 \| \tau_t^{1/2}\gamma\Lambda^s v_t \|^2 - O(\| \tau_t\Lambda^s v_t \|^2_{-1/2})$$

for $t \geqq 1$ and $s \in R^1$, where C_1 is a positive constant independent of t,s. Thus one can deduce that $p_0 v_{02} \in H^\infty(R^n \cap \{y_0 < \bar{y}_0 - 3\delta_1\})$, which yields (7.12).

We are now in a position to prove an analogue to (7.7).

<u>Proposition 7.6</u>. Let $v_{02} \in H^{-\infty}(R^n)$ be a solution of $\widetilde{\mathcal{B}} v_{02} = (1 - \tilde{\chi}_\varepsilon) f_0$ with (5.29) such that supp $\hat{v}_{02}(\eta') \subset \{|\alpha| < 2\delta_4\}$. Assume the hypotheses of Proposition 6.1 are satisfied. Then

$$(7.16) \qquad WF(v_{02}) \cap \hat{N}_0 \subset \hat{M}_0^+(f_0).$$

In particular, $v_{02} \in H^\infty(R^n \cap \{y_0 < \bar{y}_0 - \delta_1\})$ so that $\widetilde{\mathcal{B}} v_{02} = \mathcal{B} v_{02}$ mod $H^\infty(R^n)$.

<u>Proof</u>. Let $(\tilde{y}', \tilde{\eta}') \in \hat{N}_0$ be a point such that $(\tilde{y}', \tilde{\eta}') \notin \hat{M}_0^+(f_0)$. Let $q(y'', \eta') \in S^0_{1,0}$ be a symbol independent of y_0, supported in a small conic neighborhood of $(\tilde{y}'', \tilde{\eta}')$, such that $q(y'', \eta') = 1$ on a smaller one. Let $w_1, w_2 \in H^{-\infty}(R^n)$ be solutions of $\widetilde{\mathcal{B}} w_1 = q(1 - \tilde{\chi}_\varepsilon) f_0$, $\widetilde{\mathcal{B}} w_2 = (1 - q)(1 - \tilde{\chi}_\varepsilon) f_0$ such that $\hat{w}_j(\eta') \subset \{|\alpha| < 2\delta_4\}$, $j = 1,2$, respectively. Then we have $(\tilde{y}', \tilde{\eta}') \notin WF(w_2)$ analogously to (7.11). Thus it suffices to prove $(\tilde{y}', \tilde{\eta}') \notin WF(w_1)$. To do it one can assume that w_1 satisfies (7.12) and (7.14), so (5.30) and (5.31) imply

$$\widetilde{\mathcal{B}} w_1 = (\tilde{a}(1 + L\chi_1) + b\mathcal{L}) w_1.$$

Moreover, since $(y_0, \tilde{y}'', \tilde{\eta}') \notin WF(f_0)$ for $y_0 \leqq \tilde{y}_0$, there is a positive number T such that

$$q(1 - \tilde{\chi}_\varepsilon) f_0 \in H^\infty(R^n \cap \{y_0 < \tilde{y}_0 + 2T\})$$

provided supp $q(y'', \eta')$ is sufficiently small.

Let $p_0(y_0) \in C_0^\infty(R^1)$ be a cutoff function, supported in a small neighborhood of $[\bar{y}_0 - 3\delta_1, \tilde{y}_0 + 2T]$, such that $p_0 = 1$ on a smaller one. Then, setting $v_t = \Lambda_t p_0 w_1$, we have an analogue to (7.15):

$$(7.17) \qquad \tilde{\mathcal{B}}_{t,s}\Lambda^s v_t = p\Lambda^s \Lambda_t g, \text{ mod } H^\infty(R^n),$$

for any cutoff function $p(y_0) \in C_0^\infty(R^1)$ such that $p = 1$ on supp p_0, where

$$g = p_0 q(1 - \tilde{\chi}_\varepsilon) f_0 - [p_0, \tilde{\mathcal{B}}] w_1,$$

so that $g \in H^\infty(R^n \cap \{y_0 < \tilde{y}_0 + 2T\})$. Here

$$\tilde{\mathcal{B}}_{t,s} = (\tilde{a}_t + [\Lambda^s, \tilde{a}_t]\Lambda^{-s})(1 + L_t\chi_{1t}) + (b_t + [\Lambda^s, b_t]\Lambda^{-s})\mathcal{L}_t,$$

$$\tilde{a}_t(y', \eta') = \tilde{a}(y', \eta') + (\partial_{\eta_0}\tilde{a}(y', \eta'))(-it\log\langle\eta''\rangle), \text{ mod } S^{-1}_{1,0}$$

and b_t also has the property analogous to $\tilde{a}_t$. Furhtermore one can derive

$$(7.18) \qquad \|\tau_t^{1/2}\gamma\Lambda^s v_t\| \leqq C_0\|\tau_t^{-1/2}\gamma^{-1}\tilde{\mathcal{B}}_{t,s}\Lambda^s v_t\| + O(\|\tau_t^{3/2}\gamma^{-1}\Lambda^s v_t\|_{-1})$$

for $t \geqq 1$ and $s \in R^1$, where C_0 is a constant independent of t,s. To prove this one can assume

$$\tilde{\mathcal{B}}_{t,s} = \tilde{a}(1 + L_t\chi_{1t}) + b\mathcal{L}_t.$$

Therefore, analogously to (6.6), we obtain

$$\mathrm{Re}\,(\widetilde{\mathcal{B}}_{t,s}\Lambda^s v_t,\ S_t\Lambda^s v_t) \geqq C_1\|\tau_t^{1/2}\gamma\Lambda^s v_t\|^2 - O(\|\tau_t\Lambda^s v_t\|^2_{-1/2})$$

for $t \geqq 1$ and $s \in R^1$, where C_1 is a positive constant independent of t, s. Here S_t is the operator defined by (6.3) with $q = 1$, in which τ, χ_1, L and $\mathcal{L}$ are replaced by τ_t, χ_{1t}, L_t and $\mathcal{L}_t$, respectively. Moreover in the proof we use Lemmas 4.14 and 4.15 in place of Lemmas 4.8 and 4.10, respectively. Thus we obtain (7.18).

Note that (7.17) and (7.18) imply

$$\|\Lambda^{s-1/3}v_t\| \leqq C_0\|p\Lambda^{s+1/3}\Lambda_t g\| + O(\|\Lambda^{s-2/3+\delta}v_t\|)$$

for any $\delta > 0$, say, $\delta = 1/6$. Hence we have

$$\|\Lambda^{s-1/3}v_t\| \leqq 2C_0\|p\Lambda^{s+1/3}\Lambda_t g\| + C_{t,s}\|\Lambda^{s-\beta t}v_t\|$$

for each $t \geqq 1$ and $s \in R^1$, where $\beta = \tilde{y}_0 + T - (\bar{y}_0 - 3\delta_1)$. Now suppose $w_1 \in H^{\bar{s}}(R^n)$ for some $\bar{s} \in R^1$. Then for each $t \geqq 1$, taking $s = \bar{s} - 1/3 + (\tilde{y}_0 + T)t$, we see that $\|\Lambda^{s-1/3}v_t\|$ is finite, because w_1 satisfies (7.12). Consequently we have

$$p_0 w_1 \in H^\infty(R^n \cap \{y_0 < \tilde{y}_0 + T\}),$$

in particular, $(\tilde{y}_0, \tilde{y}'', \tilde{\eta}') \notin WF(w_1)$. Since $v_{02} = w_1 + w_2$ mod $H^\infty(R^n)$, we thus prove the proposition.

Now, we shall return to (5.3), restricted to the case of (1.10). Set $f_\varepsilon = \Phi_1 \tilde{\chi}_\varepsilon \Phi_1^{-1} f$, where $\tilde{\chi}_\varepsilon$ is the cutoff function described above Proposition 7.2. We also define v_{h1}, v_{e1} by (5.11) with v_0 and f replaced by v_{01} and f_ε, respectively, and v_{h2}, v_{e2} analogously with v_0 and f replaced by v_{02} and $f - f_\varepsilon$, respectively. Finally set

$$(7.19) \qquad v^{(j)} = \begin{bmatrix} v_{0j} \\ v_{hj} \\ v_{ej} \end{bmatrix}, \; j = 1,2.$$

Then from the results of this section we have easily

<u>Proposition 7.7</u>. Suppose the hypotheses of Proposition 7.2 and 7.3 are fulfilled. Then (7.5), (7.7) and (7.8) hold with v_{01} and f_0 replaved by $v^{(1)}$ and $\Phi_1^{-1} f$, respectively. In particular, $WF(v^{(1)}) \cap \hat{N}_- = \phi$.

<u>Proposition 7.8</u>. Suppose the hypotheses of Proposition 7.4 and 7.6 are fulfilled. Then (7.9), (7.10) and (7.16) hold with v_{02} nad f_0 replaced by $v^{(2)}$ and $\Phi_1^{-1} f$, respectively. In particular, $WF(v^{(2)}) \cap \hat{N}_+ = \phi$.

From these we obtain

Corollary 7.9. Suppose the hypotheses of Theorem 1.1 are satisfied. Let $v^{(1)}, v^{(2)}$ be given by (7.19) and set $v = v^{(1)} + v^{(2)}$. Then

$$BGv|_{x_n=0} - f \in C^{\infty}(X'),$$

where G is the operator defined by (3.9), (3.10) and (3.29). Moreover for $x_n = 0$ we have $WF(Gv^{(1)}) \cap N_- = \phi$, $WF(Gv^{(2)}) \cap N_+ = \phi$, $WF(Gv^{(2)}) \cap N_- \subset WF(f)$, $WF(Gv^{(j)}) \cap N_0 \subset M_0^+(f)$ for $j = 1,2$ and

$$WF(Gv^{(1)}) \cap N_+ \subset \cup_{k=0}^{\infty} \phi_+^k (WF(f) \cap N_+).$$

The proof is analogous to that of [11], Corollary 6.10.

Remark 7.10. Suppose (1.9) holds and let $\mathcal{B}$, $\tilde{\mathcal{B}}$ be as Remark 6.11. Then the existence of solutions to $\tilde{\mathcal{B}} v_0 = f_0$ follows as usual from (6.36). Moreover the conclusions of Propositions 7.2, 7.3, 7.4 and 7.6 are still valid, because of (6.37) and (6.38). Hence those of Propositions 7.7, 7.8 and Corollary 7.9 are also true.

§8. Proof of Theorem 1.1

In this section we will complete the proof of Theorem 1.1, by showing that $E(f) = Gv$ has the required properties. Here Gv is the same function as in Corollary 7.9.

Let δ_n, $V \subset X'$ and $\Sigma \subset R^n \backslash 0$ be, respectively, a positive number, a neighborhood of $\bar{x}'$ and a conic neighborhood of $\bar{\eta}'$ such that, for $(x', x_n) \in V \times [0, \delta_n)$ and $\eta' \in \Sigma$, the phase functions $\theta(x, \eta')$, $\rho(x, \eta')$ and the amplitudes $a(x, \eta')$, $b(x, \eta')$ have the properties stated in §3. In what follows we take δ_n, V and Σ small if necessary, and often write $U_n = V \times [0, \delta_n)$. Then by virtue of (3.4) one can assume

$$\rho(x, \eta') \leqq (\alpha - c_n x_n)|\eta''|^{2/3} \text{ for } (x, \eta') \in U_n \times \Sigma, \tag{8.1}$$

where c_n is a positive number.

Now, that $Gv \in H^{\infty,-\infty}_{loc}(U_n)$ is a consequence of the following.

<u>Proposition 8.1</u>. Suppose $v^{(1)}$, $v^{(2)} \in H^{\bar{s}}(R^n)$ with some $\bar{s} \in R^1$. Then $D_n^k Gv^{(j)} \in H^{0,s_k}_{loc}(U_n)$ for each integer $k \geqq 0$ and $j = 1,2$, where s_k is the minimum of 0 and $3(\bar{s} - k) - 2n - 2$.

<u>Proof</u>. From (7.5) and (7.7) through (7.11) we see analogously to the proof of [11], Proposition 7.4 that $D_n^k G_0 v_{0j} \in H^{0,s_k}_{loc}(U_n)$. One can also show as usual that $D_n^k G_h v_{hj}$, $D_n^k G_e v_{ej} \in H^{0,\bar{s}-k}_{loc}(U_n)$. (See e.g. Chap. 10, §2 of Kumano-go [14] in the references of [11]). The proof is complete.

On the singular support of Gv we have

<u>Proposition 8.2</u>. There are compact sets K_1, $K_2 \subset X$ and a positive number T such that $K_j \cap \{x_0 < T\} \subset U_n$ and $Gv^{(j)} \in C^\infty(U_n \backslash K_j)$, $j = 1,2$.

<u>Proof</u>. Set

$$K_{02} = \{(x',0) \in X;\ x' \in \text{sing supp } \Phi_1 v_{02}\}.$$

Then from Propositions 7.4 and 7.6 we see analogously to [11], Proposition 7.6 that $G_0 v_{02} \in C^\infty(U_n \backslash K_{02})$ provided δ_1 and δ_4 are small. One can also deduce as usual that $G_e v_{e2}$ is smooth outside $\{(x',0) \in X;\ x' \in \text{sing supp } \Phi_1 v_{e2}\}$ and that $G_h v_{h2}$ is smooth except the union of all outgoing bicharacteristic curves of $(Q/Q_0)(x,\xi)$ starting from $\iota^{*-1}(WF(\Phi_1 v_{h2}))$. Therefore it follows from Proposition 7.8 that there is a compact set $K_2 \subset X$ which has the required properties provided $T > 0$ is small. Similarly one can find such a compact set $K_1 \subset X$. Thus we prove the proposition.

That (1.22) holds is a consequence of the following.

<u>Proposition 8.3</u>. $PGv^{(j)} \in C^\infty(U_n)$, $j = 1,2$.

The proof is analogous to that of [11], Proposition 7.5, if we use Lemma 3.2.

<u>End of Proof of Theorem 1.1</u>. By virtue of Proposition 8.2 one can cut off $Gv^{(j)}$ outside $K_1 \cup K_2$ modulo smooth errors and extend them to X_T. Then $Gv = Gv^{(1)} + Gv^{(2)}$ satisfies (1.20), (1.21) and (1.22). Moreover (1.23) is a direct consequence of Corollary 7.9. To derive the last conclusion of the theorem, let $(\hat{x}',\hat{\xi}') \in T^*X'\backslash 0$ be a point, with $\hat{x}_0 < T$, which does not belong to the set (1.24). First suppose $\tilde{Q}(\hat{x}',0,\hat{\xi}') \neq 0$. Then it follows from Proposition 2.4 and Corollary 7.9 that Gv is smooth up to the boundary at $(\hat{x}',\hat{\xi}')$. Next suppose $\tilde{Q}(\hat{x}',0,\hat{\xi}') = 0$. In virtue of Proposition 8.2 one can also assume $\hat{x}'$ is close to $\bar{x}'$. Then we see that $\hat{\xi}'$ is far away from $\theta_{x'}(\hat{x}',0,\hat{\xi}')$ and hence from $\psi_{kx'}(x,\eta')$ provided $x = (x',x_n)$ is close to $(\hat{x}',0)$ and $(\theta_{\eta'}(x',0,\eta'), \eta') \in WF(v_0)$. Here $\psi_k(x,\eta')$ are the phase functions in [11], Lemma 7.2. Therefore by a standard integration by parts method we find that G_0v_0 is smooth up to the boundary at $(\hat{x}',\hat{\xi}')$. (See e.g. Hörmander [5], §1.2). Similarly it follows that so are G_hv_h and G_ev_e. In view of Remark 7.10 we thus prove the theorem.

§9. Proofs of Theorems 1.2 and 1.3

The proof of Theorem 1.2 is similar to that of [11], Theorem 1.2, if we use, instead of the Dirichlet problem, the following auxiliary mixed problem

$$
\begin{aligned}
&P(x, D)u = 0 \quad \text{in } X, \\
(9.1)\qquad &B_0(x)u = g \quad \text{on } \partial X, \\
&u = 0 \quad \text{in } X \cap \{x_0 << 0\}.
\end{aligned}
$$

Here $B_0(x)$ is the $d^+ \times m$ matrix defined by (1.27) and (1.28) with $S(x) = 0$. Note that the boundary condition $B_0 u|_{x_n=0} = 0$ is strictly dissipative, namely, it is not only maximally nonpositive for P but also

$$A_n(x)u \cdot u < 0 \quad \text{for } u \in \ker B_0(x) \backslash 0 \text{ and } x \in \partial X.$$

Therefore (1.9) with $B = B_0$ holds (see Georgiev [2]).

<u>Proof of Theorem 1.2</u>. Let $E(f)$ be as in Theorem 1.1 and set $w = u - E(f)$. Then

$$(9.2)\qquad w \in H^{0,s'}_{loc}(X_T) \quad \text{for some } s' \in R^1,$$

$$(9.3)\qquad w \in C^\infty(X \cap \{x_0 << 0\}),$$

$$(9.4)\qquad Pw \in C^\infty(X_T)$$

and

$$(9.5)\qquad Bw|_{x_n=0} \in C^\infty(X'_T).$$

In a special case where the boundary condition $Bw|_{x_n=0} = 0$ is maximally dissipative, we see by virtue of Proposition 2.5 that $w \in C^\infty(X_T)$. Thus it suffices to prove

$$(9.6) \qquad w|_{x_n=0} \in C^\infty(X'_T)$$

so that $B_0 w|_{x_n=0} \in C^\infty(X'_T)$.

In virtue of Theorem 1.1 applied to (9.1) with $g = B_0 w|_{x_n=0}$, there is a parametrix $Gv \in H^{\infty,-\infty}_{loc}(X_{T_0}) \cap C^\infty(X \cap \{x_0 << 0\})$ with some $T_0 > 0$ such that $PGv \in C^\infty(X_{T_0})$ and $B_0 Gv|_{x_n=0} - g \in C^\infty(X'_{T_0})$. We also see from Remark 7.10 that $v \in H^{-\infty}(R^n)$ and

$$(9.7) \qquad WF(v) \subset WF(\Phi_1^{-1}g) \cup \hat{M}_0^+(\Phi_1^{-1}g) \cup (\cup_{k=1}^\infty \phi_2^k(WF(\Phi_1^{-1}g) \cap \hat{N}_+)).$$

For simplicity we rewrite the minimum of T and T_0 as T. Then (9.2), (9.3) and (9.4) yield that $Gv - w \in H^{0,s}_{loc}(X_T) \cap C^\infty(X \cap \{x_0 << 0\})$ for some $s \in R^1$, $P(Gv - w) \in C^\infty(X_T)$ and $B_0(Gv - w)|_{x_n=0} \in C^\infty(X'_T)$. Therefore by virtue of Proposition 2.5 we have

$$(9.8) \qquad Gv - w \in C^\infty(X_T).$$

Hence we need only to prove

$$(9.9) \qquad Gv|_{x_n=0} \in C^\infty(X'_T),$$

which is analogous to [11], (8.6).

Set

$$f = BGv|_{x_n=0}.$$

Then (9.5) and (9.8) imply $f \in C^\infty(X_T')$. Moreover by virtue of (9.7) one can assume $v(y') \in H^\infty(R^n \cap \{y_0 < \bar{y}_0 - \delta_1\})$, where δ_1 is the positive number in (5.29). We also have $\mathcal{B}v_0 = f_0$, where $\mathcal{B}$ is an operator defined by one of (5.10), (5.17) and (5.22), and f_0 also defined as before. Therefore similarly to Corollary 7.9 we obtain $Gv|_{x_n=0} - f \in C^\infty(X_T')$ which yields (9.9). Thus we prove the theorem.

We shall proceed to the proof of Theorem 1.3. Since $WF(u|_{x_n=0})$ is a closed set in $T^*X'\backslash 0$, the theorem is a consequence of the following together with Proposition 2.4.

Proposition 9.1. Suppose the hypotheses of Theorem 1.3 are fulfilled. Let $(\hat{x}', \hat{\xi}') \in \Gamma(\bar{x}', \bar{\xi}')$ be a point such that $-\delta < \hat{x}_0 \leqq 0$. Assume $WF(u|_{x_n=0}) \cap \Gamma(\bar{x}', \bar{\xi}') \cap \{\hat{x}_0 - \delta' < x_0 < \hat{x}_0\} = \phi$ for some δ' with $0 < \delta' \leqq \hat{x}_0 + \delta$. Then $(\hat{x}', \hat{\xi}') \notin WF(u|_{x_n=0})$.

Proof. One can assume without loss of genelality that $(\hat{x}', \hat{\xi}') = (\bar{x}', \bar{\xi}')$ and $\delta' = \delta$, taking δ small. We shall reduce the proposition to Theorem 1.2. Since $\Gamma(\bar{x}', \bar{\xi}')$ can be parametrized by x_0, we write a point on it as $(x_0, x''(x_0), \xi'(x_0))$.

Let $\psi_1(x', D') \in OPS^0_{1,0}$ be a pseudodifferential operator, with symbol $\psi_1(x', \xi')$ homogeneous in ξ' for $|\xi'| > 1$, such that supp $\psi_1(x', \xi')$ is contained in a small conic neighborhood of $\Gamma(\bar{x}, \bar{\xi}') \cap \{-\delta/3 \leqq x_0 \leqq 0\}$, $\psi_1(x', \xi') = 1$ on a smaller one, and $\psi_1(x', \xi') = 0$ when $|x_0| + |x'' - x''(x_0)| \geqq 2\delta/3$. Then it suffices to prove

$$(9.10) \qquad WF(\psi_1 u|_{x_n=0}) \cap \Gamma(\bar{x}', \bar{\xi}') \cap \{x_0 \leqq 0\} = \phi.$$

Set

(9.11) $\Gamma_1 = \Gamma(\bar{x}', \bar{\xi}') \cap \{-2\delta/3 \leqq x_0 \leqq -\delta/3\}$.

By assumption we have $WF(u|_{x_n=0}) \cap \Gamma_1 = \phi$. Therefore by virtue of Proposition 2.4 there are a positive number ε_1 and a pseudo-differential operator $\psi_2(x', D') \in OPS^0_{1,0}$, with symbol $\psi_2(x', \xi')$ homogeneous in ξ' for $|\xi'| > 1$, such that

(9.12) $\psi_2 u \in C^\infty(X \cap \{0 \leqq x_n < \varepsilon_1\})$

and $\psi_2(x', \xi') = 1$ on a small conic neighborhood of Γ_1. Let $\varepsilon > 0$ and $\chi_\varepsilon(x_n) \in C_0^\infty(R^1)$ be a cutoff function, supported in $-\varepsilon < x_n < \varepsilon$, such that $\chi_\varepsilon(x_n) = 1$ for $|x_n| < \varepsilon/2$. Then from (9.12) we have

(9.13) $P\chi_\varepsilon\psi_1 u = [P, \chi_\varepsilon\psi_1](1 - \psi_2)u, \mod C^\infty(X)$

for $0 < \varepsilon < \varepsilon_1/2$. From now on we suppose $0 < \varepsilon < \varepsilon_1/2$ and δ, ε_1 are small. For convenience we set

$$P'(x, D') = P(x, D) - A_n(x)D_n,$$

which does not contain D_n. Then we have

$$[P, \chi_\varepsilon\psi_1] = A_n(D_n\chi_\varepsilon)\psi_1 + \chi_\varepsilon([A_n, \psi_1]D_n + [P', \psi_1]).$$

Now let F_1 be the zero extension of $A_n(D_n\chi_\varepsilon)\psi_1(1 - \psi_2)u$ to $x_n < 0$. Note that $F_1 \in H^{\infty,-\infty}(R^{n+1})$ and $F_1(x) = 0$ when $x_n < \varepsilon/2$ or $x_0 < -2\delta/3$. Next set

$$(9.14) \qquad f_2 = ([A_n, \psi_1]D_n + [P', \psi_1])(1 - \psi_2)\chi_{\varepsilon_1}u$$

and let Ef_2 be a Seeley extension of f_2 to $x_n < 0$ (see [19]). Then, since $f_2 \in H^{\infty,-\infty}(X)$, we see that $Ef_2 \in H^{\infty,-\infty}(R^{n+1})$. Finally, setting

$$(9,15) \qquad F_2 = \chi_\varepsilon Ef_2,$$

we have

$$(9.16) \qquad P\chi_\varepsilon\psi_1 u = F_1 + F_2, \text{ mod } C^\infty(X).$$

Let us consider the following Cauchy problem with zero initial data

$$(9.17)_j \qquad \begin{aligned} &Pv = F_j \quad \text{in } R^{n+1},\ j = 1, 2, \\ &v(x) = 0 \quad \text{for } x_0 < -2\delta/3. \end{aligned}$$

Then we obtain

<u>Lemma 9.2</u>. For each $j = 1, 2$, the Cauchy problem $(9.17)_j$ has a solution v_j such that $v_j \in H^{\infty,-\infty}_{loc}(\Omega_T)$ for any $T > 0$, where $\Omega_T = \{x \in R^{n+1};\ x_0 < T\}$.

<u>Proof</u>. For a nonnegative integer k and real numbers s, γ with $\gamma > 0$, we denote by $H_\gamma^{k,s}$ the set of distributions u in R^{n+1} such that $e^{-\gamma x_0}u \in H^{k,s}(R^{n+1})$. Let $j = 1$ or 2, and for simplicity drop the subscripts. Then, since $F \in H^{\infty,-\infty}(R^{n+1})$ and $F(x) = 0$ for $x_0 < -2\delta/3$, there is a decreasing sequence $\{s_k\}_{k=0}^{\infty}$ of real numbers such that $F \in H_\gamma^{k,s_k}$ for any integer $k \geqq 0$ and $\gamma > 0$. Hence, for each integer $k \geqq 0$, there is a positive number γ_k such that (9.17) has a unique solution $v \in H_\gamma^{k,s_k}$ if $\gamma \geqq \gamma_k$. Let $\bar{k}$ be an arbitrary positive integer. Then it follows from the uniqueness of the solution that v is independent of $k = 0, 1, \cdots, \bar{k}$ if γ is large according as $\bar{k}$. Hence $v \in H_{loc}^{\infty,-\infty}(\Omega_T)$ for any $T > 0$. Thus we prove the lemma.

Now let v_j, $j = 1, 2$, be such solutions of $(9.17)_j$ as in Lemma 9.2. For small $T > 0$ we take a cutoff function $\tilde{\chi}_T(x_0) \in C^\infty(R^1)$, supported in $x_0 < T$, such that $\tilde{\chi}_T(x_0) = 1$ for $x_0 < T/2$, and set

$$(9.18) \qquad w = \chi_\varepsilon \psi_1 u - \tilde{\chi}_T(v_1 + v_2)|_X.$$

We shall prove

$$(9.19) \qquad (\bar{x}', \bar{\xi}') \notin WF(w|_{x_n=0}).$$

To this end we use Theorem 1.2. Clearly $w \in H^{\infty,-\infty}_{loc}(X_T) \cap C^\infty(X \cap \{x_0 << 0\})$. Moreover (9.16) yields $Pw \in C^\infty(X_{T/2})$. Set

$$f = Bw|_{x_n=0}.$$

If we show that

$$WF(f) \cap \Gamma_2(\bar{x}',\bar{\xi}') = \phi,$$

where

$$\Gamma_2 = \Gamma(\bar{x}',\bar{\xi}') \cap \{-2\delta/3 \leqq x_0 \leqq 0\},$$

and that $WF(w|_{x_n=0})$ is contained in a small conic neighborhood of $(\bar{x}',\bar{\xi}')$, then (9.19) follows. Thus it suffices to prove the following two lemmas.

<u>Lemma 9.3</u>. There is a positive number T such that $WF(\tilde{\chi}_T(v_1 + v_2)|_{x_n=0})$ is contained in a small conic neighborhood of $(\bar{x}',\bar{\xi}')$.

<u>Lemma 9.4</u>. $WF(v_j|_{x_n=0}) \cap \Gamma_2 = \phi$ for $j = 1,2$.

<u>Proof of Lemma 9.3</u>. Set $v = v_1 + v_2$, $F = F_1 + F_2$ and $V_T = (\text{sing supp } v) \cap \{x_0 \leqq T\})$. Then, since $WF(F)$ is contained in a small conic neighborhood of $\iota^{*-1}(\bar{x}', \bar{\xi}')$, we see that V_T is a compact subset of R^{n+1} for each $T > 0$ and it is small if so are δ, ε and T. In what follows we fix T small and restrict ourselves to a small neighborhood of V_T in R^{n+1}.

Let $\phi_1(x', D') \in OPS^0_{1,0}$ be a pseudodifferential operator such that supp $\phi_1(x', \xi')$ is away from the union of supp $\psi_1(x', \xi')$ and the set

$$\Sigma_0 = \{(x', \xi') \in T^*X'\backslash 0;\ \tilde{Q}(x', 0, \xi') = 0 \text{ for some } x_n\}.$$

Then it follows from Lemma 2.3 and results on propagation of singularities in the free space that $WF(\phi_1 v)$ is contained in the union of null bichacteristics through $WF(F)$. Bearing this in mind, we take pseudodifferential operators $\phi_2(x', D')$, $\phi_3(x', D') \in OPS^0_{1,0}$ such that supp $\phi_2(x', \xi')$ and supp $\phi_3(x', \xi')$ are contained, respectively, in small conic neighborhoods of supp $\psi_1(x', \xi')$ and Σ_0, and $\phi_j(x', \xi') = 1$ on smaller ones, $j = 2, 3$. Divide v as $v = v^{(1)} + v^{(2)} + v^{(3)}$, where $v^{(1)} = (1 - \phi_3)(1 - \phi_2)v$, $v^{(2)} = \phi_2 v$ and $v^{(3)} = \phi_3(1 - \phi_2)v$. Then we see from the above observation and [5], Theorem 2.5.11' that $WF(\tilde{\chi}_T v^{(1)}|_{x_n=0})$ is contained in a small conic neighborhood of $(\bar{x}', \bar{\xi}')$ provided δ, ε and T are small, Clearly so is $WF(v^{(2)}|_{x_n=0})$ also. Finally we shall show that $v^{(3)} \in C^\infty(R^{n+1}) \cap \{x_0 < T\})$ for some $T > 0$. Since supp $\phi_3(x', \xi')$ can be assumed to be far away from supp $\phi_2(x', \xi')$, we have $v^{(3)} = \phi_3 v$ hence

$$Pv^{(3)} = \phi_3 Pv + [P, \phi_3]v, \text{ mod } C^\infty(R^{n+1}).$$

Note that $\phi_3 Pv = \phi_3 F \in C^\infty(R^{n+1})$. Moreover

$$[P, \phi_3]v = [p, \phi_3]\phi_1 v, \text{ mod } C^\infty(R^{n+1}),$$

for such $\phi_1(x', D')$ as described above. Therefore we see that $[P, \phi_3]v \in C^\infty(R^{n+1} \cap \{x_0 < T\})$ hence so is $Pv^{(3)}$. Thus $v^{(3)} \in C^\infty(R^{n+1} \cap \{x_0 < T\})$ for small $T > 0$. The proof is complete.

<u>Proof of Lemma 9.4</u>. Let $(\hat{x}', \hat{\xi}') \in \Gamma_2$ and set $\hat{x} = (\hat{x}', 0)$. Then it suffices to show that

$$(9.20)_j \qquad (\hat{x}, \xi', \xi_n) \notin WF(q_j v_j)$$

for $(\xi', \xi_n) = (0, 1)$ and some pseudodifferential operators $q_j(x', D') \in OPS^0_{1,0}$ which are elliptic at $(\hat{x}', \hat{\xi}')$, and

$$(9.21)_j \qquad (\hat{x}, \hat{\xi}', \hat{\xi}_n) \notin WF(v_j)$$

for each root $\hat{\xi}_n$ of $Q(\hat{x}, \hat{\xi}', \xi_n) = 0$. (See [5], Theorem 2.5.11').

Since $\tilde{Q}(x', 0, \xi') \neq 0$ for $(x', \xi') \in \Gamma_2$ and $F_1(x) = 0$ (for $x_n < \varepsilon/2$,) by virtue of Lemma 2.3 there is a pseudodifferential operator $q_1(x', D')$, elliptic on Γ_2, such that $(9.20)_1$ holds. Next, since

$$(9.22) \qquad \Gamma_2 \cap (\text{supp grad } \psi_1) \cap (\text{supp } 1 - \psi_2) = \phi,$$

it follows from (9.14) that $\psi_3 f_2 \in C^\infty(X)$ for some $\psi_3(x', D') \in OPS^0_{1,0}$, elliptic on Γ_2. Hence (9.15) yields $\psi_3 F_2 \in C^\infty(R^{n+1})$, because ψ_3 and the extension operator E commute. Thus, as above, there is a pseudodifferential operator $q_2(x', D')$, elliptic on Γ_2, such that $(9.20)_2$ holds.

Now we shall prove $(9.21)_j$. Let $\xi_n^+(x, \xi')$ or $\xi_n^-(x, \xi')$ be one of simple real roots of $(Q/Q_0)(x, \xi', \xi_n) = 0$ such that $\partial\xi_n^+/\partial\xi_0 < 0$ or $\partial\xi_n^-/\partial\xi_0 > 0$, respectively. If $\hat{\xi}_n = \xi_n^+(\hat{x}, \hat{\xi}')$ or $\hat{\xi}_n = \lambda(\hat{x}, \hat{\xi}')$, then $(9.21)_1$ follows from $(9.17)_1$, because $WF(F_1)$ and the bicharacteristics of $Q(x, \xi)$ through $(\hat{x}, \hat{\xi}', \hat{\xi}_n)$ do not intersect for $x_n \leqq 0$. Let $\hat{\xi}_n = \xi_n^-(\hat{x}, \hat{\xi}')$. If $\hat{x}_0 < 0$, then $(\hat{x}', \hat{\xi}') \notin WF(u|_{x_n=0})$ by assumption hence it follows from Proposition 2.4 that $WF(u)$ does not intersect the bicharacteristics through $(\hat{x}, \hat{\xi}', \hat{\xi}_n)$ for small $x_n > 0$. The same is true by assumption also when $\hat{x}_0 = 0$. Thus $WF(F_1)$ and the bicharacteristics do not intersect for $x_n \geqq 0$ provided ε is small. Hence we obtain $(9.21)_1$.

Next we shall prove $(9.21)_2$. Let $\hat{\xi}_n = \xi_n^+(x, \hat{\xi}')$ or $\hat{\xi}_n = \xi_n^-(\hat{x}, \hat{\xi}')$. Then the projection on T^*X' of a segment, with $|x_n| << 1$, of the bicharacteristics of $Q(x, \xi)$ through $(\hat{x}, \hat{\xi}', \hat{\xi}_n)$ is contained in a small conic neighborhood of Γ_2. Therefore, by (9.15) and (9.22), there is a positive number ε_2, independent of $(\hat{x}', \hat{\xi}') \in \Gamma_2$, such that $WF(F_2)$ does not intersect the bicharacteristics if $0 < \varepsilon < \varepsilon_2$. Thus $(9.21)_2$ follows.

Finally let $\hat{\xi}_n = \lambda(\hat{x}', \hat{\xi}')$. Denote by $(\hat{x}(t), \hat{\xi}(t))$ the bicharacteristics of $-Q_0(x, \xi)$ through $(\hat{x}, \hat{\xi}', \hat{\xi}_n)$, where the parameter t is taken in such a way that $dx_0(t)/dt > 0$ and $(\hat{x}(0), \hat{\xi}(0)) = (\hat{x}, \hat{\xi}', \hat{\xi}_n)$. Let t_0 be a small positive number such that

$$(\hat{x}'(t), \hat{\xi}'(t)) \notin (\text{supp grad } \psi_1) \cap (\text{supp } 1 - \psi_2)$$

for $-t_0 < t \leqq 0$ and $(\hat{x}', \hat{\xi}') \in \Gamma_2$. Then, analogously to the proof of [//], Lemma 8.3, we find a positive number ε_3, independent of $(\hat{x}', \hat{\xi}') \in \Gamma_2$, such that if $-\varepsilon_3 < \hat{x}_n(t) < 0$ and $t < 0$ then $-t_0 < t$. Let $0 < \varepsilon < \varepsilon_3$. Then $(\hat{x}(t), \hat{\xi}(t)) \notin WF(F_2)$ for $t \leqq 0$ and $(\hat{x}', \hat{\xi}') \in \Gamma_2$. Hence $(9.21)_2$ follows. Thus, taking $\varepsilon < \varepsilon_2$ and $\varepsilon < \varepsilon_3$, we complete the proof.

<u>End of proof of Proposition 9.1</u>. It follows from (9.18), (9.19) and Lemma 9.4 that $(\bar{x}', \bar{\xi}') \notin WF(\psi_1 u|_{x_n=0})$. Since ψ_1 is elliptic at $(\bar{x}', \bar{\xi}')$, we have $(\bar{x}', \bar{\xi}') \notin WF(u|_{x_n=0})$. The proof is complete.

<u>Proof of Theorem 1.3</u>. Assume the hypotheses of the theorem are fulfilled. If $(\bar{x}', \bar{\xi}') \notin WF(u|_{x_n=0})$, we see from Proposition 2.4 that u is smooth up to the boundary at $(\bar{x}', \bar{\xi}')$, in particular,

$$(\bar{x}', \bar{\xi}') \notin \cup_{k=0}^{\infty} WF(D_n^k u|_{x_n=0}).$$

If there is a point $(\hat{x}', \hat{\xi}') \in \Gamma(\bar{x}', \bar{\xi}') \cap \{-\delta < x_0 < 0\}$ such that $(\hat{x}', \hat{\xi}') \notin WF(u|_{x_n=0})$, we observe from Proposition 9.1 that $(\bar{x}', \bar{\xi}') \notin WF(u|_{x_n=0})$. Thus we prove the theorem.

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