Trade Patterns and Perpetual Youth in A Dynamic Small Open Economy∗

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In this paper, I examine the long-run specialization patterns that arise in a small open economy using a two-sector growth model in which households faced with finite but uncertain lifetimes undertake intertemporal optimization decisions. I show that in a small open economy with a positive birth rate, imperfect specialization requires that the subjective discount rate be less than the interest rate. This finding contrasts sharply with Stiglitz’s (1970) finding that a country must be completely specialized if the subjective discount rate differs from the interest rate. In addition, I show that a steady state equilibrium with incomplete specialization can be saddle-point stable.

JEL Classification: E13, F11, F41

Keywords: Two-Sector Growth Model, Small Open Economy, Perpetual Youth Model

1. Introduction

The purpose of this paper is to examine the long-run trade patterns that arise for a small country by employing the two-sector Blanchard (1985) perpetual youth model. Using a dynamic Heckscher-Ohlin model that extends Oniki and Uzawa (1965), Stiglitz (1970) establishes the well-known result that a small open country is perfectly specialized in the capital (labor)-intensive sector if it is more (less) patient. Under this framework, when a more patient country is imperfectly specialized, it faces the fact that the interest rate, which depends on world commodity prices, is larger than the subjective discount rate. This gap induces the country to accumulate capital stock. As the capital stock increases, the country becomes capital abundant and eventually specializes completely in the capital-intensive sector. Thus, a country is incompletely specialized in the long-run only when the country’s subjective discount rate is initially equal to the interest rate. Since a country’s subjective discount rate is equal to the interest rate only by chance, most countries must be completely specialized.

However, most countries are incompletely specialized. One method for solving this contradiction is to introduce endogenous time preferences. Karasawa and Yanase (2009), which extend Stiglitz’s (1970) model, study the trade patterns in a small open economy by employing endogenous time preferences that depend

∗ I would like thank Professor Itaya for helpful comments. I am also grateful to Yoshimasa Aoki, Daisuke Amano and Koichiro Sano for discussions. Comments from an anonymous referee helped me to improve the paper. Of course, all remaining errors are mine.

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both on the level of individual consumption and on the level of average consumption. They show that a country can be incompletely specialized in the long-run even if the interest rate is not equal to the country’s subjective discount rate as an increase in the capital stock changes the country’s subjective discount rate through a change in the level of consumption\(^1\).

Another method is to use a perpetual youth model. Blanchard (1985) assumes that all individuals face a positive probability of death. Since they face an uncertain lifetimes, their assets are operated by a life insurance company in the annuity markets. He examines the steady state level of capital in a small open economy with one sector and shows that even if the country is less patient, the steady state level of capital may be positive. This result occurs because aggregate savings depend both on the ratio of consumption to capital and on the gap between the interest rate and the subjective discount rate in the perpetual youth model. Recently, Hamada, Iwasa and Kikuchi (2009) employ the Blanchard (1985) perpetual youth model and analyze trade patterns in dynamic two large country model where each country has a different discount rate. They show that even with different subjective discount rates, both countries can be incompletely specialized in the long-run. Note that they assume that capital is perfectly mobile and that the countries produce a tradable good and a non-tradable good.

In this paper, I assume that capital is immobile between countries and the economy can produce two tradable goods, such as in Stiglitz (1970), and focus on a small open economy. I extend Blanchard (1985) to examine the long-run trade patterns that arise for a small country in a two-sector perpetual youth model.

I show that with a positive birth rate imperfect specialization can occur for a small country in the long-run only if the subjective discount rate is \textit{less than} the interest rate. This result stems from the fact that aggregate savings depend on the ratio of aggregate non-human wealth to aggregate consumption in the perpetual youth model. This finding contrasts sharply with Stiglitz’s (1970) finding that if the subjective discount rate differs from the interest rate, the country must be completely specialized as aggregate savings depend solely on the difference between the interest rate and the subjective discount rate in an infinite horizon model. Furthermore, I find that a steady state equilibrium with incompletely specialization can be saddle-point stable.

The paper is organized as follows. In Section 2, I consider the optimization problems of firms and households. In Section 3, I analyze the long-run trade patterns and present the contributions of this paper. Section 4 concludes. In Appendix, I prove the two Propositions.

\section*{2. The model}
\subsection*{2.1 Firms}
I consider a small open economy. Following Oniki and Uzawa (1965) and Stiglitz (1970), the countries produce two goods, a pure investment good (denoted by \(I\)) and a pure consumption good (denoted by \(C\)) which are produced using

\footnote{1) Nishimura and Shimomura (2002) present the similar result by using endogeneous time preferences.}
capital and labor. I assume that the factors (i.e., capital and labor) are immobile between countries. However, I suppose that both commodities are traded freely. I denote \( p(t) \) as the relative world price for the consumption good in terms of the investment good and assume that this world price is constant over time, i.e., \( p(t) = p \) at all times. To simplify my model, I assume that the depreciation rate of capital is zero.

The production function in sector \( i \) \((i = I, C)\) is \( Y^i = F^i(K_i(t), L_i(t)) \), where \( K_i(t) \) is the capital used in sector \( i \) and \( L_i(t) \) is the labor employed in sector \( i \) at time \( t \). This function satisfies positive and diminishing marginal products for each input, constant-returns-to-scale, and the Inada conditions. Thus, I can rewrite the production function in sector \( i \) as

\[
Y^i = F^i(K_i(t), L_i(t)) = f^i(k_i(t)L_i(t)),
\]

where \( k_i(t) = K_i(t)/L_i(t) \) is the ratio of capital to labor in sector \( i \) and \( f^i(k_i(t)) \) is defined to equal \( F^i(k_i(t),1) \).

If both goods are produced in the country, I can derive the following equations from the profit maximization of firms:

\[
\begin{align*}
    r(t) &= f^i_K(k_i(t)) = pf^C_K(k_c(t)), \\
    w(t) &= f^i_L(k_i(t)) = pf^C_L(k_c(t)),
\end{align*}
\]

where \( f^i_j(k_i(t)) \), \( r(t) \), and \( w(t) \) are the marginal product of input \( j \) \((j = K, L)\) in sector \( i \), the rental rate on capital, and the wage rate, respectively. Note that when the country is incompletely specialized, all four variables, \( k_i(t), k_c(t), r(t), w(t) \), depend solely on the world price, \( p \). The corresponding value of variable \( x \) is denoted by \( \hat{x}(p) \). In addition, I assume that the investment goods sector is more capital intensive than the consumption goods sector for all prices of the consumption good, that is, \( \hat{k}_i(p) > \hat{k}_c(p) \) for all \( p \).

I consider the value of national income per capita, \( y \), which is given by

\[
y(p, K(t), L(t)) = \max_{K_i(t), L_i(t), i, c} \left[ \frac{F^i(K_i(t), L_i(t)) + pf^C(K_c(t), L_c(t))}{L(t)} \right],
\]

s.t. \( K(t) \geq K_i(t) + K_c(t), \) \( L(t) \geq L_i(t) + L_c(t) \),

where \( K(t) \) and \( L(t) \) are aggregate capital and aggregate labor, respectively. If the country produces only the investment good, \( y \) is \( F^i(k(t)) \), while if the country produces only the consumption good, \( y \) is \( pf^C(k(t)) \), where \( k(t) \) represents the aggregate capital stock per capita. If the country produces both goods, \( y \) is given by
\[
\left[ \frac{k(t) - \hat{k}_C(p)}{k(t) - \hat{k}_C(p)} \right] f'(\hat{k}_i(p)) + \left[ \frac{\hat{k}_i(p) - k(t)}{k(t) - \hat{k}_C(p)} \right] pf^{C}(\hat{k}_C(p)).
\]

With \( \hat{k}_i(p) > \hat{k}_C(p) \), the country is incompletely specialized if \( \hat{k}_C(p) < k(t) < \hat{k}_i(p) \). If \( k(t) \leq \hat{k}_C(p) \), the country is specialized in sector C, while if \( \hat{k}_i(p) \leq k(t) \), the country is specialized in sector I.

Finally, the marginal product of capital (denoted by \( y_k \)) and the slope of the marginal product of capital (denoted by \( y_{kk} \)) are

\[
y_k = \begin{cases} 
pf_k^C(k(t)) & \text{if } k(t) \leq \hat{k}_C(p) \\
\hat{r}(p) & \text{if } \hat{k}_C(p) < k(t) < \hat{k}_i(p), \\
\hat{f}_k^i(k(t)) & \text{if } \hat{k}_i(p) \leq k(t)
\end{cases}
\]

and

\[
y_{kk} = \begin{cases} 
0 & \text{if } \hat{k}_C(p) < k(t) < \hat{k}_i(p), \\
\hat{f}_k^{ii}(k(t)) & \text{if } \hat{k}_i(p) \leq k(t)
\end{cases}
\]

where \( f_{kk}^{ii} \) is the second derivative with respect to capital in sector \( i \).

### 2.2 Households

There is a large number of identical consumers faced with finite but uncertain lifetimes. Each cohort is born at a constant rate \( \beta \) and faces a constant probability of death \( \lambda \). The total population at time \( t \) is \( L(t) \) and the size of the cohort born at time \( t \) is \( \beta L(t) \). Thus, a cohort born at time \( s \) has a size of \( \beta L(s) e^{-\lambda(t-s)} = \beta e^{-\lambda t} e^{\beta s} \) at time \( t \), where \( L(0) = 1 \). The total population is obtained by integrating over the survivors of each cohort born at time \( t \):

\[
L(t) = \beta e^{-\lambda t} \int_{-\infty}^{t} e^{\beta s} ds = e^{nt},
\]

where \( n = \beta - \lambda > 0 \) is the growth rate of the total population.

Each consumer maximizes the following expected lifetime utility at time \( t \):

\[
\int_{t}^{\infty} \log \left[ c_i(s, \nu) \right] e^{-(\rho + \lambda)(\nu - t)} d\nu,
\]

where \( c_i(s, \nu) \) and \( \rho \) are individual consumption at time \( \nu \) by a consumer born at time \( s \) and the subjective discount rate, respectively. I assume that the subjective discount rate is larger than the growth rate of the total population, i.e., \( \rho > n \).

Each consumer supplies one unit of labor inelastically and receives the wage \( w(t) \) at time \( t \). She has zero non-human wealth when she is born, i.e., \( a_i(s, s) = 0 \). When she has \( a_i(s, \nu) \) at time \( s < \nu \), she receives \( r(\nu)a_i(s, \nu) \). I follow Yaari (1965) and Blanchard (1985) in assuming the existence of actuarially
fair annuity markets. Each consumer will contract with an insurance company which gives her a premium of \(\lambda a_i(s,\nu)\) at each point in time, as long as she lives.

The flow budget constraint for each consumer born at time \(s\) is

\[
\dot{a}_i(s,\nu) = \left[r(\nu) + \lambda\right]a_i(s,\nu) + w(\nu) - pc_i(s,\nu).
\]

The transversality condition is

\[
\lim_{\nu \to \infty} \left[a_i(s,\nu)e^{-\left[\hat{r}(\nu)+\lambda\right](\nu-t)}\right] = 0,
\]
where \(\hat{r}(t,\nu) = \int_t^\nu [r(\tau)] \, d\tau\) is the average interest rate between times \(t\) and \(\nu\). Integrating (3) and using (4), I can obtain the intertemporal budget constraint:

\[
p\int_t^\infty c_i(s,\nu) \, e^{-\left[\hat{r}(\nu)+\lambda\right](\nu-t)} \, d\nu = a_i(s,t) + \tilde{w}(t),
\]
where \(\tilde{w}(t) = \int_t^\infty w(\nu) \, e^{-\left[\hat{r}(\nu)+\lambda\right](\nu-t)} \, d\nu\) is the present value of wage income.

Each consumer maximizes expected utility (2), subject to the flow budget constraint (3), initial endowments \(a_i(s,s) = 0\), and the transversality condition (4), taking the time paths of market prices (prices of the consumption good, wages, and interest rates) as given. Applying the maximal principle, I can obtain the following individual Euler equation:

\[
\dot{c}_i(s,\nu) = \left[r(\nu) - \rho\right]c_i(s,\nu).
\]

Integrating (6) forward and using the intertemporal budget constraint (5), I can obtain the individual’s consumption function:

\[
c_i(s,t) = \frac{\lambda + \rho}{p}\left[a_i(s,t) + \tilde{w}(t)\right].
\]

Aggregate consumption at time \(t\), \(C(t)\), is obtained by integrating individual consumption in each cohort, weighted by the number of consumers alive in each cohort at time \(t\):

\[
C(t) = \beta e^{-\beta t} \int_{-\infty}^t c_i(s,t) e^{\beta s} \, ds.
\]

Similarly, aggregate nonhuman wealth \(A(t)\) and aggregate wage income \(W(t)\) at time \(t\) are given by:

\[
A(t) = \beta e^{-\beta t} \int_{-\infty}^t a_i(s,t) e^{\beta s} \, ds,
\]

\[
W(t) = \beta e^{-\beta t} \int_{-\infty}^t \tilde{w}(t) e^{\beta s} \, ds.
\]
Using the above equations, I can obtain the following aggregate consumption function:

\[ C(t) = \left( \frac{\lambda + \rho}{p} \right) \left[ A(t) + W(t) \right]. \] (9)

Differentiating equations (7) and (8), the dynamics of aggregate nonhuman wealth \( A(t) \) and that of aggregate wage income \( W(t) \) can be written as

\[ \dot{A}(t) = r(t)A(t) + w(t)e^{nt} - pC(t), \] (10)

\[ \dot{W}(t) = \{r(t) + n + \lambda\}W(t) - w(t)e^{nt}, \] (11)

respectively. Using (9), (10) and (11), I can obtain the growth rate of aggregate consumption:

\[ \frac{\dot{C}(t)}{C(t)} = [r(t) + n - \rho] - \frac{(\lambda + \rho)(\lambda + n)A(t)}{pC(t)}. \]

Finally, I can show that the growth rate of aggregate consumption per capita is

\[ \frac{\dot{c}(t)}{c(t)} = [r(t) - \rho] - \frac{(\lambda + \rho)(\lambda + n)a(t)}{pc(t)}, \] (12)

where \( c(t) \) and \( a(t) \) are aggregate consumption per capita and aggregate nonhuman wealth per capita, respectively. This equation can be interpreted as the aggregate Euler equation. The difference between the aggregate Euler equation (12) and the individual Euler equation (6) is the term \( \frac{(\lambda + \rho)(\lambda + n)a(t)}{pc(t)} \). This term appears due to the presence of the birth rate (i.e., \( \lambda + n = \beta > 0 \)). Since newborns with zero assets enter the economy every period, the growth rate of aggregate consumption per capita is smaller than that of individual consumption. Hence, the growth rate of aggregate savings per capita differs from that of individual savings. Borrowing the terminology used by Heijdra (2009) I denote the term \( \frac{(\lambda + \rho)(\lambda + n)a(t)}{pc(t)} \) as the distributional effect. If the birth rate is zero, the aggregate Euler equation is equal to the individual Euler equation and thus the distributional effect is zero.

3. Long-run Specialization Patterns

In this section, I consider the specialization patterns that arise in a steady state. Since capital is internationally immobile, non-human wealth per person is equal to the capital stock per person, i.e., \( a(t) = k(t) \). The dynamics of per capita capital can be written as

\[ \dot{k}(t) = y(t) - pc(t) - nk(t). \] (13)
Using equation (12), \( r(t) = y_k(t) \) and \( a(t) = k(t) \), I can obtain the dynamics of per capita consumption as follows:

\[
\dot{c}(t) = (y_k(t) - \rho)c(t) - \frac{(\lambda + \rho)(\lambda + n)k(t)}{p}.
\]

(14)

In a steady state \( \dot{c} = \dot{k} = 0 \). Combining (13) with (14) yields

\[
y_k^* = \rho + \frac{(\lambda + \rho)(\lambda + n)k^*}{y^* - nk^*},
\]

(15)

where \( y^* \), \( k^* \), and \( y_k^* \) are the steady state levels of per capita income, per capita capital, and the marginal product of capital, respectively. Following Karasawa and Yanase (2009), I can interpret the above equation as the long-run capital market clearing condition. The left-hand side of (15) implies the long-run capital demand function as firms demand a level of capital that equates the marginal product of capital with the interest rate. The right-hand side of (15) represents the long-run capital supply function as the supply of capital depends on the sum of the discount rate, \( \rho \), and the distributional effect, \( (\lambda + \rho)(\lambda + n)k^* / (y^* - nk^*) \).

Figure 1 provides an illustration of (15) following Stiglitz (1970). The curve corresponding to the left-hand side of (15) is downward sloping if \( \hat{p}_k \leq \hat{c}_k \) or \( \hat{c}_k(p) \leq k \), and is flat if \( \hat{k}_c(p) < k < \hat{c}_k(p) \). The curve corresponding to the right-hand side of (15) is unambiguously upward sloping (see Appendix A).

I have the following Proposition.

Proposition 1: Given the world price \( p \), when the birth rate is positive (i.e., \( \lambda + n = \beta > 0 \)), the country is incompletely specialized in the long-run if and only if...
\[
\ddot{\rho}(p) - \frac{(\lambda + \rho)(\lambda + n)\dot{k}_c(p)}{pf(k_c((p))) - nk_c(p)} > \rho > \dot{\rho}(p) - \frac{(\lambda + \rho)(\lambda + n)\dot{k}_j(p)}{f(k_j((p))) - nk_j(p)},
\]  

(16)

is satisfied. When the birth rate is zero (i.e., \(\lambda + n = \beta = 0\)), the country is incompletely specialized in the long-run if and only if the subjective discount rate is equal to the interest rate (i.e., \(\rho = \dot{\rho}(p)\)).

**Proof:** See Appendix A.■

Consider the intuition behind Proposition 1. Suppose that the ratio of capital to labor in the country, \(k\), is \(\dot{k}_c(p)\). This country produces only a consumption good. If equation (16) is satisfied, per capita capital increases as the consumers find that the interest rate is larger than the subjective discount rate and are willing to accumulate more non-human wealth. However, since newborns with zero assets enter the economy every period, the growth rate of aggregate savings declines. This effect corresponds to the distributional effect in Section 2. As per capita capital increases, this distributional effect increases. Hence, eventually the interest rate is equal to the sum of the subjective discount rate and the distributional effect, and the capital stock per person stops rising. As a result, the country is incompletely specialized in the long-run. In Figure 1, the equilibrium is denoted by point A and the ratio of capital to labor in the country is \(k^*_A\).

However, if the birth rate is zero, the distributional effect vanishes. When the subjective discount rate is less than the interest rate, \(\dot{\rho}(p)\) per capita capital increases. Thus, the country eventually specializes in the capital-intensive sector, as pointed out by Stiglitz (1970). In Figure 1, the equilibrium is given by point B and the ratio of capital to labor in the country is \(k^*_B\).

Stiglitz (1970) finds that the country is incompletely specialized only if the subjective discount rate is equal to the interest rate. In contrast, in my model, imperfect specialization of the country requires that the subjective discount rate be less than the interest rate from equation (16). The difference stems from fact that the presence of the birth rate modifies the aggregate Euler equation. In other words, in my model there exists a distributional effect.

The above results are also argued by Hamada, Iwasa and Kikuchi (2009). They employ the Blanchard (1985) perpetual youth model and analyze long-run trade patterns between two large countries where each country has a different subjective discount rate. They show that if the subjective discount rate in the more patient country is smaller than the interest rate, there can exist a unique steady state equilibrium where both countries are incompletely specialized. This result stems from the distributional effect. Note that with the perpetual youth model the distribution effect appears regardless of whether there are two countries or one small open country. Thus, their results are the same as the results derived in this paper\(^2\). However, they assume that capital is perfectly moved and that one of the two goods is a non-tradable good, while I assume that capital is

\(^2\) I thank the referee for this point.
perfectly immobile and both goods are tradable goods and focus on a small open economy.

In addition, Karasawa and Yanase (2009) show that in a small open economy even if the subjective discount rate is different from the interest rate initially, there exists a steady state where the country is incompletely specialized. They do not employ the perpetual youth model but assume endogenous time preferences that rely on both individual consumption and social average consumption. Thus, even if the subjective discount rate is less than the interest rate initially, by modifying the subjective discount rate through the changes in the consumption levels, the country’s subjective discount rate is eventually equal to the interest rate.

Finally, I find that an equilibrium with incomplete specialization can be saddle-point stable. More precisely, I have the following Proposition.

**Proposition 2:** If \((\hat{r}(p) - \rho)(\hat{r}(p) - n) < (\lambda + \rho)(\lambda + n)\) is satisfied, then the long-run equilibrium with incomplete specialization is saddle-point stable.

**Proof:** See Appendix B. ■

When a positive birth rate and the subjective discount rate is less than the interest rate, a long-run equilibrium with incomplete specialization can be saddle-point stable. The result stems from the emergence of the distributional effect in the presence of a positive birth rate.

4. Concluding Remarks

In this paper, I have examined long-run specialization patterns in a small open economy by employing the two-sector Blanchard perpetual youth model. The main contribution of this paper is to clarify how the inclusion of finite lived agents influences the implication of long-run specialization patterns. When the birth rate is positive, imperfect specialization can occur only if the subjective discount rate is less than the interest rate. This result stems from the fact that the individual Euler equation differs from the aggregate Euler equation in the perpetual youth model. On the other hand, Stiglitz (1970) finds that if the subjective discount rate differs from the interest rate, the country is never incompletely specialized. That is, if the subjective discount rate is less than the interest rate, the country must specialize in the capital intensive sector. This result arises because the individual Euler equation has the same form as the aggregate Euler equation in an infinite horizon model. When the birth rate is zero, my model reduces to that of Stiglitz (1970).

Furthermore, I show that the steady state equilibrium with incomplete specialization can be saddle-point stable when the birth rate is positive and the subjective discount rate is less than the interest rate.

The most important and natural extension is to allow for sector-specific or economy wide production externalities, as shown in Benhabib and Farmer (1994). In my paper the steady state equilibrium can be the saddle-point stable. However, the inclusion of externalities would create a source for generating indeterminacy. In addition, the equilibrium could be made indeterminate by introducing the desire
for social status (e.g., Itaya and Kanamori 2010), which induces private agents to care about other’s capital levels or consumption levels. The analysis in this paper is a first step toward a more comprehensive analysis of stabilities in such settings. Finally, it would be interesting to examine stability in a two country model. This extension converts constant commodity prices into variable prices. Hence, the stabilities of the equilibrium in a two country model may be different from those in a small open economy.

References


Appendix A: Proof of Proposition 1

I define the right-hand side of (15) as $\Theta(k)$, that is,

$$\Theta(k) \equiv \rho + \left[ (\lambda + \rho)(\lambda + n)k / (y - nk) \right].$$

Since the per capita wages of workers, $y - y_{k}$, are positive,

$$\frac{\partial \Theta(k)}{\partial k} = \frac{(\lambda + \rho)(\lambda + n)(y - y_{k}k)}{(y - nk)^{2}} > 0.$$

Thus, I find that $\Theta(k)$ is monotonically increasing in the ratio of capital to labor. Note that if the birth rate is zero, $\Theta(k)$ is $\rho$, which is constant.

I have found that from equation (1) the left-hand side of (15) is decreasing in the ratio of capital to labor when $k \leq \hat{k}_{c}(p)$ or $\hat{k}_{i}(p) \leq k$ and is $\hat{r}(p)$ when $\hat{k}_{i}(p) < k < \hat{k}_{c}(p)$.

First, I consider the birth rate is positive. I suppose that there exists a steady state per capita capital $k^{*} \in (\hat{k}_{c}(p), \hat{k}_{i}(p))$ such that $\Theta(k^{*}) = \hat{r}(p)$. Since $\Theta(k)$ is monotonically increasing in the ratio of capital to labor, both $\Theta(\hat{k}_{c}(p)) > \hat{r}(p)$ and $\Theta(\hat{k}_{i}(p)) < \hat{r}(p)$ are satisfied, that is,

$$\hat{r}(p) - \frac{(\lambda + \rho)(\lambda + n)\hat{k}_{c}(p)}{pf(\hat{k}_{c}(p)) - nk_{c}(p)} > \rho > \hat{r}(p) - \frac{(\lambda + \rho)(\lambda + n)\hat{k}_{i}(p)}{f(\hat{k}_{i}(p)) - nk_{i}(p)},$$

is satisfied. Suppose that $\Theta(\hat{k}_{c}(p)) < \hat{r}(p)$ and $\Theta(\hat{k}_{i}(p)) > \hat{r}(p)$. There exists a steady state per capita capital $k^{*} \in (\hat{k}_{c}(p), \hat{k}_{i}(p))$ such that $\Theta(k^{*}) = \hat{r}(p)$ as $\Theta(k)$ is monotonically increasing in the ratio of capital to labor.

Next, I consider the birth rate is zero. I suppose that there exist a per capita capital $k^{*} \in (\hat{k}_{c}(p), \hat{k}_{i}(p))$ such that $\Theta(k^{*}) = \hat{r}(p)$. Since $\Theta(k)$ is $\rho$, $\hat{r}(p) = \rho$ is satisfied. Suppose that $\hat{r}(p) = \rho$, $\Theta(k^{*}) = \hat{r}(p)$ is satisfied as $\Theta(k)$ is $\rho$.

Appendix B: Proof of Proposition 2

I focus on the steady state equilibrium with incomplete specialization. Using equations (13) and (14), I take a linear approximation of the dynamic system around the steady state values of per capita capital, $k^{*}$, and per person consumption, $c^{*}$:

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} \frac{(\lambda + \rho)(\lambda + n)k^{*}}{pc} & [\rho - n] \\ -\frac{(\lambda + \rho)(\lambda + n)}{p} & \frac{(\lambda + \rho)(\lambda + n)k^{*}}{pc} \end{bmatrix} \begin{bmatrix} k - k^{*} \\ c - c^{*} \end{bmatrix}.$$

Note that when the country is incompletely specialized in the steady state, the slope of the marginal product of capital, $y_{k}$, is zero. The determinant of this matrix (denoted by $D$) can be written as
\[ D = (\lambda + \rho)(\lambda + n) \left\{ \frac{k^*}{pc} \left[ (\lambda + \rho)(\lambda + n) \frac{k^*}{pc} + [\rho - n] \right] - 1 \right\}. \]

I can obtain that if
\[
\frac{\tilde{r}(p) - \rho}{(\lambda + \rho)(\lambda + n)} < 1,
\]
then \( D \) is negative.

Next, the trace of this matrix (denoted by \( T \)) can be written as
\[
T = 2(\lambda + \rho)(\lambda + n) \frac{k^*}{pc} + [\rho - n].
\]

Since \( \rho - n > 0 \), \( T \) is always positive. Thus, if equation (A) is satisfied, the steady state equilibrium is saddle-point stable.