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THE FREENESS AND MINIMAL FREE RESOLUTIONS OF MODULES OF DIFFERENTIAL OPERATORS OF A GENERIC HYPERPLANE ARRANGEMENT

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ABSTRACT. Let \mathcal{A} be a generic hyperplane arrangement composed of r hyperplanes in an *n*-dimensional vector space, and S the polynomial ring in *n* variables. We consider the *S*-submodule $D^{(m)}(\mathcal{A})$ of the *n*th Weyl algebra of homogeneous differential operators of order *m* preserving the defining ideal of \mathcal{A} .

We prove that if $n \geq 3, r > n, m > r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free (Holm's conjecture). Combining this with some results by Holm, we see that $D^{(m)}(\mathcal{A})$ is free unless $n \geq 3, r > n, m < r - n + 1$. In the remaining case, we construct a minimal free resolution of $D^{(m)}(\mathcal{A})$ by generalizing Yuzvinsky's construction for m = 1. In addition, we construct a minimal free resolution of the transpose of the *m*-jet module, which generalizes a result by Rose and Terao for m = 1.

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1. INTRODUCTION

In the study of a hyperplane arrangement, its derivation module plays a central character; in particular, its freeness over the polynomial ring attracts a great interest (see, e.g., Orlik-Terao [6]). Generalizing the study of the derivation module for a hyperplane arrangement to that of the modules of differential operators of higher order was initiated by Holm [4], [5]. In particular, he studied the case of generic hyperplane arrangements in detail.

Let K denote a field of characteristic zero, and \mathcal{A} a generic hyperplane arrangement in K^n composed of r hyperplanes. Let S be the polynomial ring $K[x_1, \ldots, x_n]$, and $D^{(m)}(\mathcal{A})$ the S-module of homogeneous differential operators of order m of the hyperplane arrangement \mathcal{A} .

Among others, in [5], Holm gave a finite generating set of the S-module $D^{(m)}(\mathcal{A})$. As to the freeness of $D^{(m)}(\mathcal{A})$, Holm [4] (cf. [9]) proved the following:

- If n = 2, then $D^{(m)}(\mathcal{A})$ is free for any m.
- If $n \ge 3, r > n, m < r n + 1$, then $D^{(m)}(\mathcal{A})$ is not free.
- If $n \ge 3, r > n, m = r n + 1$, then $D^{(m)}(\mathcal{A})$ is free.

Holm also conjectured that if $n \ge 3, r > n, m > r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free.

Snellman [9] computed the Hilbert series of $D^{(m)}(\mathcal{A})$, which supported Holm's conjecture when $n \geq 3, r > n, m > r - n + 1$, and he conjectured the Poicaré-Betti series of $D^{(m)}(\mathcal{A})$ when $n \geq 3, r > n, m < r - n + 1$.

In the derivation module case, when $n \geq 3, r > n, m < r - n + 1$ with m = 1, Rose-Terao [7] and Yuzvinsky [11] independently gave a minimal free resolution of $D^{(1)}(\mathcal{A})$. In the course of the proof, Rose-Terao [7] gave minimal free resolutions of all modules of logarithmic differential forms with poles along \mathcal{A} . They also gave a minimal free resolution of S/J, where J is the Jacobian ideal of a polynomial defining \mathcal{A} . Yuzvinsky's construction [11] is more straightforward and combinatorial than [7].

In this paper, we prove Holm's conjecture, namely, we prove that if $n \ge 3, r > n, m > r-n+1$, then $D^{(m)}(\mathcal{A})$ is free. Hence, for a generic hyperplane arrangement \mathcal{A} , $D^{(m)}(\mathcal{A})$ is free unless $n \ge 3, r > n, m < r-n+1$. In the remaining case $n \ge 3, r > n, m < r-n+1$, we construct a minimal free resolution of $D^{(m)}(\mathcal{A})$ by generalizing [11] and a minimal free resolution of the transpose of the *m*-jet module generalizing that of S/J given by [7].

After we fix notation on differential operators for a hyperplane arrangement in §2, we recall the Saito-Holm criterion in §3. It was proved by Holm, and it is a criterion for a subset of $D^{(m)}(\mathcal{A})$ to form a basis, which generalizes the Saito criterion in the case of m = 1.

From §4 on, we assume that $r \ge n$ and the hyperplane arrangement \mathcal{A} is generic. In §4, we recall the finite generating set of $D^{(m)}(\mathcal{A})$ given by Holm [5]. Then we recall the case n = 2 in §5 and the case m = r - n + 1 in §6 for completeness. In §7, we consider the case $m \ge r - n + 1$ and prove Holm's conjecture (Theorem 7.1).

From §8 on, we consider the case m < r - n + 1. In §8, we give a minimal generating set of $D^{(m)}(\mathcal{A})$ (Theorem 8.3). In §9, we generalize [11] to construct a minimal free resolution of $D^{(m)}(\mathcal{A})$ (Theorem 9.10). In §10, we generalize the minimal free resolution of S/J given in [7] (Theorem 10.7). In §11, we prove that the S-module considered in §10 is the transpose of the *m*-jet module $\Omega^{[1,m]}(S/SQ)$ (Theorem 11.2), where Q is a polynomial defining \mathcal{A} .

2. The Modules of Differential Operators for a Hyperplane Arrangement

Throughout this paper, let K denote a field of characteristic zero, \mathcal{A} a central hyperplane arrangement in K^n composed of r hyperplanes, and S the polynomial ring $K[x_1, \ldots, x_n]$. We assume that $n \geq 2$.

For a hyperplane $H \in \mathcal{A}$, we fix a linear form $p_H \in S$ defining H. Set

(2.1)
$$Q := Q_{\mathcal{A}} := \prod_{H \in \mathcal{A}} p_H$$

Let $D(S) = S\langle \partial_1, \ldots, \partial_n \rangle$ denote the *n*th Weyl algebra, where $\partial_j = \frac{\partial}{\partial x_j}$. For a nonzero differential operator $P = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha}(x) \partial^{\alpha} \in D(S)$, the maximum of $|\alpha|$ with $f_{\alpha} \neq 0$ is called the *order* of P, where

$$\partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \qquad |\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$$

for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$. If *P* has no nonzero $f_{\boldsymbol{\alpha}}$ with $|\boldsymbol{\alpha}| \neq m$, it is said to be homogeneous of order *m*. We denote by $D^{(m)}(S)$ the *S*-submodule of D(S) of differential operators homogeneous of order *m*.

We denote by * the action of D(S) on S. For an ideal I of S,

(2.2)
$$D(I) := \{ \theta \in D(S) \mid \theta * I \subseteq I \}$$

is called the *idealizer* of I.

We set

(2.3)
$$D(\mathcal{A}) := D(\langle Q \rangle).$$

Holm [5, Theorem 2.4] proved

(2.4)
$$D(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} D(\langle p_H \rangle).$$

We denote by $D^{(m)}(\mathcal{A})$ the S-submodule of $D(\mathcal{A})$ of differential operators homogeneous of order m. Then Holm [5, Proposition 4.3] proved

$$D(\mathcal{A}) = \bigoplus_{m=0}^{\infty} D^{(m)}(\mathcal{A}).$$

A differential operator homogeneous of order 1 is nothing but a derivation. Hence $D^{(1)}(\mathcal{A})$ is the module of logarithmic derivations along \mathcal{A} .

The polynomial ring $S = \bigoplus_{p=0}^{\infty} S_p$ is a graded algebra, where S_p is the K-vector subspace spanned by the monomials of degree p. The *n*th Weyl algebra D(S) is a graded S-module with $\deg(x^{\alpha}\partial^{\beta}) = |\alpha| - |\beta|$. Each $D^{(m)}(\mathcal{A})$ is a graded S-submodule of D(S). An element $P = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha}(x)\partial^{\alpha} \in D^{(m)}(\mathcal{A})$ is said to be homogeneous of polynomial degree p, and denoted by $\operatorname{pdeg} P = p$, if $f_{\alpha} \in S_p$ for all α with nonzero f_{α} .

3. SAITO-HOLM CRITERION

To prove that $D^{(1)}(\mathcal{A})$ is a free *S*-module, the Saito criterion ([8, Theorem 1.8 (ii)], see also [6, Theorem 4.19]) is very useful. Holm [4] generalized the Saito criterion to the one for $D^{(m)}(\mathcal{A})$. In this section, we briefly review Holm's generalization.

Set

$$s_m := \binom{n+m-1}{m}, \qquad t_m := \binom{n+m-2}{m-1}$$
$$\{x^{\boldsymbol{\alpha}^{(1)}}, x^{\boldsymbol{\alpha}^{(2)}}, \dots, x^{\boldsymbol{\alpha}^{(s_m)}}\}$$

Let

be the set of monomials of degree
$$m$$
. For operators $\theta_1, \ldots, \theta_{s_m}$, define an $s_m \times s_m$ coefficient matrix $M_m(\theta_1, \ldots, \theta_{s_m})$ by

$$M_m(\theta_1,\ldots,\theta_{s_m}) := \begin{bmatrix} \theta_1 * \frac{x^{\alpha^{(1)}}}{\alpha^{(1)!}} & \cdots & \theta_{s_m} * \frac{x^{\alpha^{(1)}}}{\alpha^{(1)!}} \\ \vdots & \ddots & \vdots \\ \theta_1 * \frac{x^{\alpha^{(s_m)}}}{\alpha^{(s_m)!}} & \cdots & \theta_{s_m} * \frac{x^{\alpha^{(s_m)}}}{\alpha^{(s_m)!}} \end{bmatrix},$$

where $\boldsymbol{\alpha}! = (\alpha_1!)(\alpha_2!)\cdots(\alpha_n!)$ for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

The proofs of the following two propositions go similarly to those of [6, Proposition 4.12] and [6, Proposition 4.18].

Proposition 3.1 (III Proposition 5.2 in [4] (cf. Proposition 4.12 in [6])). If $\theta_1, \ldots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$, then

$$\det M_m(\theta_1,\ldots,\theta_{s_m}) \in \langle Q^{t_m} \rangle.$$

Proposition 3.2 (III Proposition 5.7 in [4] (cf. Proposition 4.18 in [6])). Suppose that $D^{(m)}(\mathcal{A})$ is a free S-module. Then the rank of $D^{(m)}(\mathcal{A})$ is s_m .

The following is a generalization of the Saito criterion. This was proved by Holm [4, II Theorem 5.8].

Theorem 3.3 (Saito-Holm criterion). Given $\theta_1, \ldots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$, the following two conditions are equivalent:

- (1) det $M_m(\theta_1, \ldots, \theta_{s_m}) = c Q_{\mathcal{A}}^{t_m}$ for some $c \in K^{\times}$,
- (2) $\theta_1, \ldots, \theta_{s_m}$ form a basis for $D^{(m)}(\mathcal{A})$ over S.

The following is an easy consequence of Theorem 3.3.

Theorem 3.4 (III Theorem 5.9 in [4] (cf. Theorem 4.23 in [6])). Let $\theta_1, \ldots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$ be linearly independent over S. Then $\theta_1, \ldots, \theta_{s_m}$ form a basis for $D^{(m)}(\mathcal{A})$ over S if and only if

$$\sum_{j=1}^{s_m} \operatorname{pdeg} \theta_j = rt_m.$$

Suppose that $D^{(m)}(\mathcal{A})$ is free over S. We denote by $\exp D^{(m)}(\mathcal{A})$ the multi-set of polynomial degrees of a basis for $D^{(m)}(\mathcal{A})$. The expression

$$\exp D^{(m)}(\mathcal{A}) = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \ldots\}$$

means that $\exp D^{(m)}(\mathcal{A})$ has e_i i's $(i = 0, 1, 2, \cdots)$.

Proposition 3.5 (cf. Proposition 4.26 in [6]). Assume that $D^{(m)}(\mathcal{A})$ is free over S, and suppose that

$$\exp D^{(m)}(\mathcal{A}) = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \ldots\}.$$

Then

$$\sum_{k} e_k = s_m, \qquad \sum_{k} k e_k = r t_m.$$

Proof. Proposition 3.2 is the first statement, and Theorem 3.4 the second.

4. Generic Arrangements

In the rest of this paper, we assume that $r \ge n$ and \mathcal{A} is generic. An arrangement \mathcal{A} is said to be *generic*, if every *n* hyperplanes of \mathcal{A} intersect only at the origin.

For a finite set \mathcal{S} , let $\mathcal{S}^{(k)} \subseteq 2^{\mathcal{S}}$ denote the set of $\mathcal{T} \subseteq \mathcal{S}$ with $\sharp \mathcal{T} = k$. Given $\mathcal{H} \in \mathcal{A}^{(n-1)}$, the vector space

$$\{\delta \in \sum_{i=1}^{n} K \partial_i \, | \, \delta * p_H = 0 \text{ for all } H \in \mathcal{H}\}$$

is one-dimensional; fix a nonzero element $\delta_{\mathcal{H}}$ of this space. Note that

(4.1)
$$\delta_{\mathcal{H}} * p_H = 0 \Leftrightarrow H \in \mathcal{H},$$

since \mathcal{A} is generic.

For $\mathcal{H}_1, \ldots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}$, put

(4.2)
$$P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}} := \prod_{H \notin \cap_{i=1}^m \mathcal{H}_i} p_H.$$

Then $P_{\{\mathcal{H}_1,\ldots,\mathcal{H}_m\}}\delta_{\mathcal{H}_1}\cdots\delta_{\mathcal{H}_m}\in D^{(m)}(\mathcal{A})$ by (2.4). In particular, for $\mathcal{H}\in\mathcal{A}^{(n-1)}$,

$$P_{\mathcal{H}}\delta^m_{\mathcal{H}} \in D^{(m)}(\mathcal{A}),$$

where $P_{\mathcal{H}} := P_{\{\mathcal{H}\}}$. Note that (4.3)

The operator

(4.4)
$$\epsilon_m := \sum_{|\boldsymbol{\alpha}|=m} \frac{m!}{\boldsymbol{\alpha}!} x^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}$$

is called the Euler operator of order m. Then ϵ_1 is the Euler derivation, and $\epsilon_m =$ $\epsilon_1(\epsilon_1 - 1) \cdots (\epsilon_1 - m + 1)$ [5, Lemma 4.9].

 $\deg P_{\mathcal{H}} = r - n + 1.$

Holm gave a finite set of generators of $D^{(m)}(\mathcal{A})$ as an S-module:

Theorem 4.1 (Theorem 4.22 in [5]).

$$D^{(m)}(\mathcal{A}) = \sum_{\mathcal{H}_1, \dots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}} SP_{\{\mathcal{H}_1, \dots, \mathcal{H}_m\}} \delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} + S\epsilon_m$$

The following lemma will be used in Sections 7, 8, and 9.

(1) The set $\{\delta_{\mathcal{H}}^{r-n+1} | \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K-basis of $\sum_{|\alpha|=r-n+1} K \partial^{\alpha}$. Lemma 4.2. (2) The set $\{P_{\mathcal{H}} \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K-basis of $\sum_{|\alpha|=r-n+1} Kx^{\alpha} = S_{r-n+1}$.

Proof. The dimensions of $\sum_{|\alpha|=r-n+1} K \partial^{\alpha}$ and S_{r-n+1} are equal to

$$s_{r-n+1} = \binom{r}{r-n+1} = \binom{r}{n-1} = \sharp \mathcal{A}^{(n-1)}.$$

Let $\mathcal{H}, \mathcal{H}' \in \mathcal{A}^{(n-1)}$. Then

(4.5)
$$\delta_{\mathcal{H}}^{r-n+1} * P_{\mathcal{H}'} = \delta_{\mathcal{H}}^{r-n+1} * \prod_{H \notin \mathcal{H}'} p_H = \begin{cases} (r-n+1)! \prod_{H \notin \mathcal{H}} (\delta_{\mathcal{H}} * p_H) & \text{if } \mathcal{H}' = \mathcal{H} \\ 0 & \text{otherwise.} \end{cases}$$

The assertions follow, since $\delta_{\mathcal{H}} * p_H = 0$ if and only if $H \in \mathcal{H}$.

5. The case n=2

In this section, we consider central arrangements with $r \geq 2$ in K^2 , which are always generic. Note that $s_m = m + 1$, and $t_m = m$.

Let $\mathcal{A} = \{H_1, H_2, \dots, H_r\}$. Put $p_i := p_{H_i}, P_i := P_{\{H_i\}}$, and $\delta_i := \delta_{\{H_i\}}$ for $i = 1, 2, \dots, r$. We may assume that there exist distinct $a_2, \ldots, a_r \in K$ such that

$$p_1 = x_1, \qquad p_i = x_2 - a_i x_1 \quad (i = 2, \dots, r).$$

Then

$$\delta_1 = \partial_2, \qquad \delta_i = \partial_1 + a_i \partial_2 \quad (i = 2, \dots, r),$$

and

$$P_i = Q/p_i \quad (i = 1, \dots, r).$$

Proposition 5.1 (Proposition 6.7 III in [4], Proposition 4.14 in [9]). The S-module $D^{(m)}(\mathcal{A})$ is free with the following basis:

- (1) $\{\epsilon_m, P_1\delta_1^m, \ldots, P_m\delta_m^m\}$ if $m \leq r-2$.
- (2) $\{P_1\delta_1^m, \dots, P_r\delta_r^m\}$ if m = r 1. (3) $\{P_1\delta_1^m, \dots, P_r\delta_r^m, Q\eta_{r+1}, \dots, Q\eta_{m+1}\}$ if $m \ge r$, where $\{\delta_1^m, \dots, \delta_r^m, \eta_{r+1}, \dots, \eta_{m+1}\}$ is a K-basis of $\sum_{i=0}^m K\partial_1^i\partial_2^{m-i}$.

Corollary 5.2.

$$\exp D^{(m)}(\mathcal{A}) = \begin{cases} \{m^1, (r-1)^m\} & (1 \le m \le r-2), \\ \{(r-1)^{m+1}\} & (m=r-1), \\ \{(r-1)^r, r^{m-r+1}\} & (m \ge r). \end{cases}$$

6. The case $m = r - n + 1$

In this section, we consider the case m = r - n + 1. In this case,

(6.1)
$$s_m = \binom{n+m-1}{m} = \binom{r}{m} = \binom{r}{n-1}.$$

Note also that deg $P_{\mathcal{H}} = r - n + 1 = m$ (4.3).

In Sections 7, 8, and 9, we use Lemma 4.2 in the case m = r - n + 1. Lemma 4.2 reads as follows in this case:

Lemma 6.1. (1) The set $\{\delta_{\mathcal{H}}^m | \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K-basis of $\sum_{|\alpha|=m} K \partial^{\alpha}$. (2) The set $\{P_{\mathcal{H}} | \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K-basis of $\sum_{|\alpha|=m} K x^{\alpha} = S_m$.

Proposition 6.2 (III Proposition 6.8 in [4]). The S-module $D^{(m)}(\mathcal{A})$ is free with a basis $\{P_{\mathcal{H}}\delta_{\mathcal{H}}^m \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}.$

Corollary 6.3. *If* m = r - n + 1*, then*

$$\exp D^{(m)}(\mathcal{A}) = \{m^{\binom{r}{m}}\}.$$

7. The case $m \ge r - n + 1$

In this section, we assume that $m \ge r - n + 1$, and we prove Holm's conjecture by giving a basis of $D^{(m)}(\mathcal{A})$.

Set

 $\tilde{r} := n + m - 1,$

and add $\tilde{r} - r$ hyperplanes to $\mathcal{A} = \{H_1, \ldots, H_r\}$ so that

(7.1)
$$\widehat{\mathcal{A}} := \mathcal{A} \cup \{H_{r+1}, \dots, H_{\tilde{r}}\}$$

is still generic.

For $\mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}$, define a homogeneous polynomial $P'_{\mathcal{H}} \in S$ by

(7.2)
$$P'_{\mathcal{H}} := \prod_{H \notin \mathcal{H}; \, H \in \mathcal{A}} p_{H}$$

Theorem 7.1. The S-module $D^{(m)}(\mathcal{A})$ is free with a basis $\{P'_{\mathcal{H}}\delta^m_{\mathcal{H}} | \mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}\}.$

Proof. By (2.4), $P'_{\mathcal{H}}\delta^m_{\mathcal{H}} \in D^{(m)}(\mathcal{A})$ for each $\mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}$.

By Lemma 6.1 (1), $\{P'_{\mathcal{H}}\delta^m_{\mathcal{H}} \mid \mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}\}$ is linearly independent over S. Since

$$\deg P'_{\mathcal{H}} = \sharp \{ H \in \mathcal{A} \, | \, H \notin \mathcal{H} \}$$

the number of $\mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}$ with deg $P'_{\mathcal{H}} = j$ is

$$\binom{r}{j}\binom{\tilde{r}-r}{n-1-(r-j)} = \binom{r}{j}\binom{m+n-r-1}{n-r+j-1}.$$

Then

$$\sum_{j} j\binom{r}{j}\binom{m+n-r-1}{n-r+j-1} = r \sum_{j} \binom{r-1}{j-1}\binom{m+n-r-1}{n-r+j-1} \\ = r \sum_{j} \binom{r-1}{j-1}\binom{m+n-r-1}{m-j} \\ = r\binom{m+n-2}{m-1} = rt_m.$$

Hence we have the assertion by Theorem 3.4.

Corollary 7.2.

$$\exp D^{(m)}(\mathcal{A}) = \{ j^{\binom{r}{j}\binom{m+n-r-1}{m-j}} | r-n+1 \le j \le \min\{r,m\} \}$$

8. The case
$$m < r - n + 1$$

Throughout this section, we assume that m < r - n + 1. Recall that $D^{(m)}(\mathcal{A})$ is generated by

(8.1)
$$\{P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}}\delta_{\mathcal{H}_1}\cdots\delta_{\mathcal{H}_m} \mid \mathcal{H}_1,\dots,\mathcal{H}_m \in \mathcal{A}^{(n-1)}\} \cup \{\epsilon_m\}$$

over S (Theorem 4.1). In this section, we choose a minimal system of generators from (8.1) (Theorem 8.3), which implies that $D^{(m)}(\mathcal{A})$ is not free (Remark 8.5).

Note that

$$\sharp \mathcal{A}^{(n-1)} = \binom{r}{n-1} > \binom{n+m-1}{n-1} = s_m.$$

Lemma 8.1. For any $\mathcal{H}_1, \ldots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}$, the following hold:

(1)
$$P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}} \in \bigcap_{\substack{\bigcap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}}} SP_{\mathcal{H}}.$$

(2) $\delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} \in \sum_{\substack{\bigcap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}}} K \delta_{\mathcal{H}}^m.$

Proof. (1) If $\bigcap_{i=1}^{m} \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}$, then $P_{\mathcal{H}} = \prod_{H \notin \mathcal{H}} p_H$ divides $\prod_{H \notin \bigcap_{i=1}^{m} \mathcal{H}_i} p_H = P_{\{\mathcal{H}_1,\ldots,\mathcal{H}_m\}}$. Hence the assertion is clear.

(2) Let $\bar{r} := n + m - 1$. Take a subarrangement $\mathcal{B} \supset \bigcap_{i=1}^{m} \mathcal{H}_i$ of \mathcal{A} with \bar{r} hyperplanes. By Lemma 6.1, there exist $c_{\mathcal{H}} \in K$ ($\mathcal{H} \in \mathcal{B}^{(n-1)}$) such that

(8.2)
$$\delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} = \sum_{\mathcal{H} \in \mathcal{B}^{(n-1)}} c_{\mathcal{H}} \delta_{\mathcal{H}}^m.$$

It suffices to show that $c_{\mathcal{H}} = 0$ for all $\mathcal{H} \not\supseteq \cap_{i=1}^{m} \mathcal{H}_{i}$. Fix $\mathcal{H} \in \mathcal{B}^{(n-1)}$ with $\mathcal{H} \not\supseteq \cap_{i=1}^{m} \mathcal{H}_{i}$, and put $\overline{P}_{\mathcal{H}} = \prod_{H \in \mathcal{B} \setminus \mathcal{H}} p_{H}$. Then deg $\overline{P}_{\mathcal{H}} = \overline{r} - (n-1) = m$. Since there exists $H_{0} \in (\cap_{i=1}^{m} \mathcal{H}_{i}) \setminus \mathcal{H}$, we have $\delta_{\mathcal{H}_{i}} * p_{H_{0}} = 0$ for all $i = 1, \ldots, m$, and hence $\delta_{\mathcal{H}_{1}} \cdots \delta_{\mathcal{H}_{m}} * \overline{P}_{\mathcal{H}} = 0$. Recall from (4.5) that

$$\delta_{\mathcal{H}'}^{m} * \overline{P}_{\mathcal{H}} = \begin{cases} m! \prod_{H \in \mathcal{B} \setminus \mathcal{H}} (\delta_{\mathcal{H}'} * p_{H}) \neq 0 & \text{if } \mathcal{H}' = \mathcal{H} \\ 0 & \text{otherwise.} \end{cases}$$

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Let the operator (8.2) act on $\overline{P}_{\mathcal{H}}$. Since

$$0 = \delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} * \overline{P}_{\mathcal{H}} = \sum_{\mathcal{H} \in \mathcal{B}^{(n-1)}} c_{\mathcal{H}} \delta_{\mathcal{H}}^m * \overline{P}_{\mathcal{H}} = c_{\mathcal{H}} \cdot m! \prod_{H \in \mathcal{B} \setminus \mathcal{H}} (\delta_{\mathcal{H}} * p_H),$$

we have $c_{\mathcal{H}} = 0$.

Proposition 8.2. If m < r - n + 1, then

$$D^{(m)}(\mathcal{A}) = \sum_{\mathcal{H} \in \mathcal{A}^{(n-1)}} SP_{\mathcal{H}} \delta_{\mathcal{H}}^m + S\epsilon_m$$

Proof. Let $\mathcal{H}_1, \ldots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}$. By Lemma 8.1,

$$P_{\{\mathcal{H}_{1},\dots,\mathcal{H}_{m}\}}\delta_{\mathcal{H}_{1}}\cdots\delta_{\mathcal{H}_{m}} \in P_{\{\mathcal{H}_{1},\dots,\mathcal{H}_{m}\}}\cdot\sum_{\substack{\bigcap_{i=1}^{m}\mathcal{H}_{i}\subset\mathcal{H}\in\mathcal{A}^{(n-1)}}}K\delta_{\mathcal{H}}^{m}$$
$$\subset \sum_{\substack{\bigcap_{i=1}^{m}\mathcal{H}_{i}\subset\mathcal{H}\in\mathcal{A}^{(n-1)}}}SP_{\mathcal{H}}\delta_{\mathcal{H}}^{m}.$$

Hence we obtain the assertion from (8.1).

The system of generators for $D^{(m)}(\mathcal{A})$ in Proposition 8.2 is still large. Next, we fix m hyperplanes H_1, \ldots, H_m , and define an S-submodule $\Xi^{(m)}(\mathcal{A})$ of $D^{(m)}(\mathcal{A})$ by

(8.3)
$$\Xi^{(m)}(\mathcal{A}) := \{ \theta \in D^{(m)}(\mathcal{A}) \mid \theta * (p_{H_1} \cdots p_{H_m}) = 0 \}.$$

For $\mathcal{H} \in \mathcal{A}^{(n-1)}$ with $\mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset$, we have $\delta_{\mathcal{H}} * p_{H_i} = 0$ for some *i*, and hence $P_{\mathcal{H}} \delta_{\mathcal{H}}^m \in \Xi^{(m)}(\mathcal{A})$. Furthermore we have the following.

Theorem 8.3. If m < r - n + 1, then

$$D^{(m)}(\mathcal{A}) = \Xi^{(m)}(\mathcal{A}) \oplus S\epsilon_m = \sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)}\\\mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset}} SP_{\mathcal{H}} \delta_{\mathcal{H}}^m \oplus S\epsilon_m$$

Moreover, the set $\{P_{\mathcal{H}}\delta_{\mathcal{H}}^m | \mathcal{H} \in \mathcal{A}^{(n-1)}, \mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset\}$ is a minimal system of generators for $\Xi^{(m)}(\mathcal{A})$ over S.

Proof. Let $\theta \in D^{(m)}(\mathcal{A})$. Then $\theta - \frac{1}{m!} \frac{\theta * (p_{H_1} \cdots p_{H_m})}{p_{H_1} \cdots p_{H_m}} \epsilon_m \in \Xi^{(m)}(\mathcal{A})$, since $\theta \in D^{(m)}(\mathcal{A}) \subset D^{(m)}(\langle p_{H_1} \cdots p_{H_m} \rangle)$ by (2.4). So we have $D^{(m)}(\mathcal{A}) = \Xi^{(m)}(\mathcal{A}) + S\epsilon_m$. Moreover, $\epsilon_m * (p_{H_1} \cdots p_{H_m}) = m! p_{H_1} \cdots p_{H_m} \neq 0$ implies that $\Xi^{(m)}(\mathcal{A}) \cap S\epsilon_m = 0$.

Next, we show the second equality. By Proposition 8.2, it suffices to show that

$$P_{\mathcal{H}_0}\delta^m_{\mathcal{H}_0} \in \sum_{\substack{\mathcal{H}\in\mathcal{A}^{(n-1)}\\\mathcal{H}\cap\{H_1,\dots,H_m\}\neq\emptyset}} SP_{\mathcal{H}}\delta^m_{\mathcal{H}} \oplus S\epsilon_m$$

for every $\mathcal{H}_0 \in \mathcal{A}^{(n-1)}$ with $\mathcal{H}_0 \cap \{H_1, \ldots, H_m\} = \emptyset$. Put $\mathcal{B} := \mathcal{H}_0 \cup \{H_1, \ldots, H_m\}$. By Proposition 6.2,

$$D^{(m)}(\mathcal{B}) = \bigoplus_{\mathcal{H} \in \mathcal{B}^{(n-1)}} S\overline{P}_{\mathcal{H}} \delta_{\mathcal{H}}^{m}$$

where $\overline{P}_{\mathcal{H}} = \prod_{H \in \mathcal{B} \setminus \mathcal{H}} p_H$. Since $\epsilon_m \in D^{(m)}(\mathcal{B})$, there exist $c_{\mathcal{H}} \in S$ $(\mathcal{H} \in \mathcal{B}^{(n-1)})$ such that (8.4) $\epsilon_m = \sum_{\mathcal{C}_{\mathcal{H}}} \overline{P}_{\mathcal{H}} \delta^m_{\mathcal{H}}.$

(8.4)
$$\epsilon_m = \sum_{\mathcal{H} \in \mathcal{B}^{(n-1)} \atop 8} c_{\mathcal{H}} \overline{P}_{\mathcal{H}} \delta_{\mathcal{H}}^m$$

(By looking at polynomial degrees, we see $c_{\mathcal{H}} \in K$.) Multiplying $q := \prod_{H \in \mathcal{A} \setminus \mathcal{B}} p_H$ from the left, we have

(8.5)
$$q\epsilon_m = \sum_{\mathcal{H}\in\mathcal{B}^{(n-1)}} c_{\mathcal{H}} P_{\mathcal{H}} \delta_{\mathcal{H}}^m.$$

Let the operator (8.4) act on $\overline{P}_{\mathcal{H}_0}$. Since

$$0 \neq m! \overline{P}_{\mathcal{H}_0} = m! c_{\mathcal{H}_0} \overline{P}_{\mathcal{H}_0} \cdot \prod_{H \in \mathcal{B} \setminus \mathcal{H}_0} \delta_{\mathcal{H}_0} * p_H = m! c_{\mathcal{H}_0} \overline{P}_{\mathcal{H}_0} \cdot \prod_{i=1}^m (\delta_{\mathcal{H}_0} * p_{H_i}),$$

we have $c_{\mathcal{H}_0} \neq 0$. Hence, we have

$$P_{\mathcal{H}_0}\delta_{\mathcal{H}_0}^m = c_{\mathcal{H}_0}^{-1} \left(q\epsilon_m - \sum_{\substack{\mathcal{H} \in \mathcal{B}^{(n-1)}\\\mathcal{H} \neq \mathcal{H}_0}} c_{\mathcal{H}} P_{\mathcal{H}}\delta_{\mathcal{H}}^m \right) \in \sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)}\\\mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset}} SP_{\mathcal{H}}\delta_{\mathcal{H}}^m \oplus S\epsilon_m.$$

Finally, we show the minimality. It suffices to show that the set $\{P_{\mathcal{H}}\delta_{\mathcal{H}}^m | \mathcal{H} \in \mathcal{A}^{(n-1)}, \mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset\}$ is linearly independent over K, since all $P_{\mathcal{H}}\delta_{\mathcal{H}}^m$ have the same polynomial degree. Suppose that

(8.6)
$$\sum_{\substack{\mathcal{H}\in\mathcal{A}^{(n-1)}\\\mathcal{H}\cap\{H_1,\dots,H_m\}\neq\emptyset}} c_{\mathcal{H}} P_{\mathcal{H}} \delta_{\mathcal{H}}^m = 0 \ (c_{\mathcal{H}}\in K).$$

Fix arbitrary hyperplanes $H_{i_1}, \ldots, H_{i_m} \in \mathcal{A}$, and put $q' := p_{H_{i_1}} \cdots p_{H_{i_m}}$ and $\mathcal{B}' := \mathcal{A} \setminus \{H_{i_1}, \ldots, H_{i_m}\}$. Let the operator (8.6) act on q'. Then we have

$$\sum_{\substack{\mathcal{H}\in\mathcal{B}'^{(n-1)}\\\mathcal{H}\cap\{H_1,\ldots,H_m\}\neq\emptyset}} c_{\mathcal{H}} P_{\mathcal{H}} \prod_{\nu=1}^m (\delta_{\mathcal{H}} * p_{H_{i_{\nu}}}) = 0.$$

By Lemma 4.2, the set $\{P_{\mathcal{H}} | \mathcal{H} \in \mathcal{B}'^{(n-1)}\}$ is linearly independent over K. Hence $c_{\mathcal{H}} = 0$ for $\mathcal{H} \in \mathcal{B}'^{(n-1)}$ with $\mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset$. For $\mathcal{H} \in \mathcal{A}^{(n-1)}$ with $\mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset$, we may take $H_{i_1}, \ldots, H_{i_m} \in \mathcal{A}$ so that $\mathcal{H} \in \mathcal{B}'^{(n-1)}$, since r > m + n - 1. Hence we have finished the proof.

Corollary 8.4 (cf. Conjecture 6.8 in [9]). The S-module $\Xi^{(m)}(\mathcal{A})$ is minimally generated by $\binom{r}{n-1} - \binom{r-m}{n-1}$ operators of polynomial degree r - n + 1.

Remark 8.5. We can show

$$\binom{r}{n-1} - \binom{r-m}{n-1} + 1 > \binom{n+m-1}{n-1},$$

supposing that m < r - n + 1. Then by Proposition 3.2 and Corollary 8.4 we see that, for $n \ge 3$ and m < r - n + 1, $D^{(m)}(\mathcal{A})$ is not free over S, which was proved by Holm [4, III Proposition 6.8].

In this section, we assume $m \leq r - n + 1$, and we construct a minimal free resolution of $\Xi^{(m)}(\mathcal{A})$ when m < r - n + 1 and $n \geq 3$. We generalize the construction in [11] step by step, and basically we succeed Yuzvinsky's notation.

Let $V := K^n$. Recall that, for $\mathcal{H} \in \mathcal{A}^{(n-1)}$, $\delta_{\mathcal{H}} \in (V^*)^* = V$ is the nonzero derivation with constant coefficients such that $\delta_{\mathcal{H}} * p_H = 0$ for all $H \in \mathcal{H}$. Under the identification $(V^*)^* = V$, $K\delta_{\mathcal{H}}$ corresponds to the linear subspace $[\mathcal{H}] := \bigcap_{H \in \mathcal{H}} H = \bigcap_{H \in \mathcal{H}} (p_H = 0)$ of V. Similarly, $\mathcal{H} \in \mathcal{A}^{(n-j)}$ corresponds to the linear subspace $[\mathcal{H}] = \bigcap_{H \in \mathcal{H}} H \in L_j$, where L_j is the set of elements of dimension j of the intersection lattice of \mathcal{A} .

For $\mathcal{H} \in \mathcal{A}^{(n-j)}$ with $1 \leq j \leq n$, set

$$\Delta_{\mathcal{H}} := \sum_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-1)}} K \delta^m_{\mathcal{H} \cup \mathcal{H}'}.$$

Note that

$$\Delta_{\mathcal{H}} = K \delta_{\mathcal{H}}^m \qquad \text{for } \mathcal{H} \in \mathcal{A}^{(n-1)},$$

and

$$\Delta_{\emptyset} = \sum_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta_{\mathcal{H}}^m.$$

Each $\Delta_{\mathcal{H}}$ is a subspace of Δ_{\emptyset} .

Example 9.1. Let m = 1. Then

$$\Delta_{\mathcal{H}} = \{ \delta \in (V^*)^* \, | \, \delta * p_H = 0 \quad \text{for all } H \in \mathcal{H} \}.$$

Hence, under the identification $(V^*)^* = V$, $\Delta_{\mathcal{H}}$ corresponds to $[\mathcal{H}] = \bigcap_{H \in \mathcal{H}} H = \bigcap_{H \in \mathcal{H}} (p_H = 0).$

Lemma 9.2. Let $1 \leq j \leq n$, and let $\mathcal{H} \in \mathcal{A}^{(n-j)}$. Take $\mathcal{A}' := \{H_1, H_2, \dots, H_{\bar{r}}\} \subseteq \mathcal{A}$ with $\bar{r} = m + n - 1$ so that $\mathcal{H} \subseteq \mathcal{A}'$. Then $\{\delta^m_{\mathcal{H} \cup \mathcal{H}'} | \mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}\}$ forms a basis of $\Delta_{\mathcal{H}}$, and $\dim \Delta_{\mathcal{H}} = {\bar{r} - (n-j) \choose j-1} = {m+j-1 \choose j-1}$.

Proof. By Lemma 6.1,

(9.1)
$$\sum_{|\alpha|=m} K \partial^{\alpha} = \bigoplus_{\mathcal{H}'' \in (\mathcal{A}')^{(n-1)}} K \delta^{m}_{\mathcal{H}''}.$$

Hence $\delta^m_{\mathcal{H}\cup\mathcal{H}'}$ $(\mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)})$ are linearly independent. Let $\mathcal{H}''' \in (\mathcal{A} \setminus \mathcal{H})^{(j-1)} \setminus (\mathcal{A}')^{(j-1)}$. Then

(9.2)
$$\delta^m_{\mathcal{H}\cup\mathcal{H}'''} = \sum_{\mathcal{H}''\in(\mathcal{A}')^{(n-1)}} \frac{\delta^m_{\mathcal{H}\cup\mathcal{H}'''} * P'_{\mathcal{H}''}}{\delta^m_{\mathcal{H}''} * P'_{\mathcal{H}''}} \delta^m_{\mathcal{H}''},$$

where

(9.3)
$$P'_{\mathcal{H}''} := \prod_{\substack{H \in \mathcal{A}' \setminus \mathcal{H}''\\10}} p_H.$$

For $\mathcal{H}'' \not\supseteq \mathcal{H}$, there exists $H \in \mathcal{H} \setminus \mathcal{H}''$. Then p_H divides $P'_{\mathcal{H}''}$, and hence $\delta^m_{\mathcal{H} \cup \mathcal{H}''} * P'_{\mathcal{H}''} = 0$. Therefore

(9.4)
$$\delta^m_{\mathcal{H}\cup\mathcal{H}'''} = \sum_{\mathcal{H}'\in(\mathcal{A}'\setminus\mathcal{H})^{(j-1)}} \frac{\delta^m_{\mathcal{H}\cup\mathcal{H}''} * P'_{\mathcal{H}\cup\mathcal{H}'}}{\delta^m_{\mathcal{H}\cup\mathcal{H}'} * P'_{\mathcal{H}\cup\mathcal{H}'}} \delta^m_{\mathcal{H}\cup\mathcal{H}'}.$$

Hence $\{\delta_{\mathcal{H}\cup\mathcal{H}'}^m | \mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}\}$ forms a basis of $\Delta_{\mathcal{H}}$, and $\dim \Delta_{\mathcal{H}} = {\binom{\bar{r}-(n-j)}{j-1}} = {\binom{m+j-1}{j-1}}.$

Let $\mathcal{A} = \{H_1, H_2, \dots, H_r\}$. We write $H_i \prec H_j$ if i < j. We define the complex $C_*(\mathcal{A}) = C_*$ as follows. For $j = 1, 2, \dots, n$, set

$$C_{n-j} := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-j)}} \Delta_{\mathcal{H}} \boldsymbol{e}_{\wedge \mathcal{H}},$$

where $e_{\wedge \mathcal{H}}$ is just a symbol. In particular,

$$C_{n-1} := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta_{\mathcal{H}}^m \boldsymbol{e}_{\wedge \mathcal{H}},$$

and

$$C_0 := \Delta_{\emptyset} \boldsymbol{e}_{\wedge \emptyset}.$$

The differential $\partial_j : C_j \to C_{j-1}$ is defined by

$$C_{j} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(j)}} \Delta_{\mathcal{H}} \boldsymbol{e}_{\wedge \mathcal{H}} \ni \boldsymbol{\xi} \boldsymbol{e}_{\wedge \mathcal{H}} \mapsto \sum_{H \in \mathcal{H}} (-1)^{l_{\mathcal{H}}(H)} \boldsymbol{\xi} \boldsymbol{e}_{\wedge (\mathcal{H} \setminus \{H\})} \in C_{j-1},$$

where

$$l_{\mathcal{H}}(H) := \sharp \{ H' \in \mathcal{H} \mid H' \prec H \}.$$

Set

$$C_n := \operatorname{Ker} \partial_{n-1}$$

Lemma 9.3 (cf. Lemma 1.1 in [11]). The sequence C_* is exact.

Proof. As in [11, Lemma 1.1], we prove the assertion by induction. Let r = m + n - 1. Then by Lemma 6.1

$$\Delta_{\mathcal{H}} = \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-1)}} K \delta^m_{\mathcal{H} \cup \mathcal{H}'} \quad \text{for } \mathcal{H} \in \mathcal{A}^{(n-j)}.$$

Hence

$$C_{n-j} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-j)}} \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-1)}} K \delta^m_{\mathcal{H} \cup \mathcal{H}'} \boldsymbol{e}_{\wedge \mathcal{H}} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta^m_{\mathcal{H}} \otimes (\bigoplus_{\mathcal{H}' \in \mathcal{H}^{(n-j)}} K \boldsymbol{e}_{\wedge \mathcal{H}'}).$$

Thus, in this case, with $C_n = 0$,

$$C_* = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta^m_{\mathcal{H}} \otimes \tilde{S}(\mathcal{H}),$$

where $\tilde{S}(\mathcal{H})$ is the augmented chain complex of the simplex with vertex set \mathcal{H} . Hence C_* is exact.

For n = 2, the sequence

is clearly exact.

Suppose that n > 2 and r > m + n - 1. Consider the arrangements $\mathcal{A} \setminus \{H_r\}$ and \mathcal{A}^{H_r} . Since r > m + n - 1, we have $\Delta_{\mathcal{H}}(\mathcal{A}) = \Delta_{\mathcal{H}}(\mathcal{A} \setminus \{H_r\})$ for $\mathcal{H} \in (\mathcal{A} \setminus \{H_r\})^{(n-j)}$ by Lemma 9.2. Hence

$$0 \to C_*(\mathcal{A} \setminus \{H_r\}) \to C_*(\mathcal{A}) \to C_*(\mathcal{A}^{H_r})(-1) \to 0$$

is exact. We thus have the assertion by induction.

Let $\mathcal{H} \in \mathcal{A}^{(n-j)}$ with j = 1, 2, ..., n, and let $C_*^{[\mathcal{H}]} := C_*(\mathcal{A}^{[\mathcal{H}]})$. For $\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-t)}$, we have

$$\Delta_{\mathcal{H}'}(\mathcal{A}^{[\mathcal{H}]}) = \sum_{\mathcal{H}'' \in (\mathcal{A} \setminus \mathcal{H} \cup \mathcal{H}')^{(t-1)}} K(\delta_{\mathcal{H}' \cup \mathcal{H}''}^{[\mathcal{H}]})^m.$$

Since we may identify $\delta_{\mathcal{H}'\cup\mathcal{H}''}^{[\mathcal{H}]}$ with $\delta_{\mathcal{H}\cup\mathcal{H}'\cup\mathcal{H}''}$, we may identify $\Delta_{\mathcal{H}'}(\mathcal{A}^{[\mathcal{H}]})$ with $\Delta_{\mathcal{H}\cup\mathcal{H}'}$. Hence

(9.5)
$$C_{j-t}^{[\mathcal{H}]} = \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-t)}} \Delta_{\mathcal{H} \cup \mathcal{H}'} \boldsymbol{e}_{\wedge \mathcal{H}'} \boldsymbol{e}_{\mathcal{H}}$$

for $t = 1, 2, \ldots, j$, where $\boldsymbol{e}_{\mathcal{H}}$ is again a symbol.

We put

$$E_{[\mathcal{H}]} := C_j^{[\mathcal{H}]} := \operatorname{Ker}(\partial_{j-1}^{[\mathcal{H}]} : C_{j-1}^{[\mathcal{H}]} \to C_{j-2}^{[\mathcal{H}]})$$

for $\mathcal{H} \in \mathcal{A}^{(n-j)}$ with $j \ge 2$, and

$$E_{[\mathcal{H}]} := K \delta_{\mathcal{H}}^m \boldsymbol{e}_{\mathcal{H}}$$

for $\mathcal{H} \in \mathcal{A}^{(n-1)}$. Then we put

$$E_j := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-j)}} E_{[\mathcal{H}]}.$$

for j = 1, 2, ..., n.

Remark 9.4 (cf. Remark 1.2 in [11]). Let $1 \leq j \leq n$ and $\mathcal{H} \in \mathcal{A}^{(n-j)}$. Then

$$\dim E_{[\mathcal{H}]} = \binom{r-m-n+j-1}{j-1}.$$

Proof. By Lemma 9.3,

dim
$$E_{[\mathcal{H}]}$$
 = dim $C_j^{[\mathcal{H}]} = \sum_{l=1}^j (-1)^{l-1} \dim C_{j-l}^{[\mathcal{H}]}$.

Then by Lemma 9.2

dim
$$C_{j-l}^{[\mathcal{H}]} = \binom{r-n+j}{j-l} \binom{m+l-1}{l-1}.$$

Hence

$$\dim E_{[\mathcal{H}]} = \sum_{l=1}^{j} (-1)^{l-1} \binom{m+l-1}{l-1} \binom{r-n+j}{j-l} \\ = \sum_{l=1}^{j} (-1)^{l-1} \binom{m+l-2}{l-1} \binom{r-n+j-1}{j-l} = \cdots \\ = \sum_{l=1}^{j} (-1)^{l-1} \binom{l-2}{l-1} \binom{r-m-n+j-1}{j-l} = \binom{r-m-n+j-1}{j-1}.$$

Let

$$\Delta_{ij} := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i+j-n)}} \Delta_{\mathcal{H} \cup \mathcal{H}'} \boldsymbol{e}_{\wedge \mathcal{H}'} \boldsymbol{e}_{\mathcal{H}}$$

for $1 \le i \le n, \ 0 \le j \le n-1$ with $i+j \ge n$, and

$$\Delta_{in} := E_i = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} E_{[\mathcal{H}]}.$$

Then

$$\Delta_{ij} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} C_{i+j-n}^{[\mathcal{H}]}, \text{ and hence } \Delta_{i\bullet} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} C_{\bullet}^{[\mathcal{H}]}(-(n-i)).$$

As differentials of $\Delta_{i\bullet}$, we take $(-1)^i$ times the differentials of $\bigoplus_{\mathcal{H}\in\mathcal{A}^{(n-i)}} C^{[\mathcal{H}]}_{\bullet}(-(n-i))$. We define a linear map $\phi(j)_i : \Delta_{ij} \to \Delta_{i-1j}$ for $0 \le j \le n-1$ by

$$\Delta_{ij} \ni \xi \boldsymbol{e}_{\wedge \mathcal{H}'} \boldsymbol{e}_{\mathcal{H}} \mapsto \sum_{H \in \mathcal{H}'} (-1)^{l_{\mathcal{H}'}(H)} \xi \boldsymbol{e}_{\wedge (\mathcal{H}' \setminus \{H\})} \boldsymbol{e}_{\mathcal{H} \cup \{H\}} \in \Delta_{i-1j}$$

for $\mathcal{H} \in \mathcal{A}^{(n-i)}, \mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i+j-n)}$, and $\xi \in \Delta_{\mathcal{H} \cup \mathcal{H}'}$. We define $\psi_i : E_i \to E_{i-1}$ as the restriction of $\phi(n-1)_i$.



We add

$$\psi_1: E_1 = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta^m_{\mathcal{H}} \boldsymbol{e}_{\mathcal{H}} \ni \delta^m_{\mathcal{H}} \boldsymbol{e}_{\mathcal{H}} \mapsto \delta^m_{\mathcal{H}} \in E_0 := \Delta_{\emptyset} = \sum_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta^m_{\mathcal{H}}.$$

Lemma 9.5 (cf. Lemma 1.3 in [11]). The sequence

$$E_*: 0 \to E_n \to E_{n-1} \to \dots \to E_1 \to E_0 \to 0$$

 $is \ exact.$

Proof. All rows of $\Delta_{\bullet\bullet}$ are exact by Lemma 9.3 and the argument in the paragraph just after the proof of Lemma 9.3.

For $1 \leq j < n$, since we have

$$\Delta_{ij} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i+j-n)}} \Delta_{\mathcal{H} \cup \mathcal{H}'} \boldsymbol{e}_{\wedge \mathcal{H}'} \boldsymbol{e}_{\mathcal{H}}$$
$$= \bigoplus_{\mathcal{H} \in \mathcal{A}^{(j)}} \Delta_{\mathcal{H}} \otimes_{K} \left(\bigoplus_{\mathcal{H}' \in \mathcal{H}^{(i+j-n)}} K \boldsymbol{e}_{\wedge \mathcal{H}'} \boldsymbol{e}_{\mathcal{H} \setminus \mathcal{H}'} \right)$$

the *j*-th column $\Delta_{\bullet j}$ is the same as $\bigoplus_{\mathcal{H} \in \mathcal{A}^{(j)}} \Delta_{\mathcal{H}} \otimes_K \tilde{S}(\mathcal{H})$, where $\tilde{S}(\mathcal{H})$ is the augmented chain complex of the simplex with vertex set \mathcal{H} :

$$0 \to K \boldsymbol{e}_{\wedge \mathcal{H}} \to \bigoplus_{B \in \mathcal{H}^{(j-1)}} K \boldsymbol{e}_{\wedge B} \to \bigoplus_{B \in \mathcal{H}^{(j-2)}} K \boldsymbol{e}_{\wedge B} \to \dots \to \bigoplus_{H \in \mathcal{H}} K \boldsymbol{e}_{H} \to K \boldsymbol{e}_{\emptyset} \to 0.$$

Thus the *j*-th columns $(1 \le j \le n-1)$ are exact. The 0-th column has the unique nonzero term $\Delta_{\emptyset} e_{\emptyset}(=E_0)$ at i=n. Hence by the spectral sequence argument we see that E_* is exact.

Let $\sigma \subseteq \{1, 2, \ldots, r\}$ and $\sigma \neq \emptyset$. Put

$$\mathcal{L}_{j}[\sigma] := \{ \mathcal{H} \in \mathcal{A}^{(n-j)} \mid \mathcal{H} \cap \{ H_{i} \mid i \in \sigma \} \neq \emptyset \}.$$

For $1 \leq j \leq n$,

$$E_j[\sigma] := \bigoplus_{\mathcal{H} \in \mathcal{L}_j[\sigma]} E_{[\mathcal{H}]}.$$

Then

$$E_n[\sigma] = 0, \qquad E_1[\sigma] = \bigoplus_{\mathcal{H} \in \mathcal{L}_1[\sigma]} K \delta_{\mathcal{H}}^m \boldsymbol{e}_{\mathcal{H}}.$$

We put

$$E_0[\sigma] := \sum_{\mathcal{H} \in \mathcal{L}_1[\sigma]} K \delta_{\mathcal{H}}^m.$$

We also put

$$E_j[\emptyset] := 0.$$

for all j. Then $\{(E_*[\sigma], \psi_*[\sigma])\}$ is a subcomplex of $\{(E_*, \psi_*)\}$.

Lemma 9.6 (cf. Lemma 1.4 in [11]). For every σ with $|\sigma| \leq n + m - 1$, $E_*[\sigma]$ is exact.

Proof. We prove the assertion by induction on $|\sigma|$. If $|\sigma| = 0$, then the assertion is trivial. When n = 2, we have

$$0 \to E_1[\sigma] = \bigoplus_{H \in \mathcal{L}_1[\sigma]} K \delta_H^m \boldsymbol{e}_H = \bigoplus_{H \in \sigma} K \delta_H^m \boldsymbol{e}_H \to E_0[\sigma] = \sum_{H \in \mathcal{L}_1[\sigma]} K \delta_H^m = \sum_{H \in \sigma} K \delta_H^m \to 0.$$

This is an isomorphism, since $|\sigma| \leq 2 + m - 1$ (see Lemma 6.1).

Now assume that $|\sigma| \ge 1$ and $n \ge 3$. Fix $j \in \sigma$ and put $\tau := \sigma \setminus \{j\}$. Then $E_*[\tau]$ and $E_*[\{j\}] = E_*(\mathcal{A}^{H_j})$ (by (9.5)) are subcomplexes of $E_*[\sigma]$, which are exact by the induction hypothesis and Lemma 9.5. Moreover there exists an exact sequence of complexes:

$$0 \to E_*[\tau] \cap E_*[\{j\}] \to E_*[\tau] \oplus E_*[\{j\}] \to E_*[\sigma] \to 0$$

Since $E_*[\tau] \cap E_*[\{j\}] = E_*[\tau](\mathcal{A}^{H_j})$ and $|\tau| \leq (n-1) + m - 1$, we are done.

Put

 $\sigma_0 := \{1, 2, \dots, m\},\$

and

$$\bar{E}_* := E_*[\sigma_0].$$

We use notation

$$\psi_j : \bar{E}_j \to \bar{E}_{j-1} \qquad (j = 1, 2, \dots, n-1).$$

Put

$$F_j := S \otimes \overline{E}_j \qquad (j = 0, 1, \dots, n-1).$$

Note that F_i is a submodule of

$$S\Delta_{i,n-1}[\sigma_0] = \bigoplus_{\mathcal{H}\in\mathcal{L}_i[\sigma_0]} \bigoplus_{\mathcal{H}'\in(\mathcal{A}\setminus\mathcal{H})^{(i-1)}} S\delta^m_{\mathcal{H}\cup\mathcal{H}'} \boldsymbol{e}_{\wedge\mathcal{H}'} \boldsymbol{e}_{\mathcal{H}}.$$

For $i \geq 2$, the morphism $d_i : F_i \to F_{i-1}$ is defined by

$$\boldsymbol{e}_{\wedge\mathcal{H}'}\boldsymbol{e}_{\mathcal{H}}\mapsto\sum_{H'\in\mathcal{H}'}(-1)^{l_{\mathcal{H}'}(H')}p_{H'}\boldsymbol{e}_{\wedge(\mathcal{H}'\setminus\{H'\})}\boldsymbol{e}_{\mathcal{H}\cup\{H'\}}.$$

Note that

$$F_0 = S \otimes_K \sum_{\mathcal{H} \in \mathcal{L}_1[\sigma_0]} K \delta_{\mathcal{H}}^m,$$

and

$$F_1 = \bigoplus_{\mathcal{H} \in \mathcal{L}_1[\sigma_0]} S \delta^m_{\mathcal{H}} \boldsymbol{e}_{\mathcal{H}}$$

We define a morphism $d_1: F_1 \to F_0$ by

$$\delta^m_{\mathcal{H}} \boldsymbol{e}_{\mathcal{H}} \mapsto P_{\mathcal{H}} \delta^m_{\mathcal{H}}.$$

Lemma 9.7. The sequence

$$0 \to F_{n-1} \stackrel{d_{n-1}}{\to} F_{n-2} \stackrel{d_{n-2}}{\to} \cdots \stackrel{d_2}{\to} F_1 \stackrel{d_1}{\to} F_0 \to 0$$

is a complex.

Proof. By the definition of d_i , clearly $d_i \circ d_{i+1} = 0$ for $i \ge 2$. We prove $d_1 \circ d_2 = 0$. Let $X = \sum_{\mathcal{H} \in \mathcal{L}_2[\sigma_0]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H}, \mathcal{H}} \delta^m_{\mathcal{H} \cup \{H\}} \boldsymbol{e}_{\wedge \mathcal{H}} \boldsymbol{e}_{\mathcal{H}} \in F_2$. Then

$$\sum_{H \notin \mathcal{H}} f_{\mathcal{H},H} \delta^m_{\mathcal{H} \cup \{H\}} = 0 \quad \text{for all } \mathcal{H} \in \mathcal{L}_2[\sigma_0].$$

We have

$$d_{1} \circ d_{2}(X) = d_{1} \left(\sum_{\mathcal{H} \in \mathcal{L}_{2}[\sigma_{0}]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H},H} p_{H} \delta^{m}_{\mathcal{H} \cup \{H\}} e_{\mathcal{H} \cup \{H\}} \right)$$

$$= \sum_{\mathcal{H} \in \mathcal{L}_{2}[\sigma_{0}]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H},H} p_{H} P_{\mathcal{H} \cup \{H\}} \delta^{m}_{\mathcal{H} \cup \{H\}}$$

$$= \sum_{\mathcal{H} \in \mathcal{L}_{2}[\sigma_{0}]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H},H} P_{\mathcal{H}} \delta^{m}_{\mathcal{H} \cup \{H\}} \quad (\text{Here } P_{\mathcal{H}} := \prod_{H \notin \mathcal{H}} p_{H}.)$$

$$= \sum_{\mathcal{H} \in \mathcal{L}_{2}[\sigma_{0}]} P_{\mathcal{H}} \sum_{H \notin \mathcal{H}} f_{\mathcal{H},H} \delta^{m}_{\mathcal{H} \cup \{H\}} = 0.$$

The following is Theorem 8.3.

Lemma 9.8 (cf. Lemma 2.1 in [11]). Assume that m < r - n + 1. Then the image of d_1 coincides with $\Xi^{(m)}(\mathcal{A})$.

By Remark 9.4, we have the following.

Remark 9.9 (cf. Remark 2.2 in [11]).

$$\operatorname{rank}_{S}(F_{j}) = \binom{r-m-n+j-1}{j-1} \left(\binom{r}{n-j} - \binom{r-m}{n-j} \right) =: w_{j}^{(m)}.$$

Under the above preparations, we can prove the following theorem. Since the proof is almost the same as that of [11, Theorem 2.3], we omit it.

Theorem 9.10 (cf. Theorem 2.3 in [11]). Assume that $n \ge 3$ and m < r - n + 1. Then the complex

$$F_*: 0 \to F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} \Xi^{(m)}(\mathcal{A}) \to 0$$

16

is a minimal free resolution of $\Xi^{(m)}(\mathcal{A})$. In particular, the projective dimensions of S-modules $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ are equal to n-2.

By Theorem 8.3, Remark 9.9, and the construction of the complex F_* in Theorem 9.10, we have the following corollary:

Corollary 9.11 (cf. Corollary 4.4.3 in [7]). Assume that $n \ge 3$ and m < r - n + 1. Then there exist exact sequences

$$0 \rightarrow S(m+1-r)^{w_{n-1}^{(m)}} \rightarrow \cdots \rightarrow S(m+n-j-r)^{w_{j}^{(m)}} \rightarrow \cdots$$

$$\rightarrow S(m+n-2-r)^{w_{2}^{(m)}} \rightarrow S(m+n-1-r)^{w_{1}^{(m)}} \rightarrow \Xi^{(m)}(\mathcal{A}) \rightarrow 0,$$

$$0 \rightarrow S(m+1-r)^{w_{n-1}^{(m)}} \rightarrow \cdots \rightarrow S(m+n-j-r)^{w_{j}^{(m)}} \rightarrow \cdots$$

$$\rightarrow S(m+n-2-r)^{w_{2}^{(m)}} \rightarrow S(m+n-1-r)^{w_{1}^{(m)}} \bigoplus S \rightarrow D^{(m)}(\mathcal{A}) \rightarrow 0$$

where $w_i^{(m)}$ were defined in Remark 9.9, and all maps are homogeneous of degree 0.

In particular, the Castelnuovo-Mumford regularities of $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ are equal to r-m-n+1.

Remark 9.12. If we use the polynomial degrees in $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ as the degrees of graded S-modules, then the degrees are shifted by m. Then the Castelnuovo-Mumford regularities of $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ are equal to r - n + 1 as stated for $D^{(1)}(\mathcal{A})$ in [1, Section 5.2], and the Poicaré-Betti series of $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ coincide with the ones conjectured by Snellman [9, Conjecture 6.8].

10. MINIMAL FREE RESOLUTION OF $J_m(\mathcal{A})$

In this section, we generalize the minimal free resolution of S/J given in [7], where J is the Jacobian ideal of Q. We retain the assumptions $n \ge 3$ and n + m - 1 < r.

Let $J_m(\mathcal{A})$ denote the S-submodule of $S^{\binom{n+m-1}{m-1}} = \bigoplus_{|\mathcal{B}| \leq m-1} S \boldsymbol{e}_{\mathcal{B}}$ generated by all

(10.1)
$$\frac{1}{\alpha!}\partial^{\alpha} \bullet Q := \left(\frac{1}{(\alpha-\beta)!}\partial^{\alpha-\beta} * Q : |\beta| \le m-1\right) = \sum_{|\beta| \le m-1} \frac{1}{(\alpha-\beta)!}\partial^{\alpha-\beta} * Qe_{\beta}$$

with $1 \leq |\boldsymbol{\alpha}| \leq m$. Here we agree $\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} = 0$ for $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$.

Example 10.1. Let m = 1. Then $J_1(\mathcal{A})$ is the S-submodule of S generated by $\partial_j * Q$ (j = 1, ..., n), i.e., $J_1(\mathcal{A})$ is nothing but the Jacobian ideal J of Q.

Lemma 10.2. For all $\alpha, \beta \in \mathbb{N}^n$,

$$\frac{1}{(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} = (-1)^{|\boldsymbol{\beta}|} \frac{(\mathrm{ad}x)^{\boldsymbol{\beta}}}{\boldsymbol{\alpha}!} (\partial^{\boldsymbol{\alpha}}).$$

Here we denote by $\operatorname{ad} x_i$ the endomorphism of D(S): $D(S) \ni P \mapsto \operatorname{ad} x_i(P) = [x_i, P] \in D(S)$. For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, we set $(\operatorname{ad} x)^{\beta} = (\operatorname{ad} x_1)^{\beta_1} \circ \cdots \circ (\operatorname{ad} x_n)^{\beta_n}$.

Proof. We prove the assertion by induction on $|\boldsymbol{\beta}|$. For all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$,

$$\operatorname{ad} x_{i}((-1)^{|\boldsymbol{\beta}|} \frac{(\operatorname{ad} x)^{\boldsymbol{\beta}}}{\boldsymbol{\alpha}!}(\partial^{\boldsymbol{\alpha}})) = \frac{1}{(\boldsymbol{\alpha} - \boldsymbol{\beta})!} \operatorname{ad} x_{i}(\partial^{\boldsymbol{\alpha} - \boldsymbol{\beta}})$$
$$= -\frac{1}{(\boldsymbol{\alpha} - \boldsymbol{\beta})!} (\alpha_{i} - \beta_{i}) \partial^{\boldsymbol{\alpha} - \boldsymbol{\beta} - \mathbf{1}_{i}}$$
$$= -\frac{1}{(\boldsymbol{\alpha} - \boldsymbol{\beta} - \mathbf{1}_{i})!} \partial^{\boldsymbol{\alpha} - \boldsymbol{\beta} - \mathbf{1}_{i}}.$$

By Lemma 10.2,

$$\frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}} \bullet Q = ((-1)^{|\boldsymbol{\beta}|} (\mathrm{ad}x)^{\boldsymbol{\beta}} (\frac{1}{\boldsymbol{\alpha}!} \partial^{\boldsymbol{\alpha}}) * Q : |\boldsymbol{\beta}| \le m-1).$$

We define an S-module morphism

$$\delta_0: F_0^{[1,m]} := D^{[1,m]}(S) := \bigoplus_{k=1}^m D^{(k)}(S) \to S^{\binom{n+m-1}{m-1}} = \bigoplus_{|\beta| \le m-1} Se_\beta$$

by

(10.2)
$$\delta_0(\theta) := \theta \bullet Q := ((-1)^{|\boldsymbol{\beta}|} (\mathrm{ad} x)^{\boldsymbol{\beta}}(\theta) * Q : |\boldsymbol{\beta}| \le m-1).$$

By definition,

(10.3)
$$\operatorname{Im} \delta_0 = J_m(\mathcal{A}).$$

Lemma 10.3. Let $\theta \in D(S)$. Then

$$\theta x^{\boldsymbol{\beta}} = \sum_{\boldsymbol{\gamma} \leq \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix} x^{\boldsymbol{\beta} - \boldsymbol{\gamma}} (\mathrm{ad} x)^{\boldsymbol{\gamma}} (\theta),$$

where $\binom{\beta}{\gamma} = \prod_{i=1}^n \binom{\beta_i}{\gamma_i}$.

Proof. We prove the assertion by induction on $|\beta|$. We have

$$\begin{aligned} \theta x_i x^{\boldsymbol{\beta}} &= -(\operatorname{ad} x_i(\theta)) x^{\boldsymbol{\beta}} + x_i \theta x^{\boldsymbol{\beta}} \\ &= -\sum_{\gamma \leq \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta} - \boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}} (\operatorname{ad} x_i(\theta)) + x_i \sum_{\gamma \leq \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta} - \boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta) \\ &= \sum_{\gamma \leq \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma} + \mathbf{1}_i|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta} + \mathbf{1}_i - \boldsymbol{\gamma} - \mathbf{1}_i} (\operatorname{ad} x)^{\boldsymbol{\gamma} + \mathbf{1}_i}(\theta) + \sum_{\gamma \leq \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta} + \mathbf{1}_i - \boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta) \\ &= \sum_{\gamma - \mathbf{1}_i \leq \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma} - \mathbf{1}_i} x^{\boldsymbol{\beta} + \mathbf{1}_i - \boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta) + \sum_{\gamma \leq \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta} + \mathbf{1}_i - \boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta) \\ &= \sum_{\gamma \leq \boldsymbol{\beta} + \mathbf{1}_i} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta} + \mathbf{1}_i}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta} + \mathbf{1}_i - \boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta). \end{aligned}$$

Let $\bar{\delta}_0$ denote the composite of δ_0 with the canonical projections of Se_β onto $(S/SQ)e_\beta$ for $\beta \neq 0$:

(10.4)
$$\bar{\delta}_0: D^{[1,m]}(S) \xrightarrow{\delta_0} \bigoplus_{|\boldsymbol{\beta}| \le m-1} S \boldsymbol{e}_{\boldsymbol{\beta}} \to S \boldsymbol{e}_{\boldsymbol{0}} \bigoplus_{0 \ne |\boldsymbol{\beta}| \le m-1} (S/SQ) \boldsymbol{e}_{\boldsymbol{\beta}}.$$

Here note that $\bar{\delta}_0$ is a graded S-module homomorphism homogeneous of degree 0 if we put $\deg(\boldsymbol{e}_{\boldsymbol{\beta}}) = -r - |\boldsymbol{\beta}|$.

In the following two lemmas, we describe the cokernel and the kernel of $\bar{\delta}_0$.

Lemma 10.4.

$$\operatorname{Coker} \bar{\delta}_0 = S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}}).$$

Proof. By (10.3), we only need to show $Q\mathbf{e}_0 \in \operatorname{Im} \overline{\delta}_0$. We have $\epsilon_1 * Q = rQ$. Since $\epsilon_1 \in D(\mathcal{A})$, we see $\delta_0(\epsilon_1) \in \bigoplus_{|\boldsymbol{\beta}| < m-1} SQ\mathbf{e}_{\boldsymbol{\beta}}$ by the definition of δ_0 (10.2). Hence

$$Q\boldsymbol{e_0} = \bar{\delta}_0(\frac{1}{r}\epsilon_1) \in \operatorname{Im}\bar{\delta}_0$$

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Lemma 10.5.

$$\operatorname{Ker} \bar{\delta}_0 = \bigoplus_{k=1}^m D^{(k)}(\mathcal{A})' =: D^{[1,m]}(\mathcal{A})',$$

where $D^{(k)}(\mathcal{A})' := \{ \theta \in D^{(k)}(\mathcal{A}) : \theta * Q = 0 \}.$

Proof. If $\theta \in D^{[1,m]}(\mathcal{A})'$, then $(\mathrm{ad} x)^{\beta}(\theta) \in D(\mathcal{A})$ for all β , and $\theta * Q = 0$. Hence $\theta \in \mathrm{Ker} \,\bar{\delta}_0$ by the definitions of $D(\mathcal{A})$ and $\bar{\delta}_0$.

Next we suppose that $\theta \in \operatorname{Ker} \overline{\delta}_0$. Then by Lemma 10.3

(10.5)
$$\theta * x^{\beta} Q \in \langle Q \rangle = QS \text{ for all } \beta \text{ with } |\beta| \le m - 1.$$

By [5, Proposition 2.3], we conclude that $\theta \in D^{[1,m]}(\mathcal{A})'$.

Lemma 10.6. Let $k \leq r$. As S-modules,

$$\Xi^{(k)}(\mathcal{A}) \simeq D^{(k)}(\mathcal{A})'.$$

Proof. It is easy to see that

$$\gamma_k : \Xi^{(k)}(\mathcal{A}) \ni \theta \mapsto \theta - \frac{\theta * Q}{Q} \frac{\epsilon_k}{r(r-1)\cdots(r-k+1)} \in D^{(k)}(\mathcal{A})'$$

and

$$D^{(k)}(\mathcal{A})' \ni \theta \mapsto \theta - \frac{\theta * (p_1 \cdots p_k)}{p_1 \cdots p_k} \frac{\epsilon_k}{k!} \in \Xi^{(k)}(\mathcal{A})$$

are inverse to each other.

For $1 \le k \le m$, let $F_*^{(k)}$ denote the minimal free resolution of $\Xi^{(k)}$ in Theorem 9.10. We consider the following complex:

(10.6)
$$0 \to \tilde{F}_{n-1} \xrightarrow{\tilde{\delta}_{n-1}} \cdots \xrightarrow{\tilde{\delta}_2} \tilde{F}_1 \xrightarrow{\tilde{\delta}_1} \tilde{F}_0 \xrightarrow{\tilde{\delta}_0} \tilde{F}_{-1} \to \operatorname{Coker}(\tilde{\delta}_0) \to 0,$$

where

$$\tilde{F}_{-1} = \bigoplus_{|\boldsymbol{\beta}| \le m-1} S \boldsymbol{e}_{\boldsymbol{\beta}},$$

$$\tilde{F}_{0} = D^{[1,m]}(S) \bigoplus_{0 \ne |\boldsymbol{\beta}| \le m-1} S \boldsymbol{e}_{\boldsymbol{\beta}},$$

$$\tilde{F}_{j} = \bigoplus_{k=1}^{m} F_{j}^{(k)} \quad (j = 1, \dots, n-1)$$

and

$$\begin{split} \tilde{\delta}_{0}(\theta, \sum_{\beta \neq \mathbf{0}} f_{\beta} \boldsymbol{e}_{\beta}) &= \delta_{0}(\theta) + \sum_{\beta \neq \mathbf{0}} f_{\beta} Q \boldsymbol{e}_{\beta} = \theta * Q \boldsymbol{e}_{\mathbf{0}} + \sum_{\beta \neq \mathbf{0}} ((-1)^{|\beta|} (\mathrm{ad} x)^{\beta}(\theta) * Q + f_{\beta} Q) \boldsymbol{e}_{\beta}, \\ \tilde{\delta}_{1}(\delta_{\mathcal{H}}^{k} \boldsymbol{e}_{\mathcal{H}}^{(k)}) &= (\gamma_{k}(P_{\mathcal{H}} \delta_{\mathcal{H}}^{k}), -\frac{1}{Q} \sum_{\beta \neq \mathbf{0}} (-1)^{|\beta|} (\mathrm{ad} x)^{\beta} (\gamma_{k}(P_{\mathcal{H}} \delta_{\mathcal{H}}^{k})) * Q \boldsymbol{e}_{\beta}), \\ \tilde{\delta}_{j} &= \bigoplus_{k=1}^{m} d_{j}^{(k)} \qquad (j \geq 2). \end{split}$$

Recall that $D^{(k)}(S) = F_0^{(k)}$, and $d_1(\delta_{\mathcal{H}}^k \boldsymbol{e}_{\mathcal{H}}^{(k)}) = P_{\mathcal{H}} \delta_{\mathcal{H}}^k$ for $1 \le k \le m$.

Theorem 10.7 (cf. Theorem 4.5.3 in [7]). The complex (10.6) is a minimal free resolution of $\operatorname{Coker}(\bar{\delta}_0) = S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}}).$

Proof. The complex (10.6) is exact by Theorem 8.3, Theorem 9.10, Lemma 10.4, Lemma 10.5, and Lemma 10.6. The operator $P_{\mathcal{H}}\delta^k_{\mathcal{H}}$ is of order k and homogeneous of polynomial degree deg $(P_{\mathcal{H}}) = r - (n-1)$. Then each term of $\gamma_k(P_{\mathcal{H}}\delta^k_{\mathcal{H}})$ is of order k and of polynomial degree greater than or equal to k. Hence each term of the operator $(\mathrm{ad} x)^{\beta}(\gamma_k(P_{\mathcal{H}}\delta^k_{\mathcal{H}}))$ is of order $k - |\beta|$ and of polynomial degree greater than or equal to k. Therefore each term of the polynomial

$$\frac{1}{Q}(-1)^{|\boldsymbol{\beta}|}(\mathrm{ad}x)^{\boldsymbol{\beta}}(\gamma_k(P_{\mathcal{H}}\delta_{\mathcal{H}}^k)) * Q$$

is of degree greater than or equal to

$$r - (k - |\beta|) + k - r = |\beta| > 0.$$

Thus the free resolution (10.6) of $\operatorname{Coker}(\tilde{\delta}_0)$ is minimal. Clearly by (10.4)

$$\operatorname{Coker}(\tilde{\delta}_0) = S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}}) = \operatorname{Coker}(\bar{\delta}_0).$$

The following corollary is clear from Theorem 10.7 and the Auslander-Buchsbaum formula.

Corollary 10.8 (cf. Corollary 4.5.5 [7]). The projective dimension of the S-module $S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A})+QS^{\binom{n+m-1}{m-1}})$ is n, and the depth is 0.

In the complex (10.6), the degrees of elements of bases are as follows:

$$deg(\boldsymbol{e}_{\boldsymbol{\beta}}) = -r - |\boldsymbol{\beta}| \qquad \text{in } \tilde{F}_{-1}$$

$$deg(\partial^{\boldsymbol{\alpha}}) = -|\boldsymbol{\alpha}| \qquad \text{in } \tilde{F}_{0},$$

$$deg(\boldsymbol{e}_{\boldsymbol{\beta}}) = -|\boldsymbol{\beta}| \qquad \text{in } \tilde{F}_{0},$$

$$deg(\delta_{\mathcal{H}}^{k}\boldsymbol{e}_{\mathcal{H}}) = -k + r - (n-1) = r - n - k + 1 \quad \text{in } \tilde{F}_{1}.$$

Hence we have the following corollary:

Corollary 10.9 (cf. Corollary 4.5.4 in [7]). Assume that $n \ge 3$ and m < r - n + 1. Then there exists an exact sequence

$$0 \rightarrow \bigoplus_{k=1}^{m} S(k+1-r)^{w_{n-1}^{(k)}} \rightarrow \cdots \rightarrow \bigoplus_{k=1}^{m} S(k+n-j-r)^{w_{j}^{(k)}} \rightarrow \cdots$$
$$\rightarrow \bigoplus_{k=1}^{m} S(k+n-1-r)^{w_{1}^{(k)}} \rightarrow \bigoplus_{k=1}^{m} S(k)^{s_{k}} \bigoplus \bigoplus_{k=1}^{m-1} S(k)^{s_{k}} \rightarrow$$
$$\bigoplus_{k=0}^{m-1} S(r+k)^{s_{k}} \rightarrow \operatorname{Coker}(\bar{\delta}_{0}) \rightarrow 0,$$

where $w_j^{(k)}$ were defined in Remark 9.9, $s_k = \binom{n+k-1}{k}$, and all maps are homogeneous of degree 0.

In particular, the Castelnuovo-Mumford regularity of $\operatorname{Coker}(\overline{\delta}_0)$ is equal to r-n-2.

Remark 10.10. In Corollary 10.9, to make the degrees of all the minimal generators of $\operatorname{Coker}(\overline{\delta}_0)$ nonnegative, we can shift the degrees by r + (m - 1) as in [7, Corollary 4.5.5]. Then the Castelnuovo-Mumford regularity of $\operatorname{Coker}(\overline{\delta}_0)$ is equal to 2r + m - n - 3.

11. Jet modules

In this section, we prove that $\operatorname{Coker}(\bar{\delta}_0) = S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$ in Section 10 is the transpose of the *m*-jet module $\Omega^{[1,m]}(S/SQ)$. For the basics of jet modules, see [2], [3], and [10].

Let $I := \langle f_1, \ldots, f_k \rangle$ be an ideal of S. Let R := S/I. Define jet modules

(11.1)
$$\Omega^{[1,m]}(S) := J_S / J_S^{m+1}, \qquad \Omega^{\leq m}(S) := S \otimes_K S / J_S^{m+1}, \\ \Omega^{[1,m]}(R) := J_R / J_R^{m+1}, \qquad \Omega^{\leq m}(R) := R \otimes_K R / J_R^{m+1},$$

where

$$J_S := \langle 1 \otimes a - a \otimes 1 | a \in S \rangle \subseteq S \otimes_K S,$$

$$J_R := \langle 1 \otimes a - a \otimes 1 | a \in R \rangle \subseteq R \otimes_K R.$$

Then $\Omega^{\leq m}(R)$ is the representative object of the functor $M \to D^m_R(R, M)$, i.e., there exists a natural isomorphism of *R*-modules:

$$D_R^m(R, M) \simeq \operatorname{Hom}_R(\Omega^{\leq m}(R), M)$$

where M is an R-module, and $D_R^m(R, M)$ is the module of differential operators of order $\leq m$ from R to M.

As S-modules,

$$\Omega^{\leq m}(S) = \Omega^{[1,m]}(S) \bigoplus S \otimes 1, \qquad \Omega^{\leq m}(R) = \Omega^{[1,m]}(R) \bigoplus R \otimes 1.$$

Here note that S acts as $S \otimes 1$. We have

$$\{P \in D_R^m(R, M) \mid P * 1 = 0\} \simeq \operatorname{Hom}_R(\Omega^{[1,m]}(R), M)$$

for an R-module M.

For $a \in S$ (or R), we denote $1 \otimes a - a \otimes 1 \mod J_S^{m+1}$ (or J_R^{m+1} , respectively) by da. Then, for $f, g \in R$, we have

(11.2)
$$d(fg) = f \, dg + g \, df + (df)(dg)$$

As an S-module

$$\Omega^{[1,m]}(S) = \bigoplus_{1 \le |\alpha| \le m} S(dx)^{\alpha}.$$

For $f \in S$, we have

(11.3)
$$df = \sum_{1 \le |\boldsymbol{\alpha}| \le m} \frac{1}{\boldsymbol{\alpha}!} (\partial^{\boldsymbol{\alpha}} * f) (dx)^{\boldsymbol{\alpha}}.$$

We have a surjective $S \otimes S$ -module homomorphism

$$\varphi: \Omega^{[1,m]}(S) \ni (dx)^{\alpha} \mapsto (d\bar{x})^{\alpha} \in \Omega^{[1,m]}(R).$$

Lemma 11.1. As an S-module,

$$\operatorname{Ker} \varphi = \sum_{i; 1 \le |\alpha| \le m} Sf_i(dx)^{\alpha} + \sum_{i; 0 \le |\alpha| \le m-1} S(df_i)(dx)^{\alpha}.$$

Proof. The inclusion (\supset) is clear. We prove the other inclusion.

First we prove that

(11.4)
$$\operatorname{Ker}\varphi = I\,dS + S\,dI.$$

Clearly the kernel of the $S \otimes S$ -module homomorphism :

$$\Omega^{\leq m}(S) \ni f \otimes g \mapsto \bar{f} \otimes \bar{g} \in \Omega^{\leq m}(R)$$

equals $(S \otimes I + I \otimes S)/J_S^{m+1}$ or $(S dI + I \otimes S)/J_S^{m+1}$. Hence, to prove (11.4), it is enough to show that

$$(11.5) (I \otimes S) \cap J_S = I \, dS.$$

Let $\sum_k i_k \otimes g_k \in J_S$ with $i_k \in I, g_k \in S$. Then $\sum_k i_k g_k = 0$. We have

$$\sum_{k} i_k \otimes g_k = \sum_{k} (i_k \otimes g_k - i_k g_k \otimes 1) + \sum_{k} i_k g_k \otimes 1 = \sum_{k} i_k dg_k + 0 \in I \, dS.$$

Hence we have proved (11.5) and in turn (11.4). Thus as an S-module

$$\operatorname{Ker} \varphi = \sum_{1 \le |\alpha| \le m} I(dx)^{\alpha} + \sum_{0 \le |\alpha| < m} SdI(dx)^{\alpha}.$$

To finish the proof, we only need to show that $d(f_i x^{\alpha})$ belongs to the right hand of the assertion for any α . This is done by (11.2):

$$d(f_i x^{\alpha}) = f_i d(x^{\alpha}) + x^{\alpha} df_i + (df_i)(d(x^{\alpha})).$$

Hence we have an S-free presentation of $\Omega^{[1,m]}(R)$:

(11.6) $(\bigoplus_{i;1\leq |\boldsymbol{\alpha}|\leq m} Sf_i(dx)^{\boldsymbol{\alpha}}) \oplus (\bigoplus_{i;0\leq |\boldsymbol{\beta}|\leq m-1} S(df_i)(dx)^{\boldsymbol{\beta}}) \to \Omega^{[1,m]}(S) \to \Omega^{[1,m]}(R) \to 0.$

Now we consider the case I = SQ:

(

(11.7)
$$\left(\bigoplus_{1\leq |\boldsymbol{\alpha}|\leq m} SQ(dx)^{\boldsymbol{\alpha}}\right) \oplus \left(\bigoplus_{0\leq |\boldsymbol{\beta}|\leq m-1} S(dQ)(dx)^{\boldsymbol{\beta}}\right) \to \Omega^{[1,m]}(S) \to \Omega^{[1,m]}(S/SQ) \to 0.$$

Hence, as an S/SQ-module, $\Omega^{[1,m]}(S/SQ)$ has a presentation:

(11.8)
$$\bigoplus_{0 \le |\boldsymbol{\beta}| \le m-1} (S/SQ) (dQ) (dx)^{\boldsymbol{\beta}} \to \bigoplus_{1 \le |\boldsymbol{\alpha}| \le m} (S/SQ) (dx)^{\boldsymbol{\alpha}} \to \Omega^{[1,m]} (S/SQ) \to 0.$$

Note that by (11.3)

$$(dQ)(dx)^{\beta} = \sum_{|\alpha+\beta| \le m, \alpha \neq 0} \frac{1}{\alpha!} (\partial^{\alpha} * Q)(dx)^{\alpha+\beta}$$
$$= \sum_{|\gamma| \le m, \gamma \neq \beta} \frac{1}{(\gamma-\beta)!} (\partial^{\gamma-\beta} * Q)(dx)^{\gamma}$$

Hence the $(\boldsymbol{\beta}, \boldsymbol{\gamma})$ -component of the matrix of (11.8) equals $\frac{1}{(\boldsymbol{\gamma} - \boldsymbol{\beta})!} (\partial^{\boldsymbol{\gamma} - \boldsymbol{\beta}} * Q).$

By Lemma 10.4, the S/SQ-module $S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$ has a presentation:

(11.9)
$$\bigoplus_{1 \le |\boldsymbol{\gamma}| \le m} (S/SQ) \frac{1}{\boldsymbol{\gamma}!} \partial^{\boldsymbol{\gamma}} \stackrel{\bullet}{\to} \bigoplus_{0 \le |\boldsymbol{\beta}| \le m-1} (S/SQ) \boldsymbol{e}_{\boldsymbol{\beta}}$$
$$\to S^{\binom{n+m-1}{m-1}} / (J_m(\boldsymbol{\mathcal{A}}) + QS^{\binom{n+m-1}{m-1}}) \to$$

and the $(\boldsymbol{\gamma}, \boldsymbol{\beta})$ -component of the matrix of the map \bullet in (11.9) (recall (10.1)) equals $\frac{1}{(\boldsymbol{\gamma}-\boldsymbol{\beta})!}(\partial^{\boldsymbol{\gamma}-\boldsymbol{\beta}}*Q).$

0,

Thus we have proved the following theorem.

Theorem 11.2. The S/SQ-module $S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$ is the transpose of $\Omega^{[1,m]}(S/SQ)$.

Corollary 11.3. The S/SQ-modules $S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$ and $\Omega^{[1,m]}(S/SQ)$ share the same Fitting ideals.

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