Parameterized Complexity of the Spanning Tree Congestion Problem

Hans L. Bodlaender[1] Fedor V. Fomin[2]
Petr A. Golovach[3] Yota Otachi[4] Erik Jan van Leeuwen[2]

[1]Utrecht University, the Netherlands
[2]University of Bergen, Norway
[3]Durham University, United Kingdom
[4]Tohoku University, Japan

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Outline

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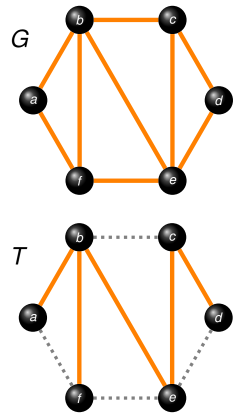
Definition: Spanning Tree Congestion



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T: a spanning tree of a graph G.

- The *detour* of $\{u, v\} \in E(G)$ is the unique u–v path in T.
- The *congestion of $e \in E(T)$*, $cng_{G,T}(e)$ is the number of edges in G whose detours contain e.
- The *congestion of T*, $cng_G(T)$ is the max. congestion over all its edges.
- The *spanning tree congestion* of G, $stc(G)$ is the min. congestion over all its spanning trees.

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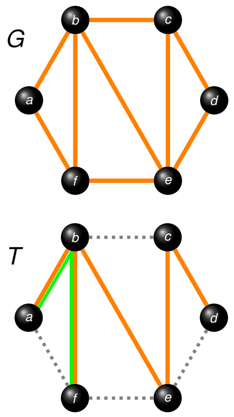

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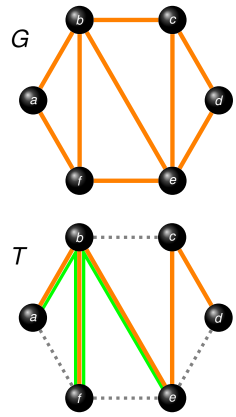

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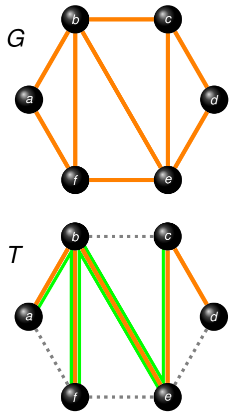

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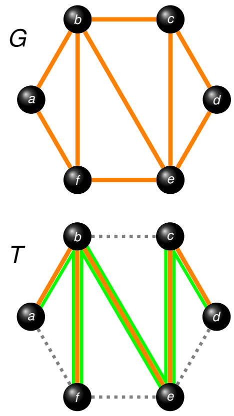

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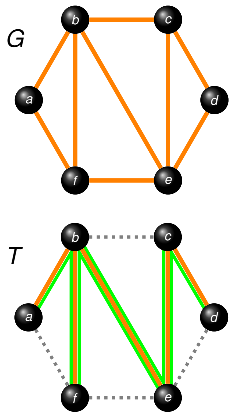

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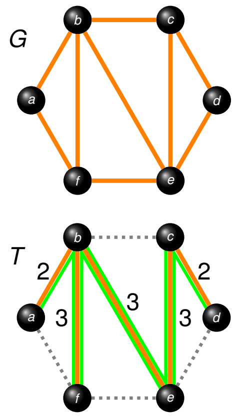

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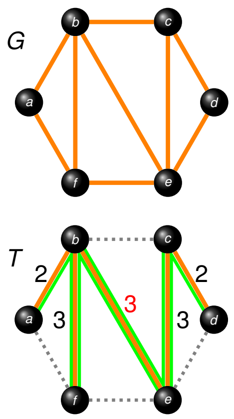

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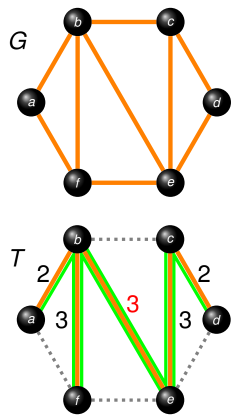

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The problems

Problem: STC
Instance: Connected graph G, positive integer k.
Question: $stc(G) \leq k$?

Problem: k-STC
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Note: k is a fixed constant.

We investigate the complexity of STC and k-STC.
(k-STC is a parameterized version of STC.)

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Previous work

S.T.C. is a relatively new graph parameter.

History

Simonson '87 (implicitly) Bounds for outerplanar graphs
1990's (implicitly) in papers on Tree Spanner problems
Ostrovskii '04 named the parameter S.T.C.
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No complexity result (to the best of my knowledge).

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Our results

Theorem (Positive results)
k-STC is linear time solvable for each of the following cases:
1. $k \leq 3$.
2. *input graphs are apex-minor-free.*
3. *input graphs have bounded maximum degree.*

Theorem (Negative results)
k-STC is NP-complete even if the following conditions hold:
- $k \geq 8$,
- *input graphs are K_6-minor-free, and*
- *input graphs have only one vertex of unbounded degree.*

STC is NP-complete for planar graphs.
No PTAS, unless P $=$ NP.

Apex-minor-free graphs

Definition (Apex graphs)
An *apex graph* is a graph that can be made planar by the removal of a single vertex.

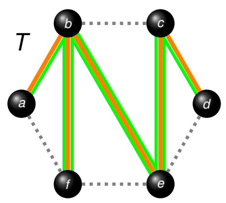
Examples of apex graphs
K_5, $K_{3,n}$ for any n (and of course all planar graphs).

Definition (Apex-minor-free graphs)
A graph class is *apex-minor-free* if it excludes a fixed apex graph as a minor.

Examples of apex-minor-free graphs
Planar graphs, bounded genus graphs, and graphs of bounded treewidth.

Outline

Related problems



stretch = 2
congestion = 3

stretch = 3
congestion = 4

stretch: the length of the longest detour.

Tree spanner problem
Minimize the stretch.

Bandwidth & Cutwidth problems
Embed a graph on a line (or, find a good linear arrangement) so that the stretch (bandwidth) or the congestion (cutwidth) is minimized.

	line	spanning tree
stretch	Bandwidth	Tree Spanner
congestion	Cutwidth	S.T.C.

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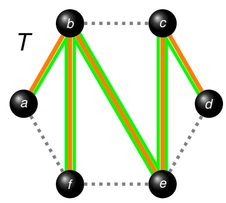

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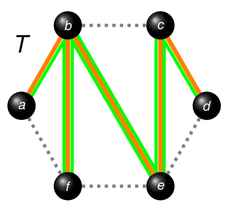

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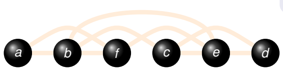

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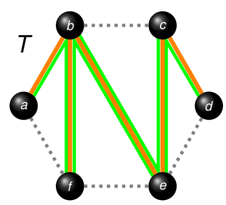

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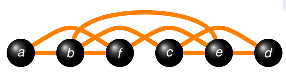

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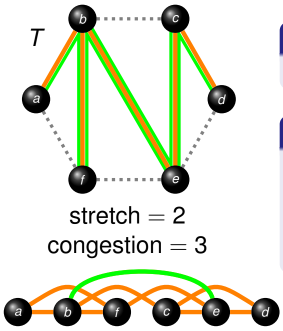

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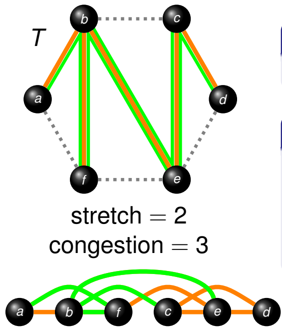

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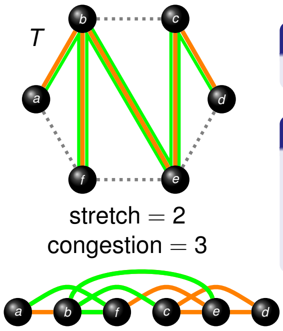

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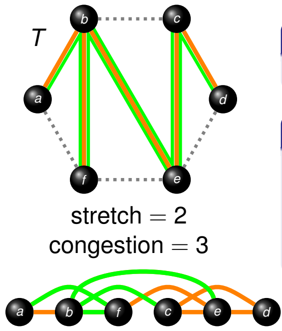

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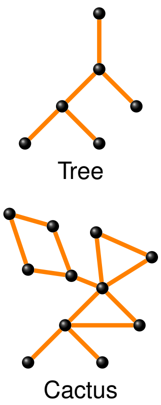
k-STC for $k \leq 3$ (1 of 2)



Theorem
$stc(G) = 1 \iff$ *G is a tree*
$stc(G) \leq 2 \iff$ *G is a cactus*

Good characterization for "$stc(G) \leq 3$" ?

Lemma
If $stc(G) \leq 3$, then G is planar.

Proof.
1. K_n subdivision $\implies stc \geq n - 1$
2. $K_{n,n}$ subdivision $\implies stc \geq n + 1$

Nonplanarity $\implies K_5$ or $K_{3,3}$ subdivision $\implies stc \geq 4$. □

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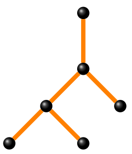
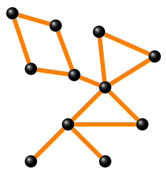


Tree



Cactus

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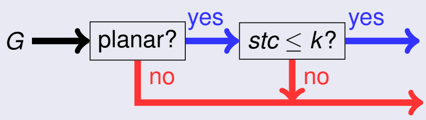
k-STC for $k \le 3$ (2 of 2)

Theorem

k-STC is solvable in linear time if $k \le 3$.

Proof.

1. Check the planarity of G in linear time.
2. If the answer is NO, then $stc(G) \ge 4$.
3. Otherwise, use our linear time algorithm for planar graphs. (Or, we can use another *practical* algorithm.)



□

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k-STC for graphs of bounded degree (1 of 2)

We need the following two lemmas. ($\Delta(G)$ is the max degree)

Lemma

$tw(G) \le \Delta(G)(stc(G) - 1)/2$.

Lemma

k-STC is linear time solvable for graphs of bounded treewidth.

Proof of the second lemma.

k-STC can be expressed in MSO logic. Courcelle's Theorem implies the lemma. □

MSO logic expression for k-STC

$$\begin{aligned}
\mathrm{Deg1}(v_1, E_1) &:= (\exists e_1 \in E_1)(\forall e_2 \in E_1)(e_1 = e_2 \iff \mathbf{inc}(v_1, e_2))\\
\mathrm{Part}(V_1, V_2, V_3) &:= V_2 \neq \emptyset \wedge V_3 \neq \emptyset \wedge (V_2 \cup V_3 = V_1) \wedge (V_2 \cap V_3 = \emptyset)\\
\mathrm{Adj}(v_1, v_2, E_1) &:= v_1 \neq v_2 \wedge (\exists e_1 \in E_1)(\mathbf{inc}(v_1, e) \wedge \mathbf{inc}(v_2, e))\\
E_1 = \mathrm{Ind}(V_1) &:= (\forall e_1)(e_1 \in E_1 \iff (\exists v_1, v_2 \in V_1)(v_1 \neq v_2 \wedge \mathbf{inc}(v_1, e_1) \wedge \mathbf{inc}(v_2, e_1)))\\
E_1 = \mathrm{Inc}_{\mathrm{E}}(v_1) &:= (\forall e_1)(e_1 \in E_1 \iff \mathbf{inc}(v_1, e_1))\\
V_1 = \mathrm{Inc}_{\mathrm{V}}(E_1) &:= (\forall v_1)(v_1 \in V_1 \iff (\exists e_1 \in E_1)(\mathbf{inc}(v_1, e_1)))\\
\mathrm{Conn}(E_1) &:= (\forall V_2, V_3)(\mathrm{Part}(\mathrm{Inc}_{\mathrm{V}}(E_1), V_2, V_3) \implies (\exists v_2 \in V_2, v_3 \in V_3)(\mathrm{Adj}(v_2, v_3, E_1)))\\
\mathrm{BiConn}(E_1) &:= (\exists v_1, v_2, v_3 \in \mathrm{Inc}_{\mathrm{V}}(E_1))(v_i \neq v_j)(1 \le i < j \le 3) \wedge (\forall v_4)(\mathrm{Conn}(E_1 \setminus \mathrm{Inc}_{\mathrm{E}}(v_4)))\\
\mathrm{Forest}(E_1) &:= (\forall V_1 \subseteq \mathrm{Inc}_{\mathrm{V}}(E_1))(\neg \mathrm{BiConn}(\mathrm{Ind}(V_1) \cap E_1))\\
\mathrm{Tree}(E_1) &:= \mathrm{Forest}(E_1) \wedge \mathrm{Conn}(E_1)\\
\mathrm{Path}(v_1, v_2, E_1) &:= \mathrm{Tree}(E_1) \wedge (\forall v_3 \in \mathrm{Inc}_{\mathrm{V}}(E_1))(\mathrm{Deg1}(v_3, E_1) \iff v_3 = v_1 \vee v_3 = v_2)\\
\mathrm{SpnTree}(E_1) &:= \mathrm{Tree}(E_1) \wedge (\forall v)(v \in \mathrm{Inc}_{\mathrm{V}}(E_1))\\
\mathrm{Detour}(e_1, E_1) &:= (\exists v_1, v_2)(v_1 \neq v_2 \wedge \mathbf{inc}(v_1, e_1) \wedge \mathbf{inc}(v_2, e_1) \wedge \mathrm{Path}(v_1, v_2, E_1))\\
\mathrm{Cong}_k(e_0, E_0) &:= \neg(\exists e_1, \ldots, e_k)((e_i \notin E_0)(1 \le i \le k) \wedge e_i \neq e_j(1 \le i < j \le k)\\
&\qquad \wedge (\exists E_i)(e_0 \in E_i \wedge E_i \subseteq E_0 \wedge \mathrm{Detour}(e_i, E_i))(1 \le i \le k))
\end{aligned}$$

$$stc(G) \le k \iff G \models (\exists E_0)(\mathrm{SpnTree}(E_0) \wedge (\forall e_0 \in E_0)(\mathrm{Cong}_k(e_0, E_0))).$$

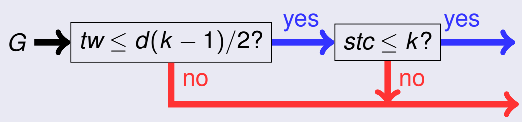
k-STC for graphs of bounded degree (2 of 2)

Theorem

k-STC is linear time solvable on graphs of bounded degree.

Proof.

1. Check $tw(G) \le d(k-1)/2$ in linear time. ($d := \Delta(G)$)
2. If the answer is NO, then $stc(G) > k$.
3. O.w., use a linear time algorithm for bounded treewidth.



We use Bodlaender's algorithm to check bounded tw. □

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Theorem (Negative results)

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- $k \geq 8$,
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- *input graphs have only one vertex of unbounded degree.*

STC is NP-complete for planar graphs.
No PTAS, unless $\mathrm{P} = \mathrm{NP}$.

Open problems

- Complexity of k-STC for $k \in \{4, 5, 6, 7\}$.
  - I think 4-STC might be NP-complete.
- Complexity of STC and k-STC for some graph classes.
  - Recent progress with Y. Okamoto, R. Uehara, and T. Uno: STC is NP-complete for split graphs and chordal bipartite graphs, and linear time solvable for trivially perfect graphs.
- Approximation. ($O(\log n)$-factor approximation?)
  - Constant factor approximation might be NP-hard.
- Is STC $\in$ FPT for k-outerplanar graphs? (parameter is k)
  - If $k = 1$, i.e., for outerplanar graphs, STC can be solved in linear time. (with Bodlaender, Kozawa, Matsushima)

Thank you for your attention!

Introduction Sketch of proofs Conclusion

Open problems

- Complexity of k-STC for $k \in \{4, 5, 6, 7\}$.
  - I think 4-STC might be NP-complete.
- Complexity of STC and k-STC for some graph classes.
  - Recent progress with Y. Okamoto, R. Uehara, and T. Uno: STC is NP-complete for split graphs and chordal bipartite graphs, and linear time solvable for trivially perfect graphs.
- Approximation. ($O(\log n)$-factor approximation?)
  - Constant factor approximation might be NP-hard.
- Is STC $\in$ FPT for k-outerplanar graphs? (parameter is k)
  - If $k = 1$, i.e., for outerplanar graphs, STC can be solved in linear time. (with Bodlaender, Kozawa, Matsushima)

Thank you for your attention!

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