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Abstract

Gel'fand, Kapranov and Zelevinsky proved, using the theory of perverse sheaves, that in the Cohen–Macaulay case an A-hypergeometric system is irreducible if its parameter vector is non-resonant. In this paper we prove, using the theory of the ring of differential operators on an affine toric variety, that in general an A-hypergeometric system is irreducible if and only if its parameter vector is non-resonant. In the course of the proof, we determine the irreducible quotients of an A-hypergeometric system.

1. Introduction

Let K be a field of characteristic 0, and let $A := (a_{ij})$ be a $d \times n$ integer matrix. We assume that $\mathbb{Z}^d$ is generated by the column vectors of A as an abelian group. Given a parameter vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)^{\mathrm{T}} \in K^d$, the *$A$-hypergeometric* (or *GKZ*) *system $M_A(\boldsymbol{\beta})$ with parameter vector $\boldsymbol{\beta}$* is defined by

$$M_A(\boldsymbol{\beta}) := D(K^n)/D(K^n)I_A(\partial) + D(K^n)\langle A\theta - \boldsymbol{\beta}\rangle, \tag{1}$$

where $D(K^n)$ is the nth Weyl algebra, i.e.

$$D(K^n) = K[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n\rangle, \tag{2}$$

$I_A(\partial)$ is the toric ideal of $K[\partial_1, \ldots, \partial_n]$ defined by A, and $D(K^n)\langle A\theta - \boldsymbol{\beta}\rangle$ is the left ideal of $D(K^n)$ generated by $\sum_{j=1}^n a_{ij}x_j\partial_j - \beta_i$, $i = 1, \ldots, d$.

The irreducibility of $M_A(\boldsymbol{\beta})$ is one of the most fundamental questions in the theory of A-hypergeometric systems. Gel'fand *et al.* proved, using the theory of perverse sheaves, that when the toric ring is Cohen–Macaulay, $M_A(\boldsymbol{\beta})$ is irreducible if its parameter vector $\boldsymbol{\beta}$ is non-resonant; see [GKZ90, Proposition 4.4 and Theorem 4.6]. Schulze and Walther have determined for which parameter vector $\boldsymbol{\beta}$ the Fourier transform of $M_A(\boldsymbol{\beta})$ is naturally isomorphic to the direct image of a simple object on the big torus of the affine toric variety defined by A (see [SW09, Corollary 3.7]), which sharpens [GKZ90, Theorem 4.6]. Walther proved in [Wal07, Theorem 3.13] that if $M_A(\boldsymbol{\beta})$ has irreducible monodromy representation, then so does $M_A(\boldsymbol{\gamma})$ for any $\boldsymbol{\gamma} \in \boldsymbol{\beta} + \mathbb{Z}^d$, using homological tools developed in [MMW05]. Naturally, an irreducible $D(K^n)$-module has irreducible monodromy representation; see Proposition 6.8.

In this paper, using the theory of the ring of differential operators on an affine toric variety, we prove that $M_A(\boldsymbol{\beta})$ is irreducible if and only if $\boldsymbol{\beta}$ is non-resonant, without assuming that the toric ring is Cohen–Macaulay. Moreover, in the course of the proof, we determine the irreducible quotients of $M_A(\boldsymbol{\beta})$.

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Let ι be the anti-automorphism of $D(K^n)$ defined by $\iota(x_j) = \partial_j$ and $\iota(\partial_j) = x_j$ for $j = 1, \dots, n$. Then ι gives rise to the equivalence between the category of left $D(K^n)$-modules and the category of right $D(K^n)$-modules; the left $D(K^n)$-module $M_A(\boldsymbol{\beta})$ corresponds to the right $D(K^n)$-module $M_{K^n}(\boldsymbol{\beta})$ (whose definition is given in (8)). Hence the irreducibility of $M_A(\boldsymbol{\beta})$ is equivalent to that of $M_{K^n}(\boldsymbol{\beta})$. In this paper, we work with the categories of right D-modules. This has two advantages: one is that the support of $M_{K^n}(\boldsymbol{\beta})$ is precisely the affine toric variety defined by A; the other is that we consider direct image functors of D-modules, and for this purpose, right D-modules work more naturally than left D-modules.

In § 2 we introduce the varieties considered in this paper, and in § 3 we briefly recall the rings of differential operators on these varieties and their $\mathbb{Z}^d$-gradings.

In § 4, for each variety X introduced in § 2 we consider the category $\mathcal{O}_X$, which is analogous to the category $\mathcal{O}$ from the theory of highest-weight modules over semisimple Lie algebras defined in [BGG76] (cf. [MV98, Sai07]). We then recall the simple objects in $\mathcal{O}_X$ for $X = X_A$, the affine toric variety defined by A (see Proposition 4.3), and for $X = T_A$, the big torus of X_A (see Proposition 4.2). Finally, we define Verma-type modules in $\mathcal{O}_X$. The right-module counterpart $M_{K^n}(\boldsymbol{\beta})$ of the A-hypergeometric system $M_A(\boldsymbol{\beta})$ is a Verma-type module in $\mathcal{O}_{K^n}$.

In § 5, we explicitly describe the direct image functors of D-modules by inclusions between the varieties under consideration. Using this description, in § 6 we show that the direct image of a simple object in $\mathcal{O}_{T_A}$ by the inclusion of T_A into K^n has a unique irreducible $D(K^n)$-submodule, and we describe it explicitly (see Theorem 6.4). We then show that each simple object in $\mathcal{O}_{K^n}$ is obtained in a similar way from a possibly smaller torus (Theorem 6.6).

In § 7, we compute the pull-back of each simple object in $\mathcal{O}_{K^n}$ by the inclusion of X_A into K^n (Theorems 7.3 and 7.4). As a consequence, we determine the irreducible quotients of $M_{K^n}(\boldsymbol{\beta})$ (Corollaries 7.5 and 7.6). In § 8, we prove that $M_{K^n}(\boldsymbol{\beta})$ is irreducible if and only if $\boldsymbol{\beta}$ is non-resonant (Theorem 8.3).

2. Varieties

Let $A := \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\}$ be a finite set of column vectors in $\mathbb{Z}^d$. We will sometimes identify A with the matrix $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) = (a_{ij})$. Let $\mathbb{Z}A$ and $\mathbb{R}_{\geqslant 0}A$ denote, respectively, the abelian group and the cone generated by A. Throughout this paper, we assume that $\mathbb{Z}A = \mathbb{Z}^d$ and that $\mathbb{R}_{\geqslant 0}A$ is strongly convex.

Let K denote a field of characteristic 0. For a face τ of the cone $\mathbb{R}_{\geqslant 0}A$, we define the following varieties:

$$\begin{aligned}
K^\tau &:= \{\boldsymbol{x} = (x_1, \dots, x_n) \in K^n : x_j = 0 \text{ when } \boldsymbol{a}_j \notin \tau\},\\
(K^\times)^\tau &:= \{\boldsymbol{x} \in K^\tau : x_j \neq 0 \text{ when } \boldsymbol{a}_j \in \tau\},\\
X_\tau &:= \{\boldsymbol{x} \in K^\tau : x^{\boldsymbol{u}} - x^{\boldsymbol{v}} = 0 \text{ for } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n \text{ such that } A\boldsymbol{u} = A\boldsymbol{v}\},\\
T_\tau &:= \{\boldsymbol{x} \in (K^\times)^\tau : x^{\boldsymbol{u}} - x^{\boldsymbol{v}} = 0 \text{ for } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n \text{ such that } A\boldsymbol{u} = A\boldsymbol{v}\}.
\end{aligned}$$

Here we have used multi-index notation, where $x^{\boldsymbol{u}}$ stands for $x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$, with $\boldsymbol{u} = (u_1, u_2, \dots, u_n)^{\mathrm{T}}$. When τ is the whole cone $\mathbb{R}_{\geqslant 0}A$, we denote the above varieties by K^n, $(K^\times)^n$, X_A and T_A, respectively. Then

$$X_A = \coprod_{\text{faces } \tau \text{ of } \mathbb{R}_{\geqslant 0}A} T_\tau \tag{3}$$

is the $(K^\times)^d$-orbit decomposition of the toric variety X_A (see, e.g., [Ful93]). Here $(K^\times)^d$ acts on K^n by

$$(K^\times)^d \times K^n \ni (t, (x_1, \dots, x_n)) \mapsto (t^{\boldsymbol{a}_1} x_1, \dots, t^{\boldsymbol{a}_n} x_n) \in K^n,$$

where $t^{\boldsymbol{a}} = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\boldsymbol{a} = (a_1, a_2, \dots, a_d)^{\mathrm{T}}$.

Let $\mathbb{N}A$ denote the monoid generated by A. The semigroup algebra $K[\mathbb{N}A] = \bigoplus_{\boldsymbol{a}\in\mathbb{N}A} Kt^{\boldsymbol{a}}$ is the ring of regular functions on the affine toric variety X_A. Then we have $K[\mathbb{N}A] \simeq K[x]/I_A$, where I_A is the ideal of the polynomial ring $K[x] := K[x_1, \dots, x_n]$ generated by all $x^{\boldsymbol{u}} - x^{\boldsymbol{v}}$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n$ with $A\boldsymbol{u} = A\boldsymbol{v}$.

3. Rings of differential operators

Let R be a commutative K-algebra, and let M and N be R-modules. We briefly recall the module $D(M, N)$ of differential operators from M to N; for details, see [SS88]. For $k \in \mathbb{N}$, the subspaces $D^k(M, N)$ of $\mathrm{Hom}_K(M, N)$ are defined inductively by

$$D^0(M, N) := \mathrm{Hom}_R(M, N)$$

and

$$D^{k+1}(M, N) := \{P \in \mathrm{Hom}_K(M, N) : [f, P] \in D^k(M, N) \text{ for all } f \in R\},$$

where $[\,,]$ denotes the commutator. Set $D(M, N) := \bigcup_{k=0}^\infty D^k(M, N)$ and $D(M) := D(M, M)$. Then $D(M)$ is a K-algebra, and $D(M, N)$ is a $(D(N), D(M))$-bimodule.

The ring $D(K^n) := D(K[x])$ of differential operators on K^n is the nth Weyl algebra (2).

The ring $D((K^\times)^n) := D(K[x_1^{\pm1}, \dots, x_n^{\pm1}])$ of differential operators on $(K^\times)^n$ is given by

$$\begin{aligned} D((K^\times)^n) &= K[x_1^{\pm1}, \dots, x_n^{\pm1}]\langle \partial_1, \dots, \partial_n\rangle \\ &= \bigoplus_{\boldsymbol{u}\in\mathbb{Z}^n} x^{\boldsymbol{u}} K[\theta_1, \dots, \theta_n], \end{aligned}$$

where $\theta_j = x_j \partial_j$.

The ring $D(T_A) := D(K[t_1^{\pm1}, \dots, t_d^{\pm1}])$ of differential operators on T_A is given by

$$\begin{aligned} D(T_A) &= K[t_1^{\pm1}, \dots, t_d^{\pm1}]\langle \partial_{t_1}, \dots, \partial_{t_d}\rangle \\ &= \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} t^{\boldsymbol{a}} K[s_1, \dots, s_d], \end{aligned}$$

where $s_i = t_i \partial_{t_i}$ and $\partial_{t_i} = \partial/\partial t_i$.

The ring $D(X_A) := D(K[\mathbb{N}A])$ of differential operators on X_A is a subalgebra of $D(T_A)$:

$$D(X_A) = \{P \in D(T_A) : P(K[\mathbb{N}A]) \subseteq K[\mathbb{N}A]\}.$$

Let X be K^n, $(K^\times)^n$, T_A or X_A. For $\boldsymbol{a} = (a_1, \dots, a_d)^{\mathrm{T}} \in \mathbb{Z}^d$, set

$$D(X)_{\boldsymbol{a}} := \{P \in D(X) : [s_i, P] = a_i P \text{ for } i = 1, \dots, d\},$$

where $s_i = \sum_{j=1}^n a_{ij} x_j \partial_j$ for $X = K^n$ or $(K^\times)^n$. Then

$$D(X) = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} D(X)_{\boldsymbol{a}}$$

is a $\mathbb{Z}^d$-graded algebra.

Let τ be a face of the cone $\mathbb{R}_{\geqslant 0}A$. Let $\mathbb{Z}(A\cap\tau)$ and $\mathbb{N}(A\cap\tau)$ denote, respectively, the abelian group and the monoid generated by $A\cap\tau$. Set

$$\mathbb{Z}^\tau := \{\boldsymbol{u}=(u_1,\dots,u_n)\in\mathbb{Z}^n : u_j=0 \text{ when } \boldsymbol{a}_j\notin\tau\}.$$

As in the case where τ is the whole cone $\mathbb{R}_{\geqslant 0}A$, for $K^\tau, (K^\times)^\tau, T_\tau$ and X_τ we consider the following rings of differential operators:

$$\begin{aligned}
D(K^\tau) &= D(K[x_j : \boldsymbol{a}_j\in\tau]) = K[x_j : \boldsymbol{a}_j\in\tau]\langle\partial_j : \boldsymbol{a}_j\in\tau\rangle,\\
D((K^\times)^\tau) &= K[x_j^{\pm1} : \boldsymbol{a}_j\in\tau]\langle\partial_j : \boldsymbol{a}_j\in\tau\rangle = \bigoplus_{\boldsymbol{u}\in\mathbb{Z}^\tau} x^{\boldsymbol{u}}K[\theta_j : \boldsymbol{a}_j\in\tau],\\
D(T_\tau) &= \bigoplus_{\boldsymbol{a}\in\mathbb{Z}(A\cap\tau)} t^{\boldsymbol{a}}K[s_{1|\tau},\dots,s_{d|\tau}],\\
D(X_\tau) &= \{P\in D(T_\tau) : P(K[X_\tau])\subseteq K[X_\tau]\},
\end{aligned}$$

where $s_{i|\tau}$ is the operator s_i restricted to $K[T_\tau]=K[t^{\pm\boldsymbol{a}_j} : \boldsymbol{a}_j\in\tau]$ and $K[X_\tau]$ is the subalgebra of $K[T_\tau]$ defined by

$$K[X_\tau]=K[\mathbb{N}(A\cap\tau)]=K[t^{\boldsymbol{a}_j} : \boldsymbol{a}_j\in\tau].$$

These rings of differential operators are graded by $\mathbb{Z}(A\cap\tau)$, and since $\mathbb{Z}(A\cap\tau)$ is a subgroup of $\mathbb{Z}A=\mathbb{Z}^d$, they are also considered to be $\mathbb{Z}^d$-graded. Note that $s_{i|\tau}=\sum_{\boldsymbol{a}_j\in\tau}a_{ij}\theta_j$ in x-coordinates.

4. The category $\mathcal{O}_X$

Take X to be K^n, $(K^\times)^n$, T_A or X_A. We shall define a full subcategory $\mathcal{O}_X$ of the category of right $D(X)$-modules (cf. [MV98]). A right $D(X)$-module M is an object of $\mathcal{O}_X$ if the support of M is contained in X_A and M has a weight decomposition $M=\bigoplus_{\boldsymbol{\lambda}\in K^d}M_{\boldsymbol{\lambda}}$, where

$$M_{\boldsymbol{\lambda}}=\{x\in M : x.f(s)=f(-\boldsymbol{\lambda})x \text{ for all } f\in K[s]\}$$

with $K[s]=K[s_1,\dots,s_d]$.

Proposition 4.1. *Let M be a simple object in $\mathcal{O}_X$. Then M is an irreducible right $D(X)$-module.*

Proof. Let N be a right $D(X)$-submodule of M. Let $x\in N$, and write $x=\sum_{\boldsymbol{b}\in S}x_{\boldsymbol{b}}$ for $x_{\boldsymbol{b}}\in M_{\boldsymbol{b}}$, where S is a finite subset of K^d. For $\boldsymbol{b}\in S$, take $f(s)\in K[s]$ such that $f(-\boldsymbol{b})\neq 0$ and $f(-\boldsymbol{c})=0$ for all $\boldsymbol{c}\in S\backslash\{\boldsymbol{b}\}$. Upon applying $f(s)$ to x, we see that $x_{\boldsymbol{b}}\in N$. Hence $N\in\mathcal{O}_X$. By the simplicity of M in $\mathcal{O}_X$, we have $N=0$ or $N=M$. □

In the rest of this section, we define objects $L_{T_A}(\boldsymbol{\lambda})$ and $L_{X_A}(\boldsymbol{\lambda})$ which are simple in the categories $\mathcal{O}_{T_A}$ and $\mathcal{O}_{X_A}$, respectively. Then we define Verma-type modules $M_{X_A}(\boldsymbol{\beta})$, $M_{K^n}(\boldsymbol{\beta})$ and $M_{(K^\times)^n}(\boldsymbol{\beta})$.

Let $\boldsymbol{\lambda}\in K^d$. We define a right $D(T_A)$-module $L_{T_A}(\boldsymbol{\lambda})$ by

$$L_{T_A}(\boldsymbol{\lambda}) := D(T_A)/\langle s-\boldsymbol{\lambda}\rangle D(T_A) := D(T_A)\Big/\sum_{i=1}^d(s_i-\lambda_i)D(T_A).$$

Let $K[t^{\pm1}]$ denote the Laurent polynomial ring $K[t_1^{\pm1},\dots,t_d^{\pm1}]$. By taking formal adjoint operators, $D(T_A)$ acts on $K[t^{\pm1}]t^{-\boldsymbol{\lambda}}\,dT_A$ from the right as follows:

$$(g(t)\,dT_A).P = P^*(g)\,dT_A,$$

where

$$P^* = \sum_{\boldsymbol{a}} f_{\boldsymbol{a}}(-s) t^{\boldsymbol{a}}$$

for $P = \sum_{\boldsymbol{a}} t^{\boldsymbol{a}} f_{\boldsymbol{a}}(s) \in \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} t^{\boldsymbol{a}} K[s] = D(T_A)$ and dT_A is simply a formal symbol. Then $K[t^{\pm 1}]t^{-\boldsymbol{\lambda}}\, dT_A$ is a realization of $L_{T_A}(\boldsymbol{\lambda})$, and we denote $K[t^{\pm 1}]t^{-\boldsymbol{\lambda}}\, dT_A$ by $L_{T_A}(\boldsymbol{\lambda})$, so that

$$L_{T_A}(\boldsymbol{\lambda}) = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} L_{T_A}(\boldsymbol{\lambda})_{-\boldsymbol{\lambda}+\boldsymbol{a}} \quad \text{with } L_{T_A}(\boldsymbol{\lambda})_{-\boldsymbol{\lambda}+\boldsymbol{a}} = K t^{-\boldsymbol{\lambda}+\boldsymbol{a}}\, dT_A. \tag{4}$$

The following proposition is clear.

Proposition 4.2. *Each $L_{T_A}(\boldsymbol{\lambda})$ is a simple object in $\mathcal{O}_{T_A}$. Each simple object in $\mathcal{O}_{T_A}$ is isomorphic to $L_{T_A}(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in K^d$, and $L_{T_A}(\boldsymbol{\lambda}) \simeq L_{T_A}(\boldsymbol{\mu})$ if and only if $\boldsymbol{\lambda} - \boldsymbol{\mu} \in \mathbb{Z}^d$.*

Recall that the ring $D(X_A)$ is described as follows (see [Mus87, Theorem 2.3]):

$$D(X_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} \mathbb{I}(\Omega(\boldsymbol{a})) \quad \text{for } \boldsymbol{a} \in \mathbb{Z}^d,$$

where

$$\begin{aligned} \Omega(\boldsymbol{a}) &:= \Omega_A(\boldsymbol{a}) := \mathbb{N}A \backslash (-\boldsymbol{a} + \mathbb{N}A), \\ \mathbb{I}(\Omega(\boldsymbol{a})) &:= \{ f(s) \in K[s] : f(\boldsymbol{c}) = 0 \text{ for all } \boldsymbol{c} \in \Omega(\boldsymbol{a}) \}. \end{aligned} \tag{5}$$

Recall also the preorder $\preceq$ defined in [MV98] (see also [ST01]):

$$\text{for } \boldsymbol{\alpha}, \boldsymbol{\beta} \in K^d, \quad \boldsymbol{\alpha} \preceq \boldsymbol{\beta} \iff \mathbb{I}(\Omega(\boldsymbol{\beta} - \boldsymbol{\alpha})) \not\subseteq \mathfrak{m}_{\boldsymbol{\alpha}}, \tag{6}$$

where $\mathfrak{m}_{\boldsymbol{\alpha}}$ is the maximal ideal of $K[s]$ at $\boldsymbol{\alpha}$. We define an equivalence relation $\sim$ by setting $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ if and only if $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$. We write $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ if $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \not\sim \boldsymbol{\beta}$.

Since the ring $D(X_A)$ is a subalgebra of $D(T_A)$, the right $D(T_A)$-module

$$L_{T_A}(\boldsymbol{\lambda}) = K[t^{\pm 1}]t^{-\boldsymbol{\lambda}}\, dT_A = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} K t^{-\boldsymbol{\lambda}+\boldsymbol{a}}\, dT_A$$

is also a right $D(X_A)$-module. Then the subquotient

$$L_{X_A}(\boldsymbol{\lambda}) := \bigoplus_{\boldsymbol{\mu}\preceq\boldsymbol{\lambda}} K t^{-\boldsymbol{\mu}}\, dT_A \Big/ \bigoplus_{\boldsymbol{\mu}\prec\boldsymbol{\lambda}} K t^{-\boldsymbol{\mu}}\, dT_A \tag{7}$$

is a right $D(X_A)$-module (see [ST01, Proposition 4.1.5]). We have the following proposition.

Proposition 4.3. *Each $L_{X_A}(\boldsymbol{\lambda})$ is a simple object in $\mathcal{O}_{X_A}$. Each simple object in $\mathcal{O}_{X_A}$ is isomorphic to $L_{X_A}(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in K^d$. Moreover, $L_{X_A}(\boldsymbol{\lambda}) \simeq L_{X_A}(\boldsymbol{\mu})$ if and only if $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$.*

(See [MV98, Proposition 3.1.7], [ST01, Theorem 4.1.6] or [Sai07, Proposition 3.6(4)].)

For $\boldsymbol{\beta} \in K^d$, we define a right $D(X_A)$-module $M_{X_A}(\boldsymbol{\beta})$, a right $D(K^n)$-module $M_{K^n}(\boldsymbol{\beta})$ and a right $D((K^\times)^n)$-module $M_{(K^\times)^n}(\boldsymbol{\beta})$ by

$$\begin{aligned} M_{X_A}(\boldsymbol{\beta}) &:= D(X_A)/\langle s - \boldsymbol{\beta} \rangle D(X_A), \\ M_{K^n}(\boldsymbol{\beta}) &:= D(K^n)/(I_A D(K^n) + \langle s - \boldsymbol{\beta} \rangle D(K^n)), \\ M_{(K^\times)^n}(\boldsymbol{\beta}) &:= D((K^\times)^n)/(I_A D((K^\times)^n) + \langle s - \boldsymbol{\beta} \rangle D((K^\times)^n)). \end{aligned} \tag{8}$$

Recall that $s_i = t_i \partial_{t_i}$ in t-coordinates and that $s_i = \sum_{j=1}^n a_{ij}\theta_j$ with $\theta_j = x_j \partial_j$ in x-coordinates. Clearly, $M_{X_A}(\boldsymbol{\beta}) \in \mathcal{O}_{X_A}$, $M_{K^n}(\boldsymbol{\beta}) \in \mathcal{O}_{K^n}$ and $M_{(K^\times)^n}(\boldsymbol{\beta}) \in \mathcal{O}_{(K^\times)^n}$.

Let τ be a face of the cone $\mathbb{R}_{\geqslant 0}A$. Similarly to the case where τ is the whole cone $\mathbb{R}_{\geqslant 0}A$, for $Y = K^\tau, (K^\times)^\tau, T_\tau$ or X_τ we consider $\mathcal{O}_Y$, replacing $\mathbb{Z}A = \mathbb{Z}^d$, $KA = K^d$ and $f(s) \in K[s]$

by $\mathbb{Z}(A \cap \tau)$, $K(A \cap \tau)$ and $f(s)_{|\tau}$, respectively, where $f(s)_{|\tau}$ is the operator $f(s)$ restricted to $K[T_\tau] = K[t^{\pm \boldsymbol{a}_j} : \boldsymbol{a}_j \in \tau]$.

5. Direct image functors

In this section, we describe direct image functors explicitly. Using them, we link some of the modules defined in § 4.

5.1 From $\mathcal{O}_{T_A}$ to $\mathcal{O}_{(K^\times)^n}$

We shall write $D((K^\times)^n, T_A)$ instead of $D(K[x^{\pm 1}], K[t^{\pm 1}])$, where $K[x^{\pm 1}]$ stands for the Laurent polynomial ring $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Since T_A is closed in $(K^\times)^n$, the direct image functor

$$\int^0_{T_A \to (K^\times)^n} : M \mapsto M \otimes_{D(T_A)} D((K^\times)^n, T_A)$$

gives a category equivalence between $\mathcal{O}_{T_A}$ and $\mathcal{O}_{(K^\times)^n}$, known as Kashiwara's equivalence (see, e.g., [Kas03, Theorem 4.30] or [HTT08, Theorem 1.6.1]). From [SS88, § 1.3, (e) and (f)], we have

$$\begin{aligned} D((K^\times)^n, T_A) &= D((K^\times)^n)/I_A D((K^\times)^n) \\ &= \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} t^{\boldsymbol{a}} K[\theta_1, \dots, \theta_n]. \end{aligned} \tag{9}$$

By definition,

$$M_{(K^\times)^n}(\boldsymbol{\beta}) = \int^0_{T_A \to (K^\times)^n} L_{T_A}(\boldsymbol{\beta}). \tag{10}$$

Hence, by Kashiwara's equivalence, Proposition 4.2 leads to the following result.

Proposition 5.1. *For each $\boldsymbol{\beta} \in K^d$, $M_{(K^\times)^n}(\boldsymbol{\beta})$ is a simple object in $\mathcal{O}_{(K^\times)^n}$. Each simple object in $\mathcal{O}_{(K^\times)^n}$ is isomorphic to some $M_{(K^\times)^n}(\boldsymbol{\beta})$. Moreover, $M_{(K^\times)^n}(\boldsymbol{\beta}) \simeq M_{(K^\times)^n}(\boldsymbol{\beta}')$ if and only if $\boldsymbol{\beta} - \boldsymbol{\beta}' \in \mathbb{Z}^d$.*

5.2 From $\mathcal{O}_{X_A}$ to $\mathcal{O}_{K^n}$

Again from [SS88, § 1.3, (e) and (f)], we have

$$D(K^n, X_A) := D(K[x], K[\mathbb{N}A]) = D(K^n)/I_A D(K^n). \tag{11}$$

Since I_A is $\mathbb{Z}^d$-homogeneous, $D(K^n, X_A)$ inherits the $\mathbb{Z}^d$-grading from $D(K^n)$.

The algebra $D(X_A)$ can be identified with

$$\{P \in D(K^n) : P I_A \subseteq I_A D(K^n)\}/I_A D(K^n)$$

(see, e.g., [MR87, Theorem 5.13]). We may therefore consider $D(X_A)$ as being contained in $D(K^n, X_A)$.

Let $\int^0_{X_A \to K^n}$ denote the functor from $\mathcal{O}_{X_A}$ to $\mathcal{O}_{K^n}$ defined by

$$\int^0_{X_A \to K^n} M := M \otimes_{D(X_A)} D(K^n, X_A).$$

Note that, in general, X_A is singular and $\int^0_{X_A \to K^n}$ does not give a category equivalence. By definition, we have

$$M_{K^n}(\boldsymbol{\beta}) = \int^0_{X_A \to K^n} M_{X_A}(\boldsymbol{\beta}). \tag{12}$$

For the following result, see [Sai07, Proposition 4.1 and Corollary 4.2].

PROPOSITION 5.2.

$$D(K^n, X_A) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} D(K^n, X_A)_{\boldsymbol{a}} \quad \textit{with } D(K^n, X_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})),$$

where

$$\begin{aligned} \widetilde{\Omega}(\boldsymbol{a}) := \widetilde{\Omega}_A(\boldsymbol{a}) &:= \{\boldsymbol{u} \in \mathbb{N}^n : A\boldsymbol{u} \notin -\boldsymbol{a} + \mathbb{N}A\}, \\ \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})) \;&=\; \{f(\theta) \in K[\theta] : f(\boldsymbol{u}) = 0 \textit{ for all } \boldsymbol{u} \in \widetilde{\Omega}(\boldsymbol{a})\} \end{aligned} \tag{13}$$

and $K[\theta] := K[\theta_1, \ldots, \theta_n]$.

5.3 From $\mathcal{O}_{K^\tau}$ to $\mathcal{O}_{K^n}$

Let τ be a face of the cone $\mathbb{R}_{\geqslant 0}A$. We consider the direct image functor $\int^0_{K^\tau \to K^n}$ from $\mathcal{O}_{K^\tau}$ to $\mathcal{O}_{K^n}$. Given $M \in \mathcal{O}_{K^\tau}$, we define $\int^0_{K^\tau \to K^n} M \in \mathcal{O}_{K^n}$ by

$$\int^0_{K^\tau \to K^n} M := M \otimes_{D(K^\tau)} D(K^n, K^\tau),$$

where

$$D(K^n, K^\tau) := D(K[x], K[x_j : \boldsymbol{a}_j \in \tau]).$$

Put

$$\begin{aligned} K^{\tau^c} &:= \{\boldsymbol{x} = (x_1, \ldots, x_n) \in K^n : x_j = 0 \text{ when } \boldsymbol{a}_j \in \tau\}, \\ \mathbb{N}^{\tau^c} &:= \{\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n : a_j = 0 \text{ when } \boldsymbol{a}_j \in \tau\}, \\ \mathbb{Z}^{\tau^c} &:= \{\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n : a_j = 0 \text{ when } \boldsymbol{a}_j \in \tau\}. \end{aligned}$$

Then

$$\begin{aligned} D(K^n, K^\tau) &= D(K^n)/\langle x_j : \boldsymbol{a}_j \notin \tau \rangle D(K^n) \\ &= D(K^\tau) \boxtimes D(K^{\tau^c})/\langle x_j : \boldsymbol{a}_j \notin \tau \rangle D(K^{\tau^c}). \end{aligned}$$

Since, as right $D(K^{\tau^c})$-modules,

$$D(K^{\tau^c})/\langle x_j : \boldsymbol{a}_j \notin \tau \rangle D(K^{\tau^c}) \simeq \bigoplus_{\boldsymbol{b} \in \mathbb{Z}^{\tau^c}} K x^{-\boldsymbol{b}}\, d(K^\times)^{\tau^c} \Big/ \bigoplus_{\boldsymbol{b} \notin \mathbb{N}^{\tau^c}} K x^{-\boldsymbol{b}}\, d(K^\times)^{\tau^c},$$

we have

$$D(K^n, K^\tau) \simeq D(K^\tau) \boxtimes \bigoplus_{\boldsymbol{b} \in \mathbb{N}^{\tau^c}} K x^{-\boldsymbol{b}}\, d(K^\times)^{\tau^c}. \tag{14}$$

Hence

$$\int^0_{K^\tau \to K^n} M \simeq M \boxtimes \bigoplus_{\boldsymbol{b} \in \mathbb{N}^{\tau^c}} K x^{-\boldsymbol{b}}\, d(K^\times)^{\tau^c}. \tag{15}$$

6. Simple objects in $\mathcal{O}_{K^n}$

In this section, we describe the simple objects in $\mathcal{O}_{K^n}$ explicitly.

By (9), (10) and the realization (4), we have the following realization of $M_{(K^\times)^n}(\boldsymbol{\beta})$.

Lemma 6.1. *Let $\boldsymbol{\beta} \in KA = K^d$. Then*

$$M_{(K^\times)^n}(\boldsymbol{\beta}) = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} Kt^{-\boldsymbol{\beta}+\boldsymbol{a}}\, dT_A \otimes_{K[s]} K[\theta].$$

The $D(K^n)$-module $\displaystyle\int_{T_A\to K^n}^0 L_{T_A}(\boldsymbol{\beta})$ is defined to be the $D((K^\times)^n)$-module

$$\int_{T_A\to (K^\times)^n}^0 L_{T_A}(\boldsymbol{\beta}) = M_{(K^\times)^n}(\boldsymbol{\beta}), \tag{16}$$

considered as a $D(K^n)$-module.

Definition 6.2. Let $\boldsymbol{\beta} \in KA = K^d$. In $\boldsymbol{\beta} + \mathbb{Z}A = \boldsymbol{\beta} + \mathbb{Z}^d$ there exists a unique minimal equivalence class with respect to $\preceq$ (see Remark 6.3), which we denote by $\boldsymbol{\beta}^{\mathrm{empty}}$. Any fixed element belonging to the class is also denoted by $\boldsymbol{\beta}^{\mathrm{empty}}$.

Remark 6.3. In [Sai01] we defined, for a face τ and a parameter vector $\boldsymbol{\alpha} \in KA = K^d$, a finite set

$$E_\tau(\boldsymbol{\alpha}) = \{\boldsymbol{\lambda} \in K(A\cap\tau)/\mathbb{Z}(A\cap\tau) : \boldsymbol{\alpha} - \boldsymbol{\lambda} \in \mathbb{N}A + \mathbb{Z}(A\cap\tau)\}. \tag{17}$$

The class $\boldsymbol{\beta}^{\mathrm{empty}}$ is given by

$$E_\tau(\boldsymbol{\beta}^{\mathrm{empty}}) = \begin{cases} E_{\mathbb{R}_{\geqslant 0}A}(\boldsymbol{\beta}) & \text{if } \tau = \mathbb{R}_{\geqslant 0}A, \\ \emptyset & \text{if } \tau \neq \mathbb{R}_{\geqslant 0}A. \end{cases} \tag{18}$$

Theorem 6.4. *Let $\boldsymbol{\beta} \in KA/\mathbb{Z}A = K^d/\mathbb{Z}^d$, and fix an element $\boldsymbol{e} := \boldsymbol{\beta}^{\mathrm{empty}}$. Then*

$$\begin{aligned} L_{K^n}(T_A, \boldsymbol{\beta}) &:= (t^{-\boldsymbol{e}}\, dT_A \otimes 1)D(K^n) \\ &= \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} Kt^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes_{K[s]} \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})) \\ &\simeq D(K^n)/(I_A D(K^n) + D(K^n) \cap \langle s - \boldsymbol{e}\rangle D((K^\times)^n)) \end{aligned}$$

is a unique simple $D(K^n)$-submodule of $\int_{T_A\to K^n}^0 L_{T_A}(\boldsymbol{\beta})$.

Moreover, $L_{K^n}(T_A, \boldsymbol{\beta}) \simeq L_{K^n}(T_A, \boldsymbol{\beta}')$ if and only if $\boldsymbol{\beta} - \boldsymbol{\beta}' \in \mathbb{Z}^d$.

Proof. Recall that $\int_{T_A\to K^n}^0 L_{T_A}(\boldsymbol{\beta})$ is the module $M_{(K^\times)^n}(\boldsymbol{\beta})$ regarded as a $D(K^n)$-module (16). Hence $L_{K^n}(T_A, \boldsymbol{\beta})$ is isomorphic to $D(K^n)/(I_A D(K^n) + D(K^n) \cap \langle s - \boldsymbol{e}\rangle D((K^\times)^n))$ by the definition of $M_{(K^\times)^n}(\boldsymbol{\beta}) = M_{(K^\times)^n}(\boldsymbol{e})$. The first equation is clear from (11) and Proposition 5.2.

Let $y \in M_{(K^\times)^n}(\boldsymbol{\beta})_{\boldsymbol{\gamma}}$ be non-zero. We prove that $yD(K^n) \supseteq L_{K^n}(T_A, \boldsymbol{\beta})$. By multiplying a suitable $x^{\boldsymbol{u}}$ from the right, we may assume that

$$y = t^{-\boldsymbol{\beta}'}\, dT_A \otimes f(\theta) \quad \text{for some } \boldsymbol{\beta}' \sim \boldsymbol{e}. \tag{19}$$

Here $f(\theta) \notin \langle A\theta - \boldsymbol{\beta}'\rangle K[\theta]$ since $y \neq 0$. We shall use the symbols s and $A\theta$ interchangeably. We claim that

$$t^{-\boldsymbol{\beta}''}\, dT_A \otimes 1 \in yD(K^n) \quad \text{for some } \boldsymbol{\beta}'' \sim \boldsymbol{e}. \tag{20}$$

We take an element of type (19) in $yD(K^n)$ such that the total degree $\deg(f)$ of f is as small as possible, and we call this element y again. If $f(\theta) \in K[s]$, then clearly we have the claim (20). Suppose $f(\theta) \notin K[s]$. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n$ satisfy $A\boldsymbol{u} = A\boldsymbol{v}$. Since

$$f(\theta)(x^{\boldsymbol{u}} - x^{\boldsymbol{v}}) = (x^{\boldsymbol{u}} - x^{\boldsymbol{v}})f(\theta + \boldsymbol{u}) + x^{\boldsymbol{v}}(f(\theta + \boldsymbol{u}) - f(\theta + \boldsymbol{v})),$$

we have

$$y.(x^{\boldsymbol{u}} - x^{\boldsymbol{v}}) = t^{-\boldsymbol{\beta}' + A\boldsymbol{v}}\, dT_A \otimes (f(\theta + \boldsymbol{u}) - f(\theta + \boldsymbol{v})).$$

By the minimality of $\deg(f)$,

$$f(\theta + \boldsymbol{u}) - f(\theta + \boldsymbol{v}) \in \langle A\theta - (\boldsymbol{\beta}' - A\boldsymbol{v})\rangle K[\theta].$$

Hence, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n$ with $A\boldsymbol{u} = A\boldsymbol{v}$,

$$f(\theta + \boldsymbol{u}) - f(\theta + \boldsymbol{v}) \in \langle A\theta - (\boldsymbol{\beta}' - A\boldsymbol{v})\rangle K[\theta].$$

Since $f(\theta) \notin \langle A\theta - \boldsymbol{\beta}'\rangle K[\theta]$, there exists $\boldsymbol{z} \in K^n$ with $A\boldsymbol{z} = \boldsymbol{\beta}'$ such that $f(\boldsymbol{z}) \neq 0$. By Lemma 6.5 below, we have

$$f(\theta) \in f(\boldsymbol{z}) + \langle A\theta - \boldsymbol{\beta}'\rangle K[\theta].$$

Hence $y = t^{-\boldsymbol{\beta}'}\, dT_A \otimes f(\boldsymbol{z})$. We have thus proved claim (20).

Since $\boldsymbol{\beta}'' \sim \boldsymbol{e}$, there exists $p(s) \in \mathbb{I}(\Omega(\boldsymbol{\beta}'' - \boldsymbol{e}))$ such that $p(\boldsymbol{\beta}'') \neq 0$. Hence $t^{\boldsymbol{\beta}'' - \boldsymbol{e}} p(s) \in D(X_A) \subseteq D(K^n)/I_A D(K^n)$, and

$$(t^{-\boldsymbol{\beta}''}\, dT_A \otimes 1)t^{\boldsymbol{\beta}'' - \boldsymbol{e}} p(s) = p(\boldsymbol{\beta}'')t^{-\boldsymbol{e}}\, dT_A \otimes 1.$$

We have thus proved that $yD(K^n) \supseteq L_{K^n}(T_A, \boldsymbol{\beta})$ and that $L_{K^n}(T_A, \boldsymbol{\beta})$ is a unique simple $D(K^n)$-submodule of $\int^0_{T_A \to K^n} L_{T_A}(\boldsymbol{\beta})$.

Next, we prove the second statement. If $\boldsymbol{\beta} - \boldsymbol{\beta}' \in \mathbb{Z}^d$, then $\boldsymbol{\beta}^{\mathrm{empty}} = \boldsymbol{\beta}'^{\mathrm{empty}}$. Hence $L_{K^n}(T_A, \boldsymbol{\beta}) = L_{K^n}(T_A, \boldsymbol{\beta}')$ by definition. If $\boldsymbol{\beta} - \boldsymbol{\beta}' \notin \mathbb{Z}^d$, then $L_{K^n}(T_A, \boldsymbol{\beta})$ and $L_{K^n}(T_A, \boldsymbol{\beta}')$ have distinct weight sets and hence are not isomorphic. □

Lemma 6.5. *Let $f(\theta) \in K[\theta]$ satisfy*

$$f(\theta + \boldsymbol{l}) - f(\theta) \in \langle A\theta - \boldsymbol{c}\rangle K[\theta]$$

for all $\boldsymbol{l}$ with $A\boldsymbol{l} = 0$. Take $\boldsymbol{\gamma} \in K^n$ such that $A\boldsymbol{\gamma} = \boldsymbol{c}$. Then

$$f(\theta) \in f(\boldsymbol{\gamma}) + \langle A\theta - \boldsymbol{c}\rangle K[\theta].$$

Proof.

$$\begin{aligned}
&f(\theta + \boldsymbol{l}) - f(\theta) \in \langle A\theta - \boldsymbol{c}\rangle K[\theta] \quad \text{for all } \boldsymbol{l} \text{ such that } A\boldsymbol{l} = \boldsymbol{0}\\
&\Longrightarrow f(\boldsymbol{l} + \boldsymbol{\gamma}) - f(\boldsymbol{\gamma}) = 0 \quad \text{for all } \boldsymbol{l} \text{ such that } A\boldsymbol{l} = \boldsymbol{0}\\
&\Longleftrightarrow f(\theta + \boldsymbol{\gamma}) \in f(\boldsymbol{\gamma}) + \langle A\theta\rangle K[\theta]\\
&\Longleftrightarrow f(\theta) \in f(\boldsymbol{\gamma}) + \langle A\theta - \boldsymbol{c}\rangle K[\theta].
\end{aligned}$$

□

Let τ be a face of $\mathbb{R}_{\geqslant 0}A$, and let $\boldsymbol{\lambda} \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$. We define a right $D(K^\tau)$-module $L_{K^\tau}(T_\tau, \boldsymbol{\lambda})$ in the same way as we defined $L_{K^n}(T_A, \boldsymbol{\beta})$ in Theorem 6.4. By Theorem 6.4, $L_{K^\tau}(T_\tau, \boldsymbol{\lambda})$ is a simple $D(K^\tau)$-module. By Kashiwara's equivalence,

$$L_{K^n}(T_\tau, \boldsymbol{\lambda}) := \int^0_{K^\tau \to K^n} L_{K^\tau}(T_\tau, \boldsymbol{\lambda}) \tag{21}$$

is a simple $D(K^n)$-module.

THEOREM 6.6. *Each simple object in $\mathcal{O}_{K^n}$ is isomorphic to $L_{K^n}(T_\tau, \boldsymbol{\lambda})$ for some face τ and some $\boldsymbol{\lambda} \in K(A\cap\tau)/\mathbb{Z}(A\cap\tau)$.*

Moreover, $L_{K^n}(T_\tau, \boldsymbol{\lambda}) \simeq L_{K^n}(T_{\tau'}, \boldsymbol{\lambda}')$ if and only if $\tau = \tau'$ and $\boldsymbol{\lambda} - \boldsymbol{\lambda}' \in \mathbb{Z}(A\cap\tau)$.

Proof. Let L be a simple object in $\mathcal{O}_{K^n}$. Suppose that $\mathrm{supp}(L) = \overline{T_A} = X_A$. There exists the following exact sequence in $\mathcal{O}_{K^n}$:

$$0 \to \Gamma_{K^n\setminus(K^\times)^n}(L) \to L \to \Gamma_{(K^\times)^n}(L),$$

where $\Gamma_{K^n\setminus(K^\times)^n}(L) = \{y \in L : \mathrm{supp}(y) \subseteq K^n\setminus(K^\times)^n\}$ and $\Gamma_{(K^\times)^n}(L)$ is the localization of L at the multiplicatively closed set $\{x_j^m : j = 1, \dots, n; m \in \mathbb{N}\}$. By the simplicity of L, $\Gamma_{K^n\setminus(K^\times)^n}(L) = 0$. Hence L is a simple submodule of $\Gamma_{(K^\times)^n}(L)$, and then $\Gamma_{(K^\times)^n}(L)$ is simple in $\mathcal{O}_{(K^\times)^n}$. Indeed, let y be a non-zero element of $\Gamma_{(K^\times)^n}(L)$; then there exists $\boldsymbol{u} \in \mathbb{N}^n$ such that $y.x^{\boldsymbol{u}} \in L$. Since L is a simple $D(K^n)$-module, we have $y.D(K^n) \supseteq L$. Since $\Gamma_{(K^\times)^n}(L)$ is generated by L as a $D((K^\times)^n)$-module, we obtain $y.D((K^\times)^n) = \Gamma_{(K^\times)^n}(L)$, and hence $\Gamma_{(K^\times)^n}(L)$ is simple in $\mathcal{O}_{(K^\times)^n}$. Then, by Proposition 5.1, $\Gamma_{(K^\times)^n}(L) \simeq M_{(K^\times)^n}(\boldsymbol{\beta})$ for some $\boldsymbol{\beta} \in KA/\mathbb{Z}A$. Since $M_{(K^\times)^n}(\boldsymbol{\beta})$ has the unique simple submodule $L_{K^n}(T_A, \boldsymbol{\beta})$, we conclude that $L \simeq L_{K^n}(T_A, \boldsymbol{\beta})$.

By the simplicity of L, the support of L is the closure of T_τ for some face τ. By the same argument as in the previous paragraph, we obtain $L \simeq L_{K^n}(T_\tau, \boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in K(A\cap\tau)/\mathbb{Z}(A\cap\tau)$.

The second statement is clear from the second statement of Theorem 6.4. □

Example 6.7. Let $A = (1)$. In this case, the cone $\mathbb{R}_{\geqslant 0}A = \mathbb{R}_{\geqslant 0}$ has only two faces: $\{0\}$ and $\mathbb{R}_{\geqslant 0}$. Then

$$L_K(T_{\{0\}}, 0) = \int^0_{\{0\}\to K} K \simeq D/xD,$$

where D is the first Weyl algebra.

Let $\beta \in K$. If $\beta \notin \mathbb{Z} = \mathbb{Z}A$, then $\beta = \beta^{\mathrm{empty}}$. If $\beta \in \mathbb{Z}$, then $\beta = \beta^{\mathrm{empty}}$ if and only if $\beta \in \mathbb{Z}_{\leqslant -1}$. The simple module $L_K(T_A, \beta)$ is the unique simple submodule of $x^{-\beta}K[x, x^{-1}]\, dT_A$ generated by $x^{-\beta^{\mathrm{empty}}}\, dT_A$. Hence

$$\begin{aligned} L_K(T_A, \beta) &= x^{-\beta}\, dT_A.D \simeq D/(x\partial - \beta)D \quad \text{for } \beta \notin \mathbb{Z},\\ L_K(T_A, \beta) &= L_K(T_A, -1) = x\, dT_A.D \simeq D/\partial D \quad \text{for } \beta \in \mathbb{Z}. \end{aligned}$$

A left $D(K^n)$-module M is said to have *irreducible monodromy representation* if $D(K^n)(x) \otimes_{D(K^n)} M$ is an irreducible left $D(K^n)(x)$-module, where $D(K^n)(x) = K(x) \otimes_{K[x]} D(K^n)$ with $K(x) = K(x_1, \dots, x_n)$ being the field of rational functions (cf. [Wal07]). We naturally have the following proposition.

PROPOSITION 6.8. *Let M be an irreducible left $D(K^n)$-module. Suppose that $D(K^n)(x) \otimes_{D(K^n)} M \neq 0$. Then M has irreducible monodromy representation.*

Proof. We can write $M = D(K^n)/I$ with I a maximal left ideal of $D(K^n)$. Then

$$D(K^n)(x) \otimes_{D(K^n)} M = D(K^n)(x)/D(K^n)(x)I.$$

Let J be a left ideal of $D(K^n)(x)$ containing $D(K^n)(x)I$. Since $J \cap D(K^n)$ is a left ideal of $D(K^n)$ containing I, we have $J \cap D(K^n) = D(K^n)$ or I. If $J \cap D(K^n) = D(K^n)$, then $1 \in J$ and thus $J = D(K^n)(x)$.

Suppose that $J \cap D(K^n) = I$. Let $P \in J$. Then there exists a non-zero polynomial $f \in K[x]$ such that $fP \in J \cap D(K^n) = I$. Hence $P \in D(K^n)(x)I$, and we have $J = D(K^n)(x)I$. □

7. Pull-back of $L_{K^n}(T_\tau, \boldsymbol{\lambda})$

Let $i^\natural$ denote the functor from $\mathcal{O}_{K^n}$ to $\mathcal{O}_{X_A}$ defined by

$$\begin{aligned} i^\natural(N) &:= \operatorname{Hom}_{D(K^n)}(D(K^n, X_A), N) \\ &= \{x \in N : x.I_A = 0\}. \end{aligned} \tag{22}$$

The following adjointness property holds:

$$\operatorname{Hom}_{D(K^n)}\left(\int^0_{X_A\to K^n} M, N\right) \simeq \operatorname{Hom}_{D(X_A)}(M, i^\natural(N)). \tag{23}$$

In this section, we compute the pull-back of $L_{K^n}(T_\tau, \boldsymbol{\lambda})$ by $i^\natural$. As a consequence, we determine the irreducible quotients of $M_{K^n}(\boldsymbol{\beta})$.

Before considering $i^\natural(L_{K^n}(T_A, \boldsymbol{\lambda}))$, we present two preparatory lemmas.

Lemma 7.1. *Let $\boldsymbol{c} \in \mathrm{ZC}(\Omega(\boldsymbol{a}))$, where $\Omega(\boldsymbol{a})$ is as defined in (5) and* ZC *stands for the Zariski closure in K^d. Then there exist $\boldsymbol{b} \in \Omega(\boldsymbol{a})$ and a face τ such that $\boldsymbol{b} + \mathbb{N}(A\cap\tau) \subseteq \Omega(\boldsymbol{a})$ and $\boldsymbol{c} \in \boldsymbol{b} + K(A\cap\tau)$.*

Proof. This follows from [ST04, Proposition 5.1]. □

Lemma 7.2. *Suppose that*

$$\mathbb{I}(\Omega(\boldsymbol{a})) \subseteq \langle s - \boldsymbol{c}\rangle K[s].$$

Then

$$\{f \in \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})) : f(\boldsymbol{\gamma}) = f(\boldsymbol{\gamma}') \text{ if } A\boldsymbol{\gamma} = A\boldsymbol{\gamma}' = \boldsymbol{c}\} \subseteq \langle A\theta - \boldsymbol{c}\rangle K[\theta], \tag{24}$$

where $\widetilde{\Omega}(\boldsymbol{a})$ is as defined in (13).

Proof. Since $\mathbb{I}(\Omega(\boldsymbol{a})) \subseteq \langle s - \boldsymbol{c}\rangle K[s]$, we have $\boldsymbol{c} \in \mathrm{ZC}(\Omega(\boldsymbol{a}))$. By Lemma 7.1 there exist $\boldsymbol{b} \in \Omega(\boldsymbol{a})$ and a face τ such that $\boldsymbol{b} + \mathbb{N}(A\cap\tau) \subseteq \Omega(\boldsymbol{a})$ and $\boldsymbol{c} \in \boldsymbol{b} + K(A\cap\tau)$. Take $\boldsymbol{u} \in \mathbb{N}^n$ such that $A\boldsymbol{u} = \boldsymbol{b}$. Then there exists $\boldsymbol{\gamma}' \in \boldsymbol{u} + K^\tau$ such that $A\boldsymbol{\gamma}' = \boldsymbol{c}$. Observe that $\boldsymbol{\gamma}' \in \mathrm{ZC}(\widetilde{\Omega}(\boldsymbol{a}))$, since $\boldsymbol{u} + \mathbb{N}^\tau \subseteq \widetilde{\Omega}(\boldsymbol{a})$.

Let $f(\theta)$ belong to the set on the left-hand side of (24). If $A\boldsymbol{\gamma} = \boldsymbol{c}$ $(= A\boldsymbol{\gamma}')$, then we have $f(\boldsymbol{\gamma}) = f(\boldsymbol{\gamma}') = 0$ since $\boldsymbol{\gamma}' \in \mathrm{ZC}(\widetilde{\Omega}(\boldsymbol{a}))$. Hence $f \in \langle A\theta - \boldsymbol{c}\rangle K[\theta]$. □

Theorem 7.3.

$$i^\natural(L_{K^n}(T_A, \boldsymbol{\beta})) = L_{X_A}(\boldsymbol{\beta}^{\mathrm{empty}}).$$

Proof. Fix $\boldsymbol{e} := \boldsymbol{\beta}^{\mathrm{empty}}$. By Theorem 6.4,

$$\begin{aligned} L_{K^n}(T_A, \boldsymbol{\beta}) &= \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes_{K[s]} (\mathbb{I}(\tilde{\Omega}(\boldsymbol{a}))/\mathbb{I}(\tilde{\Omega}(\boldsymbol{a})) \cap \langle s - \boldsymbol{e} + \boldsymbol{a}\rangle K[\theta]) \\ &\subseteq \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes_{K[s]} K[\theta]/\langle s - \boldsymbol{e} + \boldsymbol{a}\rangle K[\theta]. \end{aligned}$$

First, we claim that

$$i^\natural(L_{K^n}(T_A, \boldsymbol{\beta})) \subseteq \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} K t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A. \tag{25}$$

Let $f(\theta) \in K[\theta]$, and fix $\boldsymbol{\gamma} \in K^n$ with $A\boldsymbol{\gamma} = \boldsymbol{e} - \boldsymbol{a}$. Then

$$\begin{aligned} &t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes f(\theta).I_A = 0 \\ &\quad \Longleftrightarrow\ t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes f(\theta).(x^{\boldsymbol{u}} - x^{\boldsymbol{v}}) = 0 \quad \text{for all } \boldsymbol{u} \text{ and } \boldsymbol{v} \text{ with } A\boldsymbol{u} = A\boldsymbol{v} \\ &\quad \Longleftrightarrow\ t^{-\boldsymbol{e}+\boldsymbol{a}+A\boldsymbol{u}}\, dT_A \otimes (f(\theta+\boldsymbol{u}) - f(\theta+\boldsymbol{v})) = 0 \quad \text{for all } \boldsymbol{u} \text{ and } \boldsymbol{v} \text{ with } A\boldsymbol{u} = A\boldsymbol{v} \\ &\quad \Longleftrightarrow\ f(\theta+\boldsymbol{u}) - f(\theta+\boldsymbol{v}) \in \langle A\theta - \boldsymbol{e} + \boldsymbol{a} + A\boldsymbol{u}\rangle K[\theta] \quad \text{for all } \boldsymbol{u} \text{ and } \boldsymbol{v} \text{ with } A\boldsymbol{u} = A\boldsymbol{v} \\ &\quad \Longleftrightarrow\ f(\theta+\boldsymbol{u}-\boldsymbol{v}) - f(\theta) \in \langle A\theta - \boldsymbol{e} + \boldsymbol{a}\rangle K[\theta] \quad \text{for all } \boldsymbol{u} \text{ and } \boldsymbol{v} \text{ with } A\boldsymbol{u} = A\boldsymbol{v}. \end{aligned}$$

Hence, by Lemma 6.5, $t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes f(\theta) \in i^\natural(L_{K^n}(T_A, \boldsymbol{\beta}))$ implies

$$f(\theta) \in f(\boldsymbol{\gamma}) + \langle A\theta - \boldsymbol{e} + \boldsymbol{a}\rangle K[\theta].$$

Therefore $t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes f(\theta) = f(\boldsymbol{\gamma}) t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes 1$ and the claim (25) is proved.

Recall that

$$\begin{aligned} \boldsymbol{e} - \boldsymbol{a} \not\sim \boldsymbol{e} \ &\Longleftrightarrow\ \boldsymbol{e} - \boldsymbol{a} \not\preceq \boldsymbol{e} \\ &\Longleftrightarrow\ \mathbb{I}(\Omega(\boldsymbol{a})) \subseteq \langle s - \boldsymbol{e} + \boldsymbol{a}\rangle K[s]. \end{aligned} \tag{26}$$

Suppose $\boldsymbol{e} - \boldsymbol{a} \sim \boldsymbol{e}$. Then there exists $f(s) \in \mathbb{I}(\Omega(\boldsymbol{a}))$ such that $f(s) \notin \langle s - \boldsymbol{e} + \boldsymbol{a}\rangle K[s]$. Hence, for $\boldsymbol{\gamma} \in K^n$ with $A\boldsymbol{\gamma} = \boldsymbol{e} - \boldsymbol{a}$, we have $f(\boldsymbol{\gamma}) = f(A\boldsymbol{\gamma}) \neq 0$. Then

$$i^\natural(L_{K^n}(T_A, \boldsymbol{\beta})) \ni t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes f(A\theta) = f(\boldsymbol{\gamma}) t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes 1 \neq 0,$$

and thus the weight $-\boldsymbol{e} + \boldsymbol{a}$ appears in $i^\natural(L_{K^n}(T_A, \boldsymbol{\beta}))$.

Next, suppose $\boldsymbol{e} - \boldsymbol{a} \not\sim \boldsymbol{e}$. Then $\mathbb{I}(\Omega(\boldsymbol{a})) \subseteq \langle s - \boldsymbol{e} + \boldsymbol{a}\rangle K[s]$. By the proof of (25), if $t^{-\boldsymbol{e}+\boldsymbol{a}}\, dT_A \otimes f(\theta) \in i^\natural(L_{K^n}(T_A, \boldsymbol{\beta}))$, then $f(\boldsymbol{\gamma}) = f(\boldsymbol{\gamma}')$ for any $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in K^n$ with $A\boldsymbol{\gamma} = A\boldsymbol{\gamma}' = \boldsymbol{e} - \boldsymbol{a}$. Hence, by (7), it suffices to prove the inclusion

$$\{f \in \mathbb{I}(\tilde{\Omega}(\boldsymbol{a})) : f(\boldsymbol{\gamma}) = f(\boldsymbol{\gamma}') \text{ if } A\boldsymbol{\gamma} = A\boldsymbol{\gamma}' = \boldsymbol{e} - \boldsymbol{a}\} \subseteq \langle A\theta - \boldsymbol{e} + \boldsymbol{a}\rangle K[\theta],$$

assuming that $\mathbb{I}(\Omega(\boldsymbol{a})) \subseteq \langle s - \boldsymbol{e} + \boldsymbol{a}\rangle K[s]$. We finish the proof by invoking Lemma 7.2. $\square$

Given faces τ and τ' of $\mathbb{R}_{\geqslant 0}A$, $\boldsymbol{\lambda} \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$ and $\boldsymbol{\lambda}' \in K(A \cap \tau')/Z(A \cap \tau')$, set

$$(\tau', \boldsymbol{\lambda}') \prec (\tau, \boldsymbol{\lambda}) \overset{\text{def.}}{\Longleftrightarrow} \tau' \prec \tau \quad \text{and} \quad \boldsymbol{\lambda} - \boldsymbol{\lambda}' \in \mathbb{Z}(A \cap \tau). \tag{27}$$

Theorem 7.4. *Let $\boldsymbol{\lambda} \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$. Then*

$$\dim_K i^\natural(L_{K^n}(T_\tau, \boldsymbol{\lambda}))_{-\boldsymbol{c}} = \begin{cases} 1 & \text{if } \boldsymbol{c} \in C_{K^n}(\tau, \boldsymbol{\lambda}), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$C_{K^n}(\tau, \boldsymbol{\lambda}) = \left\{ \boldsymbol{c} \in K^d : \begin{array}{l} E_\tau(\boldsymbol{c}) \ni \boldsymbol{\lambda} \text{ and } E_{\tau'}(\boldsymbol{c}) \not\ni \boldsymbol{\lambda}' \\ \text{whenever } (\tau', \boldsymbol{\lambda}') \prec (\tau, \boldsymbol{\lambda}) \end{array} \right\}. \tag{28}$$

Proof. By (15),

$$L_{K^n}(T_\tau, \boldsymbol{\lambda}) \simeq L_{K^\tau}(T_\tau, \boldsymbol{\lambda}) \boxtimes \Bigg(\bigoplus_{\tilde{\boldsymbol{b}} \in \mathbb{N}^{\tau^c}} K x^{-\tilde{\boldsymbol{b}}}\, d(K^\times)^{\tau^c} \Bigg).$$

By the definition of $i^\natural$,

$$\begin{aligned} i^\natural(L_{K^n}(T_\tau, \boldsymbol{\lambda})) &= \{ f \in L_{K^n}(T_\tau, \boldsymbol{\lambda}) : f.I_A = 0 \} \\ &\subseteq \{ f \in L_{K^n}(T_\tau, \boldsymbol{\lambda}) : f.(x^{\boldsymbol{u}} - x^{\boldsymbol{v}}) = 0 \text{ for } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^\tau \text{ with } A\boldsymbol{u} = A\boldsymbol{v} \}. \end{aligned}$$

Hence, by Theorem 7.3,

$$i^\natural(L_{K^n}(T_\tau, \boldsymbol{\lambda})) \subseteq \Bigg(\bigoplus_{\boldsymbol{a} \sim \boldsymbol{\lambda}^{\text{empty}}} K t^{-\boldsymbol{a}}\, dT_\tau \Bigg) \boxtimes \Bigg(\bigoplus_{\tilde{\boldsymbol{b}} \in \mathbb{N}^{\tau^c}} K x^{-\tilde{\boldsymbol{b}}}\, d(K^\times)^{\tau^c} \Bigg).$$

Note that for $\boldsymbol{a} \in K(A \cap \tau)$, $\boldsymbol{a} \sim \boldsymbol{\lambda}^{\text{empty}}$ if and only if $\boldsymbol{a} \in C_{K^n}(\tau, \boldsymbol{\lambda}) \cap K(A \cap \tau) =: C_{K^\tau}(\tau, \boldsymbol{\lambda})$. Let

$$f = \sum_{(\boldsymbol{a}, \tilde{\boldsymbol{b}}) \in C} f_{\boldsymbol{a}, \tilde{\boldsymbol{b}}} t^{-\boldsymbol{a}}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}}\, d(K^\times)^{\tau^c}, \tag{29}$$

where $C = C_{K^\tau}(\tau, \boldsymbol{\lambda}) \times \mathbb{N}^{\tau^c}$. Note that the set of $(\boldsymbol{a}, \tilde{\boldsymbol{b}}) \in C$ with a fixed $\boldsymbol{a} + A\tilde{\boldsymbol{b}}$ is finite, since $\boldsymbol{a} \in \boldsymbol{\lambda} + \mathbb{Z}(A \cap \tau)$, $\tilde{\boldsymbol{b}} \in \mathbb{N}^{\tau^c}$ and $\mathbb{R}_{\geqslant 0}(A \backslash \tau) \cap \mathbb{R}\tau = \{\boldsymbol{0}\}$.

Let $\boldsymbol{u} = \boldsymbol{u}_\tau + \boldsymbol{u}_{\tau^c}$ and $\boldsymbol{v} = \boldsymbol{v}_\tau + \boldsymbol{v}_{\tau^c}$, with $\boldsymbol{u}_\tau, \boldsymbol{v}_\tau \in \mathbb{N}^\tau$ and $\boldsymbol{u}_{\tau^c}, \boldsymbol{v}_{\tau^c} \in \mathbb{N}^{\tau^c}$, satisfy $A\boldsymbol{u} = A\boldsymbol{v}$. We claim that for f as in (29),

$$f \in i^\natural(L_{K^n}(T_\tau, \boldsymbol{\lambda})) \iff \begin{cases} \text{(i)} \;\; f_{\boldsymbol{a}+A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}} = f_{\boldsymbol{a}+A\boldsymbol{v}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c}} \\ \quad\;\; \text{for } (\boldsymbol{a}, \tilde{\boldsymbol{b}}), (\boldsymbol{a}+A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}), (\boldsymbol{a}+A\boldsymbol{v}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c}) \in C, \\ \text{(ii)} \;\; f_{\boldsymbol{a}+A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}} = 0 \\ \quad\;\; \text{for } (\boldsymbol{a}, \tilde{\boldsymbol{b}}), (\boldsymbol{a}+A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}) \in C, (\boldsymbol{a}+A\boldsymbol{v}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c}) \notin C. \end{cases} \tag{30}$$

We have

$$\begin{aligned} f.(x^{\boldsymbol{u}} - x^{\boldsymbol{v}}) = & \sum_{(\boldsymbol{a}, \tilde{\boldsymbol{b}}) \in C} f_{\boldsymbol{a}, \tilde{\boldsymbol{b}}} t^{-\boldsymbol{a}+A\boldsymbol{u}_\tau}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}}\, d(K^\times)^{\tau^c} \\ & - \sum_{(\boldsymbol{a}, \tilde{\boldsymbol{b}}) \in C} f_{\boldsymbol{a}, \tilde{\boldsymbol{b}}} t^{-\boldsymbol{a}+A\boldsymbol{v}_\tau}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c}}\, d(K^\times)^{\tau^c} \\ = & \sum_{(\boldsymbol{a}, \tilde{\boldsymbol{b}}), (\boldsymbol{a}-A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}}-\boldsymbol{u}_{\tau^c}) \in C} f_{\boldsymbol{a}, \tilde{\boldsymbol{b}}} t^{-\boldsymbol{a}+A\boldsymbol{u}_\tau}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}}\, d(K^\times)^{\tau^c} \\ & - \sum_{(\boldsymbol{a}, \tilde{\boldsymbol{b}}), (\boldsymbol{a}-A\boldsymbol{v}_\tau, \tilde{\boldsymbol{b}}-\boldsymbol{v}_{\tau^c}) \in C} f_{\boldsymbol{a}, \tilde{\boldsymbol{b}}} t^{-\boldsymbol{a}+A\boldsymbol{v}_\tau}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c}}\, d(K^\times)^{\tau^c} \\ = & \sum_{(\boldsymbol{a}, \tilde{\boldsymbol{b}}), (\boldsymbol{a}+A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}) \in C} f_{\boldsymbol{a}+A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}} t^{-\boldsymbol{a}}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}}\, d(K^\times)^{\tau^c} \\ & - \sum_{(\boldsymbol{a}, \tilde{\boldsymbol{b}}), (\boldsymbol{a}+A\boldsymbol{v}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c}) \in C} f_{\boldsymbol{a}+A\boldsymbol{v}_\tau, \tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c}} t^{-\boldsymbol{a}}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}}\, d(K^\times)^{\tau^c} \end{aligned}$$

$$= \sum_{\substack{(\boldsymbol{a},\tilde{\boldsymbol{b}}),(\boldsymbol{a}+A\boldsymbol{u}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c})\in C\\ (\boldsymbol{a}+A\boldsymbol{v}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c})\in C}} (f_{\boldsymbol{a}+A\boldsymbol{u}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}} - f_{\boldsymbol{a}+A\boldsymbol{v}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c}}) t^{-\boldsymbol{a}}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}}\, d(K^\times)^{\tau^c}$$
$$+ \sum_{\substack{(\boldsymbol{a},\tilde{\boldsymbol{b}}),(\boldsymbol{a}+A\boldsymbol{u}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c})\in C\\ (\boldsymbol{a}+A\boldsymbol{v}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c})\notin C}} f_{\boldsymbol{a}+A\boldsymbol{u}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c}} t^{-\boldsymbol{a}}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}}\, d(K^\times)^{\tau^c}$$
$$- \sum_{\substack{(\boldsymbol{a},\tilde{\boldsymbol{b}}),(\boldsymbol{a}+A\boldsymbol{v}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c})\in C\\ (\boldsymbol{a}+A\boldsymbol{u}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{u}_{\tau^c})\notin C}} f_{\boldsymbol{a}+A\boldsymbol{v}_\tau,\tilde{\boldsymbol{b}}+\boldsymbol{v}_{\tau^c}} t^{-\boldsymbol{a}}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}}\, d(K^\times)^{\tau^c},$$

so (30) is established.

Let us keep $f \in i^\natural(L_{K^n}(T_\tau, \boldsymbol{\lambda}))$ as in (29) and take $(\boldsymbol{a}, \tilde{\boldsymbol{b}}), (\boldsymbol{a}', \tilde{\boldsymbol{b}}') \in C$ with $\boldsymbol{a} + A\tilde{\boldsymbol{b}} = \boldsymbol{a}' + A\tilde{\boldsymbol{b}}'$. We claim that then

$$f_{\boldsymbol{a},\tilde{\boldsymbol{b}}} = f_{\boldsymbol{a}',\tilde{\boldsymbol{b}}'}. \tag{31}$$

Indeed, let $\boldsymbol{w} \in K^\tau$ and $\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{a}}' \in \mathbb{Z}^\tau$ satisfy $\boldsymbol{\lambda} = A\boldsymbol{w}$, $\boldsymbol{a} = A(\boldsymbol{w} + \tilde{\boldsymbol{a}})$ and $\boldsymbol{a}' = A(\boldsymbol{w} + \tilde{\boldsymbol{a}}')$. Put $\boldsymbol{u}_\tau := (\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}')_+ \in \mathbb{N}^\tau$, $\boldsymbol{v}_\tau := (\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}')_- \in \mathbb{N}^\tau$, $\boldsymbol{u}_{\tau^c} := (\tilde{\boldsymbol{b}} - \tilde{\boldsymbol{b}}')_+ \in \mathbb{N}^{\tau^c}$ and $\boldsymbol{v}_{\tau^c} := (\tilde{\boldsymbol{b}} - \tilde{\boldsymbol{b}}')_- \in \mathbb{N}^{\tau^c}$. Here, $(\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}')_+$ is the non-negative part of $\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}'$, and $(\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}')_-$ is the negative of the non-positive part of $\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}'$. Then $A(\boldsymbol{u}_\tau + \boldsymbol{u}_{\tau^c}) = A(\boldsymbol{v}_\tau + \boldsymbol{v}_{\tau^c})$ and $\tilde{\boldsymbol{b}} - \boldsymbol{u}_{\tau^c} = \tilde{\boldsymbol{b}}' - \boldsymbol{v}_{\tau^c} \in \mathbb{N}^{\tau^c}$. Furthermore, $\boldsymbol{a} - A\boldsymbol{u}_\tau = \boldsymbol{a}' - A\boldsymbol{v}_\tau \in C_{K^\tau}(\tau, \boldsymbol{\lambda})$, since $\boldsymbol{a} \sim \boldsymbol{a}' \sim \boldsymbol{\lambda}^{\text{empty}}$ is the minimal class (see [Sai01, Proposition 2.2(5)]). Hence, from (30)(i) we obtain (31).

We can rewrite (30)(ii) as

$$f_{\boldsymbol{a},\tilde{\boldsymbol{b}}} = 0 \tag{32}$$

for $(\boldsymbol{a}, \tilde{\boldsymbol{b}}), (\boldsymbol{a} - A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}} - \boldsymbol{u}_{\tau^c}) \in C$ and $(\boldsymbol{a} - A\boldsymbol{u}_\tau + A\boldsymbol{v}_\tau, \tilde{\boldsymbol{b}} - \boldsymbol{u}_{\tau^c} + \boldsymbol{v}_{\tau^c}) \notin C$.

We prove next that (32) is equivalent to the following condition:

$$\begin{aligned}&\text{if there exists } (\tau', \boldsymbol{\lambda}') \prec (\tau, \boldsymbol{\lambda}) \text{ such that } E_{\tau'}(\boldsymbol{a} + A\tilde{\boldsymbol{b}}) \ni \boldsymbol{\lambda}',\\ &\text{then } f_{\boldsymbol{a},\tilde{\boldsymbol{b}}} = 0.\end{aligned} \tag{33}$$

For this purpose, when $(\boldsymbol{a}, \tilde{\boldsymbol{b}}) \in C$ we prove the equivalence

$$\text{there exists } (\tau', \boldsymbol{\lambda}') \prec (\tau, \boldsymbol{\lambda}) \text{ such that } E_{\tau'}(\boldsymbol{a} + A\tilde{\boldsymbol{b}}) \ni \boldsymbol{\lambda}' \tag{34}$$

$$\begin{aligned}\iff\ &\text{there exist } \boldsymbol{u}_\tau, \boldsymbol{v}_\tau \in \mathbb{N}^\tau \text{ and } \boldsymbol{u}_{\tau^c}, \boldsymbol{v}_{\tau^c} \in \mathbb{N}^{\tau^c} \text{ such that}\\ &A(\boldsymbol{u}_\tau + \boldsymbol{u}_{\tau^c}) = A(\boldsymbol{v}_\tau + \boldsymbol{v}_{\tau^c}),\ (\boldsymbol{a} - A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}} - \boldsymbol{u}_{\tau^c}) \in C\\ &\text{and } (\boldsymbol{a} - A\boldsymbol{u}_\tau + A\boldsymbol{v}_\tau, \tilde{\boldsymbol{b}} - \boldsymbol{u}_{\tau^c} + \boldsymbol{v}_{\tau^c}) \notin C.\end{aligned} \tag{35}$$

First, suppose that (35) holds. Then $\tilde{\boldsymbol{b}} - \boldsymbol{u}_{\tau^c} \in \mathbb{N}^{\tau^c}$, and there exists $(\tau', \boldsymbol{\lambda}') \prec (\tau, \boldsymbol{\lambda})$ such that $E_{\tau'}(\boldsymbol{a} - A\boldsymbol{u}_\tau + A\boldsymbol{v}_\tau) \ni \boldsymbol{\lambda}'$. It follows from $\tilde{\boldsymbol{b}} - \boldsymbol{u}_{\tau^c} \in \mathbb{N}^{\tau^c}$ and $A(\boldsymbol{u}_\tau + \boldsymbol{u}_{\tau^c}) = A(\boldsymbol{v}_\tau + \boldsymbol{v}_{\tau^c})$ that $A\boldsymbol{v}_\tau - A\boldsymbol{u}_\tau \in A(\tilde{\boldsymbol{b}} - \mathbb{N}^{\tau^c})$. Hence $E_{\tau'}(\boldsymbol{a} + A\tilde{\boldsymbol{b}}) \ni \boldsymbol{\lambda}'$ (cf. [Sai01, Proposition 2.2(5)]).

Conversely, suppose that (34) holds. Then $\boldsymbol{a} + A\tilde{\boldsymbol{b}} - \boldsymbol{\lambda}' \in \mathbb{N}A + \mathbb{Z}(A \cap \tau')$. Let $\boldsymbol{w}' \in K^{\tau'}$, $\tilde{\boldsymbol{a}} \in \mathbb{Z}^\tau$, $\tilde{\boldsymbol{b}}' \in \mathbb{N}^{\tau^c}$ and $\tilde{\boldsymbol{a}}' \in \mathbb{N}^{\tau\setminus\tau'} \times \mathbb{Z}^{\tau'}$ satisfy $\boldsymbol{\lambda}' = A\boldsymbol{w}'$, $\boldsymbol{a} = A(\boldsymbol{w}' + \tilde{\boldsymbol{a}})$ and $\boldsymbol{a} + A\tilde{\boldsymbol{b}} - \boldsymbol{\lambda}' = A\tilde{\boldsymbol{b}}' + A\tilde{\boldsymbol{a}}'$. As before, put $\boldsymbol{u}_\tau := (\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}')_+ \in \mathbb{N}^\tau$, $\boldsymbol{v}_\tau := (\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}')_- \in \mathbb{N}^\tau$, $\boldsymbol{u}_{\tau^c} := (\tilde{\boldsymbol{b}} - \tilde{\boldsymbol{b}}')_+ \in \mathbb{N}^{\tau^c}$ and $\boldsymbol{v}_{\tau^c} := (\tilde{\boldsymbol{b}} - \tilde{\boldsymbol{b}}')_- \in \mathbb{N}^{\tau^c}$. Then $(\boldsymbol{a} - A\boldsymbol{u}_\tau, \tilde{\boldsymbol{b}} - \boldsymbol{u}_{\tau^c}) \in C$. Furthermore, $\boldsymbol{a} - A\boldsymbol{u}_\tau + A\boldsymbol{v}_\tau = \boldsymbol{a} - A(\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}') = \boldsymbol{\lambda}' + A\tilde{\boldsymbol{a}}' \in \boldsymbol{\lambda}' + \mathbb{N}A + \mathbb{Z}(A \cap \tau')$. Hence $\boldsymbol{\lambda}' \in E_{\tau'}(\boldsymbol{a} - A\boldsymbol{u}_\tau + A\boldsymbol{v}_\tau)$, and thus $(\boldsymbol{a} - A\boldsymbol{u}_\tau + A\boldsymbol{v}_\tau, \tilde{\boldsymbol{b}} - \boldsymbol{u}_{\tau^c} + \boldsymbol{v}_{\tau^c}) \notin C$. Finally, $A(\boldsymbol{u}_\tau + \boldsymbol{u}_{\tau^c}) - A(\boldsymbol{v}_\tau + \boldsymbol{v}_{\tau^c}) = A(\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{a}}') + A(\tilde{\boldsymbol{b}} - \tilde{\boldsymbol{b}}') = \boldsymbol{a} - \boldsymbol{\lambda}' - A\tilde{\boldsymbol{a}}' + A(\tilde{\boldsymbol{b}} - \tilde{\boldsymbol{b}}') = \boldsymbol{0}$. Therefore we have established the equivalence between (34) and (35) and hence the equivalence between (32) and (33).

In summary, we have shown that

$$i^{\natural}(L_{K^n}(T_\tau, \boldsymbol{\lambda})) = \bigoplus_{\boldsymbol{c}\in C_{K^n}(\tau,\boldsymbol{\lambda})} K \sum_{(\boldsymbol{a},\tilde{\boldsymbol{b}}),\boldsymbol{c}=\boldsymbol{a}+A\tilde{\boldsymbol{b}}} t^{-\boldsymbol{a}}\, dT_\tau \otimes x^{-\tilde{\boldsymbol{b}}}\, d(K^\times)^{\tau^c}, \tag{36}$$

so the proof of Theorem 7.4 is complete. □

Corollary 7.5.

$$\dim_K \operatorname{Hom}_{D(R)}(M_{K^n}(\boldsymbol{\beta}), L_{K^n}(T_\tau, \boldsymbol{\lambda})) = \begin{cases} 1 & \textit{if } \boldsymbol{\beta} \in C_{K^n}(\tau, \boldsymbol{\lambda}), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned} &\dim_K \operatorname{Hom}_{D(K^n)}(M_{K^n}(\boldsymbol{\beta}), L_{K^n}(T_\tau, \boldsymbol{\lambda})) \\ &\quad = \dim_K \operatorname{Hom}_{D(K^n)}\left(\int^0_{X_A \to K^n} M_{X_A}(\boldsymbol{\beta}), L_{K^n}(T_\tau, \boldsymbol{\lambda})\right) \\ &\quad = \dim_K \operatorname{Hom}_{D(X_A)}(M_{X_A}(\boldsymbol{\beta}), i^{\natural}(L_{K^n}(T_\tau, \boldsymbol{\lambda}))) \\ &\quad = \dim_K (i^{\natural}(L_{K^n}(T_\tau, \boldsymbol{\lambda})))_{-\boldsymbol{\beta}}. \end{aligned}$$

The first equality comes from (12) and the second from the adjointness (23). The third follows from [MV98, Proposition 3.1.7] (see also [Sai07, Proposition 3.6]). Theorem 7.4 then finishes the proof of this corollary. □

For $\boldsymbol{\beta} \in K^d$, set

$$E(\boldsymbol{\beta}) := \{(\tau, \boldsymbol{\lambda}) : \tau \text{ a face of } \mathbb{R}_{\geqslant 0}A,\ \boldsymbol{\lambda} \in E_\tau(\boldsymbol{\beta})\}. \tag{37}$$

Then Corollary 7.5 can be rephrased as follows.

Corollary 7.6.

$$\dim_K \operatorname{Hom}_{D(R)}(M_{K^n}(\boldsymbol{\beta}), L_{K^n}(T_\tau, \boldsymbol{\lambda})) = \begin{cases} 1 & \textit{if } (\tau, \boldsymbol{\lambda}) \textit{ is minimal in } E(\boldsymbol{\beta}), \\ 0 & \textit{otherwise.} \end{cases}$$

Here the minimality is with respect to (27).

Example 7.7. Let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3].$$

Then the cone $\mathbb{R}_{\geqslant 0}A$ has exactly four faces: $\mathbb{R}_{\geqslant 0}A = \mathbb{R}^2_{\geqslant 0}$, $\sigma_1 := \mathbb{R}_{\geqslant 0}\boldsymbol{a}_1$, $\sigma_3 := \mathbb{R}_{\geqslant 0}\boldsymbol{a}_3$ and $\{\boldsymbol{0}\}$. The semigroup $\mathbb{N}A$ is shown in Figure 1.

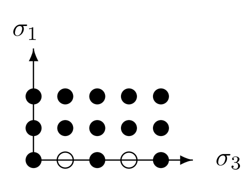

Figure 1. The semigroup $\mathbb{N}A$.

Let τ be a face of $\mathbb{R}_{\geqslant 0}A$. Then

$$|\mathbb{Z}^2 \cap K(A \cap \tau)/\mathbb{Z}(A \cap \tau)| = \begin{cases} 1 & \text{if } \tau \neq \sigma_3, \\ 2 & \text{if } \tau = \sigma_3. \end{cases}$$

Hence the category $\mathcal{O}_{K^3}$ has exactly five simple objects with weights in $\mathbb{Z}^2$, namely $L_{K^3}(T_A, \mathbf{0})$, $L_{K^3}(T_{\sigma_1}, \mathbf{0})$, $L_{K^3}(T_{\sigma_3}, \mathbf{0})$, $L_{K^3}(T_{\sigma_3}, (1,0)^{\mathrm{T}})$ and $L_{K^3}(T_{\{\mathbf{0}\}}, \mathbf{0})$. For each of these, we write down the weight set $(C_{K^n}(\tau, \boldsymbol{\lambda})$ in Theorem 7.4) of the pull-back by $i^\natural$.

(i) $i^\natural(L_{K^3}(T_A, \mathbf{0}))$: the weights in $C_{K^3}(\mathbb{R}_{\geqslant 0}A, \mathbf{0})$ are $\boldsymbol{\beta} \in \mathbb{Z}^2$ with $E_{\sigma_1}(\boldsymbol{\beta}) = \emptyset$ and $E_{\sigma_3}(\boldsymbol{\beta}) = \emptyset$, shown in Figure 2.

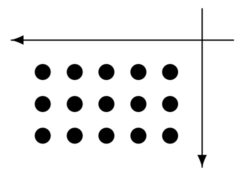

Figure 2. The weight space of $i^\natural(L_{K^3}(T_A, \mathbf{0}))$.

(ii) $i^\natural(L_{K^3}(T_{\sigma_1}, \mathbf{0}))$: the weights in $C_{K^3}(\sigma_1, \mathbf{0})$ are $\boldsymbol{\beta} \in \mathbb{Z}^2$ with $E_{\sigma_1}(\boldsymbol{\beta}) = \{\mathbf{0}\}$ and $E_{\{\mathbf{0}\}}(\boldsymbol{\beta}) = \emptyset$, shown in Figure 3.

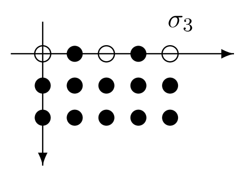

Figure 3. The weight space of $i^\natural(L_{K^3}(T_{\sigma_1}, \mathbf{0}))$.

(iii) $i^\natural(L_{K^3}(T_{\sigma_3}, \mathbf{0}))$: the weights in $C_{K^3}(\sigma_3, \mathbf{0})$ are $\boldsymbol{\beta} \in \mathbb{Z}^2$ with $E_{\sigma_3}(\boldsymbol{\beta}) \ni \mathbf{0}$ and $E_{\{\mathbf{0}\}}(\boldsymbol{\beta}) = \emptyset$, shown in Figure 4.

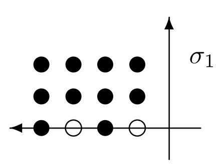

Figure 4. The weight space of $i^\natural(L_{K^3}(T_{\sigma_3}, \mathbf{0}))$.

(iv) $i^\natural(L_{K^3}(T_{\sigma_3}, (1,0)^{\mathrm{T}}))$: the weights in $C_{K^3}(\sigma_3, (1,0)^{\mathrm{T}})$ are $\boldsymbol{\beta} \in \mathbb{Z}^2$ with $E_{\sigma_3}(\boldsymbol{\beta}) \ni (1,0)^{\mathrm{T}}$, shown in Figure 5.

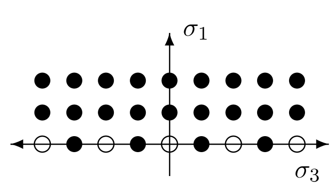

Figure 5. The weight space of $i^\natural(L_{K^3}(T_{\sigma_3}, (1,0)^{\mathrm{T}}))$.

(v) $i^\natural(L_{K^3}(T_{\{\mathbf{0}\}}, \mathbf{0}))$: the weights in $C_{K^3}(\{\mathbf{0}\}, \mathbf{0})$ are $\boldsymbol{\beta} \in \mathbb{Z}^2$ with $E_{\{\mathbf{0}\}}(\boldsymbol{\beta}) = \{\mathbf{0}\}$; hence the weight set is $\mathbb{N}A$, shown in Figure 1.

Let $\boldsymbol{\beta} \in \mathbb{Z}^2$. By Corollary 7.5, the irreducible quotients of $M_{K^3}(\boldsymbol{\beta})$ are precisely the above $L_{K^3}(T_\tau, \boldsymbol{\lambda})$ such that $\boldsymbol{\beta}$ appears in the weight set of $i^\natural(L_{K^3}(T_\tau, \boldsymbol{\lambda}))$.

Recall that $M_{K^3}(\boldsymbol{\beta}) \simeq M_{K^3}(\boldsymbol{\beta}')$ if and only if $\boldsymbol{\beta} \sim \boldsymbol{\beta}'$ (see [Sai01, Theorem 2.1]). There are eight equivalence classes in $\{M_{K^3}(\boldsymbol{\beta}) : \boldsymbol{\beta} \in \mathbb{Z}^2\}$. The following table lists the irreducible quotients for each equivalence class.

$M_{K^3}(\boldsymbol{\beta})$	Irreducible quotients
$M_{K^3}((0,1)^{\mathrm{T}})$	$L_{K^3}(T_{\{\mathbf{0}\}}, \mathbf{0})$, $L_{K^3}(T_{\sigma_3}, (1,0)^{\mathrm{T}})$
$M_{K^3}((-1,1)^{\mathrm{T}})$	$L_{K^3}(T_{\sigma_3}, \mathbf{0})$, $L_{K^3}(T_{\sigma_3}, (1,0)^{\mathrm{T}})$
$M_{K^3}((0,0)^{\mathrm{T}})$	$L_{K^3}(T_{\{\mathbf{0}\}}, \mathbf{0})$
$M_{K^3}((1,0)^{\mathrm{T}})$	$L_{K^3}(T_{\sigma_1}, \mathbf{0})$, $L_{K^3}(T_{\sigma_3}, (1,0)^{\mathrm{T}})$
$M_{K^3}((-1,0)^{\mathrm{T}})$	$L_{K^3}(T_{\sigma_3}, (1,0)^{\mathrm{T}})$
$M_{K^3}((-2,0)^{\mathrm{T}})$	$L_{K^3}(T_{\sigma_3}, \mathbf{0})$
$M_{K^3}((0,-1)^{\mathrm{T}})$	$L_{K^3}(T_{\sigma_1}, \mathbf{0})$
$M_{K^3}((-1,-1)^{\mathrm{T}})$	$L_{K^3}(T_A, \mathbf{0})$

8. The irreducibility of $M_{K^n}(\boldsymbol{\beta})$

If $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}}$, then, by Corollary 7.6, there exists a surjective homomorphism

$$M_{K^n}(\boldsymbol{\beta}) \to L_{K^n}(T_A, \boldsymbol{\beta}). \tag{38}$$

In this section, we analyze the kernel of (38) and prove that $M_{K^n}(\boldsymbol{\beta})$ is irreducible if and only if $\boldsymbol{\beta}$ is non-resonant.

Given a facet (maximal proper face) σ of $\mathbb{R}_{\geqslant 0}A$, we denote by F_σ the *primitive integral support function* of σ; that is, F_σ is the uniquely determined linear form on $\mathbb{R}^d$ satisfying:

(i) $F_\sigma(\mathbb{R}_{\geqslant 0}A) \geqslant 0$;
(ii) $F_\sigma(\sigma) = 0$;
(iii) $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

Then, by [Sai01, Proposition 2.2] and Remark 6.3, we know that $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}}$ if and only if $F_\sigma(\boldsymbol{\beta}) \notin F_\sigma(\mathbb{N}A)$ for all facets σ of $\mathbb{R}_{\geqslant 0}A$.

Let $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}}$, and let

$$\boldsymbol{v}_{-\boldsymbol{\beta}} := t^{-\boldsymbol{\beta}}\, dT_A \otimes 1 \in L_{K^n}(T_A, \boldsymbol{\beta})_{-\boldsymbol{\beta}}.$$

Then, by Theorem 6.4,

$$\mathrm{Ann}_{D(K^n)}(\boldsymbol{v}_{-\boldsymbol{\beta}}) = I_A D(K^n) + D(K^n) \cap \langle A\theta - \boldsymbol{\beta}\rangle D((K^\times)^n).$$

Let

$$N := \mathrm{Ann}_{D(K^n)}(\boldsymbol{v}_{-\boldsymbol{\beta}}) / (I_A D(K^n) + \langle A\theta - \boldsymbol{\beta}\rangle D(K^n)). \tag{39}$$

Then N is the kernel of (38). By (11) and Proposition 5.2, for $\boldsymbol{a} \in \mathbb{Z}^d$ we have

$$N_{-\boldsymbol{\beta}-\boldsymbol{a}} = t^{-\boldsymbol{a}}(\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a})) \cap \langle A\theta - \boldsymbol{\beta} - \boldsymbol{a}\rangle) / t^{-\boldsymbol{a}}(\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a}))\langle A\theta - \boldsymbol{\beta} - \boldsymbol{a}\rangle). \tag{40}$$

Since $\{\boldsymbol{u} \in \mathbb{N}^n : A\boldsymbol{u} \in \boldsymbol{a} + \mathbb{N}A\}$ is $\mathbb{N}^n$-stable, there exists a finite set $\{(\boldsymbol{u}^{(j)}, I_j) : j \in J\}$ of pairs made up of a $\boldsymbol{u}^{(j)} \in \mathbb{N}^n$ and a subset I_j of $\{1, \ldots, n\}$ (the set of so-called *standard pairs* of $\{\boldsymbol{u} \in \mathbb{N}^n : A\boldsymbol{u} \in \boldsymbol{a} + \mathbb{N}A\}$; see, e.g., [SST00, § 3.2]) such that:

- the ith coordinate of $\boldsymbol{u}^{(j)}$ is 0 for each $i\in I_j$;
- for all $i\notin I_j$, $(\boldsymbol{u}^{(j)}+\mathbb{N}^{I_j\cup\{i\}})\cap\{\boldsymbol{u}\in\mathbb{N}^n : A\boldsymbol{u}\in\boldsymbol{a}+\mathbb{N}A\}\neq\emptyset$;
- $\widetilde{\Omega}(-\boldsymbol{a})=\mathbb{N}^n\backslash\{\boldsymbol{u}\in\mathbb{N}^n : A\boldsymbol{u}\in\boldsymbol{a}+\mathbb{N}A\}=\bigcup_{j\in J}(\boldsymbol{u}^{(j)}+\mathbb{N}^{I_j})$.

Lemma 8.1. *Let $\boldsymbol{a}\in\mathbb{Z}^d$, and let $\{(\boldsymbol{u}^{(j)},I_j) : j\in J\}$ be the set of standard pairs of $\{\boldsymbol{u}\in\mathbb{N}^n : A\boldsymbol{u}\in\boldsymbol{a}+\mathbb{N}A\}$. Then for each $j\in J$ there exists a face $\tau^{(j)}$ of $\mathbb{R}_{\geqslant 0}A$ such that $I_j=\{k\in\{1,\dots n\} : \boldsymbol{a}_k\in\tau^{(j)}\}$, and either $\tau^{(j)}$ is a facet with $F_{\tau^{(j)}}(A\boldsymbol{u}^{(j)})\notin F_{\tau^{(j)}}(\boldsymbol{a}+\mathbb{N}A)$ or $F_\sigma(A\boldsymbol{u}^{(j)})\in F_\sigma(\boldsymbol{a}+\mathbb{N}A)$ for all facets $\sigma\succeq\tau^{(j)}$.*

Proof. Put $S_c=\{\boldsymbol{d}\in\mathbb{Z}^d : F_\sigma(\boldsymbol{d})\in F_\sigma(\mathbb{N}A)$ for all facets $\sigma\}$. Then there exist finitely many pairs $(\boldsymbol{b}_i,\tau_i)$ of $\boldsymbol{b}_i\in S_c$ and a face τ_i such that

$$S_c\backslash\mathbb{N}A=\bigcup_i(\boldsymbol{b}_i+\mathbb{Z}(A\cap\tau_i))\cap S_c$$

(see [ST04, proof of Proposition 5.1]). Then

$$\Omega(-\boldsymbol{a})=\Bigg(\bigcup_{\text{facets }\sigma}\ \bigcup_{m\in F_\sigma(\mathbb{N}A)\backslash F_\sigma(\boldsymbol{a}+\mathbb{N}A)}F_\sigma^{-1}(m)\cap\mathbb{N}A\Bigg)\cup\bigcup_{\boldsymbol{b}_i+\boldsymbol{a}\in\mathbb{N}A+\mathbb{Z}(A\cap\tau_i)}(\boldsymbol{b}_i+\boldsymbol{a}+\mathbb{Z}(A\cap\tau_i))\cap\mathbb{N}A.$$

Since $\widetilde{\Omega}(-\boldsymbol{a})=\{\boldsymbol{u}\in\mathbb{N}^n : A\boldsymbol{u}\in\Omega(-\boldsymbol{a})\}$ by definition, the assertion follows. □

Lemma 8.2. *Let $\boldsymbol{\beta}=\boldsymbol{\beta}^{\text{empty}}$ and $\boldsymbol{a}\in\mathbb{Z}^d$.*

(i) *If $\boldsymbol{\beta}+\boldsymbol{a}\sim\boldsymbol{\beta}$, then $N_{-\boldsymbol{\beta}-\boldsymbol{a}}=\{0\}$.*
(ii) *Suppose that there exists a facet σ such that $F_\sigma(\boldsymbol{\beta}+\boldsymbol{a})\in F_\sigma(\mathbb{N}A)$ and $F_{\sigma'}(\boldsymbol{\beta}+\boldsymbol{a})\notin F_{\sigma'}(\mathbb{N}A)$ for every facet $\sigma'\neq\sigma$. Then $N_{-\boldsymbol{\beta}-\boldsymbol{a}}\neq\{0\}$.*

Proof. (i) Suppose that $\boldsymbol{\beta}+\boldsymbol{a}\sim\boldsymbol{\beta}$. Then $\mathbb{I}(\Omega(-\boldsymbol{a}))\not\subseteq\mathfrak{m}_{\boldsymbol{\beta}+\boldsymbol{a}}$ or $\mathbb{I}(\Omega(-\boldsymbol{a}))+\mathfrak{m}_{\boldsymbol{\beta}+\boldsymbol{a}}=K[s]$. Hence $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a}))+\langle A\theta-\boldsymbol{\beta}-\boldsymbol{a}\rangle K[\theta]=K[\theta]$. Therefore $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a}))\cap\langle A\theta-\boldsymbol{\beta}-\boldsymbol{a}\rangle K[\theta]=\langle A\theta-\boldsymbol{\beta}-\boldsymbol{a}\rangle\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a}))$, or $N_{-\boldsymbol{\beta}-\boldsymbol{a}}=\{0\}$ by (40).

(ii) Since $F_\sigma(\boldsymbol{\beta}+\boldsymbol{a})\in\mathbb{N}A$, there exist $\boldsymbol{u}\in\mathbb{N}^n$ and $\boldsymbol{\gamma}\in K^\sigma$ such that $\boldsymbol{\beta}+\boldsymbol{a}=A(\boldsymbol{u}+\boldsymbol{\gamma})$. Then, for any $\boldsymbol{v}\in\mathbb{N}^\sigma$, $A(\boldsymbol{u}+\boldsymbol{v})\in\mathbb{N}A\backslash(\boldsymbol{a}+\mathbb{N}A)=\Omega(-\boldsymbol{a})$ since $F_\sigma(A(\boldsymbol{u}+\boldsymbol{v}))=F_\sigma(\boldsymbol{\beta}+\boldsymbol{a}-A\boldsymbol{\gamma}+A\boldsymbol{v})=F_\sigma(\boldsymbol{\beta}+\boldsymbol{a})\notin F_\sigma(\boldsymbol{a}+\mathbb{N}A)$. Hence $\boldsymbol{u}+\mathbb{N}^\sigma\subseteq\widetilde{\Omega}(-\boldsymbol{a})$. Put $\boldsymbol{\xi}:=\boldsymbol{u}+\boldsymbol{\gamma}$. Then $A\boldsymbol{\xi}=\boldsymbol{\beta}+\boldsymbol{a}$ and $\boldsymbol{\xi}+K^\sigma=\boldsymbol{u}+K^\sigma\subseteq\mathrm{ZC}(\widetilde{\Omega}(-\boldsymbol{a}))$. By Lemma 8.1 we have

$$\mathrm{ZC}(\widetilde{\Omega}(-\boldsymbol{a}))=\bigcup_{j\in J}(\boldsymbol{u}^{(j)}+K^{\tau^{(j)}}),$$

and we see that, by the assumption, $\boldsymbol{\xi}+K^\sigma$ is the unique irreducible component of $\mathrm{ZC}(\widetilde{\Omega}(-\boldsymbol{a}))$ containing $\boldsymbol{\xi}$. Hence, by localizing at $\boldsymbol{\xi}$, to prove the assertion it is enough to show that $\mathbb{I}(\boldsymbol{\xi}+K^\sigma)\cap\langle A\theta-(\boldsymbol{\beta}+\boldsymbol{a})\rangle\neq\mathbb{I}(\boldsymbol{\xi}+K^\sigma).\langle A\theta-(\boldsymbol{\beta}+\boldsymbol{a})\rangle$ (see (40)) or, upon translating by $\boldsymbol{\xi}$, that $\mathbb{I}(K^\sigma)\cap\langle A\theta\rangle\neq\mathbb{I}(K^\sigma).\langle A\theta\rangle$. Since it is clearly true that

$$F_\sigma(A\theta)=\sum_{j=1}^n F_\sigma(\boldsymbol{a}_j)\theta_j\in\mathbb{I}(K^\sigma)\cap\langle A\theta\rangle\backslash\mathbb{I}(K^\sigma).\langle A\theta\rangle,$$

we have finished the proof. □

THEOREM 8.3. *$M_{K^n}(\boldsymbol{\beta})$ is irreducible if and only if $\boldsymbol{\beta}$ is non-resonant, i.e. $F_\sigma(\boldsymbol{\beta}) \notin \mathbb{Z}$ for all facets σ of $\mathbb{R}_{\geqslant 0}A$.*

Proof. Suppose that $\boldsymbol{\beta}$ is non-resonant. Then $\boldsymbol{\beta} + \boldsymbol{a} \sim \boldsymbol{\beta}$ for all $\boldsymbol{a} \in \mathbb{Z}^d$. Hence, by Lemma 8.2(i), $M_{K^n}(\boldsymbol{\beta}) \simeq L_{K^n}(T_A, \boldsymbol{\beta})$.

Suppose that $\boldsymbol{\beta}$ is resonant and that $F_\sigma(\boldsymbol{\beta}) \in \mathbb{Z}$. If $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}}$, then, by Corollary 7.6, there exists a surjective homomorphism

$$M_{K^n}(\boldsymbol{\beta}) \to L_{K^n}(T_A, \boldsymbol{\beta}). \tag{41}$$

Since σ is a facet of $\mathbb{R}_{\geqslant 0}A$, there exists $\boldsymbol{b} \in \mathbb{Z}^d$ such that $F_\sigma(\boldsymbol{b}) < 0$ while $F_{\sigma'}(\boldsymbol{b}) > 0$ for every facet $\sigma' \neq \sigma$. Hence, for a sufficiently large $n \in \mathbb{N}$, $F_\sigma(\boldsymbol{\beta} - n\boldsymbol{b}) \in F_\sigma(\mathbb{N}A)$ and $F_{\sigma'}(\boldsymbol{\beta} - n\boldsymbol{b}) \notin F_{\sigma'}(\mathbb{N}A)$ for every facet $\sigma' \neq \sigma$. Thus the homomorphism (41) has a non-trivial kernel by Lemma 8.2(ii).

Let $\boldsymbol{\beta} \neq \boldsymbol{\beta}^{\text{empty}}$. Then there exists a minimal $(\tau, \boldsymbol{\lambda}) \in E(\boldsymbol{\beta})$ (see (37)) with $\tau \neq \mathbb{R}_{\geqslant 0}A$. Hence, by Corollary 7.6, $L_{K^n}(T_\tau, \boldsymbol{\lambda})$ is a quotient of $M_{K^n}(\boldsymbol{\beta})$. Since the support of $L_{K^n}(T_\tau, \boldsymbol{\lambda})$ is strictly contained in the support of $M_{K^n}(\boldsymbol{\beta})$, the kernel of the homomorphism $M_{K^n}(\boldsymbol{\beta}) \to L_{K^n}(T_\tau, \boldsymbol{\lambda})$ is non-trivial. $\square$

REFERENCES

BGG76 I. N. Bernstein, I. M. Gel'fand and S. I. Gel'fand, *A category of $\mathfrak{g}$-modules*, Funct. Anal. Appl. **10** (1976), 87–92.

Ful93 W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131 (Princeton University Press, Princeton, NJ, 1993).

GKZ90 I. M. Gel'fand, M. M. Kapranov and A. V. Zelevinskii, *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. **84** (1990), 255–271.

HTT08 R. Hotta, K. Takeuchi and T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236 (Birkhäuser, Boston, 2008).

Kas03 M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217 (American Mathematical Society, Providence, RI, 2003).

MMW05 L. F. Matusevich, E. Miller and U. Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), 919–941.

MR87 J. C. McConnel and J. C. Robson, *Noncommutative Noetherian rings* (Wiley, Chichester, 1987).

Mus87 I. M. Musson, *Rings of differential operators on invariants of tori*, Trans. Amer. Math. Soc. **303** (1987), 805–827.

MV98 I. M. Musson and M. Van den Bergh, *Invariants under tori of rings of differential operators and related topics*, Mem. Amer. Math. Soc. **650** (1998).

Sai01 M. Saito, *Isomorphism classes of A-hypergeometric systems*, Compositio Math. **128** (2001), 323–338.

Sai07 M. Saito, *Primitive ideals of the ring of differential operators on an affine toric variety*, Tohoku Math. J. **59** (2007), 119–144.

SST00 M. Saito, B. Sturmfels and N. Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, vol. 6 (Springer, New York, 2000).

ST01 M. Saito and W. N. Traves, *Differential algebras on semigroup algebras*, Contemporary Mathematics, vol. 286 (American Mathematical Society, Providence, RI, 2001), 207–226.

ST04 M. Saito and W. N. Traves, *Finite generations of rings of differential operators of semigroup algebras*, J. Algebra **278** (2004), 76–103.

SW09 M. Schulze and U. Walther, *Hypergeometric D-modules and twisted Gauss–Manin systems*, J. Algebra **322** (2009), 3392–3409.

SS88 S. P. Smith and J. T. Stafford, *Differential operators on an affine curve*, Proc. London Math. Soc. **56** (1988), 229–259.

Wal07 U. Walther, *Duality and monodromy reducibility of A-hypergeometric system*, Math. Ann. **338** (2007), 55–74.

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