Irreducible quotients of 
A-hypergeometric systems

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Abstract
Gel’fand, Kapranov and Zelevinsky proved, using the theory of perverse sheaves, that in the Cohen–Macaulay case an A-hypergeometric system is irreducible if its parameter vector is non-resonant. In this paper we prove, using the theory of the ring of differential operators on an affine toric variety, that in general an A-hypergeometric system is irreducible if and only if its parameter vector is non-resonant. In the course of the proof, we determine the irreducible quotients of an A-hypergeometric system.

1. Introduction
Let $K$ be a field of characteristic 0, and let $A := (a_{ij})$ be a $d \times n$ integer matrix. We assume that $\mathbb{Z}^d$ is generated by the column vectors of $A$ as an abelian group. Given a parameter vector $\beta = (\beta_1, \ldots, \beta_d)^T \in K^d$, the A-hypergeometric (or GKZ) system $M_A(\beta)$ with parameter vector $\beta$ is defined by

$$M_A(\beta) := D(K^n)/D(K^n)I_A(\partial) + D(K^n)\langle A\theta - \beta \rangle,$$

where $D(K^n)$ is the $n$th Weyl algebra, i.e.

$$D(K^n) = K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n],$$

$I_A(\partial)$ is the toric ideal of $K[\partial_1, \ldots, \partial_n]$ defined by $A$, and $D(K^n)\langle A\theta - \beta \rangle$ is the left ideal of $D(K^n)$ generated by $\sum_{j=1}^n a_{ij}x_j\partial_j - \beta_i$, $i = 1, \ldots, d$.

The irreducibility of $M_A(\beta)$ is one of the most fundamental questions in the theory of A-hypergeometric systems. Gel’fand et al. proved, using the theory of perverse sheaves, that when the toric ring is Cohen–Macaulay, $M_A(\beta)$ is irreducible if its parameter vector $\beta$ is non-resonant; see [GKZ90, Proposition 4.4 and Theorem 4.6]. Schulze and Walther have determined for which parameter vector $\beta$ the Fourier transform of $M_A(\beta)$ is naturally isomorphic to the direct image of a simple object on the big torus of the affine toric variety defined by $A$ (see [SW09, Corollary 3.7]), which sharpens [GKZ90, Theorem 4.6]. Walther proved in [Wal07, Theorem 3.13] that if $M_A(\beta)$ has irreducible monodromy representation, then so does $M_A(\gamma)$ for any $\gamma \in \beta + \mathbb{Z}^d$, using homological tools developed in [MMW05]. Naturally, an irreducible $D(K^n)$-module has irreducible monodromy representation; see Proposition 6.8.

In this paper, using the theory of the ring of differential operators on an affine toric variety, we prove that $M_A(\beta)$ is irreducible if and only if $\beta$ is non-resonant, without assuming that the toric ring is Cohen–Macaulay. Moreover, in the course of the proof, we determine the irreducible quotients of $M_A(\beta)$.
Let \( \iota \) be the anti-automorphism of \( D(K^n) \) defined by \( \iota(x_j) = \partial_j \) and \( \iota(\partial_j) = x_j \) for \( j = 1, \ldots, n \). Then \( \iota \) gives rise to the equivalence between the category of left \( D(K^n) \)-modules and the category of right \( D(K^n) \)-modules; the left \( D(K^n) \)-module \( M_A(\beta) \) corresponds to the right \( D(K^n) \)-module \( M_K^n(\beta) \) (which is often given in \( (8) \)). Hence the irreducibility of \( M_A(\beta) \) is equivalent to that of \( M_K^n(\beta) \). In this paper, we work with the categories of right \( D \)-modules. This has two advantages: one is that the support of \( M_K^n(\beta) \) is precisely the affine toric variety defined by \( A \); the other is that we consider direct image functors of \( D \)-modules, and for this purpose, right \( D \)-modules work more naturally than left \( D \)-modules.

In \( \S \ 2 \) we introduce the varieties considered in this paper, and in \( \S \ 3 \) we briefly recall the rings of differential operators on these varieties and their \( \mathbb{Z}^d \)-gradings.

In \( \S \ 4 \), for each variety \( X \) introduced in \( \S \ 2 \) we consider the category \( \mathcal{O}_X \), which is analogous to the category \( \mathcal{O} \) from the theory of highest-weight modules over semisimple Lie algebras defined in [BGG76] (cf. [MV98, Sai07]). We then recall the simple objects in \( \mathcal{O}_X \) for \( X = X_A \), the affine toric variety defined by \( A \) (see Proposition 4.3), and for \( X = T_A \), the big torus of \( X_A \) (see Proposition 4.2). Finally, we define Verma-type modules in \( \mathcal{O}_X \). The right-module counterpart \( M_K^n(\beta) \) of the \( A \)-hypergeometric system \( M_A(\beta) \) is a Verma-type module in \( \mathcal{O}_K^n \).

In \( \S \ 5 \), we explicitly describe the direct image functors of \( D \)-modules by inclusions between the varieties under consideration. Using this description, in \( \S \ 6 \) we show that the direct image of a simple object in \( \mathcal{O}_{T_A} \) by the inclusion of \( T_A \) into \( K^n \) has a unique irreducible \( D(K^n) \)-submodule, and we describe it explicitly (see Theorem 6.4). We then show that each simple object in \( \mathcal{O}_K^n \) is obtained in a similar way from a possibly smaller torus (Theorem 6.6).

In \( \S \ 7 \), we compute the pull-back of each simple object in \( \mathcal{O}_K^n \) by the inclusion of \( X_A \) into \( K^n \) (Theorems 7.3 and 7.4). As a consequence, we determine the irreducible quotients of \( M_K^n(\beta) \) (Corollaries 7.5 and 7.6). In \( \S \ 8 \), we prove that \( M_K^n(\beta) \) is irreducible if and only if \( \beta \) is non-resonant (Theorem 8.3).

## 2. Varieties

Let \( A := \{a_1, a_2, \ldots, a_n\} \) be a finite set of column vectors in \( \mathbb{Z}^d \). We will sometimes identify \( A \) with the matrix \( (a_1, a_2, \ldots, a_n) = (a_{ij}) \). Let \( ZA \) and \( \mathbb{R}_{\geq 0} A \) denote, respectively, the abelian group and the cone generated by \( A \). Throughout this paper, we assume that \( ZA = \mathbb{Z}^d \) and that \( \mathbb{R}_{\geq 0} A \) is strongly convex.

Let \( K \) denote a field of characteristic 0. For a face \( \tau \) of the cone \( \mathbb{R}_{\geq 0} A \), we define the following varieties:

\[
K^\tau := \{x = (x_1, \ldots, x_n) \in K^n : x_j = 0 \text{ when } a_j \notin \tau\},
\]

\[
(K^X)^\tau := \{x \in K^X : x_j \neq 0 \text{ when } a_j \in \tau\},
\]

\[
X_{\tau} := \{x \in K^\tau : x^u - x^v = 0 \text{ for } u, v \in \mathbb{N}^n \text{ such that } A u = A v\},
\]

\[
T_{\tau} := \{x \in (K^X)^\tau : x^u - x^v = 0 \text{ for } u, v \in \mathbb{N}^n \text{ such that } A u = A v\}.
\]

Here we have used multi-index notation, where \( x^u \) stands for \( x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n} \), with \( u = (u_1, u_2, \ldots, u_n)^T \). When \( \tau \) is the whole cone \( \mathbb{R}_{\geq 0} A \), we denote the above varieties by \( K^n, (K^X)^n, X_A \) and \( T_A \), respectively. Then

\[
X_A = \coprod_{\text{faces } \tau \text{ of } \mathbb{R}_{\geq 0} A} T_{\tau}
\]  \hspace{1cm} (3)
is the \((K^\times)^d\)-orbit decomposition of the toric variety \(X_A\) (see, e.g., [Ful93]). Here \((K^\times)^d\) acts on \(K^n\) by
\[
(K^\times)^d \times K^n \ni (t, (x_1, \ldots, x_n)) \mapsto (t^{a_1} x_1, \ldots, t^{a_n} x_n) \in K^n,
\]
where \(t^a = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}\) for \(a = (a_1, a_2, \ldots, a_d)^T\).

Let \(\mathbb{N}A\) denote the monoid generated by \(A\). The semigroup algebra \(K[\mathbb{N}A] = \bigoplus_{a \in \mathbb{N}A} K t^a\) is the ring of regular functions on the affine toric variety \(X_A\). Then we have \(K[\mathbb{N}A] \simeq K[x]/I_A\), where \(I_A\) is the ideal of the polynomial ring \(K[x] := K[x_1, \ldots, x_n]\) generated by all \(x^u - x^v\) for \(u, v \in \mathbb{N}^n\) with \(Au = Av\).

### 3. Rings of differential operators

Let \(R\) be a commutative \(K\)-algebra, and let \(M\) and \(N\) be \(R\)-modules. We briefly recall the module \(D(M,N)\) of differential operators from \(M\) to \(N\); for details, see [SS88]. For \(k \in \mathbb{N}\), the subspaces \(D^k(M, N)\) of \(\text{Hom}_K(M, N)\) are defined inductively by
\[
D^0(M, N) := \text{Hom}_R(M, N)
\]
and
\[
D^{k+1}(M, N) := \{ P \in \text{Hom}_K(M, N) : [f, P] \in D^k(M, N) \text{ for all } f \in R \},
\]
where \([\ ,\ ]\) denotes the commutator. Set \(D(M, N) := \bigcup_{k=0}^{\infty} D^k(M, N)\) and \(D(M) := D(M, M)\). Then \(D(M)\) is a \(K\)-algebra, and \(D(M, N)\) is a \((D(N), D(M))\)-bimodule.

The ring \(D(K^n) := D(K[x])\) of differential operators on \(K^n\) is the \(n\)th Weyl algebra (2).

The ring \(D((K^\times)^n) := D(K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])\) of differential operators on \((K^\times)^n\) is given by
\[
D((K^\times)^n) = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \langle \partial_1, \ldots, \partial_n \rangle
= \bigoplus_{u \in \mathbb{Z}^n} x^u K[\theta_1, \ldots, \theta_n],
\]
where \(\theta_j = x_j \partial_j\).

The ring \(D(T_A) := D(K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}])\) of differential operators on \(T_A\) is given by
\[
D(T_A) = K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \langle \partial_{t_1}, \ldots, \partial_{t_d} \rangle
= \bigoplus_{a \in \mathbb{Z}^d} t^a K[s_1, \ldots, s_d],
\]
where \(s_i = t_i \partial_{t_i}\) and \(\partial_{t_i} = \partial / \partial t_i\).

The ring \(D(X_A) := D(K[\mathbb{N}A])\) of differential operators on \(X_A\) is a subalgebra of \(D(T_A)\):
\[
D(X_A) = \{ P \in D(T_A) : P(K[\mathbb{N}A]) \subseteq K[\mathbb{N}A] \}.
\]

Let \(X\) be \(K^n\), \((K^\times)^n\), \(T_A\) or \(X_A\). For \(a = (a_1, \ldots, a_d)^T \in \mathbb{Z}^d\), set
\[
D(X)_a := \{ P \in D(X) : [s_i, P] = a_i P \text{ for } i = 1, \ldots, d \},
\]
where \(s_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j\) for \(X = K^n\) or \((K^\times)^n\). Then
\[
D(X) = \bigoplus_{a \in \mathbb{Z}^d} D(X)_a
\]
is a \(\mathbb{Z}^d\)-graded algebra.
Let $\tau$ be a face of the cone $\mathbb{R}_{\geq 0}A$. Let $Z(A \cap \tau)$ and $\mathbb{N}(A \cap \tau)$ denote, respectively, the abelian group and the monoid generated by $A \cap \tau$. Set
\[ Z^\tau := \{ u = (u_1, \ldots, u_n) \in \mathbb{Z}^n : u_j = 0 \text{ when } a_j \notin \tau \}. \]
As in the case where $\tau$ is the whole cone $\mathbb{R}_{\geq 0}A$, for $K^\tau, (K^\times)^\tau, T_\tau$ and $X_\tau$ we consider the following rings of differential operators:
\[
D(K^\tau) = D(K[x_j : a_j \in \tau]) = K[x_j : a_j \in \tau] \langle \partial_j : a_j \in \tau \rangle, \\
D((K^\times)^\tau) = K[x_j^{\pm 1} : a_j \in \tau] \langle \partial_j : a_j \in \tau \rangle = \bigoplus_{u \in \mathbb{Z}^\tau} a^u K[\theta_j : a_j \in \tau], \\
D(T_\tau) = \bigoplus_{a \in \mathbb{Z}(A \cap \tau)} t^a K[s_1|_\tau, \ldots, s_d|_\tau], \\
D(X_\tau) = \{ P \in D(T_\tau) : P(K[X_\tau]) \subseteq K[X_\tau] \},
\]
where $s_i|_\tau$ is the operator $s_i$ restricted to $K[T_\tau] = K[t^{\pm a_j} : a_j \in \tau]$ and $K[X_\tau]$ is the subalgebra of $K[T_\tau]$ defined by
\[
K[X_\tau] = K[\mathbb{N}(A \cap \tau)] = K[t^{a_j} : a_j \in \tau].
\]
These rings of differential operators are graded by $\mathbb{Z}(A \cap \tau)$, and since $\mathbb{Z}(A \cap \tau)$ is a subgroup of $\mathbb{Z}A = \mathbb{Z}^d$, they are also considered to be $\mathbb{Z}^d$-graded. Note that $s_i|_\tau = \sum a_j \in \tau a_j \theta_j$ in $x$-coordinates.

4. The category $\mathcal{O}_X$

Take $X$ to be $K^n$, $(K^\times)^n$, $T_A$ or $X_A$. We shall define a full subcategory $\mathcal{O}_X$ of the category of right $D(X)$-modules (cf. [MV98]). A right $D(X)$-module $M$ is an object of $\mathcal{O}_X$ if the support of $M$ is contained in $X_A$ and $M$ has a weight decomposition $M = \bigoplus_{\lambda \in K^d} M_\lambda$, where
\[
M_\lambda = \{ x \in M : x.f(s) = f(-\lambda)x \text{ for all } f \in K[s] \}
\]
with $K[s] = K[s_1, \ldots, s_d]$.

**Proposition 4.1.** Let $M$ be a simple object in $\mathcal{O}_X$. Then $M$ is an irreducible right $D(X)$-module.

**Proof.** Let $N$ be a right $D(X)$-submodule of $M$. Let $x \in N$, and write $x = \sum_{b \in S} x_b$ for $x_b \in M_b$, where $S$ is a finite subset of $K^d$. For $b \in S$, take $f(s) \in K[s]$ such that $f(-b) \neq 0$ and $f(-c) = 0$ for all $c \in S \setminus \{ b \}$. Upon applying $f(s)$ to $x$, we see that $x_b \in N$. Hence $N \in \mathcal{O}_X$. By the simplicity of $M$ in $\mathcal{O}_X$, we have $N = 0$ or $N = M$. \qed

In the rest of this section, we define objects $L_{T_A}(\lambda)$ and $L_{X_A}(\lambda)$ which are simple in the categories $\mathcal{O}_{T_A}$ and $\mathcal{O}_{X_A}$, respectively. Then we define Verma-type modules $M_{X_A}(\beta)$, $M_{K^n}(\beta)$ and $M_{(K^\times)^n}(\beta)$.

Let $\lambda \in K^d$. We define a right $D(T_A)$-module $L_{T_A}(\lambda)$ by
\[
L_{T_A}(\lambda) := D(T_A)/\langle s - \lambda \rangle D(T_A) := D(T_A)/\sum_{i=1}^d (s_i - \lambda_i) D(T_A).
\]

Let $K[t^{\pm 1}]$ denote the Laurent polynomial ring $K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. By taking formal adjoint operators, $D(T_A)$ acts on $K[t^{\pm 1}]t^{-\lambda} dT_A$ from the right as follows:
\[
(g(t) dT_A).P = P^*(g) dT_A,
\]

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where

$$P^* = \sum_a f_a (-s) t^a$$

for $P = \sum_a t^a f_a(s) \in \bigoplus_{a \in \mathbb{Z}^d} t^a K[s] = D(T_A)$ and $dT_A$ is simply a formal symbol. Then $K[t^\pm] t^{-\lambda} dT_A$ is a realization of $L_{T_A}(\lambda)$, and we denote $K[t^\pm] t^{-\lambda} dT_A$ by $L_{T_A}(\lambda)$, so that

$$L_{T_A}(\lambda) = \bigoplus_{a \in \mathbb{Z}^d} L_{T_A}(\lambda)_{-\lambda + a} \text{ with } L_{T_A}(\lambda)_{-\lambda + a} = K t^{-\lambda + a} dT_A. \quad (4)$$

The following proposition is clear.

**Proposition 4.2.** Each $L_{T_A}(\lambda)$ is a simple object in $\mathcal{O}_{T_A}$. Each simple object in $\mathcal{O}_{T_A}$ is isomorphic to $L_{T_A}(\lambda)$ for some $\lambda \in K^d$, and $L_{T_A}(\lambda) \simeq L_{T_A}(\mu)$ if and only if $\lambda - \mu \in \mathbb{Z}^d$.

Recall that the ring $D(X_A)$ is described as follows (see [Mus87, Theorem 2.3]):

$$D(X_A) = t^a \mathbb{I}((\Omega(a)) \quad \text{for} \quad a \in \mathbb{Z}^d,$$

where

$$\Omega(a) := \Omega_A(a) := \mathbb{N}A \setminus (-a + \mathbb{N}A),$$

$$\mathbb{I}((\Omega(a)) := \{ f(s) \in K[s] : f(c) = 0 \text{ for all } c \in \Omega(a) \}. \quad (5)$$

Recall also the preorder $\preceq$ defined in [MV98] (see also [ST01]):

$$\text{for } \alpha, \beta \in K^d, \quad \alpha \preceq \beta \iff \mathbb{I}((\Omega(\beta - \alpha)) \subseteq m_\alpha, \quad (6)$$

where $m_\alpha$ is the maximal ideal of $K[s]$ at $\alpha$. We define an equivalence relation $\sim$ by setting $\alpha \sim \beta$ if and only if $\alpha \preceq \beta$ and $\alpha \succeq \beta$. We write $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \nless \beta$.

Since the ring $D(X_A)$ is a subalgebra of $D(T_A)$, the right $D(T_A)$-module

$$L_{T_A}(\lambda) = K[t^\pm] t^{-\lambda} dT_A = \bigoplus_{a \in \mathbb{Z}^d} K t^{-\lambda + a} dT_A$$

is also a right $D(X_A)$-module. Then the subquotient

$$L_{X_A}(\lambda) := \bigoplus_{\mu \preceq \lambda} K t^{-\mu} dT_A / \bigoplus_{\mu \prec \lambda} K t^{-\mu} dT_A \quad (7)$$

is a right $D(X_A)$-module (see [ST01, Proposition 4.1.5]). We have the following proposition.

**Proposition 4.3.** Each $L_{X_A}(\lambda)$ is a simple object in $\mathcal{O}_{X_A}$. Each simple object in $\mathcal{O}_{X_A}$ is isomorphic to $L_{X_A}(\lambda)$ for some $\lambda \in K^d$. Moreover, $L_{X_A}(\lambda) \simeq L_{X_A}(\mu)$ if and only if $\lambda \sim \mu$.

(See [MV98, Proposition 3.1.7], [ST01, Theorem 4.1.6] or [Sai07, Proposition 3.6(4)].)

For $\beta \in K^d$, we define a right $D(X_A)$-module $M_{X_A}(\beta)$, a right $D(K^n)$-module $M_{K^n}(\beta)$ and a right $D((K^\times)^n)$-module $M_{(K^\times)^n}(\beta)$ by

$$M_{X_A}(\beta) := D(X_A)/(s - \beta) D(X_A), \quad M_{K^n}(\beta) := D(K^n)/(I_A D(K^n) + (s - \beta) D(K^n)), \quad M_{(K^\times)^n}(\beta) := D((K^\times)^n)/(I_A D((K^\times)^n) + (s - \beta) D((K^\times)^n)). \quad (8)$$

Recall that $s_i = t_i \partial_i$ in $t$-coordinates and that $s_i = \sum_{j=1}^n a_{ij} \theta_j$ with $\theta_j = x_j \partial_j$ in $x$-coordinates. Clearly, $M_{X_A}(\beta) \in \mathcal{O}_{X_A}$, $M_{K^n}(\beta) \in \mathcal{O}_{K^n}$ and $M_{(K^\times)^n}(\beta) \in \mathcal{O}_{(K^\times)^n}$.

Let $\tau$ be a face of the cone $\mathbb{R}_{\geq 0} A$. Similarly to the case where $\tau$ is the whole cone $\mathbb{R}_{\geq 0} A$, for $Y = K^\tau, (K^\times)^\tau, T_\tau$ or $X_\tau$ we consider $\mathcal{O}_Y$, replacing $\mathbb{Z} A = \mathbb{Z}^d$, $KA = K^d$ and $f(s) \in K[s]$.
by $\mathbb{Z}(A \cap \tau)$, $K(A \cap \tau)$ and $f(s)|_{\tau}$, respectively, where $f(s)|_{\tau}$ is the operator $f(s)$ restricted to $K[T_\tau] = K[t^{\pm a_j} : a_j \in \tau]$.

5. Direct image functors

In this section, we describe direct image functors explicitly. Using them, we link some of the modules defined in §4.

5.1 From $\mathcal{O}_{T_A}$ to $\mathcal{O}_{(K^\times)^n}$

We shall write $D((K^\times)^n, T_A)$ instead of $D(K[x^{\pm 1}], K[t^{\pm 1}])$, where $K[x^{\pm 1}]$ stands for the Laurent polynomial ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Since $T_A$ is closed in $(K^\times)^n$, the direct image functor $\int_0^T A \to (K^\times)^n : M \mapsto M \otimes D(T_A)$ gives a category equivalence between $\mathcal{O}_{T_A}$ and $\mathcal{O}_{(K^\times)^n}$, known as Kashiwara's equivalence (see, e.g., [Kas03, Theorem 4.30] or [HTT08, Theorem 1.6.1]). From [SS88, §1.3, (e) and (f)], we have

$$D((K^\times)^n, T_A) = D((K^\times)^n)/I_AD((K^\times)^n) = \bigoplus_{a \in \mathbb{Z}^d} a^\bullet K[\theta_1, \ldots, \theta_n].$$

(9)

By definition,

$$M_{(K^\times)^n}(\beta) = \int_0^T A \to (K^\times)^n L_{T_A}(\beta).$$

(10)

Hence, by Kashiwara's equivalence, Proposition 4.2 leads to the following result.

**Proposition 5.1.** For each $\beta \in K^d$, $M_{(K^\times)^n}(\beta)$ is a simple object in $\mathcal{O}_{(K^\times)^n}$. Each simple object in $\mathcal{O}_{(K^\times)^n}$ is isomorphic to some $M_{(K^\times)^n}(\beta)$. Moreover, $M_{(K^\times)^n}(\beta) \simeq M_{(K^\times)^n}(\beta')$ if and only if $\beta - \beta' \in \mathbb{Z}^d$.

5.2 From $\mathcal{O}_{X_A}$ to $\mathcal{O}_{K^n}$

Again from [SS88, §1.3, (e) and (f)], we have

$$D(K^n, X_A) := D(K[x], K[N_A]) = D(K^n)/I_AD(K^n).$$

(11)

Since $I_A$ is $\mathbb{Z}^d$-homogeneous, $D(K^n, X_A)$ inherits the $\mathbb{Z}^d$-grading from $D(K^n)$.

The algebra $D(X_A)$ can be identified with

$$\{ P \in D(K^n) : PI_A \subseteq I_AD(K^n) \}/I_AD(K^n) / I_AD(K^n)$$

(see, e.g., [MR87, Theorem 5.13]). We may therefore consider $D(X_A)$ as being contained in $D(K^n, X_A)$.

Let $\int_0^X A \to K^n$ denote the functor from $\mathcal{O}_{X_A}$ to $\mathcal{O}_{K^n}$ defined by

$$\int_0^X A \to K^n M := M \otimes_{D(X_A)} D(K^n, X_A).$$
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Note that, in general, $X_A$ is singular and $j_{X_A \to K^n}^0$ does not give a category equivalence. By definition, we have

$$M_{K^n}(\beta) = \int_{X_A \to K^n}^0 M_{X_A}(\beta).$$

For the following result, see [Sai07, Proposition 4.1 and Corollary 4.2].

**Proposition 5.2.**

$$D(K^n, X_A) = \bigoplus_{a \in \mathbb{Z}^d} D(K^n, X_A)_a \quad \text{with} \quad D(K^n, X_A)_a = t^a \Pi(\Omega(a)),$$

where

$$\Omega(a) := \Omega(a) := \{ u \in \mathbb{N}^n : Au \notin -a + \mathbb{N}A \},$$

$$\Pi(\Omega(a)) = \{ f(\theta) \in K[\theta] : f(u) = 0 \text{ for all } u \in \Omega(a) \}$$

and $K[\theta] := K[\theta_1, \ldots, \theta_n]$. 

**5.3 From $O_{K^\tau}$ to $O_{K^n}$**

Let $\tau$ be a face of the cone $\mathbb{R}_{\geq 0} A$. We consider the direct image functor $j_{K^\tau \to K^n}^0$ from $O_{K^\tau}$ to $O_{K^n}$. Given $M \in O_{K^\tau}$, we define $\int_{K^\tau \to K^n}^0 M \in O_{K^n}$ by

$$\int_{K^\tau \to K^n}^0 M := M \otimes_{D(K^\tau)} D(K^n, K^\tau),$$

where

$$D(K^n, K^\tau) := D(K[x], K[x_j : a_j \in \tau]).$$

Put

$$K^{\tau c} := \{ x = (x_1, \ldots, x_n) \in K^n : x_j = 0 \text{ when } a_j \in \tau \},$$

$$\mathbb{N}^{\tau c} := \{ a = (a_1, \ldots, a_n) \in \mathbb{N}^n : a_j = 0 \text{ when } a_j \in \tau \},$$

$$\mathbb{Z}^{\tau c} := \{ a = (a_1, \ldots, a_n) \in \mathbb{Z}^n : a_j = 0 \text{ when } a_j \in \tau \}.$$

Then

$$D(K^n, K^\tau) = D(K^n) / \langle x_j : a_j \notin \tau \rangle D(K^n)$$

$$= D(K^\tau) \boxtimes D(K^{\tau c}) / \langle x_j : a_j \notin \tau \rangle D(K^{\tau c}).$$

Since, as right $D(K^{\tau c})$-modules,

$$D(K^{\tau c}) / \langle x_j : a_j \notin \tau \rangle D(K^{\tau c}) \simeq \bigoplus_{b \in \mathbb{Z}^{\tau c}} Kx^{-b} d(K^{\tau c}) / \bigoplus_{b \notin \mathbb{N}^{\tau c}} Kx^{-b} d(K^{\tau c}),$$

we have

$$D(K^n, K^\tau) \simeq D(K^\tau) \bigoplus_{b \in \mathbb{N}^{\tau c}} Kx^{-b} d(K^{\tau c}).$$

Hence

$$\int_{K^\tau \to K^n}^0 M \simeq M \bigoplus_{b \in \mathbb{N}^{\tau c}} Kx^{-b} d(K^{\tau c}).$$

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6. Simple objects in $\mathcal{O}_{K^n}$

In this section, we describe the simple objects in $\mathcal{O}_{K^n}$ explicitly.

By (9), (10) and the realization (4), we have the following realization of $M_{(K^n)^n}(\beta)$.

**Lemma 6.1.** Let $\beta \in KA = K^d$. Then

$$M_{(K^n)^n}(\beta) = \bigoplus_{a \in \mathbb{Z}^d} K t^{-\beta + a} dT_A \otimes_{K[s]} K[\theta].$$

The $D(K^n)$-module $\int_{T_A \rightarrow K^n}^{0} L_{T_A}(\beta)$ is defined to be the $D((K^n)^n)$-module

$$\int_{T_A \rightarrow (K^n)^n}^{0} L_{T_A}(\beta) = M_{(K^n)^n}(\beta), \quad (16)$$

considered as a $D(K^n)$-module.

**Definition 6.2.** Let $\beta \in KA = K^d$. In $\beta + \mathbb{Z}A = \beta + \mathbb{Z}^d$ there exists a unique minimal equivalence class with respect to $\preceq$ (see Remark 6.3), which we denote by $\beta^\text{empty}$. Any fixed element belonging to the class is also denoted by $\beta^\text{empty}$.

**Remark 6.3.** In [Sai01] we defined, for a face $\tau$ and a parameter vector $\alpha \in KA = K^d$, a finite set

$$E_\tau(\alpha) = \{ \lambda \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau) : \alpha - \lambda \in \mathbb{N}A + \mathbb{Z}(A \cap \tau) \}. \quad (17)$$

The class $\beta^\text{empty}$ is given by

$$E_\tau(\beta^\text{empty}) = \begin{cases} E_{R_{\geq 0}A}(\beta) & \text{if } \tau = R_{\geq 0}A, \\ \emptyset & \text{if } \tau \neq R_{\geq 0}A. \end{cases} \quad (18)$$

**Theorem 6.4.** Let $\beta \in KA/\mathbb{Z}A = K^d/\mathbb{Z}^d$, and fix an element $e := \beta^\text{empty}$. Then

$$L_{K^n}(T_A, \beta) := (t^{-e} dT_A \otimes 1) D(K^n)$$

$$= \bigoplus_{a \in \mathbb{Z}^d} K t^{-e + a} dT_A \otimes_{K[s]} \Omega(\alpha)$$

$$\simeq D(K^n)/((I_A D(K^n) + D(K^n) \cap \langle s - e \rangle D((K^n)^n))$$

is a unique simple $D(K^n)$-submodule of $\int_{T_A \rightarrow K^n}^{0} L_{T_A}(\beta)$.

Moreover, $L_{K^n}(T_A, \beta) \simeq L_{K^n}(T_A, \beta')$ if and only if $\beta - \beta' \in \mathbb{Z}^d$.

**Proof.** Recall that $\int_{T_A \rightarrow K^n}^{0} L_{T_A}(\beta)$ is the module $M_{(K^n)^n}(\beta)$ regarded as a $D(K^n)$-module (16). Hence $L_{K^n}(T_A, \beta)$ is isomorphic to $D(K^n)/((I_A D(K^n) + D(K^n) \cap \langle s - e \rangle D((K^n)^n))$ by the definition of $M_{(K^n)^n}(\beta) = M_{(K^n)^n}(e)$. The first equation is clear from (11) and Proposition 5.2.

Let $y \in M_{(K^n)^n}(\beta)$ be non-zero. We prove that $y D(K^n) \supseteq L_{K^n}(T_A, \beta)$. By multiplying a suitable $x^u$ from the right, we may assume that

$$y = t^{-\beta'} dT_A \otimes f(\theta) \quad \text{for some } \beta' \sim e. \quad (19)$$

Here $f(\theta) \notin \langle A\theta - \beta' \rangle K[\theta]$ since $y \neq 0$. We shall use the symbols $s$ and $A\theta$ interchangeably. We claim that

$$t^{-\beta''} dT_A \otimes 1 \in y D(K^n) \quad \text{for some } \beta'' \sim e. \quad (20)$$
We take an element of type (19) in $yD(K^n)$ such that the total degree $\deg(f)$ of $f$ is as small as possible, and we call this element $y$ again. If $f(\theta) \in K[s]$, then clearly we have the claim (20). Suppose $f(\theta) \not\in K[s]$. Let $u, v \in \mathbb{N}^n$ satisfy $Au = Av$. Since
\[ f(\theta)(x^u - x^v) = (x^u - x^v)(f(\theta + u) + x^v(f(\theta + u) - f(\theta + v)), \]
we have
\[ y.(x^u - x^v) = t^{-\beta'Av} dT_A \otimes (f(\theta + u) - f(\theta + v)). \]
By the minimality of $\deg(f)$,
\[ f(\theta + u) - f(\theta + v) \in (A\theta - (\beta' - Av))K[\theta]. \]
Hence, for all $u, v \in \mathbb{N}^n$ with $Au = Av$,
\[ f(\theta + u) - f(\theta + v) \in (A\theta - (\beta' - Av))K[\theta]. \]
Since $f(\theta) \not\in (A\theta - \beta')K[\theta]$, there exists $z \in K^n$ with $Az = \beta'$ such that $f(z) \neq 0$. By Lemma 6.5 below, we have
\[ f(\theta) \in (A\theta - \beta')K[\theta]. \]
Hence $y = t^{-\beta'} dT_A \otimes f(z)$. We have thus proved claim (20).

Since $\beta'' \sim e$, there exists $p(s) \in \mathbb{I}(\Omega(\beta'' - e))$ such that $p(\beta'') \neq 0$. Hence $t^{\beta''-e}p(s) \in D(X_A) \subseteq D(K^n)/I_AD(K^n)$, and
\[ (t^{-\beta''} dT_A \otimes 1)t^{\beta''-e}p(s) = p(\beta'')t^{-e} dT_A \otimes 1. \]
We have thus proved that $yD(K^n) \supseteq L_K^n(T_A, \beta)$ and that $L_K^n(T_A, \beta)$ is a unique simple $D(K^n)$-submodule of $\int_{T_A \rightarrow K^n} L_{T_A}(\beta)$.

Next, we prove the second statement. If $\beta - \beta' \notin \mathbb{Z}^d$, then $\beta'_{\text{empty}} = \beta''_{\text{empty}}$. Hence $L_K^n(T_A, \beta) = L_K^n(T_A, \beta')$ by definition. If $\beta - \beta' \not\in \mathbb{Z}^d$, then $L_K^n(T_A, \beta)$ and $L_K^n(T_A, \beta')$ have distinct weight sets and hence are not isomorphic. □

**Lemma 6.5.** Let $f(\theta) \in K[\theta]$ satisfy
\[ f(\theta + l) - f(\theta) \in (A\theta - c)K[\theta] \]
for all $l$ with $Al = 0$. Take $\gamma \in K^n$ such that $A\gamma = c$. Then
\[ f(\theta) \in f(\gamma) + (A\theta - c)K[\theta]. \]

**Proof.**
\[ f(\theta + l) - f(\theta) \in (A\theta - c)K[\theta] \quad \text{for all } l \text{ such that } Al = 0 \]
\[ \implies f(l + \gamma) - f(\gamma) = 0 \quad \text{for all } l \text{ such that } Al = 0 \]
\[ \iff f(\theta + \gamma) \in f(\gamma) + (A\theta)K[\theta] \]
\[ \iff f(\theta) \in f(\gamma) + (A\theta - c)K[\theta]. \]

Let $\tau$ be a face of $\mathbb{R}_{>0}A$, and let $\lambda \in K(A \cap \tau)/Z(A \cap \tau)$. We define a right $D(K^\tau)$-module $L_K^\tau(T_\tau, \lambda)$ in the same way as we defined $L_K^n(T_A, \beta)$ in Theorem 6.4. By Theorem 6.4, $L_K^\tau(T_\tau, \lambda)$ is a simple $D(K^\tau)$-module. By Kashiwara’s equivalence,
\[ L_K^n(T_\tau, \lambda) := \int_{K^\tau \rightarrow K^n} L_K^\tau(T_\tau, \lambda) \quad (21) \]
is a simple $D(K^n)$-module.
Theorem 6.6. Each simple object in \(\mathcal{O}_{K^n}\) is isomorphic to \(L_{K^n}(T_\tau, \lambda)\) for some face \(\tau\) and some \(\lambda \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)\).

Moreover, \(L_{K^n}(T_\tau, \lambda) \simeq L_{K^n}(T_{\tau'}, \lambda')\) if and only if \(\tau = \tau'\) and \(\lambda - \lambda' \in \mathbb{Z}(A \cap \tau)\).

Proof. Let \(L\) be a simple object in \(\mathcal{O}_{K^n}\). Suppose that \(\text{supp}(L) = T_A = X_A\). There exists the following exact sequence in \(\mathcal{O}_{K^n}\):

\[
0 \to \Gamma_{K^n \setminus (K^\times)^n}(L) \to L \to \Gamma_{(K^\times)^n}(L),
\]

where \(\Gamma_{K^n \setminus (K^\times)^n}(L) = \{y \in L : \text{supp}(y) \subseteq K^n \setminus (K^\times)^n\}\) and \(\Gamma_{(K^\times)^n}(L)\) is the localization of \(L\) at the multiplicatively closed set \(\{x^m : j = 1, \ldots, n; m \in \mathbb{N}\}\). By the simplicity of \(L\), \(\Gamma_{K^n \setminus (K^\times)^n}(L) = 0\). Hence \(L\) is a simple module of \(\Gamma_{(K^\times)^n}(L)\), and then \(\Gamma_{(K^\times)^n}(L)\) is simple in \(\mathcal{O}_{(K^\times)^n}\). Indeed, let \(y\) be a non-zero submodule of \(\Gamma_{(K^\times)^n}(L)\); then there exists \(u \in \mathbb{N}^n\) such that \(y, x^u \in L\). Since \(L\) is a simple \(D(K^n)\)-module, we have \(y, D(K^n) \supseteq L\). Since \(\Gamma_{(K^\times)^n}(L)\) is generated by \(L\) as a \(D((K^\times)^n)\)-module, we obtain \(y, D((K^\times)^n) = \Gamma_{(K^\times)^n}(L)\), and hence \(\Gamma_{(K^\times)^n}(L)\) is simple in \(\mathcal{O}_{(K^\times)^n}\). Then, by Proposition 5.1, \(\Gamma_{(K^\times)^n}(L) \simeq M_{(K^\times)^n}(\beta)\) for some \(\beta \in KA/ZA\). Since \(M_{(K^\times)^n}(\beta)\) has the unique simple submodule \(L_{K^n}(T_A, \beta)\), we conclude that \(L \simeq L_{K^n}(T_A, \beta)\).

By the simplicity of \(L\), the support of \(L\) is the closure of \(T_\tau\) for some face \(\tau\). By the same argument as in the previous paragraph, we obtain \(L \simeq L_{K^n}(T_\tau, \lambda)\) for some \(\lambda \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)\).

The second statement is clear from the second statement of Theorem 6.4.

Example 6.7. Let \(A = (1)\). In this case, the cone \(\mathbb{R}_{\geq 0}A = \mathbb{R}_{\geq 0}\) has only two faces: \(\{0\}\) and \(\mathbb{R}_{\geq 0}\). Then

\[
L_K(T_{\{0\}}, 0) = \int_{\{0\} \rightarrow K} K \simeq D/xdD,
\]

where \(D\) is the first Weyl algebra.

Let \(\beta \in K\). If \(\beta \notin \mathbb{Z} = ZA\), then \(\beta = \beta_{\text{empty}}\). If \(\beta \in \mathbb{Z}\), then \(\beta = \beta_{\text{empty}}\) if and only if \(\beta \in \mathbb{Z}_{\leq -1}\).

The simple module \(L_K(T_A, \beta)\) is the unique simple submodule of \(x^{-\beta}K[x, x^{-1}] dT_A\) generated by \(x^{-\beta_{\text{empty}}} dT_A\). Hence

\[
L_K(T_A, \beta) = x^{-\beta} dT_A, D \simeq D/(x\partial - \beta)D \quad \text{for } \beta \notin \mathbb{Z},
\]

\[
L_K(T_A, \beta) = L_K(T_A, -1) = x dT_A, D \simeq D/\partial D \quad \text{for } \beta \in \mathbb{Z}.
\]

A left \(D(K^n)\)-module \(M\) is said to have irreducible monodromy representation if \(D(K^n)(x) \otimes_{D(K^n)} M\) is an irreducible left \(D(K^n)(x)\)-module, where \(D(K^n)(x) = K(x) \otimes_{K[x]} D(K^n)\) with \(K(x) = K(x_1, \ldots, x_n)\) being the field of rational functions (cf. [Wal07]). We naturally have the following proposition.

Proposition 6.8. Let \(M\) be an irreducible left \(D(K^n)\)-module. Suppose that \(D(K^n)(x) \otimes_{D(K^n)} M \neq 0\). Then \(M\) has irreducible monodromy representation.

Proof. We can write \(M = D(K^n)/I\) with \(I\) a maximal left ideal of \(D(K^n)\). Then

\[
D(K^n)(x) \otimes_{D(K^n)} M = D(K^n)(x)/D(K^n)(x)I.
\]

Let \(J\) be a left ideal of \(D(K^n)(x)\) containing \(D(K^n)(x)I\). Since \(J \cap D(K^n)\) is a left ideal of \(D(K^n)\) containing \(I\), we have \(J \cap D(K^n) = D(K^n)\) or \(I\). If \(J \cap D(K^n) = D(K^n)\), then \(1 \in J\) and \(D(K^n)(x) = D(K^n)(x)I\).

Suppose that \(J \cap D(K^n) = I\). Let \(P \in J\). Then there exists a non-zero polynomial \(f \in K[x]\) such that \(fP \in J \cap D(K^n) = I\). Hence \(P \in D(K^n)(x)I\), and we have \(J = D(K^n)(x)I\).

□
7. Pull-back of $L_{K^n}(T_T, \lambda)$

Let $i^\sharp$ denote the functor from $O_{K^n}$ to $O_{X_A}$ defined by

$$i^\sharp(N) := \text{Hom}_{D(K^n)}(D(K^n, X_A), N) = \{x \in N : x.I_A = 0\}. \quad (22)$$

The following adjointness property holds:

$$\text{Hom}_{D(K^n)}\left(\int_{X_A \to K^n}^\theta M, N\right) \simeq \text{Hom}_{D(X_A)}(M, i^\sharp(N)). \quad (23)$$

In this section, we compute the pull-back of $L_{K^n}(T_T, \lambda)$ by $i^\sharp$. As a consequence, we determine the irreducible quotients of $M_{K^n}(\beta)$.

Before considering $i^\sharp(L_{K^n}(T_A, \lambda))$, we present two preparatory lemmas.

**Lemma 7.1.** Let $c \in ZC(\Omega(a))$, where $\Omega(a)$ is as defined in (5) and $ZC$ stands for the Zariski closure in $K^d$. Then there exist $b \in \Omega(a)$ and a face $\tau$ such that $b + N(A \cap \tau) \subseteq \Omega(a)$ and $c \in b + K(A \cap \tau)$.

**Proof.** This follows from [ST04, Proposition 5.1]. \qed

**Lemma 7.2.** Suppose that

$$\mathbb{I}(\Omega(a)) \subseteq (s - c)K[s].$$

Then

$$\{f \in \mathbb{I}(\tilde{\Omega}(a)) : f(\gamma) = f(\gamma') \text{ if } A\gamma = A\gamma' = c\} \subseteq (A\theta - c)K[\theta], \quad (24)$$

where $\tilde{\Omega}(a)$ is as defined in (13).

**Proof.** Since $\mathbb{I}(\Omega(a)) \subseteq (s - c)K[s]$, we have $c \in ZC(\Omega(a))$. By Lemma 7.1 there exist $b \in \Omega(a)$ and a face $\tau$ such that $b + N(A \cap \tau) \subseteq \Omega(a)$ and $c \in b + K(A \cap \tau)$. Take $u \in \mathbb{N}^d$ such that $Au = b$. Then there exists $\gamma' \in u + K^n$ such that $A\gamma' = c$. Observe that $\gamma' \in ZC(\tilde{\Omega}(a))$, since $u + N^n \subseteq \tilde{\Omega}(a)$.

Let $f(\theta)$ belong to the set on the left-hand side of (24). If $A\gamma = c = A\gamma'$, then we have $f(\gamma) = f(\gamma') = 0$ since $\gamma' \in ZC(\tilde{\Omega}(a))$. Hence $f \in (A\theta - c)K[\theta].$ \qed

**Theorem 7.3.**

$$i^\sharp(L_{K^n}(T_A, \beta)) = L_{X_A}^{\emptyset}(\beta^{\emptyset}).$$

**Proof.** Fix $e := \beta^{\emptyset}$. By Theorem 6.4,

$$L_{K^n}(T_A, \beta) = \bigoplus_{a \in \mathbb{Z}^d} t^{-e+a} dT_A \otimes_{K[s]} (\mathbb{I}(\Omega(a))/\mathbb{I}(\tilde{\Omega}(a)) \cap (s - e + a)K[\theta]) \subseteq \bigoplus_{a \in \mathbb{Z}^d} t^{-e+a} dT_A \otimes_{K[s]} K[\theta]/(s - e + a)K[\theta].$$

First, we claim that

$$i^\sharp(L_{K^n}(T_A, \beta)) \subseteq \bigoplus_{a \in \mathbb{Z}^d} Kt^{-e+a} dT_A. \quad (25)$$

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Let $f(\theta) \in K[\theta]$, and fix $\gamma \in K^n$ with $A\gamma = e - a$. Then

\[ t^{-e+a} dT_A \otimes f(\theta).I_A = 0 \]
\[ \iff t^{-e+a} dT_A \otimes f(\theta).(x^u - x^v) = 0 \quad \text{for all } u \text{ and } v \text{ with } Au = Av \]
\[ \iff t^{-e+a+Au} dT_A \otimes (f(\theta + u) - f(\theta + v)) = 0 \quad \text{for all } u \text{ and } v \text{ with } Au = Av \]
\[ \iff f(\theta + u) - f(\theta + v) \in (A\theta - e + a + Au)K[\theta] \quad \text{for all } u \text{ and } v \text{ with } Au = Av \]
\[ \iff f(\theta + u - v) - f(\theta) \in (A\theta - e + a)K[\theta] \quad \text{for all } u \text{ and } v \text{ with } Au = Av. \]

Hence, by Lemma 6.5, $t^{-e+a} dT_A \otimes f(\theta) \in i^\natural(L_{K^n}(T_A, \beta))$ implies

\[ f(\theta) \in f(\gamma) + (A\theta - e + a)K[\theta]. \]

Therefore $t^{-e+a} dT_A \otimes f(\theta) = f(\gamma)t^{-e+a} dT_A \otimes 1$ and the claim (25) is proved.

Recall that

\[ e - a \not\subseteq e \iff e - a \not\subseteq e \iff \mathbb{I}(\Omega(a)) \subseteq (s - e + a)K[s]. \quad (26) \]

Suppose $e - a \not\subseteq e$. Then there exists $f(s) \in \mathbb{I}(\Omega(a))$ such that $f(s) \not\subseteq (s - e + a)K[s]$. Hence, for $\gamma \in K^n$ with $A\gamma = e - a$, we have $f(\gamma) = f(A\gamma) \neq 0$. Then

\[ i^\natural(L_{K^n}(T_A, \beta)) \ni t^{-e+a} dT_A \otimes f(A\theta) = f(\gamma)t^{-e+a} dT_A \otimes 1 \neq 0, \]

and thus the weight $-e + a$ appears in $i^\natural(L_{K^n}(T_A, \beta))$.

Next, suppose $e - a \not\supseteq e$. Then $\mathbb{I}(\Omega(a)) \subseteq (s - e + a)K[s]$. By the proof of (25), if $t^{-e+a} dT_A \otimes f(\theta) \in i^\natural(L_{K^n}(T_A, \beta))$, then $f(\gamma) = f(\gamma')$ for any $\gamma, \gamma' \in K^n$ with $A\gamma = A\gamma' = e - a$. Hence, by (7), it suffices to prove the inclusion

\[ \{f \in \mathbb{I}(\Omega(a)) : f(\gamma) = f(\gamma') \text{ if } A\gamma = A\gamma' = e - a\} \subseteq (A\theta - e + a)K[\theta], \]

assuming that $\mathbb{I}(\Omega(a)) \subseteq (s - e + a)K[s]$. We finish the proof by invoking Lemma 7.2. \qed

Given faces $\tau$ and $\tau'$ of $\mathbb{R}_{\geq 0}^A$, $\lambda \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$ and $\lambda' \in K(A \cap \tau')/\mathbb{Z}(A \cap \tau')$, set

\[ (\tau', \lambda') \prec (\tau, \lambda) \defeq \tau' \prec \tau \quad \text{and} \quad \lambda - \lambda' \in \mathbb{Z}(A \cap \tau). \quad (27) \]

**Theorem 7.4.** Let $\lambda \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$. Then

\[ \dim_K i^\natural(L_{K^n}(T_\tau, \lambda))_{-c} = \begin{cases} 1 & \text{if } c \in C_{K^n}(\tau, \lambda), \\ 0 & \text{otherwise}, \end{cases} \]

where

\[ C_{K^n}(\tau, \lambda) = \left\{ c \in K^d : E_{\tau}(c) \ni \lambda \text{ and } E_{\tau'}(c) \not\ni \lambda' \right\}. \quad (28) \]

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Proof. By (15),

\[ L_{K^n}(T_r, \lambda) \simeq L_{K^r}(T_r, \lambda) \otimes \left( \bigoplus_{b \in \mathbb{N}^r} Kx^{-\tilde{b}} d(K^X)^{\tau^c} \right) \]

By the definition of \( \tilde{i}^\dagger \),

\[ \tilde{i}^\dagger(L_{K^n}(T_r, \lambda)) = \{ f \in L_{K^n}(T_r, \lambda) : f.I_A = 0 \} \]
\[ \subseteq \{ f \in L_{K^n}(T_r, \lambda) : f.(x^u - x^v) = 0 \text{ for } u, v \in \mathbb{N}^r \text{ with } Au = Av \} \]

Hence, by Theorem 7.3,

\[ \tilde{i}^\dagger(L_{K^n}(T_r, \lambda)) \subseteq \left( \bigoplus_{a \sim \lambda \text{empty}} Kt^{-a} dT_r \right) \otimes \left( \bigoplus_{b \in \mathbb{N}^r} Kx^{-\tilde{b}} d(K^X)^{\tau^c} \right). \]

Note that for \( a \in K(A \cap \tau) \), \( a \sim \lambda \text{empty} \) if and only if \( a \in C_{K^n}(\tau, \lambda) \cap K(A \cap \tau) =: C_K(\tau, \lambda) \).

Let

\[ f = \sum_{(a,b) \in C} f_{a,b} t^{-a} dT_r \otimes x^{-\tilde{b}} d(K^X)^{\tau^c}, \]

where \( C = C_K(\tau, \lambda) \times \mathbb{N}^{\tau^c} \). Note that the set of \((a, b) \in C\) with a fixed \( a + \tilde{b} \) is finite, since \( a \in \lambda + \mathbb{Z}(A \cap \tau), b \in \mathbb{N}^{\tau^c} \) and \( \mathbb{R}_{\geq 0}(a \setminus \tau) \cap \mathbb{R} = \{0\} \).

Let \( u = u_r + u_{r^c} \) and \( v = v_r + v_{r^c} \), with \( u_r, v_r \in \mathbb{N}^r \) and \( u_{r^c}, v_{r^c} \in \mathbb{N}^{r^c} \), satisfy \( Au = Av \).

We claim that for \( f \) as in (29),

\[ f \in \tilde{i}^\dagger(L_{K^n}(T_r, \lambda)) \iff \begin{cases} 
(i) & f_{a + Au_r, \tilde{b} + u_{r^c}} = f_{a + Av_r, \tilde{b} + v_{r^c}} \\
& \text{for } (a, b), (a + Au_r, \tilde{b} + u_{r^c}), (a + Av_r, \tilde{b} + v_{r^c}) \in C, \\
(ii) & f_{a + Au_r, \tilde{b} + u_{r^c}} = 0 \\
& \text{for } (a, b), (a + Au_r, \tilde{b} + u_{r^c}) \in C, (a + Av_r, \tilde{b} + v_{r^c}) \notin C. 
\end{cases} \]

We have

\[ f.(x^u - x^v) = \sum_{(a,b) \in C} f_{a,b} t^{-a + Au_r} dT_r \otimes x^{-\tilde{b} + u_{r^c}} d(K^X)^{\tau^c} \]
\[ - \sum_{(a,b) \in C} f_{a,b} t^{-a + Av_r} dT_r \otimes x^{-\tilde{b} + v_{r^c}} d(K^X)^{\tau^c} \]
\[ = \sum_{(a,b), (a - Au_r, \tilde{b} - u_{r^c}) \in C} f_{a,b} t^{-a + Au_r} dT_r \otimes x^{-\tilde{b} + u_{r^c}} d(K^X)^{\tau^c} \]
\[ - \sum_{(a,b), (a - Av_r, \tilde{b} - v_{r^c}) \in C} f_{a,b} t^{-a + Av_r} dT_r \otimes x^{-\tilde{b} + v_{r^c}} d(K^X)^{\tau^c} \]
\[ = \sum_{(a,b), (a + Au_r, \tilde{b} + u_{r^c}) \in C} f_{a,b} t^{-a} dT_r \otimes x^{-\tilde{b}} d(K^X)^{\tau^c} \]
\[ - \sum_{(a,b), (a + Av_r, \tilde{b} + v_{r^c}) \in C} f_{a,b} t^{-a} dT_r \otimes x^{-\tilde{b}} d(K^X)^{\tau^c} \]

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and hence the equivalence between (32) and (33). We claim that then

$$f_{a + Ab} = f_{a' + Ab'}.$$  \hspace{1cm} (31)

Indeed, let $w \in K^r$ and $\tilde{a}, \tilde{a}' \in \mathbb{Z}^r$ satisfy $\lambda = Aw, a = A(\tilde{w} + \tilde{a})$ and $a' = A(\tilde{w} + \tilde{a}')$. Put $u_\tau := (\tilde{a} - \tilde{a}')_+ \in \mathbb{N}^r$, $v_\tau := (\tilde{a} - \tilde{a}')_- \in \mathbb{N}^r$, $u_{\tau c} := (\tilde{b} - \tilde{b}')_+ \in \mathbb{N}^{r c}$ and $v_{\tau c} := (\tilde{b} - \tilde{b}')_- \in \mathbb{N}^{r c}$. Here, $(\tilde{a} - \tilde{a}')_+$ is the non-negative part of $\tilde{a} - \tilde{a}'$, and $(\tilde{a} - \tilde{a}')_-$ is the negative of the non-positive part of $\tilde{a} - \tilde{a}'$. Then $A(u_\tau + u_{\tau c}) = A(v_\tau + v_{\tau c})$ and $b - u_{\tau c} = b' - v_{\tau c} \in \mathbb{N}^{r c}$. Furthermore, $a - Au_\tau = a' - Av_\tau \in C_K^u(\tau, \lambda)$, since $a \sim a' \sim \lambda^{\text{empty}}$ is the minimal class (see [Sai01, Proposition 2.2(5)]). Hence, from (30)(i) we obtain (31).

We can rewrite (30)(ii) as

$$f_{a, b} = 0$$  \hspace{1cm} (32)

for $(a, b), (a - Au_\tau, b - u_{\tau c}) \in C$ and $(a - Au_\tau + Av_\tau, b - u_{\tau c} + v_{\tau c}) \notin C$.

We prove next that (32) is equivalent to the following condition:

if there exists $(\tau', \lambda') \prec (\tau, \lambda)$ such that $E_{\tau'}(a + Ab) \ni \lambda'$, then $f_{a, b} = 0$.  \hspace{1cm} (33)

For this purpose, when $(a, b) \in C$ we prove the equivalence

there exists $(\tau', \lambda') \prec (\tau, \lambda)$ such that $E_{\tau'}(a + Ab) \ni \lambda'$  \hspace{1cm} (34)

$\iff$ there exist $u_\tau, v_\tau \in \mathbb{N}^r$ and $u_{\tau c}, v_{\tau c} \in \mathbb{N}^{r c}$ such that $A(u_\tau + u_{\tau c}) = A(v_\tau + v_{\tau c}), (a - Au_\tau, b - u_{\tau c}) \in C$ and $(a - Au_\tau + Av_\tau, b - u_{\tau c} + v_{\tau c}) \notin C$.  \hspace{1cm} (35)

First, suppose that (35) holds. Then $b - u_{\tau c} \in \mathbb{N}^{r c}$, and there exists $(\tau', \lambda') \prec (\tau, \lambda)$ such that $E_{\tau'}(a - Au_\tau + Av_\tau) \ni \lambda'$. It follows from $b - u_{\tau c} \in \mathbb{N}^{r c}$ and $A(u_\tau + u_{\tau c}) = A(v_\tau + v_{\tau c})$ that $Av_\tau - Au_\tau \in A(b - N^{r c})$. Hence $E_{\tau'}(a + Ab) \ni \lambda'$ (cf. [Sai01, Proposition 2.2(5)]).

Conversely, suppose that (34) holds. Then $a + Ab - \lambda' \in NA + Z(A \cap \tau')$. Let $w' \in K^{\tau'}$, $\tilde{a} \in \mathbb{Z}^{\tau'}$, $\tilde{b}' \in \mathbb{N}^{\tau'}$ and $\tilde{a}' \in \mathbb{N}^{\tau' r} \times \mathbb{Z}^{r c}$ satisfy $\lambda' = Aw'$, $a = A(w' + \tilde{a})$ and $a + Ab - \lambda' = Ab' + A\tilde{a}'$. As before, put $u_\tau := (\tilde{a} - \tilde{a}')_+ \in \mathbb{N}^r$, $v_\tau := (\tilde{a} - \tilde{a}')_- \in \mathbb{N}^r$, $u_{\tau c} := (\tilde{b} - \tilde{b}')_+ \in \mathbb{N}^{r c}$ and $v_{\tau c} := (\tilde{b} - \tilde{b}')_- \in \mathbb{N}^{r c}$. Then $(a - Au_\tau, b - u_{\tau c}) \in C$. Furthermore, $a - Au_\tau + Av_\tau = a - A(\tilde{a} - \tilde{a}') = \lambda' + A\tilde{a}' \in \lambda' + NA + Z(A \cap \tau')$. Hence $\lambda' \in E_{\tau'}(a - Au_\tau + Av_\tau)$, and thus $(a - Au_\tau + Av_\tau, b - u_{\tau c} + v_{\tau c}) \notin C$. Finally, $A(u_\tau + u_{\tau c}) - A(v_\tau + v_{\tau c}) = A(\tilde{a} - \tilde{a}') + A(\tilde{b} - \tilde{b}') = a - \lambda' - A\tilde{a}' + A(b - \tilde{b}') = 0$. Therefore we have established the equivalence between (34) and (35) and hence the equivalence between (32) and (33).
In summary, we have shown that
\[ i^\natural(L_{K^n}(T, \lambda)) = \bigoplus_{c \in C_K^n(\tau, \lambda)} K \sum_{(a, \tilde{b}), c = a + A\tilde{b}} t^{-a} dT \otimes x^{-\tilde{b}} d(K^c)^{\tau c}, \]
so the proof of Theorem 7.4 is complete.

**Corollary 7.5.**
\[ \dim_K \text{Hom}_{D(R)}(M_{K^n}(\beta), L_{K^n}(T, \lambda)) = \begin{cases} 1 & \text{if } \beta \in C_K^n(\tau, \lambda), \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** We have
\[
\dim_K \text{Hom}_{D(K^n)}(M_{K^n}(\beta), L_{K^n}(T, \lambda)) \\
= \dim_K \text{Hom}_{D(K^n)}(\int_{X_A - K^n} M_{X_A}(\beta), L_{K^n}(T, \lambda)) \\
= \dim_K \text{Hom}_{D(X_A)}(M_{X_A}(\beta), i^\natural(L_{K^n}(T, \lambda)) - \beta).
\]
The first equality comes from (12) and the second from the adjointness (23). The third follows from [MV98, Proposition 3.1.7] (see also [Sai07, Proposition 3.6]). Theorem 7.4 then finishes the proof of this corollary.

For \( \beta \in K^d \), set
\[ E(\beta) := \{(\tau, \lambda) : \tau \text{ a face of } R_{\geq 0}A, \ \lambda \in E_\tau(\beta)\}. \]
Then Corollary 7.5 can be rephrased as follows.

**Corollary 7.6.**
\[ \dim_K \text{Hom}_{D(R)}(M_{K^n}(\beta), L_{K^n}(T, \lambda)) = \begin{cases} 1 & \text{if } (\tau, \lambda) \text{ is minimal in } E(\beta), \\ 0 & \text{otherwise.} \end{cases} \]

*Here the minimality is with respect to (27).*

**Example 7.7.** Let
\[ A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} = [a_1, a_2, a_3]. \]
Then the cone \( R_{\geq 0}A \) has exactly four faces: \( R_{\geq 0}A = R_{\geq 0}^2, \sigma_1 := R_{\geq 0}a_1, \sigma_3 := R_{\geq 0}a_3 \) and \( \{0\} \). The semigroup \( NA \) is shown in Figure 1.

**Figure 1.** The semigroup \( NA \).

Let \( \tau \) be a face of \( R_{\geq 0}A \). Then
\[ |Z^2 \cap K(A \cap \tau)/Z(A \cap \tau)| = \begin{cases} 1 & \text{if } \tau \neq \sigma_3, \\ 2 & \text{if } \tau = \sigma_3. \end{cases} \]
Hence the category $\mathcal{O}_{K^3}$ has exactly five simple objects with weights in $\mathbb{Z}^2$, namely $L_{K^3}(T_A, 0)$, $L_{K^3}(T_{\sigma_1}, 0)$, $L_{K^3}(T_{\sigma_3}, 0)$, $L_{K^3}(T_{\sigma_3}, (1, 0)^T)$ and $L_{K^3}(T_{\{0\}}, 0)$. For each of these, we write down the weight set $(C_{K^n}(\tau, \lambda)$ in Theorem 7.4) of the pull-back by $i^\natural$.

(i) $i^\natural(L_{K^3}(T_A, 0))$: the weights in $C_{K^3}(\mathbb{R}_{\geq 0}A, 0)$ are $\beta \in \mathbb{Z}^2$ with $E_{\sigma_1}(\beta) = \emptyset$ and $E_{\sigma_3}(\beta) = \emptyset$, shown in Figure 2.

(ii) $i^\natural(L_{K^3}(T_{\sigma_1}, 0))$: the weights in $C_{K^3}(\sigma_1, 0)$ are $\beta \in \mathbb{Z}^2$ with $E_{\sigma_1}(\beta) = \{0\}$ and $E_{\{0\}}(\beta) = \emptyset$, shown in Figure 3.

(iii) $i^\natural(L_{K^3}(T_{\sigma_3}, 0))$: the weights in $C_{K^3}(\sigma_3, 0)$ are $\beta \in \mathbb{Z}^2$ with $E_{\sigma_3}(\beta) \ni 0$ and $E_{\{0\}}(\beta) = \emptyset$, shown in Figure 4.

(iv) $i^\natural(L_{K^3}(T_{\sigma_3}, (1, 0)^T))$: the weights in $C_{K^3}(\sigma_3, (1, 0)^T)$ are $\beta \in \mathbb{Z}^2$ with $E_{\sigma_3}(\beta) \ni (1, 0)^T$, shown in Figure 5.

(v) $i^\natural(L_{K^3}(T_{\{0\}}, 0))$: the weights in $C_{K^3}(\{0\}, 0)$ are $\beta \in \mathbb{Z}^2$ with $E_{\{0\}}(\beta) = \{0\}$; hence the weight set is NA, shown in Figure 1.

Let $\beta \in \mathbb{Z}^2$. By Corollary 7.5, the irreducible quotients of $M_{K^3}(\beta)$ are precisely the above $L_{K^3}(T_{\tau}, \lambda)$ such that $\beta$ appears in the weight set of $i^\natural(L_{K^3}(T_{\tau}, \lambda))$. 
Recall that $M_{K^3}(\beta) \simeq M_{K^3}(\beta')$ if and only if $\beta \sim \beta'$ (see [Sai01, Theorem 2.1]). There are eight equivalence classes in $\{M_{K^3}(\beta) : \beta \in \mathbb{Z}^2\}$. The following table lists the irreducible quotients for each equivalence class.

<table>
<thead>
<tr>
<th>$M_{K^3}(\beta)$</th>
<th>Irreducible quotients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{K^3}((0,1)^T)$</td>
<td>$L_{K^3}(T_{{0}}, 0)$, $L_{K^3}(T_{{0}}, (1,0)^T)$</td>
</tr>
<tr>
<td>$M_{K^3}((-1,1)^T)$</td>
<td>$L_{K^3}(T_{{0}}, 0)$, $L_{K^3}(T_{{0}}, (1,0)^T)$</td>
</tr>
<tr>
<td>$M_{K^3}((0,0)^T)$</td>
<td>$L_{K^3}(T_{{0}}, 0)$</td>
</tr>
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</tr>
<tr>
<td>$M_{K^3}((-1,0)^T)$</td>
<td>$L_{K^3}(T_{{0}}, 0)$</td>
</tr>
<tr>
<td>$M_{K^3}((-2,0)^T)$</td>
<td>$L_{K^3}(T_{{0}}, 0)$</td>
</tr>
<tr>
<td>$M_{K^3}((0,-1)^T)$</td>
<td>$L_{K^3}(T_{{0}}, 0)$</td>
</tr>
<tr>
<td>$M_{K^3}((-1,-1)^T)$</td>
<td>$L_{K^3}(T_{A}, 0)$</td>
</tr>
</tbody>
</table>

8. The irreducibility of $M_{K^n}(\beta)$

If $\beta = \beta^{\emptyset}$, then, by Corollary 7.6, there exists a surjective homomorphism

$$M_{K^n}(\beta) \twoheadrightarrow L_{K^n}(T_A, \beta).$$

In this section, we analyze the kernel of (38) and prove that $M_{K^n}(\beta)$ is irreducible if and only if $\beta$ is non-resonant.

Given a facet (maximal proper face) $\sigma$ of $\mathbb{R}_{>0}A$, we denote by $F_\sigma$ the primitive integral support function of $\sigma$; that is, $F_\sigma$ is the uniquely determined linear form on $\mathbb{R}^d$ satisfying:

(i) $F_\sigma(\mathbb{R}_{>0}A) \geq 0$;

(ii) $F_\sigma(\sigma) = 0$;

(iii) $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

Then, by [Sai01, Proposition 2.2] and Remark 6.3, we know that $\beta = \beta^{\emptyset}$ if and only if $F_\sigma(\beta) \notin F_\sigma(NA)$ for all facets $\sigma$ of $\mathbb{R}_{>0}A$.

Let $\beta = \beta^{\emptyset}$, and let

$$v_{-\beta} := t^{-\beta} dT_A \otimes 1 \in L_{K^n}(T_A, \beta)_{-\beta}.$$

Then, by Theorem 6.4,

$$\text{Ann}_{D(K^n)}(v_{-\beta}) = I_A D(K^n) + D(K^n) \cap \langle A\theta - \beta \rangle D((K^\times)^n).$$

Let

$$N := \text{Ann}_{D(K^n)}(v_{-\beta})/(I_A D(K^n) + \langle A\theta - \beta \rangle D(K^n)).$$

Then $N$ is the kernel of (38). By (11) and Proposition 5.2, for $a \in \mathbb{Z}^d$ we have

$$N_{-\beta - a} = t^{-a} (\mathbb{I}(\Omega(-a)) \cap \langle A\theta - \beta - a \rangle)/t^{-a} (\mathbb{I}(\Omega(-a)) \langle A\theta - \beta - a \rangle).$$

Since $\{u \in \mathbb{N}^n : Au \in a + NA\}$ is $\mathbb{N}^n$-stable, there exists a finite set $\{(u^{(j)}, I_j) : j \in J\}$ of pairs made up of a $u^{(j)} \in \mathbb{N}^n$ and a subset $I_j$ of $\{1, \ldots, n\}$ (the set of so-called standard pairs of $\{u \in \mathbb{N}^n : Au \in a + NA\}$; see, e.g., [SST00, §3.2]) such that:
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the ith coordinate of \( u^{(j)} \) is 0 for each \( i \in I_j \);

for all \( i \not\in I_j \), \( (u^{(j)}+N^{(j)}) \cap \{u \in \mathbb{N}^n: Au \in a+NA\} \neq \emptyset \);

\( \overline{\Omega}(-a) = \mathbb{N}^n \setminus \{u \in \mathbb{N}^n: Au \in a+NA\} = \bigcup_{j \in J} (u^{(j)}+N^{(j)}) \).

**Lemma 8.1.** Let \( a \in \mathbb{Z}^d \), and let \( \{(u^{(j)}, I_j): j \in J\} \) be the set of standard pairs of \( \{u \in \mathbb{N}^n: Au \in a+NA\} \). Then for each \( j \in J \) there exists a facet \( \tau^{(j)} \) of \( \mathbb{R}_{\geq 0} A \) such that \( I_j = \{k \in \{1, \ldots, n\}: a_k \in \tau^{(j)}\} \), and either \( \tau^{(j)} \) is a facet with \( F_{\tau^{(j)}}(u^{(j)}) \notin F_{\tau^{(j)}}(a+NA) \) or \( F_\sigma(Au^{(j)}) \notin F_\sigma(a+NA) \) for all facets \( \sigma \geq \tau^{(j)} \).

**Proof.** Put \( S_c = \{d \in \mathbb{Z}^d : F_\sigma(d) \in F_\sigma(NA) \text{ for all } \sigma \}. \) Then there exist finitely many pairs \( (b_i, \tau_i) \) of \( b_i \in S_c \) and a face \( \tau_i \) such that \( S_c \setminus NA = \bigcup_i (b_i + Z(A \cap \tau_i)) \cap S_c \) (see [ST04, proof of Proposition 5.1]). Then

\[
\Omega(-a) = \left( \bigcup_{\text{facets } \sigma} \bigcup_{m \in F_\sigma(NA) \setminus F_\sigma(a+NA)} F_{\sigma}^{-1}(m) \cap NA \right) \\
\cup \bigcup_{b_i+a \in NA+Z(A \cap \tau_i)} (b_i + a + Z(A \cap \tau_i)) \cap NA.
\]

Since \( \overline{\Omega}(-a) = \{u \in \mathbb{N}^n: Au \in \Omega(-a)\} \) by definition, the assertion follows.

**Lemma 8.2.** Let \( \beta = \beta^\text{empty} \) and \( a \in \mathbb{Z}^d \).

(i) If \( \beta + a \sim \beta \), then \( N_{-\beta-a} = \{0\} \).

(ii) Suppose that there exists a facet \( \sigma \) such that \( F_\sigma(\beta + a) \in F_\sigma(NA) \) and \( F_{\sigma'}(\beta + a) \notin F_{\sigma'}(NA) \) for every facet \( \sigma' \not\neq \sigma \). Then \( N_{-\beta-a} \neq \{0\} \).

**Proof.** (i) Suppose that \( \beta + a \sim \beta \). Then \( \mathbb{I}(\Omega(-a)) \not\subset m_{\beta+a} \) or \( \mathbb{I}(\Omega(-a)) + m_{\beta+a} = K[s] \). Hence \( \mathbb{I}(\Omega(-a)) + (A\theta - \beta - a)K[\theta] = K[\theta] \). Therefore \( \mathbb{I}(\Omega(-a)) \cap (A\theta - \beta - a)K[\theta] = (A\theta - \beta - a)K[\Omega(-a)] \), or \( N_{-\beta-a} = \{0\} \) by (40).

(ii) Since \( F_\sigma(\beta + a) \in NA \), there exist \( u \in \mathbb{N}^n \) and \( \gamma \in K^\sigma \) such that \( \beta + a = A(u + \gamma) \). Then, for any \( v \in \mathbb{N}^n \), \( A(u + v) \in NA \setminus (a + NA) = \Omega(-a) \) since \( F_{\sigma}(A(u + v)) = F_{\sigma}(\beta + a - A\gamma + Av) = F_{\sigma}(\beta + a) \notin F_{\sigma}(a+NA) \). Hence \( u + \mathbb{N}^\sigma \subseteq \Omega(-a) \). Put \( \xi := u + \gamma \). Then \( A\xi = \beta + a \) and \( \xi + K^\sigma = u + K^\sigma \subseteq ZC(\Omega(-a)) \). By Lemma 8.1 we have

\[
ZC(\Omega(-a)) = \bigcup_{j \in J} (u^{(j)} + K^{\tau^{(j)}}),
\]

and we see that, by the assumption, \( \xi + K^\sigma \) is the unique irreducible component of \( ZC(\Omega(-a)) \) containing \( \xi \). Hence, by localizing at \( \xi \), to prove the assertion it is enough to show that \( \mathbb{I}(\xi + K^\sigma) \cap (A\theta - (\beta + a)) \neq \mathbb{I}(\xi + K^\sigma).\langle A\theta - (\beta + a) \rangle \) (see (40)) or, upon translating by \( \xi \), that \( \mathbb{I}(K^\sigma) \cap (A\theta) \neq \mathbb{I}(K^\sigma).\langle A\theta \rangle \). Since it is clearly true that

\[
F_\sigma(A\theta) = \sum_{j=1}^n F_\sigma(a_j) \theta_j \in \mathbb{I}(K^\sigma) \cap \langle A\theta \rangle \setminus \mathbb{I}(K^\sigma).\langle A\theta \rangle,
\]

we have finished the proof. \( \square \)
IRREDUCIBLE QUOTIENTS OF $A$-HYPERGEOMETRIC SYSTEMS

Theorem 8.3. $M_K^n(\beta)$ is irreducible if and only if $\beta$ is non-resonant, i.e. $F_\sigma(\beta) \notin \mathbb{Z}$ for all facets $\sigma$ of $R_{\geq 0}A$.

Proof. Suppose that $\beta$ is non-resonant. Then $\beta + a \sim \beta$ for all $a \in \mathbb{Z}^d$. Hence, by Lemma 8.2(i), $M_K^n(\beta) \simeq L_{K^n}(T_A, \beta)$.

Suppose that $\beta$ is resonant and that $F_\sigma(\beta) \in \mathbb{Z}$. If $\beta = \beta_{\text{empty}}$, then, by Corollary 7.6, there exists a surjective homomorphism

$$M_K^n(\beta) \rightarrow L_{K^n}(T_A, \beta).$$

(41)

Since $\sigma$ is a facet of $R_{\geq 0}A$, there exists $b \in \mathbb{Z}^d$ such that $F_\sigma(b) < 0$ while $F_\sigma'(b) > 0$ for every facet $\sigma' \neq \sigma$. Hence, for a sufficiently large $n \in \mathbb{N}$, $F_\sigma(\beta - nb) \in F_\sigma(\mathbb{N}A)$ and $F_\sigma'(\beta - nb) \notin F_\sigma'(\mathbb{N}A)$ for every facet $\sigma' \neq \sigma$. Thus the homomorphism (41) has a non-trivial kernel by Lemma 8.2(ii).

Let $\beta \neq \beta_{\text{empty}}$. Then there exists a minimal $(\tau, \lambda) \in E(\beta)$ (see (37)) with $\tau \neq R_{\geq 0}A$. Hence, by Corollary 7.6, $L_{K^n}(T_\tau, \lambda)$ is a quotient of $M_K^n(\beta)$. Since the support of $L_{K^n}(T_\tau, \lambda)$ is strictly contained in the support of $M_K^n(\beta)$, the kernel of the homomorphism $M_K^n(\beta) \rightarrow L_{K^n}(T_\tau, \lambda)$ is non-trivial.

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