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A New Acceleration Factor Decision Method for ICCG Method Based on Condition Number

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The ICCG method is widely used to solve a sparse symmetric linear system which results from the finite element method. In order to improve the convergence property of the ICCG method, the introduction of the acceleration factor was proposed. The automatic acceleration factor decision method, using the incomplete Cholesky decomposition of a coefficient matrix, has been previously proposed. However, when employing the previously proposed automatic decision method, much more iterations of ICCG method are sometimes necessary, compared with using the optimum acceleration factor, which minimizes the number of ICCG iterations. In this paper, we propose a new acceleration factor decision method. The proposed method takes into account the condition number of the coefficient matrix. It is well known that the condition number represents the quality of the matrix, so the optimum acceleration factor should be decided to minimize the condition number of the coefficient matrix. However, the condition number of the coefficient matrix is not almost available due to requiring a large memory in computing. Therefore, we develop a new method using submatrix, that does not need a large memory. The procedure and demonstration of the proposed method are described in this paper.

Index Terms—Acceleration factor, condition number, finite element method, ICCG method, submatrix.

I. INTRODUCTION

FINITE ELEMENT METHOD (FEM) yields a sparse symmetric linear system. In the finite element analysis, the ICCG method [1], [2] is widely used to solve the sparse symmetric linear system. In order to solve it with short time, an acceleration factor of the ICCG method was proposed [3]. The effectiveness and usefulness of the acceleration factor was demonstrated in [3]-[5] and it has been widely and commonly used. However, it is known that a residual norm may not be converged or an unnecessarily large number of iterations of the ICCG method are required unless the acceleration factor is optimally determined. Therefore, a way to find the optimum acceleration factor, which minimizes the number of iterations, should be developed. Before an acceleration factor decision method by finding the minimum of maximum diagonal entries of the lower triangular matrix was proposed in [4]. The automatic decision method has been commonly employed. When a generated mesh is of good quality, the automatic decision method is effective in reducing the number of the ICCG iterations. However, when using a mesh of bad quality, the automatic decision method does not always yield the optimum acceleration factor and much more iterations of the ICCG method are sometimes required. Therefore, a new acceleration factor decision method has to be developed, and it is expected to be robust to the mesh quality.

The condition number of matrix commonly represents the quality of the matrix [6], and hence the acceleration factor which minimizes the condition number of the pre-conditioned matrix after the incomplete Cholesky (IC) decomposition should be explored as an optimum acceleration factor. It is, however, impossible to compute the condition number of the coefficient matrix resulting from the finite element method due to requiring a large memory to compute the eigenvalue of the coefficient matrix [7]-[9]. In this paper, we propose a new

acceleration factor decision method by minimizing the condition number of the submatrix which is small enough to compute its eigenvalue. The procedure of the proposed method is presented in this paper. In order to confirm the effectiveness of the proposed method, a few examples are also presented.

II. PRE-CONDITIONED SYSTEM

Given the linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad (1)$$

where \mathbf{A} , \mathbf{x} , and \mathbf{b} are the coefficient matrix, the solution vector, and the right hand vector, respectively. Using the incomplete Cholesky (IC) decomposition of \mathbf{A} , $\mathbf{A} \approx \mathbf{L}\mathbf{D}\mathbf{L}^T$, (1) is pre-conditioned as follows:

$$\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}, \quad (2)$$

where

$$\begin{aligned} \tilde{\mathbf{A}} &= \mathbf{M}^{-1}\mathbf{A}\mathbf{M}^{-T}, \quad \tilde{\mathbf{x}} = \mathbf{M}^T\mathbf{x}, \quad \tilde{\mathbf{b}} = \mathbf{M}^{-1}\mathbf{b}, \\ \mathbf{M} &= \mathbf{L}(\gamma)\mathbf{D}(\gamma)^{1/2}, \end{aligned} \quad (3)$$

where γ is the acceleration factor. The pre-conditioned system (2) depends on the acceleration factor γ since \mathbf{L} and \mathbf{D} are obtained using the acceleration factor. The condition number κ of $\tilde{\mathbf{A}}$ is as follows [6]:

$$\kappa = \frac{\mu_{\max}}{\mu_{\min}}, \quad (4)$$

where μ_{\max} and μ_{\min} are the maximum eigenvalue and the nonzero minimum eigenvalue of $\tilde{\mathbf{A}}$, respectively.

Note that the optimum acceleration factor of the ICCG

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method gives the minimum condition number of the pre-conditioned system and the minimum number of iterations.

III. THE PROPOSED AUTOMATIC ACCELERATION FACTOR DECISION METHOD

The condition number κ of the coefficient matrix derived from the FEM commonly represents the quality of the coefficient matrix. It is, however, almost impossible to compute κ due to requiring much large memory to compute the eigenvalue of the coefficient matrix. Then, the proposed method estimates the condition number of the full matrix from ones of the small-size submatrices, which are small enough to compute their eigenvalue. Here, it is not required to highly accurately estimate the condition number of the full coefficient matrix, since the condition number is used to decide the acceleration factor which minimizes the number of the ICCG iterations. Therefore, the condition number of the full coefficient matrix is roughly estimated from the submatrices, and then the acceleration factor is decided by minimizing the condition number of the submatrices.

The procedure of the proposed automatic acceleration factor decision method is shown as follows.

- Step 1) Make the full coefficient matrix derived from FEM.
- Step 2) Subdivide the full coefficient matrix into many submatrices, as shown in Fig. 1.
- Step 3) Calculate the condition number of all the submatrices.
- Step 4) Choose the submatrices whose condition number is more than the threshold κ_{th} . The way to decide the threshold is explained later.
- Step 5) Decide the optimum acceleration factor γ_p for each submatrix chosen on Step 4. Here, γ_p is obtained so as to minimize the condition number of each pre-conditioned system for these submatrices.

Note: For obtaining γ_p , the condition number of each pre-conditioned system is investigated with various acceleration factors.

- Step 6) Employ the maximum of the acceleration factors, $\gamma_{p\text{MAX}}$, as the acceleration factor γ_{new} to the full coefficient matrix.

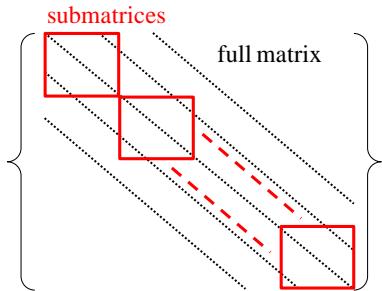


Fig. 1. Subdividing the full sparse symmetric coefficient matrix to many submatrices. The dots represent the nonzero entries of the matrix. The diagonal entries of the submatrix correspond to the part of the diagonal entries of the full matrix.

The optimum acceleration factor γ_{opt} should be larger as the convergence property of the ICCG method is worse. That is, a matrix with bad condition number requires a large optimum acceleration factor. However, the submatrix with the worst condition number does not always give the largest optimum acceleration factor. Therefore, it is necessary to decide the threshold in order to choose the submatrices with bad condition number. That is the reason why the threshold of the condition number, κ_{th} , should be decided to choose the submatrices with bad condition number. If the condition number of all the submatrices is computed, it takes a lot of time. A good example is shown later with a test model.

IV. APPLICATIONS OF THE PROPOSED METHOD

A. Simple Model Test

A simple test model consists of a permanent magnet (1.0 T), as shown in Fig. 2. The analysis region is firstly divided into regular hexahedra, and then each hexahedron is regularly divided into six tetrahedra. Table I shows the properties of the mesh. For the verification of the proposed method, the distorted mesh, which makes the convergence of the ICCG iteration bad, is generated by zigzag moving the nodes of the generated model into x -direction, as shown in Fig. 3. The moving distance of nodes is 49.99 mm.

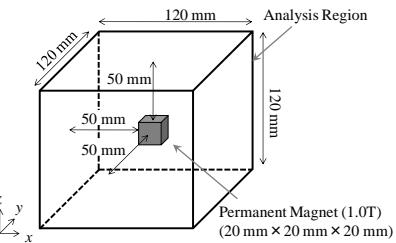


Fig. 2. Test model consisting of a permanent magnet (1.0 T).

TABLE I
MESH PROPERTIES OF TEST MODEL

Nodes	Edges	Elements
2,197	13,428	10,368

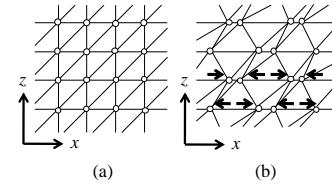


Fig. 3. Generating the distorted mesh in the test model by zigzag moving the nodes of the regular mesh, from (a) to (b).

The proposed method is applied to the simple test model with the distorted mesh. The matrix derived from FEM is divided into 8 submatrices. Fig. 4 shows the condition number κ and the acceleration factor γ_p , which minimizes the condition number of the pre-conditioned system, for all the submatrices. The acceleration factor γ_{new} is 1.09, taking the highest value among the acceleration factors, $\gamma_{p\text{MAX}}$, for all the submatrices.

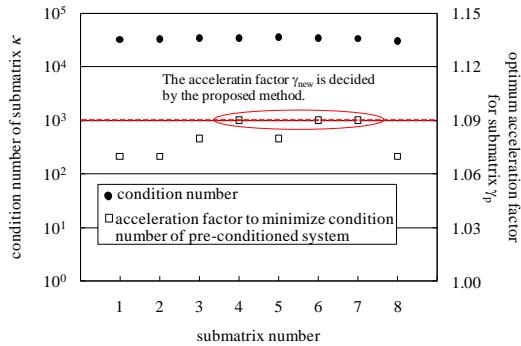


Fig. 4. Condition numbers and acceleration factors decided by minimizing the condition number of the pre-conditioned matrix for each submatrix. The largest value 1.09 of the acceleration factor is adopted as an optimum acceleration factor γ_{new} of the proposed method.

In order to investigate the optimum acceleration factor γ_{opt} , the finite element matrix of the simple test model is solved by the ICCG method with various acceleration factors γ . Fig. 5 shows the number of the ICCG iterations versus the acceleration factor. Table II shows the details of the results.

Fig. 5 shows that γ_{opt} is equal to 1.08 and near to the acceleration factor γ_{new} decided by the proposed method. Using the previous method [4], the acceleration factor γ_{old} is decided to be 1.05 by taking the minimum of the maximum diagonal entry L . The numbers of the ICCG iterations with γ_{opt} and γ_{new} are 801 and 806, respectively, as shown in Table II, and both are almost the same. On the other hand, the number of the iterations with γ_{old} is much larger than that with γ_{opt} and γ_{new} (see Table II). The effectiveness of the newly proposed method is confirmed.

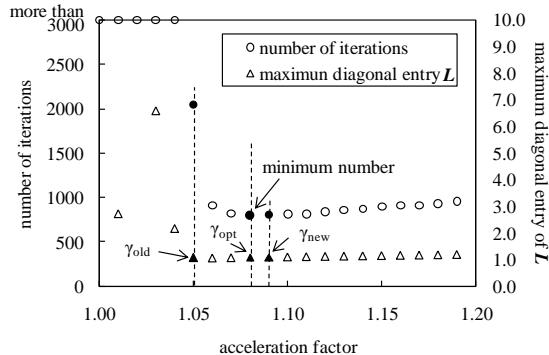


Fig. 5. Number of iterations and maximum diagonal entry versus acceleration factor. The optimum acceleration factor γ_{opt} is 1.08 and the acceleration factor γ_{new} decided by the proposed method is 1.09. The acceleration factor γ_{old} decided by the previous method [4] is 1.05.

TABLE II
ACCELERATION FACTOR AND NUMBER OF ICCG ITERATIONS
ON TEST MODEL

Acceleration factor	Number of iterations
γ_{opt}	801
γ_{new}	806
γ_{old}	2,046

B. SQUID Model

As a practical and complicated model, a SQUID model (thin plate model) [10], as shown in Fig. 6, is solved by FEM. Table III shows the mesh properties of the SQUID model. The mesh includes many ill-shaped elements, such as a flat element. Fig. 7 shows the nonzero distribution of the finite element matrix with $A\phi$ formulation.

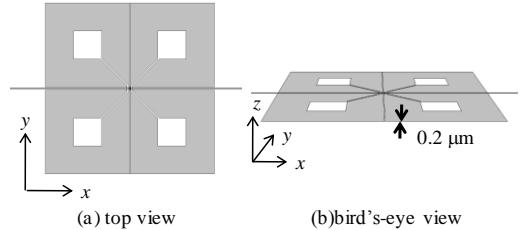


Fig. 6. A directly coupled HTS dc-SQUID model [10] with thin plate (width : 0.2 μm).

TABLE III

Nodes	Edges	Elements
525,813	3,629,542	3,098,852

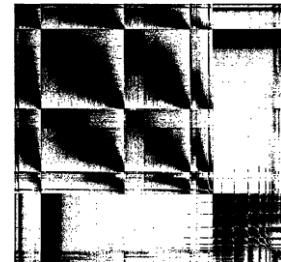


Fig. 7. Nonzero distribution of the finite element matrix derived from SQUID model [10]. The dots represent nonzero entries.

The full matrix is divided to 3,628 submatrices whose dimension is equal to about 1,000. Fig. 8 shows the condition numbers of all submatrices. From Fig. 8, the threshold κ_{th} is decided to be 10^3 for eliminating the worse condition number of the submatrices. Fig. 9 shows the acceleration factors γ_p which minimize the condition number of the pre-conditioned system for the submatrices with the condition number more than threshold κ_{th} . From Fig. 9, the acceleration factor γ_{new} is decided to be 1.37 by the proposed method.

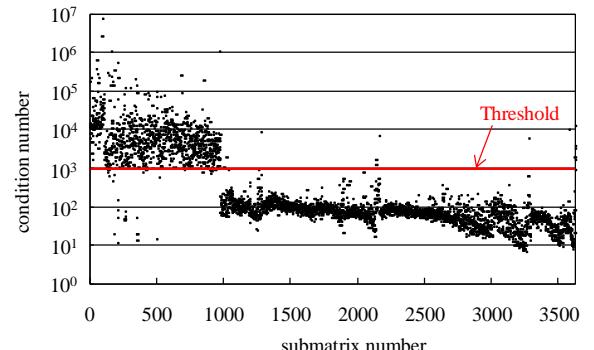


Fig. 8. The condition number distribution of submatrices.

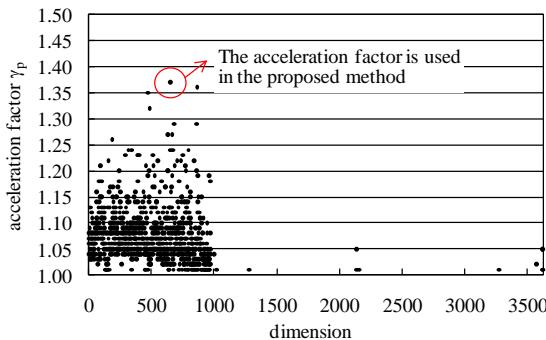


Fig. 9. The optimum acceleration factor of the submatrix.

The finite element equations are solved by ICCG method with various acceleration factors in order to investigate the effectiveness of the proposed method and the optimum acceleration factor, as shown in Fig. 10. The maximum diagonal entry of the lower triangular matrix L , used in the previous proposed method [4], is also investigated in order to compare with the proposed method. From Fig. 10, if the acceleration factor is small, such as 1.10, it is impossible to solve the equations within a proper number of the ICCG iterations. Table IV shows the acceleration factors, the number of the ICCG iterations, and the computation time (Xeon W5590 processor (3.33 GHz)).

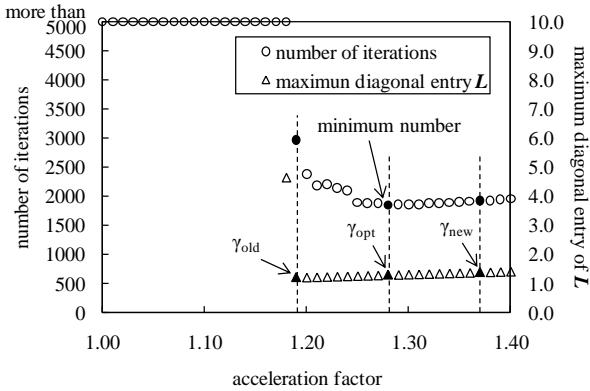


Fig. 10. Number of iterations and maximum diagonal entry versus acceleration factor. The optimum acceleration factor is 1.28 and the acceleration factor decided by the newly proposed method and the previous method [4] is 1.37 and 1.19, respectively. The acceleration factor of the proposed method needs nearly more 100 iterations than that of the optimum but less 1,000 iterations than that of the previous method.

TABLE IV
ACCELERATION FACTOR, NUMBER OF ICCG ITERATIONS
AND COMPUTATION TIME ON SQUID MODEL

Acceleration factor	Number of iterations	Computation time (Xeon W5590)
γ_{opt}	1.28	1,844
γ_{new}	1.37	1,919
γ_{old}	1.19	2,969

Fig. 10 shows that the optimum acceleration factor γ_{opt} is 1.28 and then the number of iterations is 1,844. Using the acceleration factor decided by the proposed method ($\gamma_{\text{new}} =$

1.37), the number of iterations is 1,919 whereas the acceleration factors decided by the previous method [4] ($\gamma_{\text{old}} = 1.19$) needs 2,969 iterations. Consequently, the proposed method decides the acceleration factor γ_{new} almost equal to the optimum acceleration factor γ_{opt} from viewpoint of the number of ICCG iterations. Furthermore, the acceleration factor obtained by the proposed method needed almost the same computation time as the optimal acceleration factor, and took the shorter computation time than the previous method [4].

It is impossible to compute the condition number of the full coefficient matrix due to lack of memory, however, it takes 1,538 s and 11 MB to compute the condition number of all the submatrices. This computation was executed on Xeon W5590 processor (3.33 GHz, 12 GB). It is short enough to obtain the good acceleration factor as a pre-processing.

V. CONCLUSION

In this paper, we proposed a new acceleration factor decision method so as to minimize the condition number of the submatrix which was small enough to calculate the eigenvalue. For the demonstration, the proposed method was applied to the simple test model and the SQUID model with a thin plate. The proposed method can adequately decide the acceleration factor, which needs the number of the ICCG iterations as many as the optimum acceleration factor, with short time.

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