Extensions of the mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space

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Abstract

In this paper we define one-parameter families of Legendrian double fibrations in the products of pseudo-spheres in Lorentz-Minkowski space which are the extensions of four Legendrian double fibrations in the previous research[9]. We show that these are contact diffeomorphic to each other. Moreover, we construct one-paramter families of new extrisic differential geometries on spacelike hypersurfaces in these pseudo-spheres as applications of such extensions of the Legendrian double fibrations.

1 Introduction

If we have a Legendrian double fibration, the projections of a Legendrian submanifold in the total contact manifold are said to be *Legendrian dual* to each other. The Legendrian duality is a generalization of the classical projective duality and the spherical duality. A theorem of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space was shown in [9]. It is now a fundamental tool for the study of extrinsic differential geometries on submanifolds in these pseudo-spheres from the view point of Singularity theory (cf., [9, 11, 12, 15, 17]). The theorem for these Legendrian dualities was generalized into pseudo-spheres in general semi-Euclidean space [7]. The assertion is expressed by a commutative diagram of contact diffeomorphisms among total spaces of special Legendrian double fibrations in the products of pseudo-spheres. Such the commutative diagram of contact diffeomorphisms has a similar structure to the religious picture of Buddhism called the "mandala"(cf., §3). Therefore, the diagram of contact diffeomorphisms for Legendrian double fibrations is called the *mandala of Legednrian dualities* in [7]. In this paper, we extend the mandala of Legendrian dualities which was given in [9] for continuous families of pseudo-spheres in Lorentz-Minkowski space. We do not consider semi-Euclidean space with general index here. However, we remark that by exactly the same way as in this paper we can easily generalize the results into the pseudo-spheres in semi-Euclidean space with general index, so that we omit them. The main results (cf., Theorems 3.1 and 3.2) are simple generalizations of the results in [9]. However, there are some new applications of such extended dualities. In §4, we only give some basic results on such applications. The detailed arguments on these applications have been recently appeared in the papers [3, 16].

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2 Basic notions

In this section we give basic notions and properties on Lorentz-Minkowski space. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) \mid x_i \in \mathbb{R},\ i = 0, \ldots, n\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Lorentz-Minkowski* $(n+1)$*-space* and denoted by $\mathbb{R}^{n+1}_1$. We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0, respectively. The norm of a vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike, respectively. We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}_1$:
Hyperbolic n*-space* is defined by

$$H^n(-c^2) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -c^2\},$$

de Sitter n*-space* by

$$S^n_1(c^2) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = c^2\ \}$$

and the (*open*) *lightcone* by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\ \},$$

for any real number c. Instead of $S^n_1(1)$, we usually write S^n_1.

3 Legendrian dualities

In this section we formulate theorems on Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and give their proofs. For our purpose, we briefly review some properties of contact manifolds and Legendrian submanifolds. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally, such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case, K is called a *contact structure* and α is a *contact form*. Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if $\dim L = n$ and $di_x(T_x L) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \longrightarrow M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \longrightarrow M$ is called a *Legendrian map*. The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(L)$. For any $z \in E$, it is known that there is a local coordinate system $(x, y, p) = (x_1, \ldots, x_m, y, p_1, \ldots, p_m)$ around z such that $\pi(x, y, p) = (x, y)$ and the contact structure is given by the 1-form $\alpha = dy - \sum_{i=1}^{m} p_i dx_i$ (cf. [1], 20.3). In [9], the basic duality theorem for four Legendrian double fibrations which is the fundamental tool for the study of spacelike hypersurfaces in Lorentz-Minkowski pseudo-spheres was shown. We now consider a slight extension of these dualities by the following double fibrations:

(1) (a) $H^n(-1) \times S^n_1 \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \}$,
(b) $\pi_{11} : \Delta_1 \longrightarrow H^n(-1)$, $\pi_{12} : \Delta_1 \longrightarrow S^n_1$,
(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_1$.

(2) (a) $H^n(-1) \times LC^* \supset \Delta_2^{\pm} = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \pm 1 \}$,
(b) $\pi^{\pm}_{21} : \Delta^{\pm}_2 \longrightarrow H^n(-1)$, $\pi^{\pm}_{22} : \Delta^{\pm}_2 \longrightarrow LC^*$,
(c) $\theta^{\pm}_{21} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta^{\pm}_2$, $\theta^{\pm}_{22} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta^{\pm}_2$.

(3) (a) $LC^* \times S^n_1 \supset \Delta^{\pm}_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \pm 1 \}$,
(b) $\pi^{\pm}_{31} : \Delta^{\pm}_3 \longrightarrow LC^*$, $\pi^{\pm}_{32} : \Delta^{\pm}_3 \longrightarrow S^n_1$,
(c) $\theta^{\pm}_{31} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta^{\pm}_3$, $\theta^{\pm}_{32} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta^{\pm}_3$.

(4) (a) $LC^* \times LC^* \supset \Delta^{\pm}_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \pm 2 \}$,
(b) $\pi^{\pm}_{41} : \Delta^{\pm}_4 \longrightarrow LC^*$, $\pi^{\pm}_{42} : \Delta^{\pm}_4 \longrightarrow LC^*$,
(c) $\theta^{\pm}_{41} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta^{\pm}_4$, $\theta^{\pm}_{42} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta^{\pm}_4$.

Here, $\pi_{11}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{12}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$, $\pi^{\pm}_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$ and $\pi^{\pm}_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$ (i=2,3,4). Moreover, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$ are one-forms on $\mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$. We remark that ${\theta_{11}}^{-1}(0)$ and ${\theta_{12}}^{-1}(0)$ define the same tangent hyperplane field denoted by K_1 over Δ_1. And also ${\theta^{\pm}_{i1}}^{-1}(0)$ and ${\theta^{\pm}_{i2}}^{-1}(0)$ define the same tangent hyperplane field denoted by $K^{\pm}_i$ over $\Delta^{\pm}_i$ (i=2,3,4). We have the following basic duality theorem:

Theorem 3.1 *Under the same notations as the previous paragraph, (Δ_1, K_1) and $(\Delta^{\pm}_i, K^{\pm}_i)$ $(i = 2, 3, 4)$ are contact manifolds such that π_{1j} and $\pi^{\pm}_{ij}$ $(j = 1, 2)$ are Legendrian fibrations. Moreover, these contact manifolds are contact diffeomorphic to each other.*

Proof. By definition, we can easily show that Δ_1 and $\Delta^{\pm}_i$ $(i = 2, 3, 4)$ are smooth submanifolds in $\mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$ and all of π_{1j} and $\pi^{\pm}_{ij}$ $(i = 2, 3, 4; j = 1, 2)$ are smooth fibrations.

In [9] it was shown that (Δ_1, K_1) is a contact manifold. We now give a brief review of the proof. Since $H^n(-1)$ is a spacelike hypersurface in $\mathbb{R}^{n+1}_1$, $\langle\ ,\ \rangle | H^n(-1)$ is a Riemannian metric. Let $\pi : S(TH^n(-1)) \longrightarrow H^n(-1)$ be the unit tangent sphere bundle of $H^n(-1)$. For any $\boldsymbol{v} \in H^n(-1)$, we have the local coordinates $(v_1, \dots, v_n)$ such that $\boldsymbol{v} = (\pm\sqrt{v_1^2 + \cdots + v_n^2 + 1}, v_1, \dots, v_n)$. We can represent the tangent vector $\boldsymbol{w} \in T_v H^n(-1)$ by

$$\boldsymbol{w} = (\pm \frac{1}{v_0} \sum_{i=1}^n w_i v_i, w_1, \dots, w_n).$$

It follows that $\langle \boldsymbol{w}, \boldsymbol{v} \rangle = (\pm \frac{1}{v_0} \sum_{i=1}^n w_i v_i)(\mp v_0) + \sum_{i=1}^n w_i v_i = 0$. Therefore, $\boldsymbol{w} \in S(T_v H^n(-1))$ if and only if $\langle \boldsymbol{w}, \boldsymbol{w} \rangle = 1$ and $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$. The last conditions are equivalent to the condition that $(\boldsymbol{v}, \boldsymbol{w}) \in \Delta_1$. This means that we can canonically identify $S(TH^n(-1))$ with Δ_1. Moreover, the canonical contact structure on $S(TH^n(-1))$ is given by the one-form $\theta(V) = \langle d\pi(V), \tau(V) \rangle$, where $\tau : TS(TH^n(-1)) \longrightarrow S(TH^n(-1))$ is the tangent bundle of $S(TH^n(-1))$ (cf., [4, 6]). It can be represented by $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$ through the above identification. Thus, $(\Delta_1, \theta^{-1}_{11}(0))$ is a contact manifold. For the other $\Delta^{\pm}_i$ $(i = 2, 3, 4)$, we define the smooth mappings $\Psi^{\pm}_{1i} : \Delta_1 \longrightarrow \Delta^{\pm}_i$ by $\Psi^{\pm}_{12}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, \mp\boldsymbol{v} + \boldsymbol{w})$, $\Psi^{\pm}_{13}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} \pm \boldsymbol{w}, \boldsymbol{w})$ and $\Psi^{\pm}_{14}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} \pm \boldsymbol{w}, \mp\boldsymbol{v} + \boldsymbol{w})$.

We can construct their converse mappings, so that $\Psi^{\pm}_{1i}$ are diffeomorphisms. Moreover, we have

$$
\begin{aligned}
{\Psi^{\pm}_{12}}^{*}\theta^{\pm}_{21} &= \langle d\boldsymbol{v}, \mp\boldsymbol{v}+\boldsymbol{w}\rangle|\Delta_1 \\
&= \langle d\boldsymbol{v}, \mp\boldsymbol{v}\rangle|\Delta_1 + \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_1 \\
&= \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_1 = \theta_{11}.
\end{aligned}
$$

This means that $(\Delta^{\pm}_2, K^{\pm}_2)$ is a contact manifold such that $\Psi^{\pm}_{12}$ is a contact diffeomorphism. For the other $\Delta^{\pm}_i$ $(i = 3, 4)$, we have the similar calculations, so that $(\Delta^{\pm}_i, K^{\pm}_i)$ $(i = 3, 4)$ are contact manifolds such that $\Psi^{\pm}_{1i}$ are contact diffeomorphisms. This completes the proof. $\square$

We can also give the contact diffeomorphisms $\Psi^{\pm}_{ij} : \Delta^{\pm}_i \longrightarrow \Delta^{\pm}_j$ for the other pairs (i, j) by $\Psi^{\pm}_{ij} = \Psi^{\pm}_{i1} \circ \Psi^{\pm}_{1j}$, where $\Psi^{\pm}_{i1} = (\Psi^{\pm}_{1i})^{-1}$. It follows that we have a "mandala of Legendrian dualities" by the following commutative diagram:

$$
\begin{tikzcd}
 & \Delta_1 \arrow[ddl, shift right, "\Psi^{\pm}_{12}"'] \arrow[ddr, shift left, "\Psi^{\pm}_{13}"] \arrow[d, shift left, "\Psi^{\pm}_{14}"] & \\
 & \Delta^{\pm}_4 \arrow[u, shift left, "\Psi^{\pm}_{41}"] \arrow[dl, shift right, "\Psi^{\pm}_{42}"'] \arrow[dr, shift left, "\Psi^{\pm}_{43}"] & \\
\Delta^{\pm}_2 \arrow[uur, shift right, "\Psi^{\pm}_{21}"'] \arrow[ur, shift right, "\Psi^{\pm}_{24}"'] \arrow[rr, shift left, "\Psi^{\pm}_{23}"] & & \Delta^{\pm}_3 \arrow[uul, shift left, "\Psi^{\pm}_{31}"] \arrow[ul, shift left, "\Psi^{\pm}_{34}"] \arrow[ll, shift left, "\Psi^{\pm}_{32}"]
\end{tikzcd}
$$

The above mandala is a slight extension of the mandala given by the Legendrian dualities in [9]. However, we can extend it to infinite families of Legedrian dualities as follows:

(5) (a) $H^n(-1) \times S^n_1(\cos^2\phi) \supset \Delta^{\pm}_{12}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm\sin\phi\ \}$,
(b) $\pi[\phi]^{\pm}_{(12)1} : \Delta^{\pm}_{12}(\phi) \longrightarrow H^n(-1)$, $\pi[\phi]^{\pm}_{(12)2} : \Delta^{\pm}_{12}(\phi) \longrightarrow S^n_1(\cos^2\phi)$,
(c) $\theta[\phi]^{\pm}_{(12)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{12}(\phi)$, $\theta[\phi]^{\pm}_{(12)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{12}(\phi)$.

(6) (a) $H^n(-\cos^2\phi) \times S^n_1 \supset \Delta^{\pm}_{13}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm\sin\phi\ \}$,
(b) $\pi[\phi]^{\pm}_{(13)1} : \Delta^{\pm}_{13}(\phi) \longrightarrow H^n(-\cos^2\phi)$, $\pi[\phi]^{\pm}_{(13)2} : \Delta^{\pm}_{13}(\phi) \longrightarrow S^n_1$,
(c) $\theta[\phi]^{\pm}_{(13)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{13}(\phi)$, $\theta[\phi]^{\pm}_{(13)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{13}(\phi)$.

(7) (a) $H^n(-\cos^2\phi) \times S^n_1(\cos^2\phi) \supset \Delta^{\pm}_{14}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm 2\sin\phi\ \}$,
(b) $\pi[\phi]^{\pm}_{(14)1} : \Delta^{\pm}_{14}(\phi) \longrightarrow H^n(-\cos^2\phi)$, $\pi[\phi]^{\pm}_{(14)2} : \Delta^{\pm}_{14}(\phi) \longrightarrow S^n_1(\cos^2\phi)$,
(c) $\theta[\phi]^{\pm}_{(14)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{14}(\phi)$, $\theta[\phi]^{\pm}_{(14)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{14}(\phi)$.

(8) (a) $H^n(-\cos^2\phi) \times S^n_1(\sin^2\phi) \supset \Delta^{\pm}_{23}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm(\sin\phi + \cos\phi)\ \}$,
(b) $\pi[\phi]^{\pm}_{(23)1} : \Delta^{\pm}_{23}(\phi) \longrightarrow H^n(-\cos^2\phi)$, $\pi[\phi]^{\pm}_{(23)2} : \Delta^{\pm}_{23}(\phi) \longrightarrow S^n_1(\sin^2\phi)$,
(c) $\theta[\phi]^{\pm}_{(23)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{23}(\phi)$, $\theta[\phi]^{\pm}_{(23)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{23}(\phi)$.

(9) (a) $H^n(-\cos^2\phi) \times LC^* \supset \Delta^{\pm}_{24}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm(\sin\phi + 1)\ \}$,
(b) $\pi[\phi]^{\pm}_{(24)1} : \Delta^{\pm}_{24}(\phi) \longrightarrow H^n(-\cos^2\phi)$, $\pi[\phi]^{\pm}_{(24)2} : \Delta^{\pm}_{24}(\phi) \longrightarrow LC^*$,
(c) $\theta[\phi]^{\pm}_{(24)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{24}(\phi)$, $\theta[\phi]^{\pm}_{(24)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{24}(\phi)$.

(10) (a) $LC^* \times S^n_1(\cos^2\phi) \supset \Delta^{\pm}_{34}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm(\sin\phi + 1)\ \}$,
(b) $\pi[\phi]^{\pm}_{(34)1} : \Delta^{\pm}_{34}(\phi) \longrightarrow LC^*$, $\pi[\phi]^{\pm}_{(34)2} : \Delta^{\pm}_{34}(\phi) \longrightarrow S^n_1(\cos^2\phi)$,
(c) $\theta[\phi]^{\pm}_{(34)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{34}(\phi)$, $\theta[\phi]^{\pm}_{(34)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{34}(\phi)$.

We also define the tangent hyperplane field $K[\phi]^{\pm}_{ij}$ over $\Delta^{\pm}_{ij}(\phi)$ by $K[\phi]^{\pm}_{ij} = {\theta[\phi]^{\pm}_{(ij)1}}^{-1}(0) = {\theta[\phi]^{\pm}_{(ij)2}}^{-1}(0)$. The main result in this paper is the following theorem:

Theorem 3.2 *Under the same notations as those of the previous paragraph,* (Δ_1, K_1) *and* $(\Delta^{\pm}_{ij}(\phi), K[\phi]^{\pm}_{ij})$ $((i,j) = (1,2),(1,3),(1,4),(2,3),(2,4),(3,4))$ *are contact manifolds such that* π_{1k} *and* $\pi[\phi]^{\pm}_{(ij)k}$ $(k = 1,2)$ *are Legendrian fibrations. Moreover, these contact manifolds are contact diffeomorphic to each other.*

Proof. We can construct the diffeomorphisms $\Psi^{\pm}_{(ij)1} : \Delta^{\pm}_{ij}(\phi) \longrightarrow \Delta_1$ with $d\Psi^{\pm}_{(ij)1}(K[\phi]^{\pm}_{ij}) = K_1$ as follows:
(5) We define a mapping

$$\Psi^{\pm}_{(12)1} : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \ ; \ \Psi^{\pm}_{(12)1}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v}, \pm\sin\phi\boldsymbol{v} + \boldsymbol{w}).$$

For any $(\boldsymbol{v},\boldsymbol{w}) \in \Delta^{\pm}_{12}(\phi)$, we have

$$\langle \pm\sin\phi\boldsymbol{v} + \boldsymbol{w}, \pm\sin\phi\boldsymbol{v} + \boldsymbol{w}\rangle = -\sin^2\phi + 2\sin^2\phi + \cos^2\phi = 1$$

and $\langle \boldsymbol{v}, \pm\sin\phi\boldsymbol{v} + \boldsymbol{w}\rangle = 0$. Therefore, we have $\Psi^{\pm}_{(12)1}(\Delta^{\pm}_{12}(\phi)) \subset \Delta_1$. We also define a mapping

$$\Psi^{\pm}_{1(12)} : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \ ; \ \Psi^{\pm}_{1(12)}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v}, \mp\sin\phi\boldsymbol{v} + \boldsymbol{w}).$$

We can easily calculate that $\Psi^{\pm}_{1(12)}(\Delta_1) \subset \Delta^{\pm}_{12}(\phi)$, $\Psi^{\pm}_{1(12)} \circ \Psi^{\pm}_{(12)1}|\Delta^{\pm}_{12}(\phi) = 1_{\Delta^{\pm}_{12}(\phi)}$ and $\Psi^{\pm}_{(12)1} \circ \Psi^{\pm}_{1(12)}|\Delta_1 = 1_{\Delta_1}$. Moreover, we have

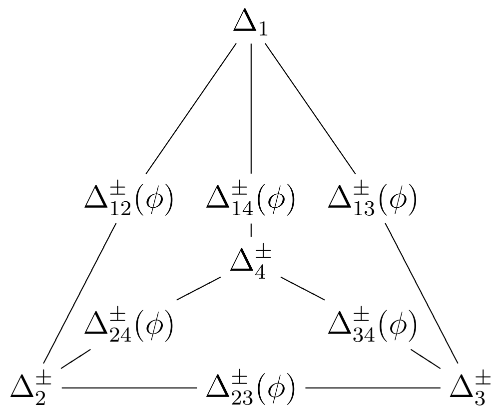
$$(\Psi^{\pm}_{(12)1})^*\theta_{11} = \langle d\boldsymbol{v}, \pm\sin\phi\boldsymbol{v} + \boldsymbol{w}\rangle|\Delta^{\pm}_{12}(\phi) = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{12}(\phi) = \theta[\phi]^{\pm}_{(12)1}.$$

Therefore, $K[\phi]^{\pm}_{12}$ is a contact structure on $\Delta^{\pm}_{12}(\phi)$ such that $\Psi^{\pm}_{1(12)}$ is a contact diffeomorphism.

For other cases, we can define the following mappings:
(6) $\Psi^{\pm}_{(13)1}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v} \mp \sin\phi\boldsymbol{w}, \boldsymbol{w})$.
(7) $\Psi^{\pm}_{(14)1}(\boldsymbol{v},\boldsymbol{w}) = \dfrac{1}{\sin^2\phi + 1}(\boldsymbol{v} \mp \sin\phi\boldsymbol{w}, \pm\sin\phi\boldsymbol{v} + \boldsymbol{w})$.
(8) $\Psi^{\pm}_{(23)1}(\boldsymbol{v},\boldsymbol{w}) = \dfrac{1}{\sin\phi\cos\phi + 1}(\boldsymbol{v} \mp \sin\phi\boldsymbol{w}, \pm\cos\phi\boldsymbol{v} + \boldsymbol{w})$.
(9) $\Psi^{\pm}_{(24)1}(\boldsymbol{v},\boldsymbol{w}) = \dfrac{1}{\sin\phi + 1}(\boldsymbol{v} \mp \sin\phi\boldsymbol{w}, \pm\boldsymbol{v} + \boldsymbol{w})$.
(10) $\Psi^{\pm}_{(34)1}(\boldsymbol{v},\boldsymbol{w}) = \dfrac{1}{\sin\phi + 1}(\boldsymbol{v} \mp \boldsymbol{w}, \pm\sin\phi\boldsymbol{v} + \boldsymbol{w})$.

By straightforward calculations, we can show that $\Psi^{\pm}_{(ij)1}|\Delta^{\pm}_{ij}(\phi) : \Delta^{\pm}_{ij}(\phi) \longrightarrow \Delta_1$, $((i,j) = (1,3),(1,4),(2,3),(2,4),(3,4))$ are diffeomorphisms such that $d\Psi^{\pm}_{(ij)1}(K[\phi]^{\pm}_{ij}) = K_1$. Therefore, $(\Delta^{\pm}_{ij}(\phi), K[\phi]^{\pm}_{ij})$ are contact manifolds which are contact diffeomorphic to (Δ_1, K_1). □

We can write the above extension of the mandala as follows:



$$\phi \in \left[0,\ \frac{\pi}{2}\right],\ \Delta_{ij}^{\pm}\left(\frac{\pi}{2}\right) = \Delta_j^{\pm},\ \Delta_{1j}^{\pm}(0) = \Delta_1,\ \Delta_{ij}^{\pm}(0) = \Delta_i^{\pm}\ (i \neq 1).$$

The extended Mandala of Legendrian Dualities

The above diagram is not a diagram for contact diffeomorphisms. If we add informations on the contact diffeomorphisms between $\Delta_{ij}^{\pm}$, the diagram might be very complicated, so that we omit the contact diffeomorphisms in the above diagram.

Remark 3.3 We can also define

$$\Delta_{ji}^{\pm}(\phi) = \Delta_{ij}^{\pm}\left(\frac{\pi}{2} - \phi\right),\ K[\phi]_{ji}^{\pm} = K\left[\frac{\pi}{2} - \phi\right]_{ij}^{\pm},\ \pi[\phi]_{(ji)k}^{\pm} = \pi\left[\frac{\pi}{2} - \phi\right]_{(ij)k}^{\pm}$$

for $(i,j) = (1,2),(1,3),(1,4),(2,3),(2,4)$ and $(3,4)$. Then these are contact manifolds with $\Delta_{ji}^{\pm}(0) = \Delta_j^{\pm}$, $\Delta_{j1}^{\pm}(\pi/2) = \Delta_1$ and $\Delta_{ji}^{\pm}(\pi/2) = \Delta_i^{\pm}$ $(i \neq 1)$. Moreover, all of them are canonically contact diffeomorphic to (Δ_1, K_1). Since these contact diffeomorphisms can be constructed by the canonical way, we omit to give the definitions here.

We can explicitly write these families of Legendrian dualities as follows:

(5*) (a) $H^n(-1) \times S_1^n(\sin^2\phi) \supset \Delta_{21}^{\pm}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm\cos\phi\ \}$,
(b) $\pi[\phi]_{(21)1}^{\pm} : \Delta_{21}^{\pm}(\phi) \longrightarrow H^n(-1)$, $\pi[\phi]_{(21)2}^{\pm} : \Delta_{21}^{\pm}(\phi) \longrightarrow S_1^n(\sin^2\phi)$,
(c) $\theta[\phi]_{(21)1}^{\pm} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_{21}^{\pm}(\phi)$, $\theta[\phi]_{(21)2}^{\pm} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_{21}^{\pm}(\phi)$.

(6*) (a) $H^n(-\sin^2\phi) \times S_1^n \supset \Delta_{31}^{\pm}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm\cos\phi\ \}$,
(b) $\pi[\phi]_{(31)1}^{\pm} : \Delta_{31}^{\pm}(\phi) \longrightarrow H^n(-\sin^2\phi)$, $\pi[\phi]_{(31)2}^{\pm} : \Delta_{31}^{\pm}(\phi) \longrightarrow S_1^n$,
(c) $\theta[\phi]_{(31)1}^{\pm} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_{31}^{\pm}(\phi)$, $\theta[\phi]_{(31)2}^{\pm} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_{31}^{\pm}(\phi)$.

(7*) (a) $H^n(-\sin^2\phi) \times S_1^n(\sin^2\phi) \supset \Delta_{41}^{\pm}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm 2\cos\phi\ \}$,
(b) $\pi[\phi]_{(41)1}^{\pm} : \Delta_{41}^{\pm}(\phi) \longrightarrow H^n(-\sin^2\phi)$, $\pi[\phi]_{(41)2}^{\pm} : \Delta_{41}^{\pm}(\phi) \longrightarrow S_1^n(\sin^2\phi)$,
(c) $\theta[\phi]_{(41)1}^{\pm} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_{41}^{\pm}(\phi)$, $\theta[\phi]_{(41)2}^{\pm} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_{41}^{\pm}(\phi)$.

(8*) (a) $H^n(-\sin^2\phi) \times S_1^n(\cos^2\phi) \supset \Delta_{32}^{\pm}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm(\cos\phi + \sin\phi)\ \}$,
(b) $\pi[\phi]_{(32)1}^{\pm} : \Delta_{32}^{\pm}(\phi) \longrightarrow H^n(-\sin^2\phi)$, $\pi[\phi]_{(32)2}^{\pm} : \Delta_{32}^{\pm}(\phi) \longrightarrow S_1^n(\cos^2\phi)$,
(c) $\theta[\phi]_{(32)1}^{\pm} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_{32}^{\pm}(\phi)$, $\theta[\phi]_{(32)2}^{\pm} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_{32}^{\pm}(\phi)$.

(9*) (a) $H^n(-\sin^2\phi) \times LC^* \supset \Delta_{42}^{\pm}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm(\cos\phi + 1)\ \}$,
(b) $\pi[\phi]_{(42)1}^{\pm} : \Delta_{42}^{\pm}(\phi) \longrightarrow H^n(-\sin^2\phi)$, $\pi[\phi]_{(42)2}^{\pm} : \Delta_{42}^{\pm}(\phi) \longrightarrow LC^*$,

(c) $\theta[\phi]^{\pm}_{(42)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{42}(\phi)$, $\theta[\phi]^{\pm}_{(42)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{42}(\phi)$.

(10^*) (a) $LC^* \times S^n_1(\sin^2\phi) \supset \Delta^{\pm}_{43}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm(\cos\phi + 1)\ \}$,

(b) $\pi[\phi]^{\pm}_{(43)1} : \Delta^{\pm}_{43}(\phi) \longrightarrow LC^*$, $\pi[\phi]^{\pm}_{(43)2} : \Delta^{\pm}_{43}(\phi) \longrightarrow S^n_1(\sin^2\phi)$,

(c) $\theta[\phi]^{\pm}_{(43)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{43}(\phi)$, $\theta[\phi]^{\pm}_{(43)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{43}(\phi)$.

4 Slant geometry of submanifolds in pseudo-spheres

In this section we consider one-parameter families of new extrinsic differential geometries on spacelike hypersurfaces in pseudo-spheres in Lorentz-Minkowski space as an application of the extended mandala of Legendrian dualities. Here, we only give some basic properties. The detailed arguments will be appeared in the forthcoming papers [3, 16].

4.1 Hyperbolic space

Let $\mathcal{L}_1 : U \longrightarrow \Delta_1$ be a Legendrian embedding with $\mathcal{L}_1(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^d(u))$ for an open subset $U \subset \mathbb{R}^{n-1}$. Suppose that $\boldsymbol{X}^h : U \longrightarrow H^n(-1)$ is an embedding. Since $\mathcal{L}_1$ is a Legendrian embedding, $\boldsymbol{X}^d : U \longrightarrow S^n_1$ can be considered as a unit normal vector field along the hypersurface $M^H = \boldsymbol{X}^h(U)$ in $H^n(-1)$. We define $\boldsymbol{X}^{\ell}_{\pm}(u) = \boldsymbol{X}^h(u) \pm \boldsymbol{X}^d(u)$. Then these are lightlike vectors. It follows that we have lightlike normal vector fields $\boldsymbol{X}^{\ell}_{\pm} : U \longrightarrow LC^*$ along M^H. We respectively call $\boldsymbol{X}^d$ and $\boldsymbol{X}^{\ell}_{\pm}$, the *de Sitter Gauss image* and the *lightcone Gauss image* of M^H. We define a map $\mathcal{L}_2 : U \longrightarrow \Delta^-_2$ by $\mathcal{L}_2(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^{\ell}_{\pm}(u))$. It is easy to check that $\mathcal{L}_2$ is a Legendrian embedding. In [8], $\boldsymbol{X}^d$ and $\boldsymbol{X}^{\ell}_{\pm}$ were constructed by an explicit way and the geometric meanings of the singularities of these Gauss images were investigated. Both of the de Sitter Gauss image $\boldsymbol{X}^d$ and the lightcone Gauss image $\boldsymbol{X}^{\ell}_{\pm}$ play similar roles with the Gauss map of a hypersurface in Euclidean space. We can interpret that $d\boldsymbol{X}^d(u)$ is a linear transformation on T_pM^H for $p = \boldsymbol{X}^h(u)$. Since the derivative $d\boldsymbol{X}^h(u)$ can be identified with the identity mapping $1_{T_pM^H}$ on the tangent space T_pM^H under the identification of U and M^H through the embedding $\boldsymbol{X}^h$, we have $d\boldsymbol{X}^{\ell}_{\pm}(u) = 1_{T_pM^H} \pm d\boldsymbol{X}^d(u)$, so that $d\boldsymbol{X}^{\ell}_{\pm}(u)$ can be also interpreted as a linear transformation on T_pM^H. We call the linear transformations $A^d_p = -d\boldsymbol{X}^d(u) : T_pM^H \longrightarrow T_pM^H$ and $(S^{\pm}_h)_p = -d\boldsymbol{X}^{\ell}_{\pm}(u) : T_pM^H \longrightarrow T_pM^H$, the *de Sitter shape operator* and the *lightcone shape operator* of $M^H = \boldsymbol{X}^h(U)$ at $p = \boldsymbol{X}^h(u)$, respectively. The *de Sitter Gauss-Kronecker curvature* and the *lightcone Gauss-Kronecker curvature* of M^H at $p = \boldsymbol{X}^h(u)$ are defined to be $K_d(u) = \det A^d_p$ and $K^{\pm}_{\ell}(u) = \det(S^{\pm}_h)_p$, respectively. In [8], the geometric meanings of the lightcone Gauss-Kronecker curvature from the contact viewpoint were investigated. The consequences of the results are that the de Sitter Gauss-Kronecker curvature (respectively, the lightcone Gauss-Kronecker curvature) estimates the contact of hypersurfaces with hyperplanes (respectively, hyperhorospheres). Here, a *hyperplane* is defined to be the intersection of $H^n(-1)$ with a timelike hyperplane through the origin and a *hyperhorosphere* is defined to be the intersection of $H^n(-1)$ with a lightlike hyperplane. We only remark here that $\boldsymbol{X}^d$ is a constant vector if and only if M^H is a part of a hyperplane. Moreover, one of $\boldsymbol{X}^{\ell}_{\pm}$ is a constant vector if and only if M^H is a part of a hyperhorosphere. These facts suggest us that there are two kinds of flat subjects in Hyperbolic space. One of them is a hyperplane and the other one is a hyperhorosphere. In the Poincaré ball model of Hyperbolic space, the hyperplane is a hypersphere as the Euclidean sense and it is orthogonal to the ideal boundary. The hyperhorosphere is also a hypersphere as the Euclidean sense, but it is tangent to the ideal boundary. We remark that the hyperplanes are totally flat hypersurfaces in the sense of Hyperbolic Geometry. What about hyperhorospheres? We emphasize that a new geometry which is called "Horospherical Geometry" in Hyperbolic space was discovered through the researches [5, 8, 10, 13, 14, 15]. Hyperhorospheres are totally flat hypersurfaces in Hyperbolic space in the sense of Horospherical Geometry.

On the other hand, an *equidistant hypersurface* is defined to be the intersection of $H^n(-1)$ with a timelike hyperplane which does not contain the origin. It is well known that a non-compact complete totally umbilic hypersurface in Hyperbolic space is a hyperplane, an equidistant hypersurface or a hyperhorosphere (cf., [8]). Here, we consider a natural question.

Question. Can we construct a geometry such that an equidistant hypersurface is a totally flat hypersurface?

In order to give an answer to this question, we consider the contact manifold $(\Delta^-_{(21)}(\phi), K[\phi]^-_{21})$ and the contact diffeomorphism $\Psi^-_{1(21)} : \Delta_1 \longrightarrow \Delta^-_{(21)}(\phi)$ defined by $\Psi^-_{1(21)}(\boldsymbol{v}, \cos\phi\boldsymbol{v} \pm \boldsymbol{w})$. We define $\mathbb{N}^d_\pm[\phi] : U \longrightarrow S^n_1(\sin^2\phi)$ by

$$\mathbb{N}^d_\pm[\phi](u) = \cos\phi\boldsymbol{X}^h(u) \pm \boldsymbol{X}^d(u),$$

for $\phi \in [0, \pi/2]$. It follows that $\mathbb{N}^d_\pm[0] = \boldsymbol{X}^\ell_\pm$, $\mathbb{N}^d_+[\pi/2] = \pm\boldsymbol{X}^d$ and $\langle \boldsymbol{X}^h(u), \mathbb{N}^d_\pm[\phi](u)\rangle = -\cos\phi$. We also define an embedding $\mathcal{L}_{21}[\phi] : U \longrightarrow \Delta^-_{21}(\phi)$ by $\mathcal{L}_{21}[\phi](u) = (\boldsymbol{X}^h(u), \mathbb{N}^d_\pm[\phi](u))$. Then we have $\mathcal{L}_{21}[\phi] = \Psi^-_{1(21)} \circ \mathcal{L}_1$, so that $\mathcal{L}_{21}[\phi]$ is a Legendrian embedding. Therefore, we have $\langle d\boldsymbol{X}^h, \mathbb{N}^d_\pm[\phi]\rangle = \mathcal{L}_{21}[\phi]^*\theta[\phi]^-_{(21)1} = 0$. This means that $\mathbb{N}^d_\pm[\phi](u)$ is a normal vector of M^H at $p = \boldsymbol{X}^h(u)$. We call $\mathbb{N}^d_\pm[\phi] : U \longrightarrow S^n_1(\sin^2\phi)$ the *ϕ-de Sitter Gauss image* of M^H. By definition, we have $d\mathbb{N}^d_\pm[\phi](u) = \cos\phi 1_{T_pM^H} \pm d\boldsymbol{X}^d(u)$ which can be considered as a linear transformation on T_pM^H. We call $S^d_\pm[\phi]_p = -d\mathbb{N}^d_\pm[\phi](u) : T_pM^H \longrightarrow T_pM^H$ a *ϕ-de Sitter shape operator* (or *ϕ-de Sitter Weingarten map*) of M^H at $p = \boldsymbol{X}^h(u)$. The *ϕ-de Sitter Gauss-Kronecker curvature* of M^H at $p = \boldsymbol{X}^h(u)$ is defined to be $K^\pm_d[\phi](u) = \det S^d_\pm[\phi]_p$. The geometry related to ϕ-de Sitter Gauss image is called a *ϕ-geometry of hypersurfaces in Hyperbolic space.* Since the 0-geometry is the horospherical geometry and $\pi/2$-geometry is the hyperbolic geometry, we call the ϕ-geometry a *slant geometry* in Hyperbolic space if $\phi \in (0, \pi/2)$. The detailed investigation of the slant geometry in Hyperbolic space will be appeared in the forthcoming paper [3]. Here, we only consider the most degenerate case.

Proposition 4.1 *For a hypersurface M^H, one of $\mathbb{N}^d_\pm[\phi](u)$ is a constant vector if and only if M^H is a part of a hyperquadric $H^n(-1) \cap HP(\boldsymbol{v}, -\cos\phi)$ with $\boldsymbol{v} \in S^n_1(\sin^2\phi)$.*

Proof. Suppose that $\mathbb{N}^d_+[\phi](u) = \text{constant} = \boldsymbol{v}$. Then we have

$$\langle \boldsymbol{X}^h(u), \boldsymbol{v}\rangle = \langle \boldsymbol{X}^h(u), \mathbb{N}^d_+[\phi](u)\rangle = 0.$$

This means that $M^H \subset H^n(-1) \cap HP(\boldsymbol{v}, -\cos\phi)$. If $\mathbb{N}^d_-[\phi](u) = \text{constant} = \boldsymbol{v}$, we have the similar result. For the converse, suppose that $M^H \subset H^n(-1) \cap HP(\boldsymbol{v}, -\cos\phi)$ with $\boldsymbol{v} \in S^n_1(\sin^2\phi)$. Since $\boldsymbol{v}$ is a normal vector of M^H, there exist real numbers λ, μ such that $\boldsymbol{v} = \lambda\boldsymbol{X}^h(u) + \mu\boldsymbol{X}^d(u)$. By definition, we have $-\cos\phi = \langle \boldsymbol{X}^h(u), \boldsymbol{v}\rangle = -\lambda$ and $\sin^2\phi = -\lambda^2 + \mu^2$. It follows that $\boldsymbol{v} = \mathbb{N}^d_+[\phi](u)$ or $\boldsymbol{v} = \mathbb{N}^d_-[\phi](u)$. □

We remark that the above proposition asserts that a totally flat hypersurface in the ϕ-geometry is a part of a hyperquadric $H^n(-1) \cap HP(\boldsymbol{v}, -\cos\phi)$ with $\boldsymbol{v} \in S^n_1(\sin^2\phi)$. We call it a *$\phi$-hyperquadric* in Hyperbolic space $H^n(-1)$. A ϕ-hyperquadric is a hyperhorosphere (respectively, a hyperplane, an equidistant hypersurface), when $\phi = 0$ (respectively, $\phi = \pi/2$, $\phi \in (0, \pi/2)$).

4.2 De Sitter space

We also consider the Legendrian embedding $\mathcal{L}_1 : U \longrightarrow \Delta_1$ and suppose that $\boldsymbol{X}^d : U \longrightarrow S^n_1$ is an embedding. In this case, all the tangent vectors of $M^D = \boldsymbol{X}^d(U)$ are spacelike, so that $\boldsymbol{X}^d$ is a spacelike embedding. In [17] Kasedou constructed the extrinsic differential geometry on the spacelike hypersurfaces in S^n_1 analogous to the theory in [8]. We can interpret his framework by using the mandala of Legendrian dualities. We consider the lightlike vectors $\pm\boldsymbol{X}^\ell_\pm(u) = \boldsymbol{X}^d(u) \pm \boldsymbol{X}^h(u)$. We call $\boldsymbol{X}^h$ and $\pm\boldsymbol{X}^\ell_\pm$, the hyperbolic Gauss image and the lightcone Gauss image of $M^D = \boldsymbol{X}^d(U)$, respectively. We also define a map $\mathcal{L}^\pm_3 : U \longrightarrow \Delta^+_3$ by $\mathcal{L}^\pm_3(u) = (\pm\boldsymbol{X}^\ell_\pm(u), \boldsymbol{X}^d(u))$. It is a Legendrian embedding and $d(\pm\boldsymbol{X}^\ell_\pm)(u) = 1_{T_pM^D} \pm d\boldsymbol{X}^h(u)$ for $p = \boldsymbol{X}^d(u)$. Since $d\boldsymbol{X}^h(u)$ is considered to be a linear transformation on T_pM^D, $d(\pm\boldsymbol{X}^\ell_\pm)(u)$ is also a linear transformation on T_pM^D. We call $(S^\pm_d)_p = -d(\pm\boldsymbol{X}^\ell_\pm)(u) : T_pM^D \longrightarrow T_pM^D$ and $A^h_p = -d\boldsymbol{X}^h(u) : T_pM^D \longrightarrow T_pM^D$, the *lightcone shape operator* and the *hyperbolic shape operator* of M^D at $p = \boldsymbol{X}^d(u)$, respectively. Geometric characterizations of the singularities of the lightcone Gauss image $\pm\boldsymbol{X}^\ell_\pm$ of M^D from the view point of the contact with model hypersurfaces (cf.,[18]) are one of the main results in [17]. Especially, Theorem 5.6 in [17] was obtained by applying the theory of Legendrian singularities for $\pm\boldsymbol{X}^\ell_\pm(u)$. For definitions and basic properties of the theory of Legendrian singularities, see (Part III, [1]). Here, we can interprete the results in [17] by using the mandala of Legendrian dualities. Let $\Phi^\pm_{23} : \Delta^-_2 \longrightarrow \Delta^+_3$ be the mappings defined by $\Phi^\pm_{23}(\boldsymbol{v}, \boldsymbol{w}) = (\pm\boldsymbol{w}, \pm(\boldsymbol{w} - \boldsymbol{v}))$. Then we have $\pi_{31} \circ \Phi^\pm_{23} = \pm\pi_{22}$. It is easy to show that $\Phi^\pm_{23}$ are contact diffeomorphisms. By definition, we have

$$\Phi^\pm_{23} \circ \mathcal{L}_2(u) = \left(\pm\boldsymbol{X}^\ell_\pm(u), \pm(\boldsymbol{X}^\ell_\pm(u) - \boldsymbol{X}^h(u))\right) = \mathcal{L}^\pm_3(u).$$

This means that Legendrian maps $\pm\pi_{22} \circ \mathcal{L}_2$ and $\pi_{31} \circ \mathcal{L}^\pm_3$ are Legendrian equivalent. We only remark here that all of the conditions in Theorem 6.3 in [8] and Theorem 5.6 in [17] are invariant under the Legendrian equivalence. Therefore, the assertions of these theorems are equivalent.

On the other hand, we consider the contact manifold $(\Delta^+_{31}(\phi), K[\phi]^+_{31})$ and the contact diffeomorphism $\Psi^+_{1(31)} : \Delta_1 \longrightarrow \Delta^+_{31}(\phi)$ defined by $\Psi^+_{1(31)}(\boldsymbol{v}, \boldsymbol{w}) = (\pm\boldsymbol{v} + \cos\phi\boldsymbol{w}, \boldsymbol{w})$. We define a map $\mathbb{N}^h_\pm[\phi] : U \longrightarrow H^n(-\sin^2\phi)$ by

$$\mathbb{N}^h_\pm[\phi](u) = \cos\phi\boldsymbol{X}^d(u) \pm \boldsymbol{X}^h(u),$$

for $\phi \in [0, \pi/2]$. It follows that $\mathbb{N}^h_\pm[0] = \pm\boldsymbol{X}^\ell_\pm$, $\mathbb{N}^h_\pm[\pi/2] = \pm\boldsymbol{X}^h$ and $\langle\boldsymbol{X}^d(u), \mathbb{N}^h_\pm[\phi](u)\rangle = \cos\phi$. We also define an embedding $\mathcal{L}_{31}[\phi] : U \longrightarrow \Delta^+_{31}(\phi)$ by $\mathcal{L}_{31}[\phi](u) = (\mathbb{N}^h_\pm[\phi](u), \boldsymbol{X}^d(u))$. Then we have $\mathcal{L}_{31}[\phi] = \Psi^+_{1(31)} \circ \mathcal{L}_1$, so that $\mathcal{L}_{31}[\phi]$ is a Legendrian embedding. Therefore, we have $\langle d\boldsymbol{X}^d, \mathbb{N}^h_\pm[\phi]\rangle = \mathcal{L}_{31}[\phi]^*\theta[\phi]^+_{(31)2} = 0$. By exactly the same way as the hyperbolic case, we can construct the *ϕ-hyperbolic shape operator* $S^h_\pm[\phi]_p = -d\mathbb{N}^h_\pm[\phi](u) : T_pM^D \longrightarrow T_pM^D$ and the *ϕ-hyperbolic Gauss-Kronecker curvature* $K^\pm_h[\phi](u)$ of M^D at $p = \boldsymbol{X}^d(u)$. The geometry related to the Gauss image $\mathbb{N}^h_\pm[\phi]$ is called a *ϕ-geometry of the spacelike hypersurfaces in de Sitter space.* We also consider the most degenerate case here.

Proposition 4.2 *For a spacelike hypersurface $M^D \subset S^n_1$, one of $\mathbb{N}^h_\pm[\phi](u)$ is a constant vector if and only if M^D is a part of a hyperquadric $S^n_1 \cap HP(\boldsymbol{v}, \cos\phi)$ with $\boldsymbol{v} \in H^n(-\sin^2\phi)$.*

Proof. Suppose that $\mathbb{N}^h_+[\phi](u) = \text{constant} = \boldsymbol{v}$. Then we have

$$\langle \boldsymbol{X}^d(u), \boldsymbol{v}\rangle = \langle \boldsymbol{X}^d(u), \mathbb{N}^h_+[\phi](u)\rangle = 0.$$

This means that $M^D \subset S^n_1 \cap HP(\boldsymbol{v}, \cos\phi)$. If $\mathbb{N}^h_-[\phi](u) = \text{constant} = \boldsymbol{v}$, we have the similar result. For the converse, suppose that $M^D \subset S^n_1 \cap HP(\boldsymbol{v}, \cos\phi)$ with $\boldsymbol{v} \in H^n(-\sin^2\phi)$. Since $\boldsymbol{v}$ is a normal vector of M^D, there exist real numbers λ, μ such that $\boldsymbol{v} = \lambda\boldsymbol{X}^h(u) + \mu\boldsymbol{X}^d(u)$. By definition, we have $\cos\phi = \langle \boldsymbol{X}^d(u), \boldsymbol{v}\rangle = \mu$ and $-\sin^2\phi = -\lambda^2 + \mu^2$. It follows that $\boldsymbol{v} = \mathbb{N}^h_+[\phi](u)$ or $\boldsymbol{v} = \mathbb{N}^h_-[\phi](u)$. □

We also remark that the above proposition asserts that a totally flat spacelike hypersurface in the ϕ-geometry is a part of a hyperquadric $S^n_1 \cap HP(\boldsymbol{v}, \cos\phi)$ with $\boldsymbol{v} \in H^n(-\sin^2\phi)$. We call it a *$\phi$-hyperquadric* in de Sitter space S^n_1. By definition, the 0-hyperquadric is $S^n_1 \cap HP(\boldsymbol{v}, 1)$ for $v \in LC^*$ and $\pi/2$-hyperquadric is $S^n_1 \cap HP(\boldsymbol{v}, 0)$ for $\boldsymbol{v} \in H^n(-1)$. The 0-hyperquadric is called a *de Sitter hyperhorosphere* which is nothing but a parabolic hyperquadric. We call the $\pi/2$-hyperquadric a *small elliptic hyperquadric*. We remak that a small elliptic hyperquadric is a spacelike geodesic, when $n = 2$. We also call the geometry related to the Gauss image $\mathbb{N}^h_\pm[\phi]$ a *slant geometry of spacelike hypersurfaces in de Sitter space* if $\phi \in (0, \pi/2)$.

4.3 The lightcone

In [9], an extrinsic differential geometry on spacelike hypersurfaces was considered in the lightcone motivated by the result of [2]. The induced metric on the lightcone is degenerate, so that we cannot apply ordinary submanifold theory of semi-Riemannian geometry. The Δ^-_4-duality is really useful in this case. Let $\mathcal{L}_4 : U \longrightarrow \Delta^-_4$ be a Legendrian embedding with $\mathcal{L}_4(u) = (\boldsymbol{X}^\ell_+(u), \boldsymbol{X}^\ell_-(u))$ for an open subset $U \subset \mathbb{R}^{n-1}$. Suppose that $\boldsymbol{X}^\ell_+ : U \longrightarrow LC^*$ is a spacelike embedding. In [9], the Legendrian embedding $\mathcal{L}_4$ was used for the construction of the extrinsic differential geometry on spacelike hypersurfaces $M^L_+ = \boldsymbol{X}^\ell_+(U)$ in the lightcone. It was shown that for any spacelike embedding $\boldsymbol{X}^\ell_+ : U \longrightarrow LC^*$, there exists a unique Legendrian embedding $\mathcal{L}_4 : U \longrightarrow \Delta^-_4$ such that $\pi^-_{41} \circ \mathcal{L}_4 = \boldsymbol{X}^\ell_+$. Since $\mathcal{L}_4$ is Legendrian, $\boldsymbol{X}^\ell_-(u)$ is a lightlike normal vector of M^L_+ at $p = \boldsymbol{X}^\ell_+(u)$. We call it a *lightcone normal vector* of M^L_+. If $\boldsymbol{X}^\ell_-$ is an embedding, then $\boldsymbol{X}^\ell_+(u)$ is called a *lightcone normal vector* of $M^L_- = \boldsymbol{X}^\ell_-(U)$ at $p = \boldsymbol{X}^\ell_-(u)$. We define two vector fields

$$\boldsymbol{X}^h(u) = \frac{\boldsymbol{X}^\ell_-(u) + \boldsymbol{X}^\ell_+(u)}{2} \quad \text{and} \quad \boldsymbol{X}^d(u) = \frac{\boldsymbol{X}^\ell_-(u) - \boldsymbol{X}^\ell_+(u)}{2}.$$

Then $\boldsymbol{X}^h(u) \in H^n(-1)$ and $\boldsymbol{X}^d(u) \in S^n_1$. Moreover, we have mappings $\mathcal{L}_1 : U \longrightarrow \Delta_1$, $\mathcal{L}^\pm_2 : U \longrightarrow \Delta^-_2$ and $\mathcal{L}^\pm_3 : U \longrightarrow \Delta^+_3$ which are defined by $\mathcal{L}_1(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^d(u))$, $\mathcal{L}^\pm_2(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^\ell_\pm(u)))$ and $\mathcal{L}^\pm_3(u) = (\pm\boldsymbol{X}^\ell_\pm(u), \boldsymbol{X}^d(u))$, respectively. It is easy to show that $\mathcal{L}_1$ and $\mathcal{L}^\pm_i$ $(i = 2, 3)$ are Legendrian embeddings. We now define mappings $\Phi^\pm_{42} : \Delta^-_4 \longrightarrow \Delta^-_2$ by $\Phi^+_{42}(\boldsymbol{v}, \boldsymbol{w}) = (\dfrac{\boldsymbol{v}+\boldsymbol{w}}{2}, \boldsymbol{v})$ and $\Phi^-_{42}(\boldsymbol{v}, \boldsymbol{w}) = (\dfrac{\boldsymbol{v}+\boldsymbol{w}}{2}, \boldsymbol{w})$. Then we have $\pi^-_{22} \circ \Phi^-_{42} = \pi^-_{42}$ and $\pi^-_{22} \circ \Phi^+_{42} = \pi^-_{41}$. We can show that $\Phi^\pm_{42}$ are contact diffeomorphisms and $\Phi^\pm_{42} \circ \mathcal{L}_4 = \mathcal{L}^\pm_2$. Therefore, $\pi^-_{41} \circ \mathcal{L}_4$ (respectively, $\pi^-_{42} \circ \mathcal{L}_4$) and $\pi^-_{22} \circ \mathcal{L}^+_2$ (respectively, $\pi^-_{22} \circ \mathcal{L}^-_2$) are Legendrian equivalent. It follows that the assertions of Theorem 6.3 in [8] and Theorem 6.6 in [9] are equivalent. By the arguments in Subsection 4.2, the assertions of Theorem 5.6 in [17] and Theorem 6.6 in [9] are also equivalent. However, we can directly define the Legendrian equivalence between $\pi^-_{4i} \circ \mathcal{L}_4$ $(i = 1, 2)$ and $\pi^+_{31} \circ \mathcal{L}^\pm_3$ as follows: Let $\Phi^\pm_{43} : \Delta^-_4 \longrightarrow \Delta^+_3$ be mappings defined by $\Phi^+_{43}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v},\dfrac{\boldsymbol{w}-\boldsymbol{v}}{2})$ and $\Phi^-_{43}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{w},\dfrac{\boldsymbol{w}-\boldsymbol{v}}{2})$. By exactly the same reasons as the above, we can show that $\Phi^\pm_{43}$ give Legendrian equivalences between $\pi^-_{4i}\circ\mathcal{L}_4$ $(i=1,2)$ and $\pm\pi^+_{31}\circ\mathcal{L}^\pm_3$.

On the other hand, we have a proposition as a special case of Proposition 3.7 in [9] as follows:

Proposition 4.3 *Let $\mathcal{L}_4 : U \longrightarrow \Delta^-_4$ be a Legendrian embedding with $\mathcal{L}_4(u)=(\boldsymbol{X}^\ell_+(u),\boldsymbol{X}^\ell_-(u))$.*
(1) Suppose that $\boldsymbol{X}^\ell_+$ is an embedding. Then $\boldsymbol{X}^d(u)$ is a constant vector if and only if M^L_+ is a part of $LC^*\cap HP(\boldsymbol{v},-1)$ with $\boldsymbol{v}\in S^n_1$.
(2) Suppose that $\boldsymbol{X}^\ell_+$ is an embedding. Then $\boldsymbol{X}^\ell_-(u)$ is a constant vector if and only if M^L_+ is a part of $LC^*\cap HP(\boldsymbol{v},-2)$ with $\boldsymbol{v}\in LC^*$.
(3) Suppose that $\boldsymbol{X}^\ell_-$ is an embedding. Then $\boldsymbol{X}^h(u)$ is a constant vector if and only if M^L_- is a part of $LC^*\cap HP(\boldsymbol{v},-1)$ with $\boldsymbol{v}\in H^n(-1)$.

We respectively call $LC^*\cap HP(\boldsymbol{v},-1)$ with $\boldsymbol{v}\in S^n_1$, $LC^*\cap HP(\boldsymbol{v},-2)$ with $\boldsymbol{v}\in LC^*$ and $LC^*\cap HP(\boldsymbol{v},-1)$ with $\boldsymbol{v}\in H^n(-1)$, a *de Sitter flat hyperbolic hyperquadric*, a *lightcone flat parabolic hyperquadric* and a *hyperbolic flat elliptic hyperquadric*. In [9], the lightcone Gauss-Kronecker curvature for a spacelike hypersurface M^L_+ was introduced by using $\boldsymbol{X}^\ell_-$ as a Gauss map. Actually, it is defined by $K^\ell(u)=\det(-d\boldsymbol{X}^\ell_-(u))$. The lightcone flat parabolic hyperquadric is totally flat in this sense. By the above proposition, we have three kinds of totally flat spacelike hypersurfaces in the lightcone. Therefore, we are interested in the relations among these flatness.

We consider the contact manifold $(\Delta^-_{43}(\phi),K[\phi]^-_{43})$ and the contact diffeomorphism $\Psi^-_{4(43)} : \Delta^-_4 \longrightarrow \Delta^-_{43}(\phi)$ defined by

$$\Psi^-_{4(43)}(\boldsymbol{v},\boldsymbol{w})=\left(\boldsymbol{v},\frac{1}{2}\left((\cos\phi-1)\boldsymbol{v}+(\cos\phi+1)\boldsymbol{w}\right)\right).$$

We define a map $\mathbb{N}^d_\ell[\phi] : U \longrightarrow S^n_1(\sin^2\phi)$ by

$$\mathbb{N}^d_\ell[\phi](u)=\frac{1}{2}((\cos\phi-1)\boldsymbol{X}^\ell_+(u)+(\cos\phi+1)\boldsymbol{X}^\ell_-(u)),$$

for $\phi\in[0,\pi/2]$. We also define an embedding $\mathcal{L}_{43}[\phi] : U \longrightarrow \Delta^-_{43}(\phi)$ by

$$\mathcal{L}_{43}[\phi](u)=(\boldsymbol{X}^\ell_+(u),\mathbb{N}^d_\ell[\phi](u)).$$

Then we have $\mathcal{L}_{43}[\phi]=\Psi^-_{4(43)}\circ\mathcal{L}_4$, so that $\mathcal{L}_{43}[\phi]$ is a Legendrian embedding. Therefore, we have $\langle d\boldsymbol{X}^\ell_+,\mathbb{N}^d_\ell[\phi]\rangle=\mathcal{L}_{43}[\phi]^*\theta[\phi]^-_{(43)1}=0$. This means that $\mathbb{N}^d_\ell[\phi](u)$ can be considered as a normal vector of M^L_+ at $p=\boldsymbol{X}^\ell_+(u)$. We remark that $\mathbb{N}^d_\ell[0](u)=\boldsymbol{X}^\ell_-(u)$ and $\mathbb{N}^d_\ell[\pi/2](u)=\boldsymbol{X}^d(u)$. Then we have the following proposition.

Proposition 4.4 *Suppose that $\boldsymbol{X}^\ell_+$ is an embedding. Then $\mathbb{N}^d_\ell[\phi](u)$ is a constant vector if and only if M^L_+ is a part of $LC^*\cap HP(\boldsymbol{v},-(\cos\phi+1))$ with $\boldsymbol{v}\in S^n_1(\sin^2\phi)$.*

Proof. Suppose that $\mathbb{N}^d_\ell[\phi](u)=\boldsymbol{v}$. Then we have $\langle\boldsymbol{X}^\ell_+(u),\boldsymbol{v}\rangle=\langle\boldsymbol{X}^\ell_+(u),\mathbb{N}^d_\ell[\phi](u)\rangle=0$. This means that $M^L_+\subset LC^*\cap HP(\boldsymbol{v},-(\cos\phi+1))$. For the converse, suppose that $M^L_+\subset LC^*\cap HP(\boldsymbol{v},-(\cos\phi+1))$ with $\boldsymbol{v}\in S^n_1(\sin^2\phi)$. Since $\boldsymbol{v}$ is a normal vector of M^L_+ in $\mathbb{R}^{n+1}_1$, there exist real numbers λ,μ such that $\boldsymbol{v}=\lambda\boldsymbol{X}^\ell_+(u)+\mu\boldsymbol{X}^\ell_-(u)$. By definition, we have $-(\cos\phi+1)=\langle\boldsymbol{X}^\ell_+(u),\boldsymbol{v}\rangle=-2\mu$ and $\sin^2\phi=-4\lambda\mu$, so that $2\lambda=\cos\phi-1$. It follows that $\boldsymbol{v}=\mathbb{N}^d_\ell[\phi](u)$. $\Box$

We call $LC^*\cap HP(\boldsymbol{v},-(\cos\phi+1))$ with $\boldsymbol{v}\in S^n_1(\sin^2\phi)$ a *ϕ-de Sitter flat hyperbolic hyperquadric.*

On the other hand, we consider the contact manifold $(\Delta^-_{42}(\phi),K[\phi]^-_{42})$ and the contact diffeomorphism $\Psi^-_{4(42)}:\Delta^-_4\longrightarrow\Delta^-_{42}(\phi)$ defined by

$$\Psi^-_{4(42)}(\boldsymbol{v},\boldsymbol{w})=\left(\frac{1}{2}\left((1+\cos\phi)\boldsymbol{v}+(1-\cos\phi)\boldsymbol{w}\right),\boldsymbol{w}\right).$$

We define a map $\mathbb{N}^h_\ell[\phi]:U\longrightarrow H^n(-\sin^2\phi)$ by

$$\mathbb{N}^h_\ell[\phi](u)=\frac{1}{2}((1+\cos\phi)\boldsymbol{X}^\ell_+(u)+(1-\cos\phi)\boldsymbol{X}^\ell_-(u)),$$

for $\phi\in[0,\pi/2]$ and have a map $\mathcal{L}_{42}[\phi]:U\longrightarrow\Delta^-_{42}(\phi)$ defined by $\mathcal{L}_{42}[\phi](u)=(\mathbb{N}^h_\ell[\phi](u),\boldsymbol{X}^\ell_-(u))$. By exactly the same reason as the above case, $\mathcal{L}_{42}[\phi]$ is a Legendrian embedding, so that $\mathbb{N}^h_\ell[\phi](u)$ can be considered as a normal vector of M^L_- at $p=\boldsymbol{X}^\ell_-(u)$. We remark that $\mathbb{N}^h_\ell[0](u)=\boldsymbol{X}^\ell_+(u)$ and $\mathbb{N}^h_\ell[\pi/2](u)=\boldsymbol{X}^h(u)$. Then we have the following proposition.

Proposition 4.5 *Suppose that $\boldsymbol{X}^\ell_-$ is an embedding. Then $\mathbb{N}^h_\ell[\phi](u)$ is a constant vector if and only if M^L_- is a part of $LC^*\cap HP(\boldsymbol{v},-(1+\cos\phi))$ with $\boldsymbol{v}\in H^n(-\sin^2\phi)$.*

Since the proof of Proposition 4.5 is given by exactly the same arguments as those of Proposition 4.4, we omit it. We call $LC^*\cap HP(\boldsymbol{v},-(1+\cos\phi))$ with $\boldsymbol{v}\in H^n(-\sin^2\phi)$ a *ϕ-hyperbolic flat elliptic hyperquadric.*

We call both the geometry related to the Gauss maps $\mathbb{N}^d_\ell[\phi]$ and $\mathbb{N}^h_\ell[\phi]$ a *slant geometry of spacelike hypersurfaces in the lightcone.* The detailed arguments on the slant geometry will be appeared in the forthcoming paper [16].

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