Generic bifurcations of framed curves in a space form and their envelopes

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1 Introduction.

A curve endowed with a frame, called *a framed curve*, in a space-form plays important roles in topology, geometry and singularity theory. For example, as it is well-known, the self-linking number in 3-space is defined via framing ([28]). The fundamental theory of curves is formulated via osculation framing. Surface boundaries have adapted framings, etc. Two kinds of frames, adapted frames and osculating frames, are considered in this paper from the viewpoint of duality. We classify the singularities of envelopes associated to framed curves. The singularities of envelopes in E^3 were studied in [12] to apply to the flat extension problem of a surface with boundary. The problem on extensions by tangentially degenerate surfaces motivates to study the envelopes associated to framings on curves in a space form. In this paper we consider framed curves in $X = E^{n+1}$, Euclidean space, S^{n+1}, the sphere or H^{n+1}, the hyperbolic space of dimension $n+1$, and we try to understand them commonly in terms of projective geometry.

Actually we work with the models

$$S^{n+1} = \{x \in \mathbf{R}^{n+2} \mid x^2 = 1\}, \quad H^{n+1} = \{x \in \mathbf{R}^{1,n+1} \mid x^2 = -1,\ x_0 > 0\},$$

where $\mathbf{R}^{1,n+1} = \mathbf{R}^{n+2}_1 = \{(x_0, x_1, \dots, x_{n+1})\}$ is the Minkowski space of index $(1, n+1)$ (See for instance [17][8]). The inner product in $\mathbf{R}^{1,n+1}$ is defined by $x \cdot y = -x_0y_0 + \sum_{i=1}^{n+1} x_iy_i$. Moreover we identify Euclidean space E^{n+1} with $\{x \in \mathbf{R}^{n+2} \mid x_0 = 1\} \subset \mathbf{R}^{n+2}$ if necessary.

Let $\gamma : I \to X$ be a C^∞ immersion from an interval or a circle I. In general, we mean by a *framing* of the immersed curve γ, an oriented orthonormal frame $(e_1, e_2, \dots, e_{n+1})$ along γ. We always pose the condition that e_{n+1} is orthogonal to the velocity vector γ'. Then the unit normal vectors e_{n+1} provide a 1-parameter family of tangent hyperplanes to γ and its envelope $E(\gamma)$.

In particular, in three dimensional case ($n = 2$), if a framed curve γ is given, then we have a 1-parameter family of planes and its envelope surface $E(\gamma)$ in three space. For a 1-parameter family of framed curves γ_λ, we have the 1-parameter family of envelopes $E(\gamma_\lambda)$. Then we will show

Theorem 1.1 *Let γ_λ be a generic 1-parameter family of framed curves in E^3, S^3 or H^3. Then the local singularity in the associated envelope $E(\gamma_\lambda)$ is given by one of following 5-classes:* (I) *the cuspidal edge,* (II) *the swallowtail,* (III) *the cuspidal beaks,* (IV) *the cuspidal butterfly, and* (V) *the full-folded-umbrella.*

In particular, the list of singularities (diffeomorphism classes), is the same for all of three geometries.

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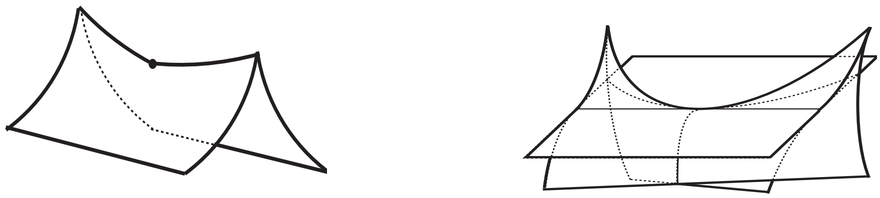
the swallowtail and the cuspidal beaks



the cuspidal butterfly and the full-folded-unbrella

The cuspidal edges and the swallowtails appear generically and stably. The swallowtails can appear in isolated positions on the envelope. The cuspidal beaks, the cuspidal butterflies, or the full-folded-umbrellas appear in isolated positions momentarily at isolated value λ. Along the parameter λ, both the cuspidal beaks and butterflies bifurcate into just cuspidal edges and swallowtails. Note that the cuspidal beaks and butterflies appear as singularities of wave fronts of codimension one in Arnold's theory ([2][3]). However, as we see later, they have different characters in our theory. The full-folded-umbrella is not a wavefront (image of a non-singular Legendre submanifold), but, a frontal surface (image of a singular Legendre variety).

The singularities of envelopes are closely related to singularities of tangent developables of curves. Tangent developables are flat in E^3. However they are not flat but "extrisically flat" or tangentially degenerate in S^3 and H^3 (cf. [1][23]). In this paper the notion of *types* (a_1, a_2, a_3) for a curve-germ is introduced and the cuspidal edge, (resp. the swallowtail, the cuspidal beaks, the cuspidal butterfly) is obtained as the tangent developable of a curve of type $(1, 2, 3)$ (resp. $(2, 3, 4)$, $(1, 3, 4)$, $(3, 4, 5)$). The cuspidal beaks are called also *Mond surfaces* [10]. (See also [26][27]). We have adopted the notations in [20]. We remark that the cuspidal butterflies bifurcate within tangent developables, however, the cuspidal beaks (Mond surfaces) do not. In fact we observe that the Mond surface is stable for the deformations of curves with *osculating frames*.

The full-folded-umbrella contains the tangent developable of a curve of type $(1, 2, 4)$. Each singularity mentioned above is given by the generating family

$$F(t, x_1, x_2, x_3) = \frac{t^{a_3}}{a_3!} + x_1 \frac{t^{a_3 - a_1}}{(a_3 - a_1)!} + x_2 \frac{t^{a_3 - a_2}}{(a_3 - a_2)!} + x_3 = 0,$$

where the normal form of the envelope is given by $\{(x_1, x_2, x_3) \mid F = \frac{\partial F}{\partial t} = 0 \text{ for some } t\}$ ([9]).

In this paper two kinds of frames are involved: one is an *adapted frame* of γ which satisfies just the condition $e_1 = \gamma'$, the unit velocity vector field, or the differential by the arc-length parameter. Then e_{n+1} is orthogonal to γ'. For the classification problem of envelops just that e_{n+1} is orthogonal to γ' is essential. Another is the Frenet-Serret frame of γ along ordinary points where the derivatives $\gamma'(t), \gamma''(t), \dots, \gamma^{(n)}(t)$, are linearly independent. Then our main idea is to introduce two kinds of distributions, or differential systems, on flag manifolds and regard framed curves as integral curves to those distributions.

Bifurcations of wavefronts based on Legendre singularity theory are established by Arnold-Zakalyukin's theory ([2][3][4][31][32]). The application of singularity theory to differential geometry has been developed by many authors (see for instance [6]). The geometric study of submanifolds in hyperbolic space H^{n+1} based on singularity theory was initiated by Izumiya et al. ([17][18][19]). The Legendre duality developed in [16][7] enables us to unify the theory of framed curves in any space form as describes in this paper.

In §2, we recall Legendre duality (see [5][15][7]) within the level we need in this paper. We understand the duality in the framework of moving frames and flags in §3. After considering non-oriented flags in §4, we introduce two distributions in §5. They are very essential to study the bifurcation problem of envelopes in this paper. In §6, the notion of type of curves is introduced and that of osculating flags is considered. Two kinds of framed curves are regarded as integral curves to two kinds of distributions. Then we prove codimension formulae for framed curves in §7. We show the classification results of singularities of envelopes including Theorem1.1 in §8, by using codimension formulae and the transversality theorem.

2 Legendre duality.

In this paper we mainly treat curves in Riemannian spaces $X = E^{n+1}, S^{n+1}, H^{n+1}$. However, regarding the duality, naturally we work in other spaces as well. In particular we are led to consider *de Sitter space*

$$S^{1,n} = \{x \in \mathbf{R}^{1,n+1} \mid x^2 = 1\},$$

which is a semi-Riemannian manifold, since any vector of a frame $(e_1, \dots, e_{n+1})$ along a curve in H^{n+1} belongs to $S^{1,n}$.

We regard $\widehat{\gamma} = e_{n+1}$ a curve in $Y = S^{n+1}$ (resp. in $Y = S^{1,n}$) if $X = S^{n+1}$ (resp. $X = H^{n+1}$). In Euclidean case, we set $\widehat{\gamma} = (-\gamma \cdot e_{n+1}, e_{n+1})$ and regard it as a curve in $Y = \mathbf{R} \times S^n$. We call $\widehat{\gamma}$ the *frame dual* to γ. Then, in any case, the "type" of the curve $\widehat{\gamma}$ in Y describes the singularities of the envelope $E(\gamma)$ in X.

We denote by $Z = \widetilde{\mathrm{Gr}}(n, TX)$ the manifold of oriented tangent hyperplanes of X and by $\pi_1 : Z \to X$ the projection which maps a hyperplane $\Pi \subset T_xX$ to $x \in X$. A framed curve $\gamma : I \to X$ with the framing $(e_1, \dots, e_{n+1})$ lifts to a curve $\widetilde{\gamma} : I \to Z$ which is defined by $\widetilde{\gamma}(t) = \langle e_1(t), \dots, e_n(t)\rangle_{\mathbf{R}}$. In each of three cases, Z is identified with the unit tangent bundle T_1X via the metric, actually with $T_1E^{n+1} = E^{n+1} \times S^n$,

$$\begin{array}{rcl} T_1S^{n+1} & = & \{(x, y) \in S^{n+1} \times S^{n+1} \mid x \cdot y = 0\}, \quad \text{and,} \\ T_1H^{n+1} & = & \{(x, y) \in H^{n+1} \times S^{1,n} \mid x \cdot y = 0\}. \end{array}$$

Then, under the above identification, the lifting $\widetilde{\gamma} : I \to Z$ is given by $\widetilde{\gamma}(t) = (\gamma(t), e_{n+1}(t))$ ([14]).

Consider the contact structure on Z: the one-form $\theta = v \cdot dx$ on $E^{n+1} \times E^{n+1}$ restricted to $Z = E^{n+1} \times S^n$, $\theta = y \cdot dx$ on $\mathbf{R}^{n+2} \times \mathbf{R}^{n+2}$ restricted to $Z = T_1S^{n+1}$ or T_1H^{n+1}, is a contact form on Z. In elliptic or hyperbolic case, let $\pi_2 : Z \to Y$ be the projection defined by $\pi_2(x, y) = y$. In Euclidean case, let $\pi_2 : Z \to Y$ be the projection defined by $\pi_2(x, y) = (-x \cdot y,\ y)(x \in E^{n+1}, y \in S^n)$. Then we see both $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$ are Legendre fibrations.

Suppose the framing of $\gamma : I \to X$ satisfies the condition $e_1 = \gamma'$. Then e_{n+1} is normal to γ'. Then we see that the lifting $\widetilde{\gamma} : I \to Z$ of $\gamma : I \to X$ turns to be *integral* in the sense that $\widetilde{\gamma}^*\theta = 0$. The lifting $\widetilde{\gamma} : I \to Z$ of a framed immersion $\gamma : I \to X$ defines a "sub-front" $\widehat{\gamma} = \pi_2 \circ \widetilde{\gamma} : I \to Y$ possibly with singularities, in the sense that the integral lifting $\widetilde{\gamma}$ with respect to π_2 is attached to the just parametrised curve $\widehat{\gamma}$ in Y.

Note that, in the case $X = H^{n+1}$, $Z = T_1 H^{n+1}$ is identified with $T_{-1}S^{1,n}$, the manifold of tangent vectors $v \in T_y S^{1,n}$ with $v^2 = -1$ to the semi-Riemannian manifold $S^{1,n}$ ([14]).

As the model of duality, we do have the projective duality ([29][15]): We set

$$Z \;=\; \mathcal{I}_{n+2} \;:=\; \{([x],[y]) \in P^{n+1} \times P^{n+1*} \mid x \cdot y = 0\}.$$

Here P^{n+1*} is the dual projective space and $\cdot$ means the natural paring. The contact structure on $\mathcal{I}_{n+2}$ is defined by $dx \cdot y = x \cdot dy = 0$ ([15]). The projections $\pi_1 : \mathcal{I}_{n+2} \to X = P^{n+1}, \pi_2 : \mathcal{I}_{n+2} \to Y = P^{n+1*}$ are both Legendre fibrations.

The following fact is basic to unify our treatment:

Proposition 2.1 ([13][14]) *All Legendre double fibrations $X \longleftarrow Z \longrightarrow Y$ constructed above are locally isomorphic to each other. In particular each of them is locally isomorphic to the double fibration of the projective duality $P^{n+1} \longleftarrow \mathcal{I}_{n+2} \longrightarrow P^{n+1*}$.*

The proof of Proposition 2.1 is achieved via the underlying flag structure that we are going to explain.

3 Moving frames and flags.

For a framed curve $\gamma : I \to X = E^{n+1}, S^{n+1}, H^{n+1}$, naturally we will associate a "moving frame" $\widetilde{\gamma} : I \to G$, for each of three cases, in a Lie subgroup G of $\mathrm{GL}_+(n+2, \mathbf{R})$, where $\mathrm{GL}_+(n+2, \mathbf{R})$ is the Lie group of regular matrices with positive determinant.

If $X = E^{n+1}$, then we set $e_0(t) = \gamma(t) \in E^{n+1}$, and we have the moving frame $\widetilde{\gamma} = (e_0, e_1, \dots, e_{n+1}) : I \to G = \mathrm{Euc}(E^{n+1}) \subset \mathrm{GL}_+(n+2, \mathbf{R})$ in the group of orientation preserving Euclidean motions on E^{n+1}.

If $X = S^{n+1}$, then we set $e_0(t) = \gamma(t) \in S^{n+1}$, and we have the moving frame $\widetilde{\gamma} = (e_0, e_1, \dots, e_{n+1}) : I \to G = SO(n+2) \subset \mathrm{GL}_+(n+2, \mathbf{R})$.

If $X = H^{n+1}$, then we set $e_0(t) = \gamma(t) \in H^{n+1}$, and we have the moving frame $\widetilde{\gamma} = (e_0, e_1, \dots, e_{n+1}) : I \to G = SO(1, n+1) \subset \mathrm{GL}_+(n+2, \mathbf{R})$.

In any of three cases, the frame manifold G is identified with an open subset of the oriented flag manifold $\widetilde{\mathcal{F}}_{n+2}$ consisting of oriented complete flags

$$V_1 \subset V_2 \subset \cdots \subset V_{n+1} \subset \mathbf{R}^{n+2}$$

in $\mathbf{R}^{n+2}$. For each $g = (e_0, e_1, \dots, e_{n+1}) \in \mathrm{GL}_+(n+2, \mathbf{R})$, we set the oriented subspace

$$V_i = \langle e_0, e_1, \dots, e_{i-1} \rangle_{\mathbf{R}} \subset \mathbf{R}^{n+2}, \ (1 \le i \le n+1).$$

This induces an open embedding $G \to \widetilde{\mathcal{F}}_{n+2}$. Note that $\widetilde{\mathcal{F}}_{n+2}$ is the quotient space of $\mathrm{GL}_+(n+2, \mathbf{R})$ by upper triangular matrices, and $\dim G = \dim \widetilde{\mathcal{F}}_{n+2} = \frac{(n+1)(n+2)}{2}$. Moreover note that the inner product restricted to each V_i is non-degenerate. Therefore G is embedded in non-degenarate flags $\widetilde{\mathcal{F}}^0_{n+2} \subset \widetilde{\mathcal{F}}_{n+2}$ consisting of flags $(V_1, \dots, V_{n+1})$ where the inner product restricted to each V_i is non-degenerate. Remark that $\widetilde{\mathcal{F}}^0_{n+2}$ is open dense in

$\widetilde{\mathcal{F}}_{n+2}$. However note that in [16][7][20], more general framings are considered to treat also *the light cone* in Minkowski space.

Thus, for a framed curve $\gamma : I \to X$ in $X = E^{n+1}, S^{n+1}, H^{n+1}$, with the frame $(e_1, \dots, e_{n+1})$, we have the curve $\widetilde{\gamma} : I \to \widetilde{\mathcal{F}}^0_{n+2}$ by setting

$$V_i(t) = \langle e_0(t), e_1(t), \dots, e_{i-1}(t)\rangle_{\mathbf{R}} \subset \mathbf{R}^{n+2}, \ (1 \leq i \leq n+1).$$

Then $\widetilde{\gamma}$ is a lifting of γ for the projection $\pi_1 : \widetilde{\mathcal{F}}^0_{n+2} \to \widetilde{\mathrm{Gr}}(1, \mathbf{R}^{n+2})$ to Grassmannian of oriented lines in $\mathbf{R}^{n+2}$. Note that there is the natural open embedding $X \subset \widetilde{\mathrm{Gr}}(1, \mathbf{R}^{n+2})$ in each of three cases.

4 Reduced Legendre duality.

The vector space $\mathbf{R}^{n+2}$ has the $\mathbf{Z}/2\mathbf{Z}$-action defined by $x \mapsto -x$. To describe the duality, it is natural to take the quotient and set

$$\begin{aligned} AG(n, n+1) \ &:= \ \{(r, y) \in \mathbf{R} \times S^n\}/(\mathbf{Z}/2\mathbf{Z}), \quad & P^{n+1} \ &:= \ \{x \in \mathbf{R}^{n+2} \mid x^2 = 1\}/(\mathbf{Z}/2\mathbf{Z}), \\ H^{n+1} \ &:= \ \{x \in \mathbf{R}^{1,n+1} \mid x^2 = -1\}/(\mathbf{Z}/2\mathbf{Z}), \ \text{and}, & P^{1,n} \ &:= \ \{x \in \mathbf{R}^{1,n+1} \mid x^2 = 1\}/(\mathbf{Z}/2\mathbf{Z}). \end{aligned}$$

We call P^{n+1} the *elliptic space* and $P^{1,n}$ the *reduced de Sitter space*. Remark that $AG(n, n+1)$ is identified with the set of affine non-oriented hyperplanes in E^{n+1}. We regard P^{n+1} (resp. $P^{1,n}$) as the double-quotient of the sphere S^{n+1} (resp. $S^{1,n}$) with the induced metric. Set $X = E^{n+1}, P^{n+1}, H^{n+1}$ in Euclidean, elliptic, hyperbolic case, respectively. Then we set $Y = AG(n, n+1), P^{n+1}, P^{1,n}$ respectively.

We consider the incidence manifold in each geometry:

$$Z \ := \ \{([x], [y]) \in X \times Y \mid x \cdot y = 0\},$$

for elliptic and hyperbolic cases, and

$$Z \ := \ \{(x, [r, y]) \in X \times Y \mid x \cdot y + r = 0\},$$

for Euclidean case. In each case, Z is regarded naturally as an open subset of PT^*X and is endowed with the standard contact structure. Then the double fibrations $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$ are Legendre. Moreover all Legendre double fibrations $X \longleftarrow Z \longrightarrow Y$ are locally isomorphic to the projective duality $P^{n+1} \longleftarrow \mathcal{I}_{n+2} \longrightarrow P^{n+1*}$ as in Proposition 2.1.

Let $\mathcal{F}_{n+2}$ be the manifold of non-oriented complete flags

$$V_1 \subset V_2 \subset \cdots \subset V_{n+1} \subset \mathbf{R}^{n+2},$$

consisting of vector subspaces V_i of dimension i in $\mathbf{R}^{n+2}$. The forgetful mapping $\pi : \widetilde{\mathcal{F}}_{n+2} \to \mathcal{F}_{n+2}$ forms a covering of order 2^{n+1}. For a framed curve in a reduced space, the lifting is a curve in a non-oriented flag manifold.

5 Pseudo-contact and canonical distributions.

We will consider two classes of curves in the frame manifold $\widetilde{\mathcal{F}}_{n+2}$, by introducing two kinds of distributions $\widetilde{\mathcal{C}} \subset \widetilde{\mathcal{D}} \subset T\widetilde{\mathcal{F}}_{n+2}$. Denote by $\pi_i : \widetilde{\mathcal{F}}_{n+2} \to \widetilde{\mathrm{Gr}}(i, \mathbf{R}^{n+2})$ the projection to

Grassmannian of oriented i-planes in $\mathbf{R}^{n+2}$ defined by $\pi_i(V_1, \dots, V_i, \dots, V_{n+1}) = V_i$. Then we define, for $v \in T\widetilde{\mathcal{F}}_{n+2}$, $v \in \widetilde{\mathcal{D}}_{(V_1,\dots,V_{n+1})}$ if $\pi_{1*}(v) \in T\widetilde{\mathrm{Gr}}(1, V_{n+1})(\subset T\widetilde{\mathrm{Gr}}(1, \mathbf{R}^{n+2}))$, while $v \in \widetilde{\mathcal{C}}_{(V_1,\dots,V_{n+1})}$ if $\pi_{i*}(v) \in T\widetilde{\mathrm{Gr}}(i, V_{i+1})(\subset T\widetilde{\mathrm{Gr}}(i, \mathbf{R}^{n+2})), (1 \leq i \leq n)$.

We call the distribution $\widetilde{\mathcal{D}}$ *pseudo-contact distribution* and $\widetilde{\mathcal{C}}$ *canonical distribution.* The canonical distribution was already used by Scherbak for the study of the projective duality ([29]). Note that the rank of $\widetilde{\mathcal{C}}$ (resp. $\widetilde{\mathcal{D}}$) is $n+1$ (resp. $\frac{(n+1)(n+2)}{2} - 1$) in $T\widetilde{\mathcal{F}}_{n+2}$. Both $\widetilde{C}$ and $\widetilde{D}$ are bracket generating. In fact, n-th bracket $\widetilde{\mathcal{C}}^n$ of $\widetilde{\mathcal{C}}$ coincides with $\widetilde{\mathcal{D}}$. Denote by $\widetilde{\mathcal{I}}_{n+2}$ the flag manifold consisting of flag $V_1 \subset V_{n+1} \subset \mathbf{R}^{n+2}$ with an oriented line V_1 and an oriented hyperplanes V_{n+1}. Consider the canonical projection $\pi_{1,n+1} : \widetilde{\mathcal{F}}_{n+2} \longrightarrow \widetilde{\mathcal{I}}_{n+2}$ defined by $\pi_{1,n+1}(V_1, V_2, \dots, V_{n+1}) = (V_1, V_{n+1})$. Then we have $\widetilde{\mathcal{D}} = (\pi_{1,n+1})_*^{-1}(D)$, the pull-back of the contact structure D on $\widetilde{\mathcal{I}}_{n+2}$: for $v \in T\widetilde{\mathcal{I}}_{n+2}$, $v \in \widetilde{D}_{(V_1,V_{n+1})}$ if $(\pi_1)_*(v) \in T\widetilde{\mathrm{Gr}}(1, V_{n+1})$. The contact structure D on $\widetilde{\mathcal{I}}_{n+2}$ is the pull-back of the contact structure on $\mathcal{I}_{n+2}$ introduced in §2.

Similar constructions go as well for non-oriented case.

Define two distributions (vector sub-bundles) $\mathcal{C} \subset \mathcal{D} \subset T\mathcal{F}_{n+2}$ on the non-oriented flag manifold $\mathcal{F}_{n+2}$ as follows: For $v \in T\mathcal{F}_{n+2}$, $v \in \mathcal{D}_{(V_1,\dots,V_{n+1})}$ if $\pi_{1*}(v) \in T\mathrm{Gr}(1, V_{n+1})(\subset T\mathrm{Gr}(1, \mathbf{R}^{n+2}))$, while $v \in \mathcal{C}_{(V_1,\dots,V_{n+1})}$ if $\pi_{i*}(v) \in T\mathrm{Gr}(i, V_{i+1})(\subset T\mathrm{Gr}(i, \mathbf{R}^{n+2})), (1 \leq i \leq n)$. We call also the distribution $\mathcal{D}$ *pseudo-contact distribution* and $\mathcal{C}$ *canonical distribution.* Clearly the forgetful covering $\pi : \widetilde{\mathcal{F}}_{n+2} \to \mathcal{F}_{n+2}$ induces a local isomorphism of $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ (resp. $\widetilde{\mathcal{C}}$ and $\mathcal{C}$). The pseudo-contact structure $\mathcal{D}$ is the pull-back of the contact structure on $\mathcal{I}_{n+2}$ via the canonical projection $\pi_{1,n+1} : \mathcal{F}_{n+2} \to \mathcal{I}_{n+2}$.

Now we describe the local structure of the canonical distribution $\mathcal{C} \subset T\mathcal{F}_{n+2}$. Since $\mathcal{C}$ is $\mathrm{GL}(n+2, \mathbf{R})$-invariant, we describe $\mathcal{C}$ in a neighbourhood of the standard flag $E \in \mathcal{F}_{n+2}$ which corresponds to the unit matrix. The flag manifold has local coordinates $x_i^{\,j}, (0 \leq j < i \leq n+1)$ near E as components of lower triangular matrices. Then $\mathcal{C}$ is defined by the system of 1-forms

$$dx_i^{\,j} - x_i^{\,j+1} dx_{j+1}^{\,j} = 0, \qquad (0 \leq j,\ j+1 < i).$$

Therefore a $\mathcal{C}$-integral curve $\Gamma(t) = (x_i^{\,j}(t))_{0 \leq j < i \leq n+1}$ through the standard flag $E \in \mathcal{F}_{n+2}$ is determined just by $x_j^{j-1}(t), (1 \leq j \leq n+1)$.

Remark 5.1 The complete flag manifold $\mathcal{F}_{n+2} = \mathcal{F}(\mathbf{R}^{n+2})$ (resp. $\widetilde{\mathcal{F}}_{n+2} = \widetilde{\mathcal{F}}(\mathbf{R}^{n+2})$) possesses the duality between $\mathcal{F}^*_{n+2} = \mathcal{F}(\mathbf{R}^{n+2*})$ (resp. $\widetilde{\mathcal{F}}^*_{n+2} = \widetilde{\mathcal{F}}(\mathbf{R}^{n+2*})$) by

$$(V_1, V_2, \dots, V_n, V_{n+1}) \mapsto (V_{n+1}^\vee, V_n^\vee, \dots, V_2^\vee, V_1^\vee)$$

where $V^\vee \subset \mathbf{R}^{n+2*}$ is the annihilator for $V \subset \mathbf{R}^{n+2}$. Then, for each metric on $\mathbf{R}^{n+2}$, the dual space $\mathbf{R}^{n+2*}$ is identified with $\mathbf{R}^{n+2}$. Thus we have the canonical involution on $\mathcal{F}(\mathbf{R}^{n+2})$ (resp. $\widetilde{\mathcal{F}}(\mathbf{R}^{n+2}), \mathcal{F}(\mathbf{R}^{1,n+1}), \widetilde{\mathcal{F}}(\mathbf{R}^{1,n+1})$). Similarly we have the canonical involution on $\mathcal{I}(\mathbf{R}^{n+2})$ (resp. $\widetilde{\mathcal{I}}(\mathbf{R}^{n+2}), \mathcal{I}(\mathbf{R}^{1,n+1}), \widetilde{\mathcal{I}}(\mathbf{R}^{1,n+1})$).

6 Osculating flags on curves of finite type.

In general we treat a curve *of finite type* and define an analogue of Frenet-Serret frame even when the curve is not an immersion. Here, since $X \subset \mathbf{R}^{n+2} \setminus \{0\}$, we regard γ as a curve in $\mathbf{R}^{n+2} \setminus \{0\}$. The metric on $\mathbf{R}^{n+2}$ does not concern here.

Let $\gamma : I \to \mathbf{R}^{n+2} \setminus \{0\}$ be a C^∞ curve. The curve γ is called of finite type at $t = t_0 \in I$ if the $(n+2) \times \infty$-matrix

$$\tilde{A}(t) = \left(\gamma(t), \gamma'(t), \gamma''(t), \cdots, \gamma^{(r)}(t), \cdots\right),$$

is of rank $n+2$ for $t = t_0$. We set $(n+2) \times (r+1)$-matrix

$$\tilde{A}_r(t) = \left(\gamma(t), \gamma'(t), \gamma''(t), \cdots, \gamma^{(r)}(t)\right).$$

Then γ is of finite type at $t = t_0$ if $\tilde{A}_r(t)$ is of rank $n+2$ for a sufficiently large r.

Let $\boldsymbol{a} = (a_1, \ldots, a_n, a_{n+1})$ be a sequence of strictly increasing natural numbers, $1 \leq a_1 < \cdots < a_n < a_{n+1}$. Then we call γ of type $\boldsymbol{a}$ at $t = t_0 \in I$ if

$$\min\{r \mid \mathrm{rank}\tilde{A}_r(t_0) = 2\} = a_1, \quad \min\{r \mid \mathrm{rank}\tilde{A}_r(t_0) = 3\} = a_2, \quad \ldots,$$
$$\min\{r \mid \mathrm{rank}\tilde{A}_r(t_0) = n+2\} = a_{n+1}.$$

We can define type for curves in the reduced space P^{n+1} as well, by just considering the double covering.

A point $\gamma(t_0)$ on γ is called an *ordinary point* if γ is of type $(1, 2, \ldots, n+1)$. Otherwise it is called a *special point*. The parameters of special points form discrete subset in I if γ is of finite type.

If γ is of type $\boldsymbol{a}$ at $t = t_0$, then we set

$$O_i(t_0) = \langle \gamma(t_0), \gamma'(t_0), \ldots, \gamma^{(a_{i-1})}(t_0) \rangle_{\mathbf{R}},$$

which is, by definition, an i-dimensional subspace of $\mathbf{R}^{n+2}$, $(1 \leq i \leq n+1)$. Then we have

Lemma 6.1 *The curve $\widetilde{\gamma} : I \to \mathcal{F}_{n+2}$ in the non-oriented flag manifold $\mathcal{F}_{n+2}$ defined by*

$$\widetilde{\gamma}(t) : O_1(t) \subset O_2(t) \subset \cdots \subset O_{n+1}(t) \subset \mathbf{R}^{n+2}$$

is a C^∞ curve. Moreover we can give an orientation on the flag locally near $t_0 \in I$. Namely we have local lifting of γ in $\widetilde{\mathcal{F}}_{n+2}$ for the forgetful covering $\pi : \widetilde{\mathcal{F}}_{n+2} \to \mathcal{F}_{n+2}$ from the manifold of oriented flags to those of non-oriented flags.

We call $\widetilde{\gamma}(t)$ the *osculating flag* of γ at $t \in I$.

Proof of Lemma 6.1. Consider $(n+2) \times (n+2)$-matrix $B(t) = (\gamma(t), \gamma^{(a_1)}, \ldots, \gamma^{(a_{n+1})})$. We may suppose, after a suitable linear transformation of γ in $\mathbf{R}^{n+2}$, that $B(t_0)$ is the unit matrix. Then the lower triangular components of the matrix $(\gamma(t), \gamma'(t), \ldots, \gamma^{(n+1)}(t))$ provides the local representation of $\widetilde{\gamma}$ in terms of local coordinates $\widetilde{\mathcal{F}}_{n+2}$ near $\widetilde{\gamma}(t_0)$. $\square$

Suppose γ is a curve in $X = E^{n+1}, S^{n+1}(\subset \mathbf{R}^{n+2})$ or $X = H^{n+1}(\subset \mathbf{R}^{1,n+1})$ and moreover suppose, in the case $X = H^{n+1}$, the restriction of the metric to each $O_i(t)$ is non-degenerate. If an oriented flag field along γ is given, then an orthonormal frame $(e_1, \ldots, e_{n+1})$ along γ is uniquely constructed by the Gram-Schmidt's orthogonalisation which depends on the given orientation. We call this frame on γ an *osculating frame*. For instance, there exists the unique unit vector $e_1(t) \in O_2(t)$ normal to $e_0(t) = \gamma(t)$ such that $(e_0(t), e_1(t))$ forms an oriented basis of $O_2(t)$. We see, by Lemma 6.1, any osculating frame constructed above is C^∞ along γ which coincides, up to sign pointwise, with Frenet-Serret frame on ordinary points.

7 Integral curves and codimension formula.

For an adapted framing, the lifting $\widetilde{\gamma} : I \to G$ is an integral curve to $\widetilde{\mathcal{D}}$. Moreover, for an osculating framing, $\widetilde{\gamma} : I \to G$ is an integral curve to $\widetilde{\mathcal{C}}$. (See §5.) Thus we regard the class of adapted framed curves as the class of $\widetilde{\mathcal{D}}$-integral curves in G or $\widetilde{\mathcal{F}}_{n+2}$ or, as being locally equivalent, the class of $\mathcal{D}$-integral curves in $\mathcal{F}_{n+2}$.

On the other hand, a curve of finite type $\gamma : I \to X$ lifts, via the osculating flag, to a $\mathcal{C}$-integral curve $\widetilde{\gamma} : I \to \mathcal{F}_{n+2}$ globally. Moreover γ lifts locally to a $\widetilde{\mathcal{C}}$-integral curve $\widetilde{\gamma} : I \to \widetilde{\mathcal{F}}_{n+2}$, which satisfies $e_1 = \pm\gamma'$ (arc-length differential) on immersive points pointwise.

Remark 7.1 A C^∞ family of curves of finite types $\gamma_\lambda : I \to X$ needs not to be liftable, even locally, as a C^∞ family of $\mathcal{C}$-integral curves $\widetilde{\gamma}_\lambda : I \to \mathcal{F}_{n+2}$. The osculating flags do not behave smoothly under arbitrary deformation of curves. This is why we consider also the class of $\mathcal{C}$-integral curves for the bifurcation problem of envelopes.

Now we consider three kinds of jet spaces of curves.

First, we recall the ordinary jet space $J^r(I, X)$ consisting of r-jets of curves $I \to X$ or $J^r(I, Y)$ for curves $I \to Y$. Their local descriptions are the same as in the case $X = Y = P^{n+1}$. O.P.Scherbak [29] shows that the codimension, called the *jet-codimension* Jet-codim($\boldsymbol{a}$), in the jet space $J^r(I, P^{n+1})$ of the set $\Sigma(\boldsymbol{a})$ of curves in P^{n+1} of type $\boldsymbol{a} = (a_1, a_2, \dots, a_{n+1})$ is given, for sufficiently large r, by

$$\text{Jet-codim}(\boldsymbol{a}) \ = \ s(\boldsymbol{a}) \ := \ \sum_{i=1}^{n+1}(a_i - i),$$

the Schubert number which appears in Schubert calculus ([25][22]).

Second, we consider the jet space of $\mathcal{D}$-integral curves, $J^r_{\mathcal{D}}(I, \mathcal{F}_{n+2}) \subset J^r(I, \mathcal{F}_{n+2})$. Each $\mathcal{D}$-integral curve $\Gamma : I \to \mathcal{F}_{n+2}$ projects to a curve $\pi_1 \circ \Gamma : I \to P^{n+1} = \mathrm{Gr}(1, \mathbf{R}^{n+2})$ by the canonical projection $\pi_1 : \mathcal{F}_{n+2} \to P^{n+1}$, $\pi_1(V_1, V_{n+1}) = V_1$. Then, given type $\boldsymbol{a}$, we have the set of jets $\Sigma_{\mathcal{D}}(\boldsymbol{a})$ in $J^r_{\mathcal{D}}(I, \mathcal{F}_{n+2})$. We denote its codimension by Jet-codim$_{\mathcal{D}}(\boldsymbol{a})$.

Then we have

Theorem 7.2 *The jet-codimension of the set of $\mathcal{D}$-integral curves $\Gamma : I \to \mathcal{F}_{n+2}$ such that $\pi_1 \circ \Gamma$ is of type $\boldsymbol{a} = (a_1, a_2, \dots, a_{n+1})$, is given by*

$$\text{Jet-codim}_{\mathcal{D}}(\boldsymbol{a}) = \sum_{i=2}^{n+1}(a_i - i) \ = \ s(\boldsymbol{a}) - (a_1 - 1).$$

Proof: We may suppose $t_0 = 0$. Consider the integral jet space $J^r_{\rm int}(1, 2n+1)$ on germs of integral curves $\Gamma : I \to \mathcal{I}_{n+2}$ to the contact structure. Denote by $\Sigma_{a_1} \subset J^r_{\rm int}(1, 2n+1)$ the set of integral jet $j^r\Gamma$ with $\pi_1 \circ \Gamma$ is of order $\geq a_1$. Then we see that Σ_{a_1} is a submanifold of $J^r_{\rm int}(1, 2n+1)$ of codimension $n(a_1 - 1)$. Take Darboux coordinates $x_1, \dots, x_n, z, p_1, \dots, p_n$ of $\mathcal{I}_{n+2}$ centred at $\Gamma(t_0)$ and so that the contact structure is given by $dz - (p_1 dx_1 + \cdots + p_n dx_n) = 0$ and $\pi_1 : \mathcal{I}_{n+2} \to \mathrm{Gr}(1, \mathbf{R}^{n+2})$ is given by $(x_1, \dots, x_n, z, p_1, \dots, p_n) \mapsto (x_1, \dots, x_n, z)$. Let $\Gamma(t) = (x_1(t), \dots, x_n(t), z(t), p_1(t), \dots, p_n(t))$. Without loss of generality, we suppose $\mathrm{ord} x_1 = a_1$. Consider the mapping $\Pi : \Sigma_{a_1}(\subset J^r_{\rm int}(1, 2n+1)) \to J^{r-a_1+1}(1, n+1)$ defined by

$$\Pi j^r(x_1, \dots, x_n, z, p_1, \dots, p_n) = j^{r-a_1+1}(x_1/t^{a_1-1}, \dots, x_n/t^{a_1-1}, z/t^{a_1-1}).$$

Take any deformation $c(t,s) = (X_1(t,s), \ldots, X_n(t,s), Z(t,s))$ of $\Pi(j^r\Gamma)$ at $s = 0$. We set

$$\begin{aligned} P_1(t,s) &:= \frac{(t^{a_1-1}Z(t,s))'}{(t^{a_1-1}X_1(t,s))'} - \textstyle\sum_{i=2}^n p_i(t)\frac{(t^{a_1-1}X_i(t,s))'}{(t^{a_1-1}X_1(t,s))'}, \\ P_i(t,s) &:= p_i(t), (i = 2, \ldots, n), \end{aligned}$$

for representatives at $(t,s) = (0,0)$. Then we get the integral deformation

$$C(t,s) = (X_1(t,s), \ldots, X_n(t,s), Z(t,s), P_1(t,s), \ldots, P_n(t,s))$$

of $\Gamma(t)$ at $s = 0$, which satisfies $\pi(C(t,s)) = c(t,s)$. This show that any curve starting at $j^k(\pi \circ \Gamma)(0)$ in $J^{r-a_1}(1, n+1)$ lifts to a curve starting at $j^r\Gamma(0)$ in $\Sigma_{a_1} \subset J^k_{\mathrm{int}}(1, 2n+1)$. Therefore Π is a submersion at $j^r\Gamma(0)$. The type $\boldsymbol{b}$ of $\Pi(j^r\Gamma)$ at $t = 0$ for sufficiently large is given by $b_i = a_i - a_1 + 1$. Then we have

$$\text{Jet-codim}_{\mathcal{D}}(\boldsymbol{a}) = \sum_{i=1}^{n+1}(b_i - i) + n(a_1 - 1) = \sum_{i=1}^{n+1}(a_i - a_1 + 1 - i) + n(a_1 - 1) = \sum_{i=2}^{n+1}(a_i - i).$$

□

Third, similarly to above, we consider the jet space of $\mathcal{C}$-integral curves, $J^r_{\mathcal{C}}(I, \mathcal{F}_{n+2}) \subset J^r(I, \mathcal{F}_{n+2})$ and $\Sigma_{\mathcal{C}}(\boldsymbol{a})$ in $J^r_{\mathcal{C}}(I, \mathcal{F}_{n+2})$. We denote its codimension by Jet-codim$_{\mathcal{C}}(\boldsymbol{a})$.

Theorem 7.3 *The jet-codimension of the set of $\mathcal{C}$-integral curves $\Gamma : I \to \mathcal{F}_{n+2}$ such that $\pi_1 \circ \Gamma$ is of type $\boldsymbol{a} = (a_1, a_2, \ldots, a_{n+1})$, is given by*

$$\text{Jet-codim}_{\mathcal{C}}(\boldsymbol{a}) = a_{n+1} - (n+1) \ = \ s(\boldsymbol{a}) - s(\boldsymbol{a}'),$$

where $\boldsymbol{a}' = (a_1, a_2, \ldots, a_n)$.

Proof: As is explained in §5, a $\mathcal{C}$-integral curve is described by the components $x_j^{j-1}(t), (1 \leq j \leq n+1)$. In fact, by projecting to these components, we have a diffeomorphism between the fiber $J^r_{\mathcal{C}}(I, \mathcal{F}_{n+2})_{(t,f)}$ over $(t,f) \in I \times \mathcal{F}_{n+2}$ of the jet bundle and the ordinary jet space $J^r(1, n+1)$. To get the formula on Jet-codim$_{\mathcal{C}}(\boldsymbol{a})$, let $\Gamma(t) = (x_{ij}(t))$ be a $\mathcal{C}$-integral curve for the coordinates introduced in §5 through the origin at $t = 0$. Then we have, for the order at $t = 0$,

$$\text{ord} x_i^{\ j} = \text{ord} x_i^{\ j+1} + \text{ord} x_{j+1}^{\ j}, \quad (0 \leq j, j+1 < i).$$

Therefore we have $\text{ord}x_i^{\ 0} = \sum_{1 \leq j \leq i} \text{ord}x_j^{\ j-1}$ and hence $\text{ord}x_i^{\ 0} - \text{ord}x_{i-1}^{\ 0} = \text{ord}x_i^{\ i-1} \geq 1, (2 \leq i \leq n+1)$. Then the type of $\pi_1 \circ \Gamma$ is of type at $t = 0$ if and only if $\text{ord}x_i^{\ 0} = a_i, (1 \leq i \leq n+1)$. The condition is equivalent to that $\text{ord}x_i^{\ i-1} = a_i - a_{i-1}, (1 \leq i \leq n+1)$. Regarding the codimension in $J^r(1, n+1)$, we have

$$\text{Jet-codim}_{\mathcal{C}}(\boldsymbol{a}) = \sum_{i=1}^{n+1}(\text{ord}x_i^{\ i-1} - 1) = a_{n+1} - (n+1).$$

□

Remark 7.4 If the type of $\gamma = (1, x_1^{\ 0}, \ldots, x_{n+1}^{\ 0}) : I \to \mathbf{R}^{n+2} \setminus \{0\}$ at $t \in I$ is $\boldsymbol{a} = (a_1, a_2, \ldots, a_{n+1})$, then the dual curve $\gamma^* = (1, x_{n+1}^{\ n}, x_{n+1}^{\ n-1}, \ldots, x_{n+1}^{\ 1}, x_{n+1}^{\ 0}) : I \to \mathbf{R}^{n+2} \setminus \{0\}$ is of type

$$\boldsymbol{a}^* = (a_{n+1} - a_n, a_{n+1} - a_{n-1}, \ldots, a_{n+1} - a_1, a_{n+1}).$$

(Arnold-Scherbak's theorem [29]).

As a consequence, we observe that

$$\text{Jet-codim}_{\mathcal{C}}(\boldsymbol{a}) \leq \text{Jet-codim}_{\mathcal{D}}(\boldsymbol{a}) \leq \text{Jet-codim}(\boldsymbol{a}).$$

By the transversality theorem ([24]), we see a curve of type $\boldsymbol{a}$ at a point appear generically if $s(\boldsymbol{a}) \leq 1$ in the class of curves in $P^{n+1} = \text{Gr}(1, \mathbf{R}^{n+2})$, and the list is given by $\boldsymbol{a} = (1, 2, \ldots, n, n+1), (1, 2, \ldots, n, n+2)$. Moreover, a curve of type $\boldsymbol{a}$ at a point appear momentarily in a generic one-parameter family of curves in P^{n+1} if $s(\boldsymbol{a}) \leq 2$. The list is given by

$$(1, 2, \ldots, n, n+1), (1, 2, \ldots, n, n+2), (1, 2, \ldots, n, n+3), (1, 2, \ldots, n+1, n+2),$$

for $n \geq 3$, and, when $n = 2$, $(2, 3, 4)$ is added to the list.

Then, for adapted framed curves, we have:

Theorem 7.5 *For a generic one-parameter family of integral curves $\Gamma_\lambda : I \to \mathcal{F}_{n+2}$ ($\lambda \in J$, J being a one-dimensional manifold) to the pseudo contact structure $\mathcal{D}$ on the flag manifold $\mathcal{F}_{n+2}$, the type of $\gamma_\lambda = \pi_1 \circ \Gamma_\lambda$ and $\widehat{\gamma}_\lambda = \pi_{n+1} \circ \Gamma_\lambda$ at any point in I for any parameter $\lambda \in J$ is one of the following list:*

$(1, 2, \ldots, n, n{+}1), (1, 2, \ldots, n, n{+}2), (1, 2, \ldots, n, n{+}3), (1, 2, \ldots, n{+}1, n{+}2), (2, 3, 4) (n = 2).$

Proof: The transversality theorem for integral curves to contact structure is given in [12]. Moreover the pseudo contact structure is the pull-back by the submersion $\pi : \mathcal{F}_{n+2} \to \mathcal{I}_{n+2}$ of the contact structure on the incident manifold $\mathcal{I}_{n+2}$. Therefore the transversality theorem holds for $\mathcal{D}$-integral curves. By Theorem 7.2, we see $\text{Jet-codim}_{\mathcal{D}}(\boldsymbol{a}) \leq 1$ if and only if $\boldsymbol{a} = (1, 2, \ldots, n, n+1), (1, 2, \ldots, n, n+2)$. Moreover $\text{Jet-codim}_{\mathcal{D}}(\boldsymbol{a}) \leq 2$ if and only if $\boldsymbol{a}$ is one of the above list. Therefore we have the result. □

For osculating framed curves we have:

Theorem 7.6 *For a generic one-parameter family of integral curves $\Gamma_\lambda : I \to \mathcal{F}_{n+2}$, ($\lambda \in J$) to the canonical structure $\mathcal{C}$, the type of $\gamma_\lambda = \pi_1 \circ \Gamma_\lambda$ and $\widehat{\gamma}_\lambda = \pi_{n+1} \circ \Gamma_\lambda$ at any point in I for any parameter $\lambda \in J$ is one of the following list:*

$$(1, 2, \ldots, n, n+1), (1, 2, \ldots, i, i+2, \ldots, n+2) (0 \leq i \leq n),$$
$$(1, 2, \ldots, i, i+2, \ldots, j, j+2, \ldots, n+3) (0 \leq i < j \leq n+1).$$

Proof: As it is stated in the proof of Theorem 7.3, the fiber $J^r_{\mathcal{C}}(I, \mathcal{F}_{n+1})_{(t,f)}$ is diffeomorphic to $J^r(1, n+1)$. Then we have the transversality theorem for $\mathcal{C}$-integral curves $I \to \mathcal{F}_{n+2}$ by the ordinary transversality theorem. Thus we have the required result by Theorem 7.3. □

8 Singularities of envelopes and their bifurcations.

Let $\gamma : I \to X$ be a framed curve with framing $(e_1, \ldots, e_{n+1})$. Then the *envelope* $E(\gamma)$ of γ, generated by the family of tangent hyperplanes $e_{n+1}^{\perp}(t) \subset T_{\gamma(t)}X$, is defined as follows ([30]): Take the frame-dual $\widehat{\gamma} = e^{n+1} : I \to Y$ and take the fiber product

$$W := \{(t, z) \in I \times Z \mid \widehat{\gamma}(t) = \pi_2(z)\}$$

of $\widehat{\gamma} : I \to Y$ and $\pi_2 : Z \to Y$. Then W is an $(n+1)$-dimensional manifold. The envelope $E(\gamma)$ is defined as the set of critical values of the projection $\Pi_1 : W \to X$ defined by $\Pi_1(t, z) = \pi_1(z)$, for $(t, z) \in W$.

The lifting $\widetilde{\gamma} : I \to G \subset \widetilde{\mathcal{F}}_{n+2}$ projects to the integral lifting $\overline{\gamma} : I \to Z \subset \widetilde{\mathcal{I}}_{n+2}$, $\overline{\gamma}(t) = (e_0(t), e_{n+1}(t))$, to $\mathcal{D}$. Note that $\pi_2 \circ \overline{\gamma} = \widehat{\gamma}$. Moreover if we consider the osculating hyperplanes to $\widehat{\gamma} : I \to Y \subset \widetilde{\mathrm{Gr}}(1, \mathbf{R}^{n+2})$, (resp. $\widetilde{\mathrm{Gr}}(1, \mathbf{R}^{1,n+1})$), we get the dual curve $\widehat{\gamma}^* : I \to P^{n+2}$ (resp. $I \to P^{1,n+1}$), forgetting the orientation if necessary. In fact, $\widehat{\gamma}^* = \pi_1 \circ \Gamma$ for the $\mathcal{C}$-integral lift $\Gamma : I \to \mathcal{F}^0_{n+2}$ of $\widehat{\gamma}$ constructed by associated osculating flags to $\widehat{\gamma}$, with respect to $\pi_{n+1} : \mathcal{F}^0_{n+2} \to P^{n+1}$ (resp. $P^{1,n}$).

Proof of Theorem 1.1. By Theorem 7.5, the type of a curve $\widehat{\gamma}_\lambda : I \to Y$ in a generic one-parameter family of framed curves at a point (t, λ) is one of (I) : $(1,2,3)$, (II) : $(1,2,4)$, (III) : $(1,3,4)$, (IV) : $(1,2,5)$ or (V) : $(2,3,4)$.

Then we have the normal forms of singularities by the classification results in [9]. □

Moreover, by Theorem 7.6, we have immediately:

Theorem 8.1 *For a generic one-parameter family γ_λ of osculating framed curves in E^3, S^3, H^3, the frame dual $\widehat{\gamma}_\lambda$ at a point (t, λ) has one of type in the list*

$$(1,2,3); (1,2,4), (1,3,4), (2,3,4); (1,2,5), (1,3,5), (1,4,5), (2,3,5), (2,4,5), (3,4,5).$$

Corresponding to each type in the above list, the dual curve $\widehat{\gamma}^*_\lambda$ turns to be of type

$$(1,2,3); (2,3,4), (1,3,4), (1,2,4); (3,4,5), (2,4,5), (1,4,5), (2,3,5), (1,3,5), (1,2,5).$$

By [9], the diffeomorphism class of the envelope $E(\gamma_\lambda)$ is determined in the case

$$(1,2,3), (2,3,4), (1,3,4), (1,2,4), (3,4,5), (1,2,5).$$

Moreover in any case the topological class of $E(\gamma_\lambda)$ is determined ([11]). We will describe in the forthcoming paper in detail, the topological bifurcations of envelopes for osculating framed curves of type Jet-codim$_{\mathcal{C}}(\boldsymbol{a}) \leq 2$, namely, for

$$(2,4,5), (1,4,5), (2,3,5), (1,3,5), (1,2,5).$$

Remark 8.2 Though the list of singularities is common for all of three geometries, the geometric characters are of course distinguished. For instance, in the case $n = 2$ and for an adapted frame $e_0' = e_1$, we have the structure equation, under the arc-length derivative,

$$\left\{ \begin{array}{lcllll} e_1' & = & -\delta e_0 & & +\kappa_1 e_2 & +\kappa_2 e_3, \\ e_2' & = & & -\kappa_1 e_1 & & +\kappa_3 e_3, \\ e_3' & = & & -\kappa_2 e_1 & -\kappa_3 e_2, & \end{array} \right.$$

where $\delta = 0, 1, -1$ for $X = E^3, S^3, H^3$ respectively ([21]). In general, we characterise the type $\boldsymbol{a}$ of the frame-dual $\widehat{\gamma}$ for a framed curve γ by polynomials, distinguished in each gemetry, of geometric invariants of γ and their derivatives up to order $a_{n+1} - 1$. For E^3, see [12].

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