Induced Norms of the Schur Multiplication Operator

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1. Introduction and notations

1.1 Definitions and basic facts

Let M_{mn} denote the space of all $m \times n$ complex matrices and M_n the algebra of all n square matrices. The *Schur product* (or *Hadamard product*) of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in M_{mn} is defined by the entrywise product multiplication and is denoted by $A \circ B$:

$$A \circ B = [a_{ij} b_{ij}]. \tag{1.1}$$

Thus the Schur product is defined for a pair of (not necessarily square) matrices of same size, which is always commutative and it is completely different from the ordinary matrix product.

Historically, it seems that the first systematic study of algebraic and analytic properties of Schur product was made by I. Schur. The research on Schur product has been done on the analogy of known results for the ordinary matrix product.

We denote the *transpose* of a matrix $A = [a_{ij}]$ by ${}^tA \equiv [a_{ji}]$; and the *adjoint* of A by $A^* \equiv [\overline{a_{ji}}]$ where for a complex number z, $\overline{z}$ means the complex conjugate of z. A matrix $A \in M_n$ is *Hermitian* if and only if $A = A^*$. An Hermitian matrix A is *positive definite* if $\langle A\vec{x}|\vec{x}\rangle > 0$ for all nonzero $\vec{x} \in \mathbb{C}^n$ and A is *positive semidefinite* if $\langle A\vec{x}|\vec{x}\rangle \geq 0$ for all $\vec{x} \in \mathbb{C}^n$ where $\langle\cdot|\cdot\rangle$ denote the inner product on $\mathbb{C}^n$. For two Hermitian matrices A and B in M_n, $A > B$ (or $B < A$) means $A - B$ is positive definite matrix, similarly $A \geq B$ (or $B \leq A$) means $A - B$ is a positive semidefinite matrix. For $A = [a_{ij}] \in M_n$ its *trace* $\mathrm{tr}(A)$ is the sum of its main diagonal entries: $\mathrm{tr}(A) \equiv a_{11} + a_{22} + \cdots + a_{nn}$. $\lambda \in \mathbb{C}$ is called an *eigenvalue* of a matrix A if there exists a nonzero vector $\vec{x} \in \mathbb{C}^n$ such that $A\vec{x} = \lambda\vec{x}$, and $\vec{x}$ is called an *eigenvector* of A corresponding to the eigenvalue λ. It is known that if A is an Hermitian matrix then every eigenvalue of A is real number and if $A \geq 0$ then every eigenvalue is nonnegative number. Hence when A is Hermitian we write the eigenvalues of A by $\vec{\lambda}(A) = (\lambda_1(A), \cdots, \lambda_n(A))$ which are arranged in nonincreasing order: $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. The *spectral radius* of a matrix $A \in M_n$ is given by $r(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$. Let A be a positive semidefinite matrix, then there exists a unique positive semidefinite matrix C such that $C^2 = A$ and C is denoted by $C \equiv A^{1/2}$. The *singular values* of a matrix $A \in M_n$ are defined as the eigenvalues of $|A| \equiv (A^*A)^{1/2}$, that is, $s_i(A) = \lambda_i(|A|) \quad (i = 1, \cdots, n)$. For a vector $\vec{x} = {}^t(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$, the matrix $\mathrm{diag}(\vec{x}) \in M_n$ is a diagonal matrix whose i th diagonal entry is x_i. Let $\|\cdot\|$ be a norm on M_n. $\|\cdot\|$ is called a *unitarily invariant norm* if $\|A\| = \|UAV\|$ for every $A \in M_n$ and all unitaries $U, V \in M_n$, and $\|\cdot\|$ is a *unitary similarity invariant norm* if $\|A\| = \|UAU^*\|$ for every $A \in M_n$ and every unitary $U \in M_n$.

For real number $p \geq 1$, we denote the *Schatten p-norm* by $\|\cdot\|_p$:

$$\|A\|_p \equiv \left(\sum_{i=1}^{n} s_i^p(A) \right)^{1/p} \qquad (A \in M_n).$$

In particular, $\|\cdot\|_1$ is called the *trace norm*:

$$\|A\|_1 \equiv \sum_{i=1}^{n} s_i(A),$$

$\|\cdot\|_2$ is called the *Frobenius norm* (or *Hilbert-Schmidt norm*):

$$\|A\|_2 \equiv \left(\sum_{i=1}^{n} s_i^2(A) \right)^{1/2} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2},$$

$\|\cdot\|_\infty$ is called the *spectral norm* (or *operator norm*):

$$\|A\|_\infty \equiv s_1(A) = \sup_{\vec{x} \in \mathbb{C}^n} \frac{\|A\vec{x}\|}{\|\vec{x}\|}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{C}^n$. $\|\cdot\|_p$ is an example of unitarily invariant norm. A nontrivial example of a unitary similarity invariant but not unitarily invariant norm is the *numerical radius* $w(\cdot)$ defined by

$$w(A) \equiv \sup_{\vec{x} \in \mathbb{C}^n} \frac{|\langle A\vec{x} | \vec{x} \rangle|}{\|\vec{x}\|^2} \qquad (A \in M_n)$$

with usual inner product $\langle \cdot | \cdot \rangle$.

For a given $A \in M_n$, consider the *Schur multiplication operator* $S_A : M_n \longmapsto M_n$ given by

$$S_A(X) \equiv A \circ X \quad (X \in M_n).$$

Since S_A is a linear map on M_n, for any norm $\|\cdot\|$ on M_n we can define the norm $\|S_A\|$ induced by $\|\cdot\|$:

$$\|S_A\| \equiv \sup_{X \in M_n} \frac{\|A \circ X\|}{\|X\|}. \tag{1.2}$$

In this paper, we shall investigate the induced norms of the Schur multiplication operator for several norms on M_n.

Historically, in his famous paper [29], I. Schur proved that if A and B are positive semidefinite matrices in M_n, then $A \circ B$ is also positive semidefinite (today usually called Schur product theorem), that is,

$$A, B \geq 0 \Longrightarrow A \circ B \geq 0. \tag{1.3}$$

and he proved the submultiplicative inequality of Schur product for the spectral norm:

$$\|A \circ B\|_\infty \leq \|A\|_\infty \|B\|_\infty \quad (A, B \in M_n). \tag{1.4}$$

1.2 Examples of Schur product

The Schur product arises naturally from several different points of view. We describe only two examples involving the Schur product. See [12] and [13] about further details of examples which arise in an application.

[Example 1]

Let $f(\theta)$ and $g(\theta)$ be continuous periodic functions of period 2π and consider their Fourier series:

$$a_k \equiv \int_0^{2\pi} e^{ik\theta} f(\theta) d(\theta) \text{ and } \quad b_k \equiv \int_0^{2\pi} e^{ik\theta} g(\theta) d(\theta) \quad (k = 0, \pm 1, \pm 2, \cdots).$$

Then the convolution

$$h(\theta) \equiv \int_0^{2\pi} f(\theta - t) g(\theta) d(t)$$

has Fourier series

$$c_k = \int_0^{2\pi} e^{ik\theta} h(\theta) d(\theta)$$

that satisfy the identities $c_k = a_k \cdot b_k,\ k = 0, \pm 1, \pm 2, \cdots$. Then the Toeplitz matrix of Fourier series of $h(\theta)$ is the Schur product of the Toeplitz matrices of Fourier series of $f(\theta)$ and $g(\theta)$:

$$[c_{i-j}] = [a_{i-j}] \circ [b_{i-j}].$$

[**Example 2**]

Let f be a continuously differentiable real-valued function on a real interval (a, b) and let $A, B \in M_n$ be given Hermitian matrices with all their eigenvalues in (a, b). Define $g(t)$ by

$$g(t) = f(tA + (1-t)B) \quad (t \in (0,1)).$$

For every $t \in (0,1)$ from the spectral theory there is a unitary matrix $U_t \in M_n$ such that $tA + (1-t)B = U_t \mathrm{diag}(\lambda_i(t)) U_t^*$ where $\lambda_i(t)$ are the eigenvalues of $tA + (1-t)B$. Then we can obtain by calculation

$$g'(t) = U_t \left[K_f(\{\lambda_i(t)\}) \circ (U_t^*(A-B)U_t) \right] U_t^*.$$

Here we write

$$K_f(\{\lambda_i(t)\})_{pq} = \begin{cases} f'(\lambda_p(t)) & (\lambda_p(t) = \lambda_q(t)) \\ \dfrac{f(\lambda_p(t)) - f(\lambda_q(t))}{\lambda_p(t) - \lambda_q(t)} & (\lambda_p(t) \neq \lambda_q(t)). \end{cases}$$

1.3 Sketch of the contents

The weak majorization relation for the usual matrix product was known by Horn [11], that is,

$$(1.5) \qquad \vec{s}(AB) \prec_w \vec{s}(A) \circ \vec{s}(B) \quad (A, B \in M_n).$$

Hölder-type norm inequalities for the usual product are derived from this relation. In section 2 we discuss a Schur product version of (1.5), and give Hölder-type inequalities for

Schur products of the form $\|A \circ B\|_{\phi_0,p_0} \leq \|A\|_{\phi_1,p_1} \cdot \|B\|_{\phi_2,p_2}$. Here for unitarily invariant norm $\|A\|_\phi$, we define $\|A\|_{\phi,p}$ by $\| \, |A|^p \, \|_\phi^{1/p}$. As a corollary, we settle, in a strong form, a conjecture of Marcus et al. [18] on submultiplicativity of a unitarily invariant norm with respect to the Schur multiplication.

For $A \in M_n$ the induced norm $\|S_A\|$ of Schur multiplication S_A with respect to a norm $\|\cdot\|$ on M_n is defined by (1.2).

For $p \geq 1$, the Schatten p-norm $\|\cdot\|_p$ is a typical example of unitarily invariant norm. If we denote the norm of S_A induced by the Schatten norm $\|\cdot\|_p$ by $\|S_A\|_p$ then $\|S_A\|_p = \|S_A\|_q$ for $p, q \geq 1$ such that $1/p + 1/q = 1$, and

$$\|S_A\|_2 = \min_{1 \leq p \leq \infty} \|S_A\|_p.$$

In section 3 we present some convex property of the function $p \longmapsto \|S_A\|_p$.

In section 4, we show that the induced norm of S_A with respect to the numerical radius norm is at most one if and only if A is factorized by $A = B^*WB$, where W is a contractive matrix and the Euclidean norms of the columns of B are at most one. We give other equivalent characterizations and derive, as a concequence, Haagerup's Theorem.

For $\rho > 0$, Holbrook [10] defined an *operator radius* $w_\rho(\cdot)$ by relating with ρ-contraction which was introduced by Sz.-Nagy (see [30]).

The set $\{w_\rho(\cdot)\}_{\rho>0}$ of operator radii is a one parameter family of unitary similarity invariant (quasi-)norms, which interpolates the spectral norm, the numerical radius and the spectral radius. For commutative matrices $A, B \in M_n$ with respect to the ordinary product it has been shown (see [23]) that

$$w_\rho(AB) \leq K_\rho \|A\|_\infty \cdot w_\rho(B)$$

where

$$K_\rho = \begin{cases} \inf_{0<\delta<1}\{\delta^{-1}(2-\delta)^{-1} + (\rho-1)^2(\rho-\delta)^{-2}\}^{1/2} & (1 < \rho < \infty), \\ K_{2-\rho} & (0 < \rho \leq 1). \end{cases}$$

Especially, when $\rho = 2$, using another tool, we proved

$$(1.6) \qquad w(AB) \leq 1.16 \cdot \|A\|_\infty \cdot w(B) \quad (A, B \in M_n, \; AB = BA).$$

It had been a long standing conjecture whether for commuting matrices the constant 1.16 in (1.6) can be replaced by 1, that is,

$$(1.7) \qquad w(AB) \le \|A\|_\infty \cdot w(B) \quad (A, B \in M_n,\ AB = BA).$$

But recently, Müller [20] and Davidson, Holbrook [7] showed that the conjecture (1.7) is not true. In section 5, we mention the analogy of (1.7) in the case of the Schur product with respect to operator radius and the Hölder-type inequalities.

The contents of this paper were published in [5],[21] and [22]. Moreover the contents of [22] was presented at the Inaugural Conference of the International Linear Algebra Society held, August 12-15, 1989, at Brigham Young University in Provo, Utah, U.S.A. while the contents of [5] was presented at the Conference "Directions in Matrix Theory", held March 20-23, 1990, at Auburn University in Auburn, Alabama, U.S.A..

2. Unitarily invariant norm

2.1 Preliminaries and weak majorization relation for Schur Product

It is known(see [8, p.78]) there is a one-to-one correspondence between the set of unitarily invariant norms $\|\cdot\|$ and the set of symmetric gauge functions ϕ on $\mathbb{R}^n_+$. This correspondence is given by the relation

$$(2.1) \qquad \|A\| = \phi(\vec{s}(A)) = \phi({}^t(s_1(A), \cdots, s_n(A))),$$

where $\vec{s}(A) = {}^t(s_1(A), \cdots, s_n(A))$ is n tuple of the singular values of A, i.e. the eigenvalues of $(A^*A)^{1/2}$, arranged in nonincreasing order with multiplicities counted. We denote this norm by $\|\cdot\|_\phi$. Recall that a nonnegative function ϕ on $\mathbb{R}^n_+$ is called a *symmetric gauge function* if it is subadditive, positive homogeneous, and invariant under every coodinate permutation:

$$\phi({}^t(\xi_1, \cdots, \xi_n)) = \phi({}^t(\xi_{\sigma_1}, \cdots, \xi_{\sigma_n}))$$

for every permutation σ of order n.

A vector $\vec{\xi} = {}^t(\xi_1, \cdots, \xi_n)$ in $\mathbb{R}^n_+$ is said to be *weakly majorized* by another vector $\vec{\eta} = {}^t(\eta_1, \cdots, \eta_n)$ (in symbol $\vec{\xi} \prec_w \vec{\eta}$) if

$$(2.2) \qquad \sum_{j=1}^{k} \xi_{[j]} \leq \sum_{j=1}^{k} \eta_{[j]} \quad (k = 1, \cdots, n),$$

where $\xi_{[1]} \geq \xi_{[2]} \geq \cdots \geq \xi_{[n]}$ and $\eta_{[1]} \geq \eta_{[2]} \geq \cdots \geq \eta_{[n]}$ are the decreasing rearrangements of the components of $\vec{\xi}$ and $\vec{\eta}$, respectively. If in addition, equality holds in (2.2) for $k = n$ then $\vec{\xi}$ is said to be *majorized* by $\vec{\eta}$ (in symbol $\vec{\xi} \prec \vec{\eta}$).

We will make frequent use of the fact that a symmetric gauge function ϕ is monotone with respect to the semiorder induced by weak majorization (see [8, p.72]):

$$\vec{\xi} \prec_w \vec{\eta} \text{ implies } \phi(\vec{\xi}) \leq \phi(\vec{\eta});$$

hence by the statement (2.1)

$$(2.3) \qquad \vec{s}(A) \prec_w \vec{s}(B) \text{ implies } \|A\|_\phi \leq \|B\|_\phi.$$

For the function

$$\phi_{(p)}(\vec{\xi}) = \begin{cases} \left(\sum_{j=1}^n \xi_j^p\right)^{1/p} & \text{if } 1 \leq p \leq \infty, \\ \max_{1 \leq j \leq n} \xi_j & \text{if } p = \infty, \end{cases}$$

the corresponding norm is simply denoted by $\|\cdot\|_p$. This is in accordance with the norm $\|\cdot\|_\infty$. When we analyze the norm inequalities for the usual (matrix) product, the following weak majorization relation is a very useful tool:

$$(2.4) \qquad \vec{s}(AB) \prec_w \vec{s}(A) \circ \vec{s}(B) \quad (A, B \in M_n)$$

between the singular values of the product AB and those of A and B. Here $\vec{\xi} \circ \vec{\eta}$ denotes the Schur product, i.e. the coordinatewise product of vector $\vec{\xi}$ and $\vec{\eta}$. Therefore, to obtain Schur product version of the norm inequalities, we will start with the following lemma, which is a Schur product version of (2.4).

LEMMA 2.1.

$$(2.5) \qquad \vec{s}(A \circ B) \prec_w \vec{s}(A) \circ \vec{s}(B) \quad (A, B \in M_n)$$

Proof: Let us first consider the case $A \geq 0$ and $B \geq 0$. Put $\varepsilon_i = s_i(A) - s_{i+1}(A)$ and $\delta_j = s_j(B) - s_{j+1}(B)(i, j = 1, \cdots, n)$, with the convention $s_{n+1}(A) = s_{n+1}(B) = 0$. Since $s_i(A)$ and $s_j(B)$ are eigenvalues of A and B respectively, there are orthogonal projections $P_i, Q_j(i, j = 1, 2, \cdots, n)$ such that

$$\text{rank}(P_i) = i \quad \text{and} \quad \text{rank}(Q_j) = j \quad (i, j = 1, 2, \cdots, n)$$

and

$$A = \sum_{i=1}^{n} \varepsilon_i P_i, \quad B = \sum_{j=1}^{n} \delta_j Q_j.$$

Since $\varepsilon_i, \delta_j \geq 0(i, j = 1, 2, \cdots n)$ and

$$A \circ B = \sum_{i,j=1}^{n} \varepsilon_i \delta_j (P_i \circ Q_j),$$

we have (see [19, p.243])

$$\vec{s}(A \circ B) \prec_w \sum_{i,j=1}^{n} \varepsilon_i \delta_j \vec{s}(P_i \circ Q_j). \tag{2.6}$$

Now the Schur product theorem (1.3) implies, with I denoting the identity,

$$P_i \circ Q_j \leq P_i \circ I = \text{the diagonal matrix of } P_i,$$

and similarly,

$$P_i \circ Q_j \leq I \circ Q_j = \text{the diagonal matrix of } Q_j.$$

Then since $\leq$ between positive semidefinite matrices implies $\prec_w$ (see [19, p.475]), we have

$$\vec{s}(P_i \circ Q_j) \prec_w \vec{s}(P_i \circ I) \text{ and } \vec{s}(P_i \circ Q_j) \prec_w \vec{s}(I \circ Q_j).$$

It is known (see [19, p.228]) that

$$\vec{s}(P_i \circ I) \prec \vec{s}(P_i) = (\overbrace{1, \cdots, 1}^{i}, 0, \cdots, 0)$$

and

$$\vec{s}(I \circ Q_j) \prec \vec{s}(Q_j) = (\overbrace{1, \cdots, 1}^{j}, 0, \cdots, 0).$$

A conclusion is that, with $i \wedge j \equiv \min\{i, j\}$,

$$\vec{s}(P_i \circ Q_j) \prec_w (\overbrace{1, \cdots 1}^{i\wedge j}, 0, \cdots, 0) = \vec{s}(P_i) \circ \vec{s}(Q_j). \tag{2.7}$$

Now it follows from (2.6) and (2.7) that

$$\vec{s}(A \circ B) \prec_w \sum_{i,j=1}^{n} \varepsilon_i \delta_j \vec{s}(P_i) \circ \vec{s}(Q_j) = \vec{s}(A) \circ \vec{s}(B).$$

This proves (2.5) for $A, B \geq 0$.

Next let us consider general $A, B \in M_n$. For $A \in M_n$ let $A^* = U|A^*|$ be the polar decomposition of A^*. Then

$$\begin{aligned} 0 &\leq \begin{bmatrix} U|A^*|^{1/2} \\ |A^*|^{1/2} \end{bmatrix} [|A^*|^{1/2}U^* \ |A^*|^{1/2}] \\ &= \begin{bmatrix} U|A^*|U^* & U|A^*| \\ |A^*|U^* & |A^*| \end{bmatrix} \\ &= \begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix} \end{aligned} \tag{2.8}$$

and similary,

$$\begin{bmatrix} |B| & B^* \\ B & |B^*| \end{bmatrix} \geq 0.$$

Hence the Schur product theorem (1.3), applied to these $2n \times 2n$ matrices, yields

$$\begin{bmatrix} |A| \circ |B| & A^* \circ B^* \\ A \circ B & |A^*| \circ |B^*| \end{bmatrix} \geq 0. \tag{2.9}$$

It follows from inequality (2.9) that there is $W \in M_n$ such that $\|W\|_\infty \leq 1$ and

$$A \circ B = (|A^*| \circ |B^*|)^{1/2} \cdot W \cdot (|A| \circ |B|)^{1/2}.$$

Then according to (2.4) we see

$$\begin{aligned} \vec{s}(A \circ B) &\prec_w \vec{s}\left((|A^*| \circ |B^*|)^{1/2}\right) \circ \vec{s}(W) \circ \vec{s}\left((|A| \circ |B|)^{1/2}\right) \\ &\leq \vec{s}(|A^*| \circ |B^*|)^{1/2} \circ \vec{s}(|A| \circ |B|)^{1/2}, \end{aligned}$$

where for $\vec{\xi} = {}^t(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n_+$ we use $\vec{\xi}^{1/2} = {}^t(\xi_1^{1/2}, \cdots, \xi_n^{1/2})$. By the arithmetic-geometric mean inequality, we have

$$\vec{s}(|A^*| \circ |B^*|)^{1/2} \circ \vec{s}(|A| \circ |B|)^{1/2} \leq \frac{\vec{s}(|A^*| \circ |B^*|) + \vec{s}(|A| \circ |B|)}{2}.$$

Now apply (2.5), already proved for positive semidefinite matrices, to the pairs $|A^*|, |B^*|$ and $|A|, |B|$, and use the relation $\vec{s}(A) = \vec{s}(A^*)$ and $\vec{s}(B) = \vec{s}(B^*)$ to get

$$\frac{\vec{s}(|A^*| \circ |B^*|) + \vec{s}(|A| \circ |B|)}{2} \prec_w \vec{s}(A) \circ \vec{s}(B).$$

This completes the proof of the lemma. ∎

Remark. Bapat and Sunder [6] showed that for positive semidefinite matrices $A, B \in M_n$ $\vec{s}(A \circ B) \prec_w \vec{s}(A) \circ \vec{\beta}$ where $\vec{\beta}$ is the diagonal entries of B, arranged in decreasing order. This is an improvement of our lemma 2.1 in the case of $A, B \geq 0$, but they did not treat the general case or Hölder-type inequalities.

2.2 Hölder-type inequalities

THEOREM 2.2. *The following conditions for symmetric gauge functions ϕ_0, ϕ_1, ϕ_2 are mutually equivalent:*

(i) $\phi_0(\vec{\xi} \circ \vec{\eta}) \leq \phi_1(\vec{\xi}) \cdot \phi_2(\vec{\eta}) \quad (\vec{\xi}, \vec{\eta} \in \mathbb{R}^n_+)$.

(ii) $\|AB\|_{\phi_0} \leq \|A\|_{\phi_1} \cdot \|B\|_{\phi_2} \quad (A, B \in M_n)$.

(iii) $\|A \circ B\|_{\phi_0} \leq \|A\|_{\phi_1} \cdot \|B\|_{\phi_2} \quad (A, B \in M_n)$.

PROOF: It is easy to see that the Schur product $A \circ B$ coincides with usual matrix product AB if A and B are diagonal, and that for a diagonal matrix $A = \mathrm{diag}(a_1, \cdots, a_n)$ and symmetric gauge function ϕ,

$$\|A\|_\phi = \phi({}^t(|a_1|, \cdots, |a_n|)).$$

These considerations lead to (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i) immediately. (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) follow respectively from the weak majorization relations (2.4) and (2.5) of usual product and Schur product, combined with the monotone relation (2.3). ∎

Theorem 2.2 admits a natural generalization to an $(m+1)$-tuple of symmetric gauge functions $\phi_0, \phi_1, \cdots, \phi_m$.

Remark. Horn and Johnson [15] showed a result which corresponds to our Theorem 2.2 with $\phi_0 = \phi_1 = \phi_2$. Their proof is also based on the weak majorization relation (2.5) of Schur product. Their proof of the key lemma 2.1 is quite different from ours, but it uses a common idea with ours in reducing the problem to the case of a Schur product of projections.

For each symmetric gauge function ϕ, its *dual function* ϕ^* is defined by

$$\phi^*(\vec{\xi}) = \sup_{\vec{\eta} \in \mathbb{R}^n_+} \frac{\sum_{j=1}^n \xi_i \eta_j}{\phi(\vec{\eta})} \quad (\vec{\xi} \in \mathbb{R}^n_+). \tag{2.10}$$

Then ϕ^* is again a symmetric gauge function and the corresponding norm $\|\cdot\|_{\phi^*}$ becomes the dual norm of $\|\cdot\|_\phi$ in the sence (see [8, p.130-135])

$$\|A\|_{\phi^*} = \sup_{B \in M_n} \frac{|\mathrm{tr}(AB^*)|}{\|B\|_\phi} \quad (A \in M_n).$$

We define $(\|\cdot\|_\phi, \|\cdot\|_1)$-norm of S_A by

$$\|S_A\|_{\phi,1} \equiv \sup_{X \in M_n} \frac{\|A \circ X\|_1}{\|X\|_\phi} \quad (A \in M_n)$$

Similary, $(\|\cdot\|_t, \|\cdot\|_r)$-norm of S_A is defined by

$$\|S_A\|_{t,r} \equiv \sup_{X \in M_n} \frac{\|A \circ X\|_r}{\|X\|_t} \quad (A \in M_n)$$

Since, by definition (2.10),

$$\sum_{j=1}^n \xi_j \eta_j \le \phi^*(\vec{\xi}) \cdot \phi(\vec{\eta}) \quad (\vec{\xi}, \vec{\eta} \in \mathbb{R}^n_+),$$

the following is immediate from Theorem 2.2.

COROLLARY 2.3. *For every symmetric gauge function* ϕ

$$(2.11) \qquad \|S_A\|_{\phi,1} \le \|A\|_{\phi^*} \quad (A \in M_n).$$

A classical Hölder inequality for numerical sequences says that

$$\left\{\sum_{j=1}^{n}(\xi_j\eta_j)^r\right\}^{1/r} \le \left\{\sum_{j=1}^{n}\xi_j^s\right\}^{1/s}\left\{\sum_{j=1}^{n}\eta_j^t\right\}^{1/t} \qquad (\vec{\xi},\vec{\eta} \in \mathbb{R}_+^n)$$

whenever $r, s, t \ge 1$ and $1/r = 1/s + 1/t$. Then it follows from Theorem 2.2 that

$$(2.12) \qquad \|S_A\|_{t,r} \le \|A\|_s \text{ whenever } 1/r = 1/s + 1/t.$$

To formulate a generalization of (2.12) in a compact form, let us introduce some notation. For a symmetric gauge function ϕ and $1 \le p < \infty$, we define $\|\cdot\|_{\phi,p}$ by

$$\|A\|_{\phi,p} = \|\,|A|^p\,\|_\phi^{1/p} \quad (A \in M_n),$$

then $\|\cdot\|_{\phi,p}$ is a unitarily invariant norm which corresponds to the symmetric gauge function $\phi(\vec{\xi^p})^{1/p}$. In fact, $\|\cdot\|_{\phi,p}$ is a norm, hence $\phi(\vec{\xi^p})^{1/p}$ is a symmetric gauge function. We have to show only the convexity property of $\|\cdot\|_{\phi,p}$. For $A, B \in M_n$ and $0 < \lambda < 1$ we know by a result of K. Fan (see [19, p.243])

$$\vec{s}(\lambda A + (1-\lambda)B) \prec_w \lambda\vec{s}(A) + (1-\lambda)\vec{s}(B).$$

Since t^p is a non-decreasing convex function of $t \ge 0$,

$$\begin{aligned}\vec{s}(\lambda A + (1-\lambda)B)^p &\prec_w \{\lambda\vec{s}(A) + (1-\lambda)\vec{s}(B)\}^p \\ &\le \lambda\vec{s}(A)^p + (1-\lambda)\vec{s}(B)^p.\end{aligned}$$

Since the symmetric gauge function ϕ is monotone with respect to weak majorization, it follows that

$$\|\lambda A + (1-\lambda)B\|_{\phi,p}^p \le \lambda\|A\|_{\phi,p}^p + (1-\lambda)\|B\|_{\phi,p}^p.$$

Therefore the unit ball $\{X : \|X\|_{\phi,p} \le 1\}$ is a convex set, that is, $\|\cdot\|_{\phi,p}$ is a norm. Then $\|A\|_{\phi,p}$ is a continuous nonincreasing function of p. If we put

$$\|A\|_{\phi,\infty} = \|A\|_\infty,$$

it is in accordence with other $\|\cdot\|_{\phi,\infty}$ in the sence that

$$\|A\|_{\phi,\infty} = \lim_{p\to\infty} \|A\|_{\phi,p}. \tag{2.13}$$

For $p, q \geq 1, \|S_A\|_{\{\phi,p\},\{\phi,q\}}$ is defined by

$$\|S_A\|_{\{\phi,p\},\{\phi,q\}} \equiv \sup_{X\in M_n} \frac{\|A\circ X\|_{\phi,q}}{\|X\|_{\phi,p}} \quad (A \in M_n), \tag{2.14}$$

then we have the following theorem.

THEOREM 2.4. *Let ϕ be a symmetric gauge function. If $p_0, p_1, p_2 \geq 1$ and $1/p_0 = 1/p_1 + 1/p_2$ then*

$$\|S_A\|_{\{\phi,p_2\},\{\phi,p_0\}} \leq \|A\|_{\phi,p_1} \quad (A \in M_n).$$

PROOF: In view of (2.13), we may assume $p_0, p_1, p_2 < \infty$. Then by Theorem 2.2 it suffices to prove that

$$\phi(\vec{\xi}^{p_0} \circ \vec{\eta}^{p_0})^{1/p_0} \leq \phi(\vec{\xi}^{p_1})^{1/p_1} \cdot \phi(\vec{\eta}^{p_2})^{1/p_2} \quad (\vec{\xi}, \vec{\eta} \in \mathbb{R}^n_+)$$

To this end, let $p = p_1/p_0$ and $q = p_2/p_0$. Since $1/p + 1/q = 1$ by assumption, we have Young's inequality (see [9, p.111])

$$\xi\eta \leq \frac{1}{p}\xi^p + \frac{1}{q}\eta^q \quad (\xi, \eta > 0);$$

hence

$$\phi(\vec{\xi}^{p_0} \circ \vec{\eta}^{p_0}) \leq \frac{1}{p}\phi(\vec{\xi}^{p_1}) + \frac{1}{q}\phi(\vec{\eta}^{p_2}). \tag{2.15}$$

Replacing $\vec{\xi}$ and $\vec{\eta}$ by $t\vec{\xi}$ and $(1/t)\vec{\eta}$ respectively and taking the infimum on the right hand side of (2.15) for $t > 0$, we arrive at the inequality

$$\phi(\vec{\xi}^{p_0} \circ \vec{\eta}^{p_0}) \leq \phi(\vec{\xi}^{p_1})^{p_0/p_1} \cdot \phi(\vec{\eta}^{p_2})^{p_0/p_2} \quad (\vec{\xi}, \vec{\eta} \in \mathbb{R}^n_+). \quad \blacksquare$$

Theorem 2.4 admits a natural generalization to an (m+1)-tuple $p_0, p_1, \cdots, p_m$ with $1/p_0 = \sum_{j=1}^m 1/p_j$.

COROLLARY 2.5. *Suppose that real numbers $p_0, p_1, \cdots p_m$ satisfy $p_i \geq 1$ $(i = 1, 2, \cdots, m)$ and $1/p_0 = 1/p_1 + \cdots + 1/p_m$. Then for any $A_1, \cdots, A_m \in M_n$*

$$\|A_1 \circ \cdots \circ A_m\|_{\phi, p_0} \leq \prod_{k=1}^{m} \|A_k\|_{\phi, p_k}.$$

PROOF: We shall prove by induction. We already showed for m=2 in Theorem 2.4. Suppose m ≥ 3 and the assertion is true for $m-1$. Define $\tilde{p}_{m-1}$ by

$$1/\tilde{p}_{m-1} = 1/p_{m-1} + 1/p_m.$$

Then since

$$1/p_0 = 1/p_1 + \cdots + 1/p_{m-2} + 1/\tilde{p}_{m-1},$$

by induction assumption we have

$$\begin{aligned} \|A_1 \circ \cdots \circ A_m\|_{\phi, p_0} &= \|A_1 \circ \cdots \circ A_{m-2} \circ (A_{m-1} \circ A_m)\|_{\phi, p_0} \\ &\leq \left\{ \prod_{k=1}^{m-2} \|A_k\|_{\phi, p_k} \right\} \cdot \|A_{m-1} \circ A_m\|_{\phi, \tilde{p}_{m-1}}. \end{aligned} \tag{2.16}$$

Again by the assertion for the case of m=2, we have

$$\|A_{m-1} \circ A_m\|_{\phi, \tilde{p}_{m-1}} \leq \|A_{m-1}\|_{\phi, p_{m-1}} \cdot \|A_m\|_{\phi, p_m},$$

which together with (2.16) completes the proof. ∎

In (2.14) if $p = q = 1$ then we write $\|S_A\|_\phi = \|S_A\|_{\{\phi,p\},\{\phi,q\}}$. Specializing Theorem 2.4 to the case $p_0 = 1, p_1 = \infty$, and $p_2 = 1$ we obtain the following.

COROLLARY 2.6. *For every symmetric gauge function ϕ,*

$$\|S_A\|_\phi \leq \|A\|_\infty \quad (A \in M_n) \tag{2.17}$$

If a symmetric gauge function ϕ is *normalized* as

$$\phi(1, 0, \cdots, 0) = 1,$$

then

$$\|A\|_\phi \geq \|A\|_\infty \quad (A \in M_n), \tag{2.18}$$

with equality for A of rank one. The following is now immediate from (2.17) and (2.18) which was conjectured by Marcus, Kidman and Sandy [18].

COROLLARY 2.7. *If a symmetric gauge function ϕ is normalized, then $\|\cdot\|_\phi$ is submultiplicative with respect to Schur multiplication:*

$$\|A \circ B\|_\phi \leq \|A\|_\phi \cdot \|B\|_\phi \quad (A, B \in M_n).$$

Our Theorem 2.4 implies that for every symmetric gauge function ϕ the following inequality holds:

$$\|A \circ B\|_\phi \leq \|\,|A|^p\,\|_\phi^{1/p} \cdot \|\,|B|^q\,\|_\phi^{1/q} \quad \text{whenever } 1/p + 1/q = 1,$$

which is an analog of the well-known Hölder inequality for numerical sequences:

$$\sum_{i=1}^{m} |\xi_i \eta_i| \leq \left\{ \sum_{i=1}^{m} |\xi_i|^p \right\}^{1/p} \cdot \left\{ \sum_{i=1}^{m} |\eta_i|^q \right\}^{1/q}$$

In particular, if we put $p_0 = 1$ and $p_1 = p_2 = 2$ we can obtain Cauchy-Schwarz type inequality: if ϕ is a symmetric gauge function, then

$$\|A \circ B\|_\phi^2 \leq \|A^* A\|_\phi \cdot \|B^* B\|_\phi.$$

An inequality of this type is discussed by Horn and Mathias [16].

3. Schatten p-norm

3.1 Preliminaries

For $A \in M_n$ and $p \geq 1$, the Schatten p-norm is defined by

$$(3.1) \qquad ||A||_p \equiv \left(\sum_{i=1}^{n} s_i^p(A) \right)^{1/p}.$$

In particular when $p = 1, 2$ and ∞, norm $||\cdot||_p$ is called the *trace norm*, the *Hilbert-Schmidt* (or *Frobenius*)*norm* and the *spectral* (or *operator*)*norm* respectively.

We denote the operator norm of S_A from the Banach space $(M_n, ||\cdot||_p)$ to the Banach space $(M_n, ||\cdot||_q)$ by $||S_A||_{p,q}$, that is,

$$(3.2) \qquad ||S_A||_{p,q} \equiv \sup_{X \in M_n} \frac{||A \circ X||_q}{||X||_p} \quad (A \in M_n),$$

and write $||S_A||_p \equiv ||S_A||_{p,p}$.

Recall a norm $||\cdot||$ on M_n is called *unitarily invariant* if $||A|| = ||UAV||$ for all $A \in M_n$ and for all unitaries $U, V \in M_n$. For arbitrary $p \geq 1$, $||\cdot||_p$ is an example of a unitarily invariant norm. It is known (see [8, p.132]) that $||\cdot||_q$ becomes the dual norm of $||\cdot||_p$ for $p, q \geq 1$ such that $1/p + 1/q = 1$. The dual map of S_A on the Banach space $(M_n, ||\cdot||_p)$ is given by $S_{\overline{A}}$ on $(M_n, ||\cdot||_q)$; in fact, for any $B, C \in M_n$ then

$$\begin{aligned} \langle S_A(B)|C \rangle &= \mathrm{tr}((A \circ B) \cdot C^*) \\ &= \mathrm{tr}((A \circ {}^tC^*) \cdot {}^tB) \\ &= \mathrm{tr}(B \cdot {}^t(A \circ {}^tC^*)) \\ &= \mathrm{tr}(B \cdot (\overline{A} \circ C)^*) \\ &= \langle B|\overline{A} \circ C \rangle \\ &= \langle B|S_{\overline{A}}(C) \rangle. \end{aligned}$$

Here to show the second equality we note the following known result (cf.[14]):

$$\mathrm{tr}((X \circ Y) \cdot Z) = \mathrm{tr}((X \circ {}^tZ) \cdot {}^tY)$$

for any $X, Y, Z \in M_n$. It is easy to show $\|S_A\|_p = \|S_{\overline{A}}\|_q = \|S_A\|_q$ if $p, q \geq 1$ and $1/p + 1/q = 1$.

In this section, we will present some convex property of the mapping $[1, \infty] \ni p \longmapsto \|S_A\|_p$ for any given $A \in M_n$.

3.2 Known results of induced norms for S_A

Before we mention the main results in this section, we state the known results about induced norms of S_A. I. Schur [29] showed, as already remarked, that

$$\|A \circ B\|_\infty \leq \|A\|_\infty \cdot \|B\|_\infty \quad (A, B \in M_n). \tag{3.3}$$

Schur's result is equivalent to the following statement:

$$\|S_A\|_\infty \leq \|A\|_\infty. \tag{3.4}$$

We denote the Euclidean norms of the columns of A, arranged in non-increasing order, by $c_1(A) \geq c_2(A) \geq \cdots \geq c_n(A)$ and the Euclidean norms of the rows of A by $r_1(A) \geq r_2(A) \geq \cdots \geq r_n(A)$. For $A \in M_n$ and $\alpha \in (0,1)$, we write $t_i(A, \alpha) = \{p_i(A, \alpha) \cdot q_i(A, 1-\alpha)\}^{1/2}$ where $p_i(A, \alpha)$ is the ith largest main diagonal entry of $(AA^*)^\alpha$ and $q_i(A, 1-\alpha)$ is the ith largest main diagonal entry of $(A^*A)^{1-\alpha}$.

Then C. Ong [24] proved that

$$\|S_A\|_\infty \leq \min\{r_1(A), c_1(A)\} \quad (A \in M_n), \tag{3.5}$$

and M. E. Walter [31] proved that for $\alpha \in (0,1)$ and $A \in M_n$

$$\|S_A\|_\infty \leq t_1(A, \alpha). \tag{3.6}$$

Next in (3.8) and (3.10),we consider a unitarily invariant norm $\|\cdot\|_\phi$ and operator norm of S_A with respect to $\|\cdot\|_\phi$, that is,

$$\|S_A\|_\phi = \sup_{X \in M_n} \frac{\|A \circ X\|_\phi}{\|X\|_\phi}. \tag{3.7}$$

As we saw in Corollary 2.6 we have that

$$\|S_A\|_\phi \le \|A\|_\infty \quad (A \in M_n). \tag{3.8}$$

Ando, Horn and Johnson [3] showed

$$\|S_A\|_\phi \le \inf\{c_1(X)\cdot c_1(Y) : X^*Y = A,\ X, Y \in M_{rn},\ r \ge 1\} \quad (A \in M_n). \tag{3.9}$$

Haagerup (cf. [25, p.119]) showed the value of the right hand side of (3.9) is equal to $\|S_A\|_\infty$, hence it follows that

$$\|S_A\|_\phi \le \|S_A\|_\infty \quad (A \in M_n). \tag{3.10}$$

3.3 Convexity of $\|S_A\|_p$

For the Hilbert-Schmidt norm $\|\cdot\|_2$ we can show the following:

LEMMA 3.1. *If $\|\cdot\|$ is any norm on M_n then*

$$\|S_A\|_2 \le \|S_A\| \quad (A \in M_n). \tag{3.11}$$

PROOF: Let $A = [a_{ij}]$. Then

$$\begin{aligned}\|S_A\|_2 &= \sup\{\|A \circ B\|_2 : \|B\|_2 = 1\}\\ &= \sup\{(\sum_{i,j=1}^{n} |a_{ij}b_{ij}|^2)^{1/2} : (\sum_{i,j=1}^{n} |b_{ij}|^2)^{1/2} = 1\}\\ &= \max_{1\le i,j\le n} |a_{ij}|\end{aligned}$$

Let $e_i = {}^t(0, \cdots, 0\overset{(i)}{1}, 0, \cdots, 0)$ for $i = 1, \cdots, n$. Since $S_A(E_{ij}) = a_{ij}E_{ij}$ where $E_{i,j} = e_i \otimes e_j^*$, a_{ij} is an eigenvalue of S_A for i, j such that $1 \le i, j \le n$. It follows from this fact that

$$\|S_A\| \ge \frac{\|A \circ E_{ij}\|}{\|E_{ij}\|} = |a_{ij}|$$

for any norm $\|\cdot\|$ on M_n and for any i, j of $1 \le i, j \le n$; hence we have

$$\|S_A\|_2 = \max_{1 \le i,j \le n} |a_{ij}| \le \|S_A\|. \qquad \blacksquare$$

Remark. It is known (see [31]) that if $A \ge 0$ then

$$\|S_A\|_\infty = \max_{1 \le i \le n} |a_{ii}| \text{ where } A = [a_{ij}].$$

Hence, as in the proof of Lemma 3.1, we can show that if $A \ge 0$ then

$$\|S_A\|_\phi = \max_{1 \le i,j \le n} |a_{ij}|$$

.

As mentioned before, we have $\|S_A\|_p = \|S_A\|_q$ for $p, q \ge 1$ such that $1/p + 1/q = 1$; in particular, $\|S_A\|_1 = \|S_A\|_\infty$, and from Lemma 3.1 it follows

$$\min_{1 \le p \le \infty} \|S_A\|_p = \|S_A\|_2.$$

We are interested in the behavior of the function: $[1,\infty] \ni p \longmapsto \|S_A\|_p$ for a given $A \in M_n$.

To show the main result in this section, we need a celebrated convexity theorem of M. Riesz (see [9, p.214]). Let $A \in M_n$ and $1 \le s, t \le \infty$ be given, then we define $\|A\|_{s,t}$ by

$$\|A\|_{s,t} \equiv \sup_{\vec{\xi} \in C^n} \frac{\|A\vec{\xi}\|_t}{\|\vec{\xi}\|_s}$$

where $\|\vec{\xi}\|_s = (\sum_{i=1}^n |\xi_i|^s)^{1/s}$ for $\vec{\xi} = {}^t(\xi_1, \cdots, \xi_n) \in \mathbb{C}^n$.

Lemma 3.2. (M. Riesz) *Let $A \in M_n$ and $\lambda \in (0,1)$ be given. If $s, s_1, s_2, t, t_1, t_2 \ge 1$ and $1/s = \lambda/s_1 + (1-\lambda)/s_2, 1/t = \lambda/t_1 + (1-\lambda)/t_2$ then*

$$\|A\|_{s,t} \le \|A\|_{s_1,t_1}{}^{\lambda} \cdot \|A\|_{s_2,t_2}{}^{1-\lambda}.$$

Theorem 3.3. *Let $A \in M_n$ and $\lambda \in (0,1)$ be given. If $p, p_1, p_2, q, q_1, q_2 \ge 1$ and $1/p = \lambda/p_1 + (1-\lambda)/p_2, 1/q = \lambda/q_1 + (1-\lambda)/q_2$ then*

$$\|S_A\|_{p,q} \le \|S_A\|_{p_1,q_1}{}^{\lambda} \cdot \|S_A\|_{p_2,q_2}{}^{1-\lambda}.$$

PROOF: By the singular value decomposition of matrix (see [13, p.157]), $||B||_s \le 1$ if and only if B is written in the form

$$B = U\,\Sigma\,V$$

where U, V are unitary and $\Sigma = \text{diag}(s_1(A), s_2(A), \cdots, s_n(A))$ such that $\sum_{i=1}^{n} s_i^s(A) \le 1$. Let t, t' be positive real numbers which satisfies $t, t' \ge 1$ and $1/t + 1/t' = 1$. By the definition of dual norm,

$$||S_A(B)||_t = \sup_{||C||_{t'} \le 1} |\text{tr}((A \circ B) \cdot C^*)|.$$

Since $||C||_{t'} \le 1$, C is written by $C = U'\,\Sigma'\,V'$ where U', V' are unitary and $\Sigma' = \text{diag}(s_1(C), s_2(C), \cdots, s_n(C))$ such that $\sum_{i=1}^{n} s_i^{t'}(C) \le 1$. Therefore

$$\begin{aligned} ||S_A||_{s,t} &= \sup_{||B||_s \le 1} ||S_A(B)||_t \\ &= \sup_{||B||_s \le 1} \sup_{||C||_{t'} \le 1} |\text{tr}((A \circ B) \cdot C^*)| \\ &= \sup_{||B||_s \le 1} \sup_{||C||_{t'} \le 1} |\text{tr}((A \circ (U \cdot \text{diag}(\vec{s}(B)) \cdot V)) \cdot V'^* \cdot \text{diag}(\vec{s}(C)) \cdot U'^*)| \end{aligned}$$

where $\vec{s}(B) = {}^t(s_1(B), s_2(B), \cdots, s_n(B))$ and $\vec{s}(C) = {}^t(s_1(C), s_2(C), \cdots, s_n(C))$. Since there exists a matrix $T = T_{U,V,U',V'}$ depending on unitaries U, V, U' and V' such that

$$\langle T(\vec{s}(B)) | \vec{s}(C) \rangle = \text{tr}((A \circ (U \cdot \text{diag}((\vec{s}(B)) \cdot V)) \cdot V'^* \cdot \text{diag}(\vec{s}(C)) \cdot U'^*),$$

we have

$$\begin{aligned} ||S_A||_{s,t} &= \sup_{U,V,U',V'} \sup_{||\vec{\xi}||_s \le 1, ||\vec{\eta}||_{t'} \le 1} |\langle T_{U,V,U',V'} \vec{\xi}, \vec{\eta} \rangle| \\ &= \sup_{U,V,U',V'} ||T_{U,V,U',V'}||_{s,t}. \end{aligned}$$

Hence according to Lemma 3.2 we have

$$\begin{aligned} ||S_A||_{s,t} &\le \sup_{U,V,U',V'} ||T_{U,V,U',V'}||_{s_1,t_1}^{\lambda} \cdot ||T_{U,V,U',V'}||_{s_2,t_2}^{1-\lambda} \\ &\le ||S_A||_{s_1,t_1}{}^{\lambda} \cdot ||S_A||_{s_2,t_2}{}^{1-\lambda}. \quad \blacksquare \end{aligned}$$

COROLLARY 3.4. *Let $A \in M_n$ be given. Then $||S_A||_p$ is a log-convex fuction of $\frac{1}{p}$* $\quad (p \ge 1)$.

PROOF: It follows from Theorem 3.3 that

$$\|S_A\|_p \leq \|S_A\|_{p_1}^{\lambda} \cdot \|S_A\|_{p_2}^{1-\lambda} \quad (A \in M_n)$$

for $p, p_1, p_2 \geq 1$ and $1/p = \lambda/p_1 + (1-\lambda)/p_2$, that is, $\|S_A\|_p$ is a log-convex function of $\frac{1}{p}$ $(p \geq 1)$. ∎

COROLLARY 3.5. *Let $A \in M_n$ be given. Then*

$$\|S_A\|_p \leq \|S_A\|_q \quad (1 \leq q \leq p \leq 2)$$

and

$$\|S_A\|_p \leq \|S_A\|_q \quad (2 \leq p \leq q < \infty)$$

PROOF: From Lemma 3.1, we know

$$\min\{\|S_A\|_p : p \geq 1\} = \|S_A\|_2,$$

and from Corollary 3.4 $\|S_A\|_p$ is a log-convex function of $\frac{1}{p}$ $(p \geq 1)$; hence we can easily show Corollary 3.5. ∎

4. Numerical Radius

4.1 Preliminaries and main result

For $A \in M_n$ the numerical radius $w(A)$ of A is defined by

$$w(A) \equiv \sup_{\vec{x} \in \mathbb{C}^n} \frac{|\langle A\vec{x}|\vec{x}\rangle|}{\|\vec{x}\|^2}$$

where $\langle \cdot|\cdot \rangle$ and $\| \cdot \|$ denote the inner product and the Euclidean norm of vector in $\mathbb{C}^n$, respectively. It is easy to show that

(4.1) $$w(A) \leq \|A\|_\infty \leq 2 \cdot w(A) \quad (A \in M_n).$$

In this section, we will give a factorization of A for the norm of S_A with respect to numerical radius to be at most one. First we mention two characterizations of a matrix X for which $w(X) \leq 1$.

The first one is almost trivial: $w(X) \leq 1$ if ond only if for any real θ the real (or Hermitian) part of $e^{i\theta}X$ is not greater than I, that is,

$$\mathrm{Re}(e^{i\theta}X) \equiv 1/2(e^{i\theta}X + e^{-i\theta}X^*) \leq I \quad (0 \leq \theta \leq 2\pi).$$

The second one is not trivial and is mentioned as a lemma. See [1] for a proof.

LEMMA 4.1. (T. Ando) *For a matrix $X \in M_n$, $w(X) \leq 1$ if and only if there is a Hermitian matrix $Z \in M_n$ such that* $\begin{bmatrix} I+Z & X \\ X^* & I-Z \end{bmatrix} \geq 0.$

The induced norm of S_A with respect to the numerical radius $w(\cdot)$ will be denoted by $\|S_A\|_w$:

$$\|S_A\|_w \equiv \sup_{X \in M_n} \frac{w(A \circ X)}{w(X)} \quad (A \in M_n).$$

It is mentioned in ([25, p.110-119]) that Haagerup succeeded in determining $\|S_A\|_\infty$ in the following form:

HAAGERUP'S THEOREM. *For $A = [a_{ij}] \in M_n$ the following assertions are mutually equivalent.*

(i) $\|S_A\|_\infty \leq 1$.

(ii) *A admits a factorization $A = B^*C$ such that*

$$B^*B \circ I \leq I \text{ and } C^*C \circ I \leq I,$$

where I is the identity (or unit) matrix.

(iii) *There are vectors $\vec{x}_i, \vec{y}_i \in \mathbb{C}^n$ $(i = 1, 2, \cdots, n)$ such that $\|\vec{x}_i\|, \|\vec{y}_i\| \leq 1$ $(i = 1, 2, \cdots, n)$ and*

$$a_{ij} = \langle \vec{x}_j | \vec{y}_i \rangle \quad (i, j = 1, 2, \cdots, n).$$

(iv) *There are* $0 \leq R_1, R_2 \in M_n$ *such that*

$$\begin{bmatrix} R_1 & A \\ A^* & R_2 \end{bmatrix} \geq 0, \; R_1 \circ I \leq I \text{ and } R_2 \circ I \leq I.$$

Now we are going to determine the norm $\|S_A\|_w$, and to derive Haagerup's Theorem as a consequence. Our main theorem in this section is the following:

THEOREM 4.2. *For* $A = [a_{ij}] \in M_n$ *the following assertions are mutually equivalent.*

$(\mathrm{i})_w$ $\|S_A\|_w \leq 1$.

$(\mathrm{ii})_w$ *A admits a factorization* $A = B^*WB$ *such that*

$$B^*B \circ I \leq I \quad \text{and} \quad W^*W \leq I.$$

$(\mathrm{iii})_w$ *There are vectors* $\vec{x}_i \in \mathbb{C}^n \quad (i = 1, 2, \cdots, n)$ *and a matrix* $W \in M_n$ *such that* $\|\vec{x}_i\| \leq 1 \quad (i = 1, 2, \cdots, n)$, $W^*W \leq I$ *and*

$$a_{ij} = \langle W\vec{x}_j | \vec{x}_i \rangle \quad (i = 1, 2, \cdots, n).$$

$(\mathrm{iv})_w$ *There is* $0 \leq R \in M_n$ *such that*

$$\begin{bmatrix} R & A \\ A^* & R \end{bmatrix} \geq 0 \text{ and } R \circ I \leq I.$$

A proof of the theorem is given after a series of lemmas.

4.2 Lemmas for the proof of Theorem

For two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{C}^n$, a rank one matrix is defined by $\vec{x} \otimes \vec{y}^* \equiv \vec{x}\vec{y}^*$.

For a subset $\mathcal{S}$ of M_n the convex hull of $\mathcal{S}$ is defined by

$$conv(\mathcal{S}) = \{\sum_{i=1}^{k} \lambda_i X_i : X_i \in \mathcal{S}, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, k \text{ is arbitrary}\}$$

and $\overline{\mathcal{S}}$ means the closure of $\mathcal{S}$. We denote the dual norm of numerical radius $w(\cdot)$ by $\|\cdot\|_{w^*}$, that is,

$$\|A\|_{w^*} = \sup_{B \in M_n} \frac{|\mathrm{tr}(AB^*)|}{w(B)} \quad (A \in M_n).$$

Then for $\vec{x} \in \mathbb{C}^n$, we have $\|\vec{x} \otimes \vec{x}^*\|_{w^*} = \|\vec{x}\|^2$. In fact,

$$\|\vec{x} \otimes \vec{x}^*\|_{w^*} = \sup_B \frac{|\mathrm{tr}(\vec{x} \otimes \vec{x}^*)B^*|}{w(B)} = \sup_B \frac{|\langle B\vec{x}|\vec{x}\rangle|}{w(B)} \leq \sup_B \frac{w(B) \cdot \|\vec{x}\|^2}{w(B)} = \|\vec{x}\|^2.$$

On the oter hand, since $w(\vec{x} \otimes \vec{x}) = \|\vec{x}\|^2$

$$\|\vec{x} \otimes \vec{x}^*\|_{w^*} \geq \frac{|\mathrm{tr}(\vec{x} \otimes \vec{x})(\vec{x} \otimes \vec{x})|}{w(\vec{x} \otimes \vec{x})} = \frac{\langle(\vec{x} \otimes \vec{x})\vec{x}|\vec{x}\rangle}{w(\vec{x} \otimes \vec{x})} = \frac{\langle \vec{x}|\vec{x}\rangle^2}{\|\vec{x}\|^2} = \|\vec{x}\|^2.$$

LEMMA 4.3. $\mathcal{S} = \{\xi\vec{x} \otimes \vec{x}^* : \ |\xi| = 1, \ \|\vec{x}\| = 1\}$ *is the set of extreme points of the unit ball of* M_n *with respect to* $\|\cdot\|_{w^*}$. *Moreover, If* $\|A\|_{w^*} = 1$ *then there exist a finite family* $\{X_i \in \mathcal{U} : i = 1, \cdots, N\}$ *and a finite family* $\{\lambda_i : i = 1 \cdots N\}$ *of nonnegative real numbers such that*

$$A = \sum_{i=1}^{N} \lambda_i X_i \ \textit{and} \ \|A\|_{w^*} = \sum_{i=1}^{N} \lambda_i .$$

PROOF: Consider M_n as a real vector space of $2n^2$ dimension.

Now by definition of $w(A)$,

$$\begin{aligned} w(A) &= \sup\{|\langle A\vec{x}|\vec{x}\rangle| : \|\vec{x}\| = 1\} \\ &= \sup\{|\mathrm{tr}(A \cdot \vec{x} \otimes \vec{x}^*)| : \|\vec{x}\| = 1\} \\ &= \sup\{|\mathrm{tr}\{A \cdot \sum_i \lambda_i \xi_i(\vec{x}_i \otimes \vec{x}_i^*)\}| : \lambda_i \geq 0, \sum_i \lambda_i = 1, |\xi_i| = 1, \|\vec{x}_i\| = 1\} \qquad (4.2) \\ &= \sup\{|\mathrm{tr}(AY)| : Y \in \overline{\mathcal{Z}}\} \end{aligned}$$

where $\mathcal{Z} = conv\mathcal{S}$.

On the other hand, from the duality theorem (see [27, p.89]),

$$w(A) = \sup\{|\mathrm{tr}(AX)| : \|X\|_{w^*} \leq 1\}. \qquad (4.3)$$

Then we have

$$\|X\|_{w^*} \leq 1 \iff X \in \overline{\mathcal{Z}} \qquad (4.4)$$

by using the separation theorem (see [27, p.58]) and (4.2),(4.3). Next, we will show the following:

(4.5) $$\mathcal{Z} \text{ is compact}$$

and

(4.6) $$\text{all elements of } \mathcal{S} \text{ are extreme points of } \mathcal{Z}.$$

For (4.5), first of all, it is easy to show that $\mathcal{S}$ is compact in M_n. By the Caratheodory theorem (see [26 p.76]), if $X \in \mathcal{Z} = conv\mathcal{S} \subset M_n \simeq \mathbb{R}^{2n^2}$ then X is represented as

$$X = \sum_{i=1}^{N} t_i X_i$$

where $N = 2n^2 + 1, t_i \geq 0, \sum_{i=1}^{N} t_i = 1$ *and* $\vec{x}_i \in \mathcal{S}$.

Let $T = \{t = (t_1, \cdots, t_N) : t_i \geq 0 \text{ and } \sum_{i=1}^{N} t_i = 1\}$. Then $\mathcal{Z}$ is the image of

$$T \times \mathcal{S} \times \cdots \times \mathcal{S}$$

($\mathcal{S}$ occurs N times) under the continuous mapping

$$(t, X_1, \cdots, X_N) \mapsto \sum_{i=1}^{N} t_i X_i$$

Hence it is proven that $\mathcal{Z}$ is compact.

As previously remarked, $\|\vec{x} \otimes \vec{x}\|_{w^*} = 1$ for a vector $\vec{x}$ such that $\|\vec{x}\| = 1$. For (4.6), we must show that if for κ such that $0 \leq \kappa \leq 1$

$$\vec{x} \otimes \vec{x}^* = \kappa Y + (1 - \kappa) Z$$

where $Y = \sum_{i=1}^{N} \lambda_i \xi_i \vec{y}_i \otimes \vec{y}_i^*$ and $Z = \sum_{i=1}^{N} \mu_i \eta_i \vec{z}_i \otimes \vec{z}_i^*$, $\lambda_i, \mu_i > 0, \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \mu_i = 1, |\xi_i| = |\eta_i| = 1$, and $\|\vec{y}_i\| = \|\vec{z}_i\| = 1$ then $\vec{x} \otimes \vec{x}^* = Y = Z$. It suffices to treet the case $N = 2$, that is,

(4.7) $$\vec{x} \otimes \vec{x}^* = \lambda \xi (\vec{y} \otimes \vec{y}^*) + (1 - \lambda) \eta (\vec{z} \otimes \vec{z}^*)$$

for λ such that $0 \leq \lambda \leq 1$ and $|\xi| = |\eta| = 1$ then $\vec{x} \otimes \vec{x}^* = \vec{y} \otimes \vec{y}^* = \vec{z} \otimes \vec{z}^*$ and $\xi = \eta = 1$. From the transformation of the base, we may assume $\vec{x} = {}^t(1, 0, \cdots, 0)$. Let $\vec{y} = {}^t(y_1, \cdots . y_n)$ and $\vec{z} = {}^t(z_1, \cdots, z_n)$; then (4.7) implies

$$\begin{pmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \lambda\xi \begin{pmatrix} |y_1|^2 & y_1\overline{y_2} & \cdots & y_1\overline{y_n} \\ y_2\overline{y_1} & |y_2|^2 & \cdots & y_2\overline{y_n} \\ \vdots & & \ddots & \vdots \\ y_n\overline{y_1} & \cdots & \cdots & |y_n|^2 \end{pmatrix} + (1-\lambda)\eta \begin{pmatrix} |z_1|^2 & z_1\overline{z_2} & \cdots & z_1\overline{z_n} \\ z_2\overline{z_1} & |z_2|^2 & \cdots & z_2\overline{z_n} \\ \vdots & \ddots & \ddots & \vdots \\ z_n\overline{z_1} & \cdots & \cdots & |z_n|^2 \end{pmatrix}$$

hence we have $|y_1|^2 = |z_1|^2 = 1$ and $\xi = \eta = 1$ therefore (4.6) is proven.

From (4.5) it is known that

$$\{X \in M_n : ||A||_{w^*} \leq 1\} = \mathcal{Z},$$

hence we complete the proof of Lemma 4.3. ∎

Given $\vec{x} \in \mathbb{C}^n$, denote by $D_{\vec{x}}$ the diagonal matrix with $\vec{x}$ on the diagonal.

LEMMA 4.4. *$||S_A||_w \leq 1$ if and only if*

$$||D_{\vec{x}} A D_{\vec{x}}^*||_{w^*} \leq ||\vec{x}||^2 \quad (\vec{x} \in \mathbb{C}^n)$$

where $||\cdot||_{w^*}$ denotes the dual norm of $w(\cdot)$

PROOF: The adjoint operator of S_A is given by $S_{\overline{A}}$ where $\overline{A}$ is a complex conjugate as we already showed. Since it is easy to see $||S_A||_w = ||S_{\overline{A}}||_w = ||S_A||_{w^*}$ and from Lemma 4.3 the unit ball for the norm $||\cdot||_{w^*}$ is the absolute convex hull of matrices of the form $\vec{x} \otimes \vec{x}^*$ with $||\vec{x}|| = 1$, the assertion follows from the relations $||\vec{x} \otimes \vec{x}^*||_{w^*} = ||\vec{x}||^2$ and

$$S_A(\vec{x} \otimes \vec{x}^*) = D_{\vec{x}} A D_{\vec{x}}^*. \quad ∎$$

Denote by J_k the k-by-k matrix with all entries equal to ones:

$$J_k = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$

LEMMA 4.5.

$$\|S_A\|_w = \|S_{A\otimes J_k}\|_w.$$

PROOF: We treat only the case $k=2$. The general case ($k>2$) can be proven by a similar argument. It is clear that

$$\|S_A\|_w \le \|S_{A\otimes J_2}\|_w.$$

Therefore it remains to show that $\|S_A\|_w \le 1$ implies $\|S_{A\otimes J_2}\|_w \le 1$.

According to Lemma 4.4 this will follow if it is shown that if

$$\|D_{\vec{x}} A D_{\vec{x}}^*\|_{w^*} \le \|\vec{x}\|^2 \quad (\vec{x}\in\mathbb{C}^n) \tag{4.8}$$

then

$$\left\| \begin{bmatrix} D_{\vec{y}} A D_{\vec{y}}^* & D_{\vec{y}} A D_{\vec{z}}^* \\ D_{\vec{z}} A D_{\vec{y}}^* & D_{\vec{z}} A D_{\vec{z}}^* \end{bmatrix} \right\|_{w^*} \le \|\vec{y}\|^2 + \|\vec{z}\|^2 \quad (\vec{y},\vec{z}\in\mathbb{C}^n). \tag{4.9}$$

Assume (4.8). Given $\vec{y},\vec{z}\in\mathbb{C}^n$, take $\vec{u}\in\mathbb{C}^n$ such that

$$\vec{u}\circ\bar{\vec{u}} = \vec{y}\circ\bar{\vec{y}} + \vec{z}\circ\bar{\vec{z}}. \tag{4.10}$$

Then obviously

$$\|\vec{u}\|^2 = \|\vec{y}\|^2 + \|\vec{z}\|^2. \tag{4.11}$$

We can define, without ambiguity, two diagonal matrices U and V by

$$U = D_{\vec{y}} \cdot D_{\vec{u}}^{-1} \text{ and } V = D_{\vec{z}} \cdot D_{\vec{u}}^{-1}. \tag{4.12}$$

It follows from (4.10) and (4.12) that

$$[U^* \; V^*] \cdot \begin{bmatrix} U \\ V \end{bmatrix} \le I,$$

and hence $\begin{bmatrix} U \\ V \end{bmatrix}$ is a contraction from $\mathbb{C}^n$ to $\mathbb{C}^{2n}$, that is, $\left\|\begin{bmatrix} U \\ V \end{bmatrix}\right\| \le 1$. Then for any $X\in M_{2n}$ we have

$$w\left([U^* \; V^*] \cdot X \cdot \begin{bmatrix} U \\ V \end{bmatrix} \right) \le w(X). \tag{4.13}$$

Now since

$$\begin{bmatrix} D_{\vec{y}}AD_{\vec{y}}^* & D_{\vec{y}}AD_{\vec{z}}^* \\ D_{\vec{z}}AD_{\vec{y}}^* & D_{\vec{z}}AD_{\vec{z}}^* \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \cdot D_{\vec{u}}AD_{\vec{u}}^* \cdot [U^* \ V^*],$$

we have by (4.13)

$$\begin{aligned} \left\| \begin{bmatrix} D_{\vec{y}}AD_{\vec{y}}^* & D_{\vec{y}}AD_{\vec{z}}^* \\ D_{\vec{z}}AD_{\vec{y}}^* & D_{\vec{z}}AD_{\vec{z}}^* \end{bmatrix} \right\|_{w^*} &= \sup_{\substack{X\in M_{2n} \\ w(X)\le 1}} \left| \operatorname{tr}\left(D_{\vec{u}}AD_{\vec{u}}^* \cdot [U^* \ V^*] X \begin{bmatrix} U \\ V \end{bmatrix} \right) \right| \\ &\le \sup_{\substack{Y\in M_n \\ w(Y)\le 1}} |\operatorname{tr}((D_{\vec{u}}AD_{\vec{u}}^*)\cdot Y)| \\ &= \|D_{\vec{u}}AD_{\vec{u}}^*\|_{w^*} \\ &\le \|\vec{u}\|^2 \quad \text{(by (4.8))} \\ &= \|\vec{y}\|^2 + \|\vec{z}\|^2 \quad \text{(by (4.11))}, \end{aligned}$$

proving (4.9). ∎

Let us recall some notions from the theory of C^*-algebras. See [25] for detail. Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras with unit. Let $\mathcal{M}$ be a subspace of $\mathcal{A}$ which contains the unit of $\mathcal{A}$ and is closed under the $*$-operation. A linear map Φ from $\mathcal{M}$ to $\mathcal{B}$ is said to be *unital* if it maps the unit of $\mathcal{A}$ to the unit of $\mathcal{B}$ while it is said to be *positive* if it maps positive elements in $\mathcal{M}$ to positive elements of $\mathcal{B}$. For each $k \ge 1$ the map Φ induces a linear map Φ_k from $M_k(\mathcal{B})$, the space of $\mathcal{M}$-valued k-by-k matrices, to $M_k(\mathcal{B})$ by

$$\Phi_k([a_{ij}]) \equiv [\Phi(a_{ij})] \quad \textit{for } a_{ij} \in \mathcal{M} \quad i,j = 1,\cdots,k.$$

Then Φ is said be *completely positive* if Φ_k is positive for $k = 1, 2, \cdots$.

Now let $\mathcal{M}$ denote the subspace of $M_2(M_n) = M_2 \otimes M_n$, defined by

$$\mathcal{M} \equiv \left\{ \begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix} : X, Y, Z \in M_n \textit{ and } \lambda \in \mathbb{C} \right\}.$$

Then $\mathcal{M}$ contains the unit of $M_2(M_n)$ and is closed under the $*$-operation.

LEMMA 4.6. *Suppose that $\|S_A\|_w \le 1$. Then the linear map Φ from $\mathcal{M}$ to M_n, defined by*

$$\Phi\left(\begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix} \right) = \lambda I + \frac{1}{2}\{A \circ X + A^* \circ Y\} \tag{4.14}$$

is unital and completely positive.

PROOF: By Lemma 4.5 we have

$$\|S_{A\otimes J_k}\|_w = \|S_A\|_w \le 1 \quad (k = 1, 2, \cdots). \tag{4.15}$$

Clearly Φ is unital. First let us prove that Φ is positive. Suppose $\begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix} \ge 0$. This positivity implies that $Y = X^*$ and $\lambda I \pm Z \ge 0$, hence $\lambda \ge 0$. We may assume $\lambda > 0$. Then we have $\begin{bmatrix} I + Z/\lambda & X/\lambda \\ X^*/\lambda & I - Z/\lambda \end{bmatrix} \ge 0$, hence by lemma 4.1 $w\left(\frac{X}{\lambda}\right) \le 1$, that is, $w(X) \le \lambda$. Then the assumption $\|S_A\|_w \le 1$ implies $w(A \circ X) \le \lambda$, and hence

$$\Phi\left(\begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix}\right) = \lambda I + \mathrm{Re}(A \circ X) \ge 0.$$

Next let us prove that the map Φ_k from $M_k(\mathcal{M})$ to $M_k(M_n)$ is positive for $k > 1$. Supposing that a $2k$-by-$2k$ block matrix

$$\begin{bmatrix} \lambda_{ij} I + Z_{ij} & X_{ij} \\ Y_{ij} & \lambda_{ij} I - Z_{ij} \end{bmatrix}_{1 \le i,j \le k} \ge 0, \tag{4.16}$$

we have to prove that

$$[\lambda_{ij} I + 1/2\{A \circ X_{ij} + A^* \circ Y_{ij}\}]_{1 \le i,j \le k} \ge 0. \tag{4.17}$$

A suitable permutation of indices $\{1, 2, \cdots, k\}$ will show that (4.16) is equivalent to

$$\begin{bmatrix} I \otimes [\lambda_{ij}] + [Z_{ij}] & [X_{ij}] \\ [Y_{ij}] & I \otimes [\lambda_{ij}] - [Z_{ij}] \end{bmatrix} \ge 0 \tag{4.18}$$

and (4.17) means

$$I \otimes [\lambda_{ij}] + 1/2\{(A \otimes J_k) \circ [X_{ij}] + (A^* \otimes J_k) \circ [Y_{ij}]\} \ge 0.$$

As in the first part of the proof, (4.18) implies that $[Y_{ij}] = [X_{ij}]^*$ and

$$I \otimes [\lambda_{ij}] \ge \mathrm{Re}\{e^{i\theta}[X_{ij}]\} \quad (0 \le \theta \le 2\pi). \tag{4.19}$$

Since $[\lambda_{ij}] \geq 0$, there is a unitary matrix $U \in M_k$ and $\rho_i \geq 0$ $(i = 1, 2, \cdots, k)$ such that

$$[\lambda_{ij}] = U^* \cdot \operatorname{diag}(\rho_1, \cdots, \rho_k) \cdot U.$$

We may assume $[\lambda_{ij}]$ is invertible. Then (4.19) implies

$$I \otimes \operatorname{diag}(\rho_1, \cdots, \rho_k) \geq \operatorname{Re}\{e^{i\theta}(I \otimes U) \cdot [X_{ij}] \cdot (I \otimes U^*)\}$$

hence we have a numerical radius inequality

$$(4.20) \quad w\left(\left(I \otimes \operatorname{diag}(\rho_1, \cdots, \rho_k)^{-1/2} \cdot U\right) \cdot [X_{ij}] \cdot \left(I \otimes U^* \cdot \operatorname{diag}(\rho_1, \cdots, \rho_k)^{-1/2}\right)\right) \leq 1.$$

Since $\|S_{A \otimes J_k}\|_w \leq 1$ by (4.15), it follows from (4.20) that

$$I \otimes [\lambda_{ij}] + \operatorname{Re}[A \circ X_{ij}] \geq 0$$

which is equivalent to (4.17). ∎

We need the following two theorems for the proof of the next lemma. Denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on a separable Hilbert space $\mathcal{H}$.

ARVESON THEOREM. *Let $\mathcal{M}$ be a subspace of a C^*-algebra $\mathcal{A}$, which contains the unit of $\mathcal{A}$ and is closed under the $*$-operation, and Φ a unital completely positive map from $\mathcal{M}$ to $\mathcal{B}(\mathcal{H})$. Then there exists a completely positive map $\tilde{\Phi}$ from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$, extending Φ:*

$$\tilde{\Phi}(a) = \Phi(a) \quad (a \in \mathcal{M}).$$

See [25, p.81] for a proof.

STINESPRING THEOREM. *Let $\mathcal{A}$ be a C^*-algebra with unit 1 and Φ a completely positive map from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Then there exists a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism π of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$, and a bounded linear map V from $\mathcal{H}$ to $\mathcal{K}$ such that $\|\Phi(1)\| = \|V\|^2$ and*

$$\Phi(a) = V^* \pi(a) V \quad (a \in \mathcal{A}).$$

See [25, p.43] for a proof.

LEMMA 4.7. *If $\|S_A\|_w \le 1$, there is a Hilbert space $\mathcal{K}$ and linear maps $\tilde{B}, \tilde{C}$ from $\mathbb{C}^n$ to $\mathcal{K}$ such that*

$$A = \tilde{B}^*\tilde{C} \tag{4.21}$$

and

$$\tilde{B}^*\tilde{B} = \tilde{C}^*\tilde{C} \text{ and } \tilde{B}^*\tilde{B} \circ I \le I. \tag{4.22}$$

PROOF: Since the linear map Φ from $\mathcal{M}$ to $M_n \simeq \mathcal{B}(\mathbb{C}^n)$, defined by (4.14), is unital and completely positive by Lemma 4.6, according to the Arveson Theorem and the Stinespring Theorem there is a Hilbert space $\mathcal{K}$, a $*$-homomorphism π of the C*-algebra $M_2(M_n)$ into $\mathcal{B}(\mathcal{K})$, and a linear map V from $\mathbb{C}^n$ to $\mathcal{K}$ such that

$$\Phi\left(\begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix}\right) = V^* \cdot \pi\left(\begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix}\right) \cdot V. \tag{4.23}$$

Then it follow from (4.23) that

$$V^* \cdot \pi\left(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\right) \cdot V = \frac{1}{2} A \circ X,$$

$$V^* \cdot \pi\left(\begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot V = V^* \cdot \pi\left(\begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}\right) \cdot V$$

and $V^*V = I$. Let $\{e_j\}$ be the canonical orthonormal basis of $\mathbb{C}^n$. Define $\tilde{B}$ and $\tilde{C}$ by

$$\tilde{B}e_j = \sqrt{2/n} \sum_{p=1}^{n} \pi\left(\begin{bmatrix} E_{pj} & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot V e_j \quad (j = 1, 2, \cdots, n) \tag{4.24}$$

$$\tilde{C}e_j = \sqrt{2/n} \sum_{p=1}^{n} \pi\left(\begin{bmatrix} 0 & E_{pj} \\ 0 & 0 \end{bmatrix}\right) \cdot V e_j \quad (j = 1, 2, \cdots, n) \tag{4.25}$$

where $E_{ij} = e_i \otimes e_j^*$. For $i, j = 1, 2, \cdots, n$ we have by (4.24) and (4.25)

$$\begin{aligned} \langle \tilde{B}^*\tilde{C}e_j | e_i \rangle &= \frac{2}{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \langle V^* \cdot \pi\left(\begin{bmatrix} E_{ip} & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \pi\left(\begin{bmatrix} 0 & E_{qj} \\ 0 & 0 \end{bmatrix}\right) \cdot V e_j | e_i \rangle \\ &= 2 \langle V^* \cdot \pi\left(\begin{bmatrix} 0 & E_{ij} \\ 0 & 0 \end{bmatrix}\right) \cdot V e_j | e_i \rangle = a_{ij}; \end{aligned}$$

hence $\tilde{B}^*\tilde{C} = A$. Further for $i, j = 1, 2, \cdots, n$

$$\langle \tilde{B}^*\tilde{B}e_j|e_i\rangle = 2\langle V^* \cdot \pi\left(\begin{bmatrix} E_{ij} & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot Ve_j|e_i\rangle$$

and

$$\langle \tilde{C}^*\tilde{C}e_j|e_i\rangle = 2\langle V^* \cdot \pi\left(\begin{bmatrix} 0 & 0 \\ 0 & E_{ij} \end{bmatrix}\right) \cdot Ve_j|e_i\rangle;$$

hence $\tilde{B}^*\tilde{B} = \tilde{C}^*\tilde{C}$. Finally

$$\begin{aligned} 2\langle \tilde{B}^*\tilde{B}e_j|e_j\rangle &= \langle \tilde{B}^*\tilde{B}e_j|e_j\rangle + \langle \tilde{C}^*\tilde{C}e_j|e_j\rangle \\ &= 2\langle V^* \cdot \pi\left(\begin{bmatrix} E_{jj} & 0 \\ 0 & E_{jj} \end{bmatrix}\right) \cdot Ve_j|e_j\rangle \\ &\le 2\langle V^*Ve_j|e_j\rangle = 2, \end{aligned}$$

hence $\tilde{B}^*\tilde{B} \circ I \le I$. ∎

LEMMA 4.8. *If $\|S_A\|_w \le 1$, there exist $B, W \in M_n$ such that*

$$A = B^*WB$$

and

$$B^*B \circ I \le I, \quad W^*W \le I$$

PROOF: By Lemma 4.7 there are linear maps $\tilde{B}, \tilde{C}$ from $\mathbb{C}^n$ to a Hilbert space $\mathcal{K}$ satisfying (4.21) and (4.22). Then $\tilde{B}$ and $\tilde{C}$ have the same *modulus*:

$$|\tilde{B}| \equiv (\tilde{B}^*\tilde{B})^{1/2} = (\tilde{C}^*\tilde{C})^{1/2} \equiv |\tilde{C}|.$$

Let $B \equiv |\tilde{B}|$. Then first $B^*B \circ I = \tilde{B}^*\tilde{B} \circ I \le I$. Next there are linear maps U, V from $\mathbb{C}^n$ to $\mathcal{K}$ such that

$$\tilde{B} = UB, \; U^*U = I \text{ and } \tilde{C} = VB, \; V^*V = I.$$

Let $W \equiv U^*V$. Then we have $W^*W \le I$ and

$$A = \tilde{B}^*\tilde{C} = B^*U^*VB = B^*WB. \quad ∎$$

LEMMA 4.9. *If* $\begin{bmatrix} R & A \\ A^* & R \end{bmatrix} \geq 0$ *for some* $0 \leq R \in M_n$ *with* $R \circ I \leq I$, *then* $\|S_A\|_w \leq 1$.

PROOF: Take $X \in M_n$ with $w(X) \leq 1$. Then by Lemma 4.1 there is $Z \in M_n$ such that

$$\begin{bmatrix} I+Z & X \\ X^* & I-Z \end{bmatrix} \geq 0.$$

Then according to the Schur product theorem (1.3), we have

$$\begin{bmatrix} R\circ(I+Z) & A\circ X \\ A^*\circ X^* & R\circ(I-Z) \end{bmatrix} \geq 0. \tag{4.26}$$

Since $R \circ I \leq I$, it follows from (4.26) that, with $U \equiv R \circ Z$,

$$\begin{bmatrix} I+U & A\circ X \\ A^*\circ X^* & I-U \end{bmatrix} \geq 0. \tag{4.27}$$

Again using Lemma 4.1 we can conclude from (4.27) that $w(A \circ X) \leq 1$, and hence $\|S_A\|_w \leq 1$. ∎

Proof of Theorem 4.2. $(\mathrm{i})_w$ implies $(\mathrm{ii})_w$ by Lemma 4.8. Equivalence of $(\mathrm{ii})_w$ and $(\mathrm{iii})_w$ is immediate by writing $B = [\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n]$. Implication $(\mathrm{ii})_w \Longrightarrow (\mathrm{iv})_w$ is seen by taking $R \equiv B^*B$. In fact, $R \circ I \leq I$ and

$$\begin{bmatrix} R & A \\ A^* & R \end{bmatrix} = \begin{bmatrix} B^* & 0 \\ B^*W^* & B^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I-W^*W \end{bmatrix} \begin{bmatrix} B & WB \\ 0 & B \end{bmatrix} \geq 0.$$

Finally implication $(\mathrm{iv})_w \Longrightarrow (\mathrm{i})_w$ follows from Lemma 4.9. ∎

Turning to Haagerup's Theorem, remark first that equivalence of (ii),(iii) and (iv) as well as implication (iv) $\Longrightarrow$ (i) is found in [25, p.110-119] and is shown just as in the proof of Theorem 4.2.

For a proof of (i) $\Longrightarrow$ (iv), we need one more lemma of independent interest.

LEMMA 4.10. *If* $\mathbf{A} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$, *then*

$$\|S_A\|_\infty = \|S_{\mathbf{A}}\|_w \quad (A, 0 \in M_n)$$

PROOF: Remark first that by (4.1) for a 2n-by-2n block matrix $\begin{bmatrix} B & D \\ C & E \end{bmatrix}$,

$$2w\left(\begin{bmatrix} B & D \\ C & E \end{bmatrix}\right) \geq \left\|\begin{bmatrix} B & D \\ C & E \end{bmatrix}\right\|_\infty \geq \left\|\begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}\right\|_\infty = \|D\|_\infty.$$

On the other hand, it is known (see [10]) that

$$w\left(\begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}\right) = \frac{1}{2}\|D\|_\infty.$$

Therefore we have

$$\begin{aligned} \|S_{\mathbf{A}}\|_w &= \sup\left\{ w\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} B & D \\ C & E \end{bmatrix}\right) : w\left(\begin{bmatrix} B & D \\ C & E \end{bmatrix}\right) \leq 1 \right\} \\ &= \sup\left\{ w\left(\begin{bmatrix} 0 & A \circ D \\ 0 & 0 \end{bmatrix}\right) : \|D\|_\infty \leq 2 \right\} \\ &= \sup\left\{ \frac{1}{2}\|A \circ D\|_\infty : \|D\|_\infty \leq 2 \right\} \\ &= \|S_A\|_\infty. \quad \blacksquare \end{aligned}$$

Proof of implication (i) $\Longrightarrow$ (iv) in Haagerup's Theorem. Suppose that $\|S_A\|_\infty = 1$. Since by Lemma 4.10 if we put $\mathbf{A} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ then $\|S_A\|_\infty = \|S_{\mathbf{A}}\|_w$, according to Theorem 4.2 there are $R_{ij} \in M_n$ $(i, j = 1, 2)$ such that $\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \geq 0$, $R_{11} \circ I \leq I$, $R_{22} \circ I \leq I$ and

$$\begin{bmatrix} R_{11} & R_{12} & 0 & A \\ R_{21} & R_{22} & 0 & 0 \\ 0 & 0 & R_{11} & R_{12} \\ A^* & 0 & R_{21} & R_{22} \end{bmatrix} \geq 0.$$

Then, with $R_1 \equiv R_{11}$ and $R_2 \equiv R_{22}$, we have (iv):

$$\begin{bmatrix} R_1 & A \\ A^* & R_2 \end{bmatrix} \geq 0, \quad R_1 \circ I \leq I, \quad R_2 \circ I \leq I. \quad \blacksquare$$

4.3 Related results

COROLLARY 4.11.

$$\|S_A\|_\infty \le \|S_A\|_w \le 2\|S_A\|_\infty \quad (A \in M_n).$$

PROOF: To see the first inequality, let $\|S_A\|_w = 1$. Then $(\mathrm{iv})_w$ implies (iv) with $R_1 = R_2 \equiv R$. The second inequality follows immediately from (4.1). ∎

Johnson [17] showed the inequality

$$w(A \circ B) \le 2w(A) \cdot w(B) \quad (A, B \in M_n)$$

which is equivalent to

$$\|S_A\|_w \le 2w(A) \quad (A \in M_n).$$

In view of (4.1) the following result gives a refinement.

COROLLARY 4.12. $\|S_A\|_w \le \|A\|_\infty \quad (A \in M_n)$.

PROOF: If $\|A\|_\infty = 1$, take $R = I$ in $(\mathrm{iv})_w$. ∎

COROLLARY 4.13. *If A is Hermitian, then* $\|S_A\|_\infty = \|S_A\|_w$.

PROOF: If $\|S_A\|_\infty = 1$, by Haagerup's theorem there are $0 \le R_1,\ R_2 \in M_n$ satisfying (iv). Since $A = A^*$, $R = 1/2(R_1 + R_2)$ satisfies $(\mathrm{iv})_w$. Therefore $\|S_A\|_\infty \ge \|S_A\|_w$. The converse inequality follows from Corollary 4.11. ∎

COROLLARY 4.14. *If $A = [a_{ij}]$ is positive semidefinite,*

$$\|S_A\|_w = \max_i a_{ii}.$$

PROOF: Since

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} \ge 0,$$

the inequality $\|S_A\|_w \leq \max_i a_{ii}$ follows from our Theorem. The converse inequality is immediate because

$$a_{ii} = w(S_A(E_{ii})) \quad (i = 1, 2, \cdots, n). \quad \blacksquare$$

Remark that since here $\|S_A\|_w = \|S_A\|_\infty$ by Corollary 4.13 and the map S_A is positive, the assertion of Corollary 4.14 is an immediate consequence of a general result that a positive linear map on a C^*-algebra attains its norm on the unit element (see [25, p.16]):

$$\|S_A\|_\infty = \|S_A(I)\|_\infty = \|A \circ I\|_\infty = \max_i a_{ii}.$$

Corollary 4.15. *If* $\mathbf{A} = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$, *then*

$$\|S_A\|_\infty = \|S_{\mathbf{A}}\|_w \quad (A \in M_n).$$

Proof: Since $\mathbf{A} = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ is Hermitian, it suffices to prove

$$\|S_A\|_\infty = \|S_{\mathbf{A}}\|_\infty \quad (A \in M_n),$$

which is, however, immediate from the definition of norm by using the obvious relation

$$\left\| \begin{bmatrix} B & D \\ C & E \end{bmatrix} \right\|_\infty \geq \left\| \begin{bmatrix} 0 & D \\ C & 0 \end{bmatrix} \right\|_\infty = \max\{\|C\|_\infty, \|D\|_\infty\}. \quad \blacksquare$$

Corollary 4.16. *If A is unitary, then* $\|S_A\|_\infty = \|S_A\|_w = 1$.

Proof: Inequality $\|S_A\|_w \leq 1$ follows from Corollary 4.12. On the other hand, since the unitarity of A implies that the Schur product $A \circ \overline{A}$ of A and its complex conjugate $\overline{A}$ is doubly stochastic, we have $\|A \circ \overline{A}\|_\infty \geq 1$ (see [24]), and hence $\|S_A\|_\infty \geq 1$ because $\|\overline{A}\|_\infty = \|A\|_\infty = 1$. Now the assertion follows from Corollary 4.11. ∎

From the proof of Lemma 2.1 we have for $A, B \in M_n$,

$$\vec{s}(A \circ B) \prec_w \frac{\vec{s}(|A| \circ |B|) + \vec{s}(|A^*| \circ |B^*|)}{2}.$$

Hence we have

$$\|A \circ B\|_\infty \le \frac{1}{2}\{\| \, |A| \circ |B| \, \|_\infty + \| \, |A^*| \circ |B^*| \, \|_\infty\}.$$

Therefore we obtain

$$\begin{aligned} \|S_A\|_\infty &= \sup\{\|A \circ X\|_\infty : X \in M_n, \|X\|_\infty \le 1\} \\ &= \sup\{\|A \circ U\|_\infty : U \text{ is a unitary}\} \\ &\le \frac{1}{2}\{\| \, |A| \circ I\|_\infty + \| \, |A^*| \circ I\|_\infty\} \\ &= \frac{1}{2}\{\|S_{|A|}\|_\infty + \|S_{|A^*|}\|_\infty\} \end{aligned}$$

Moreover, if A is normal then $\|S_A\|_\infty \le \|S_{|A|}\|_\infty$. The following result is an anology of this inequality with respect to the norm induced by numerical radius.

COROLLARY 4.17. *For any* $A \in M_n$

(4.28) $$\|S_A\|_w \le \|S_{|A|+|A^*|}\|_w$$

If A *is normal, that is,* $|A| = |A^*|$, *then*

$$\|S_A\|_w \le \|S_{|A|}\|_w.$$

PROOF: Inequality (4.28) follows from Corollary 4.14 and

$$\begin{bmatrix} |A| + |A^*| & A \\ A^* & |A| + |A^*| \end{bmatrix} \ge 0,$$

which is a consequence of the inequality described in (2.8), that is,

(4.29) $$\begin{bmatrix} |A^*| & A \\ A^* & |A| \end{bmatrix} \ge 0.$$

If A is normal, (4.29) becomes

$$\begin{bmatrix} |A| & A \\ A^* & |A| \end{bmatrix} \ge 0.$$

and we can take $R = |A|$ instead of $|A| + |A^*|$. ∎

Let us show by example that the inequality in Corollary 4.11 as well as inequality (4.28) is best possible.

Example. Consider $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Since $A = U \cdot \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \cdot U^*$ with unitary $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, we have

$$w(A) = w\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right) = 1,$$

and

$$\|A\|_\infty = \left\|\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right\|_\infty = 2.$$

Hence $\|S_A\|_w \leq \|A\|_\infty = 2$ by Corollary 4.12. Since $S_A(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $w\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2$ we can conclude that $\|S_A\|_w = 2$. Since, with $V \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$S_A(X) = VX \quad (X \in M_n),$$

we have $\|S_A\|_\infty = 1$. Further it is easy to see that $|A| = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $|A^*| = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Hence

$$\left\|S_{|A|+|A^*|}\right\|_w = \|S_{2I}\|_w = 2 = \|S_A\|_w = 2\,\|S_A\|_\infty.$$

Therefore the inequality in Corollary 4.11 and inequality (4.28) are best possible.

In closing this section, we give a remark. It would be pleasant to be able to write condition $(\text{ii})_w$ of Theorem 4.2 as

$(\text{ii})'_w$: A admits a factorization $A = B^*C$ such that $B^*B = C^*C$ and $B^*B \circ I \leq I$,

because it would then be parallel to (ii) of Haagerup's Theorem. This alternative formulation would be correct if the contraction W in $(\text{ii})_w$ could be chosen to be a unitary. Y. Nakamura has shown that it is not the case, so the alternative fomulation $(\text{ii})'_w$ is not correct.

5. Operator Radius Norm

5.1 Known results of operator radius

For $\rho > 0$, a matrix $A \in M_n$ is called a *ρ-contraction* if A has a unitary ρ-dilation, that is, there is a Hilbert space $\mathcal{K}$ containing $\mathbb{C}^n$ as a subspace, and a unitary operator U on $\mathcal{K}$ such that

(5.1) $$A^m = \rho P U^m|_{\mathbb{C}^n} \quad (m = 1, 2, \cdots)$$

where P is the projection from $\mathcal{K}$ to $\mathbb{C}^n$ (cf. [30, p.45]).

Holbrook [10] defined the *operator radius* $w_\rho(\cdot)$ $(0 < \rho < \infty)$ by

$$w_\rho(A) \equiv \inf\{r > 0 : \frac{1}{r}A \text{ is } \rho\text{-contraction}\}.$$

The spectral norm $\|\cdot\|_\infty$ and the numerical radius are incorporated in $w_\rho(\cdot)$, and the operator radius has following properties (see [4], [10]):

(i) For $0 < \rho$, $w_\rho(\cdot)$ is quasi-convex in the sence that

(5.2) $$w_\rho(A + B) \leq \max(1, \rho/2)\{w_\rho(A) + w_\rho(B)\} \quad (A, B \in M_n).$$

In particular, $w_\rho(\cdot)$ is a norm if $0 < \rho \leq 2$.

(ii) $w_\rho(\cdot)$ $(0 < \rho < \infty)$ is unitary similarity invariant in the sense that

(5.3) $$w_\rho(A) = w_\rho(UAU^*) \text{ for all unitaries } U.$$

(iii) For fixed $A \in M_n, w_\rho(A)$ is nonincreasing and convex, and $\rho w_\rho(A)$ is nondecreasing in ρ on the interval $[1,\infty)$.

(iv)

(5.4) $$w_1(\cdot) = \|\cdot\|_\infty \quad \text{(the spectral norm)}$$

(5.5) $$w_2(\cdot) = w(\cdot) \quad \text{(the numerical radius)}$$

For fixed $A \in M_n$,

(5.6) $$\lim_{\rho \to \infty} w_\rho(A) = r(A) \quad \text{(the spectral radius of } A).$$

(v) Let $0 < \rho < 2$. Then $w_\rho(A) \le 1$ if and only if

(5.7) $$\|(\rho - 1)I + (2-\rho)e^{i\theta}A\|_\infty \le 1 \; for \; all \; \theta \in [0, 2\pi].$$

(vi) If $0 < \rho < 2$, then

(5.8) $$\rho w_\rho(\cdot) = (2-\rho)\, w_{2-\rho}(\cdot).$$

5.2 Hölder-type inequalities

For $A \in M_n$ and $\sigma, \rho \ge 0$, define $\|S_A\|_{w_\rho, \sigma}$ by

$$\|S_A\|_{w_\rho, w_\sigma} \equiv \sup_{X \in M_n} \frac{w_\sigma(A \circ X)}{w_\rho(X)} \quad (A \in M_n).$$

In particular we write $\|S_A\|_{w_\rho} = \|S_A\|_{w_\rho, w_\rho}$. The following theorem can be consider as a multiplicative Hölder-type inequality for operator radii.

THEOREM 5.1. *For every $\sigma, \rho > 0$ the following inequality holds:*

(5.9) $$\|S_A\|_{w_\rho, w_{\sigma\rho}} \le max\left(1, \frac{\sigma\rho}{2}\right) w_\sigma(A) \quad (A \in M_n).$$

PROOF: We have to prove that

$$w_{\sigma\rho}(A \circ B) \le \max\left(1, \frac{\sigma\rho}{2}\right) w_\sigma(A) w_\rho(B) \quad (A, B \in M_n).$$

We may assume $w_\sigma(A) = 1$ and $w_\rho(B) = 1$ in view of positive homogeneity of operator radius. Then by the definition of operator radius there are a Hilbert space $\mathcal{K} \supset \mathcal{H} \equiv \mathbb{C}^n$ and unitary operators U and V on $\mathcal{K}$ such that

(5.10) $$A^m = \sigma P U^m|_{\mathcal{H}}, \quad B^m = \rho P V^m|_{\mathcal{H}} \quad (m = 1, 2, \cdots).$$

Then the tensor product $U \otimes V$ is unitary on the tensor-product Hilbert space $\mathcal{K} \otimes \mathcal{K}$, containing $\mathcal{H} \otimes \mathcal{H}$, and $P \otimes P$ is the orthogonal projection from $\mathcal{K} \otimes \mathcal{K}$ to $\mathcal{H} \otimes \mathcal{H}$. It follows from (5.10) that

$$(A \otimes B)^m = \sigma\rho(P \otimes P)(U \otimes V)^m|_{\mathcal{H}\otimes\mathcal{H}} \quad (m = 1, 2, \cdots),$$

which implies, again by the definition,

$$w_{\sigma\rho}(A \otimes B) \leq 1. \tag{5.11}$$

To prove (5.9), let us first consider the case $\sigma\rho \leq 2$. Apply (5.7) to $A \otimes B$, for which $w_{\sigma\rho}(A \otimes B) \leq 1$ by (5.11), to see that

$$\|(\sigma\rho - 1)I \otimes I + (2 - \sigma\rho)e^{i\theta} A \otimes B\|_\infty \leq 1 \quad (0 \leq \theta < 2\pi). \tag{5.12}$$

It is known (see [2]) that there is a positive linear map Φ from $M_n \otimes M_n \simeq M_{n^2}$ to M_n such that

$$\Phi(S \otimes T) = S \circ T \quad (S, T \in M_n);$$

hence in particular $\Phi(I \otimes I) = I$. Since such a map necessarily has norm ≤ 1 (see [28]), it follows from (5.12) that

$$\|(\sigma\rho - 1)I + (2 - \sigma\rho)e^{i\theta} A \circ B\|_\infty = \|\Phi((\sigma\rho - 1)I \otimes I + (2 - \sigma\rho)e^{i\theta} A \otimes B)\|_\infty \leq 1,$$

which implies, again by (5.7) $w_{\sigma\rho}(A \circ B) \leq 1$. Therefore we have proved (5.9) for $\sigma\rho \leq 2$.

Next let us consider the case $\sigma\rho > 2$. Since, for fixed $S \in M_n$, the function $w_\lambda(S)$ and $\lambda w_\lambda(S)$ are nonincreasing and nondecreasing, respectively, in λ on the interval $[1,\infty)$ as remarked in (iii), we have

$$\begin{aligned} w_{\sigma\rho}(A \circ B) &\leq w_2(A \circ B) \\ &\leq w_2(A \otimes B) \quad \text{(use the map } \Phi) \\ &\leq \frac{\sigma\rho}{2} w_{\sigma\rho}(A \otimes B) \\ &\leq \frac{\sigma\rho}{2} \quad (by\ (5.11)) \\ &= \max\left\{1, \frac{\sigma\rho}{2}\right\} \end{aligned}$$

This completes the proof. ∎

Remark. The constant $\max\{1, \frac{\sigma\rho}{2}\}$ in Theorem 5.1 is best possible. In fact, it is known (see [10]) that for $n = 2$

$$w_\rho\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \frac{1}{\rho} \quad (0 < \rho < \infty). \tag{5.13}$$

Put

$$A = B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then for every $\sigma, \rho > 0$ we have

$$w_{\sigma\rho}(A \circ B) = \frac{1}{\sigma\rho} = w_\sigma(A) w_\rho(B).$$

Next put $A = B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Since $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ is unitarily similar to $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$, we have from (5.13)

$$w_\sigma(A) = \frac{2}{\sigma} \text{ and } w_\rho(B) = \frac{2}{\rho}.$$

On the other hand,

$$A \circ B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is unitarily similar to $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, so that for $\sigma\rho \geq 2$,

$$w_{\sigma\rho}(A \circ B) = 2 = \frac{\sigma\rho}{2} w_\sigma(A) w_\rho(B).$$

To provide the inequality (5.9) with more Hölder-like expression, let us confine ourselves to the interval $1 \leq \rho \leq 2$ and make a change of parameter:

$$W_t(A) = (2 - t) w_{2-t}(A) \quad (0 \leq t \leq 1). \tag{5.14}$$

From (5.4), (5.5) and (5.14) we can see that $W_1(\cdot) = \|\cdot\|_\infty = w_1(\cdot)$ and $W_0(\cdot) = 2w(\cdot)$. For $0 \leq s, t \leq 1$, define $\|S_A\|_{W_s, W_t}$ by

$$\|S_A\|_{W_s, W_t} \equiv \sup_{X \in M_n} \frac{W_t(A \circ X)}{W_s(X)} \quad (A \in M_n).$$

Then we have the following:

THEOREM 5.2. *For* $0 \le s, t \le 1$

$$\|S_A\|_{W_t, W_{st}} \le W_s(A) \quad (A \in M_n).$$

PROOF: Let $st = 0$. If $s = t = 0$ then Theorem is clear therefore we may assume $s \neq 0$ and $t = 0$. Using Corollary 4.12 and $\rho \longmapsto \rho w_\rho(\cdot)$ is nondecreasing function of $\rho \ge 1$ by (iii), we obtain

$$\begin{aligned} W_{st}(A \circ B) &= W_0(A \circ B) \\ &= 2w(A \circ B) \\ &\le 2w(B) \cdot \|A\|_\infty \\ &\le 2w(B)(2-s)w_{2-s}(A) \\ &= W_s(A)W_t(B). \end{aligned}$$

Next let $st \neq 0$. Then by (5.8) and Theorem 5.1, we have

$$\begin{aligned} W_{st}(A \circ B) &= (2-st)w_{2-st}(A \circ B) \\ &= st \cdot w_{st}(A \circ B) \\ &\le st \cdot w_s(A)w_t(B) \\ &= (2-s)w_{2-s}(A) \cdot (2-t)w_{2-t}(B) \\ &= W_s(A)W_t(B). \end{aligned}$$

This completes the proof of Theorem. ∎

With $\sigma = 1$ and $\rho = 2$ in Theorem 5.1 we obtain the result which is already shown in Corollary 4.12.

COROLLARY 5.3. *For* $0 < \rho \le 2$ *the following inequality holds:*

$$\|S_A\|_{w_\rho} \le \|A\|_\infty \quad (A \in M_n); \tag{5.15}$$

in particular

$$\|S_A\|_w \le \|A\|_\infty \quad (A \in M_n).$$

We write the induced norm of S_A from $(M_n, w_\rho(\cdot))$ to $(M_n, \|\cdot\|_1)$ by $\|S_A\|_{w_\rho,1}$. Next, we will give an operator radius version of the inequality which appeared in Corollary 2.3.

THEOREM 5.4. *If $0 < \rho \leq 2$, then we have*

$$\|S_A\|_{w_\rho,1} \leq \|A\|_{w_\rho^*} \quad (A \in M_n), \tag{5.16}$$

where $\|\cdot\|_{w_\rho^*}$ is the dual norm of $w_\rho(\cdot)$; in particular

$$\|S_A\|_{w,1} \leq \|A\|_{w^*} \quad (A \in M_n)$$

where $\|\cdot\|_{w^*}$ is the dual norm of the numerical radius $w(\cdot)$.

PROOF: We have to prove that

$$\|A \circ B\|_1 \leq \|A\|_{w_\rho^*}\, w_\rho(B) \quad (A, B \in M_n).$$

We may asssume $w_\rho(B) = 1$. Then the inequality (5.15) means that the linear map $S_B : A \to A \circ B$ from the Banach space $(M_n, \|\cdot\|_\infty)$ to the Banach space $(M_n, w_\rho(\cdot))$ has norm ≤ 1; hence its dual map from $(M_n, \|\cdot\|_{w_\rho^*}))$ to $(M_n, \|\cdot\|_1)$ has norm ≤ 1. As the dual map is given by $S_{\overline{B}}$,

$$\|A \circ \overline{B}\|_1 \leq \|A\|_{w_\rho^*} = \|A\|_{w_\rho^*}\, w_\rho(B).$$

Now (5.16) follows by changing B by $\overline{B}$ and using $w_\rho(\overline{B}) = w_\rho(B)$. ∎

5.3 Relations among $\|S_A\|_\infty, \|S_A\|_{w_\rho}$ and $\|S_A\|_w$

Next we will discuss the relations among $\|S_A\|_\infty, \|S_A\|_{w_\rho}$ and $\|S_A\|_w$ for $1 \leq \rho \leq 2$. In Corollary 4.11 we proved that $\|S_A\|_\infty \leq \|S_A\|_w$ $(A \in M_n)$. We will give a proof of the following:

THEOREM 5.5. *For $1 < \rho < 2$*

$$\|S_A\|_{w_\rho} \leq \|S_A\|_w. \tag{5.17}$$

We need the following lemma to show Theorem 5.5.

Lemma 5.6. *For* $1 \leq \rho \leq 2$

$$\rho \cdot w_\rho(A) = 2w\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)} \\ 0 & 1-\rho \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\right). \tag{5.18}$$

Proof: Suppose $w_\rho(A) \leq 1$. Then (5.7) means

$$I \geq \{(\rho-1)I + (2-\rho)e^{i\theta}A\}^*\{(\rho-1)I + (2-\rho)e^{i\theta}A\}.$$

Hence

$$\begin{bmatrix} I & (\rho-1)I + (2-\rho)e^{i\theta}A \\ (\rho-1)I + (2-\rho)e^{-i\theta}A^* & I \end{bmatrix} \geq 0, \tag{5.19}$$

which is equivalent to

$$\begin{aligned} \begin{bmatrix} I & (\rho-1)I \\ (\rho-1)I & I \end{bmatrix} &\geq \frac{1}{2}\left\{e^{i\theta}\begin{bmatrix} 0 & 2(\rho-2)A \\ 0 & 0 \end{bmatrix} + e^{-i\theta}\begin{bmatrix} 0 & 0 \\ 2(\rho-2)A^* & 0 \end{bmatrix}\right\} \\ &= \operatorname{Re}\left\{e^{i\theta}\begin{bmatrix} 0 & 2(\rho-2)A \\ 0 & 0 \end{bmatrix}\right\}. \end{aligned} \tag{5.20}$$

Since

$$\begin{bmatrix} I & (\rho-1)I \\ (\rho-1)I & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ (\rho-1)I & \sqrt{\rho(2-\rho)}I \end{bmatrix}\begin{bmatrix} I & (\rho-1)I \\ 0 & \sqrt{\rho(2-\rho)}I \end{bmatrix}$$

and

$$\begin{bmatrix} I & (\rho-1)I \\ 0 & \sqrt{\rho(2-\rho)}I \end{bmatrix}^{-1} = \begin{bmatrix} I & \frac{-(\rho-1)}{\sqrt{\rho(2-\rho)}}I \\ 0 & \frac{1}{\sqrt{\rho(2-\rho)}}I \end{bmatrix},$$

$$\begin{bmatrix} I & 0 \\ (\rho-1)I & \sqrt{\rho(2-\rho)}I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ \frac{-(\rho-1)}{\sqrt{\rho(2-\rho)}}I & \frac{1}{\sqrt{\rho(2-\rho)}}I \end{bmatrix},$$

(5.20) is equivalent to

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} &\geq \operatorname{Re}\left\{e^{i\theta}\begin{bmatrix} I & 0 \\ \frac{-(\rho-1)}{\sqrt{\rho(2-\rho)}}I & \frac{1}{\sqrt{\rho(2-\rho)}}I \end{bmatrix}\begin{bmatrix} 0 & 2(2-\rho)A \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & \frac{-(\rho-1)}{\sqrt{\rho(2-\rho)}}I \\ 0 & \frac{1}{\sqrt{\rho(2-\rho)}}I \end{bmatrix}\right\} \\ &= \operatorname{Re}\left\{e^{i\theta}\begin{bmatrix} 0 & 2\sqrt{\frac{2-\rho}{\rho}}A \\ 0 & \frac{2(1-\rho)}{\rho}A \end{bmatrix}\right\}. \end{aligned}$$

Since θ is arbitrary, as already remarked in section 4, the above is equivalent to

$$w\left(\frac{2}{\rho}\begin{bmatrix}0 & \sqrt{\rho(2-\rho)}I\\ 0 & (1-\rho)I\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & A\end{bmatrix}\right) \leq 1.$$

Therefore $\rho \cdot w_\rho(A) \leq 1$ if and only if

$$2w\left(\begin{bmatrix}0 & \sqrt{\rho(2-\rho)}I\\ 0 & (1-\rho)I\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & A\end{bmatrix}\right) \leq 1$$

which is the same as the desired assertion. ∎

Proof of Theorem 5.5: Let $\mathbf{A} = \begin{bmatrix}A & A\\ A & A\end{bmatrix}$. We have to show that

$$w_\rho(A \circ B) \leq \|S_A\|_w \cdot w_\rho(B) \quad (A, B \in M_n).$$

Assume $w_\rho(B) \leq 1$. Then by Lemma 5.6

$$\begin{aligned}
w_\rho(A \circ B) &= \frac{2}{\rho} w\left(\begin{bmatrix}0 & \sqrt{\rho(2-\rho)}I\\ 0 & (1-\rho)I\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & A\circ B\end{bmatrix}\right)\\
&= \frac{2}{\rho} w\left(\begin{bmatrix}0 & \sqrt{\rho(2-\rho)}I\\ 0 & (1-\rho)I\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & B\end{bmatrix}\circ\begin{bmatrix}A & A\\ A & A\end{bmatrix}\right)\\
&\leq \frac{2}{\rho}\|S_{\mathbf{A}}\|_w \cdot w\left(\begin{bmatrix}0 & \sqrt{\rho(2-\rho)}I\\ 0 & (1-\rho)I\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & B\end{bmatrix}\right)\\
&= \|S_{\mathbf{A}}\|_w \cdot w_\rho(B)\\
&\leq \|S_{\mathbf{A}}\|_w.
\end{aligned}$$

Hence we obtain $\|S_A\|_{w_\rho} \leq \|S_{\mathbf{A}}\|_w$. Since it was shown in Lemma 4.5 that

$$\|S_{\mathbf{A}}\|_w = \|S_A\|_w,$$

the last inequality implies our assertion. ∎

In Corollary 4.13 we showed that if $A \in M_n$ is Hermitian, then $\|S_A\|_\infty = \|S_A\|_w$. The following is a generalization of this equality.

PROPOSITION 5.7. *If A is Hermitian then*

$$\|S_A\|_\infty = \|S_A\|_{w_\rho} \quad (1 \le \rho \le 2).$$

PROOF: Let $1 \le \rho \le 2$. By definition,

$$\begin{aligned}
\|S_A\|_{w_\rho} &= \sup\{w_\rho(A \circ B) : w_\rho(B) \le 1\} \\
&\ge \sup\{w(A \circ B) : w_\rho(B) \le 1\} \\
&= \sup\{|\langle (A \circ B)\vec{x}|\vec{x}\rangle| : w_\rho(B) \le 1, \|\vec{x}\| \le 1\} \\
&= \sup\{|\mathrm{tr}((A \circ B) \cdot (\vec{x} \otimes \vec{x}^*))| : w_\rho(B) \le 1, \|\vec{x}\| \le 1\} \\
&= \sup\{|\mathrm{tr}((A \circ {}^t(\vec{x} \otimes \vec{x}^*)) \cdot {}^tB)| : w_\rho(B) \le 1, \|\vec{x}\| \le 1\} \\
&= \sup\{\|A \circ (\vec{x} \otimes \vec{x}^*)\|_{w_\rho^*} \ : \|\vec{x}\| \le 1\}.
\end{aligned}$$

Now since A and $\vec{x} \otimes \vec{x}^*$ are Hermitian, $A \circ (\vec{x} \otimes \vec{x}^*)$ is Hermitian. Hence there exist real numbers $\lambda_1, \cdots, \lambda_n$ and orthogonal unit vectors $\vec{x}_1, \cdots, \vec{x}_n \in \mathbb{C}^n$ such that $A \circ (\vec{x} \otimes \vec{x}^*) = \sum_{i=1}^n \lambda_i \vec{x}_i \otimes \vec{x}_i^*$. Therefore,

$$\begin{aligned}
\|A \circ (\vec{x} \otimes \vec{x}^*)\|_{w^*} &= \left\| \sum_{i=1}^n \lambda_i \vec{x}_i \otimes \vec{x}_i^* \right\|_{w^*} \\
&\le \sum_{i=1}^n |\lambda_i| \|\vec{x}_i \otimes \vec{x}_i^*\|_{w^*} \\
&= \sum_{i=1}^n |\lambda_i| \\
&= \|A \circ (\vec{x}_i \otimes \vec{x}_i)\|_1 .
\end{aligned} \tag{5.21}$$

Here to show the second equality, we use the fact if $\vec{y}$ is a unit vector then $\|\vec{y} \otimes \vec{y}^*\|_{w^*} = 1$. On the other hand, for any $C \in M_n$ we have

$$\|C\|_1 \le \|C\|_{w_\rho^*} \le \|C\|_{w^*} \tag{5.22}$$

by definition of dual norm and the nonincreasing property of operator radii. It follows from (5.21) and (5.22) that if A is Hermitian then

$$\|A \circ (\vec{x} \otimes \vec{x}^*)\|_{w_\rho^*} = \|A \circ (\vec{x} \otimes \vec{x}^*)\|_{w^*}.$$

Hence, we have

$$\sup\{\|A\circ(\vec{x}\otimes\vec{x}^*)\|_{w_\rho^*} : \|\vec{x}\|\le 1\} = \sup\{\|A\circ(\vec{x}\otimes\vec{x}^*)\|_{w^*} : \|\vec{x}\|\le 1\}$$
$$= \|S_A\|_{w^*} = \|S_A\|_w .$$

Since we have $\|S_A\|_{w_\rho} \le \|S_A\|_w$ by Theorem 5.5 and $\|S_A\|_\infty = \|S_A\|_w$ for Hermitian matrix A by Corollary 4.13, our assertion has been proven. ∎

PROPOSITION 5.8. *If A is unitary then*

$$\|S_A\|_\infty = \|S_A\|_{w_\rho} = 1 \quad (1\le\rho\le 2).$$

PROOF: By Theorem 5.5 and Corollary 4.16 we have $\|S_A\|_{w_\rho} \le \|S_A\|_w = 1$. A proof of the reverse inequality is almost same as Corollary 4.16. ∎

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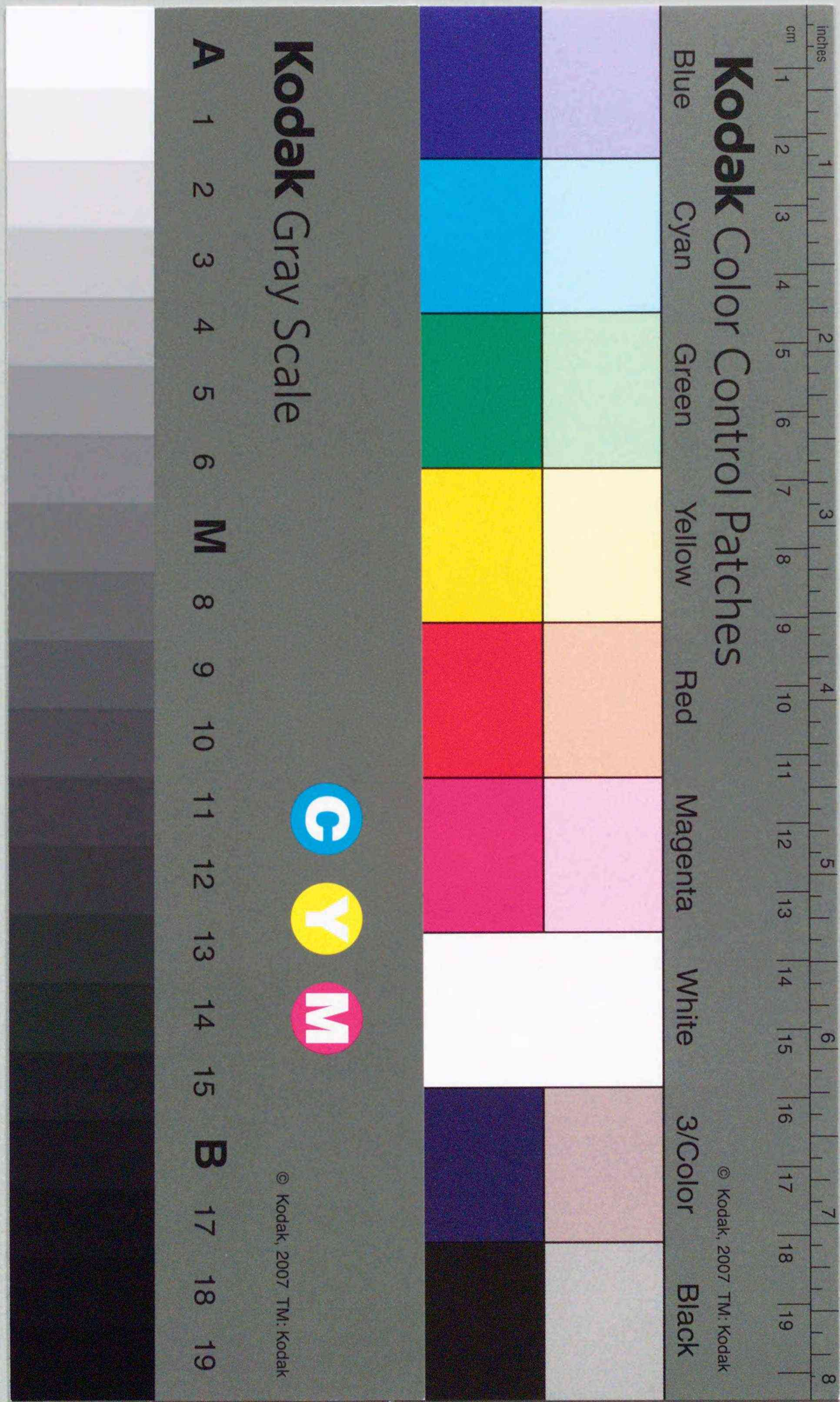
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