Cubic Harmonics and Bernoulli Numbers*

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Abstract

The functions satisfying the mean value property for an n-dimensional cube are determined explicitly. This problem is related to invariant theory for a finite reflection group, especially to a system of invariant differential equations. Solving this problem is reduced to showing that a certain set of invariant polynomials forms an invariant basis. After establishing a certain summation formula over Young diagrams, the latter problem is settled by considering a recursion formula involving Bernoulli numbers.

Keywords: polyhedral harmonics; cube; reflection groups; invariant theory; invariant differential equations; generating functions; partitions; Young diagrams; Bernoulli numbers.

1 Introduction

Let P be an n-dimensional polytope in $\mathbb{R}^n$. For $k = 0, \dots, n$, let $P(k)$ be the k-dimensional skeleton of P, that is, the union of all k-dimensional faces of P. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be $P(k)$-*harmonic* if it satisfies

$$f(x) = \frac{1}{|P(k)|} \int_{P(k)} f(x + ry)\, d\mu_k(y) \tag{1}$$

for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $r > 0$, where μ_k is the k-dimensional Euclidean measure on $P(k)$ and $|P(k)| := \mu_k(P(k))$ is the k-dimensional Euclidean volume of $P(k)$. This is an extension to a polytope of the classical notion of harmonic functions characterized by the mean value property for the sphere S^{n-1}. Let $\mathcal{H}_{P(k)}$ denote the set of all $P(k)$-harmonic functions on $\mathbb{R}^n$. Studying it should be a rich topic in discrete harmonic analysis and algebraic combinatorics, like a somewhat similar but not quite the same topic of spherical designs (see e.g. [1, 3, 16]).

A general result in Iwasaki [11] states that for any n-dimensional polytope P and any $k = 0, \dots, n$, the set $\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of polynomials. Note that $\mathcal{H}_{P(k)}$ becomes an $\mathbb{R}[\partial]$-module because equation (1) is stable under partial differentiations $\partial = (\partial_1, \dots, \partial_n)$, where $\partial_i := \partial/\partial x_i$. Moreover, if $G \subset O(n)$ is the symmetry group of P then $\mathcal{H}_{P(k)}$ is naturally an $\mathbb{R}[G]$-module since equation (1) is also stable under the action of G.

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If P has ample symmetry, that is, if G acts on $\mathbb{R}^n$ irreducibly then $\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of harmonic polynomials [11], so one may ask which portion of spherical harmonics appears as polyhedral harmonics. To discuss such a question for a variety of polytopes, it is a first step to determine $\mathcal{H}_{P(k)}$ explicitly when P is a regular convex polytope with center at the origin in $\mathbb{R}^n$. This problem is already settled unless P is an n-dimensional cube. As for the cube case, however, although the vertex problem ($k = 0$) was solved by Flatto [5] and Haeuslein [8] as early as 1970, the higher skeleton problem ($k = 1, \dots, n$) has been open up to now. The aim of this article is to give a complete solution to this problem. A characteristic feature of our work is that it provides a simultaneous resolution for all skeletons, which reveals the natural structure of this problem from the viewpoint of combinatorial analysis. Here we refer to Iwasaki [13] for a general review of the topic discussed in this article.

Let C be an n-dimensional cube with center at the origin in the Euclidean space $\mathbb{R}^n$ endowed with the standard orthonormal coordinates $x = (x_1, \dots, x_n)$. After a scale change and a rotation one may assume that the vertices of C are at $(\pm 1, \dots, \pm 1)$. The symmetry group W_n of C is the group of signed permutations, that is, the semi-direct product $W_n = S_n \ltimes \{\pm 1\}^n$ of $\{\pm 1\}^n$ and S_n acting on $\mathbb{R}^n$ by sign changes and by permutations of the variables. It is a finite reflection group of type B_n, having order $2^n \cdot n!$. Its fundamental alternating polynomial is given by

$$\mathit{\Delta}(x) = x_1 \cdots x_n \prod_{i<j} (x_i^2 - x_j^2).$$

The first main result of this article is then stated as follows.

Theorem 1.1 *$\mathcal{H}_{C(k)}$ is independent of $k = 0, \dots, n$. It is $2^n \cdot n!$-dimensional as a linear space, generated by $\mathit{\Delta}(x)$ as an $\mathbb{R}[\partial]$-module, and is the regular representation as an $\mathbb{R}[W_n]$-module.*

To illustrate the theorem we give a detailed description of the $n = 3$ case in Section 5.

For an arbitrary polytope P Iwasaki [11] introduced an infinite sequence of homogeneous polynomials $\tau_m^{(k)}(x)$ of degrees $m = 1, 2, 3, \dots$ in terms of some combinatorial data about $P(k)$ and characterized $\mathcal{H}_{P(k)}$ as the solution space to the system of partial differential equations

$$\tau_m^{(k)}(\partial) f = 0 \qquad (m = 1, 2, 3, \dots).$$

From the way in which they are defined, the polynomials $\tau_m^{(k)}(x)$ are invariant under the symmetry group G of P. This observation connects our problem to the theory of group-harmonic functions due to Steinberg [17]. A C^∞-function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be G-*harmonic* if it satisfies $\varphi(\partial) f = 0$ for any G-invariant polynomial $\varphi(x)$ without constant term. Let $\mathcal{H}_G$ denote the set of all G-harmonic functions. There is always the inclusion $\mathcal{H}_G \subset \mathcal{H}_{P(k)}$, and if $\{\tau_m^{(k)}(x)\}_{m=1}^\infty$ happens to generate the ring of G-invariant polynomials, then there occurs the coincidence $\mathcal{H}_G = \mathcal{H}_{P(k)}$. Steinberg [17] made an explicit determination of $\mathcal{H}_G$ when G is a finite reflection group: $\dim \mathcal{H}_G = |G|$; as an $\mathbb{R}[\partial]$-module $\mathcal{H}_G$ is generated by the fundamental alternating polynomial of G; and as an $\mathbb{R}[G]$-module it is the regular representation of G. If P is a regular polytope, then G is a finite reflection group and we are done if we are able to show that the sequence $\{\tau_m^{(k)}(x)\}_{m=1}^\infty$ actually generates the G-invariant ring.

According to article [11], for the n-dimensional cube C and $k = 0, \dots, n$, the polynomials $\tau_m^{(k)}(x)$ are constructed as follows. First we define

$$h_m^{(k)}(x) := h_m(x_1 + \cdots + x_n,\ x_2 + \cdots + x_n,\ \dots,\ x_{k+1} + \cdots + x_n), \tag{2}$$

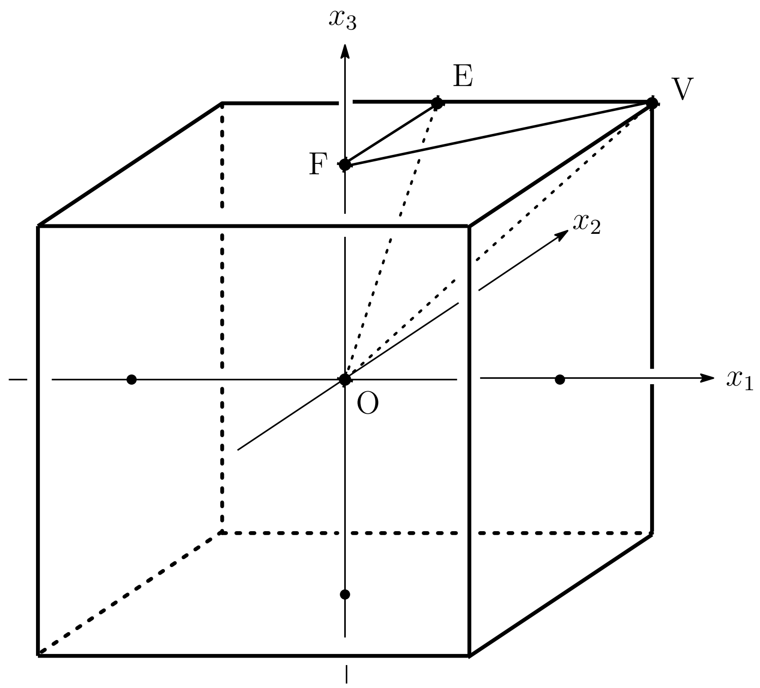

Figure 1: The cube in three dimensions with a flag V $\prec$ E $\prec$ F $\prec$ O.

where $h_m(t_1, \dots, t_{k+1})$ is the m-th complete symmetric polynomial of $k+1$ variables (see e.g. Macdonald [15]). When $k = n$ the term $x_{k+1} + \cdots + x_n$ is null so that $h_m^{(n)}(x) = h_m^{(n-1)}(x)$ since $h_m(t_1, \dots, t_k, 0) = h_m(t_1, \dots, t_k)$. For example, when $n = 3$ these polynomials are given by

$$h_m^{(k)}(x) = \begin{cases} h_m(\mathrm{V} \cdot x) & (k = 0), \\ h_m(\mathrm{V} \cdot x, \mathrm{E} \cdot x) & (k = 1), \\ h_m(\mathrm{V} \cdot x,\ \mathrm{E} \cdot x,\ \mathrm{F} \cdot x) & (k = 2, 3), \end{cases}$$

where $\mathrm{V} = (1, 1, 1)$ is a vertex, $\mathrm{E} = (0, 1, 1)$ is the midpoint of an edge and $\mathrm{F} = (0, 0, 1)$ is the center of a face of the 3-cube C (see Figure 1) and $\mathrm{V} \cdot x$ stands for the inner product of V and x, regarded as space vectors. Identify E and F with the edge and the face on which they lie. Similarly the origin O, i.e., the center of the cube C is identified with the unique 3-cell, i.e., the cube itself. Then we have a flag V $\prec$ E $\prec$ F $\prec$ O, where $* \prec **$ indicates that $*$ is a face of $**$. The simplex VEFO is a fundamental domain of the symmetry group W_3. There is a bijection between the elements of W_3 and the flags of C. These pictures carry over in n dimensions.

Finally $\tau_m^{(k)}(x)$ is defined to be the W_n-symmetrization of $h_m^{(k)}(x)$, that is, the average:

$$\tau_m^{(k)}(x) := \frac{1}{2^n \cdot n!} \sum_{\sigma \in W_n} h_m^{(k)}(\sigma x) \qquad (k = 0, \dots, n). \tag{3}$$

In other words $\tau_m^{(k)}(x)$ is the average of $h_m^{(k)}(x)$ over all flags of C, which is the original definition of $\tau_m^{(k)}(x)$ for a polytope with or without symmetry (see [11]). Note that $\tau_m^{(n)}(x) = \tau_m^{(n-1)}(x)$ since $h_m^{(n)}(x) = h_m^{(n-1)}(x)$ as mentioned earlier. It is immediate from definition (3) that $\tau_m^{(k)}(x)$ is a homogeneous W_n-invariant of degree m. Recall that the degrees of W_n are $2, 4, \dots, 2n$, which

$c_{3,1}^{(0)}$	$=$	1	$c_{3,2}^{(0)}$	$=$	4	$c_{3,3}^{(0)}$	$=$	48
$c_{3,1}^{(1)}$	$=$	$7/3$	$c_{3,2}^{(1)}$	$=$	$28/3$	$c_{3,3}^{(1)}$	$=$	64
$c_{3,1}^{(2)}$	$=$	$10/3$	$c_{3,2}^{(2)}$	$=$	$28/3$	$c_{3,3}^{(2)}$	$=$	64
$c_{3,1}^{(3)}$	$=$	$10/3$	$c_{3,2}^{(3)}$	$=$	$28/3$	$c_{3,3}^{(3)}$	$=$	64

Table 1: The values of coefficients $c_{n,m}^{(k)}$ when $n = 3$.

are all even (see e.g. Humphreys [9]). So the invariant polynomial $\tau_m^{(k)}(x)$ vanishes identically for every m odd. The second main result of this article is then stated as follows.

Theorem 1.2 *For any $k = 0, \dots, n$, the polynomials $\tau_2^{(k)}(x), \tau_4^{(k)}(x), \dots, \tau_{2n}^{(k)}(x)$ form an invariant basis of the reflection group W_n.*

For the proof of this theorem we take a standard basis of W_n-invariants. Let $e_m(x)$ be the m-th elementary symmetric polynomial of $(x_1, \dots, x_n)$ and set

$$\varphi_{2m}(x) := e_m(x_1^2, \dots, x_n^2) \qquad (m = 1, \dots, n). \tag{4}$$

It is well known that polynomials (4) form an invariant basis of W_n. Since $\tau_{2m}^{(k)}(x)$ is a homogeneous W_n-invariant of degree $2m$, there exist a unique constant $c_{n,m}^{(k)}$ and a unique weighted homogeneous polynomial $P_{n,m}^{(k)}(t_1, \dots, t_{m-1})$ of degree $2m$ with t_i being of weight $2i$ such that

$$\tau_{2m}^{(k)}(x) = c_{n,m}^{(k)}\, \varphi_{2m}(x) + P_{n,m}^{(k)}(\varphi_2(x), \dots, \varphi_{2m-2}(x)) \qquad (m = 1, \dots, n). \tag{5}$$

Here we employ the notation $c_{n,m}^{(k)}$ and $P_{n,m}^{(k)}$ to emphasize the dependence upon n. Note that $c_{n,m}^{(n)} = c_{n,m}^{(n-1)}$ since $\tau_{2m}^{(n)}(x) = \tau_{2m}^{(n-1)}(x)$. If we are able to show that the coefficient $c_{n,m}^{(k)}$ does not vanish for any $m = 1, \dots, n$, then we can invert equations (5) to express $\varphi_2(x), \dots, \varphi_{2m}(x)$ as polynomials of $\tau_2^{(k)}(x), \dots, \tau_{2m}^{(k)}(x)$. From this Theorem 1.2 follows immediately. So it is important to develop a method to calculate $c_{n,m}^{(k)}$ or at least to show that it does not vanish. For example Table 1 gives the values of $c_{n,m}^{(k)}$ for $n = 3$. Note that they are all positive.

It turns out that the coefficients $c_{n,m}^{(k)}$ exhibit a beautiful combinatorial structure upon introducing the generating polynomials

$$G_{n,m}(t) := \sum_{k=0}^{n} \frac{n!\, c_{n,m}^{(k)}}{(n-k)!\,(2m+k)!}\, t^{n-k}. \tag{6}$$

The third main result of this article is concerned with the structure of these polynomials.

Theorem 1.3 *The polynomials $G_{n,m}(t)$ are tied to $G_m(t) := G_{m,m}(t)$ by a simple relation*

$$G_{n,m}(t) = (t+1)^{n-m} G_m(t) \qquad (n \ge m \ge 1). \tag{7}$$

On the other hand the polynomials $G_m(t)$ admit a generating series representation

$$\sum_{m=1}^{\infty} (-1)^{m-1} G_m(t) \left(\frac{z^2}{t+1} \right)^m = \frac{z \coth z + t z^2 - 1}{2(t z \coth z + 1)}. \tag{8}$$

Equation (8) readily leads to a recursion formula for $G_m(t)$ involving the Bernoulli numbers B_m. There are several conventions for defining Bernoulli numbers, but the most useful one in our context is through the Maclaurin series expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{m=1}^{\infty} (-1)^{m-1} \frac{B_m}{(2m)!} z^{2m},$$

or equivalently through the formula

$$z \coth z = 1 + 2 \sum_{m=1}^{\infty} (-1)^{m-1} b_m z^{2m}, \qquad b_m := \frac{2^{2m-1}}{(2m)!} B_m. \tag{9}$$

Multiplying formula (8) by $2(tz \coth z + 1)$, expanding the resulting equation into a power series of z^2, and comparing the m-th coefficients of both sides, we obtain the following.

Corollary 1.4 *The polynomials $G_m(t)$ satisfy a recursion formula*

$$G_m(t) = b_m (t+1)^{m-1} + 2t \sum_{i=1}^{m-1} b_{m-i} (t+1)^{m-i-1} G_i(t), \qquad G_1(t) = \frac{t}{2} + \frac{1}{6}. \tag{10}$$

A polynomial of degree m is said to be *positive* if its coefficients up to degree m are all positive. Note that the product of a positive polynomial of degree i and a positive polynomial of degree j is a positive polynomial of degree $i + j$. With definition (9) we have

$$b_m = \frac{\zeta(2m)}{\pi^{2m}} = \frac{1}{\pi^{2m}} \sum_{j=1}^{\infty} \frac{1}{j^{2m}} > 0 \qquad (m = 1, 2, 3, \dots).$$

Therefore recursion formula (10) inductively implies that $G_m(t)$ is a positive polynomial of degree m. Formula (7) then tells us that $G_{n,m}(t)$ is a positive polynomial of degree n. Finally formula (6) concludes that the coefficient $c_{m,n}^{(k)}$ is positive for any $n \ge m \ge 1$ and $k = 0, \dots, n$. This establishes Theorem 1.2. The logical structure of our main results is this:

$$\text{Theorem 1.3} \longrightarrow \text{Corollary 1.4} \longrightarrow \text{Theorem 1.2} \longrightarrow \text{Theorem 1.1}.$$

Thus the main body of this article is exclusively devoted to establishing Theorem 1.3.

The plan of this article is as follows. In Section 2 we represent the coefficient $c_{n,m}^{(k)}$ in terms of a sum over some matrices (see Proposition 2.5). In Section 3 this representation is recast to a summation formula over some Young diagrams (see Proposition 3.4). After these preliminaries, Theorem 1.3 and Corollary 1.4 are established in Section 4, where some amplifications of these results and a summary on polyhedral harmonics for regular convex polytopes are also included. The final Section 5 is an appendix dealing with the $n = 3$ case as an instructive example.

2 Matrix Representation

We derive a representation of the coefficient $c_{n,m}^{(k)}$ as the sum of some quantities depending on a certain class of matrices. The main result of this section is given in Proposition 2.5.

$$
A = \begin{pmatrix}
* & * & * & * & * & * & * \\
 & * & * & * & * & * & * \\
 & & * & * & * & * & * \\
 & & & * & * & * & * \\
 & & & & * & * & *
\end{pmatrix}.
$$

Figure 2: A $(k+1)$-by-n upper quadrilateral matrix.

Various representations in this section involve those matrices as in Figure 2, namely, $(k+1)$-by-n matrices $A = (a_{ij})$ with $a_{ij} = 0$ for any $i > j$. Such a matrix is referred to as an *upper quadrilateral* matrix. Note that it becomes an upper triangular matrix when $k = n-1$, n.

Throughout this article we use the following notation and terminology. For a matrix $M = (m_{ij})$ of nonnegative integers, upper quadrilateral or not, or even for a row or column vector,

$$
M! := \prod_{i,j} m_{ij}!, \qquad |M| := \sum_{i,j} m_{ij}.
$$

A *composition* of size n is an ordered n-tuple $\nu = (\nu_1, \dots, \nu_n)$ of nonnegative integers, where each ν_i is called a *part* of ν; zero is allowed as a part (not as in the case of a partition). We say that ν is an n-composition of $j \in \mathbb{Z}$ if $\nu_1 + \cdots + \nu_n = j$. The *length* $\ell(\nu)$ of ν is the number of positive parts in ν. Put $x^\nu := x_1^{\nu_1} \cdots x_n^{\nu_n}$. Define a row vector $\vec{e}$ and a column vector $\mathbf{1}$ by

$$
\vec{e} := (\overbrace{1, \dots, 1}^{k+1}), \qquad \mathbf{1} := \left.\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\} n.
$$

The row vector $\vec{e}A$ represents the composition consisting of the consecutive column sums of A, while the column vector $A\mathbf{1}$ represents the one consisting of the consecutive row sums of A.

Lemma 2.1 *The polynomial in* (2) *is expressed as*

$$
h_m^{(k)}(x) = \sum_{|A|=m} \frac{(A\mathbf{1})!}{A!} x^{\vec{e}A}, \tag{11}
$$

where the sum is taken over all $(k+1)$-by-n quadrilateral matrices A of nonnegative integers whose entries sum up to m.

Proof. Since the m-th complete symmetric polynomial $h_m(t_1, \dots, t_{k+1})$ admits an expression

$$
h_m(t_1, \dots, t_{k+1}) = \sum_{m_1 + \cdots + m_{k+1} = m} t_1^{m_1} \cdots t_{k+1}^{m_{k+1}},
$$

using the multinomial theorem in definition (2) yields

$$
\begin{aligned}
h_m^{(k)}(x) &= \sum_{m_1 + \cdots + m_{k+1} = m} \prod_{i=1}^{k+1} (x_i + \cdots + x_n)^{m_i} \\
&= \sum_{m_1 + \cdots + m_{k+1} = m} \prod_{i=1}^{k+1} \left(\sum_{a_{ii} + \cdots + a_{in} = m_i} \frac{m_i!}{\prod_{j=i}^n a_{ij}!} \prod_{j=i}^n x_j^{a_{ij}} \right).
\end{aligned}
$$

Putting $A = (a_{ij})$ with $a_{ij} = 0$ for $i > j$ and exchanging the sums, we have

$$\begin{aligned} h_m^{(k)}(x) &= \sum_{m_1+\cdots+m_{k+1}=m} \prod_{i=1}^{k+1} \left(\sum_{a_{i1}+\cdots+a_{in}=m_i} \frac{m_i!}{\prod_{j=1}^n a_{ij}!} \prod_{j=1}^n x_j^{a_{ij}} \right) \\ &= \sum_{|A|=m} \frac{\prod_{i=1}^{k+1}(a_{i1}+\cdots+a_{in})!}{\prod_{i,j} a_{ij}!} \prod_{i,j} x_j^{a_{ij}} \\ &= \sum_{|A|=m} \frac{(A\mathbf{1})!}{A!} \prod_{j=1}^n x_j^{a_{1j}+\cdots+a_{k+1,j}} = \sum_{|A|=m} \frac{(A\mathbf{1})!}{A!} x^{\vec{e}A}, \end{aligned}$$

which establishes the lemma. □

Consider the $\{\pm 1\}^n$-symmetrization of $h_m^{(k)}(x)$, that is, the average:

$$g_m^{(k)}(x) := \frac{1}{2^n} \sum_{\varepsilon \in \{\pm 1\}^n} h_m^{(k)}(\varepsilon_1 x_1, \dots, \varepsilon_n x_n), \qquad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n. \tag{12}$$

Lemma 2.2 *The polynomial in* (12) *is expressed as*

$$g_m^{(k)}(x) = \sum_A \frac{(A\mathbf{1})!}{A!} x^{\vec{e}A}, \tag{13}$$

where the sum is taken over all $(k+1)$-by-n quadrilateral matrices A of nonnegative integers whose entries sum up to m and whose column sums are all even.

Proof. Substituting formula (11) into definition (12) yields

$$g_m^{(k)}(x) = \frac{1}{2^n} \sum_{\varepsilon \in \{\pm 1\}^n} \sum_A \frac{(A\mathbf{1})!}{A!} \varepsilon^{\vec{e}A} x^{\vec{e}A} = \frac{1}{2^n} \sum_A \frac{(A\mathbf{1})!}{A!} \left(\sum_{\varepsilon \in \{\pm 1\}^n} \varepsilon^{\vec{e}A} \right) x^{\vec{e}A}, \tag{14}$$

where the matrix A ranges in the same manner as in formula (11). Put $\vec{e}A = (\nu_1, \dots, \nu_n)$, where ν_j is the j-th column sum of A. Observe that

$$\sum_{\varepsilon \in \{\pm 1\}^n} \varepsilon^{\vec{e}A} = \sum_{\varepsilon \in \{\pm 1\}^n} \varepsilon_1^{\nu_1} \cdots \varepsilon_n^{\nu_n} = \begin{cases} 2^n & (\nu_j \text{ is even for any } j = 1, \dots, n), \\ 0 & (\nu_j \text{ is odd for some } j = 1, \dots, n). \end{cases}$$

So the sum $\sum_A$ in (14) can be restricted to those A's whose column sums are all even. □

For any matrix with even column sums, its entries must sum up to an even number, so that formula (13) implies that $g_m^{(k)}(x)$ vanishes identically for every m odd. Thus from now on m is replaced by $2m$ with m being a positive integer. This allows us to write $\vec{e}A = 2\nu(A)$ with $\nu(A) = (\nu_1(A), \dots, \nu_n(A))$ being an n-composition of m. The polynomial $\tau_{2m}^{(k)}(x)$ in formula (3) is the S_n-symmetrization of $g_{2m}^{(k)}(x)$, that is,

$$\tau_{2m}^{(k)}(x) = \frac{1}{n!} \sum_{\sigma \in S_n} g_{2m}^{(k)}(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \tag{15}$$

Putting formula (13) with m replaced by $2m$ into formula (15) yields

$$\tau_{2m}^{(k)}(x) = \frac{1}{n!} \sum_{A\in\mathcal{M}_{n,m}^{(k)}} \frac{(A\mathbf{1})!}{A!} \sum_{\sigma\in S_n} x_{\sigma(1)}^{2\nu_1(A)} \cdots x_{\sigma(n)}^{2\nu_n(A)}, \tag{16}$$

where $\mathcal{M}_{n,m}^{(k)}$ is the set of all $(k+1)$-by-n upper quadrilateral matrices of nonnegative integers whose entries sum up to $2m$ and whose column sums are all even.

Let ζ_j be a primitive j-th root of unity, say $\zeta_j = \exp(2\pi\sqrt{-1}/j)$. Since invariants (4) satisfy

$$\varphi_{2j}(\zeta_{2m}, \zeta_{2m}^2, \ldots, \zeta_{2m}^m, \overbrace{0,\ldots,0}^{n-m}) = \begin{cases} 0 & (j = 1, \ldots, m-1), \\ (-1)^{m-1} & (j = m), \end{cases}$$

substituting $x = (\zeta_{2m}, \zeta_{2m}^2, \ldots, \zeta_{2m}^m, 0, \ldots, 0)$ into equation (5) yields

$$c_{n,m}^{(k)} = (-1)^{m-1} \tau_{2m}^{(k)}(\zeta_{2m}, \zeta_{2m}^2, \ldots, \zeta_{2m}^m, 0, \ldots, 0). \tag{17}$$

For each n-composition $\nu = (\nu_1, \ldots, \nu_n)$ of m we define

$$u_{n,m}(\nu) = u_m(\nu_1, \ldots, \nu_n) := \sum_{\sigma\in S_{n,m}(\nu)} \zeta_m^{\sigma(1)\nu_1+\cdots+\sigma(n)\nu_n}, \tag{18}$$

where $S_{n,m}(\nu) := \{\sigma \in S_n \,:\, \text{for } i = 1, \ldots, n, \text{ if } \nu_i \geq 1 \text{ then } \sigma(i) \in \{1, \ldots, m\}\ \}$. Since

$$x_{\sigma(1)}^{2\nu_1} \cdots x_{\sigma(n)}^{2\nu_n} = \begin{cases} \zeta_m^{\sigma(1)\nu_1+\cdots+\sigma(n)\nu_n} & (\sigma \in S_{n,m}(\nu)), \\ 0 & (\sigma \notin S_{n,m}(\nu)), \end{cases}$$

at $x = (\zeta_{2m}, \zeta_{2m}^2, \ldots, \zeta_{2m}^m, 0, \ldots, 0)$, formulas (16) and (17) yield

$$c_{n,m}^{(k)} = \frac{(-1)^{m-1}}{n!} \sum_{A\in\mathcal{M}_{n,m}^{(k)}} u_{n,m}(\nu(A)) \frac{(A\mathbf{1})!}{A!}. \tag{19}$$

Note that the length $\ell(\nu) \in \{1, \ldots, m\}$, because $\ell(\nu) \leq \nu_1 + \cdots + \nu_n = m$.

Lemma 2.3 *The function $u_{n,m}(\nu)$ is symmetric, that is, invariant under any permutation of $\nu_1, \ldots, \nu_n$. For any n-composition $\nu = (\nu_1, \ldots, \nu_n)$ of m with $\nu_{m+1} = \cdots = \nu_n = 0$ we have*

$$u_{n,m}(\nu) = \frac{(n-\ell(\nu))!}{(m-\ell(\nu))!} u_{m,m}(\nu_1, \ldots, \nu_m). \tag{20}$$

Proof. For an element $\tau \in S_n$ we put $\nu^\tau := (\nu_{\tau(1)}, \ldots, \nu_{\tau(n)})$. Then it is easy to see that $S_{n,m}(\nu^\tau) = S_{n,m}(\nu) \cdot \tau$. Using this we show that $u_{n,m}(\nu^\tau) = u_{n,m}(\nu)$. Indeed,

$$\begin{aligned} u_{n,m}(\nu^\tau) &= \sum_{\sigma\in S_{n,m}(\nu^\tau)} \zeta_m^{\sigma(1)\nu_{\tau(1)}+\cdots+\sigma(n)\nu_{\tau(n)}} &&= \sum_{\sigma\in S_{n,m}(\nu^\tau)} \zeta_m^{(\sigma\cdot\tau^{-1})(1)\cdot\nu_1+\cdots+(\sigma\cdot\tau^{-1})(n)\cdot\nu_n} \\ &= \sum_{\sigma'\in S_{n,m}(\nu^\tau)\cdot\tau^{-1}} \zeta_m^{\sigma'(1)\nu_1+\cdots+\sigma'(n)\nu_n} &&= \sum_{\sigma'\in S_{n,m}(\nu)} \zeta_m^{\sigma'(1)\nu_1+\cdots+\sigma'(n)\nu_n} = u_{n,m}(\nu), \end{aligned}$$

as desired. This proves that $u_{n,m}(\nu)$ is a symmetric function of $\nu = (\nu_1, \dots, \nu_n)$.

We proceed to the second assertion. Suppose that ν is of the form $\nu = (\nu_1, \dots, \nu_r, 0, \dots, 0)$ with $r := \ell(\nu) \le m$. Then $S_{n,m}(\nu) = \{\sigma \in S_n \,:\, \sigma(\{1, \dots, r\}) \subset \{1, \dots, m\}\}$. We think of S_m as a subgroup of S_n by setting $S_m := \{\sigma \in S_n \,:\, \sigma(i) = i \text{ for } i = m+1, \dots, n\}$. Define a map

$$S_{n,m}(\nu) \to S_m, \quad \sigma \mapsto \tau \qquad \text{by} \quad \tau(i) := \begin{cases} \sigma(i) & (i = 1, \dots, r), \\ p(i) & (i = r+1, \dots, m), \\ i & (i = m+1, \dots, n), \end{cases} \tag{21}$$

where p is the unique bijection $p : \{r+1, \dots, m\} \to \{1, \dots, m\} \setminus \sigma(\{1, \dots, r\})$ which is "order-equivalent" to the injection $\sigma|_{\{r+1,\dots,m\}}$ in the sense that $p(i) < p(j)$ if and only if $\sigma(i) < \sigma(j)$ for every $i, j \in \{r+1, \dots, m\}$. We claim that the map (21) is $\frac{(n-r)!}{(m-r)!}$-to-one. Indeed, given any element $\tau \in S_m$, the fiber over τ has a one-to-one correspondence with the set of data (S, q):

- a subset S of cardinality $m - r$ of $T := \{1, \dots, n\} \setminus \tau(\{1, \dots, r\})$,
- a bijection $q : \{m+1, \dots, n\} \to T \setminus S$.

It is clear from definition (21) that given a data (S, q) there exists a unique element $\sigma \in S_{n,m}(\nu)$ such that $\sigma(\{r+1, \dots, m\}) = S$ and $\sigma|_{\{m+1,\dots,n\}} = q$. Since $\#T = n - r$, there are $\binom{n-r}{m-r}$ choices of S, for each of which there are $(n-m)!$ choices of q. Thus the fiber has a total of $\binom{n-r}{m-r} \cdot (n-m)! = \frac{(n-r)!}{(m-r)!}$ elements. Since $\zeta_m^{\sigma(1)\nu_1 + \cdots + \sigma(n)\nu_n} = \zeta_m^{\tau(1)\nu_1 + \cdots + \tau(m)\nu_m}$, we have

$$u_{n,m}(\nu) = \frac{(n-r)!}{(m-r)!} \sum_{\tau \in S_m} \zeta_m^{\tau(1)\nu_1 + \cdots + \tau(m)\nu_m} = \frac{(n-r)!}{(m-r)!}\, u_{m,m}(\nu_1, \dots, \nu_m),$$

where $S_m = S_{m,m}(\nu_1, \dots, \nu_m)$ is used to obtain the second equality. □

Formula (20) reduces the calculation of $u_{n,m}$ to that of $u_{m,m}$, which we now carry out.

Lemma 2.4 *For any m-composition $\nu = (\nu_1, \dots, \nu_m)$ of m we have*

$$u_m(\nu) := u_{m,m}(\nu) = m\,(-1)^{\ell(\nu)-1}\,(\ell(\nu) - 1)!\,(m - \ell(\nu))!. \tag{22}$$

Proof. When $n = m$ the function $u_{n,m}(\nu)$ in (18) becomes simpler because $S_{m,m}(\nu) = S_m$ for every m-composition ν of m. The proof is by induction on $\ell(\nu)$. When $\ell(\nu) = 1$ we may assume that ν is of the form $\nu = (m, 0, \dots, 0)$ by the symmetry of $u_m(\nu)$. Then definition (18) reads

$$u_m(\nu) = \sum_{\sigma \in S_m} \zeta_m^{\sigma(1)m} = \sum_{\sigma \in S_m} 1 = m!,$$

which verifies formula (22) for $\ell(\nu) = 1$. Let $1 \le r < m$ and assume that formula (22) is true for every m-composition $\nu = (\nu_1, \dots, \nu_m)$ of m with length $\ell(\nu) = r$. Consider the case $\ell(\nu) = r+1$. By the symmetry of $u_m(\nu)$ we may assume that ν is of the form $\nu = (\nu_1, \dots, \nu_{r+1}, 0, \dots, 0)$ with $\nu_1, \dots, \nu_{r+1} \ge 1$ and $\nu_1 + \cdots + \nu_{r+1} = m$. Note that $1 \le \nu_{r+1} < m$. Formula (18) now reads

$$u_m(\nu) = \sum_{\sigma \in S_m} \zeta_m^{\sigma(1)\nu_1 + \cdots + \sigma(r)\nu_{r+1}} = (m - r - 1)! \sum_{(p_1, \dots, p_{r+1})} \zeta_m^{p_1\nu_1 + \cdots + p_{r+1}\nu_{r+1}},$$

where $(p_1, \dots, p_{r+1})$ ranges over all permutations of distinct $r+1$ numbers in $\{1, \dots, m\}$. Thus,

$$\begin{aligned} u_m(\nu) &= (m-r-1)! \sum_{(p_1,\dots,p_r)} \zeta_m^{p_1\nu_1+\cdots+p_r\nu_r} \sum_{p_{r+1}\in\{1,\dots,m\}\setminus\{p_1,\dots,p_r\}} \zeta_m^{p_{r+1}\nu_{r+1}} \\ &= (m-r-1)! \sum_{(p_1,\dots,p_r)} \zeta_m^{p_1\nu_1+\cdots+p_r\nu_r} \left(\sum_{l=1}^{m} \zeta_m^{l\nu_{r+1}} - \sum_{j=1}^{r} \zeta_m^{p_j\nu_{r+1}} \right). \end{aligned}$$

Since $1 \le \nu_{r+1} < m$, we have $\sum_{l=1}^{m} \zeta_m^{l\nu_{r+1}} = 0$ and hence

$$u_m(\nu) = -(m-r-1)! \sum_{j=1}^{r} \sum_{(p_1,\dots,p_r)} \zeta_m^{p_1\nu_1^{(j)}+\cdots+p_r\nu_r^{(j)}} = -\frac{1}{m-r} \sum_{j=1}^{r} v(\nu^{(j)}),$$

where $\nu^{(j)} = (\nu_1^{(j)}, \dots, \nu_m^{(j)}) := (\nu_1 + \delta_{1j}\nu_{r+1}, \dots, \nu_1 + \delta_{rj}\nu_{r+1}, 0, \dots, 0)$ with δ_{ij} being Kronecker's delta. Note that $\nu^{(j)}$ is an m-composition of m with length $\ell(\nu^{(j)}) = r$ so that the induction hypothesis yields $u_m(\nu^{(j)}) = m\,(-1)^{r-1}\,(r-1)!\,(m-r)!$ for each $j = 1, \dots, r$. Hence

$$u_m(\nu) = -\frac{1}{m-r}\, r\, m\,(-1)^{r-1}\,(r-1)!\,(m-r)! = m\,(-1)^r\, r!\,(m-r-1)!,$$

which means that formula (22) is true for $\ell(\nu) = r+1$. The induction is complete. □

A column of a matrix is said to be *nontrivial* if it has at least one nonzero entry.

Proposition 2.5 *Let $\ell(A)$ denote the number of nontrivial columns in A. Then*

$$c_{n,m}^{(k)} = \frac{(-1)^{m-1}\, m}{n!} \sum_{A\in\mathcal{M}_{n,m}^{(k)}} (-1)^{\ell(A)-1}\,(\ell(A)-1)!\,(n-\ell(A))!\,\frac{(A\mathbf{1})!}{A!}, \tag{23}$$

where $\mathcal{M}_{n,m}^{(k)}$ is defined right after formula (16)*.*

Proof. First, Lemmas 2.3 and 2.4 are put together to yield a formula

$$u_{n,m}(\nu) = m\,(-1)^{\ell(\nu)-1}\,(\ell(\nu)-1)!\,(n-\ell(\nu))!, \tag{24}$$

for any n-composition $\nu = (\nu_1, \dots, \nu_n)$ of m. Indeed, by the symmetry of $u_{n,m}(\nu)$ we may assume $\nu_{m+1} = \cdots = \nu_n = 0$. So using formula (22) in formula (20) gives formula (24). Next, putting formula (24) with $\nu = \nu(A)$ into (19) yields formula (23), since $\ell(A) = \ell(\nu(A))$. □

3 Young Diagram Representation

The theme of this section is a passage from compositions to partitions. Namely we rewrite formula (23) in Proposition 2.5 as a sum over some partitions, that is, over some Young diagrams. The main result of the section is Proposition 3.4. For each composition $\nu = (\nu_1, \dots, \nu_n)$ of size n, let $\mathcal{M}_n^{(k)}(\nu)$ denote the set of all $(k+1)$-by-n upper quadrilateral matrices whose i-th column sum is equal to ν_i for $i = 1, \dots, n$. Motivated by expression (23), we consider the function

$$v_n^{(k)}(\nu) := \sum_{A\in\mathcal{M}_n^{(k)}(\nu)} \frac{(A\mathbf{1})!}{A!}. \tag{25}$$

Lemma 3.1 *For any composition $\nu = (\nu_1, \dots, \nu_n)$ of size n we have*

$$v_n^{(k)}(\nu) = \frac{(\nu_1 + \dots + \nu_n + k)!}{\prod_{j=1}^k (\nu_1 + \dots + \nu_j + j) \cdot \prod_{j=1}^n \nu_j!}. \tag{26}$$

Proof. The proof is by induction on k. For $k = 0$ there is nothing to prove. Suppose that formula (26) is true for $k-1$. We write $\nu = \vec{\nu}$ to emphasize that ν is a row vector. Put

$$\psi_n^{(k)}(\vec{a}_1, \dots, \vec{a}_{k+1}) := \frac{(A\mathbf{1})!}{A!} \quad \text{for} \quad A = \begin{pmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_{k+1} \end{pmatrix} \quad \text{with} \quad \vec{a}_i = (\overbrace{0, \dots, 0}^{i-1}, a_{ii}, \dots, a_{in}).$$

Here we also write $\vec{a}_{k+1} = \vec{a} = (0, \dots, 0, a_{k+1}, \dots, a_n)$ to simplify the notation. Observe that $\psi_n^{(k)}(\vec{a}_1, \dots, \vec{a}_k, \vec{a}) = \psi_n^{(0)}(\vec{a}) \cdot \psi_n^{(k-1)}(\vec{a}_1, \dots, \vec{a}_k)$. Using this we have

$$\begin{aligned} v_n^{(k)}(\vec{\nu}) &= \sum_{\vec{a}_1 + \dots + \vec{a}_k + \vec{a} = \vec{\nu}} \psi_n^{(k)}(\vec{a}_1, \dots, \vec{a}_k, \vec{a}) \\ &= \sum_{\vec{a} \le \vec{\nu}} \psi_n^{(0)}(\vec{a}) \sum_{\vec{a}_1 + \dots + \vec{a}_k = \vec{\nu} - \vec{a}} \psi_n^{(k-1)}(\vec{a}_1, \dots, \vec{a}_k) = \sum_{\vec{a} \le \vec{\nu}} \psi_n^{(0)}(\vec{a}) \cdot v_n^{(k-1)}(\vec{\nu} - \vec{a}), \end{aligned}$$

where $\vec{a} \le \vec{\nu}$ means that $\vec{\nu} - \vec{a} \in \mathbb{Z}_{\ge 0}^n$. Put $\mu_j := \nu_1 + \dots + \nu_j$, $\bar{\mu}_j := \nu_{j+1} + \dots + \nu_n$ and $b := a_{k+1} + \dots + a_n$. Since $a_j = 0$ for $j = 1, \dots, k$, the induction hypothesis yields

$$v_n^{(k-1)}(\vec{\nu} - \vec{a}) = \frac{1}{\prod_{j=1}^{k-1} (\mu_j + j) \cdot \prod_{j=1}^k \nu_j!} \cdot \frac{(\mu_n + k - 1 - b)!}{\prod_{j=k+1}^n (\nu_j - a_j)!}.$$

Substituting this into the previous formula and after some manipulations we have

$$\begin{aligned} v_n^{(k)}(\vec{\nu}) &= \frac{1}{\prod_{j=1}^{k-1} (\mu_j + j) \cdot \prod_{j=1}^n \nu_j!} \sum_{b=0}^{\bar{\mu}_k} b!\, (\mu_n + k - 1 - b)! \sum_{\substack{\vec{a} \le \vec{\nu} \\ a_{k+1} + \dots + a_n = b}} \prod_{j=k+1}^n \binom{\nu_j}{a_j} \\ &= \frac{1}{\prod_{j=1}^{k-1} (\mu_j + j) \cdot \prod_{j=1}^n \nu_j!} \sum_{b=0}^{\bar{\mu}_k} b!\, (\mu_n + k - 1 - b)! \binom{\bar{\mu}_k}{b} \\ &= \frac{\bar{\mu}_k!\, (\mu_k + k - 1)!}{\prod_{j=1}^{k-1} (\mu_j + j) \cdot \prod_{j=1}^n \nu_j!} \sum_{b=0}^{\bar{\mu}_k} \binom{\mu_n + k - 1 - b}{\mu_k + k - 1} \\ &= \frac{\bar{\mu}_k!\, (\mu_k + k - 1)!}{\prod_{j=1}^{k-1} (\mu_j + j) \cdot \prod_{j=1}^n \nu_j!} \binom{\mu_n + k}{\mu_k + k} = \frac{(\mu_n + k)!}{\prod_{j=1}^k (\mu_j + j) \cdot \prod_{j=1}^n \nu_j!}, \end{aligned}$$

where the following general formulas are used to obtain the second and fourth equalities.

$$\sum_{\substack{0 \le b_i \le a_i \\ b_1 + \dots + b_k = b}} \prod_{i=1}^k \binom{a_i}{b_i} = \binom{a_1 + \dots + a_k}{b}, \qquad \sum_{i=b}^a \binom{i}{b} = \binom{a+1}{b+1}.$$

Therefore formula (26) remains true for k and the induction is complete. □

Lemma 3.2 *Formula* (23) *in Proposition* 2.5 *is rewritten as*

$$c_{n,m}^{(k)} = \frac{(-1)^{m-1} m \cdot (2m+k)!}{n!} \sum_{\nu \vdash m} (-1)^{\ell(\nu)-1} \, (\ell(\nu)-1)! \, (n-\ell(\nu))! \, \bar{v}_n^{(k)}(\nu), \tag{27}$$

where the sum is taken over all n*-composition* $\nu = (\nu_1, \dots, \nu_n)$ *of* m *and*

$$\bar{v}_n^{(k)}(\nu) := \frac{1}{\prod_{j=1}^k (2\nu_1 + \cdots + 2\nu_j + j) \cdot \prod_{j=1}^n (2\nu_j)!}. \tag{28}$$

Proof. Since $\mathcal{M}_{n,m}^{(k)}$ is the disjoint union of $\mathcal{M}_n^{(k)}(2\nu)$ over all n-compositions $\nu = (\nu_1, \dots, \nu_n)$ of m and one has $\ell(A) = \ell(\nu)$ for each $A \in \mathcal{M}_n^{(k)}(2\nu)$, formula (23) and definition (25) lead to

$$\begin{aligned} c_{n,m}^{(k)} &= \frac{(-1)^{m-1} \, m}{n!} \sum_{\nu \vdash m} (-1)^{\ell(\nu)-1} \, (\ell(\nu)-1)! \, (n-\ell(\nu))! \sum_{A \in \mathcal{M}_n^{(k)}(2\nu)} \frac{(A\mathbf{1})!}{A!} \\ &= \frac{(-1)^{m-1} \, m}{n!} \sum_{\nu \vdash m} (-1)^{\ell(\nu)-1} \, (\ell(\nu)-1)! \, (n-\ell(\nu))! \, v_n^{(k)}(2\nu). \end{aligned}$$

Use formula (26) with ν replaced by 2ν and factor the common term $(2\nu_1 + \cdots + 2\nu_n + k)! = (2m+k)!$ out of the summation. Then we obtain formula (27). □

Formula (27) appears as a sum over compositions. The next task is to recast it to a sum over partitions, that is, over Young diagrams. In the sequel we follow the standard partition and Young diagram notations as in Macdonald [15]. Let $\lambda = (\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots)$ be a partition. Its length is the number $\ell(\lambda) := \#\{j : \lambda_j > 0\}$ of parts in λ while its weight is the sum $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$. It is sometimes convenient to use a notation which indicates the number of times each positive integer occurs as a part: $\lambda = (1^{r_1} \, 2^{r_2} \, \cdots \, i^{r_i} \, \cdots)$ means that i occurs exactly r_i times in λ, where r_i is called the multiplicity of i in λ. Note that $\ell(\lambda) = r_1 + r_2 + r_3 + \cdots$. A composition $\nu = (\nu_1, \nu_2, \nu_3, \dots)$ defines a unique partition denoted by ν^+ that is obtained by rearranging $\nu_1, \nu_2, \nu_3, \dots$ in descending order of magnitude.

Motivated by expression (28), for each partition μ of length at most k we define

$$w_k(\mu) := \sum_{\nu^+ = \mu} \frac{1}{\prod_{j=1}^k (2\nu_1 + \cdots + 2\nu_j + j) \cdot \prod_{j=1}^k (2\nu_j)!}, \tag{29}$$

where the sum is taken over all compositions $\nu = (\nu_1, \dots, \nu_k)$ of size k such that $\nu^+ = \mu$. Note that the denominator of the summand in (29) differs from the denominator of (28) by the factor $\prod_{j=1}^k (2\nu_j)!$ in place of $\prod_{j=1}^n (2\nu_j)!$. This function is evaluated as follows.

Lemma 3.3 *For any partition* $\mu = (1^{s_1} \, 2^{s_2} \, \cdots \, k^{s_k})$ *of length at most* k *we have*

$$w_k(\mu) = \frac{1}{\prod_{j \ge 0} s_j! \prod_{j \ge 0} ((2j+1)!)^{s_j}}, \qquad \text{where} \quad s_0 := k - \ell(\mu). \tag{30}$$

Proof. The proof is by induction on k. For $k = 1$ formula (30) holds trivially. Suppose that $k \ge 2$ and formula (30) is true for $k-1$. Consider the index set $J := \{j \ge 0 : s_j > 0\}$. Note that $\sum_{j \in J} j s_j = |\mu|$ and $\sum_{j \in J} s_j = k$. For each $i \in J$ we define a partition of length at most

$k-1$ by $\mu^{(i)} := (j^{s_j-\delta_{ij}} \,|\, j \in J \setminus \{0\})$ with δ_{ij} being Kronecker's delta. Here if $0 \in J$ then $\mu^{(0)}$ is the same as μ but regarded as a partition of length at most $k-1$. By induction hypothesis,

$$w_{k-1}(\mu^{(i)}) = \frac{1}{\prod_{j\in J}(s_j-\delta_{ij})!\prod_{j\in J}((2j+1)!)^{s_j-\delta_{ij}}} = \frac{s_i(2i+1)!}{\prod_{j\in J}s_j!\prod_{j\in J}((2j+1)!)^{s_j}} \tag{31}$$

for each $i \in J$. Consider the set of all compositions $\nu = (\nu_1, \dots, \nu_k)$ of size k such that $\nu^+ = \mu$. It is divided into disjoint subsets indexed by $i \in J$, where the i-th subset consists of those ν's with $\nu_k = i$. Compositions $\nu = (\nu_1, \dots, \nu_k)$ from the i-th subset are in one-to-one correspondence with compositions $\nu^{(i)} = (\nu_1, \dots, \nu_{k-1})$ of size $k-1$ such that $(\nu^{(i)})^+ = \mu^{(i)}$. Thus noticing $2\nu_1 + \cdots + 2\nu_k + k = 2|\mu| + k$ we have

$$\begin{aligned}
w_k(\mu) &= \sum_{i\in J}\frac{1}{(2|\mu|+k)\cdot(2i)!}\sum_{\nu^{(i)+}=\mu^{(i)}}\frac{1}{\prod_{j=1}^{k-1}(2\nu_1^{(i)}+\cdots+2\nu_j^{(i)}+j)\prod_{j=1}^{k-1}(2\nu_j^{(i)})!}\\
&= \sum_{i\in J}\frac{1}{(2|\mu|+k)\cdot(2i)!}\,w_{k-1}(\mu^{(i)})\\
&= \sum_{i\in J}\frac{1}{(2|\mu|+k)\cdot(2i)!}\cdot\frac{s_i(2i+1)!}{\prod_{j\in J}s_j!\prod_{j\in J}((2j+1)!)^{s_j}}\\
&= \frac{1}{(2|\mu|+k)}\cdot\frac{\sum_{i\in J}s_i(2i+1)}{\prod_{j\in J}s_j!\prod_{j\in J}((2j+1)!)^{s_j}}\\
&= \frac{1}{\prod_{j\in J}s_j!\prod_{j\in J}((2j+1)!)^{s_j}},
\end{aligned}$$

where the third and fifth equalities follow from formula (31) and $\sum_{i\in J} s_i(2i+1) = 2|\mu| + k$ respectively. This shows that formula (30) remains true for k and the induction is complete. □

Proposition 3.4 *For any $n \geq m \geq 1$ the polynomial $G_{n,m}(t)$ in definition* (6) *is expressed as*

$$(-1)^{m-1}\frac{G_{n,m}(t)}{(t+1)^n} = m\sum_{\lambda\vdash m}(-1)^{r_1+\cdots+r_m-1}\frac{(r_1+\cdots+r_m-1)!}{r_1!\cdots r_m!}\,T_1^{r_1}\cdots T_m^{r_m}, \tag{32}$$

where the sum is taken over all partitions $\lambda = (1^{r_1}\,2^{r_2}\,\cdots\,m^{r_m})$ of m and T_j is defined by

$$T_j := \frac{1}{(2j+1)!}\cdot\frac{(2j+1)t+1}{t+1} \qquad (j = 1, \dots, m). \tag{33}$$

In particular the rational function $(t+1)^{-n}G_{n,m}(t)$ is independent of n.

Proof. Fix $n \geq m \geq 1$ and let $k = 0, \dots, n$ (we will vary k later). Let λ be a partition of m and write $\lambda = (1^{r_1}\,2^{r_2}\,3^{r_3}\,\cdots)$. For another partition $\mu = (1^{s_1}\,2^{s_2}\,3^{s_3}\,\cdots)$ we write $\mu \prec_k \lambda$ if μ is of length at most k and $s_i \leq r_i$ for every $i = 0, 1, 2, \dots,$ where $r_0 := n - \ell(\lambda)$ and $s_0 := k - \ell(\mu)$. Fix such a pair (λ, μ) and put $\lambda/\mu := (1^{r_1-s_1}\,2^{r_2-s_2}\,3^{r_3-s_3}\,\cdots)$. For a composition $\nu = (\nu_1, \dots, \nu_n)$ of size n we write $\nu \rhd_k (\lambda, \mu)$ if $\nu^+ = \lambda$ and $(\nu')^+ = \mu$, where $\nu' := (\nu_1, \dots, \nu_k)$ is the cut-off to the first k parts of ν. Note that this is the case if and only if $(\nu')^+ = \mu$ and $(\nu'')^+ = \lambda/\mu$, where $\nu'' := (\nu_{k+1}, \dots, \nu_n)$ is the cut-off to the last $n-k$ parts. (Here $\mu \prec_k \lambda$, $\nu \rhd_k (\lambda, \mu)$ and λ/μ are notations valid only in this article.) There are $(n-k)!/\prod_{j\geq 0}(r_j - s_j)!$

compositions $\nu'' = (\nu_{k+1}, \ldots, \nu_n)$ of size $n-k$ such that $(\nu'')^+ = \lambda/\mu$, for each of which one has $\prod_{i=k+1}^n (2\nu_i)! = \prod_{j\geq 0}((2j)!)^{r_j - s_j}$. Thus definition (28) leads to

$$\begin{aligned}
\sum_{\nu \rhd_k (\lambda,\mu)} \bar{v}_n^{(k)}(\nu) &= \sum_{(\nu')^+=\mu} \frac{1}{\prod_{j=1}^k (2\nu_1 + \cdots + 2\nu_j + j) \cdot \prod_{j=1}^k (2\nu_j)!} \sum_{(\nu'')^+=\lambda/\mu} \frac{1}{\prod_{j=k+1}^n (2\nu_j)!} \\
&= \sum_{(\nu')^+=\mu} \frac{1}{\prod\limits_{1\leq j\leq k} (2\nu_1 + \cdots + 2\nu_j + j) \prod\limits_{1\leq j\leq k} (2\nu_j)!} \cdot \frac{(n-k)!}{\prod\limits_{j\geq 0} (r_j - s_j)! \prod\limits_{j\geq 0} ((2j)!)^{r_j - s_j}} \\
&= w_k(\mu) \frac{(n-k)!}{\prod_{j\geq 0} (r_j - s_j)! \prod_{j\geq 0} ((2j)!)^{r_j - s_j}} \\
&= \frac{1}{\prod_{j\geq 0} s_j! \prod_{j\geq 0} ((2j+1)!)^{s_j}} \cdot \frac{(n-k)!}{\prod_{j\geq 0} (r_j - s_j)! \prod_{j\geq 0} ((2j)!)^{r_j - s_j}} \\
&= \frac{(n-k)!}{\prod_{j\geq 0} r_j!} \prod_{j\geq 0} \binom{r_j}{s_j} \left(\frac{1}{(2j+1)!}\right)^{s_j} \left(\frac{1}{(2j)!}\right)^{r_j - s_j},
\end{aligned} \tag{34}$$

where the third and fourth equalities follow from definition (29) and formula (30) respectively.

Since any composition ν of size n with $\nu^+ = \lambda$ determines a unique partition $\mu \prec_k \lambda$ (which may depend on k) such that $\nu \rhd_k (\lambda, \mu)$ and conversely any partition $\mu \prec_k \lambda$ occurs in this manner, formula (34) yields

$$\begin{aligned}
\sum_{k=0}^n \frac{t^{n-k}}{(n-k)!} \sum_{\nu^+=\lambda} \bar{v}_n^{(k)}(\nu) &= \sum_{k=0}^n \frac{t^{n-k}}{(n-k)!} \sum_{\mu \prec_k \lambda} \sum_{\nu \rhd_k (\lambda,\mu)} \bar{v}_n^{(k)}(\nu) \\
&= \sum_{k=0}^n \frac{t^{n-k}}{(n-k)!} \sum_{\substack{0\leq s_j \leq r_j \\ s_0+s_1+\cdots=k}} \frac{(n-k)!}{\prod_{j\geq 0} r_j!} \prod_{j\geq 0} \binom{r_j}{s_j} \left(\frac{1}{(2j+1)!}\right)^{s_j} \left(\frac{1}{(2j)!}\right)^{r_j - s_j} \\
&= \frac{1}{\prod_{j=0}^m r_j!} \sum_{s_0=0}^{r_0} \cdots \sum_{s_m=0}^{r_m} \prod_{j=0}^m \binom{r_j}{s_j} \left(\frac{1}{(2j+1)!}\right)^{s_j} \left(\frac{t}{(2j)!}\right)^{r_j - s_j} \\
&= \frac{1}{\prod_{j=0}^m r_j!} \prod_{j=0}^m \left(\frac{1}{(2j+1)!} + \frac{t}{(2j)!}\right)^{r_j} \\
&= \frac{(1+t)^{r_0}}{r_0! \prod_{j=1}^m r_j!} \prod_{j=1}^m (t+1)^{r_j} T_j^{r_j} \\
&= \frac{(1+t)^n}{r_0! \prod_{j=1}^m r_j!} \prod_{j=1}^m T_j^{r_j}
\end{aligned} \tag{35}$$

where $n - k = \sum_{j=0}^m (r_j - s_j)$ and $n = \sum_{j=0}^m r_j$ are used in the third and final equalities.

On the other hand, any n-composition ν of m determines a unique partition λ of m such that $\nu^+ = \lambda$ and conversely any partition λ of m occurs in this manner as we are assuming that $m \leq n$. Since $\ell(\nu) = \ell(\lambda)$ for $\nu^+ = \lambda$, formula (27) implies

$$(-1)^{m-1} \frac{n!\, c_{n,m}^{(k)}}{(2m+k)!} = m \sum_{\lambda \vdash m} (-1)^{\ell(\lambda)-1} (\ell(\lambda)-1)!\, (n - \ell(\lambda))! \sum_{\nu^+=\lambda} \bar{v}_n^{(k)}(\nu). \tag{36}$$

Thus definition (6) together with $\ell(\lambda) = r_1 + \cdots + r_m$ and $n - \ell(\lambda) = r_0$ gives

$$\begin{aligned}
(-1)^{m-1} \frac{G_{n,m}(t)}{(t+1)^n} &= \frac{(-1)^{m-1}}{(t+1)^n} \sum_{k=0}^{n} \frac{n!\, c_{n,m}^{(k)}}{(n-k)!\,(2m+k)!}\, t^{n-k} \\
&= \frac{m}{(t+1)^n} \sum_{\lambda \vdash m} (-1)^{\ell(\lambda)-1} (\ell(\lambda)-1)!\, r_0! \sum_{k=0}^{n} \frac{t^{n-k}}{(n-k)!} \sum_{\nu^+ = \lambda} \bar{v}_n^{(k)}(\lambda) \\
&= m \sum_{\lambda \vdash m} (-1)^{r_1 + \cdots + r_m - 1} \frac{(r_1 + \cdots + r_m - 1)!}{r_1! \cdots r_m!} T_1^{r_1} \cdots T_m^{r_m},
\end{aligned}$$

where the second and third equalities follow from formulas (36) and (35) respectively. This proves formula (32). As the right-hand side of formula (32) depends only on m, the rational function $(t+1)^{-n} G_{n,m}(t)$ is independent of n. □

4 Generating Functions and Bernoulli Numbers

We are now in a position to establish Theorem 1.3 and Corollary 1.4.

Proofs of Theorem 1.3 *and Corollary* 1.4. Formula (7) is an immediate consequence of the last assertion in Proposition 3.4 that $(t+1)^{-n} G_{n,m}(t)$ is independent of n. The proof of formula (8) is based on the following general fact on generating series: if we put

$$\beta_m := m \sum_{\lambda \vdash m} (-1)^{r_1 + \cdots + r_m - 1} \frac{(r_1 + \cdots + r_m - 1)!}{r_1! \cdots r_m!} \alpha_1^{r_1} \cdots \alpha_m^{r_m} \qquad (m = 1, 2, 3, \dots),$$

where $\lambda = (1^{r_1}\, 2^{r_2} \cdots m^{r_m})$, then there exists a formal power series expansion

$$\log \left(1 + \sum_{m=1}^{\infty} \alpha_m z^{2m} \right) = \sum_{m=1}^{\infty} \frac{\beta_m}{m} z^{2m}. \tag{37}$$

We apply this formula to the situation of Proposition 3.4, where $\alpha_j = T_j$ in formula (33) and

$$\beta_m = (-1)^{m-1} (t+1)^{-m} G_m(t) \tag{38}$$

in formula (32) with $n = m$. We now find

$$\begin{aligned}
1 + \sum_{m=1}^{\infty} T_m z^{2m} &= 1 + \frac{t}{t+1} \sum_{m=1}^{\infty} \frac{z^{2m}}{(2m)!} + \frac{1}{t+1} \sum_{m=1}^{\infty} \frac{z^{2m}}{(2m+1)!} \\
&= \frac{t}{t+1} \sum_{m=0}^{\infty} \frac{z^{2m}}{(2m)!} + \frac{1}{t+1} \sum_{m=0}^{\infty} \frac{z^{2m}}{(2m+1)!} \\
&= \frac{1}{t+1} \left(t \cosh z + \frac{\sinh z}{z} \right).
\end{aligned} \tag{39}$$

Substitute this into formula (37) and apply the differential operator $\frac{z}{2} \frac{\partial}{\partial z}$ to the ensuing equation. Then after some calculations we get formula (8) and thus establish Theorem 1.3. Corollary 1.4 then follows easily from this theorem in the manner mentioned in the Introduction. □

We present some amplifications of Theorem 1.3 and Corollary 1.4. For the extremal cases of $k = 0$, $n-1$, n, the coefficients $c_{n,m}^{(k)}$ can be written explicitly in terms of Bernoulli numbers.

Lemma 4.1 *For $k = 0,\ n-1,\ n$, the coefficients $c^{(k)}_{n,m}$ are directly tied to b_m by*

$$c^{(0)}_{n,m} = (2m)!\,(2^{2m}-1)\,b_m \qquad c^{(n-1)}_{n,m} = c^{(n)}_{n,m} = \frac{(n+2m)!}{n!}\,b_m, \qquad (n \ge m \ge 1). \tag{40}$$

Proof. Substitute $t = 0$ into definition (6) to get $G_{n,m}(0) = \frac{n!}{(n+2m)!}\,c^{(n)}_{n,m}$. Similarly put $t = 0$ in formulas (7) and (10) to have $G_{n,m}(0) = G_m(0) = b_m$. These together lead to the assertion for $c^{(n)}_{n,m}$ in formula (40). The assertion for $c^{(n-1)}_{n,m}$ then follows from the identity $c^{(n-1)}_{n,m} = c^{(n)}_{n,m}$ mentioned in the Introduction. To prove the assertion for $c^{(0)}_{n,m}$ in formula (40) we consider the generating polynomial $\hat{G}_{n,m}(t) := t^n G_{n,m}(1/t)$ instead of $G_{n,m}(t)$. After the change $t \mapsto 1/t$ and multiplication by t^n, formula (6) gives $\hat{G}_{n,m}(0) = c^{(0)}_{n,m}/(2m)!$. On the other hand, formula (7) yields $\hat{G}_{n,m}(t) = (t+1)^{n-m}\hat{G}_m(t)$, where $\hat{G}_m(t) := \hat{G}_{m,m}(t)$, while formula (8) gives

$$\sum_{m=1}^{\infty} (-1)^{m-1}\hat{G}_m(t)\left(\frac{z^2}{t+1}\right)^m = \frac{t(z\coth z - 1) + z^2}{2(z\coth z + t)},$$

which upon putting $t = 0$ reduces to the equality

$$\sum_{m=1}^{\infty} (-1)^{m-1}\hat{G}_m(0) z^{2m} = \frac{z}{2\coth z} = \frac{z}{2}\tanh z.$$

Comparing it with the Maclaurin expansion $\tanh z = 2\sum_{m=1}^{\infty}(-1)^{m-1}(2^{2m}-1)\,b_m\,z^{2m-1}$, we find $\hat{G}_m(0) = (2^{2m}-1)\,b_m$. Thus $c^{(0)}_{n,m} = (2m)!\,\hat{G}_{n,m}(0) = (2m)!\,\hat{G}_m(0) = (2m)!\,(2^{2m}-1)\,b_m$. □

The first formula in (40) is already found in [8]. To deal with the intermediate coefficients $c^{(k)}_{n,m}$ for $k = 1, \dots, n-2$, another modification of the generating polynomials $G_{n,m}(t)$ is helpful.

$$F_{n,m}(t) := t^n G_{n,m}\left(\frac{1-t}{t}\right) = \sum_{k=0}^{n} \frac{n!\,c^{(k)}_{n,m}}{(n-k)!\,(2m+k)!}\,t^k(1-t)^{n-k}. \tag{41}$$

Lemma 4.2 *For $n \ge m \ge 1$, the polynomials $F_{n,m}(t)$ depend only on m, being independent of n. They satisfy the differential-difference equation*

$$2F_{n,m}(t) + \frac{t}{m}F'_{n,m}(t) + \frac{(1-t)^2}{m-1}F'_{n-1,m-1}(t) = 0 \qquad (n \ge m \ge 2). \tag{42}$$

All the $F_{n,m}(t)$ can be determined inductively by solving equation (42) with initial conditions

$$F_{n,m}(0) = (2^{2m}-1)b_m, \qquad F_{n,1}(t) = \frac{1}{2} - \frac{t}{3} \qquad (n \ge m \ge 1). \tag{43}$$

Proof. Put $F_m(t) := F_{m,m}(t)$. It readily follows from relation (7) and definition (41) that $F_{n,m}(t) = F_m(t)$ for every $n \ge m$. The substitution $t \mapsto \frac{1-t}{t}$ induces the changes

$$\beta_m \mapsto (-1)^{m-1}F_m(t), \qquad 1 + \sum_{m=1}^{\infty} T_m z^{2m} \mapsto (1-t)\cosh z + t\,\frac{\sinh z}{z}$$

in formulas (38) and (39) respectively. With these changes formula (37) reads

$$\log\left\{(1-t)\cosh z + t\,\frac{\sinh z}{z}\right\} = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}F_m(t)z^{2m}. \tag{44}$$

Denote the both sides of this equation by $\Phi = \Phi(z,t)$. A direct check using the left-hand side of equation (44) tells us that Φ satisfies the partial differential equation

$$z\frac{\partial\Phi}{\partial z} + \left\{t-(1-t)^2z^2\right\}\frac{\partial\Phi}{\partial t} = (1-t)z^2. \tag{45}$$

Next we look at this equation by means of the right-hand side of formula (44). For each $m \ge 2$ the coefficient of z^{2m} in equation (45) being zero gives the differential-difference equation

$$2F_m(t) + \frac{t}{m}F_m'(t) + \frac{(1-t)^2}{m-1}F_{m-1}'(t) = 0 \qquad (m \ge 2),$$

which can be expressed as equation (42), because $F_m(t) = F_{n,m}(t)$ and $F_{m-1}(t) = F_{n-1,m-1}(t)$ by the first assertion of the lemma. The first condition in (43) is derived from formulas (41) and (40) as $F_{n,m}(0) = c_{n,m}^{(0)}/(2m)! = (2^{2m}-1)b_m$, while the second condition follows from $F_{n,1}(t) = F_{1,1}(t)$ and the direct calculation of $F_{1,1}(t)$, which is easy. $\square$

Differential-difference equation (42) can be used to derive a recursion formula for $c_{n,m}^{(k)}$ as well as to explicitly determine $c_{n,m}^{(k)}$ for some k's near 0 or n in terms of Bernoulli numbers.

Proposition 4.3 *For $k = 0$, 1, the coefficients $c_{n,m}^{(k)}$ are given by the first formula in* (40) *and*

$$c_{n,m}^{(1)} = (2m+1)!\left\{(2^{2m}-1)\,b_m - \frac{2m}{n}(2^{2(m+1)}-1)\,b_{m+1}\right\} \qquad (n \ge m \ge 1). \tag{46}$$

For $k = n-2$, $n-1$, n, the coefficients $c_{n,m}^{(k)}$ take a common value which is given by

$$c_{n,m}^{(n-2)} = c_{n,m}^{(n-1)} = c_{n,m}^{(n)} = \frac{(n+2m)!}{n!}\,b_m \qquad (n \ge m \ge 2). \tag{47}$$

Moreover for $1 \le k \le n-2$ and $2 \le m \le n$ there exists a recursion formula

$$c_{n,m}^{(k)} - c_{n,m}^{(k-1)} = \frac{(n-k)(n-k-1)m}{n(m-1)}\left\{(2m+k-1)\,c_{n-1,m-1}^{(k)} - (k+1)\,c_{n-1,m-1}^{(k+1)}\right\}. \tag{48}$$

Proof. If $\alpha_{n,m}^{(k)} := n!\,c_{n,m}^{(k)}/\{(n-k)!(2m+k)!\}$ then formula (41) and its derivative yield

$$F_{n,m}(t) = \sum_{k=0}^{n}\alpha_{n,m}^{(k)}\,t^k(1-t)^{n-k},$$

$$tF_{n,m}'(t) = \sum_{k=1}^{n}\{k\alpha_{n,m}^{(k)} - (n-k+1)\alpha_{n,m}^{(k-1)}\}\,t^k(1-t)^{n-k},$$

$$(1-t)^2F_{n-1,m-1}'(t) = \sum_{k=0}^{n-2}\{(k+1)\alpha_{n-1,m-1}^{(k+1)} - (n-k-1)\alpha_{n-1,m-1}^{(k)}\}\,t^k(1-t)^{n-k}.$$

Thus the left-hand side of equation (42) can be expressed as $\sum_{k=0}^{n} \gamma_{n,m}^{(k)} t^k (1-t)^{n-k}$ with

$$\begin{aligned}
\gamma_{n,m}^{(0)} &= 2\alpha_{n,m}^{(0)} + \frac{1}{m-1}\left\{\alpha_{n-1,m-1}^{(1)} - (n-1)\alpha_{n-1,m-1}^{(0)}\right\} && (k=0), \\
\gamma_{n,m}^{(n-1)} &= 2\alpha_{n,m}^{(n-1)} + \frac{1}{m}\left\{(n-1)\alpha_{n,m}^{(n-1)} - 2\alpha_{n,m}^{(n-2)}\right\} && (k=n-1), \\
\gamma_{n,m}^{(n)} &= 2\alpha_{n,m}^{(n)} + \frac{1}{m}(n\alpha_{n,m}^{(n)} - \alpha_{n,m}^{(n-1)}) && (k=n), \\
\gamma_{n,m}^{(k)} &= 2\alpha_{n,m}^{(k)} + \frac{1}{m}\left\{k\alpha_{n,m}^{(k)} - (n-k+1)\alpha_{n,m}^{(k-1)}\right\} \\
&\quad + \frac{1}{m-1}\left\{(k+1)\alpha_{n-1,m-1}^{(k+1)} - (n-k-1)\alpha_{n-1,m-1}^{(k)}\right\} && (1 \le k \le n-2).
\end{aligned}$$

Since $t^k(1-t)^{n-k}$, $k = 0, \ldots, n$, form a linear basis of all polynomials in t of degree at most n, it follows from equation (42) that $\gamma_{n,m}^{(k)} = 0$ for every $k = 0, \ldots, n$. First, for $k = 0$ we find

$$\gamma_{n,m}^{(0)} = 2\frac{c_{n,m}^{(0)}}{(2m)!} + \frac{n-1}{m-1}\left\{\frac{c_{n-1,m-1}^{(1)}}{(2m-1)!} - \frac{c_{n-1,m-1}^{(0)}}{(2m-2)!}\right\} = 0 \qquad (n \ge m \ge 2),$$

where $c_{n,m}^{(0)}$ and $c_{n-1,m-1}^{(0)}$ are already known as in the first formula of (40). Thus $c_{n-1,m-1}^{(1)}$ is also known from this equation. Replacing $(n-1, m-1)$ with (n,m) we get formula (46). Secondly, for $k = n-1$, n, equations $\gamma_{n,m}^{(n-1)} = 0$ and $\gamma_{n,m}^{(n)} = 0$ lead to $c_{n,m}^{(n-2)} = c_{n,m}^{(n-1)}$ and $c_{n,m}^{(n-1)} = c_{n,m}^{(n)}$ respectively, where the latter is already mentioned in the Introduction and Lemma 4.1. Thus formula (47) follows from the second formula of (40). Finally, for $1 \le k \le n-2$ equation $\gamma_{n,m}^{(k)} = 0$ with $2 \le m \le n$ is equivalent to recursion formula (48). $\square$

Since $c_{n,m}^{(k)}$ is already known for the k's at both ends of the interval $0 \le k \le n$ as in formulas (40), (46) and (47), the recursion formula (48) can be used to inductively determine all coefficients $c_{n,m}^{(k)}$, where there are three directions in which induction works productively.

(a) $c_{n,m}^{(k)} \leftarrow c_{n,m}^{(k-1)}, c_{n-1,m-1}^{(k)}, c_{n-1,m-1}^{(k+1)}$, (b) $c_{n,m}^{(k-1)} \leftarrow c_{n,m}^{(k)}, c_{n-1,m-1}^{(k)}, c_{n-1,m-1}^{(k+1)}$,

(c) $c_{n-1,m-1}^{(k+1)} \leftarrow c_{n,m}^{(k)}, c_{n,m}^{(k-1)}, c_{n-1,m-1}^{(k)}$ (with $(m-1, n-1)$ replaced by (m,n)).

For example formula (48) with $k = n-2$ is used in direction (b) to derive

$$c_{n,m}^{(n-3)} = \frac{1}{n!}\left\{(n+2m)!\, b_m - 4m \cdot (n+2m-3)!\, b_{m-1}\right\} \qquad (n \ge 3,\ n \ge m \ge 2)$$

from formula (47). Similarly formula (48) with $k = 1$ can be applied in direction (c) to deduce a closed expression for $c_{m,n}^{(2)}$ from formulas (40) and (46), and so forth.

At the end we return to the starting point of this article, that is, to polyhedral harmonics. With Theorem 1.1 for the cube case, the determination of polyhedral harmonic functions for all skeletons of all regular convex polytopes has been completed. As a summary we have:

Theorem 4.4 *Let P be any n-dimensional regular convex polytope with center at the origin in $\mathbb{R}^n$ and G the symmetry group of P. Let Δ_G be the fundamental alternating polynomial of the reflection group G. Then $\mathcal{H}_{P(k)}$ is independent of $k = 0, \ldots, n$. It is $|G|$-dimensional as a linear space, generated by Δ_G as an $\mathbb{R}[\partial]$-module, and is the regular representation as an $\mathbb{R}[G]$-module, where $|G|$ is the order of G.*

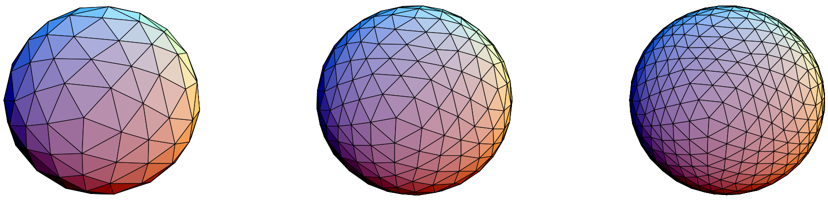

Figure 3: Approximations of the sphere by geodesic domes.

For the classification of regular convex polytopes we refer to Coxeter [2]. Theorem 4.4 is proved in article [12] for the n-dimensional regular simplex and in article [14] for the exceptional regular polytopes, that is, for the dodecahedron and icosahedron in 3-dimensions and for the 24-cell, 120-cell and 600-cell in 4-dimensions. For the n-dimensional cross polytope, namely, the analogue in n-dimensions of the octahedron, there is no detailed written proof in the literature, but a proof quite similar to the regular n-simplex case is feasible. This is because each face of an n-dimensional cross polytope is an $(n-1)$-dimensional regular simplex. Finally the n-dimensional cube case has been treated in this article (Theorem 1.1), in which case the proof is quite different from those in the other cases (as of this writing). Here we should also mention the important studies [4, 5, 6, 7, 8, 10] etc. of earlier times, which contain partial answers to our questions, referring to the survey [13] for a more extensive literature.

Apart from the regular figures for which symmetry plays a dominant role, polyhedral harmonics is largely open, for example, for such figures as geodesic domes in Figure 3. One may ask whether there exists a sequence of polytopes P_i, $i = 1, 2, 3, \ldots$, approximating the sphere, such that the sequence of finite-dimensional spaces $\mathcal{H}_{P_i(k)}$, $i = 1, 2, 3, \ldots$, is exhausting the infinite-dimensional space of all harmonic polynomials.

5 Three-Dimensional Case

This last section is an appendix to Section 1, picking out the case of 3-dimensions as an instructive example. We carry out the irreducible decomposition of $\mathcal{H}_3 := \mathcal{H}_{C(k)} = \mathcal{H}_{W_3}$ as $\mathbb{R}[W_3]$-modules by exhibiting all of the ensuing generators. There is a direct sum decomposition

$$\mathcal{H}_3 = \bigoplus_{m=0}^{9} \mathcal{H}_3(m),$$

where $\mathcal{H}_3(m)$ is the homogeneous component of degree m. Each component is yet to be decomposed into irreducible factors. We write (x, y, z) for the coordinates (x_1, x_2, x_3) of $\mathbb{R}^3$.

First we need a list of all irreducible representations of W_3. Our 3-dimensional cube C has its vertices at $(\pm 1, \pm 1, \pm 1)$ as in Figure 1. The even vertices among them, that is, those vertices with even number of -1's in their coordinates are given by

$$a = (1, 1, 1), \quad b = (1, -1, -1), \quad c = (-1, 1, -1), \quad d = (-1, -1, 1),$$

which are the vertices of a regular tetrahedron. The symmetry group T of this tetrahedron is a subgroup of W_3 isomorphic to the symmetric group S_4 on the vertices a, b, c, d. If ι denotes

	1	P	Q	R	S	ι
1	1	1	1	1	1	1
δ	1	-1	1	1	-1	-1
π	1	1	1	1	1	-1
$\delta\pi$	1	-1	1	1	-1	1

Table 2: The 1-dimensional representations (characters) of W_3.

	1	P	Q	R	S	ι
ψ	3	1	-1	0	-1	-1
θ	2	0	2	-1	0	1

Table 3: Two multi-dimensional irreducible characters of W_3.

the antipodal transformation of $\mathbb{R}^3$, then $W_3 = T \times \langle \iota \rangle \cong S_4 \times \mathbb{Z}_2$. We find that

$$\begin{aligned} (ab) &\;:\; (x, y, z) \mapsto (x, -z, -y), & (bc) &\;:\; (x, y, z) \mapsto (y, x, z), \\ (cd) &\;:\; (x, y, z) \mapsto (x, z, y), & \iota &\;:\; (x, y, z) \mapsto (-x, -y, -z). \end{aligned}$$

The conjugacy classes of T are represented by the unit element 1, $P = (ab), Q = (ab)(cd)$, $R = (abc) = (ab)(bc)$ and $S = (abcd) = (ab)(bc)(cd)$, so that each character of W_3 is determined by its values at those elements and ι. Table 2 gives all the four 1-dimensional characters of W_3, where δ corresponds to the sign representation on the fundamental alternating polynomial $\varDelta_3 = xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$, while π stems from the nontrivial character of $\langle \iota \rangle$. On the other hand there are two multi-dimensional irreducible characters, say ψ and θ, as in Table 3. The corresponding representations come from two irreducible components of $\mathcal{H}_3$, that is,

$$\begin{aligned} V_\psi &= \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}z = \mathcal{H}_3(1), & \dim V_\psi &= 3, \\ V_\theta &= \mathbb{R}(y^2 - z^2) + \mathbb{R}(z^2 - x^2) + \mathbb{R}(x^2 - y^2) \subset \mathcal{H}_3(2), & \dim V_\theta &= 2. \end{aligned}$$

The complete character table for W_3 then consists of four 1-dimensional characters in Table 2; two 2-dimensional ones $\theta = \delta\pi\theta$ and $\delta\theta = \pi\theta$; and four 3-dimensional ones ψ, $\delta\psi$, $\pi\psi$ and $\delta\pi\psi$.

Now the $\mathbb{R}[W_3]$-module $\mathcal{H}_3$ admits the irreducible decomposition as in Table 4, where the third column exhibits a generator of each irreducible factor V. Note that the cyclic permutations in x, y, z of the generator span V as a linear space; they form a linear basis if $\dim V = 3$ and sum up to zero if $\dim V = 2$. In the first column $\deg V = m$ means $V \subset \mathcal{H}_3(m)$, while in the fourth column χ stands for the character of the factor V. There is an involutive operation

$$V \mapsto V^* := \{\, f(\partial)\varDelta_3 \;:\; f \in V \,\}$$

among the irreducible factors of $\mathcal{H}_3$. This operation results in the reflection of Table 4 around the central axis ℓ. In particular one has $\deg V^* = 9 - \deg V$, $\dim V^* = \dim V$ and $\chi^* = \delta\chi$.

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$\deg V$	$\dim V$	irreducible $\mathbb{R}[W_3]$-module V generated by	χ
0	1	1	1
1	3	x	ψ
2	2	$y^2 - z^2$	θ
	3	yz	$\pi\psi$
3	1	xyz	π
	3	$x(2x^2 - 3y^2 - 3z^2)$	ψ
		$x(y^2 - z^2)$	$\delta\pi\psi$
4	2	$(y^2 - z^2)(6x^2 - y^2 - z^2)$	θ
	3	$yz(6x^2 - y^2 - z^2)$	$\pi\psi$
		$yz(y^2 - z^2)$	$\delta\psi$
5	3	$x\{2x^4 - 10x^2(y^2 + z^2) - 5(y^4 + z^4) + 60y^2z^2\}$	ψ
		$x(y^2 - z^2)(2x^2 - y^2 - z^2)$	$\delta\pi\psi$
	2	$xyz(y^2 - z^2)$	$\delta\theta$
6	3	$yz\{30x^4 - 30x^2(y^2 + z^2) - 3(y^4 + z^4) + 20y^2z^2\}$	$\pi\psi$
		$yz(y^2 - z^2)(10x^2 - y^2 - z^2)$	$\delta\psi$
	1	$(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$	$\delta\pi$
7	3	$x(y^2 - z^2)\{3x^4 - 5(y^2 + z^2) + 15y^2z^2\}$	$\delta\pi\psi$
	2	$xyz(y^2 - z^2)(10x^2 - 3y^2 - 3z^2)$	$\delta\theta$
8	3	$yz(y^2 - z^2)\{5x^4 - 3x^2(y^2 + z^2) + y^2z^2\}$	$\delta\psi$
9	1	$\mathit{\Delta}_3 = xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$	δ

Table 4: Irreducible decomposition of $\mathcal{H}_3$.

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