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FIRST SYZYGIES OF IRREDUCIBLE A-HYPERGEOMETRIC QUOTIENTS

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ABSTRACT. An A-hypergeometric system is not irreducible, if its parameter vector is resonant. In this paper, we present a way of computing a finite system of generators of the first syzygy module of an irreducible A-hypergeometric quotient. In particular, if the semigroup generated by A is simplicial and scored, then an explicit system of generators is given.

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Keywords: A-hypergeometric system, irreducible quotient, first syzygy, toric variety

1. INTRODUCTION

Let K be a field of characteristic 0, and let $A := (a_{ij})$ be a $d \times n$ integer matrix. We assume that \mathbb{Z}^d is generated by the column vectors of A as an abelian group. Given a parameter vector $\boldsymbol{\beta} = {}^t(\beta_1, \ldots, \beta_d) \in K^d$, the A-hypergeometric (or GKZ (after the systematic study by Gel'fand, Kapranov, and Zelevinskii [1]-[4])) system $M^L(\boldsymbol{\beta})$ with parameter vector $\boldsymbol{\beta}$ is defined as the left $D(K^n)$ -module

(1) $M^{L}(\boldsymbol{\beta}) := M^{L}_{A}(\boldsymbol{\beta}) := D(K^{n})/D(K^{n})I_{A}(\boldsymbol{\partial}) + D(K^{n})\langle A\boldsymbol{\theta} - \boldsymbol{\beta} \rangle,$

where $D(K^n)$ is the *n*th Weyl algebra

$$D(K^n) = K[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle,$$

 $I_A(\partial)$ is the toric ideal of $K[\partial_1, \ldots, \partial_n]$ defined by A, and $D(K^n)\langle A\theta - \beta\rangle$ is the left ideal of $D(K^n)$ generated by $\sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i$ $(i = 1, \ldots, d)$.

The A-hypergeometric system $M^{L}(\boldsymbol{\beta})$ is not irreducible in general. Indeed $M^{L}(\boldsymbol{\beta})$ is irreducible if and only if the parameter vector $\boldsymbol{\beta}$ is nonresonant (see [4] and [12]). In the paper [12], we considered a category $\mathcal{O}_{K^{n}}$ of right $D(K^{n})$ -modules appropriate for the study of A-hypergeometric systems, and we considered irreducible modules in $\mathcal{O}_{K^{n}}$. Each modules in $\mathcal{O}_{K^{n}}$ has a weight decomposition with respect

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to the torus action defined by A. We treat right $D(K^n)$ -modules in this paper as well. We remark that there exists a one-to-one correspondence between right $D(K^n)$ -modules and left $D(K^n)$ -modules by the antiautomorphism ι of $D(K^n)$ defined by

(2)
$$\iota(x_j) = \partial_j, \quad \iota(\partial_j) = x_j \quad \text{for all } j.$$

Let $\boldsymbol{\beta} \in K^d$ satisfy $F_{\sigma}(\boldsymbol{\beta}) \notin \mathbb{N}$ for every facet σ of the cone generated by A, where F_{σ} is the primitive integral support function of σ . Then

$$L(\boldsymbol{\beta}) := D(K^n) / I_A D(K^n) + D(K^n) \cap \langle A\theta - \boldsymbol{\beta} \rangle D((K^{\times})^n)$$

is irreducible [12, Theorem 6.4], and any irreducible module in \mathcal{O}_{K^n} can be described similarly [12, Theorem 6.6], where I_A is the toric ideal of $K[x_1, \ldots, x_n]$ defined by A, and

$$D((K^{\times})^n) = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \langle \partial_1, \dots, \partial_n \rangle.$$

In this paper, we describe a finite system of generators of the right ideal (the first syzygy module of $L(\boldsymbol{\beta})$)

$$I_A D(K^n) + D(K^n) \cap \langle A\theta - \beta \rangle D((K^{\times})^n),$$

apart from that of I_A , which can be computed by the commutative Gröbner basis theory. To this aim, we consider generators of the right $D(K^n)$ -module

$$N := \frac{I_A D(K^n) + D(K^n) \cap \langle A\theta - \beta \rangle D((K^{\times})^n)}{I_A D(K^n) + \langle A\theta - \beta \rangle D(K^n)}$$

Since the $D(K^n)$ -module N is finitely generated, it is generated by finite number of weight spaces. In Theorem 4.7, we specify those weights. This enables us to compute a finite system of generators of N and, in turn, that of the first syzygy module of the irreducible module $L(\beta)$.

If the semigroup is simplicial and scored, then those weights are associated to facets, and we give explicit generators of N (Theorem 7.1).

We note that Hosono et al [5] and [6] considered $L(\beta)$ (called the extended GKZ system) for the reflexive case.

2. Rings of differential operators

Let K denote a field of characteristic 0. Let R be a commutative K-algebra, and let M, N be R-modules. We briefly recall the module D(M, N) of differential operators from M to N. For details, see [16]. For $k \in \mathbb{N}$, the subspaces $D^k(M, N)$ of $\operatorname{Hom}_K(M, N)$ are inductively defined by

$$D^0(M,N) = \operatorname{Hom}_R(M,N)$$

and

 $D^{k+1}(M,N) = \{P \in \operatorname{Hom}_K(M,N) : [f,P] \in D^k(M,N) \quad (\forall f \in R)\},\$ where [,] denotes the commutator. Set $D(M,N) := \bigcup_{k=0}^{\infty} D^k(M,N),\$ and D(M) := D(M,M). Then D(M) is a K-algebra, and D(M,N) is a (D(N), D(M))-bimodule.

The ring $D(K^n) := D(K[x_1, \ldots, x_n])$ of differential operators on K^n is the *n*th Weyl algebra:

$$D(K^n) = K[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle,$$

where $\partial_j = \frac{\partial}{\partial x_j}$.

The ring $D((K^{\times})^n) := D(K[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ of differential operators on $(K^{\times})^n$ is given by

$$D((K^{\times})^{n}) = K[x_{1}^{\pm 1}, \dots, x_{n}^{\pm 1}] \langle \partial_{1}, \dots, \partial_{n} \rangle$$

= $\bigoplus_{u \in \mathbb{Z}^{n}} x^{u} K[\theta_{1}, \dots, \theta_{n}],$

where $\theta_j = x_j \partial_j$.

Let $A := \{ a_1, a_2, \ldots, a_n \}$ be a finite set of column vectors in \mathbb{Z}^d . Sometimes we identify A with the matrix $(a_1, a_2, \ldots, a_n) = (a_{ij})$. Let $\mathbb{Z}A$, and $\mathbb{Q}_{\geq 0}A$ denote the abelian group, and the cone generated by A, respectively. Throughout this paper, we assume that $\mathbb{Z}A = \mathbb{Z}^d$ and that $\mathbb{Q}_{>0}A$ is strongly convex.

Let X_A denote the affine toric variety defined by A, and T_A its big torus. Let $\mathbb{N}A$ be the semigroup generated by A. The semigroup algebra $K[\mathbb{N}A] = \bigoplus_{\boldsymbol{a} \in \mathbb{N}A} Kt^{\boldsymbol{a}}$ is the ring of regular functions on X_A . Then we have $K[\mathbb{N}A] \simeq K[x]/I_A$, where I_A is the ideal of the polynomial ring $K[x] := K[x_1, \ldots, x_n]$ generated by all $x^{\boldsymbol{u}} - x^{\boldsymbol{v}}$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n$ with $A\boldsymbol{u} = A\boldsymbol{v}$. Here we have used the multi-index notation, e.g., $x^{\boldsymbol{u}} = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ for $\boldsymbol{u} = t(u_1, u_2, \ldots, u_n)$. The ring $D(T_A) := D(K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}])$ of differential operators on T_A

The ring $D(T_A) := D(K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}])$ of differential operators on T_A is given by

$$D(T_A) = K[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \langle \partial_{t_1}, \dots, \partial_{t_d} \rangle$$

= $\bigoplus_{a \in \mathbb{Z}^d} t^a K[s_1, \dots, s_d],$

where $s_i = t_i \partial_{t_i}$ and $\partial_{t_i} = \frac{\partial}{\partial t_i}$.

The ring $D(X_A) := D(K[\mathbb{N}A])$ of differential operators on X_A is a subalgebra of $D(T_A)$:

$$D(X_A) = \{ P \in D(T_A) : P(K[\mathbb{N}A]) \subseteq K[\mathbb{N}A] \}.$$

Let $X = K^n$, $(K^{\times})^n$, T_A , or X_A . For $a \in \mathbb{Z}^d$, set $D(X)_a := \{P \in D(X) : [s_i, P] = a_i P \ (i = 1, ..., d)\},\$ where $s_i = \sum_{j=1}^n a_{ij} x_j \partial_j$ for $X = K^n$ or $(K^{\times})^n$. Then

$$D(X) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} D(X)_{\boldsymbol{a}}$$

is a \mathbb{Z}^d -graded algebra.

Recall from [8, Theorem 2.3] that the graded part of $D(X_A)$ is described by

$$D(X_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} \mathbb{I}(\Omega(\boldsymbol{a})) \text{ for all } \boldsymbol{a} \in \mathbb{Z}^d,$$

where

$$\begin{aligned} \Omega(\boldsymbol{a}) &:= \Omega_A(\boldsymbol{a}) := \mathbb{N}A \setminus (-\boldsymbol{a} + \mathbb{N}A), \\ \mathbb{I}(\Omega(\boldsymbol{a})) &:= \{f(s) \in K[s] : f(\boldsymbol{c}) = 0 \text{ for all } \boldsymbol{c} \in \Omega(\boldsymbol{a})\}, \\ K[s] &:= K[s_1, \dots, s_d]. \end{aligned}$$

We write $D(K^n, X_A)$ instead of $D(K[x], K[\mathbb{N}A])$. From [16, 1.3 (e),(f)], we have

(3)
$$D(K^n, X_A) = D(K^n)/I_A D(K^n).$$

The algebra $D(X_A)$ can be identified with

$$\{P \in D(K^n) : PI_A \subseteq I_A D(K^n)\}/I_A D(K^n).$$

(See e.g. [7].) We may thus consider that $D(X_A)$ is contained in $D(K^n, X_A)$. For the following proposition, see [11, Proposition 4.1 and Corollary 4.2].

Proposition 2.1.

$$D(K^n, X_A) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} t^{\boldsymbol{a}} \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})),$$

where

$$\widetilde{\Omega}(\boldsymbol{a}) := \widetilde{\Omega}_A(\boldsymbol{a}) := \{ \boldsymbol{u} \in \mathbb{N}^n : A\boldsymbol{u} \notin -\boldsymbol{a} + \mathbb{N}A \}, \\ \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})) = \{ f(\theta) \in K[\theta] : f(\boldsymbol{u}) = 0 \text{ for all } \boldsymbol{u} \in \widetilde{\Omega}(\boldsymbol{a}) \}, \\ K[\theta] := K[\theta_1, \dots, \theta_n].$$

In particular, $D(K^n, X_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} K[\theta]$ for all $\boldsymbol{a} \in \mathbb{N}A$.

Recall that a pair (\boldsymbol{u}, σ) with $\sigma \subseteq \{1, 2, \ldots, n\}$ and $\boldsymbol{u} \in \mathbb{N}^{\sigma^c} := \{\boldsymbol{v} \in \mathbb{N}^n \mid v_j = 0 \text{ for all } j \in \sigma\}$ is called a standard pair of a monomial ideal M of $K[\partial_1, \ldots, \partial_n]$ if the following conditions are satisfied:

- (1) for any $\boldsymbol{v} \in \mathbb{N}^{\sigma}$, the monomial $\partial^{\boldsymbol{u}+\boldsymbol{v}}$ does not belong to M;
- (2) for any $l \notin \sigma$, there exists $\boldsymbol{v} \in \mathbb{N}^{\sigma \cup \{l\}}$ such that $\partial^{\boldsymbol{u}+\boldsymbol{v}}$ belongs to M.

Let $\mathbb{I}(\boldsymbol{a})$ denote the ideal of $K[\partial_1, \ldots, \partial_n]$ generated by the monomials $\partial^{\boldsymbol{u}}$ with $A\boldsymbol{u} \in -\boldsymbol{a} + \mathbb{N}A$. Let $S(\mathbb{I}(\boldsymbol{a}))$ denote the set of standard pairs of the monomial ideal $\mathbb{I}(\boldsymbol{a})$. Then we obtain the following theorem from [13, Theorem 3.2.2, Corollary 3.2.3].

Theorem 2.2.

$$\begin{split} \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})) &= \widetilde{\mathbb{I}(\boldsymbol{a})} = \langle [\theta]_{\boldsymbol{u}} : \partial^{\boldsymbol{u}} \in \mathbb{I}(\boldsymbol{a}) \rangle \\ &= \bigcap_{(\boldsymbol{u},\sigma) \in S(\mathbb{I}(\boldsymbol{a}))} \langle \theta_i - u_i : i \notin \sigma \rangle, \end{split}$$

where

$$[\theta]_{\boldsymbol{u}} := \prod_{j=1}^{n} [\theta_j]_{u_j} := \prod_{j=1}^{n} \prod_{k=0}^{u_j-1} (\theta_j - k)$$

3. SIMPLE OBJECTS IN \mathcal{O}_{K^n}

In this section, we briefly review simple objects in \mathcal{O}_{K^n} from [12]. Let $X = K^n$, $(K^{\times})^n$, or T_A . In [12], we defined a full subcategory \mathcal{O}_X of the category of right D(X)-modules (cf. [9, 11]). A right D(X)module M is an object of \mathcal{O}_X if the support of M is contained in X_A , and M has a weight decomposition $M = \bigoplus_{\lambda \in K^d} M_{\lambda}$, where

$$M_{\lambda} = \{ x \in M : x \cdot f(s) = f(-\lambda)x \text{ for all } f \in K[s] \}.$$

Recall that the preorder \leq is defined in [9] (see also [14]):

For
$$\boldsymbol{\alpha}, \boldsymbol{\beta} \in K^d$$
, $\boldsymbol{\alpha} \preceq \boldsymbol{\beta} \Longleftrightarrow \mathbb{I}(\Omega(\boldsymbol{\beta} - \boldsymbol{\alpha})) \not\subseteq \mathfrak{m}_{\boldsymbol{\alpha}}$,

where \mathfrak{m}_{α} is the maximal ideal of K[s] at α . An equivalence relation $\alpha \sim \beta$ is defined to be $\alpha \preceq \beta$ and $\alpha \succeq \beta$.

For $\boldsymbol{\beta} \in K^d$, the right $D(K^n)$ -module

$$M_{K^n}(\boldsymbol{\beta}) := D(K^n) / (I_A D(K^n) + \langle s - \boldsymbol{\beta} \rangle D(K^n))$$

is the right *D*-module counterpart to the *A*-hypergeometric system $M_A^L(\boldsymbol{\beta})$ with parameter vector $\boldsymbol{\beta}$ (cf. (1) and (2)). Recall that $s_i = \sum_{j=1}^n a_{ij}\theta_j$, where $\theta_j = x_j\partial_j$. Clearly $M_{K^n}(\boldsymbol{\beta}) \in \mathcal{O}_{K^n}$.

Definition 3.1 (Definition 6.2 in [12]). Let $\beta \in K^d$. In $\beta + \mathbb{Z}A$, there exists a unique minimal equivalence class with respect to \preceq , which we denote by β^{empty} . An element belonging to the class is also denoted by β^{empty} .

Remark 3.2. In [10], we have defined a finite subset $E_{\tau}(\boldsymbol{\alpha})$ for a face τ and a parameter vector $\boldsymbol{\alpha} \in K^d$:

$$E_{\tau}(\boldsymbol{\alpha}) = \{\boldsymbol{\lambda} \in K(A \cap \tau) / \mathbb{Z}(A \cap \tau) : \boldsymbol{\alpha} - \boldsymbol{\lambda} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)\}.$$

The class $\boldsymbol{\beta}^{\text{empty}}$ is described as

$$E_{\tau}(\boldsymbol{\beta}^{\text{empty}}) = \begin{cases} E_{\mathbb{Q}_{\geq 0}A}(\boldsymbol{\beta}) & (\tau = \mathbb{Q}_{\geq 0}A) \\ \emptyset & (\tau \neq \mathbb{Q}_{\geq 0}A). \end{cases}$$

Theorem 3.3 (Theorem 6.4 in [12]). Let $\beta = \beta^{\text{empty}} \in K^d$. Then

$$L(\boldsymbol{\beta}) := L_{K^n}(T_A, \boldsymbol{\beta})$$

:= $D(K^n)/(I_A D(K^n) + D(K^n) \cap \langle s - \boldsymbol{\beta} \rangle D((K^{\times})^n))$
 $\simeq \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} t^{-\boldsymbol{\beta} + \boldsymbol{a}} K[s]/\langle s - \boldsymbol{\beta} + \boldsymbol{a} \rangle \otimes_{K[s]} \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a}))$

is a unique simple $D(K^n)$ -submodule of

$$D((K^{\times})^n)/(I_A D((K^{\times})^n) + \langle s - \boldsymbol{\beta} \rangle D((K^{\times})^n)).$$

Remark 3.4. Any simple object in \mathcal{O}_{K^n} is isomorphic to some $L(\boldsymbol{\beta})$ or a similar module associated to a torus constituting the toric variety X_A [12, Theorem 6.6].

Let $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}}$, and let

$$N_{K^n}(T_A,\boldsymbol{\beta}) := \frac{I_A D(K^n) + D(K^n) \cap \langle A\theta - \boldsymbol{\beta} \rangle D((K^{\times})^n)}{(I_A D(K^n) + \langle A\theta - \boldsymbol{\beta} \rangle D(K^n))}.$$

Here and hereafter we interchangeably use s and $A\theta$. The $D(K^n)$ -module $N_{K^n}(T_A, \beta)$ is the kernel of the natural surjection

$$M_{K^n}(\boldsymbol{\beta}) \to L_{K^n}(T_A, \boldsymbol{\beta}) = L(\boldsymbol{\beta}).$$

Our aim of this paper is to find a finite system of generators of the $D(K^n)$ -module $N_{K^n}(T_A, \beta)$. For a different choice of β^{empty} , we have the following proposition.

Proposition 3.5. Let $\beta = \beta^{\text{empty}} \sim \beta'$. Then there exists $P, Q \in D(X_A)$ such that

$$N_{K^n}(T_A, \boldsymbol{\beta}) = PN_{K^n}(T_A, \boldsymbol{\beta}')$$

$$N_{K^n}(T_A, \boldsymbol{\beta}') = QN_{K^n}(T_A, \boldsymbol{\beta}).$$

Proof. Since $\boldsymbol{\beta} \sim \boldsymbol{\beta}'$,

 $\mathbb{I}(\Omega(\boldsymbol{\beta}-\boldsymbol{\beta}')) \not\subseteq \mathfrak{m}_{\boldsymbol{\beta}'}, \qquad \mathbb{I}(\Omega(\boldsymbol{\beta}'-\boldsymbol{\beta})) \not\subseteq \mathfrak{m}_{\boldsymbol{\beta}}.$

Take $p(s) \in \mathbb{I}(\Omega(\boldsymbol{\beta} - \boldsymbol{\beta}')) \setminus \mathfrak{m}_{\boldsymbol{\beta}'}$ and $q(s) \in \mathbb{I}(\Omega(\boldsymbol{\beta}' - \boldsymbol{\beta})) \setminus \mathfrak{m}_{\boldsymbol{\beta}}$, and let $P := t^{\boldsymbol{\beta} - \boldsymbol{\beta}'} p(s)$ and $Q := t^{\boldsymbol{\beta}' - \boldsymbol{\beta}} q(s)$. Then clearly $PN_{K^n}(T_A, \boldsymbol{\beta}') \subseteq N_{K^n}(T_A, \boldsymbol{\beta})$ and $QN_{K^n}(T_A, \boldsymbol{\beta}) \subseteq N_{K^n}(T_A, \boldsymbol{\beta}')$.

Moreover, since $PQ = p(s + \beta' - \beta)q(s) \notin \mathfrak{m}_{\beta}$ and $QP = q(s + \beta - \beta')p(s) \notin \mathfrak{m}_{\beta'}$, $PQN_{K^n}(T_A, \beta) = N_{K^n}(T_A, \beta)$ and $QPN_{K^n}(T_A, \beta') = N_{K^n}(T_A, \beta')$. Hence the assertion follows.

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4. Weights of generating relations of $L(\beta)$

Let $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}} \in K^d$. In this section, we choose a finite set J of weights of $N := N_{K^n}(T_A, \boldsymbol{\beta})$ such that the weight spaces with weight in J generate N (Theorem 4.7). This enables us to compute a finite system of generators of N and, in turn, that of the irreducible module $L(\boldsymbol{\beta})$.

We recall the primitive integral support function of a facet (maximal proper face) of the cone $\mathbb{Q}_{\geq 0}A$. Let \mathcal{F} denote the set of facets of $\mathbb{Q}_{\geq 0}A$. Given a facet $\sigma \in \mathcal{F}$, we denote by F_{σ} the primitive integral support function of σ , i.e., F_{σ} is the uniquely determined linear form on \mathbb{Q}^d satisfying

(1) $F_{\sigma}(\mathbb{Q}_{\geq 0}A) \geq 0,$ (2) $F_{\sigma}(\sigma) = 0,$ (3) $F_{\sigma}(\mathbb{Z}^d) = \mathbb{Z}.$

Then we know, by [10, Proposition 2.2] and Remark 3.2,

(4)
$$\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}} \Leftrightarrow F_{\sigma}(\boldsymbol{\beta}) \notin F_{\sigma}(\mathbb{N}A) \text{ for all facets } \sigma \in \mathcal{F}.$$

Set

$$\mathcal{F}(\boldsymbol{\beta}) := \{ \sigma \in \mathcal{F} \, | \, F_{\sigma}(\boldsymbol{\beta}) \in \mathbb{Z} \}.$$

From now on, we fix $\boldsymbol{\beta} \in K^d$ satisfying $F_{\sigma}(\boldsymbol{\beta}) < 0$ for all $\sigma \in \mathcal{F}(\boldsymbol{\beta})$. Then $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}}$ by (4). Let $N := N(\boldsymbol{\beta}) := N_{K^n}(T_A, \boldsymbol{\beta})$. Then, for $\boldsymbol{a} \in \mathbb{Z}^d$, by the definition of N, (3), and Proposition 2.1,

$$N_{-\boldsymbol{\beta}-\boldsymbol{a}} = \frac{t^{-\boldsymbol{a}} \left(\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a})) \cap \langle A\boldsymbol{\theta} - \boldsymbol{\beta} - \boldsymbol{a} \rangle \right)}{t^{-\boldsymbol{a}} \left(\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a})) \langle A\boldsymbol{\theta} - \boldsymbol{\beta} - \boldsymbol{a} \rangle \right)}$$

Proposition 4.1 (Lemma 8.2 (1) in [12]). Let $\mathbf{a} \in \mathbb{Z}^d$. If $\boldsymbol{\beta} + \mathbf{a} \sim \boldsymbol{\beta}$, then $N_{-\boldsymbol{\beta}-\boldsymbol{a}} = \{0\}$.

Choose $\tilde{\boldsymbol{\beta}} \in \mathbb{Q}^d$ such that $\mathcal{F}(\tilde{\boldsymbol{\beta}}) = \mathcal{F}(\boldsymbol{\beta})$ and $F_{\sigma}(\tilde{\boldsymbol{\beta}}) = F_{\sigma}(\boldsymbol{\beta})$ for all $\sigma \in \mathcal{F}(\boldsymbol{\beta})$. (Such $\tilde{\boldsymbol{\beta}}$ exists by Cramer's rule.) Set

$$C(\tilde{\boldsymbol{\beta}}) := (-\tilde{\boldsymbol{\beta}} - \mathbb{Q}_{\geq 0}A) \cap \mathbb{Z}^d.$$

Proposition 4.2. The right $D(K^n)$ -module N is generated by $\bigoplus_{a \in \partial C(\tilde{\beta})} N_{-\beta-a}$, where we put $\partial C(\tilde{\beta}) = \bigcup_{\sigma \in \mathcal{F}(\beta)} \{ a \in C(\tilde{\beta}) \mid F_{\sigma}(\beta + a) = 0 \}.$

Proof. Let σ be a facet of the cone $\mathbb{Q}_{\geq 0}A$. Then

$$F_{\sigma}(\boldsymbol{\beta}+\boldsymbol{a})N_{-\boldsymbol{\beta}-\boldsymbol{a}} = N_{-\boldsymbol{\beta}-\boldsymbol{a}}(\sum_{j=1}^{n}F_{\sigma}(\boldsymbol{a}_{j})x_{j}\partial_{j}) \subseteq \sum_{\boldsymbol{a}_{j}\notin\sigma}N_{-\boldsymbol{\beta}-(\boldsymbol{a}-\boldsymbol{a}_{j})}\partial_{j}.$$

Hence N is generated by $\bigoplus_{\boldsymbol{a}\in C(\tilde{\boldsymbol{\beta}})} N_{-\boldsymbol{\beta}-\boldsymbol{a}}$. By (4) and Proposition 4.1, $N_{-\boldsymbol{\beta}-\boldsymbol{a}} = 0$ if $\boldsymbol{a}\in C(\tilde{\boldsymbol{\beta}})\setminus \partial C(\tilde{\boldsymbol{\beta}})$.

Since $\boldsymbol{a} \in C(\tilde{\boldsymbol{\beta}})$ implies that $F_{\sigma}(\boldsymbol{\beta} + \boldsymbol{a}) = F_{\sigma}(\tilde{\boldsymbol{\beta}} + \boldsymbol{a}) \leq 0$ for all $\sigma \in \mathcal{F}(\boldsymbol{\beta})$, and that $F_{\sigma'}(\tilde{\boldsymbol{\beta}} + \boldsymbol{a}) < 0$ for all $\sigma' \notin \mathcal{F}(\boldsymbol{\beta})$, we see that $\partial C(\tilde{\boldsymbol{\beta}})$ is decomposed according to the decomposition of $\mathbb{Q}_{>0}A$:

$$\partial C(\tilde{\boldsymbol{\beta}}) = \prod_{\tau} (-\tilde{\boldsymbol{\beta}} - \mathring{\tau}) \cap C(\tilde{\boldsymbol{\beta}}),$$

where τ runs over all proper faces of the cone $\mathbb{Q}_{\geq 0}A$ such that $\sigma \in \mathcal{F}(\beta)$ for all facets $\sigma \succeq \tau$, and $\mathring{\tau}$ denotes the relative interior of τ .

Notation 4.3. As in [15], let

$$\mathbb{N}A = \mathbb{Q}_{\geq 0}A \cap \mathbb{Z}^d \setminus \bigcup_i (\boldsymbol{b}_i + \mathbb{N}(A \cap \tau_i)),$$

and

(5)
$$M := \max_{\sigma,i} F_{\sigma}(\boldsymbol{b}_i) + 1$$

We agree M = 0 if $\mathbb{N}A = \mathbb{Q}_{\geq 0}A \cap \mathbb{Z}^d$ ($\mathbb{N}A$ is said to be normal (or saturated) in this case).

Lemma 4.4. Let τ be a face of $\mathbb{Q}_{\geq 0}A$. Assume $F_{\sigma}(\boldsymbol{a}) \leq -M$ for every facet $\sigma \not\succeq \tau$. Then the following hold.

The support of each minimal generator of the Nⁿ-set Nⁿ∩f_A⁻¹(**a**+ NA) is contained in τ^c := {j | **a**_j ∉ τ}, where f_A is the linear map from Zⁿ to Z^d defined by A.
 (2)

$$\mathbb{N}^n \setminus f_A^{-1}(\boldsymbol{a} + \mathbb{N}A) = \bigcup_{\boldsymbol{\sigma} \succeq \boldsymbol{\tau}} \{(\boldsymbol{u}, \boldsymbol{\sigma}) \mid F_{\boldsymbol{\sigma}}(A\boldsymbol{u}) < F_{\boldsymbol{\sigma}}(\boldsymbol{a}) \}$$
$$\cup \bigcup_{\tau_i \succeq \boldsymbol{\tau}} \{(\boldsymbol{u}, \tau_i) \mid A\boldsymbol{u} \in \boldsymbol{a} + \boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i) \},$$

where $(\boldsymbol{u}, \sigma) := \boldsymbol{u} + \mathbb{N}^{\sigma}$, and

$$\mathbb{N}^{\sigma} := \{ \boldsymbol{v} \in \mathbb{N}^n \, | \, v_j = 0 \text{ for all } \boldsymbol{a}_j \notin \sigma \}.$$

Proof. It is enough to prove (1) that, if $\boldsymbol{u} \in \mathbb{N}^n$ and $A\boldsymbol{u} \notin \boldsymbol{a} + \mathbb{N}A$, then $A(\boldsymbol{u} + \boldsymbol{v}) \notin \boldsymbol{a} + \mathbb{N}A$ for any $\boldsymbol{v} \in \mathbb{N}^{\tau}$.

Suppose that $Au \notin a + \mathbb{N}A$. Then one of the following two holds:

- (1) $F_{\sigma}(A\boldsymbol{u}) < F_{\sigma}(\boldsymbol{a})$ for some facet $\sigma \succeq \tau$.
- (2) $A\boldsymbol{u} \in \boldsymbol{a} + \boldsymbol{b}_i + \mathbb{N}(A \cap \tau_i)$ for some *i*.

In the first case, we clearly have $A(\boldsymbol{u} + \boldsymbol{v}) \notin \boldsymbol{a} + \mathbb{N}A$ for any $\boldsymbol{v} \in \mathbb{N}^{\tau}$.

Suppose that $A\mathbf{u} \in \mathbf{a} + \mathbf{b}_i + \mathbb{N}(A \cap \tau_i)$ for some *i*. Then we prove $\tau_i \succeq \tau$. For this, we prove $\sigma \succeq \tau$ for all facets $\sigma \succeq \tau_i$. Suppose that $\sigma \succeq \tau_i$ but $\sigma \succeq \tau$. Then $F_{\sigma}(A\mathbf{u}) = F_{\sigma}(\mathbf{a} + \mathbf{b}_i) \leq -M + F_{\sigma}(\mathbf{b}_i) < 0$, which is a contradiction. We have thus proved $\tau_i \succeq \tau$. Hence we have $A(\mathbf{u} + \mathbf{v}) \notin \mathbf{a} + \mathbb{N}A$ for any $\mathbf{v} \in \mathbb{N}^{\tau}$ in the second case, too. We also have proved the second statement of the lemma.

Lemma 4.5. Let $\boldsymbol{a} \in \mathbb{Z}^d$, and let τ be a face of $\mathbb{Q}_{\geq 0}A$. Let $\boldsymbol{\beta} + \boldsymbol{a}$ be in the relative interior of $-\tau$, and $\boldsymbol{b} \in \mathbb{N}(A \cap \tau)$. Assume that $F_{\sigma}(\boldsymbol{a} + \boldsymbol{b}) \leq -M$ for all facets $\sigma \not\geq \tau$. Then

- (1) $D(K^n, X_A)_{-\boldsymbol{a}-\boldsymbol{b}} x^{\boldsymbol{u}} = D(K^n, X_A)_{-\boldsymbol{a}}$ for any $\boldsymbol{u} \in \mathbb{N}^{\tau}$ with $A\boldsymbol{u} = \boldsymbol{b}$.
- (2) $N_{-\boldsymbol{\beta}-\boldsymbol{a}-\boldsymbol{b}}x^{\boldsymbol{u}} = N_{-\boldsymbol{\beta}-\boldsymbol{a}}$ for any $\boldsymbol{u} \in \mathbb{N}^{\tau}$ with $A\boldsymbol{u} = \boldsymbol{b}$.

Proof. By Theorem 2.2 and Lemma 4.4 (1), the minimal generators of $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a}))$ do not have variables θ_i for $\boldsymbol{a}_i \in \tau$. Moreover, by Lemma 4.4 (2), $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a})) = \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a}-\boldsymbol{b}))$. Hence the assertions follow. \Box

Notation 4.6. Let τ be a face of $\mathbb{Q}_{\geq 0}A$ such that all facets containing τ belong to $\mathcal{F}(\boldsymbol{\beta})$, and let $m_{\tau} := [\mathbb{Z}^{\overline{d}} \cap \mathbb{Q}\tau : \mathbb{Z}(A \cap \tau)].$

Choose $\beta_{\tau,j} \in \mathbb{Z}^d$ $(j = 1, ..., m_{\tau})$ so that $\tilde{\boldsymbol{\beta}} + \boldsymbol{\beta}_{\tau,j}$ form a set of representatives of $(\mathbb{Q}\tau \cap (\tilde{\boldsymbol{\beta}} + \mathbb{Z}^d))/\mathbb{Z}(A \cap \tau)$, and that $F_{\sigma}(\boldsymbol{\beta}_{\tau,j}) \leq -M$ for every facet σ with $\sigma \not\geq \tau$. When $m_{\tau} = 1$, we simply write $\boldsymbol{\beta}_{\tau}$ instead of $\boldsymbol{\beta}_{\tau,1}$.

Theorem 4.7. The right $D(K^n)$ -module N is generated by

$$\bigoplus_{\tau} \bigoplus_{j=1}^{m_{\tau}} N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\tau,j}}$$

where τ ranges over all faces of $\mathbb{Q}_{\geq 0}A$ such that all facets containing τ belong to $\mathcal{F}(\boldsymbol{\beta})$.

Proof. Let $\beta + a$ be in the relative interior of $-\tau$. By Proposition 4.2, it is enough to prove that $N_{-\beta-a}$ is generated by $\bigoplus_{\tau' \succeq \tau} \bigoplus_{j=1}^{m_{\tau'}} N_{-\beta-\beta_{\tau',j}}$. We prove this by induction on the codimension of τ .

There exists $\beta_{\tau,j}$ such that $\beta_{\tau,j} - a \in \mathbb{Z}(A \cap \tau)$. Take a' so that $\beta_{\tau,j} - a', a - a' \in \mathbb{N}(A \cap \tau)$.

We claim that

$$\boldsymbol{\beta} + \boldsymbol{a} \sim \boldsymbol{\beta} + \boldsymbol{a}'.$$

Since $\boldsymbol{a} - \boldsymbol{a}' \in \mathbb{Z}(A \cap \tau)$, we have, by definition, $E_{\tau'}(\boldsymbol{\beta} + \boldsymbol{a}) = E_{\tau'}(\boldsymbol{\beta} + \boldsymbol{a}')$ for $\tau' \succeq \tau$. For $\tau' \succeq \tau$, there exists a facet $\sigma \succeq \tau'$ with $\sigma \succeq \tau$. Since

$$+ \mathbf{a} \in -\overset{\circ}{\tau}$$
 and $\mathbf{a} - \mathbf{a}' \in \mathbb{N}(A \cap \tau)$, we have, for $\sigma \in \mathcal{F}(\boldsymbol{\beta})$,
 $F_{\sigma}(\boldsymbol{\beta} + \mathbf{a}) = F_{\sigma}(\tilde{\boldsymbol{\beta}} + \mathbf{a}) < 0$,
 $F_{\sigma}(\boldsymbol{\beta} + \mathbf{a}') = F_{\sigma}(\tilde{\boldsymbol{\beta}} + \mathbf{a}) - F_{\sigma}(\mathbf{a} - \mathbf{a}') < 0$.

For $\sigma \notin \mathcal{F}(\boldsymbol{\beta})$, of course $F_{\sigma}(\boldsymbol{\beta}+\boldsymbol{a}), F_{\sigma}(\boldsymbol{\beta}+\boldsymbol{a}') \notin \mathbb{N}$. Hence $E_{\sigma}(\boldsymbol{\beta}+\boldsymbol{a}) =$ $\emptyset = E_{\sigma}(\boldsymbol{\beta}+\boldsymbol{a}')$ by [10, Proposition 2.2 (3)]. Then $E_{\tau'}(\boldsymbol{\beta}+\boldsymbol{a}) = \emptyset =$ $E_{\tau'}(\boldsymbol{\beta}+\boldsymbol{a}')$ by [10, Proposition 2.2 (4)]. We have thus proved the claim (6).

By [14, Lemma 4.1.4] and Theorem 2.2, there exists a *b*-function $b_{\boldsymbol{a}-\boldsymbol{a}'}(\theta) := b_{\boldsymbol{a}-\boldsymbol{a}'}(A\theta) = \sum_{\boldsymbol{u}} a_{\boldsymbol{u}} x^{\boldsymbol{u}} \partial^{\boldsymbol{u}}$ with $b_{\boldsymbol{a}-\boldsymbol{a}'}(\boldsymbol{\beta}+\boldsymbol{a}) \neq 0$ and $A\boldsymbol{u} \in \boldsymbol{a}-\boldsymbol{a}' + \mathbb{N}A$ for $a_{\boldsymbol{u}} \neq 0$. Then

$$N_{-\beta-a} = N_{-\beta-a}b_{a-a'}(\beta+a) = N_{-\beta-a}b_{a-a'}(\theta) \subseteq \sum_{u} N_{-\beta-a+Au}\partial^{u}.$$

Here $\mathbf{a} - A\mathbf{u} \in \mathbf{a}' - \mathbb{N}A$. If $\mathbf{a} - A\mathbf{u} \notin \mathbf{a}' - \mathbb{N}(A \cap \tau)$, then $\dot{\boldsymbol{\beta}} + \mathbf{a} - A\mathbf{u}$ is in the relative interior of a larger face, and hence the induction hypothesis would do. If $\mathbf{a} - A\mathbf{u} \in \mathbf{a}' - \mathbb{N}(A \cap \tau)$, then $\boldsymbol{\beta}_{\tau,j} - (\mathbf{a} - A\mathbf{u}) \in \mathbb{N}(A \cap \tau)$, and by Lemma 4.5 $N_{-\boldsymbol{\beta}-\mathbf{a}+A\mathbf{u}}$ is generated by $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\tau,j}}$.

We may rephrase Theorem 4.7 as follows:

Corollary 4.8. The irreducible module $L(\beta)$ is described as

$$D(K^{n}) / \left(\begin{array}{c} I_{A}D(K^{n}) + \langle A\theta - \beta \rangle D(K^{n}) \\ + \bigoplus_{\tau} \bigoplus_{j=1}^{m_{\tau}} t^{-\beta_{\tau,j}} (\mathbb{I}(\tilde{\Omega}(-\beta_{\tau,j})) \cap \langle A\theta - \beta - \beta_{\tau,j} \rangle) D(K^{n}) \end{array} \right)$$

Example 4.9. Let $A = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then $\mathbb{Q}_{\geq 0}A = \mathbb{Q}_{\geq 0}^2$ has two facets: $\sigma_{12} := \mathbb{Q}_{\geq 0} a_1 = \mathbb{Q}_{\geq 0} a_2$ and $\sigma_3 := \mathbb{Q}_{\geq 0} a_3$; $F_{\sigma_{12}}(s) = s_2$ and $F_{\sigma_3}(s) = s_1$. We have $\mathbb{N}A = \mathbb{N}^2 \setminus \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, M = 2, and $m_{\sigma_{12}} = m_{\sigma_3} = m_{\{0\}} = 1$. Let $\boldsymbol{\beta} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Note that $s_1 = 2\theta_1 + 3\theta_2 + \theta_4$, $s_2 = \theta_3 + \theta_4$. Let $\boldsymbol{\beta}_{\sigma_{12}} := -\boldsymbol{a}_2 - \boldsymbol{\beta} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Then $\boldsymbol{\beta} + \boldsymbol{\beta}_{\sigma_{12}} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$, and $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma_{12}})) = \langle \theta_3, \theta_4 \rangle$. $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma_{12}})) \cap \langle s_1 + 3, s_2 \rangle \equiv \langle \theta_3 + \theta_4 \rangle$,

where \equiv denotes the equality modulo $\mathbb{I}(\widetilde{\Omega}(-\beta_{\sigma_{12}}))\langle s_1+3, s_2\rangle$. Since $-\beta_{\sigma_{12}} = a_1 - a_3 = a_2 - a_4$,

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(7)
$$t^{-\boldsymbol{\beta}_{\sigma_{12}}}(\theta_{3}+\theta_{4}) = x_{1}\partial_{3}+x_{2}\partial_{4}.$$
Let $\boldsymbol{\beta}_{\sigma_{3}} := -2\boldsymbol{a}_{3}-\boldsymbol{\beta} = \begin{pmatrix} 1\\ -1 \end{pmatrix}$. Then $\boldsymbol{\beta}+\boldsymbol{\beta}_{\sigma_{3}} = \begin{pmatrix} 0\\ -2 \end{pmatrix}$.
$$\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma_{3}})) = \langle \theta_{1},\theta_{2},\theta_{4} \rangle.$$

$$\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma_{3}})) \cap \langle s_{1},s_{2}+2 \rangle \equiv \langle 2\theta_{1}+3\theta_{2}+\theta_{4} \rangle.$$
Since $-\boldsymbol{\beta}_{\sigma_{3}} = \boldsymbol{a}_{4}-\boldsymbol{a}_{1} = \boldsymbol{a}_{1}+\boldsymbol{a}_{3}-\boldsymbol{a}_{2} = 2\boldsymbol{a}_{3}-\boldsymbol{a}_{4},$
(2) $t^{-\boldsymbol{\beta}_{\sigma_{3}}}(2\theta_{1}+2\theta_{2}+\theta_{3}) = 2\boldsymbol{a}_{3}+2\boldsymbol{a}_{3}+2\boldsymbol{a}_{3}+2\boldsymbol{a}_{3}$

(8) $t^{-\beta_{\sigma_3}}(2\theta_1 + 3\theta_2 + \theta_4) = 2x_4\partial_1 + 3x_1x_3\partial_2 + x_3^2\partial_4.$ Let $\beta_{\{0\}} := -\beta$. Then

$$\begin{split} \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\{\mathbf{0}\}})) &= \langle \theta_4, \theta_2 \theta_3, [\theta_1]_2 \theta_3, \theta_1[\theta_3]_2 \rangle. \\ \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\{\mathbf{0}\}})) \cap \langle s_1, s_2 \rangle &\equiv \left\langle \begin{array}{c} \theta_2(\theta_3 + \theta_4), \\ [\theta_1]_2(\theta_3 + \theta_4), \theta_1(\theta_3 - 1)(\theta_3 + \theta_4), \\ [\theta_3]_2(2\theta_1 + 3\theta_2 + \theta_4), \\ (\theta_1 - 1)\theta_3(2\theta_1 + 3\theta_2 + \theta_4), \\ \theta_3(2\theta_1 + 3\theta_2 + \theta_4) - 2\theta_1(\theta_3 + \theta_4) \\ &= 3\theta_2\theta_3 + (\theta_3 - 2\theta_1)\theta_4 \end{array} \right\rangle. \end{split}$$

Since $-\beta_{\{0\}} = -a_4 = a_1 - a_2 - a_3 = a_2 - 2a_1 - a_3 = a_4 - a_1 - 2a_3,$ $t^{-\beta_{\{0\}}}\theta_2(\theta_3 + \theta_4) = x_1\partial_2\partial_2 + x_2\partial_2\partial_4.$

$$t^{-\beta_{\{0\}}}[\theta_1]_2(\theta_3 + \theta_4) = x_1 \theta_2 \theta_3 + x_2 \theta_2 \theta_4,$$

$$t^{-\beta_{\{0\}}}[\theta_1]_2(\theta_3 + \theta_4) = x_2 \partial_1^2 \partial_3 + x_1^2 \partial_1^2 \partial_4,$$

$$t^{-\beta_{\{0\}}} \theta_1(\theta_3 - 1)(\theta_3 + \theta_4) = x_4 \partial_1 \partial_3^2 + \theta_1(\theta_3 - 1) \partial_4,$$

$$t^{-\beta_{\{0\}}}[\theta_3]_2(2\theta_1 + 3\theta_2 + \theta_4)$$

$$= 2x_1 \partial_1 \partial_2^2 + 3x_1 (\theta_1 - 1) \partial_1 \partial_2 + [\theta_1] \partial_1$$

$$= 2x_4\partial_1\partial_3^2 + 3x_1(\theta_3 - 1)\partial_2\partial_3 + [\theta_3]_2\partial_4,$$

$$t^{-\beta_{\{0\}}}(\theta_1 - 1)\theta_3(2\theta_1 + 3\theta_2 + \theta_4)$$

$$= 2x_2\partial_1^2\partial_3 + 3x_1(\theta_1 - 1)\partial_2\partial_3 + (\theta_1 - 1)\theta_3\partial_4,$$

$$t^{-\beta_{\{0\}}}(3\theta_2\theta_3 + (\theta_3 - 2\theta_1)\theta_4) = 3x_1\partial_2\partial_3 + (\theta_3 - 2\theta_1)\partial_4.$$

Hence N is generated by the operators (7), (8), and (9) by Theorem 4.7.

5. Scored case for a facet

Recall that a semigroup $\mathbb{N}A$ is said to be scored if

(9)

$$\mathbb{N}A = \bigcap_{\sigma \in \mathcal{F}} \{ \boldsymbol{a} \in \mathbb{Z}A : F_{\sigma}(\boldsymbol{a}) \in F_{\sigma}(\mathbb{N}A) \}.$$

(See [14].) Clearly a normal semigroup is scored. Note that, if $\mathbb{N}A$ is scored, then $m_{\tau} = 1$ for all faces τ [11, Lemma 7.11]. In this section, we assume that $\mathbb{N}A$ is scored, and we give an explicit generator of K[s]-module $N_{-\beta-\beta_{\tau}}$ for a facet σ (Theorem 5.3).

Remark 5.1. In the scored case, we can refine some previous statements without changing proofs.

In Lemma 4.4, the condition $F_{\sigma}(\boldsymbol{a}) \leq -M$ can be replaced by the condition $-F_{\sigma}(\boldsymbol{a}) \in F_{\sigma}(\mathbb{N}A)$. In Lemma 4.5, the condition $F_{\sigma}(\boldsymbol{a} + \boldsymbol{b}) \leq -M$ can be replaced by the condition $-F_{\sigma}(\boldsymbol{a} + \boldsymbol{b}) \in F_{\sigma}(\mathbb{N}A)$. In Notation 4.6, we take $\boldsymbol{\beta}_{\tau}$ so that $-F_{\sigma}(\boldsymbol{\beta}_{\tau}) \in F_{\sigma}(\mathbb{N}A)$ instead of $F_{\sigma}(\boldsymbol{\beta}_{\tau}) \leq -M$; Theorem 4.7 is valid for this choice of $\boldsymbol{\beta}_{\tau}$.

Lemma 5.2. Assume that $\mathbb{N}A$ is scored. Let $\sigma \in \mathcal{F}(\beta)$. Then

(1)
$$S(\mathbb{I}(-\boldsymbol{\beta}_{\sigma})) = \{(\boldsymbol{u},\sigma) \mid \boldsymbol{u} \in \mathbb{N}^{\sigma^{c}}, F_{\sigma}(A\boldsymbol{u}) \notin -F_{\sigma}(\boldsymbol{\beta}) + F_{\sigma}(\mathbb{N}A)\}.$$

(2) $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma})) = \bigcap_{(\boldsymbol{u},\sigma)\in S(\mathbb{I}(-\boldsymbol{\beta}_{\sigma}))} \langle \theta_{i} - u_{i} \mid i \notin \sigma \rangle.$

$$N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma}} \equiv t^{-\boldsymbol{\beta}_{\sigma}} (\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma})) : F_{\sigma}(A\theta)) \cdot F_{\sigma}(A\theta)$$
$$\equiv t^{-\boldsymbol{\beta}_{\sigma}} \bigcap_{(\boldsymbol{u},\sigma) \in S(\mathbb{I}(-\boldsymbol{\beta}_{\sigma})), \, \boldsymbol{u} \neq \boldsymbol{0}} \langle \theta_{i} - u_{i} \, | \, i \notin \sigma \rangle \cdot F_{\sigma}(A\theta).$$

Here $\mathbb{N}^{\sigma^c} = \{ \boldsymbol{u} \in \mathbb{N}^n | u_j = 0 \text{ for all } \boldsymbol{a}_j \in \sigma \}$. We sometimes write $j \in \sigma$ instead of $\boldsymbol{a}_j \in \sigma$.

Proof. Recall from Notation 4.6 and Remark 5.1 that we have chosen β_{σ} so that $-F_{\sigma'}(\beta_{\sigma}) \in F_{\sigma'}(\mathbb{N}A)$ for all $\sigma' \neq \sigma$.

(1) follows from Lemma 4.4 (2), and (2) follows from (1) and Theorem 2.2.

To prove (3), by renumbering if necessary, we assume that a_1, \ldots, a_d are linearly independent with $a_1 \notin \sigma$ and $a_2, \ldots, a_d \in \sigma$. Let $F_i(s)$ be a linear form such that, for $j \leq d$, $F_i(a_j) \neq 0$ if and only if i = j. Take $F_1 = F_{\sigma}$. Then

$$\langle A\theta - (\boldsymbol{\beta} + \boldsymbol{\beta}_{\sigma}) \rangle = \langle F_i(A\theta) - F_i(\boldsymbol{\beta} + \boldsymbol{\beta}_{\sigma}) | i = 1, \dots, d \rangle.$$

Note that $F_{\sigma}(A\theta) = F_1(A\theta) \in \langle A\theta - (\beta + \beta_{\sigma}) \rangle$ since $F_{\sigma}(\beta + \beta_{\sigma}) = 0$ by definition. Hence \supseteq of the first equality is clear, and \supseteq of the second equality follows from (2).

Suppose that

(10)
$$\sum_{i=1}^{d} f_i(F_i(\theta) - F_i(\boldsymbol{\beta} + \boldsymbol{\beta}_{\sigma})) \in \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma})).$$

Here and hereafter, we sometimes write $F(\theta)$ instead of $F(A\theta)$. Since $F_i(\theta)$ contains θ_i but not $\theta_j (j \neq i, j \leq d)$, we may assume that $f_i \in K[\theta_j \mid j \leq i \text{ or } j > d]$.

Let (\mathbf{u}, σ) satisfy $\mathbf{u} \in \mathbb{N}^{\sigma^c}$ and $F_{\sigma}(\mathbf{u}) \notin -F_{\sigma}(\boldsymbol{\beta}) + F_{\sigma}(\mathbb{N}A)$. Then

(11)
$$\sum_{i=1}^{d} f_i(\boldsymbol{u}, \theta_{\sigma}) (F_i(\boldsymbol{u}, \theta_{\sigma}) - F_i(\boldsymbol{\beta} + \boldsymbol{\beta}_{\sigma})) = 0,$$

where $F(\boldsymbol{u}, \theta_{\sigma})$ denotes the function obtained from F by replacing θ_j by u_j for $j \notin \sigma$. By looking at the variables θ_i (i = d, ..., 2), we see $f_i(\boldsymbol{u}, \theta_{\sigma}) = 0$ (i = d, ..., 2). Hence $f_i \in \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma}))$ for i = d, ..., 2. In turn, $f_1(\boldsymbol{u}, \theta_{\sigma})F_1(\boldsymbol{u}, \theta_{\sigma}) = 0$. (Note that $F_1(\boldsymbol{\beta} + \boldsymbol{\beta}_{\sigma}) = 0$.) Since $F_1(\boldsymbol{u}, \theta_{\sigma}) = F_1(\boldsymbol{u}) = F_{\sigma}(A\boldsymbol{u})$, we have

$$f_{1} \in \bigcap_{\substack{\boldsymbol{u} \in \mathbb{N}^{\sigma^{c}}; \, \boldsymbol{u} \neq 0\\ F_{\sigma}(A\boldsymbol{u}) \notin -F_{\sigma}(\boldsymbol{\beta}) + F_{\sigma}(\mathbb{N}A)}} \langle \theta_{i} - u_{i} \, | \, i \notin \sigma \rangle$$

since, for $\boldsymbol{u} \in \mathbb{N}^{\sigma^c}$, $F_{\sigma}(A\boldsymbol{u}) \neq 0$ if and only if $\boldsymbol{u} \neq \boldsymbol{0}$.

 \square

Theorem 5.3. Assume that $\mathbb{N}A$ is scored. Let $\sigma \in \mathcal{F}(\beta)$. For $j \notin \sigma$, put

$$m_j := m_{\sigma,j} = \max\{u_j \in \mathbb{N} | F_{\sigma}(\boldsymbol{a}_j)u_j \notin -F_{\sigma}(\boldsymbol{\beta}) + F_{\sigma}(\mathbb{N}A)\}.$$

Then

$$N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma}} \equiv t^{-\boldsymbol{\beta}_{\sigma}} \langle \prod_{j \notin \sigma} \prod_{k=1}^{m_{j}} (\theta_{j} - k) \rangle \cdot F_{\sigma}(A\theta)$$

Proof. By Lemma 5.2, it is enough to show

(12)
$$\bigcap_{\substack{\boldsymbol{u} \in \mathbb{N}^{\sigma^{c}}, F_{\sigma}(A\boldsymbol{u}) \neq 0\\F_{\sigma}(A\boldsymbol{u}) \notin -F_{\sigma}(\boldsymbol{\beta}) + F_{\sigma}(\mathbb{N}A)}} \langle \theta_{j} - u_{j} | j \notin \sigma \rangle = \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma})) + \langle \prod_{j \notin \sigma} \prod_{k=1}^{m_{j}} (\theta_{j} - k) \rangle.$$

We know $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma})) = \bigcap_{\boldsymbol{u} \in \mathbb{N}^{\sigma^{c}}, F_{\sigma}(A\boldsymbol{u}) \notin -F_{\sigma}(\boldsymbol{\beta}) + F_{\sigma}(\mathbb{N}A)} \langle \theta_{j} - u_{j} \mid j \notin \sigma \rangle$ by Lemma 5.2.

Since each ideal in (12) is generated by elements in $K[\theta_j | j \notin \sigma]$, we only need to check (12) in $K[\theta_j | j \notin \sigma]$. Then the zero set of each side of (12) equals the finite set $\{ \boldsymbol{u} \neq \boldsymbol{0} | (\boldsymbol{u}, \sigma) \in S(\mathbb{I}(-\boldsymbol{\beta}_{\sigma})) \}$. By localizing at each zero point, we see that both sides are equal.

Corollary 5.4. Assume that $\mathbb{N}A$ is normal. Let $\sigma \in \mathcal{F}(\beta)$.

Then

$$N_{-\beta-\beta_{\sigma}} \equiv t^{-\beta_{\sigma}} \langle \prod_{j \notin \sigma} \prod_{k=1}^{\lfloor \frac{-F_{\sigma}(\beta)-1}{F_{\sigma}(a_j)} \rfloor} (\theta_j - k) \rangle \cdot F_{\sigma}(A\theta)$$

Proof. In this case, the m_j in Theorem 5.3 equals $\lfloor \frac{-F_{\sigma}(\boldsymbol{\beta}) - 1}{F_{\sigma}(\boldsymbol{a}_j)} \rfloor$. \Box

6. Scored case for a simple face

In this section, we keep to assume that $\mathbb{N}A$ is scored. Furthermore we assume that a fixed face τ of $\mathbb{Q}_{\geq 0}A$ of codimension c satisfies

(13)
$$\{\sigma \in \mathcal{F} \mid \sigma \succeq \tau\} = \{\sigma \in \mathcal{F}(\beta) \mid \sigma \succeq \tau\} = \{\sigma_1, \sigma_2, \dots, \sigma_c\}.$$

Under these assumptions, we show that $N_{-\beta-\beta_{\tau}}$ is generated by $N_{-\beta-\beta_{\sigma_i}}$ $(1 \leq i \leq c)$ (Theorem 6.3).

Change the order if necessary, and take $a_1, \dots, a_d \in A$ so that

$$\boldsymbol{a}_i \in \bigcap_{k=1, k \neq i} \sigma_k \setminus \sigma_i \qquad (i \le c)$$

 $\boldsymbol{a}_{c+1},\ldots,\boldsymbol{a}_d\in\tau$ is linearly independent.

Put $F_i := F_{\sigma_i}$ for $i \leq c$, and take $F_{c+1}, \ldots, F_d \in \langle A\theta \rangle$ so that for $i, j \leq d$

$$F_i(\boldsymbol{a}_j) \neq 0 \Leftrightarrow i \neq j$$

We prove that $N_{-\beta-\beta_{\tau}}$ is generated by $N_{-\beta-\beta_{\sigma_i}}$ $(1 \le i \le c)$. For simplicity, put

$$\begin{split} \mathbf{i} &:= & \mathbb{I}(\Omega(-\boldsymbol{\beta}_{\tau})), \\ \mathbf{a} &:= & \langle A\boldsymbol{\theta} - \boldsymbol{\beta} - \boldsymbol{\beta}_{\tau} \rangle, \\ \mathbf{f} &:= & \sum_{i=1}^{c} \mathbf{i}_{\sigma_{i}} \langle \prod_{j \notin \sigma_{i}} \prod_{k=1}^{m_{\sigma_{i},j}} (\theta_{j} - k) \cdot F_{i} \rangle \end{split}$$

Here, for a facet $\sigma \succeq \tau$, we put

 $\mathfrak{i}_{\sigma} := \langle [\theta]_{\boldsymbol{u}} \mid \boldsymbol{u} \in \mathbb{N}^{\sigma}, F_{\sigma'}(A\boldsymbol{u}) \in F_{\sigma'}(\boldsymbol{\beta}_{\tau}) + F_{\sigma'}(\mathbb{N}A) \ (\forall \sigma' \neq \sigma) \rangle.$ We have, by Theorem 2.2 and Lemma 4.4 (2),

$$\begin{aligned}
\mathbf{i} &= \langle [\theta]_{\boldsymbol{u}} \mid A \boldsymbol{u} \in \boldsymbol{\beta}_{\tau} + \mathbb{N}A \rangle \\
&= \langle [\theta]_{\boldsymbol{u}} \mid F_i(A \boldsymbol{u}) \in -F_i(\boldsymbol{\beta}) + F_i(\mathbb{N}A) \quad (1 \le i \le c) \rangle
\end{aligned}$$

and

(14)
$$\mathbf{i}_{\sigma_i} = \left\langle \begin{bmatrix} \boldsymbol{\theta} \end{bmatrix}_{\boldsymbol{u}} \mid \begin{array}{c} \boldsymbol{u} \in \mathbb{N}^{\sigma_i}, \\ F_k(A\boldsymbol{u}) \in F_k(\boldsymbol{\beta}_{\tau}) + F_k(\mathbb{N}A) \ (\forall k \neq i, \ k \leq c) \end{array} \right\rangle,$$

since $-F_{\sigma}(\boldsymbol{\beta}_{\tau}) \in F_{\sigma}(\mathbb{N}A)$ for any $\sigma \succeq \tau$. Lemma 6.1. $t^{-\boldsymbol{\beta}_{\tau}} \mathfrak{f} \subseteq \sum_{i=1}^{c} N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma_{i}}} D(K^{n})$.

Proof. Let $\sigma = \sigma_i$ (i = 1, ..., c), and let $\boldsymbol{u} \in \mathbb{N}^{\sigma}$ satisfy $F_{\sigma'}(A\boldsymbol{u}) \in F_{\sigma'}(\boldsymbol{\beta}_{\tau}) + F_{\sigma'}(\mathbb{N}A)$ for all $\sigma' \neq \sigma$. Let $\boldsymbol{a} := \boldsymbol{\beta}_{\tau} - A\boldsymbol{u}$. Since \boldsymbol{a} satisfies the condition for ' $\boldsymbol{\beta}_{\sigma}$ ' (Notation 4.6 and Remark 5.1),

$$t^{-\boldsymbol{a}} \langle \prod_{j \notin \sigma} \prod_{k=1}^{m_{\sigma,j}} (\theta_j - k) \cdot F_{\sigma} \rangle = N_{-\boldsymbol{\beta}-\boldsymbol{a}}$$

by Theorem 5.3. Then by the proof of Theorem 4.7

$$N_{-\boldsymbol{\beta}-\boldsymbol{a}} \subseteq N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma}}D(K^n).$$

Hence we have

(15)
$$t^{-a} \langle \prod_{j \notin \sigma} \prod_{k=1}^{m_{\sigma,j}} (\theta_j - k) \cdot F_{\sigma} \rangle \subseteq N_{-\beta - \beta_{\sigma}} D(K^n).$$

Multiplying (15) by $\partial^{\boldsymbol{u}}$, we have

$$t^{-\boldsymbol{\beta}_{\tau}}[\boldsymbol{\theta}]_{\boldsymbol{u}} \prod_{j \notin \sigma} \prod_{k=1}^{m_{\sigma,j}} (\boldsymbol{\theta}_j - k) \cdot F_{\sigma} \in N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma}} D(K^n).$$

By Lemma 6.1, we only need to prove $\mathfrak{i} \cap \mathfrak{a} = \mathfrak{f} + \mathfrak{i} \cdot \mathfrak{a}$. To this aim, we prove that

(16)
$$(\mathfrak{i} \cap \mathfrak{a})_{\mathfrak{m}} = (\mathfrak{f} + \mathfrak{i} \cdot \mathfrak{a})_{\mathfrak{m}}$$

for all maximal ideals \mathfrak{m} of $R := K[\theta_1, \ldots, \theta_n]$. In this argument, we extend the field K into its algebraic closure \overline{K} . We simply write K instead of \overline{K} . For $\boldsymbol{v} \in K^d$, let $\mathfrak{i}_{\boldsymbol{v}}$ be the localization of \mathfrak{i} at the maximal ideal corresponding to \boldsymbol{v} . We have

$$\mathbb{V}(\mathbf{i}) := \{ \mathbf{v} \in K^d \, | \, \mathbf{i}_{\mathbf{v}} \neq R_{\mathbf{v}} \} \\ = \bigcup_{i=1}^c \bigcup_{F_i(A\mathbf{u}) \notin -F_i(\beta) + F_i(\mathbb{N}A)} \mathbf{u} + K^{\sigma_i}.$$

Proposition 6.2. $i \cap a = f + ia$.

Proof. If $\boldsymbol{v} \notin \mathbb{V}(\mathfrak{i}) \cap \mathbb{V}(\mathfrak{a})$, then $(\mathfrak{i} \cap \mathfrak{a})_{\boldsymbol{v}} = (\mathfrak{i}\mathfrak{a})_{\boldsymbol{v}}$.

Let $\boldsymbol{v} \in \mathbb{V}(\mathfrak{i}) \cap \mathbb{V}(\mathfrak{a})$, and let $\theta'_j := \theta_j - v_j$. By the definitions and Theorem 2.2,

$$\mathfrak{i}_{\boldsymbol{v}} = \langle \prod_{v_j \in \mathbb{N}, v_j < u_j} \theta'_j \, | \, [\theta]_{\boldsymbol{u}}: \text{ minimal in } \mathfrak{i} \rangle,$$

and

$$\mathfrak{f}_{\boldsymbol{v}} = \sum_{i=1}^{c} \langle \prod_{j \notin \sigma_{i}, v_{j} \in \mathbb{N}, 1 \leq v_{j} \leq m_{\sigma_{i}, j}} \theta_{j}' \cdot F_{i} \cdot \prod_{v_{j} \in \mathbb{N}, v_{j} < u_{j}'} \theta_{j}' \, | \, [\theta]_{\boldsymbol{u}'}: \text{ minimal in } \mathfrak{i}_{\sigma_{i}} \rangle.$$

Note that \mathfrak{i}_{v} is a monomial ideal in the variables $\theta'_{1}, \ldots, \theta'_{n}$. Let

(17)
$$\sum_{i=1}^{d} f_i F_i(\theta') \in \mathfrak{i}_{\boldsymbol{v}}$$

Note that, among $\theta_1, \ldots, \theta_d$, the variable θ_i is the unique one appearing in F_i . Hence we may assume that

$$f_1 \in K[\theta'_1, \theta'_{d+1}, \dots, \theta'_n], f_2 \in K[\theta'_1, \theta'_2, \theta'_{d+1}, \dots, \theta'_n], \dots, f_d \in K[\theta'_1, \dots, \theta'_n].$$

By looking at the variable θ'_d in (17), we see

$$f_d \theta'_d \in \mathfrak{i}_{\boldsymbol{v}}.$$

If $[\theta]_{\boldsymbol{u}}$ is minimal in \mathfrak{i} , then $u_{c+1} = \cdots = u_d = 0$. Hence, if c < d, then we have $f_d \in \mathfrak{i}_{\boldsymbol{v}}, f_d F_d(\theta') \in (\mathfrak{i}\mathfrak{a})_{\boldsymbol{v}}$, and

(18)
$$\sum_{i=1}^{d-1} f_i F_i(\theta') \in \mathfrak{i}_{\boldsymbol{v}}$$

Similarly we have $f_i F_i(\theta') \in (\mathfrak{ia})_{\boldsymbol{v}}$ for $c+1 \leq i \leq d$, and

(19)
$$\sum_{i=1}^{c} f_i F_i(\theta') \in \mathfrak{i}_{\boldsymbol{v}}.$$

By looking at the variable θ'_c in (19), we see $f_c \theta'_c \in \mathfrak{i}_v$ and

$$f_c \in \langle \prod_{v_j \in \mathbb{N}, v_j < u_j} \theta'_j | F_i(A\boldsymbol{u}) \in F_i(\boldsymbol{\beta}_\tau) + F_i(\mathbb{N}A) \, (\forall i < c) \rangle,$$

since $F_i(\boldsymbol{a}_c) = 0$ for all i < c.

Let \boldsymbol{u} satisfy $F_i(A\boldsymbol{u}) \in F_i(\boldsymbol{\beta}_{\tau}) + F_i(\mathbb{N}A)$ for all i < c, and let $h := \prod_{i=1}^{n} \theta'_i (= [\theta]_{ii})$ up to multiplication by a unit in R_{ii})

$$h := \prod_{v_j \in \mathbb{N}, v_j < u_j} \theta'_j (= [\theta]_u \text{ up to multiplication by a unit in } R_v).$$

In what follows, we omit to write 'up to multiplication by a unit in $R_{\boldsymbol{v}}$ '. Let $j \notin \sigma_c$. If $u_j > m_{\sigma_c,j}$, then $F_c(A\boldsymbol{u}) \in F_c(\boldsymbol{\beta}_{\tau}) + F_c(\mathbb{N}A)$, and $h = [\boldsymbol{\theta}]_{\boldsymbol{u}} \in \mathbf{i}_{\boldsymbol{v}}$. Suppose that $u_j \leq m_{\sigma_c,j}$. Then $F_c(A\boldsymbol{u} + (m_{\sigma_c,j} + 1 - u_j)\boldsymbol{a}_j) \in F_c(\boldsymbol{\beta}_{\tau}) + F_c(\mathbb{N}A)$. Note that for $k \geq 0$

$$h = [\theta]_{\boldsymbol{u} + k \boldsymbol{1}_i}$$

unless $u_j \leq v_j < u_j + k$.

Hence, if there exists $j \notin \sigma_c$ such that the condition $u_j \leq v_j \leq m_{\sigma_c,j}$ does not hold, then

$$h = [\theta]_{\boldsymbol{u} + (m_{\sigma_c, j} + 1 - u_j) \mathbf{1}_j} \in \mathfrak{i}_{\boldsymbol{v}}.$$

Next suppose that $u_j \leq v_j \leq m_{\sigma_c,j}$ for all $j \notin \sigma_c$. Since $\boldsymbol{v} \in \mathbb{V}(\mathfrak{a})$, we have $F_c(A\boldsymbol{v}) = 0$. Hence $v_j = 0$ for all $j \notin \sigma_c$, and in turn $u_j = 0$ for all $j \notin \sigma_c$, or $\boldsymbol{u} \in \mathbb{N}^{\sigma_c}$. By (14), $h = [\theta]_{\boldsymbol{u}} \in (\mathfrak{i}_{\sigma_c})_{\boldsymbol{v}}$. Therefore

 $hF_c \in \mathfrak{f}_v$

by noting that $v_j = 0$ for all $j \notin \sigma_c$. In all cases, we have thus proved

 $hF_c \in (\mathfrak{f} + \mathfrak{ia})_{\boldsymbol{v}}.$

Hence we have $f_c F_c \in (\mathfrak{f} + \mathfrak{i} \cdot \mathfrak{a})_{\boldsymbol{v}} \subseteq \mathfrak{i}_{\boldsymbol{v}}$, and $\sum_{i=1}^{c-1} f_i F_i \in \mathfrak{i}_{\boldsymbol{v}}$. Similarly, we obtain

$$f_i F_i \in (\mathfrak{f} + \mathfrak{i} \cdot \mathfrak{a})_{\boldsymbol{v}}$$

for i = c - 1, ..., 1. Hence $(\mathfrak{i} \cap \mathfrak{a})_{\boldsymbol{v}} \subseteq (\mathfrak{f} + \mathfrak{i}\mathfrak{a})_{\boldsymbol{v}}$. The other inclusion is clear.

Theorem 6.3. Assume that $\mathbb{N}A$ is scored, and that a face τ of $\mathbb{Q}_{\geq 0}A$ of codimension c satisfies (13). Then $N_{-\beta-\beta_{\tau}}$ is generated by $N_{-\beta-\beta_{\sigma_i}}$ $(1 \leq i \leq c)$.

Proof. This is immediate from Lemma 6.1 and Proposition 6.2. \Box

Example 6.4. Let
$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4)$$
, and
 $\boldsymbol{\beta} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = -\boldsymbol{a}_1 - \boldsymbol{a}_2 = -\boldsymbol{a}_3 - \boldsymbol{a}_4.$

This example is normal but non-simplicial.

$$F_{13} := F_{\sigma_{13}} = s_2 = \theta_2 + \theta_4,$$

$$F_{23} := F_{\sigma_{23}} = s_1 = \theta_1 + \theta_4,$$

$$F_{14} := F_{\sigma_{14}} = s_2 + s_3 = \theta_2 + \theta_3,$$

$$F_{24} := F_{\sigma_{24}} = s_1 + s_3 = \theta_1 + \theta_3.$$

Let $\beta + \beta_{14} := -a_1 - a_4$. Then $\beta_{14} = {}^t(-1, 0, 1) = a_3 - a_1 = a_2 - a_4$. By Corollary 5.4, $N_{-\beta - \beta_{14}}$ is generated by

$$t^{-\beta_{14}}F_{14} = t^{-\beta_{14}}(\theta_2 + \theta_3) = x_4\partial_2 + x_1\partial_3.$$

Each 1-dimensional face satisfies the condition (13).

We have $\boldsymbol{\beta}_0 := \boldsymbol{\beta}_{\{\mathbf{0}\}} = -\boldsymbol{\beta}$, and

$$\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_0)) = \langle \theta_2, \theta_3 \rangle \cap \langle \theta_1, \theta_3 \rangle \cap \langle \theta_2, \theta_4 \rangle \cap \langle \theta_1, \theta_4 \rangle \\ = \langle \theta_1 \theta_2, \theta_3 \theta_4 \rangle.$$

In particular,

$$\theta_1\theta_2 - \theta_3\theta_4 = \theta_1(\theta_2 + \theta_3) - \theta_3(\theta_1 + \theta_4) \in \mathbb{I}(\widetilde{\Omega}(-\beta_0)) \cap \langle A\theta \rangle,$$

and hence

$$N_{\mathbf{0}} \ni t^{-\beta_0}(\theta_1\theta_2 - \theta_3\theta_4) = t^{\beta}(\theta_1\theta_2 - \theta_3\theta_4) = \partial_1\partial_2 - \partial_3\partial_4.$$

Indeed N_0 is generated by

$$\begin{aligned} t^{\beta}\theta_{1}\theta_{4}(\theta_{2}+\theta_{3}) &= \theta_{4}\partial_{1}\partial_{2}+\theta_{1}\partial_{3}\partial_{4}, \\ t^{\beta}\theta_{2}\theta_{4}(\theta_{1}+\theta_{3}) &= \theta_{4}\partial_{1}\partial_{2}+\theta_{2}\partial_{3}\partial_{4}, \\ t^{\beta}\theta_{1}\theta_{3}(\theta_{2}+\theta_{4}) &= \theta_{3}\partial_{1}\partial_{2}+\theta_{1}\partial_{3}\partial_{4}, \\ t^{\beta}\theta_{2}\theta_{3}(\theta_{1}+\theta_{4}) &= \theta_{3}\partial_{1}\partial_{2}+\theta_{2}\partial_{3}\partial_{4}, \\ t^{\beta}(\theta_{1}\theta_{2}-\theta_{3}\theta_{4}) &= \partial_{1}\partial_{2}-\partial_{3}\partial_{4}. \end{aligned}$$

Hence by Theorems 4.7 and 6.3, N is generated by

$$t^{-\beta_{14}}(\theta_2 + \theta_3) = x_4 \partial_2 + x_1 \partial_3, t^{-\beta_{24}}(\theta_1 + \theta_3) = x_4 \partial_1 + x_2 \partial_3, t^{-\beta_{13}}(\theta_2 + \theta_4) = x_3 \partial_2 + x_1 \partial_4, t^{-\beta_{23}}(\theta_1 + \theta_4) = x_3 \partial_1 + x_2 \partial_4, t^{\beta}(\theta_1 \theta_2 - \theta_3 \theta_4) = \partial_1 \partial_2 - \partial_3 \partial_4.$$

7. SIMPLICIAL SCORED CASE

Theorem 7.1. Suppose that $\mathbb{N}A$ is scored and simplicial. Then N is generated by $N_{-\beta-\beta_{\sigma}}$ ($\sigma \in \mathcal{F}(\beta)$). More explicitly, N is generated by

$$t^{-\boldsymbol{\beta}_{\sigma}} \langle \prod_{j \notin \sigma} \prod_{k=1}^{m_{\sigma,j}} (\theta_j - k) \rangle \cdot F_{\sigma}(A\theta) \qquad (\sigma \in \mathcal{F}(\boldsymbol{\beta})).$$

Proof. This is clear from Theorems 4.7, 5.3, and 6.3.

Corollary 7.2. Suppose that $\mathbb{N}A$ is normal and simplicial. Then N is generated by

$$t^{-\beta_{\sigma}} \prod_{j \notin \sigma} \prod_{k=1}^{\lfloor \frac{-F_{\sigma}(\beta)-1}{F_{\sigma}(a_{j})} \rfloor} (\theta_{j}-k) \cdot F_{\sigma}(A\theta) \qquad (\sigma \in \mathcal{F}(\beta)).$$

Proof. This is immediate from Theorem 7.1 and Corollary 5.4.

Corollary 7.3. Suppose that NA is normal and simplicial. Assume that $F_{\sigma}(\beta) = -1$ for all $\sigma \in \mathcal{F}(\beta)$. Then N is generated by

$$t^{-\boldsymbol{\beta}_{\sigma}}F_{\sigma}(A\theta) \qquad (\sigma \in \mathcal{F}(\boldsymbol{\beta})).$$

Proof. This is immediate from Corollary 7.2.

Example 7.4. Let $A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & n-1 & n \end{pmatrix} = (\boldsymbol{a}_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1}, \boldsymbol{a}_n).$ Then

$$F_0 := F_{\sigma_0}(s) = s_2 = \sum_{i=1}^n i\theta_i$$

$$F_n := F_{\sigma_n}(s) = ns_1 - s_2 = \sum_{i=0}^{n-1} (n-i)\theta_i$$

This is normal and simplicial.

Let $\boldsymbol{\beta} = \begin{pmatrix} -2 \\ -n \end{pmatrix} = -\boldsymbol{a}_0 - \boldsymbol{a}_n, \ \boldsymbol{\beta}_0 := \boldsymbol{\beta}_{\sigma_0} = -\boldsymbol{a}_0 - \boldsymbol{\beta} = \boldsymbol{a}_n, \ \boldsymbol{\beta}_n := \boldsymbol{\beta}_{\sigma_n} = -\boldsymbol{a}_n - \boldsymbol{\beta} = \boldsymbol{a}_0, \ \boldsymbol{\beta}_0 := -\boldsymbol{\beta}.$ By Corollary 5.4, $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma_0}}$ is generated by

(20)
$$t^{-\beta_0} \prod_{i=1}^{n-1} \prod_{k=1}^{\lfloor \frac{n-1}{i} \rfloor} (\theta_i - k) F_0,$$

and $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma_n}}$ is generated by

(21)
$$t^{-\boldsymbol{\beta}_n} \prod_{i=1}^{n-1} \prod_{k=1}^{\lfloor \frac{n-1}{n-i} \rfloor} (\theta_i - k) F_n.$$

By Corollary 7.2, N is generated by (20) and (21).

Example 7.5. Let
$$A = \begin{pmatrix} & -1 & 0 \\ I_{d-1} & \vdots & \vdots \\ & & -1 & 0 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_d, \boldsymbol{a}_{d+1}),$$

and $\boldsymbol{\beta} = -\boldsymbol{a}_{d+1}$. We have $\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_d = d\boldsymbol{a}_{d+1}$. This example is normal, homogeneous, simplicial, and reflexive; $F_{\sigma}(\boldsymbol{\beta}) = -1$ for all facets σ .

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Let σ_i be the facet not containing a_i $(1 \le i \le d)$. Put $F_i := F_{\sigma_i}$. We have

$$F_{\check{d}} = s_d - (s_1 + \dots + s_{d-1}) = d\theta_d + \theta_{d+1},$$

$$F_i = s_d - (\sum_{j=1, j \neq i}^{d-1} s_j) + (d-1)s_i = d\theta_i + \theta_{d+1} \quad (i < d).$$

We have $F_{\sigma}(\boldsymbol{\beta}) = -1$ for all facets σ , and take $\boldsymbol{\beta}_{\tilde{i}} := \boldsymbol{\beta}_{\sigma_{\tilde{i}}}$ as follows:

$$\boldsymbol{\beta}_{\check{d}} = \begin{pmatrix} -1 \\ \vdots \\ -1 \\ 2-d \end{pmatrix}, \quad \boldsymbol{\beta}_{\check{1}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 2-d \end{pmatrix}, \cdots, \boldsymbol{\beta}_{\check{d-1}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 2-d \end{pmatrix}.$$

The vectors $-\boldsymbol{\beta}_{1}, \ldots, -\boldsymbol{\beta}_{d}$ are the roots (e.g. see (2.10) in [6]). Since $m_{\check{d},d} = m_{\check{d},d+1} = 0$ and $-\boldsymbol{\beta}_{\check{d}} = (d-1)\boldsymbol{a}_{d+1} - \boldsymbol{a}_{d} = \boldsymbol{a}_{1} + \cdots + \boldsymbol{a}_{d-1} - \boldsymbol{a}_{d+1}$,

$$N_{-\beta-\beta_{\bar{d}}} = t^{-\beta_{\bar{d}}} \langle F_{\bar{d}}(\theta) \rangle = t^{-\beta_{\bar{d}}} \langle d\theta_d + \theta_{d+1} \rangle$$
$$= \langle dx_{d+1}^{d-1} \partial_d + x_1 \cdots x_{d-1} \partial_{d+1} \rangle.$$

Let i < d. Since $m_{\check{i},i} = m_{\check{i},d+1} = 0$ and $-\beta_{\check{i}} = (d-1)a_{d+1} - a_i = \sum_{j=1, j \neq i}^{d} a_j - a_{d+1}$,

$$N_{-\beta-\beta_{\tilde{i}}} = t^{-\beta_{\tilde{i}}} \langle F_{\tilde{i}}(\theta) \rangle = t^{-\beta_{\tilde{i}}} \langle d\theta_{i} + \theta_{d+1} \rangle$$
$$= \langle dx_{d+1}^{d-1} \partial_{i} + (\prod_{j=1, j \neq i}^{d} x_{j}) \partial_{d+1} \rangle.$$

Hence the left module counterpart $L^{L}(\boldsymbol{\beta})$ to $L(\boldsymbol{\beta})$ is described as

$$L^{L}(\boldsymbol{\beta}) = D(K^{d+1})/D(K^{d+1}) \left\langle \begin{array}{c} d\theta_{i} + \theta_{d+1} + 1 & (i \leq d) \\ dx_{i}\partial_{d+1}^{d-1} + x_{d+1} \prod_{j=1, \ j \neq i}^{d} \partial_{j} & (i \leq d) \\ \partial_{1} \cdots \partial_{d} - \partial_{d+1}^{d} \end{array} \right\rangle,$$

which is the extended hypergeometric system considered in [5] and [6]. The rank of the A-hypergeometric system $M^L(\beta)$ equals the volume d.

Take the weight $(0, \ldots, 0, 1)$, and consider a refined monomial order. Then the exponents of $M^L(\boldsymbol{\beta})$ are

$$(-i/d, \dots, -i/d, i-1)$$
 $(i = 1, 2, \dots, d),$

and

$$\phi_i = (x_1 \cdots x_d)^{-\frac{i}{d}} x_{d+1}^{i-1} \sum_{n=0}^{\infty} \frac{[-i/d]_n^d}{[dn+i-1]_{dn}} \left(\frac{x_{d+1}^d}{x_1 \cdots x_d}\right)^n$$

(i = 1, 2, ..., d) form a fundamental basis (see [13, Chapters 2 and 3] for this argument).

Among them, $\phi_1, \ldots, \phi_{d-1}$ satisfy $L^L(\boldsymbol{\beta})$, but ϕ_d does not. Hence the rank of $L^L(\boldsymbol{\beta})$ equals d-1.

Take the weight $(1, \ldots, 1, 0)$, and consider a refined monomial order. Then the unique exponent is $(0, \ldots, 0, -1)$, and $L^{L}(\boldsymbol{\beta})$ has a fundamental basis consisting of log-series starting with

$$x_{d+1}^{-1} \left(\log \frac{x_1 \cdots x_d}{x_{d+1}^d} \right)^i \qquad (i = 0, 1, \dots, d-2).$$

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