FIRST SYZYGIES OF IRREDUCIBLE A-HYPERGEOMETRIC QUOTIENTS

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Abstract. An A-hypergeometric system is not irreducible, if its parameter vector is resonant. In this paper, we present a way of computing a finite system of generators of the first syzygy module of an irreducible A-hypergeometric quotient. In particular, if the semigroup generated by A is simplicial and scored, then an explicit system of generators is given.

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1. Introduction

Let K be a field of characteristic 0, and let $A := (a_{ij})$ be a $d \times n$ integer matrix. We assume that $\mathbb{Z}^d$ is generated by the column vectors of A as an abelian group. Given a parameter vector $\boldsymbol{\beta} = {}^t(\beta_1, \dots, \beta_d) \in K^d$, the *$A$-hypergeometric* (or *GKZ* (after the systematic study by Gel'fand, Kapranov, and Zelevinskii [1]-[4])) *system* $M^L(\boldsymbol{\beta})$ *with parameter vector* $\boldsymbol{\beta}$ is defined as the left $D(K^n)$-module

$$(1) \quad M^L(\boldsymbol{\beta}) := M_A^L(\boldsymbol{\beta}) := D(K^n)/D(K^n)I_A(\partial) + D(K^n)\langle A\theta - \boldsymbol{\beta}\rangle,$$

where $D(K^n)$ is the nth Weyl algebra

$$D(K^n) = K[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n\rangle,$$

$I_A(\partial)$ is the toric ideal of $K[\partial_1, \dots, \partial_n]$ defined by A, and $D(K^n)\langle A\theta - \boldsymbol{\beta}\rangle$ is the left ideal of $D(K^n)$ generated by $\sum_{j=1}^n a_{ij}x_j\partial_j - \beta_i$ $(i = 1, \dots, d)$.

The A-hypergeometric system $M^L(\boldsymbol{\beta})$ is not irreducible in general. Indeed $M^L(\boldsymbol{\beta})$ is irreducible if and only if the parameter vector $\boldsymbol{\beta}$ is nonresonant (see [4] and [12]). In the paper [12], we considered a category $\mathcal{O}_{K^n}$ of right $D(K^n)$-modules appropriate for the study of A-hypergeometric systems, and we considered irreducible modules in $\mathcal{O}_{K^n}$. Each modules in $\mathcal{O}_{K^n}$ has a weight decomposition with respect

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to the torus action defined by A. We treat right $D(K^n)$-modules in this paper as well. We remark that there exists a one-to-one correspondence between right $D(K^n)$-modules and left $D(K^n)$-modules by the anti-automorphism ι of $D(K^n)$ defined by

$$\iota(x_j) = \partial_j, \qquad \iota(\partial_j) = x_j \qquad \text{for all } j. \tag{2}$$

Let $\boldsymbol{\beta} \in K^d$ satisfy $F_\sigma(\boldsymbol{\beta}) \notin \mathbb{N}$ for every facet σ of the cone generated by A, where F_σ is the primitive integral support function of σ. Then

$$L(\boldsymbol{\beta}) := D(K^n)/I_A D(K^n) + D(K^n) \cap \langle A\theta - \boldsymbol{\beta}\rangle D((K^\times)^n)$$

is irreducible [12, Theorem 6.4], and any irreducible module in $\mathcal{O}_{K^n}$ can be described similarly [12, Theorem 6.6], where I_A is the toric ideal of $K[x_1, \dots, x_n]$ defined by A, and

$$D((K^\times)^n) = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]\langle \partial_1, \dots, \partial_n\rangle.$$

In this paper, we describe a finite system of generators of the right ideal (the first syzygy module of $L(\boldsymbol{\beta})$)

$$I_A D(K^n) + D(K^n) \cap \langle A\theta - \boldsymbol{\beta}\rangle D((K^\times)^n),$$

apart from that of I_A, which can be computed by the commutative Gröbner basis theory. To this aim, we consider generators of the right $D(K^n)$-module

$$N := \frac{I_A D(K^n) + D(K^n) \cap \langle A\theta - \boldsymbol{\beta}\rangle D((K^\times)^n)}{I_A D(K^n) + \langle A\theta - \boldsymbol{\beta}\rangle D(K^n)}.$$

Since the $D(K^n)$-module N is finitely generated, it is generated by finite number of weight spaces. In Theorem 4.7, we specify those weights. This enables us to compute a finite system of generators of N and, in turn, that of the first syzygy module of the irreducible module $L(\boldsymbol{\beta})$.

If the semigroup is simplicial and scored, then those weights are associated to facets, and we give explicit generators of N (Theorem 7.1).

We note that Hosono et al [5] and [6] considered $L(\boldsymbol{\beta})$ (called the extended GKZ system) for the reflexive case.

2. Rings of differential operators

Let K denote a field of characteristic 0. Let R be a commutative K-algebra, and let M, N be R-modules. We briefly recall the module $D(M, N)$ of differential operators from M to N. For details, see [16]. For $k \in \mathbb{N}$, the subspaces $D^k(M, N)$ of $\mathrm{Hom}_K(M, N)$ are inductively defined by

$$D^0(M, N) = \mathrm{Hom}_R(M, N)$$

and

$$D^{k+1}(M,N)=\{P\in \mathrm{Hom}_K(M,N)\,:\,[f,P]\in D^k(M,N)\quad(\forall f\in R)\},$$

where $[\,,\,]$ denotes the commutator. Set $D(M,N):=\bigcup_{k=0}^{\infty}D^k(M,N)$, and $D(M):=D(M,M)$. Then $D(M)$ is a K-algebra, and $D(M,N)$ is a $(D(N),D(M))$-bimodule.

The ring $D(K^n):=D(K[x_1,\dots,x_n])$ of differential operators on K^n is the nth Weyl algebra:

$$D(K^n)=K[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle,$$

where $\partial_j=\dfrac{\partial}{\partial x_j}$.

The ring $D((K^\times)^n):=D(K[x_1^{\pm1},\dots,x_n^{\pm1}])$ of differential operators on $(K^\times)^n$ is given by

$$\begin{aligned}D((K^\times)^n)&=K[x_1^{\pm1},\dots,x_n^{\pm1}]\langle\partial_1,\dots,\partial_n\rangle\\&=\textstyle\bigoplus_{\boldsymbol{u}\in\mathbb{Z}^n}x^{\boldsymbol{u}}K[\theta_1,\dots,\theta_n],\end{aligned}$$

where $\theta_j=x_j\partial_j$.

Let $A:=\{\,\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n\,\}$ be a finite set of column vectors in $\mathbb{Z}^d$. Sometimes we identify A with the matrix $(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n)=(a_{ij})$. Let $\mathbb{Z}A$, and $\mathbb{Q}_{\geq0}A$ denote the abelian group, and the cone generated by A, respectively. Throughout this paper, we assume that $\mathbb{Z}A=\mathbb{Z}^d$ and that $\mathbb{Q}_{\geq0}A$ is strongly convex.

Let X_A denote the affine toric variety defined by A, and T_A its big torus. Let $\mathbb{N}A$ be the semigroup generated by A. The semigroup algebra $K[\mathbb{N}A]=\bigoplus_{\boldsymbol{a}\in\mathbb{N}A}Kt^{\boldsymbol{a}}$ is the ring of regular functions on X_A. Then we have $K[\mathbb{N}A]\simeq K[x]/I_A$, where I_A is the ideal of the polynomial ring $K[x]:=K[x_1,\dots,x_n]$ generated by all $x^{\boldsymbol{u}}-x^{\boldsymbol{v}}$ for $\boldsymbol{u},\boldsymbol{v}\in\mathbb{N}^n$ with $A\boldsymbol{u}=A\boldsymbol{v}$. Here we have used the multi-index notation, e.g., $x^{\boldsymbol{u}}=x_1^{u_1}x_2^{u_2}\cdots x_n^{u_n}$ for $\boldsymbol{u}={}^t(u_1,u_2,\dots,u_n)$.

The ring $D(T_A):=D(K[t_1^{\pm1},\dots,t_d^{\pm1}])$ of differential operators on T_A is given by

$$\begin{aligned}D(T_A)&=K[t_1^{\pm1},\dots,t_d^{\pm1}]\langle\partial_{t_1},\dots,\partial_{t_d}\rangle\\&=\textstyle\bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d}t^{\boldsymbol{a}}K[s_1,\dots,s_d],\end{aligned}$$

where $s_i=t_i\partial_{t_i}$ and $\partial_{t_i}=\dfrac{\partial}{\partial t_i}$.

The ring $D(X_A):=D(K[\mathbb{N}A])$ of differential operators on X_A is a subalgebra of $D(T_A)$:

$$D(X_A)=\{P\in D(T_A)\,:\,P(K[\mathbb{N}A])\subseteq K[\mathbb{N}A]\}.$$

Let $X=K^n$, $(K^\times)^n$, T_A, or X_A. For $\boldsymbol{a}\in\mathbb{Z}^d$, set

$$D(X)_{\boldsymbol{a}}:=\{P\in D(X)\,:\,[s_i,P]=a_iP\ (i=1,\dots,d)\},$$

where $s_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j$ for $X = K^n$ or $(K^\times)^n$. Then

$$D(X) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} D(X)_{\boldsymbol{a}}$$

is a $\mathbb{Z}^d$-graded algebra.

Recall from [8, Theorem 2.3] that the graded part of $D(X_A)$ is described by

$$D(X_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} \mathbb{I}(\Omega(\boldsymbol{a})) \quad \text{for all } \boldsymbol{a} \in \mathbb{Z}^d,$$

where

$$\begin{array}{rcl} \Omega(\boldsymbol{a}) & := & \Omega_A(\boldsymbol{a}) := \mathbb{N}A \setminus (-\boldsymbol{a} + \mathbb{N}A), \\ \mathbb{I}(\Omega(\boldsymbol{a})) & := & \{ f(s) \in K[s] \,:\, f(\boldsymbol{c}) = 0 \text{ for all } \boldsymbol{c} \in \Omega(\boldsymbol{a}) \}, \\ K[s] & := & K[s_1, \dots, s_d]. \end{array}$$

We write $D(K^n, X_A)$ instead of $D(K[x], K[\mathbb{N}A])$. From [16, 1.3 (e),(f)], we have

$$D(K^n, X_A) = D(K^n)/I_A D(K^n). \tag{3}$$

The algebra $D(X_A)$ can be identified with

$$\{ P \in D(K^n) \,:\, P I_A \subseteq I_A D(K^n) \} / I_A D(K^n).$$

(See e.g. [7].) We may thus consider that $D(X_A)$ is contained in $D(K^n, X_A)$. For the following proposition, see [11, Proposition 4.1 and Corollary 4.2].

Proposition 2.1.

$$D(K^n, X_A) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} t^{\boldsymbol{a}} \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})),$$

where

$$\begin{array}{rcl} \widetilde{\Omega}(\boldsymbol{a}) & := & \widetilde{\Omega}_A(\boldsymbol{a}) := \{ \boldsymbol{u} \in \mathbb{N}^n \,:\, A\boldsymbol{u} \notin -\boldsymbol{a} + \mathbb{N}A \}, \\ \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})) & = & \{ f(\theta) \in K[\theta] \,:\, f(\boldsymbol{u}) = 0 \textit{ for all } \boldsymbol{u} \in \widetilde{\Omega}(\boldsymbol{a}) \}, \\ K[\theta] & := & K[\theta_1, \dots, \theta_n]. \end{array}$$

In particular, $D(K^n, X_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} K[\theta]$ *for all* $\boldsymbol{a} \in \mathbb{N}A$.

Recall that a pair $(\boldsymbol{u}, \sigma)$ with $\sigma \subseteq \{1, 2, \dots, n\}$ and $\boldsymbol{u} \in \mathbb{N}^{\sigma^c} := \{ \boldsymbol{v} \in \mathbb{N}^n \mid v_j = 0 \text{ for all } j \in \sigma \}$ is called a standard pair of a monomial ideal M of $K[\partial_1, \dots, \partial_n]$ if the following conditions are satisfied:

1. for any $\boldsymbol{v} \in \mathbb{N}^\sigma$, the monomial $\partial^{\boldsymbol{u}+\boldsymbol{v}}$ does not belong to M;
2. for any $l \notin \sigma$, there exists $\boldsymbol{v} \in \mathbb{N}^{\sigma \cup \{l\}}$ such that $\partial^{\boldsymbol{u}+\boldsymbol{v}}$ belongs to M.

Let $\mathbb{I}(\boldsymbol{a})$ denote the ideal of $K[\partial_1, \ldots, \partial_n]$ generated by the monomials $\partial^{\boldsymbol{u}}$ with $A\boldsymbol{u} \in -\boldsymbol{a} + \mathbb{N}A$. Let $S(\mathbb{I}(\boldsymbol{a}))$ denote the set of standard pairs of the monomial ideal $\mathbb{I}(\boldsymbol{a})$. Then we obtain the following theorem from [13, Theorem 3.2.2, Corollary 3.2.3].

Theorem 2.2.

$$\begin{aligned} \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})) &= \widetilde{\mathbb{I}(\boldsymbol{a})} = \langle [\theta]_{\boldsymbol{u}} \, : \, \partial^{\boldsymbol{u}} \in \mathbb{I}(\boldsymbol{a}) \rangle \\ &= \bigcap_{(\boldsymbol{u},\sigma) \in S(\mathbb{I}(\boldsymbol{a}))} \langle \theta_i - u_i \, : \, i \notin \sigma \rangle, \end{aligned}$$

where

$$[\theta]_{\boldsymbol{u}} := \prod_{j=1}^{n} [\theta_j]_{u_j} := \prod_{j=1}^{n} \prod_{k=0}^{u_j - 1} (\theta_j - k).$$

3. Simple objects in $\mathcal{O}_{K^n}$

In this section, we briefly review simple objects in $\mathcal{O}_{K^n}$ from [12]. Let $X = K^n$, $(K^\times)^n$, or T_A. In [12], we defined a full subcategory $\mathcal{O}_X$ of the category of right $D(X)$-modules (cf. [9, 11]). A right $D(X)$-module M is an object of $\mathcal{O}_X$ if the support of M is contained in X_A, and M has a weight decomposition $M = \bigoplus_{\boldsymbol{\lambda} \in K^d} M_{\boldsymbol{\lambda}}$, where

$$M_{\boldsymbol{\lambda}} = \{x \in M \, : \, x.f(s) = f(-\boldsymbol{\lambda})x \quad \text{for all } f \in K[s]\}.$$

Recall that the preorder $\preceq$ is defined in [9] (see also [14]):

$$\text{For } \boldsymbol{\alpha}, \boldsymbol{\beta} \in K^d, \qquad \boldsymbol{\alpha} \preceq \boldsymbol{\beta} \iff \mathbb{I}(\Omega(\boldsymbol{\beta} - \boldsymbol{\alpha})) \not\subseteq \mathfrak{m}_{\boldsymbol{\alpha}},$$

where $\mathfrak{m}_{\boldsymbol{\alpha}}$ is the maximal ideal of $K[s]$ at $\boldsymbol{\alpha}$. An equivalence relation $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ is defined to be $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$.

For $\boldsymbol{\beta} \in K^d$, the right $D(K^n)$-module

$$M_{K^n}(\boldsymbol{\beta}) := D(K^n)/(I_A D(K^n) + \langle s - \boldsymbol{\beta} \rangle D(K^n))$$

is the right D-module counterpart to the A-hypergeometric system $M_A^L(\boldsymbol{\beta})$ with parameter vector $\boldsymbol{\beta}$ (cf. (1) and (2)). Recall that $s_i = \sum_{j=1}^n a_{ij}\theta_j$, where $\theta_j = x_j\partial_j$. Clearly $M_{K^n}(\boldsymbol{\beta}) \in \mathcal{O}_{K^n}$.

Definition 3.1 (Definition 6.2 in [12])**.** Let $\boldsymbol{\beta} \in K^d$. In $\boldsymbol{\beta} + \mathbb{Z}A$, there exists a unique minimal equivalence class with respect to $\preceq$, which we denote by $\boldsymbol{\beta}^{\text{empty}}$. An element belonging to the class is also denoted by $\boldsymbol{\beta}^{\text{empty}}$.

Remark 3.2. In [10], we have defined a finite subset $E_\tau(\boldsymbol{\alpha})$ for a face τ and a parameter vector $\boldsymbol{\alpha} \in K^d$:

$$E_\tau(\boldsymbol{\alpha}) = \{\boldsymbol{\lambda} \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau) \, : \, \boldsymbol{\alpha} - \boldsymbol{\lambda} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)\}.$$

The class $\boldsymbol{\beta}^{\text{empty}}$ is described as

$$E_\tau(\boldsymbol{\beta}^{\text{empty}}) = \begin{cases} E_{\mathbb{Q}_{\geq 0}A}(\boldsymbol{\beta}) & (\tau = \mathbb{Q}_{\geq 0}A) \\ \emptyset & (\tau \neq \mathbb{Q}_{\geq 0}A). \end{cases}$$

Theorem 3.3 (Theorem 6.4 in [12])**.** *Let $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}} \in K^d$. Then*

$$\begin{aligned} L(\boldsymbol{\beta}) &:= L_{K^n}(T_A, \boldsymbol{\beta}) \\ &:= D(K^n)/(I_A D(K^n) + D(K^n) \cap \langle s - \boldsymbol{\beta} \rangle D((K^\times)^n)) \\ &\simeq \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} t^{-\boldsymbol{\beta}+\boldsymbol{a}} K[s]/\langle s - \boldsymbol{\beta} + \boldsymbol{a} \rangle \otimes_{K[s]} \mathbb{I}(\widetilde{\Omega}(\boldsymbol{a})) \end{aligned}$$

is a unique simple $D(K^n)$-submodule of

$$D((K^\times)^n)/(I_A D((K^\times)^n) + \langle s - \boldsymbol{\beta} \rangle D((K^\times)^n)).$$

Remark 3.4. Any simple object in $\mathcal{O}_{K^n}$ is isomorphic to some $L(\boldsymbol{\beta})$ or a similar module associated to a torus constituting the toric variety X_A [12, Theorem 6.6].

Let $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}}$, and let

$$N_{K^n}(T_A, \boldsymbol{\beta}) := \frac{I_A D(K^n) + D(K^n) \cap \langle A\theta - \boldsymbol{\beta} \rangle D((K^\times)^n)}{(I_A D(K^n) + \langle A\theta - \boldsymbol{\beta} \rangle D(K^n))}.$$

Here and hereafter we interchangeably use s and $A\theta$. The $D(K^n)$-module $N_{K^n}(T_A, \boldsymbol{\beta})$ is the kernel of the natural surjection

$$M_{K^n}(\boldsymbol{\beta}) \to L_{K^n}(T_A, \boldsymbol{\beta}) = L(\boldsymbol{\beta}).$$

Our aim of this paper is to find a finite system of generators of the $D(K^n)$-module $N_{K^n}(T_A, \boldsymbol{\beta})$. For a different choice of $\boldsymbol{\beta}^{\text{empty}}$, we have the following proposition.

Proposition 3.5. *Let $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}} \sim \boldsymbol{\beta}'$. Then there exists $P, Q \in D(X_A)$ such that*

$$\begin{aligned} N_{K^n}(T_A, \boldsymbol{\beta}) &= P N_{K^n}(T_A, \boldsymbol{\beta}') \\ N_{K^n}(T_A, \boldsymbol{\beta}') &= Q N_{K^n}(T_A, \boldsymbol{\beta}). \end{aligned}$$

Proof. Since $\boldsymbol{\beta} \sim \boldsymbol{\beta}'$,

$$\mathbb{I}(\Omega(\boldsymbol{\beta} - \boldsymbol{\beta}')) \not\subseteq \mathfrak{m}_{\boldsymbol{\beta}'}, \qquad \mathbb{I}(\Omega(\boldsymbol{\beta}' - \boldsymbol{\beta})) \not\subseteq \mathfrak{m}_{\boldsymbol{\beta}}.$$

Take $p(s) \in \mathbb{I}(\Omega(\boldsymbol{\beta} - \boldsymbol{\beta}')) \setminus \mathfrak{m}_{\boldsymbol{\beta}'}$ and $q(s) \in \mathbb{I}(\Omega(\boldsymbol{\beta}' - \boldsymbol{\beta})) \setminus \mathfrak{m}_{\boldsymbol{\beta}}$, and let $P := t^{\boldsymbol{\beta}-\boldsymbol{\beta}'} p(s)$ and $Q := t^{\boldsymbol{\beta}'-\boldsymbol{\beta}} q(s)$. Then clearly $P N_{K^n}(T_A, \boldsymbol{\beta}') \subseteq N_{K^n}(T_A, \boldsymbol{\beta})$ and $Q N_{K^n}(T_A, \boldsymbol{\beta}) \subseteq N_{K^n}(T_A, \boldsymbol{\beta}')$.

Moreover, since $PQ = p(s + \boldsymbol{\beta}' - \boldsymbol{\beta}) q(s) \notin \mathfrak{m}_{\boldsymbol{\beta}}$ and $QP = q(s + \boldsymbol{\beta} - \boldsymbol{\beta}') p(s) \notin \mathfrak{m}_{\boldsymbol{\beta}'}$, $PQ N_{K^n}(T_A, \boldsymbol{\beta}) = N_{K^n}(T_A, \boldsymbol{\beta})$ and $QP N_{K^n}(T_A, \boldsymbol{\beta}') = N_{K^n}(T_A, \boldsymbol{\beta}')$. Hence the assertion follows. □

4. Weights of generating relations of $L(\boldsymbol{\beta})$

Let $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}} \in K^d$. In this section, we choose a finite set J of weights of $N := N_{K^n}(T_A, \boldsymbol{\beta})$ such that the weight spaces with weight in J generate N (Theorem 4.7). This enables us to compute a finite system of generators of N and, in turn, that of the irreducible module $L(\boldsymbol{\beta})$.

We recall the primitive integral support function of a facet (maximal proper face) of the cone $\mathbb{Q}_{\geq 0}A$. Let $\mathcal{F}$ denote the set of facets of $\mathbb{Q}_{\geq 0}A$. Given a facet $\sigma \in \mathcal{F}$, we denote by F_σ the primitive integral support function of σ, i.e., F_σ is the uniquely determined linear form on $\mathbb{Q}^d$ satisfying

(1) $F_\sigma(\mathbb{Q}_{\geq 0}A) \geq 0$,
(2) $F_\sigma(\sigma) = 0$,
(3) $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

Then we know, by [10, Proposition 2.2] and Remark 3.2,

$$\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}} \Leftrightarrow F_\sigma(\boldsymbol{\beta}) \notin F_\sigma(\mathbb{N}A) \text{ for all facets } \sigma \in \mathcal{F}. \tag{4}$$

Set

$$\mathcal{F}(\boldsymbol{\beta}) := \{\sigma \in \mathcal{F} \mid F_\sigma(\boldsymbol{\beta}) \in \mathbb{Z}\}.$$

From now on, we fix $\boldsymbol{\beta} \in K^d$ satisfying $F_\sigma(\boldsymbol{\beta}) < 0$ for all $\sigma \in \mathcal{F}(\boldsymbol{\beta})$. Then $\boldsymbol{\beta} = \boldsymbol{\beta}^{\text{empty}}$ by (4). Let $N := N(\boldsymbol{\beta}) := N_{K^n}(T_A, \boldsymbol{\beta})$. Then, for $\boldsymbol{a} \in \mathbb{Z}^d$, by the definition of N, (3), and Proposition 2.1,

$$N_{-\boldsymbol{\beta}-\boldsymbol{a}} = \frac{t^{-\boldsymbol{a}}\left(\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a})) \cap \langle A\theta - \boldsymbol{\beta} - \boldsymbol{a}\rangle\right)}{t^{-\boldsymbol{a}}\left(\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a}))\langle A\theta - \boldsymbol{\beta} - \boldsymbol{a}\rangle\right)}.$$

Proposition 4.1 (Lemma 8.2 (1) in [12]). *Let $\boldsymbol{a} \in \mathbb{Z}^d$. If $\boldsymbol{\beta} + \boldsymbol{a} \sim \boldsymbol{\beta}$, then $N_{-\boldsymbol{\beta}-\boldsymbol{a}} = \{0\}$.*

Choose $\tilde{\boldsymbol{\beta}} \in \mathbb{Q}^d$ such that $\mathcal{F}(\tilde{\boldsymbol{\beta}}) = \mathcal{F}(\boldsymbol{\beta})$ and $F_\sigma(\tilde{\boldsymbol{\beta}}) = F_\sigma(\boldsymbol{\beta})$ for all $\sigma \in \mathcal{F}(\boldsymbol{\beta})$. (Such $\tilde{\boldsymbol{\beta}}$ exists by Cramer's rule.) Set

$$C(\tilde{\boldsymbol{\beta}}) := (-\tilde{\boldsymbol{\beta}} - \mathbb{Q}_{\geq 0}A) \cap \mathbb{Z}^d.$$

Proposition 4.2. *The right $D(K^n)$-module N is generated by $\bigoplus_{\boldsymbol{a} \in \partial C(\tilde{\boldsymbol{\beta}})} N_{-\boldsymbol{\beta}-\boldsymbol{a}}$, where we put $\partial C(\tilde{\boldsymbol{\beta}}) = \bigcup_{\sigma \in \mathcal{F}(\boldsymbol{\beta})} \{\boldsymbol{a} \in C(\tilde{\boldsymbol{\beta}}) \mid F_\sigma(\boldsymbol{\beta} + \boldsymbol{a}) = 0\}$.*

Proof. Let σ be a facet of the cone $\mathbb{Q}_{\geq 0}A$. Then

$$F_\sigma(\boldsymbol{\beta} + \boldsymbol{a})N_{-\boldsymbol{\beta}-\boldsymbol{a}} = N_{-\boldsymbol{\beta}-\boldsymbol{a}}(\sum_{j=1}^n F_\sigma(\boldsymbol{a}_j)x_j\partial_j) \subseteq \sum_{\boldsymbol{a}_j \notin \sigma} N_{-\boldsymbol{\beta}-(\boldsymbol{a}-\boldsymbol{a}_j)}\partial_j.$$

Hence N is generated by $\bigoplus_{\boldsymbol{a}\in C(\tilde{\boldsymbol{\beta}})} N_{-\boldsymbol{\beta}-\boldsymbol{a}}$. By (4) and Proposition 4.1, $N_{-\boldsymbol{\beta}-\boldsymbol{a}} = 0$ if $\boldsymbol{a} \in C(\tilde{\boldsymbol{\beta}}) \setminus \partial C(\tilde{\boldsymbol{\beta}})$. □

Since $\boldsymbol{a} \in C(\tilde{\boldsymbol{\beta}})$ implies that $F_\sigma(\boldsymbol{\beta}+\boldsymbol{a}) = F_\sigma(\tilde{\boldsymbol{\beta}}+\boldsymbol{a}) \le 0$ for all $\sigma \in \mathcal{F}(\boldsymbol{\beta})$, and that $F_{\sigma'}(\tilde{\boldsymbol{\beta}}+\boldsymbol{a}) < 0$ for all $\sigma' \notin \mathcal{F}(\boldsymbol{\beta})$, we see that $\partial C(\tilde{\boldsymbol{\beta}})$ is decomposed according to the decomposition of $\mathbb{Q}_{\ge 0}A$:

$$\partial C(\tilde{\boldsymbol{\beta}}) = \coprod_{\tau} (-\tilde{\boldsymbol{\beta}} - \overset{\circ}{\tau}) \cap C(\tilde{\boldsymbol{\beta}}),$$

where τ runs over all proper faces of the cone $\mathbb{Q}_{\ge 0}A$ such that $\sigma \in \mathcal{F}(\boldsymbol{\beta})$ for all facets $\sigma \succeq \tau$, and $\overset{\circ}{\tau}$ denotes the relative interior of τ.

Notation 4.3. As in [15], let

$$\mathbb{N}A = \mathbb{Q}_{\ge 0}A \cap \mathbb{Z}^d \setminus \bigcup_i (\boldsymbol{b}_i + \mathbb{N}(A \cap \tau_i)),$$

and

$$M := \max_{\sigma,i} F_\sigma(\boldsymbol{b}_i) + 1. \tag{5}$$

We agree $M = 0$ if $\mathbb{N}A = \mathbb{Q}_{\ge 0}A \cap \mathbb{Z}^d$ ($\mathbb{N}A$ is said to be normal (or saturated) in this case).

Lemma 4.4. *Let τ be a face of $\mathbb{Q}_{\ge 0}A$. Assume $F_\sigma(\boldsymbol{a}) \le -M$ for every facet $\sigma \not\succeq \tau$. Then the following hold.*

(1) *The support of each minimal generator of the $\mathbb{N}^n$-set $\mathbb{N}^n \cap f_A^{-1}(\boldsymbol{a}+\mathbb{N}A)$ is contained in $\tau^c := \{j \mid \boldsymbol{a}_j \notin \tau\}$, where f_A is the linear map from $\mathbb{Z}^n$ to $\mathbb{Z}^d$ defined by A.*

(2)

$$\begin{aligned}\mathbb{N}^n \setminus f_A^{-1}(\boldsymbol{a}+\mathbb{N}A) &= \bigcup_{\sigma \succeq \tau} \{(\boldsymbol{u},\sigma) \mid F_\sigma(A\boldsymbol{u}) < F_\sigma(\boldsymbol{a})\} \\ &\cup \bigcup_{\tau_i \succeq \tau} \{(\boldsymbol{u},\tau_i) \mid A\boldsymbol{u} \in \boldsymbol{a}+\boldsymbol{b}_i+\mathbb{Z}(A\cap\tau_i)\},\end{aligned}$$

where $(\boldsymbol{u},\sigma) := \boldsymbol{u} + \mathbb{N}^\sigma$, and

$$\mathbb{N}^\sigma := \{\boldsymbol{v} \in \mathbb{N}^n \mid v_j = 0 \textit{ for all } \boldsymbol{a}_j \notin \sigma\}.$$

Proof. It is enough to prove (1) that, if $\boldsymbol{u} \in \mathbb{N}^n$ and $A\boldsymbol{u} \notin \boldsymbol{a}+\mathbb{N}A$, then $A(\boldsymbol{u}+\boldsymbol{v}) \notin \boldsymbol{a}+\mathbb{N}A$ for any $\boldsymbol{v} \in \mathbb{N}^\tau$.

Suppose that $A\boldsymbol{u} \notin \boldsymbol{a}+\mathbb{N}A$. Then one of the following two holds:

(1) $F_\sigma(A\boldsymbol{u}) < F_\sigma(\boldsymbol{a})$ for some facet $\sigma \succeq \tau$.
(2) $A\boldsymbol{u} \in \boldsymbol{a}+\boldsymbol{b}_i+\mathbb{N}(A\cap\tau_i)$ for some i.

In the first case, we clearly have $A(\boldsymbol{u}+\boldsymbol{v}) \notin \boldsymbol{a}+\mathbb{N}A$ for any $\boldsymbol{v} \in \mathbb{N}^\tau$.

Suppose that $A\boldsymbol{u} \in \boldsymbol{a}+\boldsymbol{b}_i+\mathbb{N}(A\cap\tau_i)$ for some i. Then we prove $\tau_i \succeq \tau$. For this, we prove $\sigma \succeq \tau$ for all facets $\sigma \succeq \tau_i$. Suppose that $\sigma \succeq \tau_i$ but $\sigma \not\succeq \tau$. Then $F_\sigma(A\boldsymbol{u}) = F_\sigma(\boldsymbol{a}+\boldsymbol{b}_i) \leq -M + F_\sigma(\boldsymbol{b}_i) < 0$, which is a contradiction. We have thus proved $\tau_i \succeq \tau$. Hence we have $A(\boldsymbol{u}+\boldsymbol{v}) \notin \boldsymbol{a}+\mathbb{N}A$ for any $\boldsymbol{v}\in\mathbb{N}^\tau$ in the second case, too. We also have proved the second statement of the lemma. □

Lemma 4.5. *Let $\boldsymbol{a} \in \mathbb{Z}^d$, and let τ be a face of $\mathbb{Q}_{\geq 0}A$. Let $\tilde{\boldsymbol{\beta}}+\boldsymbol{a}$ be in the relative interior of $-\tau$, and $\boldsymbol{b} \in \mathbb{N}(A\cap\tau)$. Assume that $F_\sigma(\boldsymbol{a}+\boldsymbol{b}) \leq -M$ for all facets $\sigma \not\succeq \tau$. Then*

1. *$D(K^n, X_A)_{-\boldsymbol{a}-\boldsymbol{b}}x^{\boldsymbol{u}} = D(K^n, X_A)_{-\boldsymbol{a}}$ for any $\boldsymbol{u}\in\mathbb{N}^\tau$ with $A\boldsymbol{u} = \boldsymbol{b}$.*
2. *$N_{-\boldsymbol{\beta}-\boldsymbol{a}-\boldsymbol{b}}x^{\boldsymbol{u}} = N_{-\boldsymbol{\beta}-\boldsymbol{a}}$ for any $\boldsymbol{u}\in\mathbb{N}^\tau$ with $A\boldsymbol{u}=\boldsymbol{b}$.*

Proof. By Theorem 2.2 and Lemma 4.4 (1), the minimal generators of $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a}))$ do not have variables θ_i for $\boldsymbol{a}_i \in \tau$. Moreover, by Lemma 4.4 (2), $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a})) = \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{a}-\boldsymbol{b}))$. Hence the assertions follow. □

Notation 4.6. Let τ be a face of $\mathbb{Q}_{\geq 0}A$ such that all facets containing τ belong to $\mathcal{F}(\boldsymbol{\beta})$, and let $m_\tau := [\mathbb{Z}^d\cap\mathbb{Q}\tau : \mathbb{Z}(A\cap\tau)]$.

Choose $\boldsymbol{\beta}_{\tau,j} \in \mathbb{Z}^d$ $(j=1,\dots,m_\tau)$ so that $\tilde{\boldsymbol{\beta}}+\boldsymbol{\beta}_{\tau,j}$ form a set of representatives of $(\mathbb{Q}\tau\cap(\tilde{\boldsymbol{\beta}}+\mathbb{Z}^d))/\mathbb{Z}(A\cap\tau)$, and that $F_\sigma(\boldsymbol{\beta}_{\tau,j}) \leq -M$ for every facet σ with $\sigma\not\succeq\tau$. When $m_\tau = 1$, we simply write $\boldsymbol{\beta}_\tau$ instead of $\boldsymbol{\beta}_{\tau,1}$.

Theorem 4.7. *The right $D(K^n)$-module N is generated by*

$$\bigoplus_{\tau}\bigoplus_{j=1}^{m_\tau} N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\tau,j}},$$

where τ ranges over all faces of $\mathbb{Q}_{\geq 0}A$ such that all facets containing τ belong to $\mathcal{F}(\boldsymbol{\beta})$.

Proof. Let $\tilde{\boldsymbol{\beta}}+\boldsymbol{a}$ be in the relative interior of $-\tau$. By Proposition 4.2, it is enough to prove that $N_{-\boldsymbol{\beta}-\boldsymbol{a}}$ is generated by $\bigoplus_{\tau'\succeq\tau}\bigoplus_{j=1}^{m_{\tau'}} N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\tau',j}}$. We prove this by induction on the codimension of τ.

There exists $\boldsymbol{\beta}_{\tau,j}$ such that $\boldsymbol{\beta}_{\tau,j}-\boldsymbol{a} \in \mathbb{Z}(A\cap\tau)$. Take $\boldsymbol{a}'$ so that $\boldsymbol{\beta}_{\tau,j}-\boldsymbol{a}', \boldsymbol{a}-\boldsymbol{a}' \in \mathbb{N}(A\cap\tau)$.

We claim that

$$\boldsymbol{\beta}+\boldsymbol{a} \sim \boldsymbol{\beta}+\boldsymbol{a}'. \tag{6}$$

Since $\boldsymbol{a}-\boldsymbol{a}' \in \mathbb{Z}(A\cap\tau)$, we have, by definition, $E_{\tau'}(\boldsymbol{\beta}+\boldsymbol{a}) = E_{\tau'}(\boldsymbol{\beta}+\boldsymbol{a}')$ for $\tau'\succeq\tau$. For $\tau'\not\succeq\tau$, there exists a facet $\sigma\succeq\tau'$ with $\sigma\not\succeq\tau$. Since

$\tilde{\boldsymbol\beta}+\boldsymbol a \in -\overset{\circ}{\tau}$ and $\boldsymbol a-\boldsymbol a' \in \mathbb{N}(A\cap\tau)$, we have, for $\sigma\in\mathcal{F}(\boldsymbol\beta)$,

$$\begin{aligned} F_\sigma(\boldsymbol\beta+\boldsymbol a) &= F_\sigma(\tilde{\boldsymbol\beta}+\boldsymbol a)<0,\\ F_\sigma(\boldsymbol\beta+\boldsymbol a') &= F_\sigma(\tilde{\boldsymbol\beta}+\boldsymbol a)-F_\sigma(\boldsymbol a-\boldsymbol a')<0. \end{aligned}$$

For $\sigma\notin\mathcal{F}(\boldsymbol\beta)$, of course $F_\sigma(\boldsymbol\beta+\boldsymbol a), F_\sigma(\boldsymbol\beta+\boldsymbol a')\notin\mathbb{N}$. Hence $E_\sigma(\boldsymbol\beta+\boldsymbol a)=\emptyset=E_\sigma(\boldsymbol\beta+\boldsymbol a')$ by [10, Proposition 2.2 (3)]. Then $E_{\tau'}(\boldsymbol\beta+\boldsymbol a)=\emptyset=E_{\tau'}(\boldsymbol\beta+\boldsymbol a')$ by [10, Proposition 2.2 (4)]. We have thus proved the claim (6).

By [14, Lemma 4.1.4] and Theorem 2.2, there exists a b-function $b_{\boldsymbol a-\boldsymbol a'}(\theta):=b_{\boldsymbol a-\boldsymbol a'}(A\theta)=\sum_{\boldsymbol u}a_{\boldsymbol u}x^{\boldsymbol u}\partial^{\boldsymbol u}$ with $b_{\boldsymbol a-\boldsymbol a'}(\boldsymbol\beta+\boldsymbol a)\neq 0$ and $A\boldsymbol u\in\boldsymbol a-\boldsymbol a'+\mathbb{N}A$ for $a_{\boldsymbol u}\neq 0$. Then

$$N_{-\boldsymbol\beta-\boldsymbol a}=N_{-\boldsymbol\beta-\boldsymbol a}b_{\boldsymbol a-\boldsymbol a'}(\boldsymbol\beta+\boldsymbol a)=N_{-\boldsymbol\beta-\boldsymbol a}b_{\boldsymbol a-\boldsymbol a'}(\theta)\subseteq\sum_{\boldsymbol u}N_{-\boldsymbol\beta-\boldsymbol a+A\boldsymbol u}\partial^{\boldsymbol u}.$$

Here $\boldsymbol a-A\boldsymbol u\in\boldsymbol a'-\mathbb{N}A$. If $\boldsymbol a-A\boldsymbol u\notin\boldsymbol a'-\mathbb{N}(A\cap\tau)$, then $\tilde{\boldsymbol\beta}+\boldsymbol a-A\boldsymbol u$ is in the relative interior of a larger face, and hence the induction hypothesis would do. If $\boldsymbol a-A\boldsymbol u\in\boldsymbol a'-\mathbb{N}(A\cap\tau)$, then $\boldsymbol\beta_{\tau,j}-(\boldsymbol a-A\boldsymbol u)\in\mathbb{N}(A\cap\tau)$, and by Lemma 4.5 $N_{-\boldsymbol\beta-\boldsymbol a+A\boldsymbol u}$ is generated by $N_{-\boldsymbol\beta-\boldsymbol\beta_{\tau,j}}$. □

We may rephrase Theorem 4.7 as follows:

Corollary 4.8. *The irreducible module $L(\boldsymbol\beta)$ is described as*

$$D(K^n)/\begin{pmatrix} I_AD(K^n)+\langle A\theta-\boldsymbol\beta\rangle D(K^n)\\ +\bigoplus_\tau\bigoplus_{j=1}^{m_\tau}t^{-\boldsymbol\beta_{\tau,j}}(\mathbb{I}(\tilde\Omega(-\boldsymbol\beta_{\tau,j}))\cap\langle A\theta-\boldsymbol\beta-\boldsymbol\beta_{\tau,j}\rangle)D(K^n)\end{pmatrix}.$$

Example 4.9. Let $A=\begin{pmatrix}2&3&0&1\\0&0&1&1\end{pmatrix}$. Then $\mathbb{Q}_{\geq 0}A=\mathbb{Q}_{\geq 0}^2$ has two facets: $\sigma_{12}:=\mathbb{Q}_{\geq 0}\boldsymbol a_1=\mathbb{Q}_{\geq 0}\boldsymbol a_2$ and $\sigma_3:=\mathbb{Q}_{\geq 0}\boldsymbol a_3$; $F_{\sigma_{12}}(s)=s_2$ and $F_{\sigma_3}(s)=s_1$. We have $\mathbb{N}A=\mathbb{N}^2\setminus\begin{pmatrix}1\\0\end{pmatrix}$, $M=2$, and $m_{\sigma_{12}}=m_{\sigma_3}=m_{\{\boldsymbol 0\}}=1$. Let $\boldsymbol\beta=\begin{pmatrix}-1\\-1\end{pmatrix}$. Note that

$$s_1=2\theta_1+3\theta_2+\theta_4,\qquad s_2=\theta_3+\theta_4.$$

Let $\boldsymbol\beta_{\sigma_{12}}:=-\boldsymbol a_2-\boldsymbol\beta=\begin{pmatrix}-2\\1\end{pmatrix}$. Then $\boldsymbol\beta+\boldsymbol\beta_{\sigma_{12}}=\begin{pmatrix}-3\\0\end{pmatrix}$, and

$$\begin{aligned}\mathbb{I}(\widetilde\Omega(-\boldsymbol\beta_{\sigma_{12}})) &= \langle\theta_3,\theta_4\rangle.\\ \mathbb{I}(\widetilde\Omega(-\boldsymbol\beta_{\sigma_{12}}))\cap\langle s_1+3,s_2\rangle &\equiv \langle\theta_3+\theta_4\rangle,\end{aligned}$$

where $\equiv$ denotes the equality modulo $\mathbb{I}(\widetilde\Omega(-\boldsymbol\beta_{\sigma_{12}}))\langle s_1+3,s_2\rangle$.
Since $-\boldsymbol\beta_{\sigma_{12}}=\boldsymbol a_1-\boldsymbol a_3=\boldsymbol a_2-\boldsymbol a_4$,

(7) $$t^{-\boldsymbol{\beta}_{\sigma_{12}}}(\theta_3+\theta_4) = x_1\partial_3 + x_2\partial_4.$$

Let $\boldsymbol{\beta}_{\sigma_3} := -2\boldsymbol{a}_3 - \boldsymbol{\beta} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $\boldsymbol{\beta} + \boldsymbol{\beta}_{\sigma_3} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

$$\begin{aligned} \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma_3})) &= \langle \theta_1, \theta_2, \theta_4 \rangle. \\ \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\sigma_3})) \cap \langle s_1, s_2+2 \rangle &\equiv \langle 2\theta_1 + 3\theta_2 + \theta_4 \rangle. \end{aligned}$$

Since $-\boldsymbol{\beta}_{\sigma_3} = \boldsymbol{a}_4 - \boldsymbol{a}_1 = \boldsymbol{a}_1 + \boldsymbol{a}_3 - \boldsymbol{a}_2 = 2\boldsymbol{a}_3 - \boldsymbol{a}_4$,

(8) $$t^{-\boldsymbol{\beta}_{\sigma_3}}(2\theta_1 + 3\theta_2 + \theta_4) = 2x_4\partial_1 + 3x_1x_3\partial_2 + x_3^2\partial_4.$$

Let $\boldsymbol{\beta}_{\{\mathbf{0}\}} := -\boldsymbol{\beta}$. Then

$$\begin{aligned} \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\{\mathbf{0}\}})) &= \langle \theta_4, \theta_2\theta_3, [\theta_1]_2\theta_3, \theta_1[\theta_3]_2 \rangle. \\ \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_{\{\mathbf{0}\}})) \cap \langle s_1, s_2 \rangle &\equiv \left\langle \begin{array}{l} \theta_2(\theta_3+\theta_4), \\ [\theta_1]_2(\theta_3+\theta_4), \theta_1(\theta_3-1)(\theta_3+\theta_4), \\ [\theta_3]_2(2\theta_1+3\theta_2+\theta_4), \\ (\theta_1-1)\theta_3(2\theta_1+3\theta_2+\theta_4), \\ \theta_3(2\theta_1+3\theta_2+\theta_4) - 2\theta_1(\theta_3+\theta_4) \\ = 3\theta_2\theta_3 + (\theta_3 - 2\theta_1)\theta_4 \end{array} \right\rangle. \end{aligned}$$

Since $-\boldsymbol{\beta}_{\{\mathbf{0}\}} = -\boldsymbol{a}_4 = \boldsymbol{a}_1 - \boldsymbol{a}_2 - \boldsymbol{a}_3 = \boldsymbol{a}_2 - 2\boldsymbol{a}_1 - \boldsymbol{a}_3 = \boldsymbol{a}_4 - \boldsymbol{a}_1 - 2\boldsymbol{a}_3$,

(9) $$\begin{aligned} & t^{-\boldsymbol{\beta}_{\{\mathbf{0}\}}}\theta_2(\theta_3+\theta_4) = x_1\partial_2\partial_3 + x_2\partial_2\partial_4, \\ & t^{-\boldsymbol{\beta}_{\{\mathbf{0}\}}}[\theta_1]_2(\theta_3+\theta_4) = x_2\partial_1^2\partial_3 + x_1^2\partial_1^2\partial_4, \\ & t^{-\boldsymbol{\beta}_{\{\mathbf{0}\}}}\theta_1(\theta_3-1)(\theta_3+\theta_4) = x_4\partial_1\partial_3^2 + \theta_1(\theta_3-1)\partial_4, \\ & t^{-\boldsymbol{\beta}_{\{\mathbf{0}\}}}[\theta_3]_2(2\theta_1+3\theta_2+\theta_4) \\ & \qquad = 2x_4\partial_1\partial_3^2 + 3x_1(\theta_3-1)\partial_2\partial_3 + [\theta_3]_2\partial_4, \\ & t^{-\boldsymbol{\beta}_{\{\mathbf{0}\}}}(\theta_1-1)\theta_3(2\theta_1+3\theta_2+\theta_4) \\ & \qquad = 2x_2\partial_1^2\partial_3 + 3x_1(\theta_1-1)\partial_2\partial_3 + (\theta_1-1)\theta_3\partial_4, \\ & t^{-\boldsymbol{\beta}_{\{\mathbf{0}\}}}(3\theta_2\theta_3 + (\theta_3-2\theta_1)\theta_4) = 3x_1\partial_2\partial_3 + (\theta_3 - 2\theta_1)\partial_4. \end{aligned}$$

Hence N is generated by the operators (7), (8), and (9) by Theorem 4.7.

5. Scored case for a facet

Recall that a semigroup $\mathbb{N}A$ is said to be scored if

$$\mathbb{N}A = \bigcap_{\sigma\in\mathcal{F}} \{\boldsymbol{a} \in \mathbb{Z}A \,:\, F_\sigma(\boldsymbol{a}) \in F_\sigma(\mathbb{N}A)\}.$$

(See [14].) Clearly a normal semigroup is scored. Note that, if $\mathbb{N}A$ is scored, then $m_\tau = 1$ for all faces τ [11, Lemma 7.11]. In this section, we assume that $\mathbb{N}A$ is scored, and we give an explicit generator of $K[s]$-module $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\sigma}$ for a facet σ (Theorem 5.3).

Remark 5.1. In the scored case, we can refine some previous statements without changing proofs.

In Lemma 4.4, the condition $F_\sigma(\boldsymbol{a}) \leq -M$ can be replaced by the condition $-F_\sigma(\boldsymbol{a}) \in F_\sigma(\mathbb{N}A)$. In Lemma 4.5, the condition $F_\sigma(\boldsymbol{a} + \boldsymbol{b}) \leq -M$ can be replaced by the condition $-F_\sigma(\boldsymbol{a}+\boldsymbol{b}) \in F_\sigma(\mathbb{N}A)$. In Notation 4.6, we take $\boldsymbol{\beta}_\tau$ so that $-F_\sigma(\boldsymbol{\beta}_\tau) \in F_\sigma(\mathbb{N}A)$ instead of $F_\sigma(\boldsymbol{\beta}_\tau) \leq -M$; Theorem 4.7 is valid for this choice of $\boldsymbol{\beta}_\tau$.

Lemma 5.2. *Assume that* $\mathbb{N}A$ *is scored. Let* $\sigma \in \mathcal{F}(\boldsymbol{\beta})$. *Then*

(1) $S(\mathbb{I}(-\boldsymbol{\beta}_\sigma)) = \{(\boldsymbol{u}, \sigma) \mid \boldsymbol{u} \in \mathbb{N}^{\sigma^c},\ F_\sigma(A\boldsymbol{u}) \notin -F_\sigma(\boldsymbol{\beta}) + F_\sigma(\mathbb{N}A)\}$.

(2) $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_\sigma)) = \displaystyle\bigcap_{(\boldsymbol{u},\sigma)\in S(\mathbb{I}(-\boldsymbol{\beta}_\sigma))} \langle \theta_i - u_i \mid i \notin \sigma\rangle$.

(3)

$$\begin{aligned} N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\sigma} &\equiv t^{-\boldsymbol{\beta}_\sigma}(\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_\sigma)) : F_\sigma(A\theta)) \cdot F_\sigma(A\theta) \\ &\equiv t^{-\boldsymbol{\beta}_\sigma} \bigcap_{(\boldsymbol{u},\sigma)\in S(\mathbb{I}(-\boldsymbol{\beta}_\sigma)),\, \boldsymbol{u}\neq \boldsymbol{0}} \langle \theta_i - u_i \mid i \notin \sigma\rangle \cdot F_\sigma(A\theta). \end{aligned}$$

Here $\mathbb{N}^{\sigma^c} = \{\boldsymbol{u} \in \mathbb{N}^n \mid u_j = 0$ *for all* $\boldsymbol{a}_j \in \sigma\}$. *We sometimes write* $j \in \sigma$ *instead of* $\boldsymbol{a}_j \in \sigma$.

Proof. Recall from Notation 4.6 and Remark 5.1 that we have chosen $\boldsymbol{\beta}_\sigma$ so that $-F_{\sigma'}(\boldsymbol{\beta}_\sigma) \in F_{\sigma'}(\mathbb{N}A)$ for all $\sigma' \neq \sigma$.

(1) follows from Lemma 4.4 (2), and (2) follows from (1) and Theorem 2.2.

To prove (3), by renumbering if necessary, we assume that $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$ are linearly independent with $\boldsymbol{a}_1 \notin \sigma$ and $\boldsymbol{a}_2, \dots, \boldsymbol{a}_d \in \sigma$. Let $F_i(s)$ be a linear form such that, for $j \leq d$, $F_i(\boldsymbol{a}_j) \neq 0$ if and only if $i = j$. Take $F_1 = F_\sigma$. Then

$$\langle A\theta - (\boldsymbol{\beta} + \boldsymbol{\beta}_\sigma)\rangle = \langle F_i(A\theta) - F_i(\boldsymbol{\beta} + \boldsymbol{\beta}_\sigma) \mid i = 1, \dots, d\rangle.$$

Note that $F_\sigma(A\theta) = F_1(A\theta) \in \langle A\theta - (\boldsymbol{\beta}+\boldsymbol{\beta}_\sigma)\rangle$ since $F_\sigma(\boldsymbol{\beta}+\boldsymbol{\beta}_\sigma) = 0$ by definition. Hence $\supseteq$ of the first equality is clear, and $\supseteq$ of the second equality follows from (2).

Suppose that

$$\sum_{i=1}^{d} f_i(F_i(\theta) - F_i(\boldsymbol{\beta} + \boldsymbol{\beta}_\sigma)) \in \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_\sigma)). \tag{10}$$

Here and hereafter, we sometimes write $F(\theta)$ instead of $F(A\theta)$. Since $F_i(\theta)$ contains θ_i but not $\theta_j (j \neq i,\, j \leq d)$, we may assume that $f_i \in K[\theta_j \,|\, j \leq i \,\mathrm{or}\, j > d]$.

Let $(\boldsymbol{u}, \sigma)$ satisfy $\boldsymbol{u} \in \mathbb{N}^{\sigma^c}$ and $F_\sigma(\boldsymbol{u}) \notin -F_\sigma(\boldsymbol{\beta}) + F_\sigma(\mathbb{N}A)$. Then

$$\sum_{i=1}^{d} f_i(\boldsymbol{u}, \theta_\sigma)(F_i(\boldsymbol{u}, \theta_\sigma) - F_i(\boldsymbol{\beta} + \boldsymbol{\beta}_\sigma)) = 0, \tag{11}$$

where $F(\boldsymbol{u}, \theta_\sigma)$ denotes the function obtained from F by replacing θ_j by u_j for $j \notin \sigma$. By looking at the variables θ_i $(i = d, \ldots, 2)$, we see $f_i(\boldsymbol{u}, \theta_\sigma) = 0$ $(i = d, \ldots, 2)$. Hence $f_i \in \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_\sigma))$ for $i = d, \ldots, 2$. In turn, $f_1(\boldsymbol{u}, \theta_\sigma) F_1(\boldsymbol{u}, \theta_\sigma) = 0$. (Note that $F_1(\boldsymbol{\beta} + \boldsymbol{\beta}_\sigma) = 0$.) Since $F_1(\boldsymbol{u}, \theta_\sigma) = F_1(\boldsymbol{u}) = F_\sigma(A\boldsymbol{u})$, we have

$$f_1 \in \bigcap_{\substack{\boldsymbol{u} \in \mathbb{N}^{\sigma^c};\ \boldsymbol{u} \neq 0 \\ F_\sigma(A\boldsymbol{u}) \notin -F_\sigma(\boldsymbol{\beta}) + F_\sigma(\mathbb{N}A)}} \langle \theta_i - u_i \,|\, i \notin \sigma \rangle,$$

since, for $\boldsymbol{u} \in \mathbb{N}^{\sigma^c}$, $F_\sigma(A\boldsymbol{u}) \neq 0$ if and only if $\boldsymbol{u} \neq \boldsymbol{0}$. ☐

Theorem 5.3. *Assume that* $\mathbb{N}A$ *is scored. Let* $\sigma \in \mathcal{F}(\boldsymbol{\beta})$*. For* $j \notin \sigma$*, put*

$$m_j := m_{\sigma,j} = \max\{u_j \in \mathbb{N} \,|\, F_\sigma(\boldsymbol{a}_j) u_j \notin -F_\sigma(\boldsymbol{\beta}) + F_\sigma(\mathbb{N}A)\}.$$

Then

$$N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\sigma} \equiv t^{-\boldsymbol{\beta}_\sigma} \langle \prod_{j \notin \sigma} \prod_{k=1}^{m_j} (\theta_j - k) \rangle \cdot F_\sigma(A\theta)$$

Proof. By Lemma 5.2, it is enough to show

$$\bigcap_{\substack{\boldsymbol{u} \in \mathbb{N}^{\sigma^c},\ F_\sigma(A\boldsymbol{u}) \neq 0 \\ F_\sigma(A\boldsymbol{u}) \notin -F_\sigma(\boldsymbol{\beta}) + F_\sigma(\mathbb{N}A)}} \langle \theta_j - u_j \,|\, j \notin \sigma \rangle = \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_\sigma)) + \langle \prod_{j \notin \sigma} \prod_{k=1}^{m_j} (\theta_j - k) \rangle. \tag{12}$$

We know $\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_\sigma)) = \bigcap_{\boldsymbol{u} \in \mathbb{N}^{\sigma^c},\, F_\sigma(A\boldsymbol{u}) \notin -F_\sigma(\boldsymbol{\beta}) + F_\sigma(\mathbb{N}A)} \langle \theta_j - u_j \,|\, j \notin \sigma \rangle$ by Lemma 5.2. Since each ideal in (12) is generated by elements in $K[\theta_j \,|\, j \notin \sigma]$, we only need to check (12) in $K[\theta_j \,|\, j \notin \sigma]$. Then the zero set of each side of (12) equals the finite set $\{\boldsymbol{u} \neq \boldsymbol{0} \,|\, (\boldsymbol{u}, \sigma) \in S(\mathbb{I}(-\boldsymbol{\beta}_\sigma))\}$. By localizing at each zero point, we see that both sides are equal. ☐

Corollary 5.4. *Assume that* $\mathbb{N}A$ *is normal. Let* $\sigma \in \mathcal{F}(\boldsymbol{\beta})$*.*

Then

$$N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\sigma} \equiv t^{-\boldsymbol{\beta}_\sigma}\langle \prod_{j\notin\sigma} \prod_{k=1}^{\lfloor \frac{-F_\sigma(\boldsymbol{\beta})-1}{F_\sigma(\boldsymbol{a}_j)}\rfloor} (\theta_j - k)\rangle \cdot F_\sigma(A\theta)$$

Proof. In this case, the m_j in Theorem 5.3 equals $\lfloor\dfrac{-F_\sigma(\boldsymbol{\beta})-1}{F_\sigma(\boldsymbol{a}_j)}\rfloor$. □

6. Scored case for a simple face

In this section, we keep to assume that $\mathbb{N}A$ is scored. Furthermore we assume that a fixed face τ of $\mathbb{Q}_{\geq 0}A$ of codimension c satisfies

$$\{\sigma\in\mathcal{F}\,|\,\sigma\succeq\tau\} = \{\sigma\in\mathcal{F}(\boldsymbol{\beta})\,|\,\sigma\succeq\tau\} = \{\sigma_1,\sigma_2,\dots,\sigma_c\}. \tag{13}$$

Under these assumptions, we show that $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\tau}$ is generated by $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma_i}}$ $(1\leq i\leq c)$ (Theorem 6.3).

Change the order if necessary, and take $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_d\in A$ so that

$$\boldsymbol{a}_i \in \bigcap_{k=1,\,k\neq i}^{c} \sigma_k\setminus\sigma_i \qquad (i\leq c)$$

$$\boldsymbol{a}_{c+1},\dots,\boldsymbol{a}_d\in\tau \text{ is linearly independent.}$$

Put $F_i := F_{\sigma_i}$ for $i\leq c$, and take $F_{c+1},\dots,F_d\in\langle A\theta\rangle$ so that for $i,j\leq d$

$$F_i(\boldsymbol{a}_j)\neq 0 \Leftrightarrow i\neq j.$$

We prove that $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\tau}$ is generated by $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma_i}}$ $(1\leq i\leq c)$. For simplicity, put

$$\begin{aligned}
\mathfrak{i} &:= \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_\tau)),\\
\mathfrak{a} &:= \langle A\theta-\boldsymbol{\beta}-\boldsymbol{\beta}_\tau\rangle,\\
\mathfrak{f} &:= \sum_{i=1}^{c}\mathfrak{i}_{\sigma_i}\langle\prod_{j\notin\sigma_i}\prod_{k=1}^{m_{\sigma_i,j}}(\theta_j-k)\cdot F_i\rangle.
\end{aligned}$$

Here, for a facet $\sigma\succeq\tau$, we put

$$\mathfrak{i}_\sigma := \langle[\theta]_{\boldsymbol{u}}\,|\,\boldsymbol{u}\in\mathbb{N}^\sigma,\ F_{\sigma'}(A\boldsymbol{u})\in F_{\sigma'}(\boldsymbol{\beta}_\tau)+F_{\sigma'}(\mathbb{N}A)\,(\forall\sigma'\neq\sigma)\rangle.$$

We have, by Theorem 2.2 and Lemma 4.4 (2),

$$\begin{aligned}
\mathfrak{i} &= \langle[\theta]_{\boldsymbol{u}}\,|\,A\boldsymbol{u}\in\boldsymbol{\beta}_\tau+\mathbb{N}A\rangle\\
&= \langle[\theta]_{\boldsymbol{u}}\,|\,F_i(A\boldsymbol{u})\in -F_i(\boldsymbol{\beta})+F_i(\mathbb{N}A)\quad(1\leq i\leq c)\rangle
\end{aligned}$$

and

$$\mathfrak{i}_{\sigma_i} = \left\langle[\theta]_{\boldsymbol{u}}\,\middle|\,\begin{array}{l}\boldsymbol{u}\in\mathbb{N}^{\sigma_i},\\ F_k(A\boldsymbol{u})\in F_k(\boldsymbol{\beta}_\tau)+F_k(\mathbb{N}A)\,(\forall k\neq i,\,k\leq c)\end{array}\right\rangle, \tag{14}$$

since $-F_\sigma(\boldsymbol{\beta}_\tau) \in F_\sigma(\mathbb{N}A)$ for any $\sigma \not\preceq \tau$.

Lemma 6.1. $t^{-\boldsymbol{\beta}_\tau}\mathfrak{f} \subseteq \sum_{i=1}^{c} N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma_i}} D(K^n)$.

Proof. Let $\sigma = \sigma_i$ $(i = 1, \dots, c)$, and let $\boldsymbol{u} \in \mathbb{N}^\sigma$ satisfy $F_{\sigma'}(A\boldsymbol{u}) \in F_{\sigma'}(\boldsymbol{\beta}_\tau) + F_{\sigma'}(\mathbb{N}A)$ for all $\sigma' \neq \sigma$. Let $\boldsymbol{a} := \boldsymbol{\beta}_\tau - A\boldsymbol{u}$. Since $\boldsymbol{a}$ satisfies the condition for '$\boldsymbol{\beta}_\sigma$' (Notation 4.6 and Remark 5.1),

$$t^{-\boldsymbol{a}}\langle \prod_{j\notin\sigma} \prod_{k=1}^{m_{\sigma,j}} (\theta_j - k) \cdot F_\sigma \rangle = N_{-\boldsymbol{\beta}-\boldsymbol{a}}$$

by Theorem 5.3. Then by the proof of Theorem 4.7

$$N_{-\boldsymbol{\beta}-\boldsymbol{a}} \subseteq N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\sigma} D(K^n).$$

Hence we have

$$t^{-\boldsymbol{a}}\langle \prod_{j\notin\sigma} \prod_{k=1}^{m_{\sigma,j}} (\theta_j - k) \cdot F_\sigma \rangle \subseteq N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\sigma} D(K^n). \tag{15}$$

Multiplying (15) by $\partial^{\boldsymbol{u}}$, we have

$$t^{-\boldsymbol{\beta}_\tau}[\theta]_{\boldsymbol{u}} \prod_{j\notin\sigma} \prod_{k=1}^{m_{\sigma,j}} (\theta_j - k) \cdot F_\sigma \in N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\sigma} D(K^n).$$

□

By Lemma 6.1, we only need to prove $\mathfrak{i} \cap \mathfrak{a} = \mathfrak{f} + \mathfrak{i} \cdot \mathfrak{a}$. To this aim, we prove that

$$(\mathfrak{i} \cap \mathfrak{a})_\mathfrak{m} = (\mathfrak{f} + \mathfrak{i} \cdot \mathfrak{a})_\mathfrak{m} \tag{16}$$

for all maximal ideals $\mathfrak{m}$ of $R := K[\theta_1, \dots, \theta_n]$. In this argument, we extend the field K into its algebraic closure $\overline{K}$. We simply write K instead of $\overline{K}$. For $\boldsymbol{v} \in K^d$, let $\mathfrak{i}_{\boldsymbol{v}}$ be the localization of $\mathfrak{i}$ at the maximal ideal corresponding to $\boldsymbol{v}$. We have

$$\begin{aligned} \mathbb{V}(\mathfrak{i}) &:= \{\boldsymbol{v} \in K^d \mid \mathfrak{i}_{\boldsymbol{v}} \neq R_{\boldsymbol{v}}\} \\ &= \bigcup_{i=1}^{c} \bigcup_{F_i(A\boldsymbol{u})\notin -F_i(\boldsymbol{\beta})+F_i(\mathbb{N}A)} \boldsymbol{u} + K^{\sigma_i}. \end{aligned}$$

Proposition 6.2. $\mathfrak{i} \cap \mathfrak{a} = \mathfrak{f} + \mathfrak{i}\mathfrak{a}$.

Proof. If $\boldsymbol{v} \notin \mathbb{V}(\mathfrak{i}) \cap \mathbb{V}(\mathfrak{a})$, then $(\mathfrak{i} \cap \mathfrak{a})_{\boldsymbol{v}} = (\mathfrak{i}\mathfrak{a})_{\boldsymbol{v}}$.

Let $\boldsymbol{v} \in \mathbb{V}(\mathfrak{i}) \cap \mathbb{V}(\mathfrak{a})$, and let $\theta'_j := \theta_j - v_j$. By the definitions and Theorem 2.2,

$$\mathfrak{i}_{\boldsymbol{v}} = \langle \prod_{v_j\in\mathbb{N},\, v_j<u_j} \theta'_j \mid [\theta]_{\boldsymbol{u}}\text{: minimal in } \mathfrak{i} \rangle,$$

and

$$\mathfrak{f}_{\boldsymbol{v}} = \sum_{i=1}^{c} \langle \prod_{j\notin\sigma_i, v_j\in\mathbb{N}, 1\le v_j\le m_{\sigma_i,j}} \theta'_j \cdot F_i \cdot \prod_{v_j\in\mathbb{N}, v_j<u'_j} \theta'_j \mid [\theta]_{\boldsymbol{u}'}\colon \text{ minimal in } \mathfrak{i}_{\sigma_i}\rangle.$$

Note that $\mathfrak{i}_{\boldsymbol{v}}$ is a monomial ideal in the variables $\theta'_1, \dots, \theta'_n$.

Let

$$\sum_{i=1}^{d} f_i F_i(\theta') \in \mathfrak{i}_{\boldsymbol{v}}. \tag{17}$$

Note that, among $\theta_1, \dots, \theta_d$, the variable θ_i is the unique one appearing in F_i. Hence we may assume that

$f_1 \in K[\theta'_1, \theta'_{d+1}, \dots, \theta'_n], f_2 \in K[\theta'_1, \theta'_2, \theta'_{d+1}, \dots, \theta'_n], \dots, f_d \in K[\theta'_1, \dots, \theta'_n].$

By looking at the variable θ'_d in (17), we see

$$f_d\theta'_d \in \mathfrak{i}_{\boldsymbol{v}}.$$

If $[\theta]_{\boldsymbol{u}}$ is minimal in $\mathfrak{i}$, then $u_{c+1} = \cdots = u_d = 0$. Hence, if $c < d$, then we have $f_d \in \mathfrak{i}_{\boldsymbol{v}}$, $f_d F_d(\theta') \in (\mathfrak{i}\mathfrak{a})_{\boldsymbol{v}}$, and

$$\sum_{i=1}^{d-1} f_i F_i(\theta') \in \mathfrak{i}_{\boldsymbol{v}}. \tag{18}$$

Similarly we have $f_i F_i(\theta') \in (\mathfrak{i}\mathfrak{a})_{\boldsymbol{v}}$ for $c+1 \le i \le d$, and

$$\sum_{i=1}^{c} f_i F_i(\theta') \in \mathfrak{i}_{\boldsymbol{v}}. \tag{19}$$

By looking at the variable θ'_c in (19), we see $f_c\theta'_c \in \mathfrak{i}_{\boldsymbol{v}}$ and

$$f_c \in \langle \prod_{v_j\in\mathbb{N},\, v_j<u_j} \theta'_j \mid F_i(A\boldsymbol{u}) \in F_i(\boldsymbol{\beta}_\tau) + F_i(\mathbb{N}A)\ (\forall i < c)\rangle,$$

since $F_i(\boldsymbol{a}_c) = 0$ for all $i < c$.

Let $\boldsymbol{u}$ satisfy $F_i(A\boldsymbol{u}) \in F_i(\boldsymbol{\beta}_\tau) + F_i(\mathbb{N}A)$ for all $i < c$, and let

$$h := \prod_{v_j\in\mathbb{N},\, v_j<u_j} \theta'_j (= [\theta]_{\boldsymbol{u}} \text{ up to multiplication by a unit in } R_{\boldsymbol{v}}).$$

In what follows, we omit to write 'up to multiplication by a unit in $R_{\boldsymbol{v}}$'. Let $j \notin \sigma_c$. If $u_j > m_{\sigma_c,j}$, then $F_c(A\boldsymbol{u}) \in F_c(\boldsymbol{\beta}_\tau) + F_c(\mathbb{N}A)$, and $h = [\theta]_{\boldsymbol{u}} \in \mathfrak{i}_{\boldsymbol{v}}$. Suppose that $u_j \le m_{\sigma_c,j}$. Then $F_c(A\boldsymbol{u} + (m_{\sigma_c,j} + 1 - u_j)\boldsymbol{a}_j) \in F_c(\boldsymbol{\beta}_\tau) + F_c(\mathbb{N}A)$.

Note that for $k \ge 0$

$$h = [\theta]_{\boldsymbol{u}+k\boldsymbol{1}_j}$$

unless $u_j \le v_j < u_j + k$.

Hence, if there exists $j \notin \sigma_c$ such that the condition $u_j \leq v_j \leq m_{\sigma_c,j}$ does not hold, then

$$h = [\theta]_{\boldsymbol{u}+(m_{\sigma_c,j}+1-u_j)\mathbf{1}_j} \in \mathfrak{i}_{\boldsymbol{v}}.$$

Next suppose that $u_j \leq v_j \leq m_{\sigma_c,j}$ for all $j \notin \sigma_c$. Since $\boldsymbol{v} \in \mathbb{V}(\mathfrak{a})$, we have $F_c(A\boldsymbol{v}) = 0$. Hence $v_j = 0$ for all $j \notin \sigma_c$, and in turn $u_j = 0$ for all $j \notin \sigma_c$, or $\boldsymbol{u} \in \mathbb{N}^{\sigma_c}$. By (14), $h = [\theta]_{\boldsymbol{u}} \in (\mathfrak{i}_{\sigma_c})_{\boldsymbol{v}}$. Therefore

$$hF_c \in \mathfrak{f}_{\boldsymbol{v}}$$

by noting that $v_j = 0$ for all $j \notin \sigma_c$. In all cases, we have thus proved

$$hF_c \in (\mathfrak{f} + \mathfrak{i}\mathfrak{a})_{\boldsymbol{v}}.$$

Hence we have $f_cF_c \in (\mathfrak{f} + \mathfrak{i} \cdot \mathfrak{a})_{\boldsymbol{v}} \subseteq \mathfrak{i}_{\boldsymbol{v}}$, and $\sum_{i=1}^{c-1} f_iF_i \in \mathfrak{i}_{\boldsymbol{v}}$. Similarly, we obtain

$$f_iF_i \in (\mathfrak{f} + \mathfrak{i} \cdot \mathfrak{a})_{\boldsymbol{v}}$$

for $i = c-1, \dots, 1$. Hence $(\mathfrak{i} \cap \mathfrak{a})_{\boldsymbol{v}} \subseteq (\mathfrak{f} + \mathfrak{i}\mathfrak{a})_{\boldsymbol{v}}$. The other inclusion is clear. □

Theorem 6.3. *Assume that $\mathbb{N}A$ is scored, and that a face τ of $\mathbb{Q}_{\geq 0}A$ of codimension c satisfies* (13). *Then $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\tau}$ is generated by $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma_i}}$ $(1 \leq i \leq c)$.*

Proof. This is immediate from Lemma 6.1 and Proposition 6.2. □

Example 6.4. Let $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4)$, and

$$\boldsymbol{\beta} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = -\boldsymbol{a}_1 - \boldsymbol{a}_2 = -\boldsymbol{a}_3 - \boldsymbol{a}_4.$$

This example is normal but non-simplicial.

$$\begin{aligned} F_{13} &:= F_{\sigma_{13}} = s_2 = \theta_2 + \theta_4, \\ F_{23} &:= F_{\sigma_{23}} = s_1 = \theta_1 + \theta_4, \\ F_{14} &:= F_{\sigma_{14}} = s_2 + s_3 = \theta_2 + \theta_3, \\ F_{24} &:= F_{\sigma_{24}} = s_1 + s_3 = \theta_1 + \theta_3. \end{aligned}$$

Let $\boldsymbol{\beta}+\boldsymbol{\beta}_{14} := -\boldsymbol{a}_1 - \boldsymbol{a}_4$. Then $\boldsymbol{\beta}_{14} = {}^t(-1, 0, 1) = \boldsymbol{a}_3 - \boldsymbol{a}_1 = \boldsymbol{a}_2 - \boldsymbol{a}_4$. By Corollary 5.4, $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{14}}$ is generated by

$$t^{-\boldsymbol{\beta}_{14}} F_{14} = t^{-\boldsymbol{\beta}_{14}}(\theta_2 + \theta_3) = x_4\partial_2 + x_1\partial_3.$$

Each 1-dimensional face satisfies the condition (13).

We have $\boldsymbol{\beta}_0 := \boldsymbol{\beta}_{\{\mathbf{0}\}} = -\boldsymbol{\beta}$, and

$$\begin{aligned}\mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_0)) &= \langle \theta_2, \theta_3\rangle \cap \langle \theta_1, \theta_3\rangle \cap \langle \theta_2, \theta_4\rangle \cap \langle \theta_1, \theta_4\rangle \\ &= \langle \theta_1\theta_2, \theta_3\theta_4\rangle.\end{aligned}$$

In particular,

$$\theta_1\theta_2 - \theta_3\theta_4 = \theta_1(\theta_2+\theta_3) - \theta_3(\theta_1+\theta_4) \in \mathbb{I}(\widetilde{\Omega}(-\boldsymbol{\beta}_0)) \cap \langle A\theta\rangle,$$

and hence

$$N_{\mathbf{0}} \ni t^{-\boldsymbol{\beta}_0}(\theta_1\theta_2 - \theta_3\theta_4) = t^{\boldsymbol{\beta}}(\theta_1\theta_2 - \theta_3\theta_4) = \partial_1\partial_2 - \partial_3\partial_4.$$

Indeed $N_{\mathbf{0}}$ is generated by

$$\begin{aligned}t^{\boldsymbol{\beta}}\theta_1\theta_4(\theta_2+\theta_3) &= \theta_4\partial_1\partial_2 + \theta_1\partial_3\partial_4,\\ t^{\boldsymbol{\beta}}\theta_2\theta_4(\theta_1+\theta_3) &= \theta_4\partial_1\partial_2 + \theta_2\partial_3\partial_4,\\ t^{\boldsymbol{\beta}}\theta_1\theta_3(\theta_2+\theta_4) &= \theta_3\partial_1\partial_2 + \theta_1\partial_3\partial_4,\\ t^{\boldsymbol{\beta}}\theta_2\theta_3(\theta_1+\theta_4) &= \theta_3\partial_1\partial_2 + \theta_2\partial_3\partial_4,\\ t^{\boldsymbol{\beta}}(\theta_1\theta_2 - \theta_3\theta_4) &= \partial_1\partial_2 - \partial_3\partial_4.\end{aligned}$$

Hence by Theorems 4.7 and 6.3, N is generated by

$$\begin{aligned}t^{-\boldsymbol{\beta}_{14}}(\theta_2+\theta_3) &= x_4\partial_2 + x_1\partial_3,\\ t^{-\boldsymbol{\beta}_{24}}(\theta_1+\theta_3) &= x_4\partial_1 + x_2\partial_3,\\ t^{-\boldsymbol{\beta}_{13}}(\theta_2+\theta_4) &= x_3\partial_2 + x_1\partial_4,\\ t^{-\boldsymbol{\beta}_{23}}(\theta_1+\theta_4) &= x_3\partial_1 + x_2\partial_4,\\ t^{\boldsymbol{\beta}}(\theta_1\theta_2 - \theta_3\theta_4) &= \partial_1\partial_2 - \partial_3\partial_4.\end{aligned}$$

7. Simplicial scored case

Theorem 7.1. *Suppose that $\mathbb{N}A$ is scored and simplicial. Then N is generated by $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_\sigma}$ $(\sigma \in \mathcal{F}(\boldsymbol{\beta}))$. More explicitly, N is generated by*

$$t^{-\boldsymbol{\beta}_\sigma}\langle \prod_{j\notin\sigma}\prod_{k=1}^{m_{\sigma,j}}(\theta_j - k)\rangle \cdot F_\sigma(A\theta) \qquad (\sigma \in \mathcal{F}(\boldsymbol{\beta})).$$

Proof. This is clear from Theorems 4.7, 5.3, and 6.3. □

Corollary 7.2. *Suppose that $\mathbb{N}A$ is normal and simplicial. Then N is generated by*

$$t^{-\boldsymbol{\beta}_\sigma}\prod_{j\notin\sigma}\prod_{k=1}^{\lfloor\frac{-F_\sigma(\boldsymbol{\beta})-1}{F_\sigma(\boldsymbol{a}_j)}\rfloor}(\theta_j - k)\cdot F_\sigma(A\theta) \qquad (\sigma \in \mathcal{F}(\boldsymbol{\beta})).$$

Proof. This is immediate from Theorem 7.1 and Corollary 5.4. □

Corollary 7.3. *Suppose that $\mathbb{N}A$ is normal and simplicial. Assume that $F_\sigma(\boldsymbol{\beta}) = -1$ for all $\sigma \in \mathcal{F}(\boldsymbol{\beta})$. Then N is generated by*

$$t^{-\boldsymbol{\beta}_\sigma} F_\sigma(A\theta) \qquad (\sigma \in \mathcal{F}(\boldsymbol{\beta})).$$

Proof. This is immediate from Corollary 7.2. □

Example 7.4. Let $A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & n-1 & n \end{pmatrix} = (\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}, \boldsymbol{a}_n)$. Then

$$\begin{aligned} F_0 := F_{\sigma_0}(s) &= s_2 = \sum_{i=1}^{n} i\theta_i \\ F_n := F_{\sigma_n}(s) &= ns_1 - s_2 = \sum_{i=0}^{n-1} (n-i)\theta_i. \end{aligned}$$

This is normal and simplicial.

Let $\boldsymbol{\beta} = \begin{pmatrix} -2 \\ -n \end{pmatrix} = -\boldsymbol{a}_0 - \boldsymbol{a}_n$, $\boldsymbol{\beta}_0 := \boldsymbol{\beta}_{\sigma_0} = -\boldsymbol{a}_0 - \boldsymbol{\beta} = \boldsymbol{a}_n$, $\boldsymbol{\beta}_n := \boldsymbol{\beta}_{\sigma_n} = -\boldsymbol{a}_n - \boldsymbol{\beta} = \boldsymbol{a}_0$, $\boldsymbol{\beta}_{\boldsymbol{0}} := -\boldsymbol{\beta}$.

By Corollary 5.4, $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma_0}}$ is generated by

$$t^{-\boldsymbol{\beta}_0} \prod_{i=1}^{n-1} \prod_{k=1}^{\lfloor \frac{n-1}{i} \rfloor} (\theta_i - k) F_0, \tag{20}$$

and $N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\sigma_n}}$ is generated by

$$t^{-\boldsymbol{\beta}_n} \prod_{i=1}^{n-1} \prod_{k=1}^{\lfloor \frac{n-1}{n-i} \rfloor} (\theta_i - k) F_n. \tag{21}$$

By Corollary 7.2, N is generated by (20) and (21).

Example 7.5. Let $A = \begin{pmatrix} & & & -1 & 0 \\ & I_{d-1} & & \vdots & \vdots \\ & & & -1 & 0 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d, \boldsymbol{a}_{d+1})$, and $\boldsymbol{\beta} = -\boldsymbol{a}_{d+1}$. We have $\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_d = d\boldsymbol{a}_{d+1}$. This example is normal, homogeneous, simplicial, and reflexive; $F_\sigma(\boldsymbol{\beta}) = -1$ for all facets σ.

Let $\sigma_{\check{i}}$ be the facet not containing $\boldsymbol{a}_i$ $(1 \le i \le d)$. Put $F_{\check{i}} := F_{\sigma_{\check{i}}}$. We have

$$\begin{aligned} F_{\check{d}} &= s_d - (s_1 + \cdots + s_{d-1}) = d\theta_d + \theta_{d+1}, \\ F_{\check{i}} &= s_d - (\sum_{j=1, j\neq i}^{d-1} s_j) + (d-1)s_i = d\theta_i + \theta_{d+1} \quad (i < d). \end{aligned}$$

We have $F_\sigma(\boldsymbol{\beta}) = -1$ for all facets σ, and take $\boldsymbol{\beta}_{\check{i}} := \boldsymbol{\beta}_{\sigma_{\check{i}}}$ as follows:

$$\boldsymbol{\beta}_{\check{d}} = \begin{pmatrix} -1 \\ \vdots \\ -1 \\ 2-d \end{pmatrix}, \quad \boldsymbol{\beta}_{\check{1}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 2-d \end{pmatrix}, \cdots, \boldsymbol{\beta}_{\check{d-1}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 2-d \end{pmatrix}.$$

The vectors $-\boldsymbol{\beta}_{\check{1}}, \dots, -\boldsymbol{\beta}_{\check{d}}$ are the roots (e.g. see (2.10) in [6]). Since $m_{\check{d},d} = m_{\check{d},d+1} = 0$ and $-\boldsymbol{\beta}_{\check{d}} = (d-1)\boldsymbol{a}_{d+1} - \boldsymbol{a}_d = \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_{d-1} - \boldsymbol{a}_{d+1}$,

$$\begin{aligned} N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\check{d}}} &= t^{-\boldsymbol{\beta}_{\check{d}}}\langle F_{\check{d}}(\theta)\rangle = t^{-\boldsymbol{\beta}_{\check{d}}}\langle d\theta_d + \theta_{d+1}\rangle \\ &= \langle dx_{d+1}^{d-1}\partial_d + x_1 \cdots x_{d-1}\partial_{d+1}\rangle. \end{aligned}$$

Let $i < d$. Since $m_{\check{i},i} = m_{\check{i},d+1} = 0$ and $-\boldsymbol{\beta}_{\check{i}} = (d-1)\boldsymbol{a}_{d+1} - \boldsymbol{a}_i = \sum_{j=1,\, j\neq i}^{d} \boldsymbol{a}_j - \boldsymbol{a}_{d+1}$,

$$\begin{aligned} N_{-\boldsymbol{\beta}-\boldsymbol{\beta}_{\check{i}}} &= t^{-\boldsymbol{\beta}_{\check{i}}}\langle F_{\check{i}}(\theta)\rangle = t^{-\boldsymbol{\beta}_{\check{i}}}\langle d\theta_i + \theta_{d+1}\rangle \\ &= \langle dx_{d+1}^{d-1}\partial_i + (\prod_{j=1,\, j\neq i}^{d} x_j)\partial_{d+1}\rangle. \end{aligned}$$

Hence the left module counterpart $L^L(\boldsymbol{\beta})$ to $L(\boldsymbol{\beta})$ is described as

$$L^L(\boldsymbol{\beta}) = D(K^{d+1})/D(K^{d+1}) \left\langle \begin{array}{ll} d\theta_i + \theta_{d+1} + 1 & (i \le d) \\ dx_i\partial_{d+1}^{d-1} + x_{d+1}\prod_{j=1,\, j\neq i}^{d} \partial_j & (i \le d) \\ \partial_1 \cdots \partial_d - \partial_{d+1}^d & \end{array} \right\rangle,$$

which is the extended hypergeometric system considered in [5] and [6]. The rank of the A-hypergeometric system $M^L(\boldsymbol{\beta})$ equals the volume d.

Take the weight $(0, \dots, 0, 1)$, and consider a refined monomial order. Then the exponents of $M^L(\boldsymbol{\beta})$ are

$$(-i/d, \dots, -i/d, i-1) \qquad (i = 1, 2, \dots, d),$$

and

$$\phi_i = (x_1 \cdots x_d)^{-\frac{i}{d}} x_{d+1}^{i-1} \sum_{n=0}^{\infty} \frac{[-i/d]_n^d}{[dn+i-1]_{dn}} \left(\frac{x_{d+1}^d}{x_1 \cdots x_d}\right)^n$$

$(i = 1, 2, \ldots, d)$ form a fundamental basis (see [13, Chapters 2 and 3] for this argument).

Among them, $\phi_1, \ldots, \phi_{d-1}$ satisfy $L^L(\boldsymbol{\beta})$, but ϕ_d does not. Hence the rank of $L^L(\boldsymbol{\beta})$ equals $d-1$.

Take the weight $(1, \ldots, 1, 0)$, and consider a refined monomial order. Then the unique exponent is $(0, \ldots, 0, -1)$, and $L^L(\boldsymbol{\beta})$ has a fundamental basis consisting of log-series starting with

$$x_{d+1}^{-1}\left(\log \frac{x_1 \cdots x_d}{x_{d+1}^d}\right)^i \qquad (i = 0, 1, \ldots, d-2).$$

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