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§1. Introduction and notations

Let m denote the normalized Lebesgue measure on the unit circle $T = \{z \, ; \, |z| = 1\}$, that is $dm(e^{ix}) = dx/2\pi$. Let A be the disc algebra, that is, A is the algebra of all continuous functions on T whose negative Fourier coefficients vanish. For $0 < p < \infty$, the Hardy space H^p is the closure of A in the Lebesgue space $L^p = L^p(m)$, and H^∞ is the weak*-closure of A in $L^\infty = L^\infty(m)$. Let S be the singular integral operator defined by

$$Sf(\zeta) = \frac{1}{\pi i} \int_T \frac{f(z)}{z - \zeta}\, dz,$$

the integral being a Cauchy principal value (cf. [3, p.38]). If f is in L^1, then $Sf(\zeta)$ exists for almost everywhere ζ of T. We shall define the analytic projection P_+ and the co-analytic projection P_- by

$$P_+ = (I + S)/2, \quad P_- = (I - S)/2,$$

where I denotes the identity operator. Then,

$$(P_+ + P_-)f(\zeta) = f(\zeta), \quad (P_+ - P_-)f(\zeta) = Sf(\zeta).$$

For a function f in L^1, we shall define $\tilde{f}$ by

$$\tilde{f}(\zeta) = -i\{Sf(\zeta) - \int_T f\, dm\}.$$

By the calculation,

$$\tilde{f}(e^{i\theta}) = \int_T \cot\left(\frac{\theta - x}{2}\right) f(e^{ix})\, dm(e^{ix}),$$

the integral being a Cauchy principal value. A function Q in H^∞ is an inner function if $|Q| = 1$ a.e.. A function h is an outer function if there exists a real function t in L^1 and a real constant c such that $h = e^{t + i\tilde{t} + ic}$. For functions α and β in L^∞,

$$S_{\alpha,\beta} = \alpha P_+ + \beta P_- = \frac{\alpha+\beta}{2} I + \frac{\alpha-\beta}{2} S$$

is called a singular integral operator (cf.[17]). We shall define subspaces A_0 and H_0^p, $1 \leqq p \leqq \infty$, by

$$A_0 = \{f ; f \text{ is in } A, \text{ and } \int_T f\, dm = 0\},$$

$$H_0^p = \{f ; f \text{ is in } H^p, \text{ and } \int_T f\, dm = 0\}.$$

By $\bar{f}$ we denote the complex conjugate function of f. We shall define subspaces $\bar{A}_0$ and $\bar{H}_0^p$, by

$$\bar{A}_0 = \{\bar{f} ; f \text{ is in } A_0\}, \quad \bar{H}_0^p = \{\bar{f} ; f \text{ is in } H_0^p\}.$$

Suppose $1 \leqq p < \infty$ and W is a non-negative function in L^1. Then $L^p(W)$ is a weighted L^p space of m-measurable functions equipped with the norm

$$||f||_{p,W} = \{\int_T |f|^p\, Wdm\}^{1/p}.$$

The weighted Hardy space $H^p(W)$ (resp. $\bar{H}_0^p(W)$) is the norm closure of A (resp. $\bar{A}_0$) in $L^p(W)$. In this paper, we shall consider the case $p = 2$, and remain entirely in Hilbert spaces. We shall wright $||\cdot||_{2,W}$ as $||\cdot||_W$ for short. $L^2(W)$ is a Hilbert space equipped with the inner product

$$(f, g)_W = \int_T f\bar{g}\, Wdm.$$

We shall define the Helson-Szegö class (HS) as follows(cf.[18]).

$$(HS) = \{e^{u+\tilde{v}} ;\ u, v \in L^\infty, \text{ real},$$
$$|v| \leqq \pi/2 - \varepsilon, \text{ for some } \varepsilon > 0\}.$$

Historically, in the famous paper [18], H.Helson and G.Szegö proved that P_+ is continuous in the norm of $L^2(W)$ if and only if W is in (HS) or $W \equiv 0$. Since $S = S_{1,-1} = 2P_+ - I$, P_+ is continuous if and only if S is continuous. If $W \in$

(HS), then $W^{-1} \in$ (HS), and hence $W^{-1} \in L^1$. In the paper [21], P.Koosis proved that P_+ becomes a continuous operator from $L^2(W)$ to $L^2(U)$ for some non-zero and non-negative function U if and only if W^{-1} is in L^1. In this paper, we shall not distinguish between an operator's being bounded and being densely defined and extendable by continuity to a bounded operator.

M.Cotlar and C.Sadosky [7] got their lifting theorem, which is called the Cotlar-Sadosky theorem, and consider the condition of the operator $S = S_{1,-1}$ to be a bounded operator on $L^2(W)$ whose operator norm is equal to or less than M, for a given constant M. We gave the another proof of the Cotlar-Sadosky theorem in [44] and considered the condition of the singular integral operator $S_{\alpha,\beta}$ to be a contraction operator on $L^2(W)$ under the strong condition that α and $\bar{\beta}$ belong to H^∞. We have used the Hilbert space methods and the Cotlar-Sadosky theorem.

Prof.T.Nakazi and the author [28] gave the more satisfactory necessary and sufficient condition of $S_{\alpha,\beta}$ to be a contraction operator on $L^2(W)$ when α, β and W satisfy some weak condition.

In Section 2, we shall give the necessary and sufficient condition of $S_{\alpha,\beta}$ to be a contraction operator on $L^2(W)$ completely in general. We consider the weighted norm inequality

$$\| S_{\alpha,\beta} f \|_W \le \| f \|_W \quad (f \in A + \bar{A}_0),$$

where $A + \bar{A}_0 = \{f_1 + f_2 \; ; \; f_1 \in A, \; f_2 \in \bar{A}_0\}$, and get a class (HS)(r) with

$$r = r_{\alpha,\beta} = \left|\frac{\alpha - \beta}{1 - \alpha\bar{\beta}}\right|.$$

We define (HS)(r) in Section 2. Since the necessary and sufficient condition for $S_{\alpha,\beta}$ to be a contraction operator on $L^2(W)$ is not simple in general, we use a class (HS)(r) to describe it. Even when $W \equiv 1$ or $\beta \equiv 0$, our results contain new results. When $\beta \equiv 0$, we get a very simple necessary and sufficient condition for $S_{\alpha,\beta}$ to be a contraction operator on $L^2(W)$. That is, $S_{\alpha,0} = \alpha P_+$ is a contraction operator on $L^2(W)$ if and only if W belongs to $(HS)(|\alpha|)$ with $|\alpha| \leqq 1$ or $\alpha W \equiv 0$. This result was essentially given by Prof.T.Nakazi and the author in [28]. If r is a non-zero constant, then (HS)(r) becomes a subset of the union of the Helson-Szegö class (HS) and $\{0\}$.

In Section 3, we consider the (left) invertibility of $S_{\alpha,\beta}$ on $L^2(W)$. By the same method which we use to consider the weighted norm inequality

$$\|S_{\alpha,\beta}f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_0)$$

in Section 2, we shall give the necessary and sufficient condition of functions α, β and W to satisfy the reverse weighted norm inequality

$$\|S_{\alpha,\beta}f\|_W \geqq \|f\|_W \quad (f \in A + \bar{A}_0)$$

completely in general. By way of this inequality, we consider the (left) invertibility of $S_{\alpha,\beta}$ on $L^2(W)$. $S_{\alpha,\beta}$ is (left) invertible if and only if ess inf $\min\{|\alpha|, |\beta|\} > 0$ and $S_{\phi,1}$ is (left) invertible where $\phi = \alpha/\beta$. Hence we may assume $\beta = 1$. For a ϕ in L^∞, the singular integral operator $S_{\phi,1} = \phi P_+ +$

P_- is denoted by S_ϕ for short. When $W = 1$, H.Widom [43] and A.Devinatz [8] considered the (left) invertibility of T_ϕ and S_ϕ (cf. [9], [10, p.187], [31, p.371]). M.Shinbrot [39] considered the invertibility of S_ϕ on L^2 and derived the method for finding the inverse operator of S_ϕ. When W is in (HS), R.Rochberg [35] defined the Toeplitz operator T_ϕ on $H^p(W)$ by $T_\phi f = P_+(\phi f)$ for all f in $H^p(W)$, and got the necessary and sufficient condition for the invertibility of T_ϕ on $L^p(W)$ (cf. [3, p.216], [4]). Many generalizations of these results have been considered (cf. [3], [4], [6], [16], [17], [25], [27], [28], [29], [44], [45], [46]). If P_+ is continuous in the norm of $L^p(W)$, then T_ϕ is (left) invertible in $H^p(W)$ if and only if S_ϕ is (left) invertible in $L^p(W)$ (cf. [15, p.124], [31, p.393]). Prof.T.Nakazi and the author [29] gave a simple necessary and sufficient condition for α, β and W which satisfy the weighted norm inequality

$$\|S_{\alpha,\beta}f\|_W \geqq \delta\|f\|_W \quad (f \in A + \bar{A}_0),$$

for some positive constant δ, when $\operatorname{ess\,inf}|\alpha - \beta| > 0$. We shall study the weighted norm inequality

$$\|S_\phi f\|_W \geqq \delta\|f\|_W \quad (f \in A + \bar{A}_0),$$

by way of the weighted norm inequality

$$\|S_{\alpha,\beta}f\|_W \geqq \|f\|_W \quad (f \in A + \bar{A}_0).$$

Under the assumption that $W \in (HS)$ or $\operatorname{ess\,inf}|1 - \phi| > 0$, we can give a simple necessary and sufficient condition for the operator S_ϕ to be bounded below w.r.t. W. In spite of this assumption, the results in Section 3 cover the Widom-Devinatz-Rochberg theorem for $p = 2$.

In Section 4, we do not assume $W \in (HS)$ or $\operatorname{ess\,inf}|1 - \phi| > 0$. It is remarkable in this case that the condition of ϕ and W satisfying the weighted norm inequality

$$\min\{||S_\phi f||_W, \ ||S_{-\phi} f||_W\} \geq \delta ||f||_W \quad (f \in A + \bar{A}_0),$$

which implies that both S_ϕ and $S_{-\phi}$ are bounded below w.r.t. W, becomes simple. If $W \in (HS)$ and S_ϕ is bounded below w.r.t. W, then $S_{-\phi}$ is also bounded below w.r.t. W. Hence the results in Section 4 also cover the Widom-Devinatz-Rochberg theorem for $p = 2$. We studied the L^2-type (left) invertibility of S_ϕ in [45].

This paper is based on the author's papers [45] and [46].

§2. Boundedness of $S_{\alpha,\beta}$ and its norm

We shall consider the condition of the singular integral operator $S_{\alpha,\beta}$ to be a contraction operator on the weighted space $L^2(W)$. Even when $W = 1$ or $\beta = 0$, Theorem A involves new results. Theorem A involves not only the Helson-Szegö theorem but also new results. In this paper, the Cotlar-Sadosky theorem (cf. [2], [7]) is essential and is used several times. We have given the another proof of this theorem in [44]. If $W_1 = W_2 = W_3 \geqq 0$, then (2) becomes the Schwarz inequality, and (3) holds with $k = 0$. We shall use the equivalence of (1) and (3).

Cotlar-Sadosky theorem. Suppose W_1 and W_2 are real functions in L^1, and W_3 is a complex function in L^1. Then the following conditions are mutually equivalent.

(1) For all $f_1 \in A$ and $f_2 \in \bar{A}_o$,

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2\mathrm{Re}(f_1 \bar{f}_2 W_3)\} dm \geqq 0.$$

(2) $W_1 \geqq 0$, $W_2 \geqq 0$, and for all $f_1 \in A$ and $f_2 \in \bar{A}_o$,

$$\left|\int_T f_1 \bar{f}_2 W_3 dm\right| \leqq \left\{\int_T |f_1|^2 W_1 dm\right\}^{1/2} \left\{\int_T |f_2|^2 W_2 dm\right\}^{1/2}.$$

(3) $W_1 \geqq 0$, $W_2 \geqq 0$, and there exists a k in H^1 such that

$$|W_3 - k|^2 \leqq W_1 W_2.$$

We shall consider the boundedness of the singular integral operator $S_{\alpha,\beta}$ on the weighted space $L^2(W)$. The following classes (HS) and (HS)(r) are useful to consider this problem. Some properties of (HS) and (HS)(r) are given in series of Propositions 2.1-2.5.

Definition 2.1. We shall define the Helson-Szegö class (HS) as follows (cf. [14, p.147], [22, p.226], [31, p.197]).

$$(HS) = \{e^{u+\tilde{v}} \ ;\ u, v \in L^\infty \text{ real}, \ |v| \leqq \pi/2 - \varepsilon, \text{ for some } \varepsilon > 0\}.$$

Definition 2.2. For a non-negative function r, we shall define (HS)(r) as follows.

$$(HS)(r) = \{Ce^{u+\tilde{v}} \ ;\ C \text{ is a non-negative constant}, \ u, v : \text{real functions}, \ u \in L^1, \ |v| \leqq \pi/2, \text{ and } r^2e^u + e^{-u} \leqq 2(\cos v)\}.$$

When $0 < r \leqq 1$, by the calculation, we have

$$(HS)(r) = \{Ce^{u+\tilde{v}} \ ;\ C \text{ is a non-negative constant}, \ u, v : \text{real functions}, \ u \in L^1, \ |v| \leqq \cos^{-1} r, \text{ and } |u| \leqq \cosh^{-1}\{(\cos v)/r\} \}.$$

where $y = \cos^{-1}x$ implies $x = \cos y$, and

$$\cosh^{-1}x = \log\{x + (x^2 - 1)^{1/2}\}.$$

In the following propositions, we shall give the basic properties of the set $(HS)(r)$.

Proposition 2.1. The following statements are true.

(a) If $(HS)(r)$ is not empty, then $r \leqq 1$.

(b) If r_1 and r_2 are measurable functions satisfying $0 \leqq r_1 \leqq r_2 \leqq 1$, then

$$(HS)(1) \subset (HS)(r_2) \subset (HS)(r_1) \subset (HS)(0).$$

(c) $(HS) \subset (HS)(0)$.

Proof. We shall prove (a). Suppose W is in $(HS)(r)$. Then there exists a non-negative constant C and real functions u, v such that $W = Ce^{u + \tilde{v}}$, $u \in L^1$, $|v| \leqq \pi/2$, and

$$r^2e^u + e^{-u} \leqq 2(\cos v).$$

Since $2r \leqq r^2e^u + e^{-u}$, $r \leqq \cos v \leqq 1$. We shall prove (b). Suppose W is in $(HS)(r_2)$. Then there exists a non-negative constant C and real functions u, v such that $u \in L^1$, $|v| \leqq \pi/2$, $W = Ce^{u + \tilde{v}}$ and

$$r_2{}^2e^u + e^{-u} \leqq 2(\cos v).$$

Since $r_1 \leqq r_2$,

$$r_1{}^2e^u + e^{-u} \leqq 2(\cos v).$$

Hence W is in $(HS)(r_1)$. We shall prove (c). Suppose W is in (HS). Then there exists a non-negative constant C and real functions u, v such that $u \in L^\infty$, $|v| \leqq \pi/2 - \varepsilon$, for some ε

> 0, and $W = Ce^{u + \tilde{v}}$. Let $\delta = ||u||_\infty - \log(\cos ||v||_\infty)$, so that

$$-(u + \delta) \leqq ||u||_\infty - \delta = \log(\cos ||v||_\infty)$$
$$\leqq \log(\cos v) \leqq \log(2 \cos v).$$

Hence $W = (Ce^{-\delta})e^{(u + \delta) + \tilde{v}}$, and $e^{-(u + \delta)} \leqq 2(\cos v)$. Hence W is in $(HS)(0)$.

Proposition 2.2. The following statements are true.

(a) If W is in $(HS)(r)$ for some function r satisfying $0 \leqq r \leqq 1$, then r^2W is in L^1.

(b) If W is a non-zero function in $(HS)(0)$, then $\log W$ and W^{-1} are in L^1.

Proof. We shall prove (a). Suppose W is in $(HS)(r)$. Then there exists a non-negative constant C and real functions u, v such that $u \in L^1$, $|v| \leqq \pi/2$, $W = Ce^{u + \tilde{v}}$ and

$$r^2e^u + e^{-u} \leqq 2(\cos v).$$

Hence,

$$r^2W = Cr^2e^{u + \tilde{v}} \leqq 2Ce^{\tilde{v}}(\cos v).$$

Since $|v| \leqq \pi/2$, $e^{\tilde{v}}(\cos v)$ is in L^1 (cf. [14, p.161]). Hence r^2W is in L^1. We shall prove (b). Suppose W is a non-zero function in $(HS)(0)$. Then there exists a positive constant C

and real functions u, v such that $u \in L^1$, $|v| \leqq \pi/2$, $W = Ce^{u+\tilde{v}}$ and

$$e^{-u} \leqq 2(\cos v).$$

Since

$$\log W = \log C + u + \tilde{v},$$

$\log W$ is in L^1. Since

$$W^{-1} = C^{-1}e^{-u-\tilde{v}} \leqq 2C^{-1}e^{-\tilde{v}}(\cos v),$$

and $e^{-\tilde{v}}(\cos v)$ is in L^1 (cf. [14, p.161]), W^{-1} is in L^1.

Proposition 2.3. The following conditions are mutually equivalent.

(1) W is in (HS) or $W \equiv 0$.

(2) W is in (HS)(r) for some constant r satisfying $0 < r \leqq 1$.

Proof. We shall show that (1) implies (2). Suppose W is in (HS). Then there exists a non-negative constant C and real functions u, v such that $u \in L^\infty$, $|v| \leqq \pi/2 - \varepsilon$, for some $\varepsilon > 0$, and $W = Ce^{u+\tilde{v}}$. Let $r = \text{ess inf}(e^{-|u|} \cos v)$, so that r is a positive constant satisfying

$$e^{u} + e^{-u} \leqq 2r^{-1}(\cos v).$$

Let $u' = u - \log r$, so that

$$r^2e^{u'} + e^{-u'} \leqq 2(\cos v), \text{ and}$$

$$W = (Cr)e^{u'+\tilde{v}}.$$

Hence W is in (HS)(r). We shall show that (2) implies (1). Suppose W is in (HS)(r). Then there exists a non-negative constant C and real functions u, v such that $u \in L^1$, $|v| \leqq \pi/2$, $W = Ce^{u+\tilde{v}}$ and

$$r^2 e^u + e^{-u} \leqq 2(\cos v).$$

Since $2r \leqq r^2e^u + e^{-u}$, $r \leqq \cos v$. Since $|v| \leqq \pi/2$, $|v| \leqq \cos^{-1} r$. Since r is a positive constant, $|v| \leqq \pi/2 - \varepsilon$ for some $\varepsilon > 0$. Since $r^2e^u + e^{-u} \leqq 2$ and r is a positive constant,

$$-\log 2 \leqq u \leqq \log 2 - 2(\log r).$$

Hence u is in L^∞.

Proposition 2.4. For a non-negative function W in L^1, the following conditions are mutually equivalent.

(1) W is in (HS)(0).

(2) $W = 0$ or W^{-1} is in L^1.

(3) W is in (HS)(r) for some function r satisfying $0 < r \leqq 1$.

Proof. By Proposition 2.2(b), (1) implies (2). By Proposition 2.1(b), (3) implies (1). We shall show that (2) implies (1). Suppose W and W^{-1} are in L^1. Then $\log W$ is in L^1. Let

$$f = \{W^{-1} + i(W^{-1})^{\sim}\}^{-1},$$

so that $0 \leqq \mathrm{Re}\, f \leqq |f| \leqq W$. Since W is in L^1, f is in H^1. Hence there exists a positive constant C and a function v such that $|v| \leqq \pi/2$ and

$$f = Ce^{\tilde{v} - iv}.$$

Then $W = Ce^{\tilde{v}}/(\cos v)$. Let $u = \log W - \tilde{v} - \log C$, so that u is in L^1, and $W = Ce^{u + \tilde{v}}$. Hence $e^{-u} = \cos v \leqq 2(\cos v)$. We shall show that (1) implies (3). Suppose W is a non-zero function in (HS)(0). Then there exists a positive constant C and real functions u, v such that $u \in L^1$, $|v| \leqq \pi/2$, $W = Ce^{u + \tilde{v}}$ and $e^{-u} \leqq 2(\cos v)$. Since

$$(e^{-u}/2)^2 \leqq e^{-u}(\cos v)/2 \leqq (\cos v)^2,$$

there exists a function r such that $r^2 = e^{-u}(\cos v)/2$ and $0 < e^{-u}/2 \leqq r \leqq \cos v \leqq 1$. Let $u' = u + \log 2$, so that

$$r^2 e^{u'} + e^{-u'} = \cos v + e^{-u}/2 \leqq 2(\cos v),$$

and $W = Ce^{u + \tilde{v}} = (C/2)e^{u' + \tilde{v}}$.

Proposition 2.5. The following conditions are mutually equivalent.

(1) W is in (HS)(1).

(2) W is a non-negative constant.

(3) W is in (HS)(r) for any constant r satisfying $0 \leqq r < 1$.

Proof. By Proposition 2.1(b), (1) implies (3). It is clear that (2) implies (1). We shall show that (3) implies (2). We shall assume that W is not identically zero. By Proposition

2.1(b) and Proposition 2.2(b), $\log W$ is in L^1. Hence $W > 0$. For any positive constant ε, there exists a constant r satisfying $1 - \varepsilon < r < 1$ and

$$\cosh^{-1}(r^{-1}) + \log r^{-1} + \cos^{-1} r < \varepsilon .$$

By (3), W is in (HS)(r). Hence there exists a positive constant C and real functions u, v such that $W = Ce^{u + \tilde{v}}$, $u \in L^1$, $|v| \leqq \pi/2$, and

$$r^2 e^u + e^{-u} \leqq 2(\cos v).$$

Hence,

$$|u + \log r| \leqq \cosh^{-1}(r^{-1}\cos v) \leqq \cosh^{-1}(r^{-1}),$$

and $|v| \leqq \cos^{-1} r \leqq \pi/2$. Hence u and v are in L^∞. Then,

$$\int_T |\tilde{v}|^2 dm = \int_T |v|^2 dm - |\int_T v\, dm|^2 \leqq \int_T |v|^2 dm \leqq (\cos^{-1} r)^2,$$

(cf. [14, p.108]). Hence,

$$\{\int_T |u + \tilde{v}|^2 dm\}^{1/2} \leqq \{\int_T |u|^2 dm\}^{1/2} + \{\int_T |\tilde{v}|^2 dm\}^{1/2}$$

$$\leqq \cosh^{-1}(r^{-1}) + \log r^{-1} + \cos^{-1} r < \varepsilon .$$

Since $\log W - \log C = u + \tilde{v}$, we have

$$\int_T |\log W - \log C|^2 dm < \varepsilon^2 ,$$

for any positive constant ε. This implies $W = C$.

We shall consider the the condition of the operator $S_{\alpha,\beta}$ to be a contraction operator on $L^2(W)$, that is,

$$\|S_{\alpha,\beta} f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_o).$$

Suppose α, β are functions in L^∞, and W is a non-negative function in L^1 such that $\alpha W = \beta W$. In this case, $S_{\alpha,\beta}$ becomes a multiplication operator on $L^2(W)$, and hence the condition of the boundedness of $S_{\alpha,\beta}$ on $L^2(W)$ is simple as follows. Since

$$|S_{\alpha,\beta} f|^2 W = |\alpha W^{1/2} P_+ f + \beta W^{1/2} P_- f|^2 = |\alpha f|^2 W,$$

we have

$$\|S_{\alpha,\beta} f\|_W = \|\alpha f\|_W \quad (f \in A + \bar{A}_o).$$

Then the following conditions are mutually equivalent.

(1) $\|S_{\alpha,\beta} f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_o)$.

(2) $\|\alpha f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_o)$.

(3) $|\alpha| W \leqq W$.

Suppose α, β are functions in L^∞, and W is a non-negative function in L^1 such that $(\alpha - \beta)W$ is not identically zero. In this case, the condition of the operator $S_{\alpha,\beta}$ to be a contraction operator on $L^2(W)$ is not simple. We shall use a class $(HS)(r_{\alpha,\beta})$.

Definition 2.3. For functions α and β in L^∞ satisfying $|1 - \alpha\bar{\beta}| > 0$,

$$r_{\alpha,\beta} = \left|\frac{\alpha - \beta}{1 - \alpha\bar{\beta}}\right|.$$

We shall give some lemmas to prove Theorem A. We shall use Lemma A to prove Lemma B. Lemma A is is a original result in this paper. In it's proof, we use the inner-outer factorization theorem. Similar results are given in [46].

Main Lemma A.([46]) Suppose r and F are measurable functions such that $r \geqq 0$, F is in L^1, and rF is not identically zero. Then the following conditions are mutually equivalent.

(1) There exists a function k in H^1 such that

$$|F - k|^2 \leqq (1 - r^2)|F|^2.$$

(2) $|F| > 0$, $r \leqq 1$, and there exists an inner function Q and a real function t in L^1 such that

$$F/|F| = Qe^{i\tilde{t}}, \quad \text{and} \quad |F|e^{-t} \in (HS)(r).$$

Proof. We shall show that (1) implies (2). If $k = 0$, then (1) implies $rF = 0$. This contradiction implies that k is non-zero. Hence $\log|k|$ is in L^1. Since $|F - k| \leqq |F|$, $|k| \leqq 2|F|$. Since $\log|k|$ is in L^1, $\log|F|$ is also in L^1. Let

$$u = \log|F/k|, \quad \text{and} \quad v = \operatorname{Arg}(F/k),$$

where $-\pi \leqq \operatorname{Arg} z < \pi$. Then u is in L^1, and

$$e^{u + iv} = F/k.$$

Since $|F| > 0$, by (1),

$$0 \leqq |1 - k/F|^2 \leqq 1 - r^2.$$

Hence $r \leqq 1$, and

$$|1 - e^{-(u + iv)}|^2 \leqq 1 - r^2.$$

Hence,

$$r^2 e^u + e^{-u} \leqq 2(\cos v).$$

Since $2r \leqq r^2 e^u + e^{-u}$, this implies $r \leqq \cos v$. Since $|v| \leqq \pi$, $|v| \leqq \cos^{-1} r \leqq \pi/2$. Hence $e^{u + \tilde{v}}$ is in (HS)(r). Since k is a non-zero function in H^1, by the inner-outer factorization theorem, there exists an inner function Q such that

$$k = Qe^{\log|k| + i(\log|k|)^{\sim}}.$$

Since $F/k = e^{iv}|F/k|$,

$$F/|F| = Qe^{i\{(\log|k|)^{\sim} + v\}}.$$

Let $t = \log|k| - \tilde{v}$, so that t is in L^1, and $\tilde{t} = (\log|k|)^{\sim} + v - c$, for some real constant c. Hence

$$F/|F| = (Qe^{ic})e^{i\tilde{t}}.$$

Since $u = \log|F/k|$ and $t = \log|k| - \tilde{v}$,

$$|F| = e^u|k| = e^{t + u + \tilde{v}}$$

Since $e^{u + \tilde{v}}$ is in (HS)(r), $|F|e^{-t}$ is in (HS)(r). We shall show that (2) implies (1). Since $|F|e^{-t}$ is in (HS)(r), there exists a positive constant C, real functions u, v such that $u \in L^1$, $|v| \leqq \pi/2$,

$$r^2e^u + e^{-u} \leqq 2(\cos v), \quad \text{and} \quad |F|e^{-t} = Ce^{u + \tilde{v}}.$$

Let $k = Fe^{-(u + iv)}$, so that

$$\begin{aligned} &|F - k|^2 - (1 - r^2)|F|^2 \\ &= |F|^2\{|1 - e^{-(u + iv)}|^2 - (1 - r^2)\} \\ &= |F|^2e^{-u}\{r^2e^u + e^{-u} - 2(\cos v)\} \leqq 0. \end{aligned}$$

Since $|k| \leqq 2|F|$, k is in L^1. Since $F = Qe^{i\tilde{t}}|F|$,

$$k = Qe^{i\tilde{t}}|F|e^{-(u + iv)} = CQe^{t + i\tilde{t} + \tilde{v} - iv}.$$

Hence k is in H^1. This completes the proof.

We shall use Lemma 2.1 and Lemma B to prove Theorem A.

Lemma 2.1. If $\max\{|\alpha|, |\beta|\} \leqq 1$ or $\min\{|\alpha|, |\beta|\} \geqq 1$, then $r_{\alpha,\beta} \leqq 1$.

Proof. $|1 - \alpha\beta|^2 - |\alpha - \beta|^2 = (1 - |\alpha|^2)(1 - |\beta|^2) \geqq 0$.

Main Lemma B.([46]) Suppose α, β are functions in L^∞, and W is a non-negative function in L^1 such that $(\alpha - \beta)W$ is not identically zero. Then the following conditions are mutually equivalent.

(1) There exists a function k in H^1 such that

$$|(1-\alpha\bar{\beta})W - k|^2 \leqq (1-|\alpha|^2)(1-|\beta|^2)W^2.$$

(2) $|1 - \alpha\bar{\beta}|W > 0$, $r_{\alpha,\beta} \leqq 1$, and there exists an inner function Q and a real function t in L^1 such that

$$(1-\alpha\bar{\beta})/|1-\alpha\bar{\beta}| = Qe^{i\tilde{t}}, \text{ and}$$

$$|1-\alpha\bar{\beta}|We^{-t} \in (HS)(r_{\alpha,\beta}).$$

Proof. Let $r = r_{\alpha,\beta}$ and $F = (1-\alpha\bar{\beta})W$. Since $r|F| = |\alpha - \beta|W$, rF is not identically zero. By Lemma A, (1) holds if and only if $|F| > 0$, $r \leqq 1$, and there exists an inner function Q and a real function t in L^1 such that

$$F/|F| = Qe^{i\tilde{t}},$$

and $|F|e^{-t}$ is in $(HS)(r)$. Since $|F| > 0$, $|1-\alpha\bar{\beta}|W > 0$. Since $W > 0$,

$$(1-\alpha\bar{\beta})/|1-\alpha\bar{\beta}| = Qe^{i\tilde{t}}.$$

Hence (1) and (2) are equivalent. This completes the proof.

The following theorem is the main theorem in this section. Even when W is a constant function, Theorem A contains new results. When $(\alpha - \beta)W$ is not identically zero, we shall consider the problem of finding the condition of $S_{\alpha,\beta}$ to be a contraction operator in $L^2(W)$. Theorem A follows immediately from the Cotlar-Sadosky theorem, Lemma 2.1 and Lemma B.

Main Theorem A.([46]) Suppose α, β are functions in L^∞, and W is a non-negative function in L^1 such that $(\alpha - \beta)W$ is not identically zero. Then the following conditions are mutually equivalent.

(1) $\|S_{\alpha,\beta} f\|_W \leq \|f\|_W \quad (f \in A + \bar{A}_0)$.

(2) $|1 - \alpha\bar{\beta}|W > 0$, $\max\{|\alpha|, |\beta|\} \leq 1$, and there exists an inner function Q and a real function t in L^1 such that

$$(1 - \alpha\bar{\beta})/|1 - \alpha\bar{\beta}| = Qe^{i\tilde{t}}, \text{ and}$$

$$|1 - \alpha\bar{\beta}|We^{-t} \in (HS)(r_{\alpha,\beta}).$$

Proof. Let $W_1 = (1 - |\alpha|^2)W$, $W_2 = (1 - |\beta|^2)W$, and $W_3 = (1 - \alpha\bar{\beta})W$. By the Cotlar-Sadosky theorem, (1) is equivalent to the condition that $W_1 \geq 0$, $W_2 \geq 0$ and there exists a k in H^1 such that $|W_3 - k|^2 \leq W_1 W_2$, that is,

$$|(1 - \alpha\bar{\beta})W - k|^2 \leq (1 - |\alpha|^2)(1 - |\beta|^2)W^2.$$

By Lemma 2.1 and Lemma B, this is equivalent to (2). This completes the proof.

By Theorem A, we prove Theorem 2.1. By Theorem 2.1, we prove Theorem 2.2. By Theorem 2.2, we prove Theorem 2.3. Theorem 2.4 follows immediately from Theorem 2.3. Theorem 2.3 and Theorem 2.4 were essentially given by Prof.T.Nakazi and the author in [28].

Theorem 2.1. Suppose α, β are functions in L^∞, and W is a non-negative function in L^1 such that $(\alpha - \beta)W$ is not identically zero. Suppose there exists a real function s in L^2 such that

$$1 - \alpha\bar{\beta} = e^{is}|1 - \alpha\bar{\beta}|,$$

and $|1 - \alpha\bar{\beta}|We^{\tilde{s}}$ is in L^1. Then, the following conditions are mutually equivalent.

(1) $\|S_{\alpha,\beta} f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_0)$.

(2) $|1 - \alpha\bar{\beta}|W > 0$, $\max\{|\alpha|, |\beta|\} \leqq 1$, and

$$|1 - \alpha\bar{\beta}|We^{\tilde{s}} \in (HS)(r_{\alpha,\beta}).$$

Proof. We shall show that (1) implies (2). By Theorem A, $|1 - \alpha\bar{\beta}|W > 0$, $\max\{|\alpha|, |\beta|\} \leqq 1$, and there exists an inner function Q and a real function t in L^1 such that

$$(1 - \alpha\bar{\beta})/|1 - \alpha\bar{\beta}| = Qe^{i\tilde{t}}, \text{ and}$$

$$|1 - \alpha\bar{\beta}|We^{-t} \in (HS)(r_{\alpha,\beta}).$$

Since $Qe^{i\tilde{t}} = e^{is}$, $Qe^{t + \tilde{s} + i(\tilde{t} - s)} = e^{t + \tilde{s}}$. By Proposition 2.2(c), $e^t/(|1 - \alpha\bar{\beta}|W)$ is in L^1. Hence,

$$\int_T e^{(t + \tilde{s})/2}\, dm$$

$$\leqq \left\{\int_T |1 - \alpha\bar{\beta}|We^{\tilde{s}}\, dm\right\}^{1/2}\left\{\int_T e^t/(|1 - \alpha\bar{\beta}|W)\, dm\right\}^{1/2} < \infty.$$

Hence $e^{t+\tilde{s}}$ is a non-negative function in $H^{1/2}$. By the Neuwirth-Newman theorem (cf. [30]), there exists a positive constant C such that $e^{t+\tilde{s}} = C$. Hence,

$$|1-\alpha\bar{\beta}|We^{\tilde{s}} = C|1-\alpha\bar{\beta}|We^{-t} \in (HS)(r_{\alpha,\beta}).$$

We shall show that (2) implies (1). Let $t = -\tilde{s}$, so that t is in L^1, and there exists a constant c such that $s = \tilde{t} + c$. Let $Q = e^{ic}$, so that Q is an inner function. Then,

$$(1-\alpha\bar{\beta})/|1-\alpha\bar{\beta}| = e^{is} = Qe^{i\tilde{t}}, \text{ and}$$

$$|1-\alpha\bar{\beta}|We^{-t} = |1-\alpha\bar{\beta}|We^{\tilde{s}} \in (HS)(r_{\alpha,\beta}).$$

By Theorem A, this implies (1).

Theorem 2.2 was given by Prof. T. Nakazi and the author (cf. [28]). We shall give the another proof of Theorem 2.2 using Theorem 2.1.

Theorem 2.2. Suppose α, β are functions in L^∞, and W is a non-negative function in L^1 such that $(\alpha-\beta)W$ is not identically zero. Suppose there exists a real function s in L^2 such that

$$1-\alpha\bar{\beta} = e^{is}|1-\alpha\bar{\beta}|,$$

and $|1-\alpha\bar{\beta}|We^{\tilde{s}}$ is in L^1. Then, the following conditions are mutually equivalent.

(1) $\|S_{\alpha,\beta} f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_0)$.

(2) $|1 - \alpha\bar{\beta}|W > 0$, $\max\{|\alpha|, |\beta|\} \leqq 1$, and there exists a positive constant C, and real functions u', v such that

$$W = C\{|1 - \alpha\bar{\beta}|^{-1}\chi_{\{\alpha=\beta\}} + |\alpha - \beta|^{-1}\chi_{\{\alpha\neq\beta\}}\}e^{u' + \tilde{v} - \tilde{s}},$$

$$|v| \leqq \cos^{-1} r_{\alpha,\beta},$$

$$|u'| \leqq \cosh^{-1}\{(\cos v)/r_{\alpha,\beta}\} \quad \text{on} \quad \{\alpha \neq \beta\}, \quad \text{and}$$

$$-\log(2\cos v) \leqq u' \quad \text{on} \quad \{\alpha = \beta\}.$$

Proof. We shall show that (1) implies (2). By Theorem 2.1, $|1 - \alpha\bar{\beta}|W > 0$, $\max\{|\alpha|, |\beta|\} \leqq 1$, and $|1 - \alpha\bar{\beta}|We^{\tilde{s}} \in (HS)(r_{\alpha,\beta})$. Hence there exists a positive constant C, real functions u, v such that $u \in L^1$, $|v| \leqq \pi/2$,

$$r^2 e^u + e^{-u} \leqq 2(\cos v), \quad \text{and}$$

$$|1 - \alpha\bar{\beta}|We^{\tilde{s}} = Ce^{u + \tilde{v}}.$$

Since $2r \leqq r^2 e^u + e^{-u}$, $|v| \leqq \cos^{-1} r_{\alpha,\beta}$. Let

$$u' = u + \log r_{\alpha,\beta} \quad \text{on} \quad \{\alpha \neq \beta\}, \quad \text{and}$$

$$u' = u \quad \text{on} \quad \{\alpha = \beta\}.$$

Then,

$$e^{u'} + e^{-u'} \leqq 2(\cos v)/r_{\alpha,\beta} \quad \text{on} \quad \{\alpha \neq \beta\}.$$

Hence

$$|u'| \leqq \cosh^{-1}\{(\cos v)/r_{\alpha,\beta}\} \quad \text{on} \quad \{\alpha \neq \beta\}, \quad \text{and}$$

$$-\log(2\cos v) \leqq u' \quad \text{on} \quad \{\alpha = \beta\}.$$

Hence

$$e^{u+\tilde{v}} = (1/r_{\alpha,\beta})e^{u'+\tilde{v}} \quad \text{on} \quad \{\alpha \neq \beta\}, \text{ and}$$

$$e^{u+\tilde{v}} = e^{u'+\tilde{v}} \quad \text{on} \quad \{\alpha = \beta\}.$$

Since $|1 - \alpha\bar{\beta}|We^{\tilde{s}} = Ce^{u+\tilde{v}}$, we have

$$|1 - \alpha\bar{\beta}|We^{\tilde{s}} = (C/r_{\alpha,\beta})e^{u'+\tilde{v}} \quad \text{on} \quad \{\alpha \neq \beta\}, \text{ and}$$

$$|1 - \alpha\bar{\beta}|We^{\tilde{s}} = Ce^{u'+\tilde{v}} \quad \text{on} \quad \{\alpha = \beta\}.$$

This implies (1). We shall show that (2) implies (1). Let

$$u = u' - \log r_{\alpha,\beta} \quad \text{on} \quad \{\alpha \neq \beta\}, \text{ and}$$

$$u = u' \quad \text{on} \quad \{\alpha = \beta\}.$$

Then, by (2), $r^2e^u + e^{-u} \leq 2(\cos v)$, and $|1 - \alpha\bar{\beta}|We^{\tilde{s}} = Ce^{u+\tilde{v}}$. Hence $|1 - \alpha\bar{\beta}|We^{\tilde{s}}$ is in $(HS)(r_{\alpha,\beta})$. By Theorem 2.1, this implies (1). This completes the proof.

Although the condition of the operator $S_{\alpha,\beta}$ to be a contraction operator on $L^2(W)$ is not simple as we have shown in Theorem A, Theorem 2.1 and Theorem 2.2, it becomes simple when $\alpha\bar{\beta}$ belongs to H^∞. Theorem 2.3 and Theorem 2.4 were essentially given by Prof.T.Nakazi and the author in [28]. We shall give an another proof of Theorem 2.3 using Theorem 2.1.

Theorem 2.3. Suppose α, β are functions in L^∞ such that $\alpha\bar{\beta}$ belongs to H^∞, and W is a non-negative function in L^1 such that $(\alpha - \beta)W$ is not identically zero. Then the following conditions are mutually equivalent.

(1) $||S_{\alpha,\beta} f||_W \leqq ||f||_W \quad (f \in A + \bar{A}_o)$.

(2) $|1 - \alpha\bar{\beta}|W > 0$, $\max\{|\alpha|, |\beta|\} \leqq 1$, and $W \in (HS)(r_{\alpha,\beta})$.

(3) $|1 - \alpha\bar{\beta}|W > 0$, $\max\{|\alpha|, |\beta|\} \leqq 1$, and

$||r_{\alpha,\beta}P_+ f||_W \leqq ||f||_W \quad (f \in A + \bar{A}_o)$.

Proof. By Theorem 2.1, (1) implies that $|1 - \alpha\bar{\beta}|W > 0$ and $\max\{|\alpha|, |\beta|\} \leqq 1$. Hence $1 - \alpha\bar{\beta}$ is an outer function in H^∞ satisfying $\mathrm{Re}(1 - \alpha\bar{\beta}) \geqq 0$. Since $\alpha - \beta$ is not identically zero, there exists a positive constant C and a function s such that $|s| \leqq \pi/2$ and $1 - \alpha\bar{\beta} = Ce^{is-\tilde{s}}$. Hence,

$$1 - \alpha\bar{\beta} = e^{is}|1 - \alpha\bar{\beta}|,$$

and $|1 - \alpha\bar{\beta}|We^{\tilde{s}}$ is in L^1. By Theorem 2.1, (1) and (2) are equivalent. By the equivalence of (1) and (2), (3) and (2) are equivalent. This completes the proof.

The condition of the operator $S_{\alpha,0} = \alpha P_+$ to be a contraction operator on $L^2(W)$ is more simple as follows. Theorem 2.4 follows immediately from Theorem 2.3.

Theorem 2.4. Suppose α is in L^{∞}, and W is a non-negative function in L^1 such that αW is not identically zero. Then the following conditions are mutually equivalent.

(1) $\quad ||\alpha P_+ f||_W = ||S_{\alpha,0} f||_W \leqq ||f||_W \quad (f \in A + \bar{A}_0)$.

(2) $\quad W > 0,\ |\alpha| \leqq 1,$ and $W \in (HS)(|\alpha|)$.

For a given positive constant M and a non-negative function W in L^1, M.Cotlar and C.Sadosky [7] consider the boundedness of the operator $S = S_{1,-1}$ on $L^2(W)$ with norm M. Using our notation $(HS)(r)$, their result can be written as follows. The following conditions are equivalent.

(1) $\quad ||Sf||_W \leqq M||f||_W \quad (f \in A + \bar{A}_0)$.

(2) $\quad M \geqq 1,$ and W is in $(HS)(2M/(M^2 + 1))$.

We shall consider the operator P_+ since the condition is more simple as follows which is a corollary of Theorem 2.3 since $P_+ = S_{1,0}$. The following conditions are equivalent.

(1) $\quad ||P_+ f||_W \leqq M||f||_W \quad (f \in A + \bar{A}_0)$.

(2) $\quad M \geqq 1,$ and W is in $(HS)(M^{-1})$.

By this equivalence and Proposition 2.3, we have the Helson-Szegö theorem which is the first characterization of the non-negative function W in L^1, defined on the circle T, such that the Riesz projection P_+ acts continuously in $L^2(W)$.

By Proposition 2.3 and Theorem 2.3, we have the following well known result (cf. [18], [14, p.149], [22, p.226]).

Helson-Szegö theorem. For a non-negative function W in L^1, the following conditions are mutually equivalent.

(1) There exists a positive constant C such that

$$\|P_+ f\|_W \leqq C\|f\|_W \quad (f \in A + \bar{A}_0).$$

(2) $W \in (HS)$ or $W \equiv 0$.

If $P_+ = S_{1,0}$ or $S = S_{1,-1}$ acts continuously in $L^2(W)$, then W is in (HS). Moreover we have the following result from the Helson-Szegö theorem immediately.

Proposition 2.6. For a non-negative function W in L^1, the following conditions are mutually equivalent.

(1) There exist functions α and β in L^∞ such that

$$\text{ess inf } |\alpha - \beta| > 0, \text{ and}$$

$$\|S_{\alpha,\beta} f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_0).$$

(2) W is in (HS).

Proof. Since $S_{\alpha,\beta} f = (\alpha - \beta)P_+ f + \beta f$, (1) implies that

$$(\text{ess inf } |\alpha - \beta|)\|P_+ f\|_W \leqq \|(\alpha - \beta)P_+ f\|_W$$

$$\leq \|S_{\alpha,\beta} f\|_W + \|\beta f\|_W \leq (1 + \|\beta\|_\infty)\|f\|_W.$$

By the Helson-Szegö theorem, W is in (HS). Since

$$\|S_{\alpha,\beta}\|_W \leq \|\alpha - \beta\|_\infty \|P_+ f\|_W + \|\beta\|_\infty,$$

$\|S_{\alpha,\beta}\|_W$ is finite when $\|P_+\|_W$ is finite. By the Helson-Szegö theorem, (2) implies (1).

We shall consider the weighted norm inequality having two weights. We shall give the another proof of the Koosis theorem. If W is in (HS), then by the Zygmund theorem (cf. [22, p.138]), $W^{-p} \in L^1$ for some $p > 1$ and hence $W^{-1} \in L^1$ (cf. [14, p.178]).

Koosis theorem. For a non-negative function W in L^1, the following conditions are mutually equivalent.

(1) There exists a non-zero and non-negative measurable function U such that

$$\|P_+ f\|_U \leq \|f\|_W \quad (f \in A + \bar{A}_0).$$

(2) W^{-1} is in L^1.

Proof. We shall show that (1) implies (2). Let $\alpha = (U/W)^{1/2}$ on $\{W > 0\}$, and $\alpha = 0$ on $\{W = 0\}$, and let $\beta \equiv 0$. By (1), $U \leqq W$ and

$$\|S_{\alpha,0} f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_0).$$

Since $U \leqq W$ and $m\{U > 0\} > 0$, we have αW is not identically zero. By Theorem 2.3, we have $W \in (HS)(\alpha)$. By Proposition

2.1(b) and Proposition 2.2(b), $W^{-1} \in L^1$. We shall show that (2) implies (1). By Proposition 2.4, $W \in (HS)(r)$ for some positive function r. By Theorem 2.3 with $\alpha = r$, we have

$$\|rP_+ f\|_W \leq \|f\|_W \quad (f \in A + \bar{A}_0).$$

Let $U = r^2 W$, so that

$$\|P_+ f\|_U \leq \|f\|_W \quad (f \in A + \bar{A}_0).$$

If $U \equiv 0$, then $W \equiv 0$. This completes the proof.

Example 2.1. Suppose W is a non-negative function in L^1 such that W^{-1} is in L^1. Let

$$U = (W^{-1})/\{W^{-2} + (W^{-1})^{\sim 2}\}.$$

Then,

$$\|P_+ f\|_U \leq \|f\|_W \quad (f \in A + \bar{A}_0).$$

Proof. Let $k = 1/\{W^{-1} + i(W^{-1})^{\sim}\}$, so that

$$|W - k|^2 = W(W - U),$$

$k \in H^1$ and $U \leq W$. By the Cotlar-Sadosky theorem,

$$\|f\|_W^2 - \|P_+ f\|_U^2$$

$$= \|P_+ f + P_- f\|_W^2 - \|P_+ f\|_U^2 \geq 0.$$

If there exists a non-zero weight U such that P_+ is a continuous operator from $L^2(W)$ to $L^2(U)$, then W^{-1} is in L^1. Moreover we have the following result.

Proposition 2.7. For a non-negative function W in L^1, the following conditions are mutually equivalent.

(1) There exist functions α and β in L^∞ such that $(\alpha - \beta)W$ is not identically zero, and

$$||S_{\alpha,\beta} f||_W \leqq ||f||_W \quad (f \in A + \bar{A}_0).$$

(2) There exist functions α and β in L^∞ such that $(\alpha - \beta)W$ is not identically zero, and

$$||S_{\alpha,\beta} f||_W = ||f||_W \quad (f \in A + \bar{A}_0).$$

(3) W^{-1} is in L^1.

Proof. We shall show that (1) implies (3). Since $S_{\alpha,\beta} f = (\alpha - \beta)P_+f + \beta f$, (1) implies that

$$||(\alpha - \beta)P_+f||_W \leqq ||S_{\alpha,\beta} f||_W + ||\beta f||_W \leqq (1 + ||\beta||_\infty)||f||_W.$$

Let $U = |\alpha - \beta|^2 W/(1 + ||\beta||_\infty)^2$, so that U satisfies

$$||P_+f||_U \leqq ||f||_W.$$

By the Koosis theorem, W^{-1} is in L^1. We shall show that (3) implies (2). Let $k = 2/\{W^{-1} + i(W^{-1})^\sim\}$, so that k is in H^1. Let $\alpha = k/\bar{k}$ and $\beta = -1$, so that

$$(1 - \alpha\bar{\beta})W = 2W(\mathrm{Re}\ k)/\bar{k} = k.$$

Hence $(1 - \alpha\bar{\beta})W$ is in H^1. Hence, for all f in $A + \bar{A}_0$,

$$||f||_W^2 - ||S_{\alpha,\beta} f||_W^2 = \int_T (P_+f)(\overline{P_-f})(1 - \alpha\bar{\beta})W\, dm = 0.$$

It is clear that (2) implies (1).

Example 2.2. Suppose W is a non-negative function in L^1 such that W^{-1} is in L^1. Suppose c is a constant satisfying $-1 \leqq c \leqq 1$. Let

$$\alpha = \{cW^{-1} + i(W^{-1})^{\sim}\}/\{W^{-1} + i(W^{-1})^{\sim}\},$$

and $\beta = 0$. Then,

$$\|S_{\alpha,\beta} f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_o).$$

Proof. Since $(\alpha - \beta)W = (c - 1)/\{W^{-1} + i(W^{-1})^{\sim}\}$, $(\alpha - \beta)W$ is in H^1. Hence,

$$\int_T (P_+ f)(\overline{P_- f})(\alpha - \beta)W \, dm = 0 \quad (f \in A + \bar{A}_o).$$

Since $-1 \leqq c \leqq 1$, $|\alpha| \leqq 1$. Hence,

$$\|f\|_W^2 - \|S_{\alpha,\beta} f\|_W^2 = \int_T (1 - |\alpha|^2)|f|^2 \geqq 0 \quad (f \in A + \bar{A}_o).$$

By Theorem 2.1, Corollary 2.1 follows immediately.

Corollary 2.1. Suppose α and β are functions in L^∞ satisfying $m\{\alpha \neq \beta\} > 0$ and $1 - \alpha\bar{\beta} = e^{ic}|1 - \alpha\bar{\beta}|$ for some real constant c. Suppose W is a positive function in L^1. Then the following conditions are mutually equivalent.

(1) $\|S_{\alpha,\beta} f\|_W \leqq \|f\|_W \quad (f \in A + \bar{A}_o)$.

(2) $|1 - \alpha\bar{\beta}|W > 0$, $\max\{|\alpha|, |\beta|\} \leqq 1$, and $|1 - \alpha\bar{\beta}|W$ is in $(HS)(r_{\alpha,\beta})$.

Suppose α, β are functions in L^∞ such that $\alpha\bar{\beta}$ belongs to H^∞, and W is a non-negative function in L^1 such that $(\alpha - \beta)W$ is not identically zero. Suppose M is a positive constant satisfying $|M^2 - \alpha\bar{\beta}|W > 0$ and $\max\{|\alpha|, |\beta|\} \leqq M$. By Theorem 2.3, the following conditions are mutually equivalent.

(1) $\|S_{\alpha,\beta}\|_W \leqq M$.

(2) $\|r_{\alpha/M,\beta/M} P_+\|_W \leqq 1$.

When α and β are different complex constants and W is in (HS), we have

$$0 < \max\{|\alpha|, |\beta|\} \leqq \|S_{\alpha,\beta}\|_W < \infty,$$

and $1 \leqq \|P_+\|_W < \infty$. We shall show in the proof of Corollary 2.3 that

$$r_{\alpha/M,\beta/M} = \frac{1}{\|P_+\|_W}$$

where $M = \|S_{\alpha,\beta}\|_W$ (cf.[46]). This seems to be not written in any paper. This is equivalent to the result of I.Gohberg-N.Krupnik [17] and I.Feldman-N.Krupnik-A.Marcus [12] which appears in the statement of Corollary 2.3. In Corollary 2.2, we do not assume $\alpha\bar{\beta}$ is in H^∞. Then we shall consider the norm of the operator P_+ on the Hilbert space $L^2(|1 - \alpha\bar{\beta}|W)$ when $S_{\alpha,\beta}$ is a contraction operator on $L^2(W)$. We use Theorem 2.3 to prove Corollary 2.2. We use Corollary 2.2 to prove Corollary 2.3.

Corollary 2.2. Suppose α and β are functions in L^∞ satisfying ess inf $|\alpha - \beta| > 0$ and $1 - \alpha\bar{\beta} = e^{ic}|1 - \alpha\bar{\beta}|$ for some real constant c. Suppose W is a positive function in L^1 satisfying

$$||S_{\alpha,\beta} f||_W \leqq ||f||_W \quad (f \in A + \bar{A}_0).$$

Suppose N is a constant and k is a function satisfying

$$N = ||P_+||_{|1 - \alpha\bar{\beta}|W}, \text{ and}$$

$$k = 2^{-1}(\{|\alpha - \beta|^2N^2 + 2\mathrm{Re}(\alpha\bar{\beta}) + 2|\alpha\beta|\}^{1/2}$$

$$+ \{|\alpha - \beta|^2N^2 + 2\mathrm{Re}(\alpha\bar{\beta}) - 2|\alpha\beta|\}^{1/2}).$$

Then, $\max\{|\alpha|, |\beta|\} \leqq k$, $|1 - \alpha\bar{\beta}|W \in (HS)(N^{-1})$ and

$$\text{ess inf } r_{\alpha,\beta} \leqq N^{-1} = r_{\alpha/k,\beta/k} \leqq 1.$$

Proof. By Corollary 2.1, $\max\{|\alpha|, |\beta|\} \leqq 1$, $|1 - \alpha\bar{\beta}| > 0$, and $|1 - \alpha\bar{\beta}|W$ is in $(HS)(r_{\alpha,\beta})$. By Theorem 2.3,

$$||r_{\alpha,\beta}P_+f||_{|1 - \alpha\bar{\beta}|W} \leqq ||f||_{|1 - \alpha\bar{\beta}|W}.$$

Since ess inf $|\alpha - \beta| > 0$, ess inf $r_{\alpha,\beta} > 0$. Hence,

$$1 \leqq N = ||P_+||_{|1 - \alpha\bar{\beta}|W} \leqq \{\text{ess inf } r_{\alpha,\beta}\}^{-1} < \infty.$$

Since

$$||P_+f||_{|1 - \alpha\bar{\beta}|W} \leqq N||f||_{|1 - \alpha\bar{\beta}|W},$$

by Theorem 2.3, $N \geqq 1$ and $|1 - \alpha\bar{\beta}|W$ is in $(HS)(N^{-1})$. Let

$$g = |\alpha - \beta|^2N^2 + 2\mathrm{Re}(\alpha\bar{\beta}),$$

so that g is in L^∞,

$$g^2 - 4|\alpha\beta|^2 \geqq (|\alpha|^2 - |\beta|^2)^2 \geqq 0, \quad \text{and}$$

$$k = 2^{-1/2}\{g + (g^2 - 4|\alpha\beta|^2)^{1/2}\}^{1/2}.$$

Hence k is in L^∞ and $k^4 - gk^2 + |\alpha\beta|^2 = 0$. This implies $N^{-1} = r_{\alpha/k,\beta/k}$. Since

$$|\alpha|^4 - g|\alpha|^2 + |\alpha\beta|^2 = |\alpha|^2|\alpha - \beta|^2(1 - N^2) \leqq 0,$$

and

$$|\beta|^4 - g|\beta|^2 + |\alpha\beta|^2 = |\beta|^2|\alpha - \beta|^2(1 - N^2) \leqq 0,$$

we have $\max\{|\alpha|, |\beta|\} \leqq k$. This completes the proof.

We shall consider the connection between the norms of the operators $S_{\alpha,\beta}$ and P_+ on $L^2(W)$. Corollary 2.3 is the special case of the Feldman-Krupnik-Marcus theorem (cf. [12], [17, Chapter 13, Lemma 5.3]). We shall give the another proof using Corollary 2.2.

Corollary 2.3. Suppose α and β are constants, and W is a positive function in L^1. Then

$$2M = \{|\alpha - \beta|^2N^2 + 2\mathrm{Re}(\alpha\bar{\beta}) + 2|\alpha\beta|\}^{1/2} + \{|\alpha - \beta|^2N^2 + 2\mathrm{Re}(\alpha\bar{\beta}) - 2|\alpha\beta|\}^{1/2},$$

where

$$M = ||S_{\alpha,\beta}||_W \quad \text{and} \quad N = ||P_+||_W.$$

Proof. Suppose $\alpha \neq \beta$. Then, by the Helson-Szegö theorem and Proposition 2.6, $M = ||S_{\alpha,\beta}||_W$ is finite if and only if $N = ||P_+||_W$ is finite. If M and N are infinite, then $N^{-1} = r_{\alpha/M,\beta/M} = 0$ and

$$\begin{aligned} 2M &= \{|\alpha - \beta|^2 N^2 + 2\mathrm{Re}(\alpha\bar{\beta}) + 2|\alpha\beta|\}^{1/2} \\ &\quad + \{|\alpha - \beta|^2 N^2 + 2\mathrm{Re}(\alpha\bar{\beta}) - 2|\alpha\beta|\}^{1/2} = \infty. \end{aligned}$$

Suppose M and N are finite. Then,

$$||S_{\alpha/M,\beta/M}\, f||_W \leqq ||f||_W.$$

Let

$$g = |\alpha - \beta|^2 N^2 + 2\mathrm{Re}(\alpha\bar{\beta}), \text{ and}$$

$$k = 2^{-1}(\{g + 2|\alpha\beta|\}^{1/2} + \{g - 2|\alpha\beta|\}^{1/2}).$$

Then by Corollary 2.2, $\max\{|\alpha/k|, |\beta/k|\} \leqq 1$, $W \in (HS)(N^{-1})$, $W \in (HS)(r_{\alpha/k,\beta/k})$ and

$$r_{\alpha/M,\beta/M} \leqq N^{-1} = r_{\alpha/k,\beta/k}.$$

By Corollary 2.1,

$$||S_{\alpha,\beta}\, f||_W = k||S_{\alpha/k,\beta/k}\, f||_W \leqq k||f||_W.$$

Hence $M = ||S_{\alpha,\beta}||_W \leqq k < \infty$. By the calculation, this implies $r_{\alpha/k,\beta/k} \leqq r_{\alpha/M,\beta/M}$. Hence, $r_{\alpha/k,\beta/k} = r_{\alpha/M,\beta/M} = N^{-1}$. Since $r_{\alpha/M,\beta/M} = N^{-1}$, $M^4 - gM^2 + |\alpha\beta|^2 = 0$. Since $M \geqq \max\{|\alpha|, |\beta|\}$, $M^2 \geqq g/2$. Since $(2M^2 - g)^2 = g^2 - 4|\alpha\beta|^2$, we have

$$\begin{aligned} 4M^2 &= 2g + 2(g^2 - 4|\alpha\beta|^2)^{1/2} \\ &= (\{g + 2|\alpha\beta|\}^{1/2} + \{g - 2|\alpha\beta|\}^{1/2})^2 = 4k^2. \end{aligned}$$

Hence $M = k$. This equality holds even when $\alpha = \beta$, since $||S_{\alpha,\alpha}||_W = ||\alpha I||_W = |\alpha|$. This completes the proof.

§3. Invertibility of S_ϕ

$S_{\alpha,\beta}$ is (left) invertible in $L^2(W)$ if and only if $S_{\phi,1}$ is (left) invertible in $L^2(W)$ with $\phi = \alpha/\beta$. Hence we shall consider the (left) invertibility of the singular integral operators $S_\phi = S_{\phi,1}$ on $L^2(W)$. We use Theorem B to prove Theorem 3.1. We use Theorem 3.1 to prove the Widom-Devinatz-Rochberg theorem.

Suppose α, β are functions in L^∞, and W is a non-negative function in L^1 such that $\alpha W = \beta W$. In this case, $S_{\alpha,\beta}$ becomes a multiplication operator on $L^2(W)$, and hence the condition of the operator $S_{\alpha,\beta}$ to become a bounded below operator on $L^2(W)$ is simple as follows. The following conditions are mutually equivalent.

(1) $||S_{\alpha,\beta}\, f||_W \geq ||f||_W$ $(f \in A + \bar{A}_o)$.

(2) $||\alpha f||_W \geq ||f||_W$ $(f \in A + \bar{A}_o)$.

(3) $|\alpha| W \geq W$.

Suppose α, β are functions in L^∞, and W is a non-negative function in L^1 such that $\alpha W = \beta W$. Then the following conditions are mutually equivalent.

(1) $S_{\alpha,\beta}$ is a bounded operator on $L^2(W)$ which has a bounded inverse operator.

(2) $S_{\alpha,\beta}$ is a bounded operator on $L^2(W)$ which has a bounded left inverse operator.

(3) There exists a positive constant C_1 and C_2 such that $C_1 W \leqq |\alpha| W \leqq C_2 W$.

The following theorem (cf. [29]) is essentially the same as Theorem A. We shall give it's proof for the sake of completeness.

Main Theorem B. Suppose α, β are functions in L^∞, and W is a non-negative function in L^1 such that $(\alpha - \beta)W$ is not identically zero. Then the following conditions are mutually equivalent.

(1) $\|S_{\alpha,\beta} f\|_W \geqq \|f\|_W \quad (f \in A + \bar{A}_0)$.

(2) $|1 - \alpha\bar{\beta}|W > 0$, $\min\{|\alpha|, |\beta|\} \geqq 1$, and there exists an inner function Q and a real function t in L^1 such that

$$(1 - \alpha\bar{\beta})/|1 - \alpha\bar{\beta}| = Qe^{i\tilde{t}},$$

and $|1 - \alpha\bar{\beta}|We^{-t}$ is in $(HS)(r_{\alpha,\beta})$.

Proof. Let $W_1 = (|\alpha|^2 - 1)W$, $W_2 = (|\beta|^2 - 1)W$, $W_3 = (\alpha\bar{\beta} - 1)W$. We shall show that (1) implies (2). By (1), for all $f_1 \in A$ and $f_2 \in \bar{A}_0$,

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2\mathrm{Re}(f_1\bar{f}_2 W_3)\}dm \geqq 0.$$

By the Cotlar-Sadosky theorem, $W_1 \geqq 0$, $W_2 \geqq 0$, and there exists a k_o in H^1 such that $|W_3 - k_o|^2 \leqq W_1 W_2$. Hence,

$$|(\alpha\bar{\beta} - 1)W - k_o|^2 \leqq (|\alpha|^2 - 1)(|\beta|^2 - 1)W^2.$$

Let $k = -k_o$, so that

$$|(1 - \alpha\bar{\beta})W - k|^2 \leqq (1 - |\alpha|^2)(1 - |\beta|^2)W^2.$$

By Lemma B, $\log|k|$ and $\log(|1 - \alpha\bar{\beta}|W)$ are in L^1. Hence $W > 0$. Since $W_1 \geqq 0$ and $W_2 \geqq 0$, we have $\min\{|\alpha|, |\beta|\} \geqq 1$. Hence $r_{\alpha,\beta} \leqq 1$. By Lemma B, (2) holds. We shall show that (2) implies (1). Since $\min\{|\alpha|, |\beta|\} \geqq 1$, we have $W_1 \geqq 0$, $W_2 \geqq 0$ and $r_{\alpha,\beta} \leqq 1$. By Lemma B, (2) implies that there exists a function k in H^1 such that

$$|(1 - \alpha\bar{\beta})W - k|^2 \leqq (1 - |\alpha|^2)(1 - |\beta|^2)W^2.$$

Hence $|W_3 + k|^2 \leqq W_1 W_2$. By the Cotlar-Sadosky theorem, for all $f_1 \in A$ and $f_2 \in \bar{A}_o$,

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2\mathrm{Re}(f_1 \bar{f}_2 W_3)\} dm \geqq 0.$$

This implies (1). This completes the proof.

If $S_{\alpha,\beta}$ is left invertible, then

$$\text{ess inf } |\alpha| > 0, \text{ and ess inf } |\beta| > 0.$$

Let $\phi = \alpha/\beta$, so that

$$S_{\alpha,\beta} = \beta S_{\alpha/\beta,1} = \beta S_{\phi,1}$$

By Lemma 2, $\beta(\phi P_+ + P_-) = \beta(P_+\phi P_+ + P_-)(I + P_-\phi P_+)$.

If a weight function W is in (HS), P_+ and P_- becomes bounded operators, and hence $I + P_-\phi P_+$ has a bounded inverse operator in $L^2(W)$. Then, $(I + P_-\phi P_+)^{-1} = I - P_-\phi P_+$. When W is in (HS) and β is invertible in L^∞, the following conditions are mutually equivalent.

(1) $S_{\alpha,\beta}$ is (left) invertible.

(2) $S_\phi = S_{\phi,1}$ is (left) invertible.

(3) $P_+\phi P_+ + P_-$ is (left) invertible.

(4) T_ϕ is (left) invertible.

Lemma 3.1. Suppose ϕ is in L^∞, and W is a non-negative function in L^1 such that ϕW is not identically zero. Then the following conditions are mutually equivalent.

(1) There exists a k in H^1 and a constant ε such that $0 < \varepsilon \leqq 1$, and

$$|\phi W - k| \leqq (1 - \varepsilon)|\phi|W.$$

(2) $|\phi|W > 0$, and there exists an inner function Q and a real function t in L^1 such that $\phi/|\phi| = Qe^{i\tilde{t}}$, and $|\phi|We^{-t}$ is in (HS).

Proof. We shall show that (1) implies (2). By (1),

$$|\phi W - k|^2 \leqq (1 - \varepsilon)^2|\phi W|^2 \leqq (1 - \varepsilon^2)|\phi W|^2.$$

By Lemma A, $|\phi|W > 0$, and there exists an inner function Q and a real function t in L^1 such that $\phi/|\phi| = \phi W/|\phi W| = Qe^{i\tilde{t}}$, and $|\phi|We^{-t}$ is in (HS)(ε). By Proposition 2.3, $|\phi|We^{-t}$ is in (HS). We shall show that (2) implies (1). By Proposition 2.3, $|\phi W|e^{-t}$ is in (HS)(r) for some constant r satisfying $0 < r \leqq 1$. Since $|\phi W| > 0$ and $\phi W/|\phi W| = Qe^{i\tilde{t}}$, by Lemma A, there exists a function k in H^1 such that

$$|\phi W - k|^2 \leqq (1 - r^2)|\phi W|^2.$$

This completes the proof.

When W is in (HS), then a simple necessary and sufficient condition for that there exists a positive constant δ such that

$$||S_\phi f||_W \geqq \delta ||f||_W \qquad (f \in A + \bar{A}_0)$$

is known as one of the Widom-Devinatz-Rochberg theorem. If ess inf$|1 - \phi| = 0$, W is not in (HS), and W is not identically zero, then it becomes too complicated to wright. When ess inf $|1 - \phi| > 0$, Prof. T. Nakazi and the author could give it as the following theorem (cf. [29]). We do not assume $W \in$ (HS) instead of the assumption: ess inf $|1 - \phi| > 0$. When $\phi \equiv 1$, S_ϕ becomes an identity operator, and hence S_ϕ is left invertible. We shall give the another proof of Theorem 3.1 using Theorem B.

Theorem 3.1. Suppose ϕ is in L^∞ such that ess inf $|1 - \phi| > 0$. Suppose W is a positive function in L^1. Then the following conditions are mutually equivalent.

(1) There exists a positive constant δ such that

$$||S_\phi f||_W \geqq \delta ||f||_W \quad (f \in A + \bar{A}_o).$$

(2) ess inf $|\phi| > 0$, and there exists an inner function Q and a real function t in L^1 such that

$$\phi/|\phi| = Qe^{i\tilde{t}},$$

and We^{-t} is in (HS).

Proof. We shall show that (1) implies (2). Let

$$c = (\text{ess inf } |\phi|)(\text{ess inf } |1 - \phi|)/(||\phi||_\infty + 1).$$

Suppose $0 < \delta < \min\{c, 1\}$. By Theorem B, (1) implies that $|\phi| \geqq \delta$, and there exists an inner function Q and a real function t' in L^1 such that

$$(\phi - \delta^2)/|\phi - \delta^2| = Qe^{i\tilde{t}'},$$

and $|\phi - \delta^2|We^{-t'}$ is in $(HS)(r_{\phi/\delta,1/\delta})$. Hence there exists a positive constant C and real functions u, v such that $|\phi - \delta^2|We^{-t'} = Ce^{u + \tilde{v}}$, u is in L^1, $|v| \leqq \pi/2$, and

$$r_{\phi/\delta,1/\delta}^{\,2}e^u + e^{-u} \leqq 2(\cos v).$$

Since ess inf $|1 - \phi| > 0$, ess inf $r_{\phi/\delta,1/\delta} > 0$. Since

$$2r_{\phi/\delta,1/\delta} \leqq r_{\phi/\delta,1/\delta}^{\,2}e^u + e^{-u},$$

$|v| \leqq \cos^{-1} r_{\phi/\delta,1/\delta}$. Hence u is in L^∞. Since $|\phi| \geqq \delta$ and $0 < \delta < 1$, $|\phi - \delta^2| \geqq \delta - \delta^2 > 0$. Since $\delta^2/|\phi| \leqq \delta < 1$, there exists a real function s in L^∞ such that

$$(\phi - \delta^2)/|\phi - \delta^2| = (\phi/|\phi|)e^{is}, \quad \text{and}$$
$$|s| \leqq \sin^{-1}(\delta^2/|\phi|).$$

Let $t = t' + \tilde{s}$, so that there exists a real constant c such that

$$\phi/|\phi| = (Qe^{ic})e^{i\tilde{t}}, \quad \text{and}$$

$$We^{-t} = Ce^{(u - \log|\phi - \delta^2|) + (v - s)^{\sim}},$$

where $u - \log|\phi - \delta^2|$ is in L^∞. Since $|v| \leqq \cos^{-1} r_{\phi/\delta,1/\delta}$,

$$|v - s| \leqq |v| + |s|$$
$$\leqq \pi/2 - \{\sin^{-1} r_{\phi/\delta,1/\delta} - \sin^{-1}(\delta^2/|\phi|)\}.$$

Let $\varepsilon = \delta(c - \delta)/(\text{ess inf } |\phi|)$, so that

$$0 < \varepsilon \leqq r_{\phi/\delta,1/\delta} - \delta^2/|\phi| < 1.$$

Hence,

$$|v - s| \leqq \pi/2 - \sin^{-1}(\varepsilon\{1 - (1 - \varepsilon)^2\}^{1/2}).$$

Hence We^{-t} is in (HS). We shall show that (2) implies (1). Since ϕ and ϕ^{-1} belong to L^∞, $|\phi|We^{-t}$ is in (HS). Since $|\phi|W > 0$, by Lemma 3.1, there exists a k in H^1 and a constant ε such that $0 < \varepsilon \leqq 1$, and

$$|\phi W - k| \leqq (1 - \varepsilon)|\phi|W.$$

Then,

$$(1 - \varepsilon^2)(|\phi|^2 - \varepsilon^2) - (1 - \varepsilon)^2|\phi|^2$$
$$= \varepsilon(1 - \varepsilon)\{2|\phi|^2 - \varepsilon(1 + \varepsilon)\}$$
$$\geqq 2\varepsilon(1 - \varepsilon)(|\phi|^2 - \varepsilon) \geqq 0.$$

Hence,

$$|\phi W - k|^2 \leqq (1 - \varepsilon^2)(|\phi|^2 - \varepsilon^2)W^2.$$

For all $f_1 \in A$ and $f_2 \in \bar{A}_o$,

$$\int_T \{(|\phi|^2 - \varepsilon^2)|f_1|^2 W + (1 - \varepsilon^2)|f_2|^2 W + 2(\mathrm{Re}\ f_1\bar{f}_2\phi W)\}dm \geqq 0.$$

Hence,

$$||S_\phi f||_W^2 - (\varepsilon^2/2)||f||_W^2$$
$$\int_T |\phi f_1 + f_2|^2 W dm - (\varepsilon^2/2)\int_T |f_1 + f_2|^2 W dm$$
$$\geqq \int_T |\phi f_1 + f_2|^2 W dm - \varepsilon^2 \int_T (|f_1|^2 + |f_2|^2) W dm$$
$$= \int_T \{(|\phi|^2 - \varepsilon^2)|f_1|^2 W + (1 - \varepsilon^2)|f_2|^2 W + 2(\mathrm{Re}\ f_1\bar{f}_2\phi W)\}dm$$
$$= \int_T \{(|\phi|^2 - \varepsilon^2)|f_1|^2 W + (1 - \varepsilon^2)|f_2|^2 W$$
$$+ 2(\mathrm{Re}\ f_1\bar{f}_2(\phi W - k)\}dm$$
$$\geqq 2\int_T |f_1\bar{f}_2|\{(|\phi|^2 - \varepsilon^2)^{1/2}(1 - \varepsilon^2)^{1/2}W - |\phi W - k|\}dm \geqq 0.$$

This completes the proof.

If $W \in (HS)$, then S_ϕ is left invertible in $L^2(W)$ if and only if $S_{\varepsilon\phi}$ is left invertible in $L^2(W)$ for some positive constant $\varepsilon > 0$. We shall give the another proof of the Widom-Devinatz-Rochberg theorem (I) (cf. [35]) using Theorem 3.1. We shall give the another proof of the Widom-Devinatz-Rochberg theorem (II) using the Widom-Devinatz-Rochberg theorem (I). The condition (3) of the Widom-Devinatz-Rochberg theorem (I) seems to be not written in any paper.

Widom-Devinatz-Rochberg theorem (I). Suppose ϕ is in L^∞. Suppose W is a non-zero function in (HS). Then the following conditions are mutually equivalent.

(1) S_ϕ is left invertible in $L^2(W)$, that is, there exists a positive constant δ such that

$$||S_\phi f||_W \geq \delta||f||_W \quad (f \in A + \bar{A}_0).$$

(2) T_ϕ is left invertible in $H^2(W)$, that is, there exists a positive constant δ such that

$$||T_\phi f||_W \geq \delta||f||_W \quad (f \in A).$$

(3) ϕ^{-1} is in L^∞, and there exists an inner function Q and a real function t in L^1 such that $\phi/|\phi| = Qe^{i\tilde{t}}$, and We^{-t} is in (HS).

Proof. We shall show that (1) implies (2) (cf. [15, p.123]). Since W is in (HS), $||P_+||_W < \infty$, and hence $||I + P_-\phi P_+||_W < c < \infty$ for some constant c. Since,

$$S_\phi = \phi P_+ + P_- = (P_+\phi P_+ + P_-)(I + P_-\phi P_+), \text{ and}$$

$$(I + P_-\phi P_+)(I - P_-\phi P_+) = I,$$

by (1), for all f in A,

$$\delta\|f\|_W \leqq \delta\|(I + P_-\phi P_+)(I - P_-\phi P_+)f\|_W$$

$$\leqq \delta c\|(I - P_-\phi P_+)f\|_W \leqq c\|S_\phi(I - P_-\phi P_+)f\|_W$$

$$= c\|(P_+\phi P_+ + P_-)f\|_W = c\|P_+(\phi f)\|_W = c\|T_\phi f\|_W.$$

We shall show that (2) implies (1). By (2), for all f in $A + \bar{A}_0$,

$$\delta\|f\|_W \leqq \delta\|P_+f\|_W + \delta\|P_-f\|_W$$

$$\leqq \|T_\phi P_+f\|_W + \delta\|P_-f\|_W$$

$$= \|P_+\phi P_+f\|_W + \delta\|P_-f\|_W$$

$$\leqq \|P_+S_\phi f\|_W + \delta\|P_-S_\phi f\|_W$$

$$\leqq \max\{\|P_+\|_W, \delta\|P_-\|_W\}\|S_\phi f\|_W.$$

We shall show that (2) implies (3). Since ϕ is in L^∞, there exists a positive constant ε such that ess inf $|1 - \varepsilon\phi| > 0$. Since

$$\|T_{\varepsilon\phi} f\|_W \geqq \delta\|f\|_W \quad (f \in A),$$

we have

$$\|S_{\varepsilon\phi} f\|_W \geqq \delta\|f\|_W \quad (f \in A + \bar{A}_0).$$

By Theorem 3.1, this implies (3). By the proof of Theorem 3.1, (3) implies (1). This completes the proof.

Widom-Devinatz-Rochberg theorem (II). Suppose ϕ is in L^∞. Suppose W is a non-zero function in (HS). Then the following conditions are mutually equivalent.

(1) S_ϕ has a bounded inverse operator on $L^2(W)$.

(2) T_ϕ has a bounded inverse operator on $H^2(W)$.

(3) ϕ^{-1} is in L^∞, and there exists a real constant c and a real function t in L^1 such that $\phi/|\phi| = e^{i(c + \tilde{t})}$, and We^{-t} is in (HS).

Proof. Since W is in (HS), P_+ and P_- are bounded in $L^2(W)$. Since

$$(I + P_-\phi P_+)(I - P_-\phi P_+) = I,$$

$I + P_-\phi P_+$ is invertible in $L^2(W)$, and

$$(I + P_-\phi P_+)^{-1} = I - P_-\phi P_+.$$

Since

$$S_\phi = \phi P_+ + P_- = (P_+\phi P_+ + P_-)(I + P_-\phi P_+),$$

S_ϕ is (left) invertible in $L^2(W)$ if and only if $P_+\phi P_+ + P_-$ is (left) invertible in $L^2(W)$. Then $P_+\phi P_+ + P_-$ is invertible in $L^2(W)$ if and only if T_ϕ is invertible in $H^2(W)$. Hence, S_ϕ is (left) invertible in $L^2(W)$ if and only if T_ϕ is (left) invertible in $H^2(W)$. Hence (1) and (2) are equivalent. We shall show that (2) implies (3). Since W is in (HS), W^{-1} is in L^1. By (2), T_ϕ is left invertible in $H^2(W)$ and $T_{\bar\phi}$ is left invertible in $H^2(W^{-1})$. Hence S_ϕ is left invertible in $L^2(W)$ and $S_{\bar\phi}$ is left invertible in $L^2(W^{-1})$. By the Widom-Devinatz-Rochberg theorem (I), there exist inner functions Q, Q', and real functions t, t' in L^1 such that

$$\phi/|\phi| = Qe^{i\tilde{t}}, \quad We^{-t} \text{ is in (HS)},$$

$$\bar\phi/|\phi| = Q'e^{i\tilde{t}'}, \quad W^{-1}e^{-t'} \text{ is in (HS)}.$$

Hence $QQ'e^{i(t + t')^\sim} = 1$, and $e^{t + t'}$ is in $L^{1/2}$. Since

$$QQ'e^{t + t' + i(t + t')^\sim} = e^{t + t'},$$

$e^{t + t'}$ is a non-negative function in $H^{1/2}$. By the Neuwirth-Newman theorem, $e^{t + t'}$ is a constant, and hence $QQ' = 1$. Hence Q and Q' are constants. We shall show that (3) implies (2). Since $\phi/|\phi| = e^{i(c + \tilde{t})}$ and We^{-t} is in (HS), by the Widom-Devinatz-Rochberg theorem (I), T_ϕ is left invertible in $H^2(W)$. Since $\bar\phi/|\phi| = e^{i(-c - \tilde{t})}$ and $W^{-1}e^{-(-t)}$ is in (HS), $T_\phi{}^* = T_{\bar\phi}$ is left invertible in $H^2(W^{-1})$. Hence T_ϕ is invertible in $H^2(W)$. This completes the proof.

§4. Invertibility and $L^2((W))$.

As we have shown in Section 3, if we assume one of conditions $W \in (HS)$ or $\operatorname{ess\,inf}|1 - \phi| > 0$, then the condition of the operator S_ϕ to be bounded below in $L^2(W)$ becomes simple. In the case that W is not in (HS) and $\operatorname{ess\,inf}|1 - \phi| = 0$, we can not give a simple necessary and sufficient condition for an operator S_ϕ to be bounded below in $L^2(W)$. In Theorem 4.2, we shall show that the necessary and sufficient condition of two operators S_ϕ and $S_{-\phi}$ to be bounded below in $L^2(W)$ becomes simple even in this case. When $W \in (HS)$, if S_ϕ is bounded below in $L^2(W)$, then $S_{-\phi}$ is also bounded below in $L^2(W)$. Hence the equivalence of conditions (5) and (2) of Theorem 4.2 covers the Widom-Devinatz-Rochberg theorem (I). Since we can not give a simple necessary and sufficient conditions for an operator S_ϕ to be bounded below in $L^2(W)$, we introduce a new space and get a necessary and sufficient conditions for an operator S_ϕ to be a left invertible operator from $L^2((W))$ to $L^2(W)$. In this section, we shall say that W is a weight function when W is a positive function in L^1. Let $L^2((W))$ denote the Hilbert space which is the completion of the pre-Hilbert space $A + \bar{A}_0$ with the inner product

$$(f, g)_{(W)} = (P_+f, P_+g)_{(W)} + (P_-f, P_-g)_{(W)},$$

and the norm

$$\|f\|_{(W)} = (f, f)_{(W)}^{1/2} = \{\|P_+f\|_W{}^2 + \|P_-f\|_W{}^2\}^{1/2}.$$

Then $\|f\|_W \leqq 2^{1/2}\|f\|_{(W)}$. If $W \in (HS)$, then two norms $\|f\|_{(W)}$ and $\|f\|_W$ are equivalent. Hence the equivalence of conditions (1) and (2) of Theorem 4.2 covers the Widom-Devinatz-Rochberg theorem (I). We shall say that S_ϕ is left invertible, when S_ϕ is bounded above and below as an operator from $L^2((W))$ to $L^2(W)$. For every weight function W and every function ϕ in L^∞, an operator S_ϕ becomes a bounded operator from $L^2((W))$ to $L^2(W)$. Hence S_ϕ is left invertible as an operator from $L^2((W))$ to $L^2(W)$ if and only if S_ϕ is bounded below as an operator from $L^2((W))$ to $L^2(W)$. In Theorem 4.1 and Theorem 4.2, we shall give necessary and sufficient conditions for S_ϕ to be left invertible as an operator from $L^2((W))$ to $L^2(W)$. We use the Cotlar-Sadosky theorem to prove Theorem 4.1 (cf. [45]). We use Theorem 4.1 to prove Theorem 4.2. Each theorem involves the Helson-Szegö theorem (cf. [18]). We shall consider weighted L^2 norm inequalities. When $p \neq 2$, our technique is not useful to study the weighted L^p norm inequality. Prof.T.Nakazi [27] could give a simple necessary and sufficient condition for the L^p-type (left) invertibility of the operator T_ϕ from some new space to $H^2(W)$ in general, and he applied it to the operator $S_{\alpha,\beta}$. His results cover the Widom-Devinatz-Rochberg theorem for any p satisfying $1 < p < \infty$.

Theorem 4.1. Suppose $|\phi| = 1$, W is a weight, δ is a constant satisfying $0 < \delta \leqq 1$, and let

$$r = \delta(2 - \delta^2)^{1/2}.$$

Then the following conditions are mutually equivalent.

(1) $\delta\|f\|_{(W)} \leqq \|S_\phi f\|_W \quad (f \in A + \bar{A}_0).$

(2) There exists an inner function Q and a real function t in L^1 such that $\phi = Qe^{i\tilde{t}}$, and We^{-t} is in $(HS)(r)$.

(3) There exists an inner function Q, a real function t in L^1, u and v in L^∞ such that

$$\phi = Qe^{i\tilde{t}}, \quad We^{-t} = e^{u + \tilde{v}},$$

$$\|v\|_\infty \leqq \cos^{-1} r, \quad |u| \leqq \cosh^{-1}\{(\cos v)/r\}.$$

Proof. We shall use the idea of R.Rochberg (cf. [35]) and the idea of R.Arocena, M.Cotlar and C.Sadosky (cf. [2],[7]). We shall show that (1) implies (2). By (1),

$$\delta^2 \int_T (|f_1|^2 + |f_2|^2)\, W\, dm \leqq \int_T |\phi f_1 + f_2|^2\, W\, dm,$$

for all f_1 in A and f_2 in $\bar{A}_0$. Hence

$$\int_T \{(1 - \delta^2)(|f_1|^2 + |f_2|^2) + 2(\mathrm{Re}\ \phi f_1 \bar{f}_2)\} W dm \geqq 0.$$

By the Cotlar-Sadosky theorem, $\delta \leqq 1$ and there exists a k in H^1 such that $|\phi W - k| \leqq (1 - \delta^2)W$. Hence,

$$|\phi W - k|^2 \leqq (1 - r^2)W^2.$$

By Lemma A, there exists an inner function Q and a real function t in L^1 such that $\phi = \phi W/|\phi W| = Qe^{i\tilde{t}}$, and $|\phi W|e^{-t}$ is in (HS)(r). Hence We^{-t} is in (HS)(r). We shall show that (2) implies (3). By (2), there exists a positive constant C, real functions u and v such that $W = Ce^{u + \tilde{v}}$, $u \in L^1$, $|v| \leqq \pi/2$, and $r^2e^u + e^{-u} \leqq 2(\cos v)$. Hence,

$$2 \leqq e^{u + \log r} + e^{-(u + \log r)} \leqq 2(\cos v)/r.$$

Hence, $|v| \leqq \cos^{-1} r$, $|u + \log r| \leqq \cosh^{-1}\{(\cos v)/r\}$, and $W = (C/r)e^{(u + \log r) + \tilde{v}}$. This implies (3). The proof is reversible. This completes the proof.

Corollary 4.1.([45]) Suppose ϕ is in L^∞ and W is a weight. Then the following conditions are mutually equivalent.

(1) S_ϕ is an isometry from $L^2((W))$ to $L^2(W)$, that is,

$$||S_\phi f||_W = ||f||_{(W)} \quad (f \in A + \bar{A}_o).$$

(2) $|\phi| = 1$, and,

$$||f||_{(W)} \leqq ||S_\phi f||_W \quad (f \in A + \bar{A}_o).$$

(3) There exists an inner function Q and a real function t in L^1 such that $\phi = Qe^{i\tilde{t}}$ and $W = e^t$.

Proof. By (1), for all f_1 in A,

$$\int_T (|\phi|^2 - 1)|f_1|^2 W\, dm = 0.$$

This implies $|\phi| = 1$. Hence (1) implies (2). By Theorem 4.1, (2) and (3) are equivalent. By (3),

$$\phi W = Qe^{t + i\tilde{t}}$$

and ϕW is in L^1. This implies ϕW is in H^1, and hence

$$\int_T |\phi f_1 + f_2|^2 W\, dm - \int_T (|f_1|^2 + |f_2|^2) W\, dm$$

$$= 2\mathrm{Re} \int_T \phi f_1 \bar{f}_2 W\, dm = 0 \quad (f_1 \in A, \ f_2 \in \bar{A}_0).$$

This implies (1). This completes the proof.

Let $H^2(W) \oplus \bar{H}_0^2(W)$ denote the algebraic direct sum of $H^2(W)$ and $\bar{H}_0^2(W)$ (cf.[10, p.78]). Then $H^2(W) \oplus \bar{H}_0^2(W)$ is the Hilbert space equipped with the inner product

$$(<f_1, f_2>, <g_1, g_2>)_{<W>} = (f_1, g_1)_W + (f_2, g_2)_W ,$$

and the norm

$$||<f_1, f_2>||_{<W>} = (<f_1, f_2>, <f_1, f_2>)_{<W>} .$$

For any f in $L^2((W))$, there exists a sequence f_{1n} in A and a sequence f_{2n} in $\bar{A}_0$ such that $f_{1n} + f_{2n}$ converges to f in the norm of $L^2((W))$. Then there exists an f_1 in $H^2(W)$ and an f_2 in $\bar{H}_0^2(W)$ such that $<f_{1n}, f_{2n}>$ converges to

$\langle f_1, f_2\rangle$ in the norm of $H^2(W) \oplus \bar{H}^2_o(W)$.

Let J denote the isometry from $L^2((W))$ onto $H^2(W) \oplus \bar{H}^2_o(W)$ defined by

$$Jf = \langle f_1, f_2\rangle.$$

This definition is correct in the sense that it does not depend on the particular choice of the Cauchy sequence which defines f_1 and f_2.

Let $R_{\phi,W}$ denote the operator from $H^2(W) \oplus \bar{H}^2_o(W)$ to $L^2(W)$ defined by

$$R_{\phi,W}\langle f_1, f_2\rangle = \phi f_1 + f_2 .$$

For all $\langle f_1, f_2\rangle$ in $H^2(W) \oplus \bar{H}^2_o(W)$,

$$\|R_{\phi,W}\langle f_1, f_2\rangle\|_W \leq \max\{\|\phi\|_\infty, 1\}(\|f_1\|_W + \|f_2\|_W)$$

$$\leq 2^{1/2}\max\{\|\phi\|_\infty, 1\}\|\langle f_1, f_2\rangle\|_{\langle W\rangle}.$$

Hence $R_{\phi,W}$ is bounded.

Sometimes we shall wright $S_\phi = S_{\phi,(W)}$ when we consider S_ϕ as an operator from $L^2((W))$ to $L^2(W)$. Since $S_{\phi,(W)} = R_{\phi,W}J$ and J is an isometry, we have the following lemma (cf. [45]). The proof of Lemma 4.1 is clear.

Lemma 4.1. Suppose ϕ is in L^∞, and W is a weight. Then $R_{\phi,W}$ is a bounded operator from $H^2(W) \oplus \bar{H}^2_o(W)$ to $L^2(W)$.

$R_{\phi,W}$ is (left) invertible if and only if $S_\phi = S_{\phi,(W)}$ is (left) invertible as an operator from $L^2((W))$ to $L^2(W)$.

Lemma 4.2.([45]) Suppose ϕ, ϕ^{-1} are in L^∞ and W is a weight. If there exists an inner function Q, outer functions α, β such that $|\alpha|^2W$, $|\beta|^2W$ are in (HS), and $\phi = Q\bar{\beta}/\alpha$, then $R_{\phi,W}$ becomes a left invertible operator from $H^2(W) \oplus \bar{H}_0^2(W)$ to $L^2(W)$, and S_ϕ becomes a left invertible operator from $L^2((W))$ to $L^2(W)$. If an operator T is defined by

$$Tf = \langle \alpha P_+(\bar{Q}f/\bar{\beta}),\ Q\bar{\beta}P_-(\bar{Q}f/\bar{\beta})\rangle,$$

for all f in $L^2(W)$, then T is the left inverse to $R_{\phi,W}$, and $J^{-1}T$ is the left inverse to S_ϕ. Then

$$J^{-1}Tg = (\alpha P_+ + Q\bar{\beta}P_-)(\bar{Q}g/\bar{\beta}),$$

for all g in $\phi A + \bar{A}_0$.

Proof. Since $|\alpha|^2W$, $|\beta|^2W$ are in (HS), $(|\alpha|^2W)^{-1}$, $(|\beta|^2W)^{-1}$ are also in (HS). Hence $(|\alpha|^2W)^{-1}$, $(|\beta|^2W)^{-1}$ are in L^1. For all f in $L^2(W)$, by the Schwarz inequality, $f/\bar{\beta}$ is in L^1. By the Helson-Szegö theorem (cf. [18]), there exist constants γ, γ' such that

$$\|Tf\|_{\langle W\rangle}^2 = \int_T |\alpha P_+(\bar{Q}f/\bar{\beta})|^2 W\,dm + \int_T |Q\bar{\beta}P_-(\bar{Q}f/\bar{\beta})|^2 W\,dm$$

$$\leq \gamma \int_T |\bar{Q}f/\bar{\beta}|^2|\alpha|^2 W\, dm + \gamma' \int_T |\bar{Q}f/\bar{\beta}|^2|\beta|^2 W\, dm$$

$$\leq (\gamma||\phi^{-1}||_\infty^2 + \gamma') \int_T |f|^2 W\, dm.$$

For all f_1 in $H^2(W)$ and f_2 in $\bar{H}_o^2(W)$, by the Schwarz inequality, f_1/α is in H^1 and $\bar{Q}f_2/\bar{\beta}$ is in $\bar{H}_o^1$. Let $f = \phi f_1 + f_2$, so that

$$\alpha P_+(\bar{Q}f/\bar{\beta}) = \alpha P_+(f_1/\alpha + \bar{Q}f_2/\bar{\beta}) = \alpha P_+(f_1/\alpha) = f_1,$$

$$Q\bar{\beta}P_-(\bar{Q}f/\bar{\beta}) = Q\bar{\beta}P_-(f_1/\alpha + \bar{Q}f_2/\bar{\beta}) = Q\bar{\beta}P_-(\bar{Q}f_2/\bar{\beta}) = f_2.$$

Hence $\alpha P_+(\bar{Q}f/\bar{\beta}) \in H^2(W)$, $Q\bar{\beta}P_-(\bar{Q}f/\bar{\beta}) \in \bar{H}_o^2(W)$, and

$$TR_{\phi,W}\langle f_1, f_2\rangle = Tf = \langle f_1, f_2\rangle.$$

Hence T is the left inverse to $R_{\phi,W}$. By Lemma 4.1, $J^{-1}T$ is the left inverse to $S_{\phi,(W)}$. For any g in $\phi A + \bar{A}_o$, there exists a g_1 in A and a g_2 in $\bar{A}_o$ such that $g = \phi g_1 + g_2$. By the calculation, $\alpha P_+(\bar{Q}g/\bar{\beta}) = g_1$, and $Q\bar{\beta}P_-(\bar{Q}g/\bar{\beta}) = g_2$. Hence $\alpha P_+(\bar{Q}g/\bar{\beta})$ is in A, and $Q\bar{\beta}P_-(\bar{Q}g/\bar{\beta})$ is in $\bar{A}_o$. Hence

$$J^{-1}Tg = J^{-1}\langle \alpha P_+(\bar{Q}g/\bar{\beta}),\ Q\bar{\beta}P_-(\bar{Q}g/\bar{\beta})\rangle$$

$$= \alpha P_+(\bar{Q}g/\bar{\beta}) + Q\bar{\beta}P_-(\bar{Q}g/\bar{\beta}) = (\alpha P_+ + Q\bar{\beta}P_-)(\bar{Q}g/\bar{\beta}).$$

This completes the proof.

Theorem 4.2.([45]) Suppose ϕ is in L^∞ and W is a weight. Then the following conditions on ϕ and W are mutually equivalent.

(1) S_ϕ is left invertible as an operator from $L^2((W))$ to $L^2(W)$.

(2) ϕ^{-1} is in L^∞, and there exists an inner function Q and a real function t in L^1 such that $\phi/|\phi| = Qe^{i\tilde{t}}$ and We^{-t} is in (HS).

(3) ϕ^{-1} is in L^∞, and there exists an inner function Q, outer functions α, β such that $|\alpha|^2 W$, $|\beta|^2 W$ are in (HS), and $\phi = Q\bar{\beta}/\alpha$.

(4) ϕ^{-1} is in L^∞, and there exists a k in H^1 and a positive constant ε such that

$$|\phi W - k| \leqq (1 - \varepsilon)|\phi|W.$$

(5) Both of two operators S_ϕ and $S_{-\phi}$ are bounded below as an operator on $L^2(W)$. That is, there exists a positive constant δ such that

$$\delta||f||_W \leqq \min\{||S_\phi f||_W , ||S_{-\phi} f||_W\} \quad (f \in A + \bar{A}_o).$$

Proof. We shall show that (1) implies (4). By (1), there exists a positive constant δ such that

$$\delta||f||_{(W)} \leqq ||S_\phi f||_W ,$$

for all f in $A + \bar{A}_o$. Hence,

$$\int_T \{(|\phi|^2 - \delta^2)|f_1|^2 + (1 - \delta^2)|f_2|^2 + 2\mathrm{Re}(\phi f_1 \bar{f}_2)\} W dm \geqq 0,$$

for all f_1 in A and f_2 in $\bar{A}_o$. By the Cotlar-Sadosky theorem, $0 < \delta \leqq 1$, $\delta \leqq |\phi|$ and there exists a k in H^1 such that

$$|\phi W - k| \leqq (1 - \delta^2)^{1/2}(|\phi|^2 - \delta^2)^{1/2} W$$

$$\leqq (1 - \delta^2)^{1/2}|\phi| W.$$

This implies (4). By Lemma 3.1, (2) and (4) are equivalent. We shall show that (2) implies (3). Put $u = \log |\phi|$, then u is in L^∞. Put

$$\alpha = e^{-\{u + t + i(u + t)^\sim\}/2},$$

$$\beta = e^{\{u - t + i(u - t)^\sim\}/2},$$

then α, β are outer functions, and $\phi = Q\bar{\beta}/\alpha$. Since We^{-t} is in (HS), $|\alpha|^2 W$ and $|\beta|^2 W$ are in (HS). This implies (3). By Lemma 4.2, (3) implies (1). We shall show that (4) implies (1). By (4), there exists a constant δ and a k in H^1 such that $0 < \delta \leqq 1$, $\delta \leqq |\phi|^2$, and $|\phi W - k| \leqq (1 - \delta)|\phi| W$. Then

$$(1 - \delta^2)(|\phi|^2 - \delta^2) - (1 - \delta)^2|\phi|^2$$

$$= \delta(1 - \delta)\{2|\phi|^2 - \delta(1 + \delta)\}$$

$$\geqq 2\delta(1 - \delta)(|\phi|^2 - \delta) \geqq 0.$$

Hence

$$|\phi W - k|^2 \leqq (1 - \delta^2)(|\phi|^2 - \delta^2) W^2.$$

By the Cotlar-Sadosky theorem, for all $f_1 \in A$ and $f_2 \in \bar{A}_0$,

$$\int_T \{(|\phi|^2 - \delta^2)|f_1|^2 + (1 - \delta^2)|f_2|^2 + 2\mathrm{Re}(\phi f_1 \bar{f}_2)\} W dm \geqq 0.$$

This implies (1). We shall show that (1) implies (5). Since $||f||_W \leq ||f||_{(W)}$, we have

$$||Sf||_W \leqq ||Sf||_{(W)} = ||f||_{(W)}.$$

Hence,

$$||f||_W + ||Sf||_W \leqq 2||f||_{(W)}.$$

Since $S_{\phi,(W)}$ is left invertible, there exists a positive constant δ such that for all $f \in A + \bar{A}_0$,

$$\delta(||f||_W + ||Sf||_W) \leqq 2\delta||f||_{(W)} \leqq ||S_\phi f||_W .$$

Hence ,

$$\delta||f||_W \leqq ||S_\phi f||_W, \text{ and}$$

$$\delta||Sf||_W \leqq ||S_\phi f||_W .$$

Since $S^2 f = f$ and $S_\phi Sf = S_\phi(P_+ - P_-)f = \phi P_+ f - P_- f = -S_{-\phi} f$, we have

$$S_\phi f = S_\phi S^2 f = -S_{-\phi} Sf.$$

Hence

$$\delta||Sf||_W \leqq ||S_{-\phi} Sf||_W \quad (f \in A + \bar{A}_0).$$

Since $f \in A + \bar{A}_0$ if and only if $Sf \in A + \bar{A}_0$, we have

$$\delta||f||_W \leqq ||S_{-\phi} f||_W \quad (f \in A + \bar{A}_0).$$

We shall show that (5) implies (1). Since

$$\delta ||f||_W \leq ||S_{-\phi} f||_W,$$

we have

$$\delta ||f||_W \leq ||S_\phi Sf||_W \quad (f \in A + \bar{A}_o).$$

Hence,

$$\delta ||Sf||_W \leq ||S_\phi f||_W \quad (f \in A + \bar{A}_o).$$

Hence,

$$\delta(||f||_W + ||Sf||_W) \leq 2||S_\phi f||_W \quad (f \in A + \bar{A}_o).$$

Since

$$||f||_W^2 + ||Sf||_W^2 = ||P_+f + P_-f||_W^2 + ||P_+f - P_-f||_W^2$$

$$= 2(||P_+f||_W^2 + ||P_-f||_W^2) = 2||f||_{(W)}^2,$$

we have

$$2^{1/2}||f||_{(W)} \leq ||f||_W + ||Sf||_W.$$

Hence,

$$2^{1/2}\delta||f||_{(W)} \leq 2||S_\phi f||_W \quad (f \in A + \bar{A}_o).$$

This completes the proof.

Remark. (a) If $S_{\phi,(W)}$ is left invertible, then $\log W$ is in L^1, and there exists an inner function Q, real functions u, v in L^∞ such that $||v||_\infty < \pi/2$ and

$$\phi/|\phi| = Qe^{i\{v - (u - \log W)\sim\}}.$$

(b) By condition (2), $S_{\phi,(W)}$ is left invertible if and only if ϕ^{-1} is in L^∞ and $S_{\phi/|\phi|,(W)}$ is left invertible.

The equivalence of conditions (3) and (4) in Corollary 4.2 is the Helson-Szegö theorem (cf. [18]). Since $||f||_W{}^2 + ||Sf||_W{}^2 = 2||f||_{(W)}{}^2$, we have

$$||f||_W \leqq 2^{1/2}||f||_{(W)} .$$

Corollary 4.2.([45]) For a weight W, the following conditions are mutually equivalent.

(1) $||S_{1,(W)}|| < 2^{1/2}$. That is, there exists a positive constant ε such that

$$||f||_W \leqq (2^{1/2} - \varepsilon)||f||_{(W)} \quad (f \in A + \bar{A}_o).$$

(2) $S_{1,(W)}$ is left invertible. That is, there exists a positive constant δ such that

$$\delta||f||_{(W)} \leqq ||f||_W \quad (f \in A + \bar{A}_o).$$

(3) There exists a positive constant γ such that

$$||P_+f||_W \leqq \gamma||f||_W \quad (f \in A + \bar{A}_o).$$

(4) W is in (HS).

(5) There exists a k in H^1 and a positive constant ε such that

$$|W - k| \leqq (1 - \varepsilon)W.$$

Proof. We shall show that (1) implies (2). By (1), there exists a positive constant δ such that

$$\|f_1 + f_2\|_W^2 \leqq (2 - \delta^2)(\|f_1\|_W^2 + \|f_2\|_W^2),$$

for all $f_1 \in A$ and $f_2 \in \bar{A}_o$. Hence,

$$(1 - \delta^2)\|f_1\|_W^2 + (1 - \delta^2)\|f_2\|_W^2 + 2\mathrm{Re}(f_1, f_2)_W \geqq 0.$$

Hence,

$$\delta(\|f_1\|_W^2 + \|f_2\|_W^2)^{1/2} \leqq \|f_1 + f_2\|_W .$$

This implies (2). Since $\|P_+ f\|_W \leqq \|f\|_{(W)}$, (2) implies (3). We shall show that (3) implies (2). By (3),

$$\|P_- f\|_W \leqq \|P_+ f\|_W + \|f\|_W \leqq \gamma'\|f\|_W,$$

for some constant γ'. Hence

$$\|f\|_{(W)}^2 = \|P_+ f\|_W^2 + \|P_- f\|_W^2 \leqq (\gamma^2 + \gamma'^2)\|f\|_W^2.$$

This implies (2). We shall show that (2) implies (4). By Theorem 4.2, there exists an inner function Q and a real function t in L^1 such that We^{-t} is in (HS) and $Qe^{i\tilde{t}} = 1$. Since $W^{-1}e^t$ is also in (HS), $W^{-1}e^t$ is in L^1. By the Schwarz inequality, $e^{t/2}$ is in L^1. Since $Qe^{t + i\tilde{t}} = e^t$, a positive function e^t is in $H^{1/2}$. By the Neuwirth-Newman theorem (cf.[30]), V is a constant. Hence W is in (HS). Conversely when W is in (HS), we can choose $Q = 1$, $t = 0$, and $\phi = 1$ in the condition (2) of Theorem 4.2. Hence (4) implies (2). By Theorem 4.2 with $\phi = 1$, (2) and (5) are equivalent. This completes the proof.

Put $W(e^{i\theta}) = |1 - e^{i\theta}|^2$, $\phi(e^{i\theta}) = e^{i\theta}$ and $k(e^{i\theta}) = (1 - e^{i\theta})^2$, then k is in H^1 and $\phi W + k = 0$. By Theorem 4.2, this implies $S_{\phi,(W)}$ is left invertible. Hence there exists a positive constant δ such that

$$2^{-1/2}\delta||f||_W \leqq \delta||f||_{(W)} \leqq ||S_\phi f||_W \quad (f \in A + \bar{A}_o).$$

Since W^{-1} is not in L^1, W is not in (HS). By Corollary 4.2, $S_{1,(W)}$ is not left invertible. $S_{1,W}$ is an isometry. But we have the following result.

Corollary 4.3.([29],[45]) Suppose $\phi \in L^\infty$ and ess $\inf|1 - \phi| > 0$, and W is a weight. If there exists a positive constant δ such that

$$\delta||f||_W \leqq ||S_\phi f||_W \quad (f \in A + \bar{A}_o),$$

then for any constant ε satisfying $0 \leqq \varepsilon \leqq \delta^2$ there exists a positive constant δ' such that

$$\delta'||f||_{(W)} \leqq ||S_{\phi-\varepsilon} f||_W \quad (f \in A + \bar{A}_o),$$

that is $S_{\phi-\varepsilon}$ is left invertible as an operator from $L^2((W))$ to $L^2(W)$.

Proof. If $\varepsilon = 0$, then by Theorem 3.1 and Theorem 4.2 give the result. Suppose $0 < \varepsilon \leqq \delta^2$. Then,

$$\int_T \{(|\phi|^2 - \varepsilon)|f_1|^2 + (1 - \varepsilon)|f_2|^2 + 2\mathrm{Re}((\phi - \varepsilon)f_1\bar{f}_2)\}Wdm \geqq 0.$$

By the Cotlar-Sadosky theorem, $\varepsilon \leqq |\phi|^2$, $\varepsilon \leqq 1$ and there exists a k in H^1 such that

$$|(\phi - \varepsilon)W - k|^2 \leqq (1 - \varepsilon)(|\phi|^2 - \varepsilon)W^2$$

$$= \{1 - \varepsilon(|\phi - 1|/|\phi - \varepsilon|)^2\}|(\phi - \varepsilon)W|^2.$$

Since ϕ and $(\phi - 1)^{-1}$ are in L^∞, there exists a constant ρ, $0 \leqq \rho < 1$ such that

$$\varepsilon(|\phi - 1|/|\phi - \varepsilon|)^2 \geqq 1 - \rho^2.$$

Hence

$$|(\phi - \varepsilon)W - k| \leqq \rho|\phi - \varepsilon|W.$$

Since

$$|\phi - \varepsilon| \geqq |\phi| - \varepsilon \geqq \varepsilon^{1/2}(1 - \varepsilon^{1/2}) > 0,$$

$(\phi - \varepsilon)^{-1}$ is in L^∞, and

$$|(\phi - \varepsilon)W - k| \leqq \rho|\phi - \varepsilon|W.$$

By Theorem 4.2, this implies $S_{\phi-\varepsilon,(W)}$ is left invertible.

Corollary 4.4.([45]) Suppose ϕ is in L^∞ and W is a weight. If there exists a real function s in L^1 such that $\phi = e^{is}|\phi|$, and $We^{\tilde{s}}$ is in L^1, then the following conditions are mutually equivalent.

(1) S_ϕ and $S_{-\phi}$ are bounded below operators on $L^2(W)$.

(2) ϕ^{-1} is in L^∞, and $We^{\tilde{s}}$ is in (HS).

Proof. By Theorem 4.2, (1) implies ϕ^{-1} is in L^∞ and there exists a k in H^1 such that

$$||1 - k/(\phi W)||_\infty < 1.$$

Hence

$$||1 - (ke^{\tilde{s} - is})/(|\phi|We^{\tilde{s}})||_\infty < 1.$$

Since $|\phi|We^{\tilde{s}}$ is in L^1, $ke^{\tilde{s} - is}$ is in H^1. By Corollary 4.2, $|\phi|We^{\tilde{s}}$ is in (HS) and hence $We^{\tilde{s}}$ is in (HS). Conversely, (2) implies $|\phi|We^{\tilde{s}}$ is in (HS). By Corollary 4.2, there exists a k in H^1 such that

$$||1 - k/(|\phi|We^{\tilde{s}})||_\infty < 1.$$

Hence $||1 - ke^{is - \tilde{s}}/(\phi W)||_\infty < 1$. By Theorem 4.2, this implies (1). This completes the proof.

Corollary 4.5.([45]) Suppose ϕ is in L^∞ and W is a weight. Suppose the argument of ϕ is in L^1 and it's harmonic conjugate function is in L^∞. (This condition is satisfied if ϕ is invertible in H^∞, or the argument of ϕ is Dini continuous.) Then the following conditions are mutually equivalent.

(1) S_ϕ and $S_{-\phi}$ are bounded below operators on $L^2(W)$.

(2) ϕ^{-1} is in L^∞, and W is in (HS).

Proof. There exists a real function s in L^1 such that $\phi = e^{is}|\phi|$ and $\tilde{s}$ is in L^∞. Hence $We^{\tilde{s}}$ is in L^1. By Corollary 4.4, ϕ and W satisfy (1) if and only if ϕ^{-1} is in L^∞, and $We^{\tilde{s}}$ is in (HS). Since $e^{\tilde{s}}$ is invertible in L^∞, $We^{\tilde{s}}$ is in (HS) if and only if W is in (HS). This completes the proof.

When P_+ is continuous in the norm of $L^p(W)$, $1 < p < \infty$, R.Rochberg [35] solved the invertibility problem of the Toeplitz operator on the weighted Hardy space $H^p(W)$. Even when P_+ is not continuous in the norm of $L^p(W)$, T.Nakazi [27] solved the invertibility problem of the Toeplitz operator on some new spaces. We shall consider the case $p = 2$, and use the Hilbert space argument. When $S_{\phi,(W)}$ has a bounded inverse operator, we shall say $S_{\phi,(W)}$ is invertible.

Prof.T.Nakazi privately communicated me the equivalence of simple conditions (1) and (2) in Theorem 4.3 (cf. [45]). We shall prove Theorem 4.3 using Theorem 4.2. In Theorem 4.3, we shall give the form of the inverse to $S_{\phi,(W)}$.

Theorem 4.3. Suppose ϕ is in L^∞ and W is a weight. Then the following conditions are mutually equivalent.

(1) S_ϕ is invertible as an operator from $L^2((W))$ to $L^2(W)$.

(2) ϕ^{-1} is in L^∞, and there exists a real constant c and a real function t in L^1 such that We^{-t} is in (HS), and

$$\phi/|\phi| = e^{i(c + \tilde{t})}.$$

(3) ϕ^{-1} is in L^∞, and there exist outer functions α, β such that $|\alpha|^2 W$, $|\beta|^2 W$ are in (HS), and $\phi = \bar{\beta}/\alpha$.

(4) ϕ^{-1} is in L^∞, and there exists an outer function k in H^1 and a positive constant ε such that

$$|\phi W - k| \leqq (1 - \varepsilon)|\phi|W.$$

Suppose one of these conditions are satisfied. Let T be the operator defined in Lemma 4.2 with $Q = 1$. Then $S_{\phi,(W)}^{-1} = J^{-1}T$, and

$$(\alpha P_+ + \bar{\beta} P_-)(1/\bar{\beta})(S_{\phi,(W)} g) = f \quad (f \in A + \bar{A}_0).$$

(This formula is essentially the same as one of H.Widom, A.Devinatz, R.Rochberg and M.Shinbrot.)

Proof. We shall show that (1) implies (2). Since $S_{\phi,(W)}$ is invertible, by Theorem 4.2, there exists an inner function Q and a real function t in L^1 such that We^{-t} is in (HS), and $\phi/|\phi| = Qe^{i\tilde{t}}$. Since $S_{\phi,(W)}$ is invertible, there exists an f in $L^2((W))$ such that $S_{\phi,(W)}f = 1$. Hence there exists an f_1 in $H^2(W)$ and an f_2 in $\bar{H}_o^2(W)$ such that $\phi f_1 + f_2 = 1$. Then,

$$Qf_1(1 - \bar{f}_2)e^{i\tilde{t} + t} = |1 - f_2|^2 W/(|\phi|We^{-t}) \geq 0.$$

Since ϕ is invertible in L^∞ and We^{-t} is in (HS), $(|\phi|We^{-t})^{-1}$ is in L^1. Since f_2 in $\bar{H}_o^2(W)$, $|1 - f_2|^2 W$ is in L^1. Hence the left hand side is a non-negative function in $H^{1/2}$. By the Neuwirth-Newman theorem, $Q = e^{ic}$ for some real constant c. Hence $\phi/|\phi| = e^{i(c + \tilde{t})}$. This implies (2). By Theorem 4.2 and it's proof with $Q = e^{ic}$, (2) implies (3). We shall show that (3) implies (1). By Lemma 4.1 and Lemma 4.2, it is sufficient to show that $R_{\phi,W}$ is right invertible. Let T be the operator defined in Lemma 4.2 with $Q = 1$. By (3), $\log W$ is in L^1. Hence there exists an outer function h in H^2 such that $W = |h|^2$. Since $|\beta|^2 W$ is in (HS), $(|\beta|^2 W)^p$ is also in (HS) for some p, $p > 1$. Hence $(|\beta|^2 W)^{-p}$ is in L^1. For all f in $L^2(W)$,

$$\int_T |f/\bar{\beta}|^{2p/(p+1)} dm$$

$$\leq \{\int_T |f|^2 W dm\}^{p/(p+1)}\{\int_T (|\beta|^2 W)^{-p} dm\}^{1/(p+1)} < \infty.$$

Since $2p/(p + 1) > 1$, by the Riesz theorem (cf.[22, p.132]), $P_+(f/\bar{\beta})$ is in $H^{2p/(p+1)}$. Since $|\alpha|^2W$ is in (HS), by the Helson-Szegö theorem, there exists a constant γ such that for all f in $L^2(W)$,

$$\int_T |\alpha h P_+(f/\bar{\beta})|^2 \, dm = \int_T |P_+(f/\bar{\beta})|^2 |\alpha|^2 W \, dm$$

$$\leqq \gamma \int_T |f/\bar{\beta}|^2 |\alpha|^2 W \, dm \leqq \gamma \|\phi^{-1}\|_\infty^2 \int_T |f|^2 W \, dm < \infty.$$

Hence $\alpha h P_+(f/\bar{\beta})$ is in H^2. Similarly, $\bar{\beta}\bar{h}P_-(f/\bar{\beta})$ is in $\bar{H}_o^2$. By the Beurling theorem (cf.[22, p.110]), there exists a sequence g_n in A such that hg_n converges to $\alpha h P_+(f/\bar{\beta})$ in the norm of L^2. Hence g_n converges to $\alpha P_+(f/\bar{\beta})$ in the norm of $L^2(W)$. This implies $\alpha P_+(f/\bar{\beta})$ is in $H^2(W)$. Similarly, $\bar{\beta}P_-(f/\bar{\beta})$ is in $\bar{H}_o^2(W)$. Hence

$$R_{\phi,W}Tf = R_{\phi,W}\langle \alpha P_+(f/\bar{\beta}),\ \bar{\beta}P_-(f/\bar{\beta})\rangle$$
$$= \phi\alpha P_+(f/\bar{\beta}) + \bar{\beta}P_-(f/\bar{\beta}) = \bar{\beta}(P_+ + P_-)(f/\bar{\beta}) = f.$$

Hence $T = R_{\phi,W}^{-1}$. By the proof of Lemma A, (2) and (4) are equivalent. This completes the proof.

Remark. (a) R.Rochberg[35] showed that if We^{-t} and $We^{-t'}$ are in (HS), and $e^{i(c + \tilde{t})} = e^{i(c' + \tilde{t}')}$, then $t - t'$ is a constant.

(b) If $|\alpha|^2W$, $|\beta|^2W$, $|\alpha'|^2W$ and $|\beta'|^2W$ are in (HS) and $\bar{\beta}/\alpha = \bar{\beta}'/\alpha'$, then there exists a constant c such that $\alpha' = c\alpha$ and $\beta' = \bar{c}\beta$, since α'/α, β'/β and their complex conjugate functions are in H^1, and hence they are constants.

(c) If W^{-1} is in L^1 and $S_{\phi,(W)}$ is invertible, then $S_{\phi,W}$ and $S_{-\phi,W}$ have a dense range, and there exists a positive constant δ such that

$$\delta||f||_W \leqq \min\{||S_\phi f||_W \ , \ ||S_{-\phi} f||_W\} \quad (f \in A + \bar{A}_0).$$

Corollary 4.6.([45]) Suppose ϕ is in L^∞ and W is a weight such that W^{-1} is in L^1. Then, $S_{\phi,(W)}$ is invertible if and only if $S_{\phi,(W)}$ and $S_{\bar{\phi},(W^{-1})}$ are left invertible.

Proof. Suppose $S_{\phi,(W)}$ and $S_{\bar{\phi},(W^{-1})}$ are left invertible. By Theorem 4.2, there exist inner functions Q, Q' and real functions t, t' in L^1 such that We^{-t}, $W^{-1}e^{-t'}$ are in (HS), and $\phi/|\phi| = Qe^{i\tilde{t}}$, $\bar{\phi}/|\phi| = Q'e^{i\tilde{t}'}$. Hence

$$QQ'e^{(t + t') + i(t + t')^\sim} = e^{t + t'} \geqq 0.$$

Since $W^{-1}e^t$, $We^{t'}$ are in L^1, $e^{(t + t')/2}$ is in L^1. By the Neuwirth-Newman theorem, Q and Q' are constants. By Theorem 4.3, $S_{\phi,(W)}$ is invertible. Suppose $S_{\phi,(W)}$ is invertible. By Theorem 4.3, there exists a real constant c and a real function t in L^1 such that We^{-t} is in (HS), and $\phi/|\phi| =$

$e^{i(c + \tilde{t})}$. Hence $W^{-1}e^{t}$ is in (HS), and $\bar{\phi}/|\phi| = e^{-i(c + \tilde{t})}$.

By Theorem 4.2, this implies $S_{\bar{\phi},(W^{-1})}$ is left invertible.

This completes the proof.

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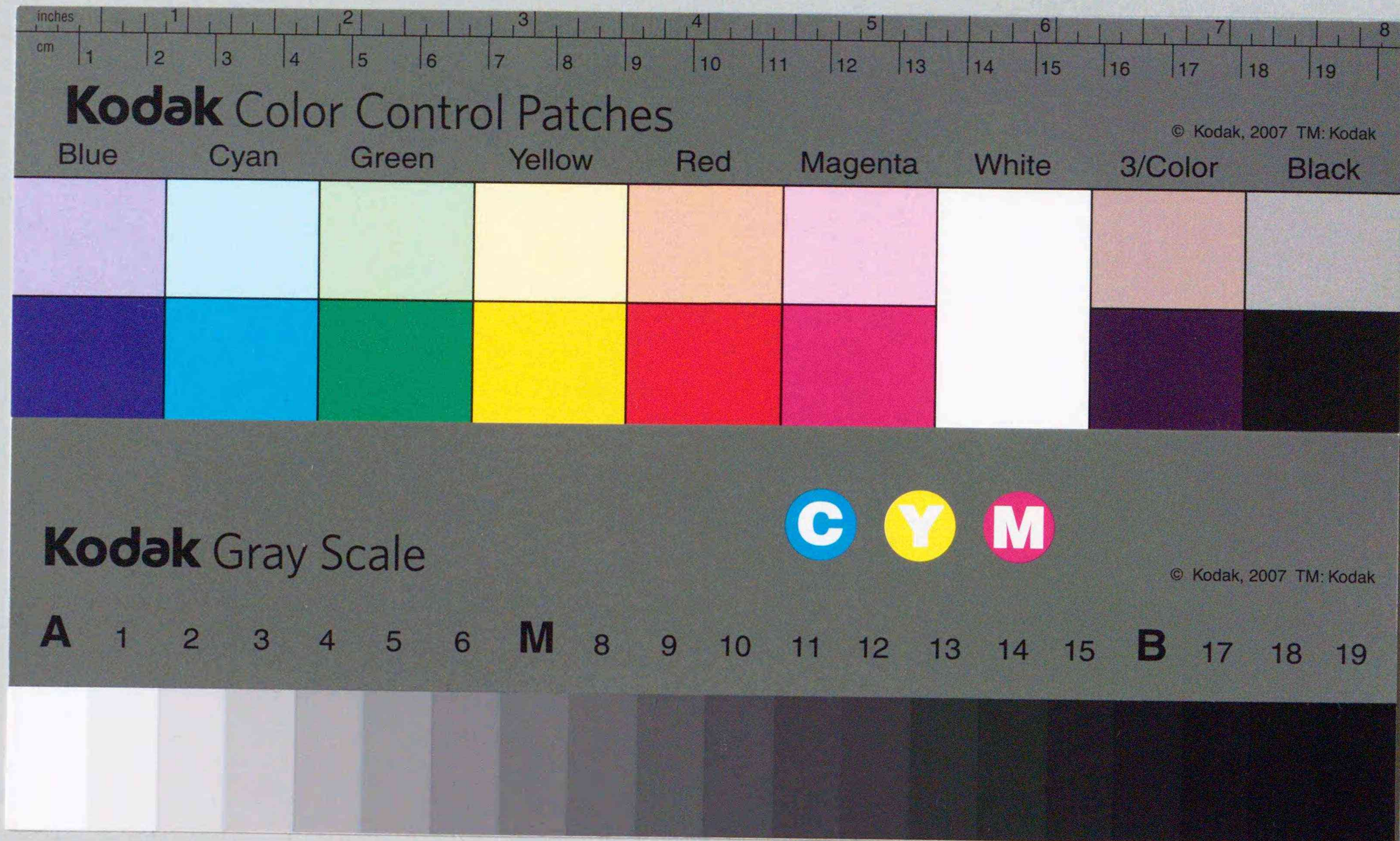
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