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0. Introduction

Let p be a prime number. When A and B are non-degenerate symmetric matrices over $\mathbb{Z}_p$ of degree m and n $(m \geq n)$, respectively, we define the local density $\alpha_p(B, A)$ by

$$\alpha_p(B,A) = \lim_{e\to\infty} p^{-te} \# A_{p^e}(B,A), \tag{0.1}$$

where $t = mn - n(n+1)/2$, $A_{p^e}(B,A) = \{\bar{X} \in M_{mn}(\mathbb{Z}_p)/p^e M_{mn}(\mathbb{Z}_p); A[X] \equiv B \bmod p^e\}$, and $A[X] = {}^tXAX$.

In spite of the importance of local densities, it is not easy to calculate their explicit values.

In [9], Kitaoka introduced the formal power series

$$P(B,A;x) = \sum_{r=0}^{\infty} \alpha_p(p^r B, A)x^r. \tag{0.2}$$

He proved it is a rational function of x and obtained its denominator for $A = \begin{pmatrix} O_k & E_k \\ E_k & O_k \end{pmatrix}$ where E_k (resp. O_k) is the identity matrix of degree k(resp. the zero matrix of degree k). Moreover he conjectured the rationality for an arbitrary symmetric A. Its rationality and the denominator have been investigated by Böcherer-Sato, Hironaka, and Katsurada in special cases.

In [2], Hironaka proved the rationality and the determined its denominator for arbitrary A and B in the case $p \neq 2$.

Böcherer and Sato ([1]) showed the rationality of the series for arbitrary A and B. They also determined its denominator in the following two cases by applying Denef's theory of p-adic integrals;

(1) m is even and A is unimodular,

(2) B is anisotropic.

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When B is decomposed into the form $B = B_0 \perp \cdots \perp B_s$, they generalized (0.2) and defined a formal power series of several variables

$$\begin{aligned} & P((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s) \\ (0.3) \qquad & = \sum_{r_1, \ldots, r_s = 0}^{\infty} \alpha_p(B_0 \perp p^{r_1} B_1 \perp \cdots \perp p^{r_s} B_s, A) x_1^{r_1} \ldots x_s^{r_s}. \end{aligned}$$

If $s = 1$ and $\deg(B_0) = 0$, then this is nothing but the previous one. They proved that this represents a rational function of $x_1, \ldots, x_l$ and that every factor of the denominator is of the form

$$1 - p^{-c} x_1^{a_1} \ldots x_s^{a_s} \quad (c, a_1, \ldots, a_s \in \mathbb{Z}).$$

In [3], Katsurada introduced the local density with a congruence condition and proved the rationality of (0.3) in the case that $\deg(B_0) = 0$. He also obtained the explicit denominator for any diagonal matrix B in the case $p \neq 2$ and $\deg(B_0) = 0$. Moreover, in [4], he modified formal power series into

$$\begin{aligned} & R((B_1, \ldots, B_s), A; x_1, \ldots, x_s) \\ (0.4) \qquad & = \sum_{r_1 \geq \cdots \geq r_s \geq 0} \alpha_p(p^{r_1} B_1 \perp \cdots \perp p^{r_s} B_s, A) x_1^{r_1} \ldots x_s^{r_s}, \end{aligned}$$

where $B = B_1 \perp \cdots \perp B_s$, $\deg(B_i) = n_i$. He and the present author proved the rationality of the series when A is even unimodular and B is diagonal in [5].

In the present paper, we show the rationality and determine the denominators of this series when A and B are arbitrary matrices. We would like to emphasize the method for $p \neq 2$ in [3] cannot be directly applied to the odd unimodular case.

We define the following formal power series;

$$\begin{aligned} & Q((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s) \\ (0.5) \qquad & = \sum_{r_1 \geq \cdots \geq r_s \geq 0} \alpha_p(B_0 \perp p^{r_1} B_1 \perp \cdots \perp p^{r_s} B_s, A) x_1^{r_1} \ldots x_s^{r_s}. \end{aligned}$$

Note that

$$Q((\varnothing, B_1, \ldots, B_s), A; x_1, \ldots, x_s) = R((B_1, \ldots, B_s), A; x_1, \ldots, x_s).$$

Our main results are as follows:

Theorem 1. *Let A and B be are arbitrary non-degenerate symmetric matrices as above. Let t be the Witt index of A. Put*

$$u_k = \min(n_1 + \cdots + n_k - 1, t)$$

and

$$v_k = \begin{cases} 1 & t \geq n_1 + \cdots + n_k \\ 0 & \textit{otherwise}. \end{cases}$$

Then $P((B_0, B_1, \dots, B_s), A; x_1, \cdots, x_s)$ *is a rational function of* $x_1, \dots, x_s$ *with a denominator*

$$\prod_{k=1}^{s}(1 - x_1 \dots x_k)^{v_k} \prod_{i=0}^{u_k}(1 - p^{(n_1+\cdots+n_k-i)(-m+n+i+1)}(x_1 \dots x_k)^2).$$

Theorem 1, in the case A is even unimodular and $\deg(B_0) = 0$, has been proved by Katsurada and the present author in [6]. The crucial part of the present paper is to prove Theorem 1 when A is odd unimodular.

Remark that the value of the local density coincides with the special value of singular series, hence it is meaningful to study not only the even unimodular case but also the arbitrary case.

If $s = 1$ and $B_0 = \varnothing$, then by definition $P(B, A; x) = Q((\varnothing, B), A; x)$.

Corollary. *The series* $P(B, A; x)$ *is a rational function of* x *with a denominator*

$$(1 - x)^v \prod_{i=0}^{u}(1 - p^{(n-i)(-m+n+i+1)}x^2),$$

where $u = \min(n - 1, t)$ *and* $v = 1$ *or* 0 *according as* $t \geq n$ *or not.*

Since $P((B_0, B_1, \dots, B_s), A; x_1, \cdots, x_s)$ is a linear combination of the series similar to $Q((B_0, B_1, \dots, B_s), A; x_1, \cdots, x_s)$, we have the following:

Theorem 2. *Let* A, B, t, u_k *and* v_k *be as above. Then* $P((B_0, B_1, \dots, B_s), A; x_1, \cdots, x_s)$ *is a rational function of* $x_1, \dots, x_s$ *with a denominator*

$$\prod_{k=1}^{s}(1 - x_1 \dots x_k)^{v_k} \prod_{i=0}^{u_k}(1 - p^{(n_1+\cdots+n_k-i)(-m+n+i+1)}(x_1 \dots x_k)^2).$$

Notation. For a set $Z, \#Z$ denotes the cardinality of Z. For a commutative ring with an identity element R, we denote by $M_{mn}(R)$ the ring of (m, n)-matrices with entries in R. Put $M_m(R) = M_{mm}(R)$. When $m = 0$ or $n = 0$, we understand $M_{mn}(R) = \varnothing$. $GL_m(R)$ is the group of all invertible elements of $M_m(R)$. We call elements of $GL_m(R)$ unimodular. Further let $S_n(R)$ denote the set of all symmetric matrices in $M_n(R)$. Put $\mathbb{Z}_{\geq 0} = \{x \in \mathbb{Z}; x \geq 0\}$. By $\mathbb{Q}_p$ and $\mathbb{Z}_p$ we denote the p-adic number field and the ring of p-adic integers, respectively. For any square matrix A, $\mathrm{tr}(A)$ denotes the trace of A. For a symmetric unimodular matrix A with entries in $\mathbb{Z}_p$, we say A is even unimodular if all the diagonal elements belong to $2\mathbb{Z}_p$, otherwise we say A is odd unimodular.

Let M be a R-module and Q a mapping from M to R which satisfies the conditions (1) $Q(rm) = r^2Q(m)$ for $r \in R, m \in M$, (2) $2B(m, n) := Q(m + n) - Q(m) - Q(n)$ is a symmetric bilinear form. We call (M, Q) or simply M a quadratic module over R, Q a quadratic form and B an associated symmetric bilinear form. For quadratic modules M and N over $\mathbb{Z}_p$, if they are isometric, we write $M \simeq N$. For square matrices U and V, we denote $\begin{pmatrix} U & O \\ O & V \end{pmatrix}$ by $U \perp V$. For an element a of R and an element X of $M_{mn}(R)$, we often use the same symbol X for the class X modulo $aM_{mn}(R)$. We denote by 1_{mn} (resp. 0_{mn}) the (m,n)-matrices whose components are all 1 (resp. 0). For any matrices $A, B \in M_n(\mathbb{Z}_p)$, we say A and B are $\mathbb{Z}_p$-equivalent if there exists a matrix $X \in GL_n(\mathbb{Z}_p)$ satisfying $A[X] = B$. Finally, for an element a of $\mathbb{Q}_p$, ν denotes the integer n such that $a \in p^n\mathbb{Z}_p$ and $a \notin p^{n+1}\mathbb{Z}_p$.

1. Preliminary results

For $S \in S_m(\mathbb{Z}_p), T \in S_n(\mathbb{Z}_p), \Xi = (\xi_i) \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^m$, and a non-negative integer e, we put (cf. [2])

$$A_e(T,S;\Xi) = \{X = (x_{ij}) \in A_e(T,S); x_{i1} \equiv \xi_i \bmod p^e \text{ for } 1 \le i \le m\}$$

and

$$a_e(T,S;\Xi) = \# A_e(T,S;\Xi).$$

For a non-negative integer $r < m+n$, put

$$A_e(r;T,S) = \{X \in M_{m+n-r}(\mathbb{Z}_p/p^e\mathbb{Z}_p); (S \perp -T)[\begin{pmatrix} X \\ E_r \end{pmatrix}] \equiv O_r \mod p^e\},$$

where E_r is the identity matrix of degree r. Put

$$a_e(r;T,S) = \# A_e(r;T,S).$$

Moreover, put

$$\alpha_p(r;T,S) = \lim_{e\to\infty} p^{-e(((m+n-r)r-<r>)} a_e(r;T,S).$$

This limit exists as will be shown in Theorem 3.2. Note that if $T = t \perp T_1$, then we have $\alpha_p(n-1;T,S) = \alpha_p(T_1, S \perp (-t))$.

Put $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. If $p \ne 2$, it is well known that a non-degenerate symmetric matrix is equivalent over $\mathbb{Z}_p$ to a unique matrix of the following form:

$$\perp_{i=0}^{r} p^i(V_i \perp U_i),$$

where

$$V_i = \overbrace{H \perp \cdots \perp H}^{l_i} \text{ with } l_i \ge 0$$

and

$$U_i = \varnothing, (c) \text{ or } c_1 \perp c_2 \text{ with } c, c_1, c_2 \in \mathbb{Z}_p^\times \text{ and } -c_1c_2 \notin (\mathbb{Z}_p^\times)^2.$$

Here $\varnothing$ denotes the empty matrix.

If $p=2$, we review the result of Watson [12] in a modified way (cf.[6]). A non-degenerate symmetric matrix with entries in $\mathbb{Z}_2$ is equivalent over $\mathbb{Z}_2$ to a unique matrix of the following form:

$$\perp_{i=0}^{r} 2^i(V_i \perp U_i),$$

where

$$V_i = \overbrace{H \perp \cdots \perp H}^{l_i} \perp Y_i \text{ with } l_i \geq 0 \text{ and } Y_i = \varnothing \text{ or } Y$$

and

$$U_i = \perp_{j=1}^{k_i} c_{ij} \text{ with } c_{ij} \in \mathbb{Z}_2^\times, 0 \leq k_i \leq 2$$

satisfying the following conditions:

(C.1) $c_{i1}=1$ or 3 if $k_i=1$, and $(c_{i1},c_{i2})=(1,\pm 1),(1,\pm 3),(-1,-1)$, or $(-1,3)$ if $k_i=2$,

(C.2) $k_i=k_{i-2}=0$ if $Y_{i-1}=Y$,

(C.3) $-\det U_i \equiv 1 \mod 4$ if $Y_i=Y$ and $k_i=2$,

(C.4) $(-1)^{k_i-1}\det U_i \equiv 1 \mod 4$ if $k_{i+1}\neq 0, k_i \neq 0$,

(C.5) $U_i \neq -1 \perp -c_{i2}$ if $k_{i-1}\neq 0$,

(C.6) $U_i = \varnothing, (1), 1\perp \pm 1, -1 \perp -1$ if $k_{i+2}\neq 0$.

The matrix satisfying the above conditions (C.1) $\sim$ (C.6) is said to be canonical form. Let the notation be as above, and $i_1,\ldots,i_s$ be non-negative integers such that $i_1<\cdots<i_s\leq r$ and $\deg(U_{i_k}\perp V_{i_k})\geq 1$ for $k=1,\ldots,s$. Then we call $(i_1,\ldots,i_s)$ the exponent.

For $\mathbb{Z}_p$-modules V and W, let $\mathrm{Hom}_{\mathbb{Z}_p}(W,V)$ be the set of $\mathbb{Z}_p$-homomorphisms from W to V. For an element ϕ of $\mathrm{Hom}_{\mathbb{Z}_p}(W,V)$, we use the same symbol ϕ for the image of the natural projection $\pi: \mathrm{Hom}_{\mathbb{Z}_p}(W,V) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(W,V)\otimes \mathbb{Z}_p/p^e\mathbb{Z}_p$. For quadratic modules V,W over $\mathbb{Z}_p$ and an integer $e\geq 0$, we set

$$L_e(W,V)$$
$$=\{\phi \in \mathrm{Hom}_{\mathbb{Z}_p}(W,V)\otimes \mathbb{Z}_p/p^e\mathbb{Z}_p; B(\phi(w),\phi(w'))\equiv B(w,w') \mod p^e \text{ for } w,w'\in W\}.$$

Further, for a quadratic submodule W_1 of W and $\phi_1 \in L_e(W_1,V)$ put

$$L_e(W,V;\phi_1)=\{\phi\in L_e(W,V); \phi|W_1 \equiv \phi_1 \mod p^eV\},$$

where $\phi|W_1$ denotes the restriction of ϕ to W_1.

Let $V=\mathbb{Z}_p[v_1,\ldots,v_m]$ and $W=\mathbb{Z}_p[w_1,\ldots,w_n]$ be quadratic modules over $\mathbb{Z}_p$ satisfying $(B(v_i,v_j))=S,(B(w_i,w_j))=T$. Put $\phi_1(w_1)=\sum_{i=1}^m \xi_i v_i$ with $\xi_i\in\mathbb{Z}_p$. Then it is easy to see that

$$\#L_e(W,V)=\#A_e(T,S),$$

and

$$\#L_e(W,V;\phi_1)=\#A_e(T,S;(\xi_i)).$$

Let V and W be as above, and V_1, V_2 be quadratic submodules of V such that $V = V_1 \oplus V_2$. Assume that the $\mathbb{Z}_p$-rank of V_2 and W is equal. Let $\mathbf{v}= \{v_1, \ldots, v_n\}$, $\mathbf{w}= \{w_1, \ldots, w_n\}$ be the $\mathbb{Z}_p$-basis of V_2 and W, respectively. Then we set

$$L_e(W, V, V_1, V_2; \mathbf{w},\mathbf{v}) = \{\phi \in L_e(W, V); \mathrm{Pr}_{V \to \mathbb{Z}_p[v_i]}\phi(w_j) = \delta_{ij}v_i \text{ for } 1 \leq i, j \leq n\},$$

where δ_{ij} is the Kronecker's delta and $\mathrm{Pr}_{V \to \mathbb{Z}_p[v_i]}$ denotes the projection of V to $\mathbb{Z}_p[v_i]$. Further for $W_1 = \mathbb{Z}_p[w_1]$ and $\phi_1 \in L_e(W_1, V)$ such that $\mathrm{Pr}_{V \to \mathbb{Z}_p[v_i]}\phi_1(w_1) = \delta_{i1}v_i$ $(1 \leq i \leq n)$ put

$$L_e(W, V; V_1, V_2; \mathbf{w},\mathbf{v}; W_1, \phi_1) = \{\phi \in L_e(W, V; V_1, V_2; \mathbf{w},\mathbf{v}); \phi|W_1 = \phi_1\}.$$

The above two sets depend on $\mathbf{w},\mathbf{v}$ as well as W, V, V_1 and V_2. However we have the following([6]).

Proposition 1.1. *Assume that $W = \mathbb{Z}_p[w_1, \ldots, w_n]$ is totally singular and $V = V_1 \perp V_2$ where $V_2 = \mathbb{Z}_p[v_1, \ldots, v_n]$. Let $V_2' = \mathbb{Z}_p[v_1', \ldots, v_n']$ be a quadratic module such that*

$$B(v_i', v_j') = -B(v_i, v_j) \ \ (1 \leq i, j \leq n).$$

Then there exists a bijection from $L_e(W, V; V_1, V_2; \mathbf{w},\mathbf{v})$ to $L_e(V_2', V_1)$. Further this induces a bijection from $L_e(W, V; V_1, V_2; \mathbf{w},\mathbf{v}; \mathbb{Z}_p[v_1'], \phi_1)$ to $L_e(V_2', V_1; \phi_1')$ where $\phi_1 \in L_e(\mathbb{Z}_p[w_1], V)$ such that $\mathrm{Pr}_{V \to \mathbb{Z}_p[v_i]}\phi_1(w_1) = \delta_{i1}v_i$ for $1 \leq i \leq n$ and $\phi_1' \in L_e(\mathbb{Z}_p[v_1'], V_1)$ such that $\phi_1'(v_1') = \mathrm{Pr}_{V \to V_1}\phi_1(w_1)$.

2. Some propositions

In this section we will prove an essential proposition to get a recursion formula.

Definition. *We call an element T of $S_n(\mathbb{Z}_p)$ or $< T >$ of level p^l if l is the least integer such that $p^l T^{-1}$ is even integral.*

Proposition 2.1([10, p91]). *Let $L = \mathbb{Z}_p[v_1, \ldots, v_n]$ and $M = \mathbb{Z}_p[w_1, \ldots, w_n]$ be quadratic lattices over $\mathbb{Z}_p$ with $\mathrm{rank}L = \mathrm{rank}M = n$ and suppose that $\mathbb{Q}_p L$ is regular. Let h be the level of the dual lattice L^* of L, i.e. $L^* = \{x \in \mathbb{Q}_p L; B(x, L) \subset \mathbb{Z}_p\}$. If $B(v_i, v_j) \equiv B(w_i, w_j) \mod p^{h+1}\mathbb{Z}_p$ and $Q(v_i) \equiv Q(w_i) \mod 2p^{h+1}\mathbb{Z}_p$, then there exists an isometry $\eta : L \simeq M$ satisfying $\eta(v_i) \equiv w_i \mod p^{h+1}M^*$.*

Let $A = V \perp U$ be a symmetric unimodular matrix with canonical form. Let $V = \perp_l H \perp Y'$, where $Y' = Y$ or $\varnothing$ and $U = \perp_j c_j$ as above. Note that if $p \neq 2$, we have always $Y' = \varnothing$. Then for $0 \leq r \leq l$ set

$$A^{(r)} = \perp_{l-r} H \perp Y' \perp U \text{ and } A^* = V.$$

Lemma 2.2. *Let k_1, k_2, k_3 and k be non-negative integers such that $k = \sum_{l=1}^{3} k_l$. Let $K = \mathbb{Z}_p[z_1, \ldots, z_k]$ be a quadratic module over $\mathbb{Z}_p$ such that*

$$(B(z_i, z_j))_{1 \le i,j \le k} = S \perp T,$$

with $S = V \perp U$ be a unimodular matrix as above such that deg $V = k_1$ *and* deg $U = k_2$ *and $T = p^s T_1 \perp T_2 \in M_{k_3}(\mathbb{Z}_p)$, where $deg(T_1) \le 2$ and $s \ge 2 + \delta_{2p}$. Put $K_1 = \mathbb{Z}_p[z_1, \ldots, z_{k_1}], K_2 = \mathbb{Z}_p[z_{k_1+1}, \ldots, z_{k_1+k_2}]$ and $K_3 = \mathbb{Z}_p[z_{k_1+k_2+1}, \ldots, z_k]$.*

For any elements $u_l \in K_l$ $(1 \le l \le 2)$ such that $u_1 + u_2$ is primitive, we put $z'_1 = u_1 + u_2 + z_{k_1+k_2+1}$.

(1)Assume that u_1 is primitive and $Q(u_1 + u_2) \equiv 0 \mod 2p$, then there exist elements $\{z'_i\}_{2 \le i \le k}$ of K such that

(a) $z'_2 \in K_1$ and $\begin{pmatrix} B(z'_1, z'_1) & B(z'_1, z'_2) \\ B(z'_2, z'_1) & B(z'_2, z'_2) \end{pmatrix} = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_2 & 0 \end{pmatrix}$ with $\eta_1 \in 2\mathbb{Z}_p, \eta_2 \in \mathbb{Z}_p^\times$,

(b)$\mathbb{Z}_p[z'_3, \ldots, z'_{k_1+k_2}]$ is isometric to $< S^{(1)} >$,

(c)$\mathbb{Z}_p[z'_{k_1+k_2+1}, \ldots, z'_k]$ is isometric to K_3,

(d) $K = \mathbb{Z}_p[z'_1, z'_2] \perp \mathbb{Z}_p[z'_3, \ldots, z'_{k_1+k_2}] \perp \mathbb{Z}_p[z'_{k_1+k_2+1}, \ldots, z'_k]$,

(e) $z'_j - z_j \in \mathbb{Z}_p[z_1, \ldots, z_{k_1+k_2+1}]$ for $k_1 + k_2 + 1 \le j \le k$.

(2)Let $p = 2$. Assume that $u_1 \equiv 0 \mod p$ and $Q(u_1 + u_2) \equiv 0 \mod p^3$, then there exist elements $\{z'_i\}_{2 \le i \le k}$ of K such that

(a) $z'_2 \in K_2$ and $\begin{pmatrix} B(z'_1, z'_1) & B(z'_1, z'_2) \\ B(z'_2, z'_1) & B(z'_2, z'_2) \end{pmatrix} = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_2 & \eta_3 \end{pmatrix}$ with $\eta_1 \in p^3\mathbb{Z}_p, \eta_2, \eta_3 \in \mathbb{Z}_p^\times$,

(b)$\mathbb{Z}_p[z'_3, \ldots, z'_{k_1+k_2}]$ is a submodule of $K_1 \perp K_2$ isometric to $< V >$,

(c)$\mathbb{Z}_p[z'_{k_1+k_2+1}, \ldots, z'_k]$ is isometric to K_3,

(d) $K = \mathbb{Z}_p[z'_1, z'_2] \perp \mathbb{Z}_p[z'_3, \ldots, z'_{k_1+k_2}] \perp \mathbb{Z}_p[z'_{k_1+k_2+1}, \ldots, z'_k]$,

(e) $z'_j - z_j \in 2\mathbb{Z}_p[z_1, \ldots, z_{k_1+k_2+1}]$ for $k_1 + k_2 + 1 \le j \le k$.

Proof.

(1) This can be proved by the same argument as that in the proof of [3, Lemma 2.3] so we omit the proof.

(2) In this case, we may assume that $a_{k_1+1} \not\equiv 0 \mod p$. Then put $z'_2 = z_{k_1+1}$. Then such a z'_2 satisfies the statement (a). It is easy to see that

$$K_1 \perp K_2 = \mathbb{Z}_p[u_1 + u_2, z'_2, z_1, \ldots, z_{k_1}].$$

Then there exist elements $\{\mu_j\}_{1 \le j \le k_1+k_2}, \{\nu_j\}_{1 \le j \le k_1+k_2}$ of $\mathbb{Z}_p$ such that

$$B(z_j + \mu_j(u_1 + u_2) + \nu_j z'_2, z'_i) = 0 \quad (1 \le j \le k_1 + k_2, 1 \le i \le 2).$$

Put $z'_{j+2} = z_j + \mu_j(u_1 + u_2) + \nu_j z'_2$ for $1 \le j \le k_1$. Then by Proposition 2.1, we have (b).

Moreover there exist elements $\{\mu_j\}_{k_1+k_2+1 \le j \le k_1+k_2+2}, \{\nu_j\}_{k_1+k_2+1 \le j \le k_1+k_2+2}$, of $\mathbb{Z}_p$ such that

$$B(z_j + \mu_j z'_1 + \nu_j z'_2, z'_i) = 0 \quad (k_1 + k_2 + 1 \le j \le k_1 + k_2 + 2, 1 \le i \le 2).$$

Put

$$z'_j = \begin{cases} z_j + \mu_j z'_1 + \nu_j z'_2, & k_1 + k_2 + 1 \le j \le k_1 + k_2 + 2 \\ z_j, & k_1 + k_2 + 3 \le j \le k. \end{cases}$$

Then we can easily see that the statement (c),(d), and (e) are satisfied by such z'_j's. This completes the proof. □

By applying Lemma 2.2, we have the following proposition.

Proposition 2.3. *Let m_1, m_2 and n be non-negative integers such that $m = m_1 + m_2$. Let $K = \mathbb{Z}_p[z_1, \ldots, z_{m+n}]$ and $W = \mathbb{Z}_p[w_1, \ldots, w_n]$ be quadratic modules over $\mathbb{Z}_p$ such that*

$$(B(z_i, z_j))_{1 \le i,j \le m+n} = A \perp B, \quad (B(w_i, w_j))_{1 \le i,j \le n} = O_n,$$

where $A = V \perp U$ is unimodular and a canonical form, deg(V) $= m_1$, deg(U) $= m_2$ and $B = (b_{ij})$ is a non-degenerate matrix of degree n and assume that $b_{11} \in 4p\mathbb{Z}_p$. Put $M_1 = \mathbb{Z}_p[z_1, \ldots, z_{m_1}], M_2 = \mathbb{Z}_p[z_{m_1+1}, \ldots, z_m], M = M_1 \perp M_2, N = \mathbb{Z}_p[z_{m+1}, \ldots z_{m+n}]$ and $W_1 = \mathbb{Z}_p[w_1]$. Let $\phi_1 \in L_e(W_1, K)$ such that $\mathrm{Pr}_{K \to N}\phi_1(w_1) = z_{m+1}$.

(1) *Let $p \neq 2$. Assume that $\mathrm{Pr}_{K \to M_1}\phi_1(W_1) \not\subset pM_1$. Then there exist elements $\{z'_i\}_{3 \le i \le m+n}$ of K such that*

$$(B(z'_i, z'_j))_{3 \le i,j \le m+n} = A^{(1)} \perp B$$

and

$$\#L_e(W, K; M, N; \mathbf{w}, \mathbf{v}; W_1, \phi_1) = \#L_e(\hat{W}, K'; M', N'; \mathbf{w'}, \mathbf{v'}),$$

where $\hat{W} = \mathbb{Z}_p[w_2, \ldots, w_n]$, $M' = \mathbb{Z}_p[z'_3, \ldots, z'_{m+1}]$, $N' = \mathbb{Z}_p[z'_{m+2}, \ldots, z'_{m+n}]$, $\mathbf{w'} = \{w_2, \ldots, w_n\}$ and $\mathbf{v'} = \{z'_{m+2}, \ldots, z'_{m+n}\}$.

(2) *Let $p = 2$. Assume that $\mathrm{Pr}_{K \to M_1}\phi_1(W_1) \not\subset pM_1$ or $U \neq 1 \perp -1$. Then there exist elements $\{z'_i\}_{3 \le i \le m+n}$ of K such that*

$$(B(z'_i, z'_j))_{3 \le i,j \le m+n} = A^{(1)} \perp B$$

and

$$\#L_e(W, K; M, N; \mathbf{w}, \mathbf{v}; W_1, \phi_1) = \#L_e(\hat{W}, K'; M', N'; \mathbf{w'}, \mathbf{v'}),$$

where the notations are the same as in (1).

(3) *Let $p = 2$. Assume that $\mathrm{Pr}_{K \to M}\phi_1(W_1) \not\subset pM$ and $\mathrm{Pr}_{K \to M_1}\phi_1(W_1) \subset pM_1$ and $U = 1 \perp -1$, then there exist elements $\{z'_i\}_{3 \le i \le m+n}$ of K such that*

$$(B(z'_i, z'_j))_{3 \le i,j \le m+n} = A^* \perp B$$

and

$$\#L_e(W, K; M, N; \mathbf{w}, \mathbf{v}; W_1, \phi_1) = \#L_e(\hat{W}, K'; M', N'; \mathbf{w'}, \mathbf{v'}),$$

where the notations are the same as in (1).

Proof. The statement (1) is treated in Katsurada and the present author in [6]. (2) is a slight modification of (1), therefore we shall prove only (3).

Put $z_1' = \phi_1(w_1)$. Then we have

$$B(z_1', z_1') = 0 \bmod p^e. \tag{2.1}$$

Put $K_i = M_i$ for $i = 1, 2$ and $K_3 = N$. Then for the modules $K_1, \ldots, K_3$, and the elements $z_1', \{z_i\}_{1 \le i \le m+n}$ stated above, there exist elements $\{z_i'\}_{2 \le i \le m+n}$ satisfying the conditions (a) $\sim$ (e) of Lemma 2.2 (2). Let

$$z_j' = \sum_{i=1}^{m+n} \xi_{ij} z_i \ \ (1 \le j \le m+n) \text{ with } (\xi_{ij}) \in GL_{m+n}(\mathbb{Z}_p). \tag{2.2}$$

Then by (a),(d) and (e) of Lemma 2.2 (2), the matrix $(\xi_{ij})_{1 \le i,j \le m+n}$ can be expressed as

$$(\xi_{ij}) = \begin{pmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \Xi_{21} & 0_{n-1,m} & \Xi_{23} \end{pmatrix}, \tag{2.3}$$

where $\Xi_{23} \equiv E_{n-1} \mod 2$.

The assumption that $\mathrm{Pr}_{K \to N} \phi_1(w_1) = z_{m+1}$ implies

$$\Xi_{21} = \begin{pmatrix} 1 \\ 0_{n-1,1} \end{pmatrix}. \tag{2.4}$$

Then by (a) $\sim$ (e) of Lemma 2.2 (2), we have

$$(B(z_i', z_j'))_{3 \le i,j \le m+n} = A^* \perp B.$$

Thus, it suffices to prove that there exists a bijection of $L_e(W, K; M, N; \mathbf{w}, \mathbf{v}; W_1, \phi_1)$ to $L_e(\hat{W}, K'; M', N'; \mathbf{w}', \mathbf{v}')$. To prove this for an element ϕ' of $\mathrm{Hom}_{\mathbb{Z}_p}(\hat{W}, K')$ by $\phi' = \mathrm{Pr}_{K \to K'} \phi | \hat{W}$. Then ϕ' belongs to $L_e(\hat{W}, K')$. In fact for $1 \le j \le n$

$$\phi(w_j) = \sum_{i=1}^{m+n} x_{ij} z_i \text{ with } (x_{ij}) \in M_{m+n,n}(\mathbb{Z}_p). \tag{2.5}$$

Then for $1 \le j \le n$ we have

$$\phi(w_j) = \sum_{i=1}^{m+n} x_{ij}' z_i' \tag{2.6}$$

with

$$x_{ij} = \sum_{a=1}^{m+n} \xi_{ia} x_{aj}'. \tag{2.7}$$

Since $\phi(w_1) \equiv z_1' \bmod p^e$ and $\phi'(w_j) = \phi(w_j) - x_{1j}' z_1' - x_{2j}' z_2'$ $(2 \le j \le n)$, we have $B(\phi(w_1), \phi(w_j)) \equiv B(z_1', x_{1j}' z_1' + x_{2j}' z_2') \equiv 0 \bmod p^e$ for $2 \le j \le n$. Thus by (2.3) and (a) of Lemma 2.2 (1) we have $x_{2j}' \equiv 0 \bmod p^e$ for $2 \le j \le n$. Thus we have

$$B(\phi'(w_i), \phi'(w_j)) \equiv B(\phi(w_i), \phi(w_j)) \bmod p^e \text{ for any } 2 \le i, j \le n. \tag{2.8}$$

Since $B(\phi(w_i), \phi(w_j)) \equiv B(w_i, w_j) \bmod p^e$ for any $2 \le i, j \le n$, ϕ' belongs to $L_e(\hat{W}, K')$. Furthermore by (2.3),(2.4) and (2.7), we have $x_{ij} \equiv x'_{ij} \bmod p^e$ $(m + 2 \le i \le m+n, 2 \le j \le n)$. Thus we have for $1 \le i \le n-1, 1 \le j \le n-1$,

(2.9) $\quad \mathrm{Pr}_{K \to N}\phi(w_j) = \delta_{ij} z_{m+i}$ if and only if $\mathrm{Pr}_{K' \to N'}\phi'(w_{j+1}) = \delta_{ij} z'_{m+i+1}$.

Thus ϕ' belongs to $L_e(\hat{W}, K'; M', N'; \mathbf{w}', \mathbf{v}')$. Thus we can define a mapping π from $L_e(W, K; M, N; \mathbf{w}, \mathbf{v}; W_1, \phi_1)$ to $L_e(\hat{W}, K'; M', N'; \mathbf{w}', \mathbf{v}')$ by

$$\pi(\phi) = \phi'.$$

Let ϕ be an element of $L_e(W, K; M, N; \mathbf{w}, \mathbf{v}; W_1, \phi_1)$. Then by (2.2), (2.4) and (2.7), we have

(2.10) $$x_{m+1,j} = x'_{1j} + \xi_{m+1,m+1} x'_{m+1,j} + \xi_{m+1,m+2} x'_{m+2,j}.$$

Then we have for $2 \le j \le n$

(2.11)

$\mathrm{Pr}_{K \to N}(\phi(w_j)) = \delta_{1j} z_{m+1}$ if and only if $x'_{1j} \equiv -\xi_{m+1,m+1} x'_{m+1,j} - \xi_{m+1,m+2} x'_{m+2,j} \bmod p^e$,

this shows the injectivity of π.

Lastly we prove π is surjective. Take an element ϕ' in $L_e(\hat{W}, K'; M', N'; \mathbf{w}', \mathbf{v}')$ and put

$$\phi'(w_j) = \sum_{i=3}^{m+n} y'_{ij} z'_i \ (2 \le j \le n).$$

Put $y'_{1j} = -\xi_{m+1,m+1} y'_{m+1,j} - \xi_{m+1,m+2} y'_{m+2,j}$ and define a mapping ϕ from W to K by

$$\phi(w'_1) = z'_1, \ \phi(w_j) = -y'_{1j} z'_1 + \sum_{i=3}^{m+n} y'_{ij} z'_i \ (2 \le j \le n).$$

Then by construction and (2.8),(2.9) and (2.10), we have

$$\phi \in L_e(W, K; M, N; \mathbf{w}, \mathbf{v}; W_1, \phi_1) \text{ and } \pi(\phi) = \phi'.$$

Thus we complete the proof. □

Remark. (1) *Under the above notaion we see easily that*

$$\# L_e(\hat{W}, K'; M', N'; \mathbf{w}', \mathbf{v}') = \# A_e(n-1, -B, A^{(1)})$$

or

$$\# L_e(\hat{W}, K'; M', N'; \mathbf{w}', \mathbf{v}') = \# A_e(n-1, -B, A^{*})$$

according as in (1), (2) or in (3).

(2) *In the above proposition, assume that $N = N_1 \perp \hat{N}$, where N_1 and $\hat{N}$ are quadratic submodules of rank* 1 *and $n-1$, respectively, and $M = < A >$ is unimodular and $\phi_1 \in L_e(N'_1, M)$, where for a quadratic module $L = < S >$ we write $L' = < -S >$. Then by Proposition 1.3 and Prorosition 2.3 we have*

$\# L_e(N', M; \phi_1) = \# L_e(\hat{N}', M^{(1)} \perp N_1)$ *or* $\# L_e(N', M; \phi_1) = \# L_e(\hat{N}', M^{*} \perp N_1)$ *where $M^{(1)} = < A^{(1)} >$ and $M^{*} = < A^{*} >$ according as the case* (1),(2) *or* (3) *in Proposition 2.3. A part of the former case is nothing but [2, Proposition 2.2 (1.b)].*

Summarizing Proposition 1.1 and Proposition 2.3, we have the following proposition.

Proposition 2.4.. *Let $A = V \perp U$ and B be as the same in Proposition 2.3. Let $\Xi = {}^t(\Xi_1 \quad \Xi_2 \quad \Xi_3) \in A_e(b_{11}, A)$, where $\Xi_1 \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^{m_1}, \Xi_2 \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^{m_2}$ and $e \geq 2$. Assume that $\begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} \not\equiv 0 \mod p$.*

(1) *Let $p \neq 2$. Assume that $\Xi_1 \not\equiv 0 \mod p$. Then we have*

$$a_e(B, A; \Xi) = a_e(n-1; B, A^{(1)}).$$

(2) *Let $p = 2$. Assume that $\Xi_1 \not\equiv 0 \mod p$ or $U \neq 1 \perp -1$. Then we have*

$$a_e(B, A; \Xi) = a_e(n-1; B, A^{(1)}).$$

(3) *Let $p = 2$. Assume that $\Xi_1 \equiv 0 \mod p$ and $U = 1 \perp -1$. Then we have*

$$a_e(B, A; \Xi) = a_e(n-1; B, A^*).$$

Remark. *Assume that the level of $\hat{T} \in S_{t-1}(\mathbb{Z}_p)$ is equal to or smaller than $\nu(2^{-1}b_1) - 1$. Then the matrix $\hat{T} + b_1(y_iy_j)_{1 \leq i,j \leq t-1}$ is equivalent to $\hat{T}$ for any $(y_i) \in \mathbb{Z}_p^{t-1}$. Thus for any symmetric matrix S we have*

$$\begin{aligned} a_e(t-1; b_1 \perp \hat{T}, S) &= \sum_{(y_i) \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^{t-1}} a_e(\hat{T} + b_1(y_iy_j)_{1 \leq i,j \leq t-1}, S) \\ &= p^{e(t-1)} a_e(\hat{T}, S). \end{aligned}$$

Thus we have

$$\alpha_p(t-1; b_1 \perp \hat{T}, S) = \alpha_p(\hat{T}, S).$$

3. A recursion formula

For $S \in S_m(\mathbb{Z}_2), T \in S_n(\mathbb{Z}_2), \bar{\Theta} = (\bar{\theta}_i) \in (\mathbb{Z}_p/p\mathbb{Z}_p)^m$, and a non-negative integer e, put (cf. [3])

$$\bar{A}_e(T, S; \bar{\Theta}) = \{(x_{ij}) \in A_e(T, S); x_{i1} \equiv \theta_i \mod 2 \text{ for any } 1 \leq i \leq m\},$$

and

$$\bar{a}_e(T, S; \bar{\Theta}) = \#\bar{A}_e(T, S; \bar{\Theta}).$$

We note that

$$\bar{a}_e(T, S; \bar{\Theta}) = \sum_{\Xi} a_e(T, S; \Xi),$$

where Ξ runs through all representatives of $(\mathbb{Z}_p/p^e\mathbb{Z}_p)^m$ such that $\Xi \equiv \Theta \mod p$.

For symmetric matrices A and B of degree m and $n (m \geq n \geq 1)$, respectively, with entries in the ring $\mathbb{Z}_p$ of p-adic integers, define a primitive local density $\beta_p(T, S)$ by

$$\beta_p(B, A) = \lim_{e \to \infty} p^{-te} \#B_e(B, A),$$

where $t = mn - n(n+1)/2$ and

$$B_e(B, A) = \{X \in A_e(B, A); X \text{ is primitive }\}.$$

Here X is said to be primitive if its reduction modulo p has the maximal rank.

Lemma 3.1. *Let $A = V \perp U \in S_m(\mathbb{Z}_2) \cap GL_m(\mathbb{Z}_2)$ be a unimodular matrix such that V (resp. U) is an even unimodular (resp. odd unimodular) part of degree m_1 (resp. m_2). Let $\bar{\Theta}_1 \in (\mathbb{Z}_2/2\mathbb{Z}_2)^{m_1}, \bar{\Theta}_2 \in (\mathbb{Z}_2/2\mathbb{Z}_2)^{m_2}$ and $b \in 2\mathbb{Z}_2$.*

(1) *Assume that $e \geq 2, \Theta_1 \not\equiv 0_{m_1,1} \bmod 2, \Theta_2 \equiv 0_{m_2,1}$ or $1_{m_2,1} \bmod 2$. Then we have*

$$2^{(-m+1)e}\bar{a}_e(b, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{\Theta}_2 \end{pmatrix})$$

$$= \begin{cases} 2^{-m+2}, & \text{either if } \Theta_2 \equiv 0_{m_2,1} \bmod 2 \text{ and } V[\Theta_1] \equiv b \bmod 4 \\ & \text{or if } \Theta_2 \equiv 1_{m_2,1} \bmod 2 \text{ and } V[\Theta_1] \equiv -tr(U) + b \bmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *Assume that $e \geq 3, U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Theta_1 \equiv 0_{m_1,1} \bmod 2$. Then we have*

$$2^{(-m+1)e}\bar{a}_e(0, A; \bar{\Theta}) = \begin{cases} 2^{-m+3}, \text{ if } U[\Theta_2] \equiv 0 \bmod 8, \\ 0, \text{ otherwise.} \end{cases}$$

Proof. For $e \geq 1$, put

$$\bar{\mathcal{A}}'_e(\Theta) = \{\bar{X} = (\bar{x}_{i1}) \in (\mathbb{Z}_2/2^e\mathbb{Z}_2)^m; A[X] \equiv b \bmod 2^{e+1}, x_{i1} \equiv \theta_i \bmod 2\}.$$

For $e \geq 2$, consider the map $\Phi : \bar{A}_e(b, A; \bar{\Theta}) \to \bar{\mathcal{A}}'_{e-1}(\Theta)$ defined by X mod $2^e \mapsto X$ mod 2^{e-1} for $e \geq 2$. It is a surjection, and we have $\#\Phi^{-1}(\bar{X}) = 2^m$ for any $\bar{X} \in \bar{\mathcal{A}}'_{e-1}(\Theta)$. Next consider the map $\Psi : \bar{\mathcal{A}}'_e(\Theta) \to \bar{\mathcal{A}}'_{e-1}(\Theta)$ defined by X mod $2^e \mapsto X$ mod 2^{e-1}. Obviously it is a surjection.

(1) It is easy to see that if $\Theta_2 \not\equiv 0_{m_2,1}, 1_{m_2,1}$ mod 2, then

$$\bar{a}_e(b, A; \bar{\Theta}) = 0.$$

First assume that $\Theta_2 \equiv 0_{m_2,1}$ mod 2. For any $\bar{Y} = \begin{pmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{pmatrix} \in \bar{\mathcal{A}}'_{e-1}(\Theta)$, put $\bar{X} = \bar{Y} + 2^{e-1}\bar{Z} \in \bar{\mathcal{A}}'_e(\Theta), \bar{Z} = \begin{pmatrix} \bar{Z}_1 \\ \bar{Z}_2 \end{pmatrix} \in (\mathbb{Z}_2/2\mathbb{Z}_2)^m$. Then

$$A[X] \equiv A[Y] + 2^e \cdot {}^tZ_1A^*Y_1 \equiv b \mod 2^{e+1}.$$

Since $\Theta_1 \not\equiv 0_{m_1,1}$ mod 2, we have $A^*Y_1 \equiv A^*\theta_1 \not\equiv 0_{m_1,1}$ mod 2, and $\#\{Z_1 \in (\mathbb{Z}_2/2\mathbb{Z}_2)^{m_1}; {}^tZ_1A^*Y_1 \equiv 2^{-e}(b - A[Y]) \bmod 2\} = 2^{m_1-1}$. Thus $\Psi^{-1}(\bar{Y}) = 2^{m-1}$. Moreover $\#\bar{\mathcal{A}}'_1(\Theta) = 1$. After all, we obtain that

$$\begin{aligned} \bar{a}_e(b, A; \bar{\Theta}) &= 2^m\#\bar{\mathcal{A}}'_{e-1}(\Theta) = \cdots = 2^m2^{(e-2)(m-1)}\#\bar{\mathcal{A}}'_1(\Theta) \\ &= 2^m2^{(e-2)(m-1)} \end{aligned}$$

for $e \geq 2$.
By similar argument as above we get $\bar{a}_e(b, A; \bar{\Theta}) = 2^m2^{(e-2)(m+1)}$ for $e \geq 2$ when $\Theta_2 \equiv 1_{m_2,1}$ mod 2.

(2) Similar argument implies

$$\bar{a}_e(0, A; \bar{\Theta}) = 2^m \# \bar{\mathcal{A}}'_{e-1}(\Theta) = \cdots = 2^m 2^{(e-3)(m-1)} \# \bar{\mathcal{A}}'_2(\Theta).$$

Since

$$\#\bar{\mathcal{A}}'_2(\Theta) = \#\{\bar{X} \in (\mathbb{Z}_2/4\mathbb{Z}_2)^m; A^*[X_1] + V[X_2] \equiv 0 \bmod 8,$$

$$X_1 \equiv 0 \bmod 2, X_2 \equiv 1 \bmod 2\}$$

$$= 2^{m-2}\#\{(\bar{x_i}) \in (\mathbb{Z}_2/4\mathbb{Z}_2)^2; x_1^2 - x_2^2 \equiv 0 \bmod 8, x_1 \equiv x_2 \equiv 1 \bmod 2\},$$

we have $\bar{\mathcal{A}}'_2(\Theta) = 2^m$, this completes the proof. □

We shall make a distinction of the following types as above.
(case I) $p \neq 2$,
(case II) $p = 2$ and $U \neq 1 \perp -1$,
(case III) $p = 2$ and $U = 1 \perp -1$.

The next theorem in the case for $p \neq 2$ and that B is diagonal is given in [3] and in the case that A is even unimodular is given in [6],[7]. We get here the generalized formula without such restrictions such that we shall prove the odd unimodular case.

Theorem 3.2. *Let $A = A^* \perp U$ be a symmetric unimodular matrix of degree m and B be a non-degenerate matrix of degree n as above. Then we have*

$$\alpha_p(B, A) - p^{-m+n+1}\alpha_p(B[p^{-1} \perp E_{n-1}], A)$$
$$= \begin{cases} \beta_p(0, A)\alpha_p(n-1; B, A^{(1)}), & \text{(case I)}, \\ p^{-2}\{\beta_p(0, A^*) + \beta_p(tr(-U), A^*)\}\alpha_p(n-1; B, A^{(1)}), & \text{(case II)}, \\ p^{-1}\beta_p(0, A^*)\alpha_p(n-1; B, A^{(1)}) + p^{-m+3}\alpha_p(n-1; B, A^*), & \text{(case III)}, \end{cases}$$

where we make the convention that $\alpha_p(n-1; B, A^{(1)}) = 0$ if the Witt index of A is 0 and $\beta_p(tr(-U), A^*) = 0$ if $U = \varnothing$.

Proof. The proof in the case A is even unimodular is completed in [6] and [7], so we shall prove the case that A is odd unimodular and $p = 2$.

We assume that $e \geq 3$. Then we have

$$a_e(B, A) = \sum_{\bar{\Theta} \in (\mathbb{Z}_2/2\mathbb{Z}_2)^m} \bar{a}_e(B, A; \bar{\Theta}).$$

Let $\deg(A^*) = m_1$ and $\deg(U) = m_2$.

Put for $a \in \mathbb{Z}_2$,

$$\mathcal{S}(a) = \{\bar{\Theta} \in (\mathbb{Z}_2/2\mathbb{Z}_2)^{m_1}; \Theta \not\equiv 0_{m_1,1} \bmod 2, A^*[\Theta] \equiv a \bmod 4\}.$$

Further put

$$\mathcal{S} = \{\bar{\Theta} \in (\mathbb{Z}_2/2\mathbb{Z}_2)^{m_2}; U[\Theta] \equiv 0_{m_2,1} \bmod 8, \Theta \not\equiv 0 \bmod 2\}.$$

and

$$\mathcal{S}' = \{(\bar{\Theta}_1, \bar{\Theta}_2) \in (\mathbb{Z}_2/2\mathbb{Z}_2)^{m_1} \times (\mathbb{Z}_2/2\mathbb{Z}_2)^{m_2}; A^*[\Theta_1] + U[\Theta_2] \equiv 0 \bmod 4,$$
$$\Theta_1 \not\equiv 0_{2d,1} \bmod 2, \Theta_2 \not\equiv 0_{m_2,1} \bmod 2\}.$$

Then we have

$$\begin{aligned} a_e(B, A) &= \bar{a}_e(B, A; \bar{0}_{m1}) \\ &+ \sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \bar{a}_e(B, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m-2d,1} \end{pmatrix}) \\ &+ \sum_{\bar{\Theta}_2 \in \mathcal{S}} \bar{a}_e(B, A; \begin{pmatrix} \bar{0}_{2d,1} \\ \bar{\Theta}_2 \end{pmatrix}) \\ &+ \sum_{(\bar{\Theta}_1, \bar{\Theta}_2) \in \mathcal{S}'} \bar{a}_e(B, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{\Theta}_2 \end{pmatrix}). \end{aligned} \tag{3.1}$$

(case II) Since $U \neq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we have $\mathcal{S} = \varnothing$. If $\deg(U) = 1$ then it is to see that $\mathcal{S}' = \varnothing$. Next we assume $\deg(U) = 2$. If $(\bar{\Theta}_1, \bar{\Theta}_2) \in \mathcal{S}'$, then $\bar{\Theta}_2 = \bar{1}_{21}$. In fact, since $A^*[\Theta_1] \equiv 0 \bmod 2$ for any $\bar{\Theta}_1 \in (\mathbb{Z}_2/2\mathbb{Z}_2)^{m_1}$, so that we have $U[\Theta_2] \equiv 0 \bmod 2$, therefore $\bar{\Theta}_2 = \bar{1}_{21}$. Then $U[\Theta_2] \equiv \mathrm{tr}(U) \bmod 4$, and we have

$$\begin{aligned} a_e(B, A) &= \bar{a}_e(B, A; \bar{0}_{m1}) \\ &+ \sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \bar{a}_e(B, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m_2,1} \end{pmatrix}) \\ &+ \sum_{\Theta_1 \in \mathcal{S}(-\mathrm{tr}(U))} \bar{a}_e(B, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{1}_{m_2,1} \end{pmatrix}). \end{aligned}$$

Note that the last term on the right-hand side is 0 if $\deg(U) \neq 2$.
We can show

$$2^{e(-mn+n(n+1)/2)} \bar{a}_e(B, A; \bar{0}_{m1}) = 2^{-m+1} \alpha_2(B[2^{-1} \perp E_{n-1}], A) \tag{3.2}$$

for a sufficiently large e in the same way as in the proof of [3, Proposition 3.6(2)].
Next we show that

$$\begin{aligned} 2^{e(-mn+n(n+1)/2)} &\sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \bar{a}_e(B, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m_2,1} \end{pmatrix}) \\ &= \beta_2(0, A^*) \alpha_2(n-1; B, A^{(1)}), \end{aligned} \tag{3.3}$$

for a sufficient large e. By (2) of Proposition 2.4, for any $\Xi \in \mathbb{Z}_2^m$ such that $\Xi \equiv \begin{pmatrix} \Theta_1 \\ 0_{m_2,1} \end{pmatrix} \bmod 2$, we have

$$a_e(B, A; \Xi) = a_e(n-1; B, A^{(1)}).$$

Thus we have

$$\sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \bar{a}_e(B, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m_2,1} \end{pmatrix})$$
$$= \sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \sum_{\Xi} a_e(n-1; B, A^{(1)})$$
$$= \sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \bar{a}_e(b_1, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m_2,1} \end{pmatrix}) a_e(n-1; B, A^{(1)}),$$

where Ξ runs over elements of $(\mathbb{Z}_2/2^e\mathbb{Z}_2)^m$ satisfying that $\Xi \equiv \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m_2,1} \end{pmatrix} \bmod 2$. Therefore we have

$$2^{e(-mn+n(n+1)/2)} \sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \bar{a}_e(B, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m_2,1} \end{pmatrix})$$
$$= 2^{(-m+1)e} \sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \bar{a}_e(b_1, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m_2,1} \end{pmatrix})$$
$$\times 2^{e(-(m-1)(n-1)+(n-1)n/2)} a_e(n-1; B, A^{(1)}).$$

By Lemma 3.1 (1), for any $\bar{\Theta}_1 \in \mathcal{S}(0)$, we have

$$2^{(-m+1)e} \bar{a}_e(b_1, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m_2,1} \end{pmatrix}) = 2^{-m+2}.$$

Thus we have

$$2^{e(-mn+n(n+1)/2)} \sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \bar{a}_e(B, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{0}_{m_2,1} \end{pmatrix})$$
$$= 2^{-m+2} \# \mathcal{S}(0) 2^{e(-(m-1)(n-1)+(n-1)n/2)} a_e(n-1; B, A^{(1)})$$
$$= 2^{-2} \beta_2(0, A^*) \alpha_2(n-1; B, A^{(1)})$$

for a sufficiently large e, which implies (3.3). Similarly, we have

$$(3.4) \qquad 2^{e(-mn+n(n+1)/2)} \sum_{\bar{\Theta}_1 \in \mathcal{S}(-\mathrm{tr}(U))} \bar{a}_e(B, A; \begin{pmatrix} \bar{\Theta}_1 \\ \bar{1}_{m_2,1} \end{pmatrix})$$
$$= 2^{-2} \beta_2(-\mathrm{tr}(U), A^*) \alpha_2(n-1; B, A^{(1)}),$$

for a sufficiently large e. Thus the assertion (case II) follows from (3.2), (3.3), and (3.4).

Next we prove (case III). We have

$$a_e(B,A) = \bar{a}_e(B,A;\bar{0}_{m1}) + \sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \{\bar{a}_e(B,A;\begin{pmatrix}\bar{\Theta}_1 \\ \bar{0}_{21}\end{pmatrix}) + \bar{a}_e(B,A;\begin{pmatrix}\bar{\Theta}_1 \\ \bar{1}_{21}\end{pmatrix})\} + \sum_{\bar{\Theta}_2} \bar{a}_e(B,A;\begin{pmatrix}\bar{0}_{m_1,1} \\ \bar{\Theta}_2\end{pmatrix}),$$

where $\bar{\Theta}_2$ runs over the set

$$\mathcal{S}'' = \{\bar{\Theta} \in (\mathbb{Z}_2/2\mathbb{Z}_2)^2; \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}[\Theta] \equiv 0 \bmod 8, \Theta \not\equiv 0_{21} \bmod 2\}.$$

Since $\mathrm{tr}(U) = 0$, we see similarly as in (3.4) that

$$2^{e(-mn+n(n+1)/2)} \sum_{\bar{\Theta}_1 \in \mathcal{S}(0)} \bar{a}_e(B,A;\begin{pmatrix}\bar{\Theta}_1 \\ \bar{1}_{21}\end{pmatrix}) = 2^{-1}\beta_2(0,A^*)\alpha_2(n-1;B,A^{(1)})$$

for a sufficiently large e. After all, the assertion (case III) reduces to (3.5) below.

$$2^{e(-mn+n(n+1)/2)} \sum_{\bar{\Theta}_2 \in \mathcal{S}''} \bar{a}_e(B,A;\begin{pmatrix}\bar{0}_{m_1,1} \\ \bar{\Theta}_2\end{pmatrix}) = 2^{-m+3}\alpha_2(n-1;B,A^*) \tag{3.5}$$

for a sufficiently large e. By (3) of Proposition 2.4, for any $\Xi \in \mathbb{Z}_2^m$ such that $\Xi \equiv \begin{pmatrix}0_{m_1,1} \\ \Theta_2\end{pmatrix} \bmod 2$, we have

$$a_e(B,A;\Xi) = a_e(n-1;B,A^{(1)}).$$

Thus we have

$$\sum_{\bar{\Theta}_2 \in \mathcal{S}''} \bar{a}_e(B,A;\begin{pmatrix}\bar{0}_{m_1,1} \\ \bar{\Theta}_2\end{pmatrix}) = \sum_{\bar{\Theta}_2 \in \mathcal{S}''} \sum_{\Xi} a_e(n-1;B,A^*) = \sum_{\bar{\Theta}_2 \in \mathcal{S}''} \bar{a}_e(b_1,A;\begin{pmatrix}\bar{0}_{m_1,1} \\ \bar{\Theta}_2\end{pmatrix}) a_e(n-1;B,A^*),$$

where Ξ runs over elements of $(\mathbb{Z}_2/2^e\mathbb{Z}_2)^m$ satisfying that $\Xi \equiv \begin{pmatrix}0_{m_1,1} \\ \Theta_2\end{pmatrix} \bmod 2$. Thus we have

$$2^{e(-mn+n(n+1)/2)} \sum_{\bar{\Theta}_2 \in \mathcal{S}''} \bar{a}_e(B,A;\begin{pmatrix}\bar{0}_{m_1,1} \\ \bar{\Theta}_2\end{pmatrix}) = 2^{(-m+1)e} \sum_{\bar{\Theta}_2 \in \mathcal{S}''} \bar{a}_e(B,A;\begin{pmatrix}\bar{0}_{m_1,1} \\ \bar{\Theta}_2\end{pmatrix}) \times 2^{e(-(m-1)(n-1)+n(n+1)/2)} a_e(n-1;B,A^*).$$

Then by Lemma 3.1 (2), for any $\bar{\Theta}_2 \in \mathcal{S}''$, we have

$$2^{(-m+1)e}\bar{a}_e(b_1, A; \begin{pmatrix} \bar{0}_{m_1,1} \\ \bar{\Theta}_2 \end{pmatrix}) = 2^{-m+3}.$$

Noting that $\#\mathcal{S}'' = 1$, we have

$$\begin{aligned} & 2^{e(-mn+n(n+1)/2)} \sum_{\bar{\Theta}_2 \in \mathcal{S}''} \bar{a}_e(B, A; \begin{pmatrix} \bar{0}_{m_1,1} \\ \bar{\Theta}_2 \end{pmatrix}) \\ & = 2^{-m+3}\#\mathcal{S}'' 2^{e(-(m-1)(n-1)+(n-1)n/2)} a_e(n-1; B, A^*) \\ & = 2^{-m+3}\alpha_2(n-1; B, A^*) \end{aligned}$$

for a sufficiently large e, which proves (3.5) and the assertion (case III). □

Remark. *It can be easily proved that*

$$\beta_2(0, A^*) + \beta_2(-tr(U), A^*) = 2^{\deg U}\beta_2(0, A).$$

Propositon 3.3([6]). *Let e_0 be an integer. Let S and T be non-degenerate symmetric matrices of degree $s, t-1$ respectively, and T be of level p^l. Assume that $l \geq e_0 + 1 - \delta_{2p}$. Then we have for a sufficiently large e*

$$\begin{aligned} & w(1 - p^{-s+t-1})p^{e(-st+t(t+1)/2)}p^{e_0-e} \sum_{a \in \mathbb{Z}_p/p^{e-e_0}\mathbb{Z}_p} \#A_e(p^{e_0}a \perp T, S) \\ = & \sum_{a \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2} \{(1 - p^{-1} - p^{-s+t-1})\alpha_p(p^{e_0}a \perp T, S) \\ & + (1 - p^{-1})p^{-1}\alpha_p(p^{e_0+1}a \perp T, S) + p^{-2}\alpha_p(p^{e_0+2}a \perp T, S)\}, \end{aligned}$$

where a runs over all elements of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ and $w = \#(\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2)$.

4. Properties

In this section we give some concrete formulae for local densities to get an interesting result. Note that the results in this section are treated by Katsurada and the present author in [6].

We set

$$f(B) = \begin{cases} \beta_p(0, A)\alpha_p(B, A^{(1)}), & \text{(case I)}, \\ p^{-2}\{\beta_p(0, A^*) + \beta_p(\mathrm{tr}(-U), A^*)\}\alpha_p(B, A^{(1)}), & \text{(case II)}, \\ p^{-1}\beta_p(0, A^*)\alpha_p(B, A^{(1)}) + p^{-m+3}\alpha_p(B, A^*), & \text{(case III)}. \end{cases}$$

First assume that $p \neq 2$.

Thorem 4.1. *Let A be as above. Let $B = p^r B_1 \perp B_2$ where $\deg B_1 \le 2$ and p^s the level of B_2.*

(1) *Let $r \ge s+1$ and $B_1 = (c)$ with $c \in \mathbb{Z}_p^\times$. Then we have*

$$\begin{aligned}\alpha_p(B, A) &- p^{-m+n+1}\alpha_p(B[p^{-1} \perp E_{n-1}], A)\\ &= f(B_2).\end{aligned}$$

(2) *Let $r \ge s+1$ and $B_1 = c_{11} \perp c_{12}$ with $-c_{11}c_{12}^{-1} \notin (\mathbb{Z}_p^\times)^2$. Then we have*

$$\begin{aligned}\alpha_p(B, A) &- p^{-m+n+1}\alpha_p(B[p^{-1} \perp E_{n-1}], A)\\ &= p^{-1}\sum_{a=0}^{p-1} f(p^r(c_{11}a^2 + c_{12}) \perp B_2).\end{aligned}$$

(3) *Let $r \ge s$ and $B_1 = H$. Then we have*

$$\begin{aligned}\alpha_p(B, A) &- p^{-m+n+1}\alpha_p(B[p^{-1} \perp E_{n-1}], A)\\ &= \frac{1}{2(1-p^{-m+n})}\sum_a \{(1 - p^{-1} - p^{-m+n})f(p^r a \perp B_2)\\ &+ (1-p^{-1})p^{-1}f(p^{r+1}a \perp B_2) + p^{-2}f(p^{r+2}a \perp B_2)\},\end{aligned}$$

where a runs over a complete set of representatives of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$.

Proof. (1)The assertion can be easily proved by Theorem 3.2.

(2)By Theorem 3.2, we have

$$\begin{aligned}\alpha_p(B, A) &- p^{-m+n+1}\alpha_p(B[p^{-1} \perp E_{n-1}], A)\\ &= \beta_p(0, A)\alpha_p(p^r c_{12} \perp B_2, A^{(1)} \perp -p^r c_{11}).\end{aligned}$$

By assumption, $c_{11}y^2 + c_{12}$ is a unit for any $y \in \mathbb{Z}_p$.

Thus we have

$$\begin{aligned}\alpha_p(p^r c_{12} &\perp B_2, A^{(1)} \perp -p^r c_{11})\\ &= p^{-1}\sum_{a=0}^{p-1}\alpha_p(p^r(c_{11}a^2 + c_{12}) \perp B_2, A^{(1)}).\end{aligned}$$

Thus the assertion holds.

(3) For a sufficiently large e we have

$$\begin{aligned}a_e(n&-1; B, A^{(1)})\\ &= \sum_{(y_j) \in M_{1,n-1}(\mathbb{Z}_p/p^e\mathbb{Z}_p)} a_e(\begin{pmatrix} 2p^r y_1 & p^r \mathbf{y}' \\ p^r \cdot {}^t\mathbf{y}' & B_2 \end{pmatrix}, A^{(1)})\\ &= \sum_{(y_j) \in M_{1,n-1}(\mathbb{Z}_p/p^e\mathbb{Z}_p)} a_e(\begin{pmatrix} 2p^r y_1 - p^{2r}B_2^{-1}[{}^t\mathbf{y}'] & 0 \\ 0 & B_2 \end{pmatrix}, A^{(1)}),\end{aligned}$$

where ${}^t\mathbf{y}' = (y_2, \ldots, y_{n-1})$. By assumption, $2p^r y_1 - p^{2r}B_2^{-1}[{}^t\mathbf{y}'] \in p^r\mathbb{Z}_p/p^e\mathbb{Z}_p$ for any $(y_1, \ldots, y_{n-1}) \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^{n-1}$. Thus the map Φ from $(\mathbb{Z}_p/p^e\mathbb{Z}_p)^{n-1}$ to $p^r\mathbb{Z}_p/p^e\mathbb{Z}_p$ given by $\Phi((y_1, \ldots, y_{n-1})) = 2p^r y_1 - p^{2r}B_2^{-1}[{}^t\mathbf{y}']$ is a surjection and for any $y \in p^r\mathbb{Z}_p$, we have $\#\Phi^{-1}(y) = p^{e(n-2)+r}$. Thus we have

$$a_e(n-1; B, A^{(1)}) = p^{e(n-2)+r} \sum_{a \in \mathbb{Z}_p/p^{e-r}\mathbb{Z}_p} a_e(p^r a \perp B_2, A^{(1)}).$$

Thus the assertion holds by applying Theorem 3.2 and Proposition 3.3. □

Corollary 4.2. *Let B_1, B_2 and B_3 be non-degenerate symmetric matrices of level p^{l_1}, p^{l_2} and p^{l_3}, respectively, with entries in $\mathbb{Z}_p$. Put $B = p^r B_1 \perp p^{r-2} B_2 \perp B_3$. Assume that B is a canonical form, B_1 is a unimodular matrix of degree $\leq 2, l_2 \leq 1$ and $r \geq l_3 + 1$. Put $\tilde{B}_1 = (p^{-1}b_1), p^{-1}b_1 \perp p b_2$ or H according as $B_1 = (b_1), B_1 = b_1 \perp b_2$ with $-b_1 b_2 \notin (\mathbb{Z}_p^\times)^2$, or H. Then we have*

$$\alpha_p(B, A) - p^{-m+n+1}\alpha_p(p^{r-1}\tilde{B}_1 \perp p^{r-2}B_2 \perp B_3, A)$$
$$= \sum_{B'} c(B'; B_1, B_2) f(p^{r-2}B' \perp B_3),$$

where where B' runs over a certain finitely many symmetric matrices of degree $\deg B_1 + \deg B_2 - 1$, and $c(B'; B_1, B_2)$ is a rational number determined by B_1, B_2 and A such that

$$\sum_{B'} c(B'; B_1, B_2) = 1.$$

Next we consider the case for $p = 2$. We have only to analize $\alpha_2(n-1; B, A^{(1)})$ since the same argument holds for $\alpha_2(n-1; B, A^*)$ without any modification. Moreover, if $\deg(B) = 1$, then a recursion formula is obtained easily so we assume assume $\deg(B) > 1$.

Theorem 4.3. *Let $B = \perp_{i=0}^r 2^i(U_i \perp V_i)$ be a canonical form, and $\deg U_i = k_i$, as above.*

(1) *Assume that $U_r = (c_r)$ and $V_r = \varnothing$. Put*

$$B_{r-1} = 2U_{r-1} \perp U_{r-2} \perp 2V_{r-1},$$

and

$$B_{r-2} = \perp_{i=0}^{r-3} 2^i U_i \perp \perp_{i=0}^{r-2} 2^i V_i.$$

Then we have

$$\alpha_2(B, A) - 2^{-m+n+1}\alpha_2(2^{r-2}c_r \perp 2^{r-2}B_{r-1} \perp B_{r-2}, A)$$
$$= 8^{-l_{r-1}-k_{r-1}-k_{r-2}} \sum_{(y_i)} f(2^{r-2}B_{r-1}((y_i)) \perp B_{r-2}),$$

where (y_i) runs over all elements of $(\mathbb{Z}_2/8\mathbb{Z}_2)^{l_r+k_{r-1}+k_{r-2}}$ and $B_{r-1}((y_i))$ is a certain symmetric matrix of degree $l_r + k_{r-1} + k_{r-2}$ with entries in $\mathbb{Z}_2$ determined by B_{r-1} and (y_i) such that $\nu(\det B_{r-1}((y_i))) \leq 3$.

(2) *Assume that* $U_r = c_{r1} \perp c_{r2}$ *with* $c_{r1}, c_{r2} \in \mathbb{Z}_2^\times$ *and* $V_r = \varnothing$. *Put*

$$B_{r-1} = U_{r-1}$$

and

$$B_{r-2} = \perp_{i=0}^{r-2} 2^i U_i \perp \perp_{i=1}^{r-1} 2^i V_i.$$

For U_r *put* $\hat{U}_r = U_r, -U_r, Y,$ *or* H *according as* $-c_{r1}c_{r2} \equiv -1 \mod 8, -c_{r1}c_{r2} \equiv 3 \mod 8, -c_{r1}c_{r2} \equiv 5 \mod 8,$ *or* $-c_{r1}c_{r2} \equiv 1 \mod 8$. *Then we have*

$$\alpha_2(B, A) - 2^{-m+n+1}\alpha_2(2^{r-1}\hat{U}_r \perp 2^{r-1}B_{r-1} \perp B_{r-2}, A)$$
$$= 8^{-k_{r-1}-1} \sum_{(y_i)} f(2^{r-1}B_{r-1}((y_i)) \perp B_{r-2}),$$

where (y_i) *runs over all elements of* $(\mathbb{Z}_2/8\mathbb{Z}_2)^{k_{r-1}+1}$ *and* $B_{r-1}((y_i))$ *is a certain symmetric matrix of degree* $k_{r-1}+1$ *determined by* B_{r-1} *and* $((y_i))$ *such that* $\nu(\det B_{r-1}((y_i)))$
≤ 3.

(3) *Assume that* $V_r = Y$. *Let*

$$B_{r-1} = 2U_r,$$

and

$$B_{r-2} = \perp_{i=0}^{r-1} 2^i (U_i \perp V_i).$$

Then we have

$$\alpha_2(B, A) - 2^{-m+n+1}\alpha_2(2^{r-1}(1 \perp 3) \perp 2^{r-1}B_{r-1} \perp B_{r-2}, A)$$
$$= 2^{-k_r-1} \sum_{(y_i)} f(B_{r-1}((y_i)) \perp B_{r-2}),$$

where (y_i) *runs over all elements of* $(\mathbb{Z}_2/8\mathbb{Z}_2)^{k_r+1}$ *and* $B_{r-1}((y_i))$ *is a certain symmetric matrix of degree* k_r+1 *determined by* B_{r-1} *and* $((y_i))$ *such that* $\nu(\det B_{r-1}((y_i))) \leq 3$.

(4) *Let* $V_r = \overbrace{H \perp \cdots \perp H}^{l_r - 1} \perp Y_r$ *with* $l_r \geq 1$. *We set*

$$g = \alpha_2(2^r H \perp B_r, A) - 2^{-m+n+1}\alpha_2(2^{r-1}H \perp B_r, A).$$

(4.1) *Assume that* $U_{r-1} = \varnothing$. *Then*

$$g = (1 - 2^{-m+n})^{-1} \sum_a \{(2^{-3} - 2^{-m+n-2}) f(2^{r+1}a \perp B_r)$$
$$+ 2^{-4} f(2^{r+2}a \perp B_r) + 2^{-4} f(2^{r+3}a \perp B_r)\},$$

where a *runs over complete set of representatives of* $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2$.

(4.2) *Assume that* $U_r = (c_r)$ *or* $c_{r1} \perp c_{r2}$. *Then*

$$g = \sum_a \{2^{-3} f(2^r a \perp B_r) + (1 - 2^{-m+n})^{-1}(2^{-4} - 2^{-m+n+3}) f(2^{r+1} a \perp B_r)$$
$$+ 2^{-5} f(2^{r+2} a \perp B_r) + 2^{-5} f(2^{r+3} a \perp B_r)\},$$

where a *runs over complete set of representatives of* $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2$.

Proof. (1) First remark that $V_{r-1} = \perp_{i=1}^{l_{r-1}} H$ by (C.2) in the definition of canonical form. Put $B'_{r-1} = 2^{r-1}(U_{r-1} \perp V_{r-1}) \perp 2^{r-2} U_{r-2} \perp B_{r-2}$. By Theorem 3.2, we have

$$\alpha_2(2^r c_r \perp B'_{r-1}, A) = 2^{-m+n+1} \alpha_2(2^{r-2} c_r \perp B'_{r-1}, A) + f(2^r c_r \perp B'_{r-1}).$$

Further we have

$$a_e(n-1; 2^r c_r \perp B'_{r-1}, A^{(1)}) = \sum_{(y_i) \in (\mathbb{Z}_2/2^e\mathbb{Z}_2)^{n-1}} a_e(B'_{r-1} + 2^r(c_r y_i y_j)_{2 \le i,j \le n}, A^{(1)}),$$

and $B'_{r-1} + 2^r(c_r y_i y_j)_{2 \le i,j \le n}$ is non-degenerate over $\mathbb{Z}_p/p^e\mathbb{Z}_p$ and its determinant does not depend on e for a sufficiently large e for any $(y_2, \ldots, y_n) \in \mathbb{Z}_2^{n-1}$. Thus we have

$$\alpha_2(n-1; 2^r c_r \perp B'_{r-1}, A^{(1)})$$
$$= 2^{-e(n-1)} \sum_{(y_i) \in (\mathbb{Z}_2/2^e\mathbb{Z}_2)^{n-1}} \alpha_2(B'_{r-1} + 2^r(c_r y_i y_j)_{2 \le i,j \le n}, A^{(1)}).$$

In particular, $B_{r-2} + 2^r(c_r y_i y_j)_{l_{r-1}+k_{r-1}+k_{r-2}+2 \le i,j \le n}$ is equivalent, over $\mathbb{Z}_2$, to B_{r-2} for any $(y_i)_{2 \le i \le l_{r-1}+k_{r-1}+k_{r-2}} \in \mathbb{Z}_2^{l_{r-1}+k_{r-1}+k_{r-2}-1}$. Thus the assertion holds.

(2) For the matrix U_r, put $\tilde{U}_r = \begin{pmatrix} c_{r1} + c_{r2} & c_{r2} \\ c_{r2} & 2c_{r2} \end{pmatrix}$. Then U_r is equivalent, over $\mathbb{Z}_2$ to $\tilde{U}_r$, and $2^r \tilde{U}_r[p^{-1} \perp 1] = 2^{r-1} \hat{U}_r$. Thus, by applying Theorem 3.2 to $2^r U_r \perp B_{r-1} \perp B_{r-2}$ the assertion can be proved similalry to (1).

(3) The assertion can be proved similalry to (1) noting that $2^r Y[p^{-1} \perp 1]$ is equivalent to $2^{r-1}(1 \perp 3)$.

(4) Similarly to (3) of Theorem 4.1, we have

$$a_e(n-1; 2^r H \perp B_r, A^{(1)}) = \#\{\begin{pmatrix} (y_j) \\ X \end{pmatrix} \in M_{m-1,n-1}(\mathbb{Z}_2/2^e\mathbb{Z}_2);$$

$$A^{(1)}[X] \equiv \begin{pmatrix} 2^{r+1} y_1 - 2^{2r} B_r^{-1}[{}^t\mathbf{y}'] & 0 \\ 0 & B_r \end{pmatrix} \mod 2^e\},$$

where ${}^t\mathbf{y}' = (y_2, \ldots, y_{n-1})$.

(4.1) First let $\deg(U_r) = 2$. By assumption, $2^{r+1} y_1 - 2^{2r} B_r^{-1}[{}^t\mathbf{y}'] \in 2^{r+1}\mathbb{Z}_2/2^e\mathbb{Z}_2$ for any $(y_2, \ldots, y_{n-1}) \in \mathbb{Z}_2^{n-1}$. Thus similarly to (3) of Theorem 3.1, we have

$$a_e(n-1; 2^r H \perp B_r, A^{(1)}) = 2^{e(n-2)+r+1} \sum_{a \in \mathbb{Z}_2/2^{e-r-1}\mathbb{Z}_2} a_e(2^{r+1} a \perp B_r, A^{(1)}),$$

thus the assertion holds.

(4.2) By assumption $2^{2r}B_r^{-1}[{}^t\mathbf{y}'] = 2^r\sum_{j=1}^{2} c_{rj}y_{2l_r+j}^2 + 2^{r+1}b$ with $b \in \mathbb{Z}_2$ for any $(y_1,\ldots,y_n) \in \mathbb{Z}_2^n$. Thus $2^{r+1}y_1 - 2^{2r}B_r^{-1}[{}^t\mathbf{y}'] \in 2^r\mathbb{Z}_2^\times$ or $\in 2^{r+1}\mathbb{Z}_2$ according as exactly one of y_{2l_r+1} and y_{2l_r+2} is unit, or not. In the former case, for each such $(y_{2l_r+1}, y_{2l_r+2}) \in (\mathbb{Z}_2/2^e\mathbb{Z}_2)^2$ and a fixed $c \in (2^r\mathbb{Z}_2^\times + 2^e\mathbb{Z}_2)/2^e\mathbb{Z}_2$ the mapping Φ_c from $(\mathbb{Z}_2/2^e\mathbb{Z}_2)^{n-3}$ to $\{c\}$ defined by

$$\Phi_c((y_1,\ldots,y_{2l_r},y_{2l_r+3},\ldots,y_{n-1})) = 2^{r+1}y_1 - 2^{2r}B_r^{-1}[{}^t\mathbf{y}']$$

is a surjection and $\Phi_c^{-1}(c) = 2^{e(n-4)+r+1}$. In the latter case, for each such $(y_{2l_r+1}, y_{2l_r+2}) \in (\mathbb{Z}_2/2^e\mathbb{Z}_2)^2$ the mapping Φ' from $(\mathbb{Z}_2/2^e\mathbb{Z}_2)^{n-3}$ to $2^{r+1}\mathbb{Z}_2/2^e\mathbb{Z}_2$ defined by

$$\Phi'((y_1,\ldots,y_{2l_r},y_{2l_r+3},\ldots,y_{n-1})) = 2^{r+1}y_1 - 2^{2r}B_r^{-1}[{}^t\mathbf{y}']$$

is a surjection and for each $x \in 2^{r+1}\mathbb{Z}_2/2^e\mathbb{Z}_2$ we have $\Phi'^{-1}(x) = 2^{e(n-4)+r+1}$. Thus we have

$$\begin{aligned}
&a_e(n-1; 2^rH \perp B_r, A^{(1)}) \\
&= 2^{2e-1+e(n-4)+r+1}\{\sum_{c\in(2^r\mathbb{Z}_2^\times+\mathbb{Z}_2)/2^e\mathbb{Z}_2} a_e(c \perp B_r, A^{(1)}) \\
&\quad + \sum_{a\in\mathbb{Z}_2/2^{e-r-1}\mathbb{Z}_2} a_e(2^{r+1}a \perp B_r, A^{(1)})\} \\
&= 2^{e(n-1)-3}\sum_{a\in\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2} a_e(2^ra \perp B_r, A^{(1)}) \\
&\quad + 2^{e(n-2)+r}\sum_{a\in\mathbb{Z}_2/2^{e-r-1}\mathbb{Z}_2} a_e(2^{r+1}a \perp B_r, A^{(1)}),
\end{aligned}$$

thus the assertion follows similarly to (1.1) in this theorem. Similarly, the assertion holds for $\deg(U_r) = 1$. □

Corollary 4.4. *Let B_1, B_2 and B_3 be non-degenerate symmetric matrices of level $2^{l_1}, 2^{l_2}$ and 2^{l_3}, respectively, with entries in $\mathbb{Z}_2$. Put $B = 2^rB_1 \perp 2^{r-2}B_2 \perp B_3$. Assume that B is a canonical form, B_1 is a unimodular matrix of degree ≤ 2, $l_2 \leq 1$ and $r \geq l_3+1$. Put $\tilde{B}_1 = 2^{-1}B_1, \hat{B}_1, 1 \perp 3$, or H according as $B_1 = (b_1), b_1 \perp b_2, Y$, or H, where " $\hat{}$ " is the same as in Theorem 4.3 (2). Then we have*

$$\begin{aligned}
&\alpha_2(B,A) - 2^{-m+n+1}\alpha_2(2^{r-1}\tilde{B}_1 \perp 2^{r-2}B_2 \perp B_3, A) \\
&= \sum_{B_2'} c(B; B_1, B_2) f(2^{r-2}B' \perp B_3),
\end{aligned}$$

where B' runs over a certain finitely many symmetric matrices of degree $deg(B_1) + deg(B_2) - 1$, and $c(B'; B_1, B_2)$ is a rational number determined by B_1, B_2 and A such that

$$\sum_{B'} c(B'; B_1, B_2) = 1.$$

5. Proof of the main theorems

For non-degenerate symmetric matrices $A, B_0, B_1, \ldots, B_s$ of degree $m, n_0, n_1, \ldots, n_s$, respectively, with entries in $\mathbb{Z}_p$, define the formal power series

$$P((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s) = \sum_{r_1, \ldots, r_s = 0}^{\infty} \alpha_p(B_0 \perp p^{r_1} B_1 \perp \cdots \perp p^{r_s} B_s, A) x_1^{r_1} \ldots x_s^{r_s}.$$

If $s = 1$ and $\deg(B_0) = 0$, the above series is nothing but the ones defined by Kitaoka in [9].

Let Δ_p be the complete set of representatives of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$, and put

$$\Lambda_{n,p} = \{(b_0, b_1, \ldots, b_n); b_i \in \Delta_p\}$$

or

$$\{(B_0, B_1, \ldots, B_s); \deg(B_0) + \deg(B_1) + \cdots + \deg(B_s) = n,$$
$$B_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ or } \deg(B_i) = 1 \text{ and } B_i \in \Delta_p\}$$

according as $p \neq 2$ or $= 2$. Then the set of power series

$$\{P((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s)\}_{(B_0, B_1, \ldots, B_s) \in \Lambda_{n,p}}$$

gives complete information on the local densities $\alpha_p(B, A)$ for all B of degree n and A of degree m. Therefore it is important to study these power series.

Definition. *Let B be a non-degenerate symmetric matrix with entries in $\mathbb{Z}_p$. B is said to be maximal if there is no square matrix X such that det(X) $\in p\mathbb{Z}_p$ and $B[X^{-1}]$ is a symmetric matrix with entries in $\mathbb{Z}_p$.*

To determine the denominator of formal power series, we need the followings.

Proposition 5.1(cf.[6]). *Let A be a symmetric unimodular matrix of degree m, B_1 and B_2 be symmetric matrices of degree n_1, n_2, respectively. Put $n = n_1 + n_2$. Let p^{b_2} be the level of B_2 if $n_2 > 0$. Put $l = \min(n_1 - 1, r)$, where r is the Witt index of A, and $n = n_1 + n_2$. Let e be an integer such that $e \geq b_2 + 2l + \delta_{2p} + 2$.*

(1) *Assume that B_1 is not maximal. Then there exists a maximal matrix $\tilde{B}_1$ such that $\deg B_1 = \deg \tilde{B}_1$ and $\nu(\det B_1) - \nu(\det \tilde{B}_1)$ is a positive even integer, and*

$$\alpha_p(p^e B_1 \perp B_2, A) - p^{(-m+n+1)b_1} \alpha_p(p^e \tilde{B}_1 \perp B_2, A)$$
$$= \sum_{B_1'} c(B_1', B_1, A) f(p^{e-2} B_1' \perp B_2),$$

where $b_1 = (\nu(\det B_1) - \nu(\det \tilde{B}_1))/2$, B_1' runs over finitely many symmetric matrices of degree $n_1 - 1$, and $c(B_1', B_1, A)$ is a rational number determined by B_1', B_1 and A such that

$$\sum_{B_1'} c(B_1', B_1, A) = \frac{1 - p^{(-m+n+1)b_1}}{1 - p^{-m+n+1}}.$$

(2) *Assume that* B_1 *is maximal. Then we have*

$$\alpha_p(p^e B_1 \perp B_2, A) - p^{(-m+n+1)n_1}\alpha_p(p^{e-2}B_1 \perp B_2, A)$$
$$= \sum_{B_1'} c(B_1', B_1, A) f(p^{e-1}B_1' \perp B_2),$$

where B_1' *runs over finitely many symmetric matrices of degree* n_1-1, *and* $c(B_1', B_1, A)$ *is a rational number determined by* B_1', B_1 *and* A *such that*

$$\sum_{B_1'} c(B_1', B_1, A) = \frac{1-p^{(-m+n+1)n_1}}{1-p^{-m+n+1}}.$$

Proof. We note that a maximal matrix is equivalent, over $\mathbb{Z}_p$, to a matrix of the following form:

$$pU_1 \perp U_2 \perp V_1,$$

where U_1, U_2 and V_1 are the ones in section 1, and in particular $\deg(U_1) \le 1$ if $p = 2$. Thus the assertion can be easily proved by using a Corollarry 4.2 or 4.4, repeatedly. □

For each non-negative integers i, j and k such that $1 \le k \le i$, put (cf. [4])

$$\gamma(i,j,k) = (-1)^k \sum_{0 \le l_1 < \cdots < l_k \le i-1} p^{(i-l_1)(j+l_1)} \cdots p^{(i-l_k)(j+l_k)}.$$

Here we understand that $\gamma(i,j,0) = 1$. It is easy to see that

$$\prod_{k=0}^{i-1}(1-p^{(i-k)(j+k)}x) = \sum_{k=0}^{i}\gamma(i,j,k)x^k.$$

Theorem 5.2. *Let the notations and the assumptions be the same as in Proposition 5.1. Then we have*

$$\sum_{i=0}^{l+1}\gamma(n_1, -m+n+1, i)\alpha_p(p^{e-2i}B_1 \perp B_2, A) = \phi \prod_{i=0}^{l} \frac{1-p^{(n_1-i)(-m+n+i+1)}}{1-p^{-m+n+i+1}}\psi(i),$$

where

$$\phi = \begin{cases} \beta_p(O_{l+1}, A)\alpha_p(B_2, A^{(l+1)}), & \text{(case I)} \\ p^{-2(l+1)}\alpha_p(B_2, A^{(l+1)}), & \text{(case II)} \\ p^{-(l+1)}\beta_p(O_{l+1}, A^*)\{\alpha_p(B_2, A^{(l+1)}) & \\ \quad +p^{-m-l+4}(p^{3(l+1)}-1)(p^3-1)^{-1}\beta_p(0, A^{*(l)})^{-1}\alpha_p(B_2, A^{*(l)})\}, & \text{(case III)} \end{cases}$$

and

$$\psi(i) = \begin{cases} 1, & \text{(case I,III)} \\ \beta_p(0, A^{*(i)}) + \beta_p(tr(-U), A^{*(i)}), & \text{(case II)}\ . \end{cases}$$

Here we make the convention that $\alpha_p(B_2, A^{(l+1)}) = 1$ or 0 if $n = n_1$ or $l = r$, respectively.

Proof. We shall prove (case III) only since the proof of (case I) is completed in [6] and (case II) can be proved in the same manner as in (case I).

We prove the assertion by induction on n_1. The assertion clearly holds if $n_1 = 1$. Assume that the assertion holds for $n_1 - 1$. First assume that B_1 is maximal. Then by (2) of Proposition 5.1, we have

$$\begin{aligned}
&\alpha_p(p^e B_1 \perp B_2, A) - p^{(-m+n+1)n_1}\alpha_p(p^{e-2}B_1 \perp B_2, A) \\
&= p^{-1}\beta_p(0, A^*)\sum_{B_1'} c(B_1', B_1, A)\alpha_p(p^{e-1}B_1' \perp B_2, A^{(1)}) \\
&+ p^{-m+3}\sum_{B_1'} c(B_1', B_1, A)\alpha_p(p^{e-1}B_1' \perp B_2, A^*),
\end{aligned}$$

where B_1' and the others are ones in (2) of Proposition 5.1. By the inductive hypothesis, we have

$$\begin{aligned}
&\sum_{i=0}^{l}\gamma(n_1-1, -m+n+2, i)\alpha_p(p^{e-2i-1}B_1' \perp B_2, A^{(1)}) \\
&= p^{-l}\beta_p(O_l, A^{*(1)})\{\alpha_p(B_2, A^{(l+1)}) + p^{-m-l+7}(p^{3l}-1)(p^3-1)^{-1} \\
&\cdot\beta_p(0, A^{*(1)})^{-1}\alpha_p(B_2, A^{*(l)}\}\prod_{i=0}^{l-1}\frac{1-p^{(n_1-i-1)(-m+n+i+2)}}{1-p^{-m+n+i+2}}.
\end{aligned}$$

By applying (case II) and by noting that

$$\beta_p(0, A)\beta_p(O_i, A^{(1)}) = \beta_p(O_{i+1}, A),$$

we have

$$\begin{aligned}
&\sum_{i=0}^{l}\gamma(n_1-1, -m+n+2, i)\alpha_p(p^{e-2i-1}B_1' \perp B_2, A^*) \\
&= p^{-2l}\beta_p(O_l, A^*)\alpha_p(B_2, A^{*(l)})\prod_{i=0}^{l-1}\frac{1-p^{(n_1-i-1)(-m+n+i+2)}}{1-p^{-m+n+i+2}}
\end{aligned}$$

for any such B_1'. Thus we have

$$\begin{aligned}
&\sum_{i=0}^{l+1}\gamma(n_1, -m+n+1, i)\alpha_p(p^{e-2i}B_1 \perp B_2, A) \\
&= \sum_{i=0}^{l}\gamma(n_1-1, -m+n+2, i)\{\alpha_p(p^{e-2i}B_1' \perp B_2, A) - p^{-m+n+1}\alpha_p(p^{e-2i-2}B_1' \perp B_2, A)\}
\end{aligned}$$

$$= \sum_{i=0}^{l} \gamma(n_1-1, -m+n+2, i)\{p^{-1}\beta_p(0, A^*) \sum_{B_1'} c(B_1', B_1, A)\alpha_p(p^{e-2i-1}B_1' \perp B_2, A^{(1)})$$

$$+ p^{-m+3} \sum_{B_1'} c(B_1', B_1, A)\alpha_p(p^{e-2i-1}B_1' \perp B_2, A^*)\}$$

$$= p^{-1}\beta_p(0, A^*) \sum_{B_1'} c(B_1', B_1, A)p^{-l}\beta_p(O_l, A^{*(1)})\{\alpha_p(B_2, A^{(l+1)})$$

$$+ p^{-m-l+7}(p^{3l}-1)(p^3-1)^{-1}\beta_p(0, A^{*(1)})^{-1}\alpha_p(B_2, A^{*(l)}\} \prod_{i=0}^{l-1} \frac{1-p^{(n_1-i-1)(-m+n+i+2)}}{1-p^{-m+n+i+2}}$$

$$+ p^{-m+3} \sum_{B_1'} c(B_1', B_1, A)p^{-2l}\beta_p(O_l, A^*)\alpha_p(B_2, A^{*(l)}) \prod_{i=0}^{l-1} \frac{1-p^{(n_1-i-1)(-m+n+i+2)}}{1-p^{-m+n+i+2}}.$$

Thus the assertion holds.

Next assume that B_1 is not maximal. Then by (1) of Proposition 5.1 there exists a maximal matrix $\tilde{B}_1$ such that we have

$$\alpha_p(p^e B_1 \perp B_2, A) - p^{(-m+n+1)b_1}\alpha_p(p^{e-2}\tilde{B}_1 \perp B_2, A)$$

$$= p^{-1}\beta_p(0, A^*) \sum_{B_1'} c(B_1', B_1, A)\alpha_p(p^{e-2}B_1' \perp B_2, A^{(1)})$$

$$+ p^{-m+3} \sum_{B_1'} c(B_1', B_1, A)\alpha_p(p^{e-2}B_1' \perp B_2, A^*),$$

where B_1' runs over finitely many even matrices of degree n_1-1, and $c(B_1', B_1, A)$ is a rational number determined by B_1' and A such that

$$\sum_{B_1'} c(B_1', B_1, A) = \frac{1-p^{(-m+n+1)b_1}}{1-p^{-m+n+1}}.$$

Thus we have

$$\sum_{i=0}^{l+1} \gamma(n_1, -m+n+1, i)\alpha_p(p^{e-2i}B_1 \perp B_2, A)$$

$$= p^{(-m+n+1)b_1} \sum_{i=0}^{l+1} \gamma(n_1, -m+n+1, i)\alpha_p(p^{e-2i}\tilde{B}_1 \perp B_2, A)$$

$$+ p^{-1}\beta_p(0, A^*) \sum_{B_1'} c(B_1', B_1, A) \sum_{i=0}^{l+1} \gamma(n_1, -m+n+1, i)\alpha_p(p^{e-2i-2}B_1' \perp B_2, A^{(1)})$$

$$+ p^{-m+3} \sum_{B_1'} c(B_1', B_1, A) \sum_{i=0}^{l+1} \gamma(n_1, -m+n+1, i)\alpha_p(p^{e-2i-2}B_1' \perp B_2, A^*).$$

By the inductive hypothesis the second term on the right-hand side of the above is

$$(1-p^{(-m+n+1)b_1})\prod_{i=0}^{l}\frac{1-p^{(n_1-i)(-m+n+i+1)}}{1-p^{-m+n+i+1}}p^{-(l+1)}\beta_p(O_{l+1},A^*)\{\alpha_p(B_2,A^{(l+1)})$$
$$+p^{-m-l+7}(p^{3l}-1)(p^3-1)^{-1}\beta_p(0,A^{*(l)})^{-1}\alpha_p(B_2,A^{*(l)})\},$$

and the third term on the right-hand side of the above is

$$(1-p^{(-m+n+1)b_1})\prod_{i=0}^{l}\frac{1-p^{(n_1-i)(-m+n+i+1)}}{1-p^{-m+n+i+1}}p^{-2l_m+3}\beta_p(O_l,A^*)\alpha_p(B_2,A^{(l)}).$$

Thus the assertion holds. □

Using Theorem 5.2, we obtain the following proposition.

Proposition 5.3. *Let A be as above. Let $B_0, B_1, \ldots, B_s$ be non-degenerate symmetric matices of degree $n_0, n_1, \ldots, n_s$, respectively, with entries in $\mathbb{Z}_p$. Let p^{a_i} be the level of B_i and $a_1 < \cdots < a_s$. Put $l = \min(n_1, r)$ and $l_0 = l + \delta_{2p}$. Then we have*

$$\prod_{i=0}^{l}(1-p^{(n_1-i)(-m+n+i+1)}x_1^2)Q((B_0,B_1,\ldots,B_s),A;x_1,\ldots,x_s)$$
$$=\sum_{i=0}^{l_0}x_1^{2i}\sum_{j=0}^{i}\gamma(n_1,-m+n+1,i-j)$$
$$\cdot Q((B_0,p^{2j}B_1\perp B_2,B_3,\ldots,B_s),A;x_1x_2,x_3,\ldots,x_s)$$
$$+\sum_{i=0}^{l_0}x_1^{2i+1}\sum_{j=0}^{i}\gamma(n_1,-m+n+1,i-j)$$
$$\cdot Q((B_0,p^{2j+1}B_1\perp B_2,B_3,\ldots,B_s),A;x_1x_2,x_3,\ldots,x_s)$$
$$+\prod_{i=0}^{l}\frac{1-p^{(n_1-i)(-m+n+i+1)}}{1-p^{-m+n+i+1}}\frac{x_1^{2l_0+2}}{1-x_1}\delta,$$

where

$$\delta=\begin{cases}\beta_p(O_{l+1},A)Q((B_0,B_2,\ldots,B_s),A^{(l+1)};x_1x_2,x_3,\ldots,x_s), & \text{(case I)}\\ p^{-2l_0}\{\beta_p(0,A^{*(i)})+\beta_p(tr(-U),A^{*(i)})\} & \\ \quad\cdot Q((B_0,B_2,\ldots,B_s),A^{(l_0)};x_1x_2,x_3,\ldots,x_s), & \text{(case II)}\\ p^{-l_0}\beta_p(O_{l_0},A^*)\{Q((B_0,B_2,\ldots,B_s),A^{(l_0)};x_1x_2,x_3,\ldots,x_s) & \\ \quad+p^{-m-l_0+4}(p^{3(l_0+1)}-1)(p^3-1)^{-1}\beta_p(0,A^{*(l_0)})^{-1} & \\ \quad\cdot R((B_0,B_2,\ldots,B_s),A^{*(l)};x_1x_2,x_3,\ldots,x_s)\}, & \text{(case III)}.\end{cases}$$

Here we make the convention that $Q((B_0,p^jB_1\perp B_2,B_3,\ldots,B_s),A;x_1x_2,x_3,\ldots,x_s)=\alpha_p(B_0\perp p^jB_1,A)$ if $s=1$ and $Q((B_2,\ldots,B_s),A^{(l+1)};x_1x_2,x_3,\ldots,x_s)=\alpha_p(B_0,A^{(l)})$ or 0 if $s=1$ or $r=l$, respectively, and $j_k'=j_k$ or j_k+1 according as $p\neq 2$ or not, respectively.

Theorem 5.4. *Let A be a symmetric unimodular matrix of degree m and B be a non-degenerate symmetric matrix of degree n. We assume that B has the following decomposition: $B = B_0 \perp B_1 \perp \cdots \perp B_s$, $\deg(B_i) = n_i$. Let t be the Witt index of A. Then $Q((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s)$ is a rational function of $x_1, \ldots, x_s$ with a denominator*

$$\prod_{k=1}^{s}\prod_{i=0}^{u_k}(1 - p^{(n_1+\cdots+n_k-i)(-m+n+i+1)}(x_1 \ldots x_k)^2) \prod_{k=1}^{s}(1 - x_1 \ldots x_s)^{v_k},$$

where $u_k = \min(n_1 + \cdots + n_k - 1, t)$ and $v_k = 1$ or 0 according as $t \geq n_1 + \cdots + n_k$ or not.

Proof. It is directly proved by Proposition 5.3. □

To prove the main Theorems, we need the following proposition due to Katsurada[5];

Proposition 5.5. *Let k, t and l be non-negative integers such that $2k \geq t \geq l \geq 1$, $k \geq l$ and $T = T_1 \perp T_2$. T_1 and T_2 are non-degenerate even integral symmetric matrices of degree l and $t - l$, respectively. Assume that $T_1 \equiv O_l \mod p$. Then we have*

$$\begin{aligned}&\beta_p(T_1, H_k)\alpha_p(T_2, H_{k-l} \perp (-T_1)) \\ &= \sum_{i=0}^{l}(-1)^i p^{i(i-1)+i(t+1-2k)} \sum_{g_i} \alpha_p(T_1[g_i^{-1}] \perp T_2, H_k),\end{aligned}$$

where $H_k = H \perp \cdots \perp H(k$ copies$)$ and g_i runs over the representatives of $GL_l(\mathbb{Z}_p)\backslash GL_l(\mathbb{Z}_p)\lambda(i)GL_l(\mathbb{Z}_p)$ with $\lambda(i) = pE_i \perp E_{l-i}$.

Proof of Therem 1. Put $-p^2A = T_1$ and $p^2B = T_2$. Then by applying Proposition 5.4 with $k = l$, we have

$$\beta_p(-p^2A, H_m)\alpha_p(p^2B, p^2A) = \sum_{i=0}^{m}(-1)^i p^{i(i-1)+i(-m+n+1)} \sum_{g_i} \alpha_p(-p^2A[g_i^{-1}] \perp p^2B, H_m).$$

Noting that $\beta_p(-p^2A, H_m) \neq 0$ and $\alpha_p(p^2B, p^2A) = p^{n(n+1)}\alpha_p(B, A)$, we have

$$\begin{aligned}&Q((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s) \\ &= \beta_p(-p^2A, H_m)^{-1} \sum_{i=0}^{m}(-1)^i p^{i(-m+n+i)-n(n+1)} \\ &\cdot \sum_{g_i} Q((-p^2A[g^{-1}] \perp p^2B_0, p^2B_1, \ldots, p^2B_s), H_m; x_1, \ldots, x_s).\end{aligned}$$

Then by applying Theorem 5.4 to $Q((-p^2A[g^{-1}] \perp p^2B_0, p^2B_1, \ldots, p^2B_s), H_m; x_1, \ldots, x_s)$'s, we complete the proof. □

To prove Theorem 2, define

$$\tau(s) = \{(r_{i_1}, \ldots, r_{i_s}) \in (\mathbb{Z}_{\geq 0})^s; r_{i_a} \geq r_{i_b} \text{ or } r_{i_a} > r_{i_b}$$

$$\text{according as } i_a < i_b \text{ or not for } 1 \leq a, b \leq s.\}.$$

Let A be the non-degenerate matrix of degree m with the Witt index t, $B_0, B_1 \ldots, B_s$ be as above. Define

$$Q_{i_1,\ldots,i_s}((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s) = \sum_{(r_{i_1},\ldots r_{i_s}) \in \tau(s)} \alpha_p(B_0 \perp p^{r_{i_1}} B_1 \perp \cdots \perp p^{r_{i_s}} B_s, A) x_1^{r_{i_1}} \ldots x_s^{r_{i_s}}.$$

Theorem 5.6. *Let the notation be as above. Then* $Q_{i_1,\ldots,i_s}((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s)$ *is a rational function of* $x_1, \ldots, x_s$ *with a denominator*

$$\prod_{k=1}^{s} \prod_{i=0}^{u_k} (1 - p^{(n_1+\cdots+n_k-i)(-m+n+i+1)}(x_1 \ldots x_k)^2) \prod_{k=1}^{s} (1 - x_1 \ldots x_s)^{v_k},$$

where $u_k = \min(n_1 + \cdots + n_k - 1, t)$ *and* $v_k = 1$ *or* 0 *according as* $t \geq n_1 + \cdots + n_k$ *or not.*

Proof. This can be easily proved by induction on s as that in Theorem 1 by noting that Proposition 5.3 is effective for $Q_{i_1,\ldots,i_s}((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s)$ instead of $Q((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s)$. $\square$

Now, we can prove Theorem 2.

proof of Theorem 2. Identifying $(i_1, \ldots, i_s)$ and $\begin{pmatrix} 1 & \ldots & s \\ i_1 & \ldots & i_s \end{pmatrix} \in \mathfrak{S}_s$(symmetric group of degree s), we can write

$$P((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s) = \sum_{(i_1,\ldots,i_s) \in \mathfrak{S}_s} Q_{i_1,\ldots,i_s}((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s),$$

that is, $P((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s)$ is a finite linear combination of $Q_{i_1,\ldots,i_s}((B_0, B_1, \ldots, B_s), A; x_1, \ldots, x_s)$, which proves the assertion. $\square$

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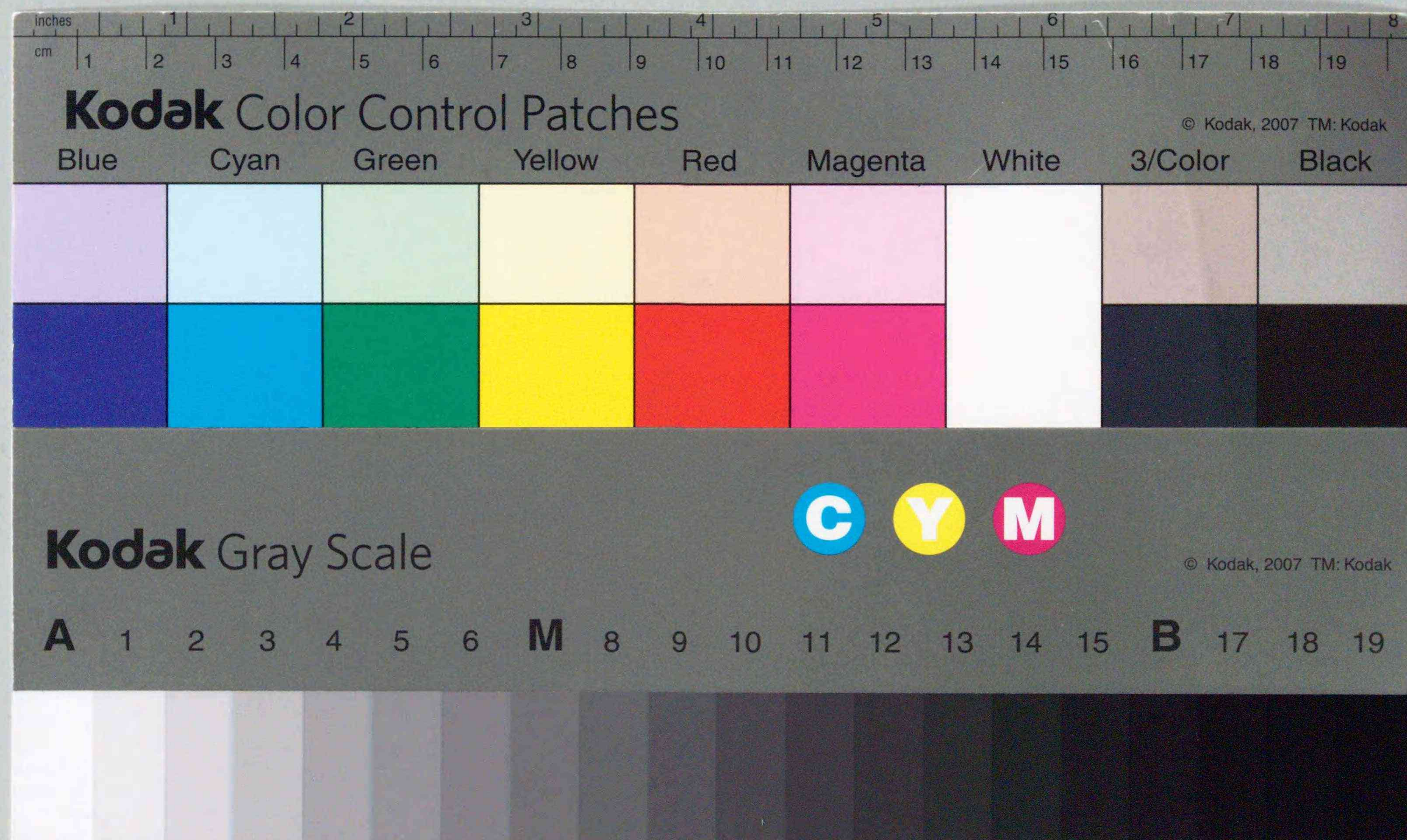
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