Singularities in Witten type Topological Field Theory

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ABSTRACT

We investigate relation between Witten type Topological Field Theory(TFT) and some singularities to find the nature of TFT and to search new possibility. Singularities that we will study are classified into two types. The first type is represented as moduli space singularity. In any gauge theories, if gauge transformation is not effective, singularities exist in moduli space. Especially we study it in Topological QCD, and we find some new relation between the Abelian Seiberg-Witten invariants and non-Abelian Seiberg-Witten invariants and Donaldson invariants. The second type of singularity appear necessitously when we treat topological invariants in field theory. We use this singularity to break topological symmetry in topological gravity. As a result we get a gravitational theory whose relevant contribution is given from Einstein gravity.

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Chapter 1

Introduction

Many problems of Topology and manifolds are solved in this century, but two big subjects are left until now. They are topological calassification of 3 and 4 dimensional manifolds. Strictly speaking, 4-dimensional topology is conquered by Freedman [46], but differential topology has not been solved yet. Differential topology have been studied since 50's, (since Milnor discovered exotic S^7). In the definition of the differential topology, identification is not only up to homeomorphism but also up to diffeomorphism. In this paper we study the relations between field theory and differential topology. We treat only Rieman manifolds, then "Topological invariant" means "Rieman metric independent" .

To study differential topology, physical(especially field theoretical) technique have been used. The toward the understanding of 4-dimensional differential topology, there is a progress with useing non-Abelian gauge theory by Donaldson in 80's. Witten reconstruct the Donaldson theory as topological field theory(TFT). Donaldson theory has solved many problems of 4-manifolds but it has difficulties for calculations. Recently new developements have been obtained by Seiberg-Witten monopole theory [1] . They conquered a difficulty of Donaldson theory . The Seiberg-Witten theory is easy for its computation since it is an Abelian gauge theory. And the Seiberg-Witten topological invariants are expected equivalent to Donaldson invariants by weak-strong duality relation. This Seiberg-Witten topological theory is reconstructed as topological field theory too. Thus TFT give us various knowledge of topology.

We investigate relation between TFT and some singularities to find the nature of TFT and to search new possibility. Singularities that we will study are classified two types. The first type is represented as moduli space singularities. In any gauge theories, if gauge transformation is not effective, singularities exist in moduli space. Especially we will study it in Topological QCD, and we will make clear the relation between the Abelian Seiberg-Witten invariants and non-Abelian Seiberg-Witten invariants and Donaldson invariants. The second type of singularity appears necessitously when we treat topological invariants in field theory. We use this singularity to break topological symmetry in topological gravity. As a result we get gravitational theory whose relevant contribution is given from Einstein gravity.

This paper is organized as follows. In chapter 2, general formalism of Witten type TFT is given. It is mentioned by Mathai-Quillen formalism. It is possible to describe Witten type TFT by any other formalism, for example N=2 SUSY twisting , Atiyah system of axiom, and so on. For physicists twisting procedure is easiest way to construct TFT. But we adopt Mathai-Quillen formalism to emphases geometrical pictures. In chapter 3, moduli space singularities in Topological QCD is discussed. Reducible connection cause this singularities. A role of reducible connections in Non-Abelian Seiberg-Witten invariants is analyzed where monopole is extended to non-Abelian groups version. By giving small external fields, we found that vacuum expectation value can be separated into a part from Donaldson theory , a part from Abelian Monopole theory and a part from non-Abelian monopole theory. As a by-product, we find identities of U(1) topological invariants. In the derivation, the weak-strong duality relation and Higgs mechanism are not necessary. In chapter 4, we discuss singularities which necessitously appeare in field theoretical description. Cause of these singularities are understood as Gribov zero modes, field theoretically. It is known that if gauge conditions have Gribov zero modes, then topological symmetry can be broken. We apply it to the Witten type topological gravitational theory in dimension $n \geq 3$. Our choice of the gauge condition for conformal invariance is $R + \alpha = 0$, where R is the Ricci scalar curvature. We find when $\alpha \neq 0$, topological symmetry is not broken, but when $\alpha = 0$ and solutions of the Einstein equations exist then topological symmetry is broken. This conditions connect to the Yamabe conjecture. Namely negative constant scalar curvature exist on manifolds of any topology, but existence of nonnegative constant scalar curvature is restricted by topology. This fact is easily seen in this theory. Topological symmetry breaking means that BRS symmetry breaking in cohomological field theory. But it is found that another BRS symmetry can be defined and physical states are redefined. The divergence due to the Gribov zero modes is regularized, and the theory after topological symmetry breaking become semiclassical Einstein gravitational theory under a special definition of observables. In the last chapter we give a summary and some discussions.

Chapter 2

General formalism

2.1 Introduction

We give general formalism of Witten type TFT. Witten type TFT is given by various ways. Twisting of N=2 Super symmetry is familiar way. Twisting is given by rotating simultaneously $SU(2)$ of Lorents and N=2 SUSY. The other ways to obtain Witten type TFT are gauge fixing construction, Atiyah system of axiom, Morse theoretical approach, and so on. We apply the Mathai-Quillen formalism to construct the Witten type TFT. The reason to apply it is that this formalism make geometrical picture clear. Especially if we do not see the singularities from a geometrical view point, we can not distinguish the singularities which appear in chapter 3 and 4.

This chapter is organized as follows. In section 2, we start discussion from Gauss-Bonnet theorem and review the Mathai-Quillen formalism as TFT. Most part of this section is found in J.M.F.Labastida and C.Lozano's lecture note and S.Cordes, G.Moore and S.Ramgoolam's lecture note [43] [44]. In section 3, relation of fermionic zero mode and integral area of topological invariants which is given as form integral are mentioned. In the section 4, we extend the theory given in section 2 and 3 to gauge theory. When the local symmetry exist, some modification is demanded. In the section 5, we study two types of singularity there. We can understand the geometrical origins of the singularities.

2.2 Witten Type Topological Field Theory as Euler Class

We give the convention of the paper first. M is $2m$ dimensional orientable compact manifolds. We denote $\chi(M)$ as the Euler number of M . Euler number $\chi(M)$ is defined with cohomology class $H^i(M;\mathbf{R})$ as

$$\chi(M) = \sum_{i=0}^{2m} \dim H^i(M;\mathbf{R}). \tag{2.1}$$

Euler form $e(\nabla,\varepsilon)$ with a fiber bundle ε and connection ∇ of $2m$-dim structure group is define as

$$\begin{aligned} e(\nabla,\varepsilon) &= \frac{(-1)^m}{2^m \pi^m m!} \varepsilon_{\alpha_1\alpha_2\ldots\alpha_{2m}} \Omega_{\alpha_1\alpha_2} \wedge \Omega_{\alpha_3\alpha_4} \wedge \ldots \wedge \Omega_{\alpha_{2m-1}\alpha_{2m}} \\ &= \frac{m!}{2^m \pi^m} \det(\Omega), \end{aligned} \tag{2.2}$$

where Ω is a curvature 2-form.

It is possible to construct Topological Field Theory(TFT) as Euler form. We chose the Gauss-Bonnet theorem ,

$$\chi(M) = \int_M e(\nabla,\varepsilon), \tag{2.3}$$

as the start point of construction of TFT. Where the fiber bundle ε is a tangent bundle of M, TM. Eq.(2.2) is denoted with fermionic integral as

$$e(\nabla,\varepsilon) = (2\pi)^{-m} \int d\chi e^{\frac{1}{2}\chi_a \Omega^{ab} \chi_b}. \tag{2.4}$$

where χ_a is a Grassmann-odd real number.

We introduce the Mathai-Quillen formalism. With the section of the fiber bundle," s", we define $e_s(\nabla,\varepsilon)$ as

$$e_s(\nabla,\varepsilon) = \frac{1}{(2\pi)^m} \int d\chi e^{-\frac{1}{2}|s|^2 + \frac{1}{2}\chi_a \Omega^{ab} \chi_b i \nabla s^a \chi_a}. \tag{2.5}$$

$e_s(\nabla,\varepsilon)$ is first introduced as pull back of Thom class by Mathai and Quillen [45]. When s is zero section, i.e. $s=0$, $e_s(\nabla,\varepsilon)$ is identified with $e(\nabla,\varepsilon)$. In general any form integral is rewritten with fermionic integral as

$$\int_M \frac{1}{n!} f_{\mu_1\ldots\mu_2} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_2} = \int_M \frac{1}{n!} f_{\mu_1\ldots\mu_2} \psi^{\mu_1} \cdots \psi^{\mu_2} d^{2m}x d^{2m}\psi, \tag{2.6}$$

where ψ is a fermionic vector. With this representation, Eq.(2.3) is

$$\int_M e_s(\nabla,\varepsilon) = \int_M dx d\psi d\chi \frac{1}{(2\pi)^m} e^{-\frac{1}{2}|s|^2+\frac{1}{2}\chi_a\Omega^{ab}_{\mu\nu}\psi^\mu\psi^\nu\chi_b+i\nabla s^a(\psi)\chi_a}. \tag{2.7}$$

Where $s^a(\psi)$ is put as s whose dx^μ is replaced with ψ^μ.

Now, we can understand Euler number as TFT. When we regard Eq.(2.7) as partition function and the action is obtained as

$$S = \frac{1}{2}|s|^2 - \frac{1}{2}\chi_a\Omega^{ab}_{\mu\nu}\psi^\mu\psi^\nu\chi_b - i\nabla s^a(\psi)\chi_a. \tag{2.8}$$

This action has a symmetry which defined as

$$\hat{\delta}x_\mu = \psi_\mu \quad \hat{\delta}\psi_\mu = 0 \quad \chi_a = is_a. \tag{2.9}$$

This transformation low is on-shell nilpotent. If we introduce auxiliary fields b, action is rewritten as

$$S = \frac{1}{2}b^2 - bs - \frac{1}{2}\chi_a\Omega^{ab}_{\mu\nu}\psi^\mu\psi^\nu\chi_b, -i\nabla s^a(\psi)\chi_a \tag{2.10}$$

and symmetry (2.9) is changed as

$$\hat{\delta}x_\mu = \psi_\mu \quad \hat{\delta}\psi_\mu = 0 \quad \hat{\delta}\chi_a = b \quad \hat{\delta}b = 0. \tag{2.11}$$

This is off-shell nilpotent transformation and regard it as a BRST transformation low. Moreover, the action (2.10) is given by BRS exact form as

$$\begin{aligned} S &= \hat{\delta}\Psi \\ &= \hat{\delta}\frac{1}{2}\chi_a(2s^a + A^{ab}_\mu\psi^\mu\chi_b + b^a). \end{aligned} \tag{2.12}$$

Where A^{ab}_μ is a connection. Therefore we find that it is possible to regard Euler number as Witten-type TFT.

Let us extend this formalism to infinite dimensional case. The fiber bundle is also extended to any bundles. Naive extension is given by following exchanging.

$$x \to \phi(x), \quad \psi \to \psi(x), \quad \chi \to \chi(x), \quad b \to b(x). \tag{2.13}$$

Where $\phi(x)$, $\psi(x)$, $\chi(x)$ and $b(x)$ is a some tensor(or spinor) fields. That is $\phi(x) = \phi(x)^i_{\mu_1,\dots,\mu_n}$, $\chi(x) = \chi(x)^i_{\mu_1,\dots,\mu_n}$ and so on, but for the simplicity we omit the space-time indices "μ" and variety of field indices " i". $s(x)$ is replaced by a functional $s(\phi(x))$ in this extension. The action of the TFT is represented with these fields as

$$\begin{aligned} S &= \int_M \hat{\delta}\Psi \\ &= \hat{\delta}\langle \chi, (2s(\phi(x)) + b(x))\rangle \\ &= \int_M \hat{\delta}\{\frac{1}{2}\chi(x)\,(2s(\phi(x)) + b(x))\} \qquad (2.14) \\ &= \int_M b(x)\,(2s(\phi(x)) + b(x)) - \chi(x)\left(\left(\psi\frac{\delta}{\delta\phi}\right)s\right). \end{aligned}$$

Where $\langle , \rangle$ is a scalar product i.e. we contract the all indices " i" and "μ". Under comparing (2.14) with (2.12) we notice that connection A^{ab}_μ do not exist in (2.14). In this case, the path integral is defined as a functional integral. In general, connections are not introduced in functional space. So we omit the A^{ab}_μ proportional terms in (2.14). In other words, we regard that $A^{ab}_\mu\psi^\mu\chi_b + b^a$ is redefined as $b^a(x)$.

2.3 Fermionic Zero modes

In this section, we study the relation between fermionic zero modes and manifolds which defined by zero locus of the section "s". We treat the case which have no local symmetry in this section 3. The theory with gauge symmetry will be discussed in the next section.

We investigate the path-integral given as

$$\langle O\rangle = \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\chi\mathcal{D}b\; O\; e^{-gS} \qquad (2.15)$$

where the S is (2.14) action, g is a coupling constant and O is some observable. We define the observable as BRS invariant object. This path-integral is independent from the coupling g because variation of (2.15) by g is vacuum expectation value of BRS exact object. When we chose the BRS invariant measure as the path-integral measure, we obtain

$$\frac{\delta}{\delta g}\langle O\rangle = -\langle\hat{\delta}(O\Psi)\rangle = 0. \qquad (2.16)$$

Therefore we can get exact result even in the weak coupling limit. In the weak coupling limit. (see the Eq.(2.7)). the relevant integral is given from

$$\phi_0 \in \{s^{-1}(0)\} = \{\phi_c | s(\phi_c) = 0\} \equiv \mathcal{A} \tag{2.17}$$

$$\psi_0 \in \{\psi_0 | \left(\psi \frac{\delta}{\delta\phi}\right) s \Big|_{\psi=\psi_0} = 0\}. \tag{2.18}$$

Where the Eq.(2.18) is given as a χ equation.

Ghost number anomaly occur when there are zero modes of Eq.(2.18) and we have to intrduce some observable whose ghost number is equal to the number of the zero mode in order to get non-zero result of $\langle O \rangle$. In the following we consider such case and we suppose the observable O has ghost number that is equal to number of fermionic zero modes. After integrate out of b and χ, Eq.(2.15) is written as

$$\langle O \rangle = \int \mathcal{D}\phi_0 \mathcal{D}\psi_0 O(\phi_0, \psi_0) \int \mathcal{D}\phi' \mathcal{D}\psi' \mathcal{D}\chi \mathcal{D}b e^{-gS}. \tag{2.19}$$

We denote ϕ' and ψ' as the other fields of ϕ_0 and ψ_0. From SUSY($\hat{\delta}$ symmetry), the Euler form part integral , $\int \mathcal{D}\phi' \mathcal{D}\psi' e^{-gS}$ is given as ± 1. This result is easily ascertained in a weak coupling limit. The sign of $\pm$ is determined by each zero modes ϕ_0 and ψ_0. We put $O(\phi, \psi) = O(\phi)^{1 \ldots \dim \mathcal{A}} \psi_1 \ldots \psi_{\dim \mathcal{A}}$. The reason why the rank of the tensor is "$\dim \mathcal{A}$" will be made clear soon. Then Eq.(2.19) is given as

$$\begin{aligned} \langle O \rangle &= \int \mathcal{D}\phi_0 \mathcal{D}\psi_0 O(\phi_0, \psi_0)(\pm 1) \\ &= \int_{\mathcal{A}} d\phi_1 \wedge \ldots \wedge d\phi_{dim \mathcal{A}} (\pm 1) O(\phi)^{1 \ldots \dim \mathcal{A}}. \end{aligned} \tag{2.20}$$

Where we use Eq.(2.6) i.e. ψ is replaced by form $d\phi$. Note that "$d\phi$" is defined not on the manifold "M" but on "$\mathcal{A}$". We sanction this replacement as follows. From the $\mathcal{A}$ is defined by $s(\phi) = 0$, the form $d\phi$ on $\mathcal{A}$ satisfies

$$ds(\phi) = \left(d\phi \frac{\delta}{\delta\phi}\right) s(\phi) = 0. \tag{2.21}$$

From Eq.(2.18) we find that the fermionic zero modes satisfy the same equation Eq(2.21). And the number of the fermionic zero mode is equal to the dimension of $\mathcal{A}$.

$$\sharp(\psi_0) = \dim \mathcal{A}. \tag{2.22}$$

Therefore we obtain the form integral (2.20) from vacuum expectation value of (2.15). Note that Topological invariants represented as form integral like (2.20) is interpreted as Poincare dual of Euler form of infinite dimensional space from (2.19) and (2.20).

We have considered the case which there are no local symmetry. When we treat gauge theories we have to modify these formalism a little. In the next section we study such case. To justify the Eq.(2.20) some problems are left. For example, what is a condition to regard $\mathcal{A}$ as a manifold? We investigate these problems in the last section in this chapter and following chapters.

2.4 Topological Gauge Theory

In this section we study the TFT with local gauge symmetry. When the gauge group and gauge transformation of ϕ are given as G and $\mathcal{G}$, goal of this section is to get a form integral like (2.20) on moduli space $\mathcal{M} \equiv \mathcal{A}/\mathcal{G}$. We have to modify theory we had seen until above section to arrive the goal.

At first, we reform the $\hat{\delta}$ transformation (2.11) as

$$\hat{\delta}\phi(x) = \psi(x) \quad \hat{\delta}\psi(x) = \delta_g(c)\phi(x) \quad \hat{\delta}\chi = b \quad \hat{\delta}b = \delta_g(c)\chi. \tag{2.23}$$

Where we denote $\delta_g(c)$ as gauge transformation with gauge parameter fields "c" which assign ghost number 2. For example, the Topological Yang-Mills case, this $\hat{\delta}$ is defined as

$$\hat{\delta}A_\mu = i\lambda_\mu, \quad \hat{\delta}\lambda_\mu = \delta_g(\phi)A_\mu = -D_\mu\phi, \quad \hat{\delta}\phi = 0. \tag{2.24}$$

From these definitions, i.e. $\hat{\delta}^2 = \delta_g$, nilpotency of $\hat{\delta}$ is lost in general. But if we treat only gauge invariants object, $\hat{\delta}$ is still nilpotent operator. In the following we regard Ψ in (2.14) and observable O as gauge invariant and we treat $\hat{\delta}$ as nilpotent operator without gauge fixing terms.

To obtain the form integral on $\mathcal{M} \equiv \mathcal{A}/\mathcal{G}$ we want restriction map to horizontal space, i.e.

$$\int_{\mathcal{A}/\mathcal{G}} O(\phi, d\phi) = \int_{\mathcal{A}} O(\phi, d\phi)\, e^{-\int \hat{\delta}\Psi_p}. \tag{2.25}$$

Where $e^{-\int \hat{\delta}\Psi_p}$ play a role which project the $\mathcal{A}$ into gauge horizontal space. We have to find this $\hat{\delta}\Psi_p$ and add this to the action.

$$S = \int_M (\hat{\delta}\Psi + \hat{\delta}\Psi_p). \tag{2.26}$$

With this modified action, we get the form integral on the moduli space.

To obtain the projective gauge fermion Ψ_p we introduce two operators named C and $C^\dagger$. We define C with the gauge parameter θ as

$$\delta_g \phi = C\theta. \tag{2.27}$$

For the economizing of symbols, we use δ_g as gauge transformation operator, but here gauge parameter θ is not necessary to assign ghost number 2 like (2.23) and (2.24). For example the case of Yang-Mills gauge connection, (2.27) is given as

$$\delta_g A^a_\mu = D_\mu \theta = C\theta, \qquad C = D_\mu. \tag{2.28}$$

$C^\dagger$ is defined as conjugate operator of C. In the case of (2.28), with the gauge invariant scalar production $\langle\ ,\ \rangle$, $C^\dagger$ is given as

$$\langle \psi_\mu, D_\mu \theta \rangle = \langle C^\dagger \psi_\mu, \theta \rangle, \qquad C^\dagger = D^\mu. \tag{2.29}$$

Note that the kernel of $C^\dagger$ have a horizontal direction of gauge transition because

$$\langle C^\dagger \psi, \theta \rangle = \langle \psi, C\theta \rangle = \langle \psi, \delta_g \phi \rangle. \tag{2.30}$$

From this character we construct the projective gauge fermion.

We introduce the projective gauge fermion which restrict the path-integral to the kernel of $C^\dagger$. To realize this, anti-ghosts η and Nakanishi-Lautrup(NL) fields $\bar{\phi}$ is introduced here.

$$\hat{\delta}\bar{\phi} = \eta, \quad \hat{\delta}\eta = \delta_g \bar{\phi}. \tag{2.31}$$

In Eqs.(2.31), δ_g is same as (2.23) and (2.24) and is assigned ghost number 2. With these fields, Ψ_p is defined as

$$\int_M \Psi_p = \langle C^\dagger \psi, \bar{\phi} \rangle. \tag{2.32}$$

From this definition, we obtain the additional action as

$$\int_M \hat{\delta}\Psi_p = \langle \hat{\delta}C^\dagger\psi, \bar{\phi}\rangle + \langle C^\dagger\psi, \eta\rangle. \tag{2.33}$$

From the η equation,

$$C^\dagger\psi = 0, \tag{2.34}$$

We find that the path-integral is restricted to moduli space integral. The moduli space dimension is obtained as similar as Eq.(2.22).

$$\dim\mathcal{M} = \sharp(\psi_0) = \dim kerC^\dagger - \dim Coker\frac{\delta s}{\delta\phi}. \tag{2.35}$$

Note that this manner is right when the gauge transformation is effective. We consider the case that δ_g do not act effectively in the next section and next chapter.

Then the vacuum expectation value is obtained as form integral on $\mathcal{M}$ as

$$\langle O\rangle = \int_{\mathcal{M}} d\phi_1\wedge\ldots\wedge d\phi_{dim\mathcal{M}} O(\phi)^{1\ldots\dim\mathcal{M}}. \tag{2.36}$$

Note that this integral is well defined only the case that $\mathcal{M}$ is a manifold. The condition to be a manifolds is discussed in the next section.

Note that this projection is different from Faddeev-Popov gauge fixing. Faddeev-Popov method introduce some gauge slice but we separate the Harr measure from path-integral measure. After we add (2.33) to the Lagrangian, there is local gauge symmetry. So we have to fix the gauge symmetry as usual.

2.5 Singularities

In this section some singularities are considered. The first type of singularities has a geometrical origin. The form integral (2.20) (or (2.36)) are well-defined when the $\mathcal{A}$(or $\mathcal{M}$) become a manifold. We study the condition for them to be manifolds. The second type of singularities has a field theoretical origin. We treat topological invariants as TFT. As we saw in the above section, we regard the tangent space of moduli space as space of fermionic zero modes. In the field theory, these zero mode cause singularities like infrared singularities. We mention about these singularities in this section and we study some example in the following chapter.

We study the conditions for $\mathcal{A}$ and $\mathcal{M}$ to be manifolds. The sufficient condition for $\mathcal{A} = s^{-1}(0)$ to be a manifold is given from the theorem on implicit function.

- The map $\frac{\delta s}{\delta \phi}$ from cotangent space to the vector space on which s is defined is surjection map.

From the definition of $\mathcal{M}$ we get the next condition,

- The gauge transformation G of gauge group G act effectively, i.e. there is no fixed point of G action.

Further more, there is a problem of compactness of $\mathcal{A}$ and $\mathcal{M}$. But we do not discuss about compactness in this paper.

These conditions is demanded when we choose the gauge fermions Ψ and Ψ_p. To avoid these singularities or problems, the gauge fermion is chosen carefully or we demand some condition to the base manifold "M". But often we choose unavoidably or intentionally gauge fermions Ψ and Ψ_p with singularities. For example in Yang-Mills theory, the second condition, i.e. there is no fixed point of $\mathcal{G}$ action, means that there is no reducible connection (see the appendix of chapter 3). In the Topological QCD, as we will see in the next chapter, we obtain more information from the theory with reducible connection than the theory whose connection is only irreducible. So we do not avoid these condition, and rather we use the singularities to get the relation of topological invariants.

There is another type of singularity in TFT. Many theories regardless of topological theory have singularities like infrared singularities, which is caused by zero mode. Witten type TFT is defined as merely action is topological invariant action plus $\hat{\delta}$-exact action.

$$\mathcal{L} = \mathcal{L}_{cl} + \hat{\delta}\Psi \tag{2.37}$$

Where $\hat{\delta}$ is nilpotent ($\hat{\delta}^2 = 0$) or up to δ_g nilpotent (or $\hat{\delta}^2 = \delta_g$). From only this definition without (2.23), it is impossible to interpret the TFT as the Mathai-Quillen formalism, but we can find the theory is topological. When we regard $\hat{\delta}$ as BRS operator, $\hat{\delta}\Psi$ is the term of gauge fixing plus Faddeev-Popov determinant term. Then it may be interpreted as existence of Gribov zero mode that the moduli space have a non-zero dimension, because "det $\frac{\delta s}{\delta \phi}$" is a Faddeev-Popov determinant. This Gribov zero mode some times cause singularities. From Eq.(2.21), we found that this zero mode always exist when dimension of $\mathcal{A}$ or $\mathcal{M}$ is non-zero. It is known that the Gribov zero mode have possibility to break the BRS symmetry i.e. Topological symmetry. We may get physical theory which is broken topological symmetry. We will study it in chapter 4 in the case of Topological Gravity.

Chapter 3

Reducible Connections and Topological QCD

3.1 Seiberg-Witten invariants and 4-manifolds

In this chapter, we study moduli space singularities and we use these singularities to find some relations of topological invariants in 4-manifolds [22].

Recently in differential topology of four manifolds there have been new developements by Seiberg-Witten monopole theory [1] . They conquered a difficulty of Donaldson theory . Donaldson theory solved many problems of differential topology of 4-manifolds about intersection form, polynomial invariants and so on [3][4]. Donaldson theory is described by non-Abelian gauge theory, hence calculations are difficult. Seiberg-Witten theory is easy for its computation since it is Abelian gauge theory. Donaldson invariants which is written by Kronheimer-Mrowka structure formula are related with Seiberg-Witten invariants, and the relation has been proved in several ways [5][6] . Hyun-J.Park-J.S.Park show the relation in path-integral formalism using massive Topological QCD [7]. Both Donaldson and Seiberg-Witten theory are understood as Topological field theory [8] [9]. Hyun-J.Park-J.S.Park computed path-integrals of massive Topological QCD and they found the way of separating it into two brunches that is Donaldson part and Seiberg-Witten part. Symbolically the result of their computation is

$$\langle massive\ \ Topological\ \ QCD\rangle = \frac{a}{m^k}\left\{\langle Donaldson\rangle - \frac{b}{m^l}\langle Seiberg - Witten\rangle\right\} \quad (3.1)$$

Where a and b are some suitable constants and m is mass of hyper- multiplets. k and l are determined by indices of some elliptic operator. $\langle massive\ Topological\ QCD \rangle$ is a vacuum expectation value of an observable with the action of massless Topological QCD. $\langle Donaldson \rangle$ stands for vacuum expectation value of an observable with the action of Donaldson-Witten theory [8], and is called Donaldson invariants. $\langle Seiberg - Witten \rangle$ stands for vacuum expectation value of an observable with the action of Abelian Seiberg-Witten topological field theory [9], and is called Seiberg-Witten invariants. The left hand side of the above equation is regular in the massless limit, $m \rightarrow 0$, so the Donaldson part and Seiberg-Witten part in the right hand side have to cancel each other. In this theory, mass terms lead a relation between vacuum expectation value of Higgs and matter fields. All computations in this paper are done in a weak coupling limit (or large scaling limit). If the weak-strong duality relation is necessary for understanding the relation of Donaldson invariants and Seiberg-Witten invariants, it is natural to think that massive particles decouple in the weak coupling limit as Witten mentioned in [1] [2] . But, in the proof of Hyun-J.Park-J.S.Park mass terms do not decouple , and the duality relation is not used. Mathematicians did not use the duality relation similarly in their proofs [5] [6]. This fact implies that mass terms of matter fields do not play essential roles in the relation between Seiberg-Witten and Donaldson invariants . The only important thing is to separate the path-integral into the Donaldson's irreducible part and Seiberg-Witten's reducible part in their theory.

In this chapter, we investigate reducibility of the gauge connections in massless Topological QCD. As we mentioned in the above chapter, reducible connections make moduli space singularities. We use these singularities for two purposes. The first one is that we cull the Abelian Seiberg-Witten part from massless Topological QCD without Higgs mechanism and weak-strong duality relation. The Abelian Seiberg-Witten part appear in massless Topological QCD as reducible connection part. The second purpose is to obtain the new relations of massless Topological QCD and their topological invariants. Especially, we will get a result that insists some topological invariants which contain the Abelian Seiberg-Witten invariants. This topological invariants is provided as reducible connection part of this Topological QCD. As a result of these relations, we find that the path-integral from non-Abelian extended monopoles [9][6][10] is separated into Donaldson parts and non-Abelian Seiberg-Witten parts. We use regularization for zero mode of a scalar field, which do not remove zero-modes but shift them to infinitesimal eigenstates without BRS-like SUSY breaking with giving perturbation by external fields. As a result of this perturbation, we obtain new relations between Seiberg-Witten invariants extended to non-Abelian gauge and Donaldson Seiberg-Witten invariants. And some identities of U(1) topological invariants from the relation are given from the relation of SUSY symmetry of the Topological QCD and external fields.

This chapter is organized as follows. We set up massless topological QCD whose action do not have Higgs potential, in section 2. Separation of vacuum expectation value of observables into reducible connection part and irreducible connection part is done in section 3. We get identities of Abelian Seiberg-Witten invariants and obtain the relation of massless Topological QCD and reducible connection part in section 4. We will find new formulas in there. In the last section of this chapter, we summarize and discuss our conclusions.

3.2 Massless Topological QCD

In this section we set up massless Topological QCD modified slightly to separate a correlation function into a reducible connection part and a irreducible connection part and study the relations between non-Abelian and Abelian Seiberg-Witten theory with no Higgs mechanism. Hence Higgs potential like $[\bar{\phi}, \phi]^2$ do not appear here. We will find later in this section how the Donaldson invariants are embedded in massless Topological QCD.

Topological QCD were already constructed by Hyun-J.Park-J.S.Park J.M.F. Labastida and Mariño by twisting N=2 SUSY QCD [9] . Donaldson theory and Seiberg-Witten theory are analyzed as topological field theory in references [8][9][11][7] . Basically we use the Hyun-J.Park-J.S.Park theory and notation in [7] . In the following, we only consider SU(2) gauge group and 4-dimensional compact Rieman manifolds with $b_2^+ > 2$ as a back ground manifold.

The action is

$$S_{QCD} = -\hat{\delta} V \tag{3.2}$$

where

$$\begin{aligned} V = & \int d^4 x g^{\frac{1}{2}} [\chi^{\mu\nu}{}_a (H^a_{\mu\nu} - i(F^{a+}_{\mu\nu} + q^\dagger \sigma_{\mu\nu} T^a q)) \\ - & \frac{1}{2} g^{\mu\nu} (D_\mu \bar{\phi})_a \lambda^a_\nu + (X^\alpha_{\tilde{q}} \psi_{q\alpha} - \psi^\alpha_{\tilde{q}} X_{q\alpha})] \end{aligned} \tag{3.3}$$

and under SUSY(BRS like) transformations, Yang-Mills fields are transformed as

$$\begin{aligned} &\hat{\delta} A_\mu = i\lambda_\mu, \quad \hat{\delta}\chi_{\mu\nu} = H_{\mu\nu}, \qquad \hat{\delta}\bar{\phi} = i\eta, \\ &\hat{\delta}\lambda_\mu = -D_\mu \phi, \quad \hat{\delta} H_{\mu\nu} = i[\phi, \chi_{\mu\nu}], \quad \hat{\delta}\eta = [\phi, \bar{\phi}], \\ &\hat{\delta}\phi = 0, \end{aligned} \tag{3.4}$$

and the matter fields are transformed as,

$$\begin{aligned} \hat{\delta} q^{\dot{\alpha}} &= -\bar{\psi}^{\dot{\alpha}}_{\tilde{q}}, \quad \hat{\delta}\bar{\psi}^{\dot{\alpha}}_{\tilde{q}} = -i\phi^a T_a q^{\dot{\alpha}}, \\ \hat{\delta} q^\dagger{}_{\dot{\alpha}} &= -\bar{\psi}_{q\dot{\alpha}}, \quad \hat{\delta}\bar{\psi}_{q\dot{\alpha}} = i q^\dagger{}_{\dot{\alpha}} \phi^a T_a, \\ \hat{\delta}\psi_{q\alpha} &= -i\sigma^\mu_{\alpha\dot{\alpha}} D_\mu q^{\dot{\alpha}} + X_{q\alpha}, \\ \hat{\delta} X_{q\alpha} &= i\phi^a T_a \psi_{q\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}} D_\mu \bar{\psi}^{\dot{\alpha}}_{\tilde{q}} + \sigma^\mu_{\alpha\dot{\alpha}} \lambda^a_\mu T_a q^{\dot{\alpha}} \\ \hat{\delta}\psi^\alpha_{\tilde{q}} &= i D_\mu q^\dagger{}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} - X^\alpha_{\tilde{q}}, \\ \hat{\delta} X^\alpha_{\tilde{q}} &= i\psi^\alpha_{\tilde{q}} \phi^a T_a - i D_\mu \bar{\psi}_{q\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} + q^\dagger{}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \lambda^a_\mu T_a. \end{aligned} \tag{3.5}$$

These transformation laws are obtained by the usual way of twisting N=2 SUSY QCD. The matter fields q are sections of $W^+_c \otimes E$ where W^+_c is *spin*^{c} bundle and E is a vector bundle whose fiber is a vector space of a representation of the gauge group $SU(2)$.

We make correspondence between the action (3.2) and Mathai-Quillen formalism. From (3.3) we can easily make correspondence of

$$\Psi = \chi^{\mu\nu}{}_a (H^a_{\mu\nu} - i(F^{a+}_{\mu\nu} + q^\dagger \sigma_{\mu\nu} T^a q)), \quad \Psi_p = \frac{1}{2} g^{\mu\nu} (D_\mu \bar{\phi})_a \lambda^a_\nu.$$

But the correspondence of $(X_{\bar{q}}^{\alpha}\psi_{q\alpha} - \psi_{\bar{q}}^{\alpha}X_{q\alpha})$ in V is not clear. As a matter of fact it correspond to gauge fermion of the section $\sigma^{\mu}D_{\mu}q$. That is we chose the section s as

$$s = (F_{\mu\nu}^{a+} + q^{\dagger}\sigma_{\mu\nu}T^{a}q,\ \ \sigma^{\mu}D_{\mu}q)$$

If you want more clearer correspondence to Mathi-Quillen formalism you should refer [43].

The topological action is given after integrating out the auxiliary fields $H_{\mu\nu}$, X_q and $X_{\bar{q}}$ as

$$\begin{aligned} S_{QCD} &= \int d^4x g^{\frac{1}{2}} \Big(\frac{1}{4}\Big|F_{\mu\nu}^{a+} + q^{\dagger}\sigma_{\mu\nu}T^{a}q\Big|^2 + \frac{1}{2}|\sigma^{\mu}D_{\mu}q|^2 - \frac{1}{2}g^{\mu\nu}(D_{\mu}\bar{\phi})_a(D_{\nu}\phi)^a \\ &-\frac{i}{2}g^{\mu\nu}[\lambda_{\mu},\bar{\phi}]_a\lambda^{a}{}_{\nu} - i\chi^{\mu\nu}{}_{a}[\phi,\chi_{\mu\nu}]^a + \chi_a^{\mu\nu}(d_A\lambda)^{+a}{}_{\mu\nu} + \frac{i}{2}g^{\mu\nu}(D_{\mu}\eta)_a\lambda^{a}{}_{\nu} \\ &-i\chi^{\mu\nu}{}_{a}\bar{\psi}_q\bar{\sigma}_{\mu\nu}T^{a}q + i\chi^{\mu\nu}{}_{a}q^{\dagger}\bar{\sigma}_{\mu\nu}T^{a}\bar{\psi}_{\bar{q}} - iD_{\mu}\bar{\psi}_{q\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{q\alpha} - i\psi_{\bar{q}}{}^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}D_{\mu}\bar{\psi}_{\bar{q}}^{\dot{\alpha}} \\ &+2i\psi_{\bar{q}}{}^{\alpha}\phi_a T^{a}\psi_{q\alpha} + q^{\dagger}{}_{\dot{\alpha}}\lambda_{\mu a}T^{a}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{q\alpha} + \psi_{\bar{q}}{}^{\alpha}\sigma_{\mu\alpha\dot{\alpha}}\lambda_{\mu a}T^{a}q^{\dot{\alpha}} \Big). \end{aligned} \tag{3.6}$$

Where the indices α and $\dot{\alpha}$ are omitted and we do not change the position of these indices to keep the sign of each terms in the following. This action is constructed in order to lead the most important fixed points,

$$F_{\mu\nu}^{a+} + q^{\dagger}\sigma_{\mu\nu}T^{a}q = 0, \quad \sigma^{\mu}D_{\mu}q = 0. \tag{3.7}$$

The Eqs.(3.7) are monopole equations extended to non-Abelian gauge group and often they are called non-Abelian Seiberg-Witten monopoles [9][10].

Usually, Seiberg-Witten theory has a Higgs potential $[\phi,\bar{\phi}]^2$. Spontaneous symmetry breakdown occur if the vacuum expectation value of Higgs fields $\langle\phi\rangle$ is non-zero. Then we get $U(1)$ monopole(Seiberg-Witten) equations. But one of our purposes is to clarify whether relations of topological invariants can be understood without Higgs mechanism and weak-strong duality relations. So the Higgs potential is not included in our theory. (But if we add these potential, they do not disturb following discussion.)

Later in this section, we find how the Donaldson invariants are embedded in this theory. We study three kind of fixed points, which give important contribution to vacuum expectation value. The path-integral is expressed as a sum of three parts. We call a part of the path-integral from fixed points determined by $F^+ = 0$ and $q = 0$ as Donaldson part. We denote a part whose gauge connections A of fixed points determined by Eqs.(3.7) are reducible as Abelian Seiberg-Witten part and a part which has irreducible connections at the fixed point as Non-Abelian Seiberg-Witten part. The observable is

$$\exp\left(\frac{1}{4\pi^2}\int_\gamma Tr\left(i\phi F + \frac{1}{2}\lambda\wedge\lambda\right) - \frac{1}{8\pi^2}Tr\phi^2\right), \tag{3.8}$$

where γ is in a 2-dimensional homology class i.e. $\gamma \in H_2(M;\mathbf{Z})$. Now let us separates vacuum expectation value with Donaldson invariants from non-Abelian Seiberg-Witten invariants. If fixed points $\langle q\rangle$ from Eqs.(3.7) is zero then the contribution to the expectation value of (3.8) is from only Donaldson theory because now our fixed point equation is simply as

$$F^+_{\mu\nu} = 0. \tag{3.9}$$

We know that we can estimate exactly the vacuum expectation values of this observables by one-loop approximation around the fixed point determined by the Eqs.(3.7) [11][12]. We can decompose the action of Eq.(3.6) into two parts [7],as

$$S_{QCD} = S_D + S_M, \tag{3.10}$$

where S_D is action of Donaldson-Witten theory [8],

$$\begin{aligned} S_D \;=\; & \int d^4x g^{\frac{1}{2}} Tr\,[\,\frac{1}{4}F^+_{\mu\nu}F^{+\mu\nu} - \frac{1}{2}g^{\mu\nu}D_\nu\bar{\phi}D_\mu\phi - i\chi^{\mu\nu}[\phi,\chi_{\mu\nu}] \\ & +\chi^{\mu\nu}(d_A\lambda)^+ + \frac{i}{2}g^{\mu\nu}(D_\mu\eta)\lambda_\nu - \frac{i}{2}g^{\mu\nu}\left[\lambda,\bar{\phi}\right]\lambda_\nu\,] \end{aligned} \tag{3.11}$$

and S_M is matter part,

We find that the quadratic part of the matter part action S_M is given by

$$\begin{aligned} S^{(2)}_M \;=\; & \int d^4x g^{\frac{1}{2}}\,[\,-3X^\alpha_{\tilde{q}}X_{q\alpha} + iX^\alpha_{\tilde{q}}\sigma^\mu_{\alpha\dot{\alpha}}D_\mu q^{\dot{\alpha}} + iD_\mu q^\dagger{}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}X_{q\alpha} \\ & -iD_\mu\bar{\psi}_{q\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{q\alpha} - i\psi_{\tilde{q}}{}^\alpha\bar{\sigma}^\mu_{\alpha\dot{\alpha}}D_\mu\bar{\psi}^{\dot{\alpha}}_{\tilde{q}}\,]. \end{aligned} \tag{3.12}$$

Note that the gauge field A_μ used in D_μ is an external field. We expand the gauge field around the solution of Eqs.(3.7) denoted as A_c, as

$$A = A_c + A_q. \tag{3.13}$$

,where A_q is a quantum fluctuation around the A_c. So the covariant derivative in the $S_M^{(2)}$ is written by external fields A_c as $d_A = d + A_c$. Note that A_c is a irreducible connection because we set $b_2^+ \geq 1$. After the Gaussian integrals of X, q, ψ and $\bar{\psi}_q$, from Eq.(3.12) we get

$$det\,(-\pi)_{W^-} det\,(-4\pi)_{W^+}. \tag{3.14}$$

The indices W^- and W^+ of determinants (3.14) show that the determinants are defined in the subspaces of W^-, W^+, and we use the similar notations also in the following. Therefore we can understand that the Donaldson invariants appear with the determinants (3.14) in massless Topological QCD and this fact is used in section 4.

In the next step, we want to evaluate the other part which correspond to the Seiberg-Witten theory and separate the path-integral into a reducible connection part and an irreducible part. But, as we will see it soon, it is impossible to separate the Seiberg-Witten part into the non-Abelian Seiberg-Witten brunch and the Abelian Seiberg-Witten brunch in the same manner as above. In this case fixed points value $\langle q \rangle$ is non zero. We consider the SU(2) gauge group. Our massless theory has no spontaneous symmetry break down, so we have to treat separately reducible gauge connections and irreducible connection by some way. Judgments whether the connections are irreducible or not can be done by examination of the existence of D_μ zero-mode. (See appendix A.) Strictly speaking, the following two propositions are same.

- $d_A : Ad\,\xi \bigotimes \bigwedge_0 \rightarrow Ad\,\xi \bigotimes \bigwedge_1$ is injection.
- A is a irreducible connection.

Where we represent a vector bundle with a structure group SU(2) as ξ. So we use this condition to divide the contribution to vacuum expectations into one from Abelian

Seiberg-Witten theory and another from non-abelian Seiberg-Witten theory. At first, we pay attention to $\bar{\phi}$ equation.

$$-\frac{1}{2}D^{\mu}D_{\mu}\phi - \frac{i}{2}[\lambda_{\mu}, \lambda^{\mu}] = 0. \tag{3.15}$$

λ in this equation is determined by η and χ equations like appendix B or references [11] . When there is no zero mode solution of λ then this equation is a distinction formula of reducibility i.e. $-\frac{1}{2}D^{\mu}D_{\mu}\phi = 0$. So we can conclude that the connection is reducible and the contribution to vacuum expectation value is from the U(1) Abelian Seiberg-Witten theory when $\langle\phi\rangle \neq 0$. But unfortunately we can not distinguish connection by this method when there are λ zero mode solutions. In the next section we find new approach to separate the contribution of Abelian Seiberg-Witten branch from non-Abelian Seiberg-Witten part.

3.3 Separation of reducible connection part

In this section we construct a new theory in which vacuum expectation value are separated into three parts i.e. Donaldson , Abelian Seiberg-Witten and non-Abelian Seiberg-Witten part. We use the determinant obtained by integration of ϕ and $\bar{\phi}$ to find whether connections are reducible or not. Therefore we consider the case when the determinant vanishes. Usually we avoid this case and remove zero eigenvalue states with various ways. For example in [20], zero-modes give a symmetry to whole action and they are removed by BRS method. In our case this method can not be used because zero-mode dose not give the existence of a local symmetry. But in this section we take zero-modes into account , and we find that this zero-modes play a essential role to distinguish between reducible connections and irreducible connections.

In the same way as section 2, we pay attention to Eq.(3.15). From Eq.(3.15), we get the vacuum expectation value of the scalar fields ϕ in 1-loop order as

$$\langle\phi\rangle = -\frac{i}{D^{\mu}D_{\mu}}[\lambda_{\nu}, \lambda^{\nu}] \tag{3.16}$$

Now we have to recall that if $b_2^+ > 0$, anti-selfdual connections which satisfy the equation (3.9) do not contain the reducible connections with U(1) isotolopy group [3][13] , but the connections which satisfy the monopole equations (3.7) contain such reducible connections in general. When the connection is reducible then D_μ has zero-mode on 0-form. (see the appendix A). Then the Green function $\frac{1}{D^\mu D_\mu}$ has singularities. Normally we avoid this kind of singularity to define a meaningful theory. However, in the present case, this singularity pays the important role. Reducibility of the gauge connection is judged by the zero-mode. The theory should keep holding characteristic properties of zero-modes while it is regularized. Such problem doesn't exist in the Donaldson theory. Or we can say that we set the condition $b_2^+ > 2$ to avoid the complication of reducible connections in Donaldson theory. But in the Seiberg-Witten theory it is the most important merit that the gauge group is U(1) Abelian group. So, if we take away such reducible connections, then this Topological QCD has almost no value. Therefore we have to manage the singularities in the Green function. Usually we dispose of singularities of this kind by removing zero-modes, inducing mass terms and so on. The following way makes it possible.

Before considering regularization we ascertain that these singularities make topological symmetry break. To see it concretely, we introduce a BRS exact observable, $\hat{\delta}\left(\bar{\phi}\lambda_\mu\right)$. If there are no singularities, topological symmetry is not broken and vacuum expectation value of this observable vanishes. But as we saw before, the propagator $\left\langle\bar{\phi}\phi\right\rangle \sim \frac{1}{D^\mu D_\mu}$ is singular at least in tree level. To avoid these singularities we add regularization term $i\epsilon\bar{\phi}\phi$ to our Lagrangian and the propagator is changed to $\frac{1}{D^\mu D_\mu - i\epsilon}$, naively. Because the $\bar{\phi}\phi$ is not invariant under the BRS-like SUSY transformation (3.4)(3.5), the vacuum expectation value is

$$\begin{aligned}
\left\langle\hat{\delta}\left(\bar{\phi}\lambda_\mu\right)\right\rangle &= \int \mathcal{D}X\hat{\delta}\left(\bar{\phi}\lambda_\mu\right)e^{-S} \\
&= \epsilon\int \mathcal{D}X\left(\bar{\phi}\lambda_\mu\right)\hat{\delta}\left(\bar{\phi}\phi\right)e^{-S} \\
&= \epsilon\left\langle\bar{\phi}\lambda_\mu\eta\phi\right\rangle \\
&= \epsilon\left(\left\langle\bar{\phi}\phi\right\rangle\left\langle\lambda_\mu\eta\right\rangle + \cdots\right).
\end{aligned} \tag{3.17}$$

(See references [14] or chapter 4.) The last equality is from the Wick's theorem. It make clear that the singularities of the propagator $\left\langle\bar{\phi}\phi\right\rangle \simeq \frac{1}{\epsilon}$ cause the right hand side of Eq.(3.7) non-zero in a limit as $\epsilon \to 0$. This fact means the vacuum expectation value of the BRS exact observable is non-zero. This is topological symmetry breaking.

It is equivalent to a phenomenon observed in chapter 4 and references [14][15][16]. This is seen after integration by $\bar{\phi}$. We get a delta function.

$$\delta\left(-\frac{1}{2}D^{\mu}D_{\mu}\phi - \frac{i}{2}\left[\lambda_{\mu},\lambda^{\mu}\right] + i\epsilon\phi\right) = \sum_{\text{all }\hat{\phi}_{\epsilon}} \frac{\delta\left(\phi - \hat{\phi}_{\epsilon}\right)}{\left|det\left(D^{\mu}D_{\mu} - i\epsilon\right)\right|}. \tag{3.18}$$

where $\hat{\phi}_{\epsilon}$ is the solutions of $-\frac{1}{2}D^{\mu}D_{\mu}\phi - \frac{i}{2}\left[\lambda_{\mu},\lambda^{\mu}\right] + i\epsilon\phi = 0$ and λ is a zero-mode which is determined by η and χ equations. (see the Appendix B). We know it from seeing the delta function of (3.18) that the determinant $\left|det\left(D^{\mu}D_{\mu} - i\epsilon\right)\right|$ is zero in a limit as ϵ approaches zero when the connection is reducible, and these singularities break the topological symmetry. From this consideration, it seems that the regularization term is not suitable. For our purpose we do not hope topological symmetry breaking, so we have to choose the regularization term to be invariant under BRS-like SUSY transformations.

Here we define the determinant by adding infinitesimal sift terms to our Lagrangian. Our problem is that we could not separate vacuum expectation value into a reducible connection part and a irreducible connection part with the way in the previous section. In this section, the method used in [14] adapt to the separation. We pay attention to the determinant $det\left(D^{\mu}D_{\mu}\right)$ which is obtained by integration of ϕ and $\bar{\phi}$. We saw above that vanishing of this determinant cause topological symmetry breaking or at least cause some singularity. Usually we used to modify determinants by removing zero eigenvalues. But these zero-modes are necessary to judge the reducibility of the gauge connection. So we do not remove but shift them by infinitesimal perturbation. This infinitesimal perturbation is given by adding some shift terms to our Lagrangian. We denote this shift term as $\bar{\phi}f_{\epsilon}$ where f_{ϵ} is a some functional and $f_{\epsilon} = 0$ as the limit of $\epsilon \to 0$. Similarly we also shift the determinant which is obtained by η,χ and λ integrations.(See the appendix B.) This shift term is represented to $\eta\zeta_{\epsilon}$ in the following where ζ_{ϵ} is a some fermionic functional and ζ_{ϵ} vanishes in the limit, $\epsilon \to 0$. We mention a little more about this shift. The η integral of $g^{\mu\nu}\left(D_{\mu}\eta\right)\lambda_{\nu}$ in action (3.6) vanishes, if the covariant derivative D_{μ} acting to the η has zero-modes.(Note that λ_{μ} zero-modes define the dimension of the moduli space, but η zero-modes do not have such a roles in usual case.) Usually we take away this zero-modes, but now instead of doing so we add infinitesimal shift terms ζ_ϵ to the Lagrangian, in order to shift the fermionic zero-eigenstates into infinitesimal eigenvalue states. Total Lagrangian is now written as

$$S = S_{QCD} + \int d^4 x g^{\frac{1}{2}} (\bar{\phi} f_\epsilon + \eta \zeta_\epsilon) \tag{3.19}$$

Here let us enumerate conditions for the infinitesimal terms.

(a) Shift terms are invariant under SUSY transformations (3.4) and (3.5).

This condition is necessary to avoid topological symmetry breaking as we saw it above.

(b) It is desirable that non-zero solutions, ϕ and λ, of the following $\bar{\phi}$ and η equations do not vanish by adding shift terms:

$$-\frac{1}{2} D^\mu D_\mu \phi - \frac{i}{2} [\lambda_\mu, \lambda^\mu] - f_\epsilon = 0 \tag{3.20}$$

$$\frac{i}{2} D_\mu \lambda^\mu - \zeta_\epsilon = 0. \tag{3.21}$$

From this condition, observable contain non-trivial one. The λ in Eq.(3.20) is determined by Eq.(3.21) and fermionic field equations, (refer [11] and see appendix B). When f_ϵ contains the field ϕ like

$$f_\epsilon = \epsilon g \phi + \epsilon h \tag{3.22}$$

where g and h are some functionals which do not contain scalar field ϕ. $\langle \phi \rangle$ is represented as

$$\langle \phi \rangle = -\frac{1}{D^\mu D_\mu + \epsilon g} \left(i [\lambda_\nu, \lambda^\nu] + 2\epsilon h \right). \tag{3.23}$$

Hence $\langle \phi \rangle$ becomes order ϵ i.e.$\langle \phi \rangle = -\frac{1}{D^\mu D_\mu + \epsilon g} (+2\epsilon h) = O(\epsilon)$ if there is only $\lambda = 0$ solution of (3.21) and D_μ has no zero-mode. Then the vacuum expectation of observable (3.8) is mere $1 + O(\epsilon)$ and this is not suitable. Note that it is not desirable that f is independent from ϕ. This reason is made clear by next condition (c). So we can not put g zero in Eq.(3.22). The right-hand side of Eq.(3.22) has at most linear in the field ϕ, however f_ϵ may contain higher power terms on ϕ. For simplicity, we treat only case Eq.(3.22). The 3rd condition for infinitesimal shift terms is

(c) following two operators have no zero-modes regardless of whether gauge connections are reducible or not.

$$\frac{1}{2}D^\mu D_\mu + \epsilon\frac{\delta f}{\delta\phi} \equiv \frac{1}{2}D_\epsilon^2. \tag{3.24}$$

$$\frac{i}{2}D_\mu - \epsilon'\frac{\delta\zeta}{\delta\lambda^\mu} \equiv \frac{i}{2}D_{\mu\epsilon}. \tag{3.25}$$

Where the operators (3.24) and (3.25) operate on $Ad(E)$ valued 0-form. This condition is just to shift determinants from zero to infinitesimal finiteness when gauge connections are reducible and D_μ has zero-modes. From these conditions, we can take count of η and ϕ zero-modes when calculate $detD_\epsilon^2$ obtained by ϕ and $\bar\phi$ integrations and $detT_\epsilon$ (see appendix.B) whose T_ϵ has $D_{\mu\epsilon}$ in first row and is obtained by fermionic fields integrations.

Example of the additional terms $\bar\phi f_\epsilon$ and $\eta_a\zeta_\epsilon^a$ which satisfy the conditions (a)(b)(c) are

$$\begin{aligned}\bar\phi f_\epsilon + \eta_a\zeta_\epsilon^a &= \hat\delta\left(\epsilon'\bar\phi_a\lambda^a{}_\mu m^\mu + \epsilon n(q^\dagger\bar\phi\bar\psi_{\bar q} - \bar\psi_q\bar\phi q)\right)\\ &= \epsilon'\bar\phi_a\left(D_\mu\phi\right)^a m^\mu + \epsilon\bar\phi_a\left(nq^\dagger(\phi T^a + T^a\phi)q - 2in\bar\psi_q T^a\bar\psi_{\bar q}\right)\\ &\quad+\epsilon'\eta_a\left(\lambda^a{}_\mu m^\mu\right) - \epsilon\eta_a\left(n\bar\psi_q T^a q + nq^\dagger T^a\bar\psi_{\bar q}\right),\end{aligned} \tag{3.26}$$

where m_μ and n are some back ground fields which are chosen to satisfy the conditions (b) and (c) and are gauge singlet. These external fields do not break SU(2) gauge symmetry and topological symmetry because they are gauge singlet fields. Infinitesimal constant number ϵ and ϵ' are independent each other. We can add shift terms (3.26) to our action since (3.26) are BRS-like SUSY invariant and have 0 U-number. Note that sometime the determinant $detD_\mu D^\mu$ obtained by ϕ and $\bar\phi$ integrations is ignored in other papers because this determinant can be countervailed by a Fadeev-Popov determinant of the SU(2) gauge fixing. But in our theory, this determinant plays a essential role for separating reducible connections.

Next we consider the zero limit of ϵ and ϵ'. We take this limit after functional integrals. When we calculate around fixed points of irreducible connections which have no zero modes of D_μ, additional terms like (3.26) play no role and there is no modification in Donaldson and non-Abelian Seiberg- Witten theory in the limit as

ϵ approaches zero. But if a connection of a fixed point is reducible some different points appear. At first, we can estimate the order of the delta function obtained by $\bar{\phi}$ integrations as

$$\delta\left(-\frac{1}{2}D^\mu D_\mu\phi - \frac{i}{2}[\lambda_\mu, \lambda^\mu] + f_\epsilon\right) = \sum_{\hat{\phi}_\epsilon} \frac{\delta\left(\phi - \hat{\phi}_\epsilon\right)}{\left|det\left(D^\mu D_\mu - \frac{\delta f_\epsilon}{\delta\phi}\right)\right|}$$

$$\frac{1}{\left|det\left(D^\mu D_\mu - \frac{\delta f_\epsilon}{\delta\phi}\right)\right|} \sim \begin{cases} 1 & \text{(for irreducible connections } A_\mu) \\ \epsilon^{-l} & \text{(for reducible connections } A_\mu) \end{cases} \tag{3.27}$$

where l is a dimension of 0-cohomology H^0_A i.e. $l = \dim \ker(D_\mu)$. We denoted ϕ that do not make delta function vanish as $\hat{\phi}_\epsilon$. Especially it is important to notice that this determinant can be chosen to depend on not ϵ' but ϵ like the case (3.26). Since the $\epsilon'\bar{\phi}_a(D_\mu\phi)^a m^\mu$ term in shift terms (3.26) vanishes under D_μ zero-modes integral. For infinitesimal ϵ and ϵ', ϵ' terms in the determinant in Eqs.(3.27) are negligible. Note that the shift terms which break the balance of bosonic and fermionic shift terms in zero-mode integral like Eq.(3.26) have less variations. In general, ϵ or ϵ' appear in the both coefficients of $\bar{\phi}$ term and η term since $\bar{\phi}$ is a super partner of η.

The equations to obtain vacuum expectation value of ϕ change also as

$$\begin{aligned}
\lim_{\epsilon\to 0}\langle\phi\rangle &= -\lim_{\epsilon\to 0}\frac{i}{D^\mu D_\mu - \epsilon g}\left([\lambda_\nu, \lambda^\nu] + \epsilon h\right) \\
&= \begin{cases} \phi_c & \text{(for irreducible connections } A_\mu) \\ \phi_c + \epsilon\phi' & \text{(for reducible connections } A_\mu) \end{cases} \\
\epsilon\phi' &\equiv \lim_{\epsilon\to 0}\frac{i\epsilon}{D^\mu D_\mu - \epsilon g}(g\phi_c + h).
\end{aligned} \tag{3.28}$$

Where ϕ_c is a solution of the equation (3.15) of each connections A_μ. Note that $\epsilon\phi'$ may be finite in the 0-limit of ϵ since D_μ has zero-modes.

In a similar manner, we estimate a delta function which obtained by fermionic field η integration as

$$\prod_{\eta_0}\zeta_\epsilon\prod_{\eta'}\delta\left(\frac{i}{2}D_\mu\lambda^\mu - \zeta_\epsilon\right) \sim \begin{cases} \prod_\eta \delta\left(\frac{i}{2}D_\mu\lambda^\mu - \zeta_\epsilon\right) \\ \text{(for irreducible connections } A_\mu) \\ O(\epsilon,\epsilon')^l \prod_{\eta'}\delta\left(\frac{i}{2}D_\mu\lambda^\mu - \zeta_\epsilon\right) \\ \text{(for reducible connections } A_\mu). \end{cases} \tag{3.29}$$

Where η_0 are D_μ zero-modes and η' are other non-zero modes and $O(\epsilon,\epsilon')^l$ is order $\epsilon^n\epsilon'^m$ where $n + m = l, (n, m = 0, 1, 2, .., l)$. Or we can say it as follows. The first row of T_ϵ that is $D_{\mu\epsilon}$ goes back to D_μ in the zero limit of ϵ and ϵ' when connections of fixed points are irreducible. On the other hand when connections of fixed points are reducible, η and λ integral can be written by separation D_μ zero-modes η_0

$$\int \mathcal{D}\eta'\mathcal{D}\lambda \ \ \exp\left(-\int d^4xg^{\frac{1}{2}}((D_\mu\eta')\lambda^\mu - \zeta_\epsilon\eta')\right)\int \mathcal{D}\eta_0 \ \ \exp\left(-\int d^4xg^{\frac{1}{2}}\eta_0\zeta_\epsilon\right). \tag{3.30}$$

So we can estimate $det\,|T|$ as

$$\lim_{\epsilon,\epsilon'\to 0} det\,|T_\epsilon| = \begin{cases} det\,|T| \ \ \text{(for irreducible connections } A_\mu) \\ \lim_{\epsilon,\epsilon'\to 0}(O(\epsilon,\epsilon'))^l \ \ \text{(for reducible connections } A_\mu) \end{cases} \tag{3.31}$$

Where $O(\epsilon,\epsilon')^l$ is a determinant of a matrix in which the η_0 row λ_μ column of T is replaced with $\frac{\delta\zeta_\epsilon}{\delta\lambda}$. In the example (3.26), the η_0 row λ_μ column element in T_ϵ is

$$\frac{\delta\zeta^a_\epsilon}{\delta\lambda_{\mu b}} = \epsilon'\delta^{ab}m^\mu. \tag{3.32}$$

From Eqs.(3.27) and (3.29), we conclude an order of vacuum expectation values is given by,

$$\begin{cases} 1 \ \ \text{(for irreducible connections } A_\mu) \\ \left(\frac{O(\epsilon,\epsilon')}{\epsilon}\right)^l \ \ \text{(for reducible connections } A_\mu). \end{cases} \tag{3.33}$$

We can conclude from (3.33) that if we set ϵ and ϵ' as same order, then our path-integral are sums over reducible and irreducible connection parts with an equal weight.

Let us consider changing the ratio ϵ'/ϵ and changing the contributions from reducible connection part in our path-integral. The ratio of ϵ and ϵ' can be changed without changing of vacuum expectation value. This fact is seen as follows. We put ϵ' as $\epsilon' = k\epsilon$ and k is some positive real number. When an action is given as

$$S = \int d^4xg^{\frac{1}{2}}\hat{\delta}V + \epsilon\hat{\delta}F + \epsilon'\hat{\delta}G \tag{3.34}$$

where V, F and G are any functionals, then the change in a vacuum expectation value of any observable O which satisfy $\hat{\delta}O = 0$ under an infinitesimal deformation of k is

$$\begin{aligned} \frac{\delta}{\delta k}\langle O\rangle &= \int \mathcal{D}X\ O\ \frac{\delta}{\delta k}\left(\epsilon'\hat{\delta}G\right)e^{-S} \\ &= \int \mathcal{D}X\ \hat{\delta}\left(O(\epsilon G)\right)e^{-S} = 0. \end{aligned} \tag{3.35}$$

So we can change k without changing vacuum expectation value. In our case, only reducible connection part depend on k. We denote $\langle O\rangle_{IR}$ as a irreducible connection (non-Abelian connection) part of $\langle O\rangle$ and denote $\langle O\rangle_R$ as a reducible connection (Abelian connection) part, then the only $\langle O\rangle_R$ depend on the ratio k. When the power expansion of k of $\langle O\rangle_R$ is written as $\langle O\rangle_R = \sum_{n=0}^{l} \langle O\rangle_{R,n} \ k^n$, vacuum expectation value $\langle O\rangle$ is expressed as

$$\begin{aligned}\langle O\rangle &= \langle O\rangle_{IR} + \langle O\rangle_R \\ &= \langle O\rangle_{IR} + \sum_{n=0}^{l} \langle O\rangle_{R,n} \ k^n \end{aligned} \tag{3.36}$$

Since $\langle O\rangle$ is k independent, we obtain

$$\langle O\rangle_{R,n} = \left(\frac{\delta}{\delta k}\right)^n \langle O\rangle_R \,|_{k=0} = 0, \quad (n = 1, ..., l). \tag{3.37}$$

This fact means that we can remove the contribution to vacuum expectation value from reducible connection without a k^0 proportional term $\langle O\rangle_{R,0}$. Note that $\langle O\rangle_{R,0} = \langle O\rangle_R \,|_{\epsilon'=0}$. This is the most important fact to derive the relation of Abelian Seiberg-Witten invariants in the next section. Note that the vacuum expectation values $\langle O\rangle$ is invariant under changing of ϵ. This is easily ascertained by a similar way of (3.35).

In the next section, we explicitely investigate a relation of the non-Abelian Seiberg-Witten, Donaldson and the Abelian Seiberg-Witten invariants with the information obtained in this section.

3.4 The relation of Topological Invariants

In the previous sections, we prepared the tools for investigation of relations of the topological invariants i.e. Donaldson, Abelian Seiberg-Witten and non-Abelian Seiberg-Witten invariants. Now we actually construct the formulas of these invariants. We treat three parts of vacuum expectation value of observable (3.8) separately. The first part is Donaldson part which is defined as fixed point $q = 0$ and gauge connections of the fixed points are irreducible connections. There is no solution of the instanton equation when $b_2^+ \geq 1$ and connections are reducible. The second part is Abelian part whose fixed points $q \neq 0$ and connections are reducible. The third part is non-Abelian part that has irreducible connections and $q \neq 0$. In our theory, vacuum expectation value of O is defined by

$$
\begin{aligned}
\langle O \rangle &= \lim_{\epsilon,\epsilon' \to 0} \int \mathcal{D}X \ O \ \exp(-S) \qquad (3.38) \\
&= \begin{cases} \int \mathcal{D}X \ O \ \exp(-S_{QCD}) \\ \text{(for Donaldson and non} - \text{Abelian Seiberg} - \text{Witten part)} \\ \\ \lim_{\epsilon,\epsilon' \to 0} \int \mathcal{D}X \ O \ \exp(-S_{QCD} - \int d^4x g^{\frac{1}{2}} (\bar{\phi} f_\epsilon + \eta \zeta_\epsilon)) \\ \text{(for Abelian Seiberg} - \text{Witten part).} \end{cases}
\end{aligned}
$$

The fact that the ϵ terms do not influence irreducible connection part was seen in previous section. In noting this point we advance analysis in the following.

Donaldson part

We already saw the transvers path-integral of this part in section2. We represent common factors from $H_{\mu\nu}$ and X integrations as $\mathcal{N}$. Then we can write Donaldson part as,

$$
\mathcal{N} det(-4\pi) \langle \exp(\tilde{v} + \tau \tilde{u}) \rangle_D \qquad (3.39)
$$

where

$$
\begin{aligned}
\tilde{v} &= \frac{1}{4\pi^2} \int_\gamma Tr \left(i\phi F + \frac{1}{2} \lambda \wedge \lambda \right) \\
\tilde{u} &= -\frac{1}{8\pi^2} Tr \left(\phi^2 \right) \qquad (3.40)
\end{aligned}
$$

and $\langle O \rangle_D$ means vacuum expectation value of O with the action (3.11) of Donaldson-Witten theory. τ is a parameter.

Abelian Seiberg-Witten part

In this part, A_μ on fixed points are reducible connections. When we calculate in the large scaling limit it is possible that we choose back ground fields A_c in (3.13) as U(1) connections. The vector bundle E reduces to the sum of line bundle i.e. $E = \zeta \oplus \zeta^{-1}$.(See the Appendix A.) We set the direction of back ground gauge fields to T_3 i.e. $A_{c\mu} = A_{3\mu} T_3$. Since we take q as fundamental representation of SU(2), we can write q as

$$
q^{\dot{\alpha}} = \begin{pmatrix} q_1^{\dot{\alpha}} \\ q_2^{\dot{\alpha}} \end{pmatrix}. \qquad (3.41)
$$

We can reduce line bundle $\partial_\mu \zeta$ to $+\zeta$ and $q_2^{\dot{\alpha}} = 0$ because there is a symmetry defined by $A_3 \to -A_3$ and $q_1 \to q_2$ in this part. So we can estimate the vacuum expectation value of (3.38) around fixed points A_c and q_c by considering quadratic action of quantum fields A, q and others. We expand A and q as

$$A = A_c + A, \quad q = q_c + q \tag{3.42}$$

where the each second term of right hand side is a quantum field and $A_c$and q_c is chosen as

$$A_{\mu c} = A_{\mu 3} T_3, \quad q_c^{\dot{\alpha}} = \begin{pmatrix} q_1^{\dot{\alpha}} \\ 0 \end{pmatrix}. \tag{3.43}$$

Next we decompose the action S_{QCD} into two parts as we did in section 2 as

$$S_{QCD} \approx S_c + S_t, \tag{3.44}$$

where S_c is the action which consists of the Caltan part of adjoint fields and the first component of fundamental representation fields. The S_c is written as

$$\begin{aligned} S_c \ = & \int d^4x g^{\frac{1}{2}} \{ \frac{1}{4} \left| F^+_{\mu\nu 3} + \frac{1}{2} q_1^\dagger \bar{\sigma}_{\mu\nu} q_1 \right|^2 + \frac{1}{2} \left| \tilde{\mathcal{D}} q_1 \right|^2 - \frac{1}{2} \partial^\mu \bar{\phi}_3 \partial_\mu \phi_3 + \chi_3^{\mu\nu} (\partial^\mu \lambda_{\nu 3})^+ \\ & + \frac{i}{2} (\partial^\mu \eta_3) \lambda_3^\mu - \frac{i}{2} \chi_3^{\mu\nu} \bar{\psi}_{q1} \bar{\sigma}_{\mu\nu} q_1 + \frac{i}{2} \chi_3^{\mu\nu} q_1^\dagger \bar{\sigma}_{\mu\nu} \bar{\psi}_{\bar{q}1} - i (\tilde{\mathcal{D}} \bar{\psi}_{q1}) \psi_{q1} \\ & - i \psi_{\bar{q}1} \tilde{\mathcal{D}} \bar{\psi}_{\bar{q}1} + \frac{1}{2} q_1^\dagger \lambda_{\mu 3} \bar{\sigma}^\mu \psi_{q1} + \frac{1}{2} \psi_{\bar{q}1} \sigma_\mu \lambda_3^\mu q_1 \ \}, \end{aligned} \tag{3.45}$$

where $\tilde{\mathcal{D}}_\mu = \partial^\mu - i\frac{1}{2} A_{\mu 3}$ and spinor indices α and $\dot{\alpha}$ are omitted. This action S_c is surely the action of topological Abelian Seiberg-Witten theory [7][9]. This Caltan part action is Witten type topological action as follows

$$\begin{aligned} S_c = \int d^4x g^{\frac{1}{2}} \quad \hat{\delta}_c \quad & [\frac{1}{2} \chi_3^{\mu\nu} ((F^+_{3\mu\nu} + \frac{1}{2} q_1^\dagger \bar{\sigma} q_1) - H_{3\mu\nu}) \\ & - \frac{1}{2} (\partial_\mu \bar{\phi}_3) \lambda_3^\mu + (X_{\bar{q}1} \psi_{q1} - \psi_{\bar{q}1} X_{q1})], \end{aligned} \tag{3.46}$$

where we define the BRS-like SUSY as

$$\begin{aligned} & \hat{\delta}_c A_{3\mu} = i\lambda_{3\mu}, \quad \hat{\delta}_c \chi_{3\mu\nu} = H_{3\mu\nu}, \quad \hat{\delta}_c \bar{\phi}_3 = i\eta_3, \\ & \hat{\delta}_c \lambda_{3\mu} = -\partial_\mu \phi_3, \quad \hat{\delta}_c H_{3\mu\nu} = 0, \qquad \hat{\delta}_c \eta = 0, \\ & \hat{\delta}_c \phi = 0. \end{aligned}$$

$$
\begin{aligned}
\hat{\delta}_c q_1^{\dot{\alpha}} &= -\bar{\psi}^{\dot{\alpha}}_{\tilde{q}1}, \qquad \hat{\delta}_c \bar{\psi}^{\dot{\alpha}}_{\tilde{q}1} = -\frac{i}{2}\phi_3^a q_1^{\dot{\alpha}}, \\
\hat{\delta}_c q^{\dagger}{}_{\dot{\alpha}1} &= -\bar{\psi}_{q1\dot{\alpha}}, \qquad \hat{\delta}_c \bar{\psi}_{q1\dot{\alpha}} = \frac{i}{2} q^{\dagger}{}_{1\dot{\alpha}}\phi_3, \\
\hat{\delta}_c \psi_{q1\alpha} &= -\sigma^{\mu}_{\alpha\dot{\alpha}}\tilde{\mathcal{D}}_{\mu} q_1^{\dot{\alpha}} + X_{q1\alpha}, \\
\hat{\delta}_c X_{q1\alpha} &= \frac{i}{2}\phi_3\psi_{q1\alpha} - i\sigma^{\mu}_{\alpha\dot{\alpha}}\tilde{\mathcal{D}}_{\mu}\bar{\psi}^{\dot{\alpha}}_{\tilde{q}1} + \frac{1}{2}\sigma^{\mu}_{\alpha\dot{\alpha}}\lambda_{3\mu} q_1^{\dot{\alpha}} \\
\hat{\delta}_c \psi^{\alpha}_{\tilde{q}1} &= i\tilde{\mathcal{D}}_{\mu} q^{\dagger}{}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} - X^{\alpha}_{\tilde{q}1}, \\
\hat{\delta}_c X^{\alpha}_{\tilde{q}1} &= \frac{i}{2}\psi^{\alpha}_{\tilde{q}1}\phi_3 - i\tilde{\mathcal{D}}_{\mu}\bar{\psi}_{q1\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} + \frac{1}{2} q^{\dagger}{}_{1\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\lambda_{3\mu}.
\end{aligned} \tag{3.47}
$$

Another terms in S_{QCD} have to be expanded around fixed points A_{3c} and q_{1c}. The quadratic action of S_t is obtained as

$$
\begin{aligned}
S_t^{(2)} &= \int d^4x g^{\frac{1}{2}}[\frac{1}{4}(\mathcal{D}_{\mu}A_{\nu+} - \mathcal{D}_{\nu}A_{\mu+})^{+}(\mathcal{D}^*_{\mu}A_{\nu-} - \mathcal{D}^*_{\nu}A_{\mu-})^{+} \\
&+ \frac{1}{8}(\mathcal{D}_{\mu}A_{\nu+} - \mathcal{D}_{\nu}A_{\mu+})^{+} q_2^{\dagger}\bar{\sigma}^{\mu\nu} q_{1c} + \frac{1}{8}(\mathcal{D}^*_{\mu}A_{\nu-} - \mathcal{D}^*_{\nu}A_{\mu-})^{+} q_{1c}^{\dagger}\bar{\sigma}^{\mu\nu} q_2 \\
&+ \frac{1}{4}\left|q_{1c}^{\dagger}\bar{\sigma}_{\mu\nu} q_2\right|^2 + \frac{1}{2}\left|\tilde{\not{\mathcal{D}}}^* q_2\right|^2 - \frac{1}{4}(\tilde{\not{\mathcal{D}}}^* q_2)^{\dagger}\sigma^{\mu}A_{\mu+}q_{1c} - \frac{1}{4}(\sigma^{\mu}A_{\mu+}q_{1c})^{\dagger}(\tilde{\not{\mathcal{D}}}^* q_2) \\
&+ \frac{1}{8}\left|\sigma^{\mu}A_{\mu+}q_{1c}\right|^2 - \frac{1}{2}(\mathcal{D}_{\mu}\bar{\phi}_+)(\mathcal{D}^{\mu *}\phi_-) - \frac{1}{2}(\mathcal{D}^*_{\mu}\bar{\phi}_-)(\mathcal{D}^{\mu}\phi_+) + \chi_+^{\mu\nu}(\mathcal{D}^*_{\mu}\lambda^-_{\nu})^{+} \\
&+ \chi_-^{\mu\nu}(\mathcal{D}_{\mu}\lambda^+_{\nu})^{+} + \frac{i}{2}(\mathcal{D}_{\mu}\eta_+)\lambda^{\mu}_- + \frac{i}{2}(\mathcal{D}^*_{\mu}\eta_-)\lambda^{\mu}_+ - \frac{i}{2}\chi_-^{\mu\nu}\bar{\psi}_{q2}\bar{\sigma}_{\mu\nu}q_{1c} \\
&+ \frac{i}{2}\chi_+^{\mu\nu}q_{1c}^{\dagger}\bar{\sigma}_{\mu\nu}\bar{\psi}_{\tilde{q}2} - i(\tilde{\not{\mathcal{D}}}^*\bar{\psi}_{q2})\psi_{q2} - i\psi_{\tilde{q}2}(\tilde{\not{\mathcal{D}}}^*\bar{\psi}_{\tilde{q}2}) \\
&+ \frac{1}{2}q_{1c}^{\dagger}\lambda_{\mu+}\bar{\sigma}^{\mu}\psi_{q2} + \frac{1}{2}\psi_{\tilde{q}2}\bar{\sigma}^{\mu}\lambda_{\mu+}q_{1c}].
\end{aligned} \tag{3.48}
$$

Where $\mathcal{D}_{\mu} = \partial_{\mu} - iA_{3\mu}$ and we put $T^{\pm}$ as $T^{\pm} = \frac{1}{2}(T_1 \pm iT_2)$. $S_t^{(2)}$ is transformed with matrices in the same way of the appendix B as

$$
\begin{aligned}
S_t^{(2)} &= (A_{\rho-}\ q_2^{\dagger})\begin{pmatrix} M_{A,A} & M_{A,q} \\ M_{q,A} & M_{q,q} \end{pmatrix}\begin{pmatrix} A_{\tau+} \\ q_2 \end{pmatrix} \\
&\quad + \bar{\phi}_+(\frac{1}{2}\mathcal{D}^*_{\mu}\mathcal{D}^{\mu *})\phi_- + \bar{\phi}_-(\frac{1}{2}\mathcal{D}_{\mu}\mathcal{D}^{\mu})\phi_+ \\
&\quad + (\eta_-\ \eta_+\ \chi_{0i-}\ \chi_{0i+}\ \psi_{\tilde{q}2}\ \psi_{q2})(\mathrm{T_t})\begin{pmatrix} \lambda_{0+} \\ \lambda_{j+} \\ \lambda_{0-} \\ \lambda_{j-} \\ \bar{\psi}_{\tilde{q}2} \\ \bar{\psi}_{q2} \end{pmatrix}.
\end{aligned} \tag{3.49}
$$

Where we chose space elements of self-dual field χ, i.e. χ_{0i}, as substantial elements. Matrix M elements are given as follows.

$$
\begin{aligned}
& (A_{\rho-}\ row\ A_{\tau+}\ column) \\
M_{A.A} &= -\frac{1}{4}(\mathcal{D}_\mu\delta_{\nu\rho} - \mathcal{D}_\nu\delta_{\mu\rho})^+(\mathcal{D}_\mu\delta_{\nu\tau} - \mathcal{D}_\nu\delta_{\mu\tau}) + \frac{1}{8}q_{1c}^\dagger\sigma^\mu\delta_{\mu\rho}\sigma_{\nu\tau}q_{1c} \\
& (A_{\rho-}\ row\ q_2\ column) \\
M_{A.q} &= -\frac{1}{8}(\mathcal{D}_\mu\delta_{\nu\rho} - \mathcal{D}_\nu\delta_{\mu\rho})^+ q_{1c}^\dagger\bar{\sigma}_{\mu\nu} - \frac{1}{4}(q_{1c}^\dagger\bar{\sigma}_\mu\delta_{\nu\rho})\,\tilde{\not{\mathcal{D}}}^* \\
& (q_2^\dagger\ row\ A_{\tau+}\ column) \\
M_{q.A} &= \frac{1}{8}\bar{\sigma}_{\mu\nu}q_{1c}(\mathcal{D}_\mu\delta_{\nu\tau} - \mathcal{D}_\nu\delta_{\mu\tau})^+ + \frac{1}{4}(\tilde{\not{\mathcal{D}}})\sigma^\mu\delta_{\mu\tau}q_{1c} \\
& (q_2^\dagger\ row\ q_2\ column) \\
M_{q.q} &= -\frac{1}{2}\,\tilde{\not{\mathcal{D}}}\,\tilde{\not{\mathcal{D}}}^* + \frac{1}{4}\bar{\sigma}_{\mu\nu}q_{1c}q_{1c}^\dagger\sigma^{\mu\nu}.
\end{aligned}
\tag{3.50}
$$

Where the index + means anti-selfdual about indices μ and ν. The elements of the matrix T_t is obtained in the same process in appendix B. We write it as

$$
\begin{pmatrix}
-\frac{i}{2}\mathcal{D}_0 & 0 & -\frac{i}{2}\mathcal{D}_j & 0 & 0 & 0 \\
0 & -\frac{i}{2}\mathcal{D}_0^* & 0 & -\frac{i}{2}\mathcal{D}_j^* & 0 & 0 \\
-\frac{1}{2}\mathcal{D}_i & 0 & \frac{1}{2}(\mathcal{P}\mathcal{D})_{ij}^+ & 0 & 0 & -\frac{i}{2}q_{1c}\bar{\sigma}_{0i} \\
0 & -\frac{1}{2}\mathcal{D}_i^* & 0 & \frac{1}{2}(\mathcal{P}\mathcal{D})_{ij}^+ & -\frac{i}{2}q_{1c}^\dagger\bar{\sigma}_{0i} & 0 \\
-\frac{1}{2}q_{1c}^\dagger\bar{\sigma}_0 & 0 & -\frac{1}{2}q_{1c}^\dagger\bar{\sigma}_j & 0 & 0 & i\,\tilde{\not{\mathcal{D}}}^* \\
0 & \frac{1}{2}\sigma_0 q_{1c} & 0 & \frac{1}{2}\sigma_j q_{1c} & -i\,\tilde{\not{\mathcal{D}}}^* & 0
\end{pmatrix}.
\tag{3.51}
$$

where $\mathcal{D}_\mu$ is defined as $\partial_\mu - iA_{3\mu}$ and we defined $(\mathcal{P}\mathcal{D})_{ij}^+$ as

$$
(\mathcal{P}\mathcal{D})_{ij}^+ = (\mathcal{D}_0\delta_{ij} - \frac{1}{2}\epsilon_{0ilk}\mathcal{D}_l\delta_{jk}).
\tag{3.52}
$$

Only in this Abelian part, we must not forget the ϵ terms. As we saw in section 3, $\frac{i}{2}D_\mu$ of η rows λ columns should be replaced by $D_{\mu\epsilon}$, and the components of η rows $\bar{\psi}$ columns are replaced by order ϵ operators $O(\epsilon)$ as we find the example (3.26). We introduce $\mathcal{D}_{\mu\epsilon'}$ as $\mathcal{D}_\mu$ shifted by ϵ' term like (3.25). Then we write down $T_{t\epsilon}$ which defined as the T_t added shift terms from $\bar{\phi}f_\epsilon + \eta\zeta_\epsilon$ in (3.19) as

$$
\begin{pmatrix}
-\frac{i}{2}\mathcal{D}_{0\epsilon'} & 0 & -\frac{i}{2}\mathcal{D}_{j\epsilon'} & 0 & O(\epsilon) & O(\epsilon) \\
0 & -\frac{i}{2}\mathcal{D}_{0\epsilon'}^* & 0 & -\frac{i}{2}\mathcal{D}_{j\epsilon'}^* & O(\epsilon) & O(\epsilon) \\
-\frac{1}{2}\mathcal{D}_i & 0 & \frac{1}{2}(\mathcal{P}\mathcal{D})_{ij}^+ & 0 & 0 & -\frac{i}{2}q_{1c}\bar{\sigma}_{0i} \\
0 & -\frac{1}{2}\mathcal{D}_i^* & 0 & \frac{1}{2}(\mathcal{P}\mathcal{D})_{ij}^+ & -\frac{i}{2}q_{1c}^\dagger\bar{\sigma}_{0i} & 0 \\
-\frac{1}{2}q_{1c}^\dagger\bar{\sigma}_0 & 0 & -\frac{1}{2}q_{1c}^\dagger\bar{\sigma}_j & 0 & 0 & i\,\tilde{\not{\mathcal{D}}}^* \\
0 & \frac{1}{2}\sigma_0 q_{1c} & 0 & \frac{1}{2}\sigma_j q_{1c} & -i\,\tilde{\not{\mathcal{D}}}^* & 0
\end{pmatrix}.
\tag{3.53}
$$

Let us path-integrate out the transversal part. To carry out this Gaussian integration, we decide the determinant of $T_{t\epsilon}$ as

$$det(T_{t\epsilon}) \equiv det(T^*_{t\epsilon}T_{t\epsilon})^{1/2}. \tag{3.54}$$

(See the reference [17].) When we denote the $T^*_{t\epsilon}T_{t\epsilon}$ as

$$T^*_{t\epsilon}T_{t\epsilon} = \begin{pmatrix} T^*T^{++}_{\mu\nu} & T^*T^{+-}_{\mu\nu} & T^*T^{+}_{\mu\bar{q}} & T^*T^{+}_{\mu q} \\ T^*T^{-+}_{\mu\nu} & T^*T^{--}_{\mu\nu} & T^*T^{-}_{\mu\bar{q}} & T^*T^{-}_{\mu q} \\ T^*T_{\bar{q}\nu}^{\ +} & T^*T_{\bar{q}\nu}^{\ -} & T^*T_{\bar{q}\bar{q}} & T^*T_{\bar{q}q} \\ T^*T_{q\nu}^{\ +} & T^*T_{q\nu}^{\ -} & T^*T_{q\bar{q}} & T^*T_{qq} \end{pmatrix}, \tag{3.55}$$

elements of $T^*_{t\epsilon}T_{t\epsilon}$ is obtained as follows.

$$\begin{aligned}
& (\lambda_{\mu+}\ row\ \lambda_{\nu+}\ column) \\
T^*T^{++}_{\mu\nu} &= (-\frac{1}{4}\mathcal{D}^2\delta_{\mu\nu} - F^-_{3\mu\nu}) + \frac{1}{4}q^\dagger_{1c}\sigma_\mu\sigma_\nu q_{1c} + O(\epsilon') \\
& (\lambda_{\mu+}\ row\ \lambda_{\nu-}\ column) \\
T^*T^{+-}_{\mu\nu} &= 0 \\
& (\lambda_{\mu-}\ row\ \lambda_{\nu+}\ column) \\
T^*T^{-+}_{\mu\nu} &= 0 \\
& (\lambda_{\mu-}\ row\ \lambda_{\nu-}\ column) \\
T^*T^{--}_{\mu\nu} &= (-\frac{1}{4}\mathcal{D}^{*2}\delta_{\mu\nu} - F^-_{3\mu\nu}) + \frac{1}{4}q^\dagger_{1c}\sigma_\mu\sigma_\nu q_{1c} + O(\epsilon') \\
& (\bar{\psi}_{\bar{q}2}\ row\ \lambda_{\nu+}\ column) \\
T^*T_{\bar{q}\nu}^{\ +} &= O(\epsilon) \\
& (\bar{\psi}_{\bar{q}2}\ row\ \lambda_{\nu-}\ column) \\
T^*T_{\bar{q}\nu}^{\ -} &= (\frac{i}{4}\bar{\sigma}_{\nu\rho}q_{1c})^+\mathcal{D}^\rho + \frac{i}{2}\tilde{\not{\mathcal{D}}}\sigma_\nu q_{1c} + O(\epsilon,\epsilon') \\
& (\bar{\psi}_{q2}\ row\ \lambda_{\nu+}\ column) \\
T^*T_{q\nu}^{\ +} &= (-\frac{i}{4}\bar{\sigma}_{\nu\rho}q^\dagger_{1c})^+\mathcal{D}^\rho + \frac{i}{2}\tilde{\not{\mathcal{D}}}q^\dagger_{1c}\bar{\sigma}_\nu + O(\epsilon,\epsilon') \\
& (\bar{\psi}_{q2}\ row\ \lambda_{\nu-}\ column) \\
T^*T_{q\nu}^{\ -} &= O(\epsilon) \\
& (\lambda_{\mu+}\ row\ \bar{\psi}_{\bar{q}2}\ column) \\
T^*T^{+}_{\mu\bar{q}} &= O(\epsilon)
\end{aligned}$$

$$
\begin{aligned}
& (\lambda_{\mu-}\ row\ \bar{\psi}_{\bar{q}2}\ column) \\
T^*T^-_{\mu\bar{q}} &= -\frac{i}{4}\mathcal{D}^\rho(q^\dagger_{1c}\bar{\sigma}_{\rho\mu})^+ - \frac{i}{2}q^\dagger_{1c}\sigma_\mu\tilde{\mathcal{D}}^* + O(\epsilon,\epsilon') \\
& (\bar{\psi}_{\bar{q}2}\ row\ \bar{\psi}_{\bar{q}2}\ column) \\
T^*T_{\bar{q}\bar{q}} &= \frac{1}{4}(\sigma_{\mu\nu}q_{1c})^+(q^\dagger_{1c}\bar{\sigma}^{\mu\nu})^+ + \tilde{\mathcal{D}}^{*2} + O(\epsilon^2) \\
& (\bar{\psi}_{q2}\ row\ \bar{\psi}_{\bar{q}2}\ column) \\
T^*T_{q\bar{q}} &= O(\epsilon^2) \\
& (\lambda_{\mu+}\ row\ \bar{\psi}_{q2}\ column) \\
T^*T^+_{\mu q} &= -\frac{i}{4}\mathcal{D}^\rho(q_{1c}\bar{\sigma}_{\rho\mu})^+ - \frac{i}{2}\sigma_\mu q_{1c}\tilde{\mathcal{D}}^* + O(\epsilon,\epsilon') \\
& (\lambda_{\mu-}\ row\ \bar{\psi}_{q2}\ column) \\
T^*T^-_{\mu q} &= O(\epsilon) \\
& (\bar{\psi}_{\bar{q}2}\ row\ \bar{\psi}_{q2}\ column) \\
T^*T_{\bar{q}q} &= O(\epsilon^2) \\
& (\bar{\psi}_{q2}\ row\ \bar{\psi}_{q2}\ column) \\
T^*T_{qq} &= \frac{1}{4}(\bar{\sigma}_{\mu\nu}q^\dagger_{1c})^+(q_{1c}\bar{\sigma}^{\mu\nu})^+ + \tilde{\mathcal{D}}^{*2} + O(\epsilon^2). \qquad (3.56)
\end{aligned}
$$

When we integrate ϕ and $\bar{\phi}$ we have to pay attention to the shift from ϵ terms. We introduce $\mathcal{D}_{\mu\epsilon}$ which include $\mathcal{D}_\mu$ and shifts from ϵ terms , like (3.24). After these transverse fields path-integration of (3.48) we can write this part with matrices M and $T_{t\epsilon}$ as,

$$
\sum_{reducible\ A_\mu} \mathcal{N} det(M)^{-1/2} det(T_{t\epsilon}) det(\frac{1}{2}\mathcal{D}^*_{\mu\epsilon}\mathcal{D}^{\mu*}_\epsilon) det(\frac{1}{2}\mathcal{D}_{\mu\epsilon}\mathcal{D}^\mu_\epsilon)\langle \exp(\tilde{v}+\tau\tilde{u})\rangle_A. \qquad (3.57)
$$

Where $\langle O\rangle_A$ means vacuum expectation value of O with Abelian Seiberg-Witten theory whose action is composed by S_c and Caltan part from $\bar{\phi}f_\epsilon + \eta\zeta_\epsilon$ i.e.

$$
S_c + \int d^4xg^{\frac{1}{2}}(\bar{\phi}f_\epsilon + \eta\zeta_\epsilon)_c. \qquad (3.58)
$$

Where we denote the Caltan part of $\bar{\phi}f_\epsilon + \eta\zeta_\epsilon$ as $(\bar{\phi}f_\epsilon + \eta\zeta_\epsilon)_c$.

If we put $\bar{\phi}f_\epsilon + \eta\zeta_\epsilon$ as an example (3.26), this Caltan part is

$$
(\bar{\phi}f_\epsilon + \eta\zeta_\epsilon)_c = \epsilon'\bar{\phi}_3\partial_\mu\phi_3 m^\mu + \epsilon'\eta_3\lambda_{3\mu}m^\mu - \frac{1}{2}\epsilon n q^\dagger_1\eta_3\bar{\psi}_{\bar{q}1}
$$

$$
\begin{aligned}
& -\frac{1}{2}\epsilon n\bar{\psi}_{q1}\eta_3 q_1 + \frac{1}{2}\epsilon n q_1^\dagger \bar{\phi}_3\phi_3 q_1 - i\epsilon n\bar{\psi}_{q1}\bar{\phi}_3\bar{\psi}_{\bar{q}1} \\
= \; & \epsilon'\hat{\delta}_c(\bar{\phi}_3\lambda_{3\mu}m^\mu) + \epsilon\frac{i}{2}\hat{\delta}_c(nq_1^\dagger\bar{\phi}_3\bar{\psi}_{\bar{q}1} - n\bar{\psi}_{q1}\bar{\phi}_3 q). \qquad (3.59)
\end{aligned}
$$

This is the same case of Eqs.(3.35)(3.36). We find $\sum_{A_\mu}\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_A$ is independent from k. In (3.57), $\sum_{A_\mu}$ is sum with weight $\mathcal{N}det(M)^{-1/2}det(T_{t\epsilon})\left|det(\frac{1}{2}\mathcal{D}_{\mu\epsilon}\mathcal{D}^\mu_\epsilon)\right|^2$.

Let us consider the case that there is only one solution of A_μ and q_1, there is not sum $\sum_{A_\mu}$ and we find $\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_A$ is independent from k. Therefore only $det(T_{t\epsilon})det(\frac{1}{2}\mathcal{D}^*_{\mu\epsilon}\mathcal{D}^{\mu*}_\epsilon)det(\frac{1}{2}\mathcal{D}_{\mu\epsilon}\mathcal{D}^\mu_\epsilon)$ term in (3.57) depend on k. As we saw in section 3, the coefficients of k have to be zero. Then we get the non-trivial result,

$$\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_A = 0. \qquad (3.60)$$

But if there are several solutions of A_μ and q_1, it is unclear whether Eq.(3.60) is correct or not.

non-Abelian Seiberg-Witten part

Finally we denote non-Abelian part as

$$\mathcal{N}\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_{nA}. \qquad (3.61)$$

Where the connections of fixed point are restricted within irreducible connections. (3.61) is a pure non-Abelian Seiberg-Witten invariants.

Now the vacuum expectation value is separately written as

$$
\begin{aligned}
\langle\exp(\tilde{v}+\tau\tilde{u})\rangle \; = \; & \mathcal{N}det(-4\pi)\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_D + \mathcal{N}\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_{nA} \qquad (3.62) \\
& + \lim_{\epsilon,\epsilon'\to 0}\sum_{reducible A_\mu}\mathcal{N}det(M)^{-1/2}det(T_{t\epsilon})det(\frac{1}{2}\mathcal{D}^*_{\mu\epsilon}\mathcal{D}^{\mu*}_\epsilon)det(\frac{1}{2}\mathcal{D}_{\mu\epsilon}\mathcal{D}^\mu_\epsilon)\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_A
\end{aligned}
$$

As we saw in section 3 that this vacuum expectation value is invariant under changing the ratio of ϵ and ϵ', so we found that the Abelian part vanishes without $\langle O\rangle_{R,0} = \langle O\rangle_R\,|_{\epsilon'=0}$ in Eq.(3.60). From this fact, we find following formulas.

$$
\begin{aligned}
\langle\exp(\tilde{v}+\tau\tilde{u})\rangle = \; & \mathcal{N}det(-4\pi)\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_D + \mathcal{N}\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_{nA} \qquad (3.63) \\
& + \left[\lim_{\epsilon\to 0}\sum_{reducible A_\mu}\frac{\mathcal{N}}{det(M)^{1/2}}det(T_{t\epsilon})\left|det(\frac{1}{2}\mathcal{D}_{\mu\epsilon}\mathcal{D}^\mu_\epsilon)\right|^2\langle\exp(\tilde{v}+\tau\tilde{u})\rangle_A\right]\Bigg|_{\epsilon'=0}.
\end{aligned}
$$

From Eq.(3.37), identities are obtained as

$$\left(\frac{\delta}{\delta k}\right)^n \left[\lim_{\epsilon \to 0} \sum_{reducible A_\mu} \frac{\mathcal{N}}{det(M)^{1/2}} det(T_{t\epsilon}) \left| det(\frac{1}{2}\mathcal{D}_{\mu\epsilon}\mathcal{D}^\mu_\epsilon) \right|^2 \langle \exp(\tilde{v} + \tau\tilde{u}) \rangle_A \right] \Bigg|_{k=0} = 0. \quad (3.64)$$

where $n = 1, ..., dim(kerd_A)$. These formulas are non-trivial. These identities of U(1) topological invariants are obtained from SU(2) Topological QCD. Note that vacuum expectation value of ϕ was changed as (3.28), but detail character of the shift terms did not need to get above formulas. We comment on the Eq.(3.60) little more. This formula may imply that Abelian Seiberg-Witten invariants vanish in general. Indeed it is possible to apply our methods for massive topological QCD with no obstacle. But there are some problems to identify the Abelian part and usual Seiberg-Witten invariants, for example Eq.(3.28) and there are problems to extend to the case which has plural monopole solutions. This subjects are discussed in [21].

3.5 Summary

We have studied massless topological QCD in detail and found new relations (3.63) and (3.64). One of the results of excluding mass terms is that our theory does not have spontaneous gauge symmetry breaking phase in usual meaning. Hence, we could not distinguish between reducible and irreducible connections without no modification. We gazed $det(D^\mu D_\mu) = 0$ when connections are reducible. In other words gauge generator act not effectively and this fact cause moduli space singularities. Infinitesimal shift terms are introduced to the Lagrangian to account zero-eigenvalue states of $D^\mu D_\mu$. For this terms we could contain the Abelian part and treat separately reducible and irreducible connections. That determinant are obtained from ϕ and $\bar{\phi}$ integral. Fermionic integral of η, which is super partner of $\bar{\phi}$, λ, χ etc. cause the determinant of T. This determinant offset $det(D^\mu D_\mu) = 0$. Infinitesimal shift terms is added to the Lagrangian to shift the both zero-determinants. We could change the ratio of infinitesimal shift terms without changing vacuum expectation value. As a result of this, we got formulas (3.63) and (3.64) (and especially case Eq.(3.60)). These are non-trivial relations between topological invariants. The identities of U(1) topological invariants are obtained from BRS symmetry of Topological QCD like

Ward-Takahashi identity.

When we interpret Topological Field Theory as a gauge fixing theory like [19], the zero-modes of D_μ is interpreted as Gribov zero-modes. As we saw in section 4 and we will see in the next chapter, Gribov zero-modes break BRS symmetry often and topological symmetry breaking occur. So the external fields in the section 4 avoid topological symmetry breaking from Gribov zero-modes.

The next subjects we have to investigate are to calculate actually in some models and to ascertain these formulas. We studied only SU(2) gauge theories in the present paper, so we want to extend it more general case. To carry it out in our formalism, we have to construct the tool that embed the equations to classify the reducible connections in equations of motion from some topological action. This is one of our future problems.
The relation between the $\langle \exp(\tilde{v} + \tau \tilde{u}) \rangle_A$ and usual Seiberg-Witten invariants have to be studied more carefully. $\langle \exp(\tilde{v} + \tau \tilde{u}) \rangle_A$ is topological invariants but have some difference from usual Seiberg-Witten invariants. It is important problem to make the difference clear.

Appendix.A Reducible connection

In this appendix, we summarize some basic of reducible connections for physicists who is unfamiliar with this words. For the convenience, we treat the only case of SU(2) gauge group. We consider a connection on a point p on a back ground manifold M. A holonomy group is defined as subgroup of SU(2) which transform the connection as parallel trnsformations around any loop with the start point p. Therefore, holonomy groups are understood as groups which is needed actually to introduce each connection. We put H as the holonomy group. We can define the reducible connection with the holonomy group which is classified in following two cases.

(1) $H \subseteq \{\pm 1\}$.

(2) H is conjugate to U(1) .

In the first case connections are flat and this case is realized when the Chern number is zero. In our theory, case (1) is ignored. When connections do not satisfy above each conditions then we call them irreducible connections, and centralizer of H is $\{\pm 1\}$.

We can understand the relation, which is used many times in this chapter, between reducibility and zero-modes of $d_A : Ad\eta \bigotimes \bigwedge_0 \to ad\eta \bigotimes \bigwedge_1$ as follows. If d_A has a zero-mode ϕ_0, we obtain an one parameter group $\{\exp(t\phi_0) \mid t \in \mathbf{R}\ d_A\phi_0 = 0\}$ whose elements transform the connection identically because

$$g^{-1}dg + g^{-1}Ag = td_A\phi_0 = 0. \tag{3.65}$$

This means that centralizer of H is not $\{\pm 1\}$ and this connection is reducible. Oppositely if a connection is reducible, then there is an one parameter group whose elements satisfy

$$g^{-1}dg + g^{-1}Ag = 0 \tag{3.66}$$

and $g \neq \pm 1$. We obtain

$$d_A\left(\frac{dg}{dt}\right) = 0 \tag{3.67}$$

after differentiate (3.60) by the parameter t. Therefore we understand $\frac{dg}{dt}$ is a zero mode of d_A.

Next we will see the process of reducible connection defined as

$$A = \begin{pmatrix} A_L & 0 \\ 0 & A_L^* \end{pmatrix} \tag{3.68}$$

where A_L is a connection on a complex line bundle L and A_L^* is a its complex conjugate. We can put the orthogonal basis of 2-dimensional complex vector bundle,

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{3.69}$$

as

$$e_1 = \begin{pmatrix} q_1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ q_2 \end{pmatrix}. \tag{3.70}$$

We introduce complex line bundle L and L' with a parallel trnsformation operator P_l where l is represented as some loop as

$$L = \{cP_l(e_1)|c \in \mathbf{C}\} \tag{3.71}$$

$$L' = \{cP_l(e_2)|c \in \mathbf{C}\}. \tag{3.72}$$

In our theory, reducible connections means U(1) connections and we can take P_l as diagonal matrices. So we find that the definitions of L and L' are unrelated of loop l. We could established L and L' in this way. Therefore connection A can be represented as

$$A = A_L \oplus A_{L'} \tag{3.73}$$

where A_L and $A_{L'}$ are connections of L and L' respectively. Since A have to be valued in u(1) now, A is pure imaginary i.e. $A_{L'} = A_L^* = -A_L$. We obtain (3.61) and

$$F_A = \begin{pmatrix} dA_L & 0 \\ 0 & -dA_L \end{pmatrix}. \tag{3.74}$$

In this chapter, we used these results in section 4.

Appendix.B Gaussian Integral

In the section 4, we integrate fermionic fields in the Abelian Seiberg-Witten part. The methods of integration used there are studied in reference [11][17]. It is possible to do this integration in not only Abelian case but also other case. For the non-Abelian parts, we can do it more general. i.e. we put classical back ground fields as $A^a_{c\mu} T_a$ and q_c in generally and expand S_{QCD} to second order of quantum fields. We write down the result of fermionic part as follows.

$$(\eta_a \chi_{0ia} \psi_q^t \psi_{\tilde{q}})\,(\mathrm{T}) \begin{pmatrix} \lambda_{0b} \\ \lambda_{jb} \\ \bar{\psi}_{\tilde{q}} \\ \bar{\psi}_q^t \end{pmatrix},$$

and T is concretely

$$\begin{pmatrix} -iTr(T_a D_0 T_b) & -iTr(T_a D_j T_b) & 0 & 0 \\ -Tr(T_a D_i T_b) & Tr(T_a (PD)^+_{ij} T_b) & iq_c^\dagger \bar{\sigma}_{0i} T_a & -iq_c^t \bar{\sigma}_{0i} T_a^t \\ -T_a^t \bar{\sigma}_0 (q_c^\dagger)^t & -T_b^t \bar{\sigma}_j (q_c^\dagger)^t & 0 & i\,\not{D} \\ \sigma_0 T_b q_c & \sigma_j T_b q_c & -i\,\not{D} & 0 \end{pmatrix}.$$

Where we denote $(D_0\delta_{ij} - \frac{1}{2}\epsilon_{0ilk}D_l\delta_{jk})$ as $(PD)^+_{ij}$. We chose space elements of self dual field χ as substantial elements. When we take account of ϵ shift terms, then the first row of T change in order ϵ. We call this shifted T as T_ϵ. For example, we obtained T_ϵ of the case concretely as

$$\begin{pmatrix} -iTr(T_a(D_0 - \epsilon' m_0)T_b) & -iTr(T_a(D_j - \epsilon' m_j)T_b) & -\epsilon n q_c^\dagger T_a & -\epsilon n q_c^t T_a^t \\ -Tr(T_a D_i T_b) & Tr(T_a (PD)^+_{ij} T_b) & i q_c^\dagger \bar{\sigma}_{0i} T_a & -i q_c^t \bar{\sigma}_{0i} T_a^t \\ -T_a^t \bar{\sigma}_0 (q_c^\dagger)^t & -T_b^t \bar{\sigma}_j (q_c^\dagger)^t & 0 & i\,\not{D} \\ \sigma_0 T_b q_c & \sigma_j T_b q_c & -i\,\not{D} & 0 \end{pmatrix}.$$

T is not a map from a space into itself. So we have to pay attention to define its determinant. We put adjoint operator of T as T^*, and define $det(T)$ as

$$det(T) \equiv det(T^*T)^{1/2}. \tag{3.75}$$

(See the references [11] [17].) Now we can treat (T^*T) with not space indices i but space-time indices μ. We name the elements of (T^*T) by

$$(\ \lambda_\mu^c\ \bar{\psi}_{\bar{q}}\ \bar{\psi}_q)\,(\mathrm{T^*T})\begin{pmatrix} \lambda_\nu^b \\ \bar{\psi}_{\bar{q}} \\ \bar{\psi}_q \end{pmatrix},$$

$$T^*T = \begin{pmatrix} T^*T_{\mu\nu}^{cb} & T^*T_{\mu\bar{q}}^{c} & T^*T_{\mu q}^{c} \\ T^*T_{\bar{q}\nu}^{\ b} & T^*T_{\bar{q}\bar{q}} & T^*T_{\bar{q}q} \\ T^*T_{q\nu}^{\ b} & T^*T_{q\bar{q}} & T^*T_{qq} \end{pmatrix} \tag{3.76}$$

The elements of (T^*T) is obtained as follows.

$(\lambda_\mu^c\ row\ \lambda_\nu^b\ column)$

$$T^*T_{\mu\nu}^{cb} = -\frac{1}{2}Tr\ (T_c(D^2\delta_{\mu\nu} - F_{\mu\nu}^-)T_b) + q_c^\dagger T_b T_c \sigma_\mu \sigma_\nu q_c + q_c^\dagger T_c T_b \sigma_\mu \sigma_\nu q_c$$

$(\bar{\psi}_{\bar{q}}\ row\ \lambda_\nu^b\ column)$

$$T^*T_{\bar{q}\nu}^{\ b} = \frac{i}{2}(T_a\bar{\sigma}_{\nu\rho}q_c)^+(2Tr\ T_a D^\rho T_b) + i\,\not{D}\sigma_\nu T_b q_c$$

$(\bar{\psi}_q^t\ row\ \lambda_\nu^b\ column)$

$$T^*T_{q\nu}^{\ b} = (-\frac{i}{2}\bar{\sigma}_{\nu\rho}T_a^t(q_c^\dagger)^t)^+ D^\rho + i\,\not{D}T_a^t\bar{\sigma}_\nu(q_c^\dagger)^t$$

$(\lambda_\mu^c\ row\ \bar{\psi}_{\bar{q}}\ column)$

$$T^*T_{\mu\bar{q}}^{c} = i(Tr\ T_c D^\rho T_a)(q_c^\dagger \bar{\sigma}_{\rho\mu} T_a)^+ - i q_c^\dagger T_c \sigma_\mu\ \not{D}$$

$(\bar{\psi}_{\bar{q}}\ row\ \bar{\psi}_{\bar{q}}\ column)$

$$
\begin{aligned}
T^*T_{\tilde{q}\tilde{q}} &= (T_a\bar{\sigma}_{\mu\nu}q_c)^+(q_c^\dagger\bar{\sigma}^{\mu\nu}T_a)^+ + \not{D}^2 \\
&\quad (\bar{\psi}_q \ row \ \psi_{\tilde{q}} \ column) \\
T^*T_{q\tilde{q}} &= (T_a^t\bar{\sigma}_{\mu\nu}(q_c^\dagger)^t)^+(q_c^t\bar{\sigma}^{\mu\nu}T_a^t)^+ \\
&\quad (\lambda_\mu^c \ row \ \bar{\psi}_q^t \ column) \\
T^*T_{\mu q}^c &= -i(Tr\ T_cD^\rho T_a)(q_c^t\bar{\sigma}_{\rho\mu}T_a^t)^+ - i(q_c)^tT_c^t\bar{\sigma}_\mu \not{D} \\
&\quad (\bar{\psi}_{\tilde{q}} \ row \ \bar{\psi}_q^t \ column) \\
T^*T_{\tilde{q}q} &= -(T_a\sigma_{\mu\nu}q_c)^+(q_c^t\sigma^{\mu\nu}T_a^t)^+ \\
&\quad (\bar{\psi}_q^t \ row \ \bar{\psi}_q^t \ column) \\
T^*T_{qq} &= (\bar{\sigma}_{\mu\nu}T_a^t(q_c^\dagger)^t)^+(q_c^tT_a^t\bar{\sigma}^{\mu\nu})^+ + \not{D}^2. \qquad (3.77)
\end{aligned}
$$

When we compares this to the matrices of the section 4, we understand that differences are in only covariant derivative D_μ and $\tilde{\mathcal{D}}_\mu$. Note that matrix obtained from bosonic part is almost same as T^*T.

Chapter 4

Topological Symmetry Breaking on Einstein Manifolds

4.1 Introduction

In this chapter we use the relation between stretch of moduli space and singularities which are caused by Gribov zero mode. As it was mensioned in chapter 1, when dimension of moduli space is non-zero Fadeev-Popov determinant is zero. We use this singularities for Topological symmetry breaking.

There are few reports in which topological symmetry realize in our world [47]. Hence, the symmetry should be broken in order to have some connection with our world [25] [26] . In ref.[26], Zhao and Lee add a infinitesimal breaking term to a Lagrangian. They found, that if gauge conditions are well defined and there is no Gribov zero mode then topological symmetry is hold in the limit of zero breaking term, but when there is a zero mode, BRS symmetry, i.e. topological symmetry, is broken. (Fujikawa conjectured "BRS-symmetry could be broken as a consequence of the Gribov ambiguity"[27].) But the theory has such problems that physical meaning is lost after BRS symmetry breaking and divergence appear from Gribov zero modes. We construct a Witten type topological gravitational theory of dimension $n \geq 3$ that has breaking phase with the method of Zhao and Lee. Topological gravitational theory we treat is fixed by $R+\alpha=0$ for the conformal symmetry. In the Mathai-Quillen formalism, this fixing condition correspond to the case of the section $s = R+\alpha$. If we change the gauge condition as $\alpha \to 0$, then the theory become ill-defined by

Gribov zero modes on some manifolds. Gribov zero modes appear when Fadeev-Popov matrix has zero eigen values. In our theory, this zero eigenvalue equations are Einstein equations. Hence, topological symmetry is broken on only Einstein manifolds. Strictly speaking, solutions of the Einstein equations exist, then topological symmetry is broken and smaller symmetry, diffeomorphism-invariance, are left in this theory. Further, we solve above problems. We recover the physical meaning to define 2nd BRS operator that is constructed by the remaining symmetry, diffeomorphism-invariance. And one method of regularization to avoid divergence from zero modes is introduced in a general case. By using this regularization for gravitational theory, we get semiclassical Einstein gravitational theory in some case.
On the physical point of view, our purposes are to treat Einstein manifolds and to construct quantum gravity of which classical limit become Einstein gravity. When cosmological constant is zero, then scalar curvature R is zero on Einstein manifolds. If a gauge condition is $R = 0$, Gribov zero modes might appear on some manifolds and theory would be ill-defined in former theory [28] [29]. In our theory, we are able to make the theory well-defined and having broken phase of topological symmetry at $R = 0$. And this broken phase is interpreted as semi-classical gravity in a sense.
The symmetry breaking conditions are connected to the Yamabe conjecture. We can easily find topological restriction to scalar curvature changing by conformal mode.

This chapter is organized as follows. We review Zhao-Lee symmetry breaking theory[26] in a general case, in section 2. The core of this theory is to use singularity of Gribov zero modes. As we mention in chapter 2, this Gribov zero relate to Moduli space dimension. Usually we introduce observable with ghost number which is equal to the number of zero modes, but they do not do so. To avoid the zero modes, they added an infinitesimal breaking term to a Lagrangian. Even though the breaking term is infinitesimal, it influence physical amplitude. In section 3, we realize it in topological gravity. As same as some other Witten type topological field theories[31] , topological gravity [23][28][29] is constructed by BRS formalism . We fix the conformal symmetry by $R + \alpha = 0$, where R is scalar curvature and α is cosmological constant. Then if the Einstein equations and $R = \alpha = 0$ are simultaneously satisfied, topological symmetry is broken. These conditions connect to the Yamabe conjecture [32]. Topological symmetry breaking means BRS symmetry breaking i.e. physical structure lose its meaning, then. In section 4 , we define 2nd BRS transformation. So we redefine physical states, then physical states recover with 2nd BRS operator after topological symmetry breaking. In section 5, one method of reguralization for the zero modes is given in the general case including our gravitational case. In section 6, we discuss about mathematical meaning. The topological symmetry breaking conditions give some topological information to scalar curvature, which connect with the Yamabe conjecture here. This information is due to the fact that absence of Gribov zero modes become a sufficient condition for functional subspace be a infinite dimensional manifold. In last section, we mention some conclusions, difficulties and prospects.

4.2 General formalism

Zhao, Lee showed that topological symmetry can be broken by singularity of Gribov zero mode [26]. We review it and extend further to a general case in this section. Let $\phi_i(x)$ represent all fields which include unphysical fields like ghost fields. A total lagrangian $\mathcal{L}$ is represented by a classical Lagrangian $\mathcal{L}_{cl}$, BRS operator $\hat{\delta}$, and gauge fermions Ψ, as

$$\mathcal{L} = \mathcal{L}_{cl} + \hat{\delta}\Psi_{g.f} \tag{4.1}$$

Ψ is constructed with antighosts $\bar{c}^i$, N-L fields b_i, and gauge fixing functions s_i. Namely our gauge conditions are $s_i = 0$. In the words of chapter 2, s_i is a section. In general, it is possible to write $\Psi = i\bar{c}^i(\alpha b_i + s_i)$, where α is a gauge parameter. BRS transformation for $\bar{c}^i$ and b_i is defined by $\hat{\delta}\bar{c}_i = b_i, \hat{\delta}b_i = 0$ or $\hat{\delta}b_i = \delta_g\bar{c}_i$ where δ_g is a generator of gauge transformation which is used in chapter 2. Then we have

$$\mathcal{L} = \mathcal{L}_{cl} + ib^i(\alpha b_i + s_i) + i\bar{c}^i(\hat{\delta}\phi_j\frac{\delta}{\delta\phi_j})s_i. \tag{4.2}$$

We chose Landau gauge, $\alpha = 0$, for simplicity. We demand the total Lagrangian $\mathcal{L}$ has some BRS symmetry . And we assume functional integral major

$\mathcal{D}\phi_k$ and a physical observable O is invariant under BRS transformation. If there are Gribov zero modes, naive gauge fixing is not correct, so we need some regularization to avoid it. All gauge conditions s_i do not have to include Gribov zero modes for symmetry breaking, but for simple notation we put regularization terms $i\epsilon b^i f_i$ for each s_i into the Lagrangian ,

$$\mathcal{L}_\epsilon = \mathcal{L} + i\epsilon b^i f_i = \mathcal{L}_{cl} + ib^i(s_i + \epsilon f_i) + i\overline{c}^i(\hat{\delta}\phi_j \frac{\delta}{\delta\phi_j})s_i, \tag{4.3}$$

and demand that $i\epsilon b^i f_i$ is not invariant under BRS transformation. The vacuum expectation value of any observable O is defined by

$$\langle O\rangle_\epsilon \equiv \lim_{\epsilon\to 0}\int \mathcal{D}\phi_k \, Oe^{-\int dx^D \mathcal{L}_\epsilon}. \tag{4.4}$$

If there is no singularity, then $i\epsilon b^i f_i$ never influence,

$$\langle O\rangle_\epsilon = \langle O\rangle = \int \mathcal{D}\phi_k \, Oe^{-\int dx^D \mathcal{L}}. \tag{4.5}$$

In the following, I omit the index ϵ of $\langle O\rangle_\epsilon$ and $\lim_{\epsilon\to 0}$ for convention. Now we estimate the vacuum expectation value of BRS exact functional SO as,

$$\langle \hat{\delta}O\rangle = \int \mathcal{D}\phi_k \, \hat{\delta}Oe^{-\int dx^D \mathcal{L}_\epsilon} = \int \mathcal{D}\phi_k \, O\hat{\delta}(i\epsilon b^i f_i)e^{-\int dx^D \mathcal{L}_\epsilon}. \tag{4.6}$$

Since $\hat{\delta}O$ is BRS exact, we usually expect it vanishing in the limit as ϵ approaches zero. But if there are Gribov zero modes, $\langle\hat{\delta}O\rangle \neq 0$ is realized as follows. After b_i integration in(4.5), we get

$$\begin{aligned}\langle \hat{\delta}O\rangle \;=\; & -\epsilon\int \mathcal{D}\phi_k \;\; Oe^{-\int \mathcal{L}_{cl} + i\overline{c}^i(\hat{\delta}\phi_j \frac{\delta}{\delta\phi_j}s_i)} \\ & (\hat{\delta}\phi_j\frac{\delta}{\delta\phi_j}f_k)(\frac{\delta}{\delta s_k + \epsilon f_k}\prod_k \delta(s_k+\epsilon f_k)) \\ = & -\epsilon\int \mathcal{D}\phi_k \;\; Oe^{-\int dx^D \mathcal{L}_{cl}}\prod_k(\hat{\delta}\phi_j\frac{\delta}{\delta\phi_j}f_k) \\ & (\frac{\delta}{\delta s_l + \epsilon f_l}\prod_m \delta(s_m+\epsilon f_m))(i)^n(\hat{\delta}\phi_j\frac{\delta}{\delta\phi_j}s_l).\end{aligned} \tag{4.7}$$

Where we assume that the observable O does not contain b_i fields. The second equality in Eq.(4.7) was gotten by $\overline{c^i}$ integration, and ϕ_i represent all fields which were not still integrated. Here, it turn out that when $\frac{\delta s_i}{\delta \phi_j} = 0$ and $s_i = 0$ are simultaneously satisfied then $\langle \hat{\delta} O \rangle \neq 0$ from $\frac{\delta}{\delta s_l + \epsilon f_l} \prod_m \delta(s_m + \epsilon f_m)$. It is easily understood by that $\delta(x^2 + \epsilon)$ has strong divergence as the limit $\epsilon \to 0$ and derivative $\frac{\delta}{\delta s_l + \epsilon f_l}$ go up the power of divergence. This is an essence of symmetry breaking. To see this apparently, next we change some of ϕ_i to gauge functions $s_i + \epsilon f_i$, and carry out their integral by using $\delta(s_k + \epsilon f_k)$. We find

$$\langle \hat{\delta} O \rangle = i^n \epsilon \int \mathcal{D}\phi_k \prod_k \frac{\delta}{\delta f_k + \epsilon f_k} \{ \frac{1}{|Det \frac{\delta(s_i + \epsilon f_i)}{\delta \phi_j}|}$$

$$\prod_{l,m} (\hat{\delta}\phi_j \frac{\delta}{\delta \phi_j} f_l)(\hat{\delta}\phi_j \frac{\delta}{\delta \phi_j} F_m) O e^{-\int dx^D \mathcal{L}_{cl}} \} \mid_{s_i + \epsilon f_i = 0} . \quad (4.8)$$

If $b^i f_i$ did not break BRS symmetry i.e.$\hat{\delta}\phi_j \frac{\delta}{\delta \phi_j} f_i = 0$ or $\mathcal{D}^i{}_{j\epsilon}$ defined by $\mathcal{D}^i{}_{j\epsilon} = \frac{\delta(s_i + \epsilon f_i)}{\delta \phi_j}$ had no zero mode, then $\langle \hat{\delta} O \rangle = 0$ in the limit $\epsilon \to 0$. But now $b_i f^i$ breaks the BRS symmetry , and we assume that there are some $\mathcal{D}^i{}_j$ zero modes at $s_i = 0$. Then ϵ and $\mathcal{D}^i{}_{j\epsilon}$ cancel each other. We get some non-zero value $\langle \hat{\delta} O \rangle \neq 0$. This means that BRS symmetry is broken. Note that the condition that $\mathcal{D}^i{}_j$ have zero modes at $s_i = 0$ is equal to the $\mathcal{A}$ or $\mathcal{M}$ have non-zero dimension (see (2.21) in chapter 2).

For example,

$$s_i + \epsilon f_i \mid_{\phi_j = \phi_j^c + \Delta\phi_j^c} = 0 \quad s_i \mid_{\phi_j = \phi_j^c} = 0 \quad j = 1 \sim n \quad (4.9)$$

$$\mathcal{D}^i{}_j \mid_{\phi_j^c} = \frac{\delta(s_i)}{\delta \phi_j} \mid_{\phi_j^c} = 0 \quad j = 1 \sim n \quad (4.10)$$

for only one ϕ^c. Where n is a number of conditions s_j and index "i" is fixed. In this case, as we will see in section 3, $\prod_i \phi_j \frac{\delta s_i}{\delta \phi_j} \sim \Delta\phi^c \sim \epsilon^{\frac{1}{2}}$, and $\mathcal{D}^i{}_{j\epsilon}{}^{-1} \sim \Delta\phi^c \sim \epsilon^{-\frac{1}{2}}$ when $f_i(\phi^c) \neq 0$. Then the most divergent term of $\frac{\delta}{\delta F_k + \epsilon f_k}(Det \mathcal{D}^i{}_{j\epsilon}{}^{-1})$ is order ϵ^{-1} . After all we get the order of the $\langle \hat{\delta} O \rangle$ as,

$$\langle \hat{\delta} O \rangle \sim \frac{\epsilon}{\Delta\phi^{c2}} = 1 \quad (4.11)$$

We have seen some BRS symmetry is broken by Gribov zero modes in a general case (see for example ref.[27]). If we use this way for Witten type topological field theories

[23][31], then topological symmetry breaking may be realized.

There are some problems of this method. First, after BRS symmetry was broken, physical states lost their meaning. To solve this, we prepare 2nd-BRS operator for topological gravity in section 4. Second problem is whether partition function Z is finite or not. At first sight it will be divergent. But we will see that it isn't true.

$$\begin{aligned} Z &= \int \mathcal{D}\phi_k e^{-\int dx^D \mathcal{L}_\epsilon} \\ &= \int \mathcal{D}\phi_k e^{-\int dx^D \mathcal{L}_{cl}} \left(\frac{\delta(\phi_j - \phi_k^c - \Delta\phi_k^c)}{|Det\mathcal{D}^i{}_{j\epsilon}|} \prod_l S\phi_j \frac{\delta}{\delta\phi_j} F_l \right) \end{aligned} \tag{4.12}$$

Since $\mathcal{D}^i{}_{j\epsilon} \sim \prod_l \hat{\delta}\phi_j \frac{\delta}{\delta\phi_j} F_l \sim \epsilon^{\frac{1}{2}}$, they cancel each other. So, the partition function keep finite. Third, as we see in (4.9), amplitude of some observable is divergent. This fact demand the theory to be regularized. We will give one method of regularization in section 5.

4.3 Case of the Witten type topological gravity

We show here the theory of the previous section will be realized in the Witten type topological gravity [33][23]. We use Myers theory [29] that treat spin connection and vierbein as independent fields. Without this property, this theory is almost same as Myers-Periwal theory [28]. The quantum fields are considered on D-dim compact manifolds. In our theory, dimension of the manifold is not essential as far as dimension $D \geq 3$, but 4-dim case is quoted often for a simple example.

In these theories, there are BRS operator S and non nilpotent BRS-like operator $\hat{\delta}$ which is reduced local orthogonal-transformations and diffeomorphism from S. The S is defined as

$$\begin{aligned} Se_\mu^a &= -(\mathrm{W}^a{}_b + P^a{}_b)e_\mu^b + L_c e_\mu^a, \\ S\mathrm{W}^a{}_b &= L_c \mathrm{W}^a{}_b - \mathrm{W}^a{}_c \mathrm{W}^c{}_b - P^a{}_c \mathrm{W}^c{}_b - \mathrm{W}^a{}_c P^c{}_b - (L_\phi e_b^\mu)e_\mu^a - Q^a{}_b, \\ Sw_\mu{}^a{}_b &= \Lambda_\mu{}^a{}_b + \nabla_\mu P^a{}_b + c^\nu R_{\nu\mu}{}^a{}_b, \end{aligned}$$

$$
\begin{aligned}
S\Lambda_{\mu}{}^{a}{}_{b} &= L_c\Lambda_{\mu}{}^{a}{}_{b} - \nabla_{\mu}Q^{a}{}_{b} - \Lambda_{\mu}{}^{a}{}_{c}P^{c}{}_{b} - P^{a}{}_{c}\Lambda_{\mu}{}^{c}{}_{b} - \phi^{\nu}R_{\nu\mu}{}^{a}{}_{b}. \\
Sc^{\mu} &= c^{\nu}\partial_{nu}c^{\mu} + \phi^{\mu}. \\
S\phi^{\mu} &= L_c\phi^{\mu}, \\
SP^{a}{}_{b} &= -P^{a}{}_{c}P^{c}{}_{b} + Q^{a}{}_{b} - c^{\mu}\Lambda_{\mu}{}^{a}{}_{b} + \frac{1}{2}c^{\mu}c^{\nu}R_{\mu\nu}{}^{a}{}_{b}. \\
SQ^{a}{}_{b} &= L_cQ^{a}{}_{b} + Q^{a}{}_{c}P^{c}{}_{b} - P^{a}{}_{c}Q^{c}{}_{b} + \phi^{\mu}\Lambda_{\mu}{}^{a}{}_{b}, \\
Sx &= y + L_cx - \delta_Px. \\
Sy &= L_cy - \delta_Py - L_{\phi}x + \delta_Qx. \qquad (4.13)
\end{aligned}
$$

Where L_c, L_ϕ denotes the Lie derivative for a fermionic vector field c^μ and for a bosonic vector field ϕ^μ. Also, δ_P, δ_Q denote local orthogonal transformations by P and Q. ϕ^μ and $Q^a{}_b$ are second stage ghosts for ghosts c^μ and $P^a{}_b$ in the Batalin, Fradkin and Vilkovisky formalism [34]. And x and y stand for all antighosts and N-L fields.

Myers and Periwal induce $\hat{\delta} = S - L_c + \delta_P$ and then $\hat{\delta}$ cohomology represent physical states. After straight forward calculation, we get $\hat{\delta}^2 = L_\phi + \delta_Q$. Then, for any scalar functional h up to a total derivative,

$$Sh = \hat{\delta}h. \qquad (4.14)$$

We will show that this S symmetry is broken by the way of section 2. We fix the GL transformation up to diffeomorphism and local orthogonal transformations at the same conditions as Myers and Periwal [28][29], in 4-dim case.

$$
\begin{aligned}
R + \alpha &= 0 \qquad (4.15) \\
W^{+}{}_{abcd} &= 0 \qquad (4.16) \\
\nabla_{\mu}e^{a}_{\nu} - \nabla_{\nu}e^{a}_{\mu} &= 0 \qquad (4.17)
\end{aligned}
$$

While we choose the constraints to fix the redundant diffeomorphism and orthogonal transformations

$$\nabla^a t_{ab} - \frac{1}{2}\nabla_b t = 0\ ,\ r^a{}_b = 0, \qquad (4.18)$$

where $t_{ab} = \frac{1}{2}(w_{ab} + w_{ba})$, $t = tr(t_{ab})$, $r_{ab} = \frac{1}{2}(w_{ab} - w_{ba})$. These conditions are all covariant. So diffeomorphism and local orthogonal transformations are still unfixed.

In ref.[29], to fix these symmetries, they imposed harmonic condition $\partial_\mu(e\, e^\mu_a e^a_\nu) = 0$ and algebraic constraint $\tilde{e}^\mu_{[a} e_{b]\mu} = 0$ where $\tilde{e}^\mu_a$ is some fixed back ground tetrad. But now, we do not adopt these conditions and the reason will appear in the next section. So, we assume these symmetries were fixed by some appropriate conditions.

Let us adapt this topological gravity to the way of section 2. On the Landau gauge, we get delta functions after N-L fields integration. According to the previous section, some symmetry breaking term should be added. We take it $i\epsilon e\tau f(e^a_\mu, w_\mu{}^{ab})$, where f is some functional of e^a_μ and $w_\mu{}^{ab}$ that satisfy $Sf \neq 0$, and τ is an N-L field used to fix on condition (4.15). We are able to regard eq.(4.15) as a fixing condition for conformal mode. Additional gauge fixing Lagrangian for eq.(4.15) can be written with antighost ρ and its N-L fields $\tau = \hat{\delta}\rho$ as,

$$\begin{aligned} \mathcal{L}_\alpha &= Se\rho(R+\alpha) \\ &= \hat{\delta} e\rho(R+\alpha) \\ &= e\tau(R+\alpha) - \rho\hat{\delta}(e(R+\alpha)) \end{aligned} \tag{4.19}$$

The formula (4.14) was used for second equality of (4.19), and total divergence was ignored. Due to the additional symmetry breaking term, the delta function changes from $\delta(e(R+\alpha))$ to $\delta(e(R+\alpha+\epsilon f))$. So, in this case, symmetry breaking is only connected to the condition $R+\alpha = 0$. We abbreviate another conditions for redundant diffeomorphism and local orthogonal symmetry, like eq.(4.18), and GL symmetry without conformal mode , like eq.(4.16) and (4.17) to "(GL)". The total Lagrangian is written as below by using appropriate gauge fermions $x_1(diffeo.) + x_2(ortho.)$ for fixing the diffeomorphism and local orthogonal transformations.

$$\begin{aligned} \mathcal{L}_\epsilon &= \mathcal{L}_{cl} + \mathcal{L}_\alpha + \epsilon e\tau f + \hat{\delta} x_0(\ GL\ ) + Sx_1(diffeo.) + Sx_2(ortho.) \\ &= \mathcal{L}_{cl} + e\tau(R+\alpha+\epsilon f) + \rho\hat{\delta}(e(R+\alpha)) \\ &\quad + \hat{\delta} x_0(\ GL\ ) + Sx_1(diffeo.) + Sx_2(ortho.) \end{aligned} \tag{4.20}$$

where x_0, x_1 and x_2 are antighost fields and their tensor property is determined by gauge functions (GL), $(diffeo.)$ and $(ortho.)$. We get delta functions after N-L fields y integration

$$\delta(e(R+\alpha+\epsilon f)\prod\delta(\ GL\ )\delta(diffeo.)\delta(orth.) \tag{4.21}$$

For simplicity, all delta functions from $\hat{\delta}x_0(GL) + Sx_1(diffeo.) + Sx_2(ortho.)$ are denoted by $\prod\delta(\ GL\ )\delta(diffeo.)\delta(orth.)$, here. The number of these delta functions is the same number of components of e^a_μ and w^{ab}_μ, because topological symmetry permit to transform each components arbitrary. So, if $(\ GL\ ).(diffeo.)$ and $(orth.)$ contain only e^a_μ and w^{ab}_μ , we can rewrite (4.21) as

$$\begin{aligned}&\delta(e(R+\alpha+\epsilon f)\prod\delta(\ GL\ )\delta(diffeo.)\delta(orth.) \\ &= \mathcal{J}^{-1}\prod_{a,b,c,\mu,\nu}\delta(e^a_\mu - e^{a(c)}_\mu - \Delta e^{a(c)}_\mu)\delta(w^{bc}_\nu - w^{bc(c)}_\nu - \Delta w^{bc(c)}_\nu).\end{aligned} \tag{4.22}$$

Where $\mathcal{J}$ is Jacobian,

$$\mathcal{J} = \begin{vmatrix} \frac{\delta e(R+\alpha+\epsilon f)}{\delta e^a_\mu} & \frac{\delta(GL,diffeo.,orth.)}{\delta e^a_\mu} \\ \frac{\delta e(R+\alpha+\epsilon f)}{\delta w^{ab}_\mu} & \frac{\delta(GL,diffeo.,orth.)}{\delta w^{ab}_\mu}\end{vmatrix},$$

$e^{a(c)}_\mu$, and $w^{bc(c)}_\nu$ are solution i.e.

$$\begin{aligned} e(R+\alpha)|_{e^a_\mu = e^{a(c)}_\mu, w^{bc}_\nu = w^{bc(c)}_\nu} &= 0 \\ (GL, diffeo., ortho.)|_{e^a_\mu = e^{a(c)}_\mu, w^{bc}_\nu = w^{bc(c)}_\nu} &= 0,\end{aligned} \tag{4.23}$$

and $\Delta e^{a(c)}_\mu$ $\Delta w^{bc(c)}_\nu$ are variation from inducing ϵf. Note that they are depend on ϵ . As we saw in section 2, if the Jacobian $\mathcal{J}$ has zero modes, then BRS symmetry, i.e. topological symmetry, is broken.

Let us analyze these broken conditions further. We analyze the situation that each component of the first column of the Jacobian matrix vanishes.

$$\frac{\delta e(R+\alpha)}{\delta e^a_\mu}|_{e^{a(c)}_\mu} = e\, e^\mu_a(R+\alpha) + eR^\mu{}_a|_{e^{a(c)}_\mu} = 0. \tag{4.24}$$

$$\frac{\delta e(R+\alpha)}{\delta w^{ab}_\mu} = -6e\, e^\mu_{[a} e^\nu_l e^\lambda_{b]}(D_\nu e^l_\lambda) = 0 \tag{4.25}$$

Eq.(4.24) is Einstein equation with cosmological constant α. From eq.(4.23), eq.(4.24) become

$$R^a{}_\mu|_{e^{a(c)}_\mu} = 0. \tag{4.26}$$

So we conclude that only if $R = \alpha = 0$ and eq.(4.25) are satisfied, the topological symmetry is broken. In Calabi-Yau manifolds in 6-dim the conditions are satisfied, for example. But in many manifolds, they are not satisfied. This condition connects to Yamabe conjecture, and it will be discussed in section 6.

The equations (4.25) mean torsion free conditions and these are not contradictory to gauge conditions if we adopt (4.17).

When these conditions are satisfied, Jacobian is of order $\epsilon^{\frac{1}{2}}$. Indeed $e(R+\epsilon f)$ can be expanded around $e_\mu^{a(c)}$ as follows,

$$\begin{aligned} 0 &= e(R+\epsilon f)|_{e^{(c)}+\Delta e^{(c)},\, w^{(c)}+\Delta w^{(c)}} \qquad (4.27)\\ &= \epsilon e f + \epsilon \frac{\delta e f}{\delta e_\mu^a}\Delta e_\mu^{a(c)} + \epsilon \frac{\delta e f}{\delta w_\mu^{ab}}\Delta w_\mu^{ab(c)} + \frac{\delta^2 e(R+\epsilon f)}{\delta e_\mu^a \delta e_\nu^b}\Delta e_\mu^{a(c)}\Delta e_\nu^{b(c)} \\ &+ \frac{\delta^2 e(R+\epsilon f)}{\delta e_\mu^a \delta w_\nu^{bc}}\Delta e_\mu^{a(c)}\Delta w_\nu^{bc(c)} + \frac{\delta^2 e(R+\epsilon f)}{\delta w_\mu^{ab} \delta w_\nu^{cd}}\Delta w_\mu^{ab(c)}\Delta w_\nu^{cd(c)} + O(\Delta e^3). \end{aligned}$$

To leading order in ϵ we have ,

$$\begin{aligned} \epsilon &= \frac{\delta^2\, eR}{\delta e_\mu^a \delta e_\nu^b}(ef)^{-1}\Delta e_\mu^{a(c)}\Delta e_\nu^{b(c)} + \frac{\delta^2\, eR}{\delta e_\mu^a \delta w_\nu^{bc}}(ef)^{-1}\Delta e_\mu^{a(c)}\Delta w_\nu^{bc(c)} \\ &+ \frac{\delta^2\, eR}{\delta w_\mu^{ab} \delta w_\nu^{cd}}(ef)^{-1}\Delta w_\mu^{ab(c)}\Delta w_\nu^{cd(c)} \qquad (4.28) \end{aligned}$$

in $f|_{e_\mu^{a(c)}} \neq 0$ case. Hence, $\Delta e_\mu^{a(c)}$ and Δw_μ^{ab} should be order $\epsilon^{\frac{1}{2}}$.

We can estimate (4.24) as,

$$\begin{aligned} &\frac{\delta e(R+\epsilon f)}{\delta e_\mu^a}|_{e^{(c)}+\Delta e^{(c)},\, w^{(c)}+\Delta w^{(c)}} \qquad (4.29)\\ &= \epsilon \frac{\delta e f}{\delta e_\mu^a} + \frac{\delta^2 e(R+\epsilon f)}{\delta e_\mu^a \delta e_\nu^b}\Delta e_\nu^{b(c)} + \frac{\delta^2 e(R+\epsilon f)}{\delta e_\mu^a \delta w_\nu^{bc}}\Delta w_\nu^{bc(c)} + O(\Delta e^2) \end{aligned}$$

Now, we get $\frac{\delta e(R+\epsilon f)}{\delta e_\mu^a} \sim \epsilon^{\frac{1}{2}}$,and similarly $\frac{\delta e(R+\epsilon f)}{\delta w_\mu^{ab}} \sim \epsilon^{\frac{1}{2}}$. From the estimation described above, it is concluded that Jacobian $\mathcal{J}$ is order $\epsilon^{\frac{1}{2}}$ when α is 0 and a solution of the eq.(4.26), $R_\mu^a = 0$, exist without contradiction to gauge conditions, then topological symmetry is broken. Note that the singularity from the Gribov zero modes, Jacobian matrix zero eigen values, is contribution from only $\delta(e(R+\epsilon f))$. Even if we did not use Jacobian $\mathcal{J}$ to estimate the singularity, we could find that the symmetry breaking occur by only this delta function.

In this section, we have studied only about vacuum condensation in topological gravity by the method of section 2. We found that the topological symmetry breaking was appeared in the process of conformal changing $R \to 0$ if Ricci flat (4.26) is realized on the background manifold. Of course, for this theory being well defined as physical theory, some other BRS symmetry should be present. This is a subject of the next section.

4.4 Two-BRS formalism

In section 2, we saw topological symmetry was broken at $R = 0$. However this means that BRS quantization is ill-defined. To clear this problem, we introduce another BRS transformation. L_c operates as diffeomorphism to all fields except anti ghosts. Anti ghost fields were transformed as scalar fields, regardless of those tensor property. As a result of this, in our Lagrangian $\mathcal{L} = \mathcal{L}_{cl} + \hat{\delta}\Psi + Sx_1(diffeo.) + Sx_2(ortho.)$ where $\hat{\delta}\Psi = \mathcal{L}_\alpha + \hat{\delta}x_0(GL)$, $\mathcal{L}_{cl}$ and $\hat{\delta}$ exact gauge fixing term $\hat{\delta}\Psi_{g.f.}$ are L_c invariant up to total divergence, because these terms are scalar, ref.[35][28][30]. So, if we can chose $Sx_1(diffeo.) + Sx_2(ortho.)$ to be invariant under L_c and redefine L_c to be nilpotent, then we adopt L_c as new BRS operator and physical states can be redefined with it.

Now one defines appropriate anti ghosts x and Lagrange multipliers y , where

$$Sx_i = y_i \quad Sy_i = 0 \quad i = 1, 2. \tag{4.30}$$

The tensor properties of these x_i and y_i are determined by $(diffeo.)$ and $(ortho.)$. One can require $(diffeo.)$ is scalar under local orthogonal transformations but it has no general covariant property, and $(ortho.)$ is scalar under general coordinate transformations. Under this choice, $Sx_1(diffeo.) + Sx_2(ortho.)$ is

$$\begin{aligned}
&Sx_1(diffeo.) + Sx_2(ortho.)\\
&= (L_c + \hat{\delta})x_1(diffeo.) + (\delta_P + \hat{\delta})x_2(ortho.)\\
&= y_1(diffeo.) + (-)^{|x_1|}x_1(L_c + \hat{\delta})(diffeo.)\\
&\quad + y_2(ortho.) + (-)^{|x_2|}x_2(\delta_P + \hat{\delta})(ortho.)
\end{aligned} \tag{4.31}$$

If we want to regard L_c (or δ_P) as a new BRS operator, it is seen in (4.31) that gauge fermions $(diffeo.)$ (or $(ortho.)$) should be $\hat{\delta}$ cohomology, and $L_c x_i = y_i$ (or $\delta_P x_i = y_i$). But $\hat{\delta}$ cohomological gauge conditions are not known at least to us. Only we know $\hat{\delta}\phi_\mu = 0$ where ϕ_μ was induced as ghost for ghost c_μ in (4.13), ref.[28]. So we adopt some functional of only ϕ_μ for gauge conditions $G_{\widetilde{\mu\nu}}(\phi_\mu) = 0$, where tileder means $G_{\widetilde{\mu\nu}}$ is not general covariant. Note that $\hat{\delta}G_{\widetilde{\mu\nu}} = \frac{\delta}{\delta\phi_\mu}G_{\widetilde{\mu\nu}}\hat{\delta}\phi_\mu = 0$, and $\delta_P G_{\widetilde{\mu\nu}} = 0$ since $G_{\widetilde{\mu\nu}}$ has no local coordinate index. To fix the diffeomorphism, we add to Lagrangian with anti ghost $\bar{c}^{\mu\nu}$ and Lagrange multiplier $b^{\mu\nu} = S\bar{c}^{\mu\nu}$

$$Se\,\bar{c}^{\mu\nu}G_{\widetilde{\mu\nu}} = et\bar{c}^{\mu\nu}G_{\widetilde{\mu\nu}} + (L_c e)\bar{c}^{\mu\nu}G_{\widetilde{\mu\nu}} + etb^{\mu\nu}G_{\widetilde{\mu\nu}} + et\bar{c}^{\mu\nu}L_c G_{\widetilde{\mu\nu}} \tag{4.32}$$

Next step, we fix the local orthogonal symmetry. Myers-Periwal fixed it at $\tilde{e}^\mu_{[a}e_{b]\mu} = 0$, where $\tilde{e}^\mu_a$ is some back ground tetrad. This condition is not suitable for our purpose. Because, under L_c, $\tilde{e}^\mu_a$ do not transform , so $\tilde{e}^\mu_{[a}e_{b]\mu}$ is not invariant. i.e. $L_c e\bar{P}^{ab}\tilde{e}^\mu_{[a}e_{b]\mu} \neq 0$, where $\bar{P}^{ab}$is anti ghost. For this reason, another condition that include no back ground field, should be induced, here. For example, we fix it at $\nabla_\mu w^\mu{}_{ab} = 0$ [36]. The gauge fixing terms,

$$Se\bar{P}^{ab}\nabla_\mu w^\mu{}_{ab} = et\bar{P}^{ab}\nabla_\mu w^\mu{}_{ab} + eq^{ab}\nabla_\mu w^\mu{}_{ab} - e\bar{P}^{ab}(\hat{\delta} + \delta_p)\nabla_\mu w^\mu{}_{ab}, \tag{4.33}$$

where q^{ab} is a Lagrange multiplier, $q^{ab} = S\bar{P}^{ab}$, is added to Lagrangian.
Everything is ready, for introducing new BRS symmetry. Let us define a new fermionic Lie derivative L'_c for some BRS operator.

definition 1 (new BRS operator L'_c) *The new BRS operator is defined as follows.*

$$L'_c = L_c \tag{4.34}$$

for all fields $e^a_\mu, w^a{}_b, w^{ab}_\mu, \ldots$ *except* $\bar{c}_{\mu\nu}$, $b_{\mu\nu}$

$$L'_c\bar{c}_{\mu\nu} = t\bar{c}_{\mu\nu} + b_{\mu\nu} \tag{4.35}$$

$$L'_c b_{\mu\nu} = tb_{\mu\nu} - (L_c t)\bar{c}_{\mu\nu} \tag{4.36}$$

for $\bar{c}_{\mu\nu}$, $b_{\mu\nu}$.

Here L'_c is nilpotent. From (4.34), $\mathcal{L}_{cl}$, $\hat{\delta}\Psi$ and $Se\bar{P}^{ab}\nabla_\mu w^\mu{}_{ab}$ are transformed as scalar by L'_c, that is

$$\int d^D x L'_c(\mathcal{L}_{cl} + \hat{\delta}\Psi + Se\bar{P}^{ab}\nabla_\mu w^\mu{}_{ab}) = 0. \tag{4.37}$$

And by using (4.36),

$$Se\,\bar{c}^{\mu\nu}G_{\widetilde{\mu\nu}} = L'_c e\,\bar{c}^{\mu\nu}G_{\widetilde{\mu\nu}}, \tag{4.38}$$

we get

$$L'_c\, Se\,\bar{c}^{\mu\nu}G_{\widetilde{\mu\nu}} = (L'_c)^2 e\,\bar{c}^{\mu\nu}G_{\widetilde{\mu\nu}} = 0, \tag{4.39}$$

from nilpotency of L'_c. Our total action is rewritten with L'_c, as

$$\int d^D x\,(\mathcal{L}_{cl} + \hat{\delta}\Psi + Se\bar{P}^{ab}\nabla_\mu w^\mu{}_{ab}) + L'_c e\,\bar{c}^{\mu\nu}G_{\widetilde{\mu\nu}} \tag{4.40}$$

and it is invariant under nilpotent operator L'_c as we saw in (4.37) and (4.39). Now it is possible to regard L'_c as a new BRS operator, and $\mathcal{L}_{cl} + \hat{\delta}\Psi + Se\bar{P}^{ab}\nabla_\mu w^\mu{}_{ab}$ is a new classical Lagrangian. In this form, $t^a{}_b$, ϕ_μ and others except $\bar{c}^{\mu\nu}$ and $b^{\mu\nu}$ become physical fields in addition to e^a_μ and w^{ab}_μ, as a result of changing the physical states conditions from $S|phys>=0$ to $L'_c|phys>=0$.

Note that using ϕ_μ to fix diffeomorphism disturbs rewriting δ functions with Jacobian like the previous section. Because the number of δ functions of e^a_μ, w^{ab}_μ is less than one of independent components of fields. But singularity of the $\delta(eR)$ is not changed. It is apparent that topological symmetry is broken.

The symmetry breaking condition is in fact the Einstein equation $R_{ab} = 0$. The solution of a classical Einstein equation exists and it contributes to the path integral , hence the theory of broken topological symmetry is realized in the above. In other words, non-topological phase is defined around classical gravity.

4.5 Regularization

In the previous section, we got the new BRS operator to define physical states again. It means that the vacuum expectation value of some BRS exact operator is zero.

But our theory is still singular due to $\delta(e(R+\epsilon f))$ in (4.21). So we have to remove this divergence. In this section, we make the prescription to regularize this divergence.

First, we discuss reguralization in general formalism. Here, we use same symbols as ones of section 2. There is strong divergence in $\delta(s_i+\epsilon f_i)$ in eq.(4.7) as $\epsilon \to 0$ when $\mathcal{D}^i{}_j(\phi^c)=0$. This delta function appeared as a result of b_i integration, where b_i is a Lagrange multiplier of the gauge function s_i. So it is evident that more stronger divergence will appear if an observable contains b_i fields. To get finite vacuum expectation value, we redefine b_i fields as

$$b'_i = \epsilon^{\frac{1}{2}} b_i \tag{4.41}$$

in $f_i(\phi^c) \neq 0$ case when

$$\frac{\delta s_i}{\delta \phi_j}|_{\phi^c} = \mathcal{D}^i{}_j(\phi^c) = 0 \quad \frac{\delta s_k}{\delta \phi_j}|_{\phi^c} \neq 0 \quad : k \neq i \ , \ j = 1 \sim n \tag{4.42}$$

are satisfied for some ϕ^c. A fixed index "i" means that functional derivative of s_i on some ϕ^c vanish like eq.(4.42) in the following. ϕ^c, i.e. solutions of $s_j=0$, do not always satisfy eq.(4.42), then we put ϕ^z and $\bar{\phi^z}$ as

$$\phi^z \in \{\phi^c \mid \mathcal{D}^i{}_j(\phi^c)=0\} \quad \bar{\phi^z} \in \{\phi^c \mid \mathcal{D}^i{}_j(\phi^c) \neq 0\} \tag{4.43}$$

In other words, ϕ^z is a kernel of $\mathcal{D}^i{}_j$. By using this b'_i, we rewrite the gauge fixing term in the Lagrangian (4.3) as

$$b^i(s_i+\epsilon f_i) \to \epsilon^{-\frac{1}{2}} b'^i(s_i+\epsilon f_i). \tag{4.44}$$

Then the delta function is changed as

$$\delta(s_i+\epsilon f_i) \to \delta(\epsilon^{-\frac{1}{2}}(s_i+\epsilon f_i)) = \epsilon^{\frac{1}{2}}\delta(s_i+\epsilon f_i) \tag{4.45}$$

Due to eq.(4.45), delta functions are order 1 for ϕ^z, but for $\phi^c=\bar{\phi^z}$ they are order $\epsilon^{\frac{1}{2}}$. In the count of the term $\prod_m \delta\phi_j \frac{\delta}{\delta\phi_j} F_m$ as similar in eq.(4.7) all amplitude of observables is order $\epsilon^{\frac{1}{2}}$ whether there are zero modes or not, as it is. So, also $\bar{c}^i$ fields have to be redefined as

$$\bar{c'}_i = \epsilon^{\frac{1}{2}} \bar{c}_i \tag{4.46}$$

and the Fadeev Popov terms in the Lagrangian change as

$$c^i\hat{\delta}\phi_j\frac{\delta}{\delta\phi_j}s_i = c'^i(\epsilon^{-\frac{1}{2}}\hat{\delta}\phi_j\frac{\delta}{\delta\phi_j}s_i). \tag{4.47}$$

As a result of these redefinition, order of $\epsilon^{\frac{1}{2}}\delta(s_i+\epsilon f_i)$ and $\prod_m \hat{\delta}\phi_j\frac{\delta}{\delta\phi_j}s_m$ are found as

$$\epsilon^{\frac{1}{2}}\delta(s_i+\epsilon f_i) \sim \begin{cases} 1 \ : \ \text{for} \ \phi^z \\ \epsilon^{\frac{1}{2}} \ : \ \text{for} \ \bar{\phi}^z \end{cases} \tag{4.48}$$

$$\epsilon^{-\frac{1}{2}}\hat{\delta}\phi_j\frac{\delta}{\delta\phi_j}s_i \sim \begin{cases} 1 \ : \ \text{for} \ \phi^z \\ \epsilon^{-\frac{1}{2}} \ : \ \text{for} \ \bar{\phi}^z. \end{cases} \tag{4.49}$$

Before these redefinitions, if an observable contains b_i fields, then singularity is stronger. Indeed from the same reason, in section 2, $\langle\hat{\delta}O\rangle$ was non zero, and yet partition function was finite. It is easy to understand by a following formulation of a delta function

$$\lim_{\epsilon\to 0}\int db\ be^{ib(x^2-\epsilon^2)} = \lim_{\epsilon\to 0}\frac{1}{2x}\frac{\partial}{\partial x}\left(\frac{\delta(x-\epsilon)+\delta(x+\epsilon)}{|2x|}\right). \tag{4.50}$$

Where existence of b induced a derivative and power of divergence went up. But now, because of redefinition (4.41) and (4.46), when an observable contains b'_i fields, i.e. $O = b'_iO^i$, then the vacuum expectation value of O is

$$\langle b'_iO^i\rangle = \int \mathcal{D}\phi_k\ O^i\epsilon^{\frac{1}{2}}\left(\frac{\partial\ \ \delta(s^i+\epsilon f^i)}{\partial(s^i+\epsilon f^i)}\right)\prod_{k\neq i}\delta(s_k)\prod_m\hat{\delta}\phi_j\frac{\delta}{\delta\phi_j}s_m\ e^{\int dx^D\mathcal{L}_{cl}}. \tag{4.51}$$

Where the power of divergence is unchanged for $\phi^c = \bar{\phi}^z$ and the power of ϵ go up for non zero mode $\phi^c \neq \phi^z$. So that, contribution for amplitude is from only Gribov zero modes ϕ^z, and other contribution from $\bar{\phi}^z$ vanish. After all, the amplitude (4.51) is sum over ϕ^z and it is order one because

$$\frac{\partial\ \ \delta(s^i+\epsilon f^i)}{\partial(s^i+\epsilon f^i)} \sim \epsilon^{-1} + \ \ less\ \ divergence. \tag{4.52}$$

Hence, we have done the regularization for all observable in general case. Especially there is remarkable property that if an observable contains Lagrange multipliers of s_i then contribution from ϕ_i path integration to the amplitude of this observable is only from ϕ^z. Next we try this regularization in topological gravity case.

We carry out this regularization in the gravitational theory in the same way as the general case. The only things we have to do is to redefine ρ and τ as follows.

$$\rho' = \epsilon^{\frac{1}{2}}\rho \qquad \tau' = \epsilon^{\frac{1}{2}}\tau \tag{4.53}$$

Then all amplitude is regularized. Note that, if an observable contains τ', its vacuum expectation value is the sum of solution of $R^a_\mu = 0$. This fact is very interesting. In the theory of section 4, we define observables as L'_c closed. If we change this definition to L'_c closed and containing τ' fields,

$$Z = \int \mathcal{D}\phi_k \tau' e^{\int dx^D \mathcal{L}_\epsilon} \ : \ \text{partitionfunction} \tag{4.54}$$

$$L'_c O = 0 \ \text{and} \ O = \tau' O' \ : \ \text{definition of observable.} \tag{4.55}$$

Then the theory is semiclassical, i.e.path integral contribution for vacuum expectation value is from only solution of the Einstein equation. Note that our theory have many constraints for fixing topological symmetry like eq.(4.16) and (4.17). So, after symmetry breaking, these constraints are left as equations of motion. In this meaning, sence of semiclassical gravity is different from usual case.

4.6 Mathematical interpretation

As we saw in section 3, our theory has broken phase on the condition $R_{ab} = 0$. Let us clarify mathematical meanings of this.

For simplicity, we omit the symmetry breaking term $\epsilon e \tau f$ in this section. As is mentioned in section 3, the Yamabe conjecture is concerned with our theory. We are going to see this fact, as follows. In topological gravity, it is trivial that any physical amplitude is invariant under changing α because gauge condition does not affect physical amplitude. Indeed, a derivative of the partition function with respect α is given by,

$$\frac{\partial}{\partial \alpha} Z = \int \mathcal{D}\phi_k \, (\hat{\delta} i e \rho) e^{\int dx^D \mathcal{L}} = 0 \tag{4.56}$$

To get the second equality, we use that $\hat{\delta}$ exact vacuum expectation value vanishes as same as S exact one, ref. [28] [30]. This means that variation of scalar curvature is not restricted by topology. Strictly speaking, our topological gravity may not classify the topology of manifolds perfectly, so we can only say that scalar curvature can be varied without changing class which is classified by our topological theory, as far as the theory is well defined. But our theory is broken at $R_{ab} = R = 0$ as we saw in section 3. From a mathematical viewpoint, gauge conditions restrict back ground manifolds to submanifolds. We identify this manifolds as moduli space $\mathcal{M}$ in chapter 2. This fact is mentioned several times in chapter 2 and section 2

Yamabe conjectured constant scalar curvature R exist on any compact Rieman manifolds with arbitrary topology of dimension $n \geq 3$ [32]. But it has been corrected by Aubin [39] , Schoen [40] and so on. Especially, Kazdan and Warner [41] gave the following theorem that

Theorem 1 (Kazdan Warner theorem) *Compact manifolds M of dimension $n \geq 3$ can be divided into three classes,*

(A) Any (C^∞)function on M is the scalar curvature of some (C^∞)metric.

(B) A function on M is the scalar curvature of some metric if and only if it is either identically zero or strictly negative somewhere, further more, any metric with vanishing scalar curvature is Ricci-flat.

(C) A function on M is a scalar curvature if and only if it is strictly negative somewhere.

This theorem says that existence of negative scalar curvature do not demand any topological condition. And there is a barrier at $R = 0$. This is consistent with our theory. We are able to classify the type (C) manifolds from the type (A) and (B) manifolds, in our theory. Let us take $R = -\alpha < 0$ first, and makes α to zero. On the type (A)(B) manifolds, topological symmetry is broken , on the other hand, on the type (C) it's not broken. In other words, our theory may classify manifolds to type (C) and other type, by calculating some vacuum expectation value of $\hat{\delta}$ exact terms on $R = 0$. If it vanish, the back ground manifold is type(C), and if it's not zero, then the manifold is type (A)or(B).

Note that in Myers and Periwal [28] observables are topological on $\alpha \neq 0$. On the other hand, they are topological-like but are non topological in a strict sense in our theory. On type (C), they are independent of metric, but on type(A) or (B) they are non topological.

4.7 Conclusion and discussion

We have constructed a topological-like gravitational theory. Its feature is that the topological symmetry is broken when gauge condition chose $R = 0$ and the Einstein equation $R_{ab} = 0$ has a solution on the back ground manifolds.

Now, the question flowing up naturally is how matter couples. For example, one change gauge condition to $R + \tilde{S}_{matter}(\Psi, e^a_\mu) = 0$, where $\tilde{S}_{matter}$ is matter action of background fields ($L_c\tilde{\Psi} = S\tilde{\Psi} = 0$). Then the condition of topological symmetry breaking is the Einstein equation with matter,

$$\frac{\delta e(R + \tilde{S}_{matter})}{\delta e^a_\mu} = e\, e^a_\mu(R + \tilde{S}_{matter}) + e(R^a{}_\mu - \tilde{T}^a_\mu) = 0 \qquad (4.57)$$

and the torsion equation ,

$$\frac{\delta e(R + \tilde{S}_{matter})}{\delta w^{ab}_\mu} = -6e\, e^\mu_{[a} e^\nu_l e^\lambda_{b]}(D_\nu e^l_\lambda) - \tilde{S}^\mu_{ab} = 0 \qquad (4.58)$$

,where $\tilde{T}^a_\mu = -\frac{\delta \tilde{S}_{matter}}{\delta e^\mu_a}$ is a energy momentum tensor and $\tilde{S}^\mu_{ab} = \frac{\delta \tilde{S}_{matter}}{\delta w^{ab}_\mu}$ is a spin density. To satisfy these (4.57) and (4.58), we have to change the gauge conditions of the torsion free condition, eq.(4.15), and instead of eq.(4.14) $R_{ab} = 0$, we need

$$R^a{}_\mu - \tilde{T}^a_\mu = 0 \qquad (4.59)$$

Since we have fixed $R + \tilde{S}_{matter}(\tilde{\Psi}, e^a_\mu) = 0$, then the following equation is necessary for symmetry breaking,

$$\tilde{S}_{matter} = \tilde{T}^a_\mu e^\mu_a = tr(\tilde{T}^a_\mu). \qquad (4.60)$$

For example, Dirac field $\tilde{S}_{matter} = \bar{\tilde{\Psi}}\gamma^\mu\nabla_\mu\tilde{\Psi}$ satisfy this condition. Then the topological symmetry breaking occurs depending upon matter fields. We may construct the theory that break topological symmetry by dynamics of matter fields.

This study has constructed topological-like field theory that has broken phase when the solution of the Einstein equations exist. This symmetry breaking is caused by Gribov zero modes, in other words by zero eigen values of Jacobian matrix, that appear as solutions of the Einstein equations. And we found that if one can take gauge fermion cohomological of reduced BRS operator $\hat{\delta}$, then we can induce new BRS operator by the reduced symmetry. In our case, we got the $L_c{}'$ as a new BRS operator and only symmetry of diffeomorphism is left. To take away the divergence from zero modes, we gave one method of regularization. Hence we could extend the topological gravity which was fixed on $R + \alpha = 0$ ($\alpha > 0$) for conformal symmetry to the topological-like gravity on $R = 0$. It has nontrivial broken phase on some manifolds, and especially when we chose the theory as eq.(4.54) and (4.55) then the theory describe semiclassical Einstein gravity. Of course, this theory dose not described the real gravity perfectly, as it is. But the property that breaking topological symmetry depend on back ground manifolds encourages us to apply our theory to other theories. For example, we will have to examine the same methods to Weyl gravity. In our theory, only the conformal symmetry was needed to break the BRS symmetry. So, we might carry out the same methods easily in the Weyl gravity without the problem of the regularization for UV-divergence. We might construct quantum gravitational theory that have phase of more real semiclassical Einstein gravity. While the same phenomena will be realized in other theory as well, for example in topological Yang-Mills theory. (Baulieu and Schaden have studied this problem with a different method from ours [42].)

In this formalism, the gauge condition directly reflect broken phase physics. There could be some criticisms. First, the way of breaking has many ambiguities of selecting breaking terms. Second, it is unnatural that we have to adopt $\hat{\delta}$ cohomological gauge fermion for inducing the 2nd BRS operator. But it may be interpreted as follows. If many BRS theory is found in real world, as we had seen in section 3, it's gauge condition will be restricted as cohomology of reduced BRS. It implies that internal space, i.e.gauge space, have some mechanism or kinematics. It may be that as a result of it, gauge condition is non free from physics. Further, ghost fields become physical fields after symmetry breaking. We have to adjust and interpretate these new physical fields to real world.

Chapter 5

Summary

Summary

We have studied singularities in Witten type TFT. In the field theoretical view point, these singularities are understood by Gribov zero mode. But in the geometrical view point, they are separated into two types. The first type is caused by fixed point of gauge transformation. If gauge transformation is effective then the moduli space are smooth, but if there are fixed points then singularities appear. The second type singularities when the solution space (or moduli space) have non-zero dimension. The origin of this singularities are not in geometry but in the manner of field theory. The dimmension of fixed point locus (or moduli space) is equal to the number of Gribov zero mode. Gribov zero modes may cause singularities. This singularities are avoidable in general but we used them to break the topological symmetry. We could clarify the difference of two types of singularity by Mathai-Quillen formalism. It is possible to adapt this knowledge to understand the singularities of any Witten type TFT.

First type of singularities were used to investigate the topological invariants on 4-manifolds with Topological QCD. By giving perturbation we studied the behavior of Topological QCD around the moduli space singularities. This moduli space singularities occur when there are reducible connections. This made us possible to separate Abelian Seiberg-Witten part from Topological QCD. Then the relation between vacuum expectationvalue of Topological QCD , Donaldson invariants, non-Abelian Seiberg-Witten invariants and Seiberg-Witten invariants were obtained. Furthermore we got some identities of Abelian Seiberg-Witten invariants. These identities were obtained from the behavior around the singularities with the method of Ward-Takahashi identities.

Next type of singularities were used to break the topological symmetry. Especially we did it in Witten type topological gravity. Topological symetry breaking is equivarent to BRS symmetry breaking in Witten type TFT. Therefore we had to introduce 2nd BRS operator to redefine physical states. These procedures were carried out with the section R, which is understood as gauge fixing condition $R = 0$ for conformal mode. Then we found that the necessary condition for topological symmetry breaking to occur is the existence of Einstein metric. Especially we got the semiclassical gravitational theory after symmetry breaking.

As we found in this paper, nature of singularities in TFT is very useful. The applications of the TFT have been left for future works. They are not only to study the mathematical subject. Physical theory may be given by low energy limit of TFT after topological symmetry breaking. For example, there is a report that F-theory action is written as topological Matrix theory [47]. We expect that study of singularities in TFT play more essential role in the future.

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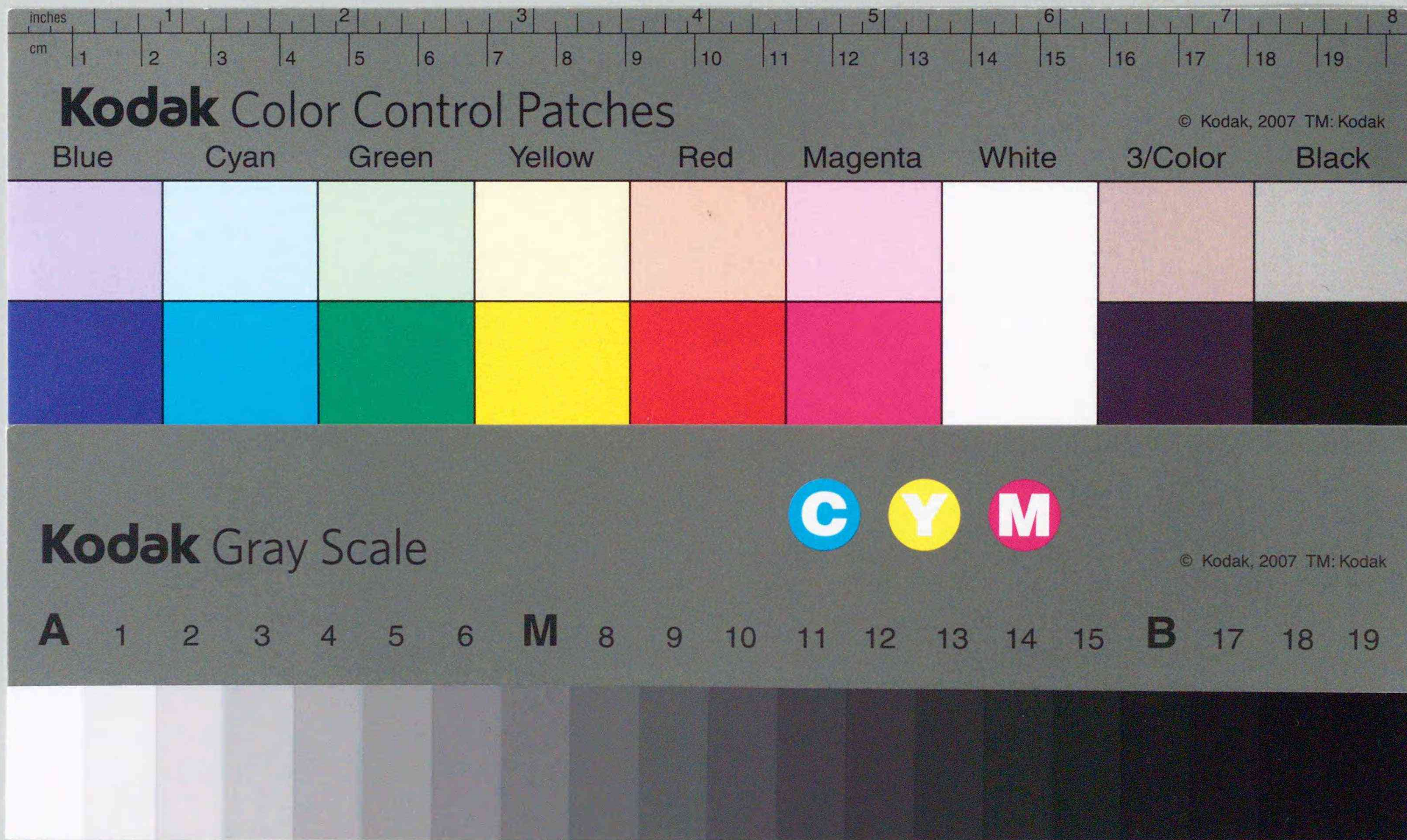
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