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学　位　論　文

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ABSTRACT. We prove Abel-Tauber theorems for Hankel and Fourier transforms. For example, let f be a locally integrable function on $[0, \infty)$ which is eventually decreasing to zero at infinity. Let $\rho = 3, 5, 7, \cdots$ and ℓ be slowly varying at infinity. We characterize the asymptotic behavior $f(t) \sim \ell(t)t^{-\rho}$ as $t \to \infty$ in terms of the Fourier cosine transform of f. Similar results for sine and Hankel transforms are also obtained. As an application, we can give an answer to a problem of R. P. Boas on Fourier series.

1. Introduction and results

As a prototype, we use Fourier cosine transforms to explain our problem. Let f be a locally integrable, eventually decreasing function on $[0, \infty)$ which tends to zero at infinity, and let F_c be its Fourier cosine transform. Let $\rho > 0$ and ℓ be slowly varying at infinity (see below). We are concerned with Abel-Tauber theorems which characterize the asymptotic behavior $f(t) \sim \ell(t)t^{-\rho}$ as $t \to \infty$ in terms of F_c. It turns out that the values $1, 3, 5, \cdots$ of ρ are exceptional. For $\rho \neq 1, 3, 5, \cdots$, one can obtain the desired Abel-Tauber theorems using regular variation — or Karamata theory. See Bingham-Goldie-Teugels [BGT, Ch. 4], where references to earlier work by Hardy and Rogosinski, Aljančić, Bojanić and Tomić, Vuilleumier, Zygmund and others are given. However the same theorems do not hold for $\rho = 1, 3, 5, \cdots$. These exceptional values are related to the power series expansion of the kernel $\cos x$ (see Soni-Soni [SS]).

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In [I1], one of the authors showed that one could use Π-variation — or de Haan theory in the terminology of [BGT] — to obtain the desired Abel-Tauber theorem for cosine transforms when $\rho = 1$. For theorems of the same type, we refer to [I1] (cosine series and integrals), [I2] (sine series and integrals), [I3] (Fourier-Stieltjes coefficients), and Bingham-Inoue [BI] (Hankel transforms).

In this paper, we consider the remaining exceptional values, e.g., $\rho = 3, 5, \cdots$ for cosine transforms. In fact, as in [BI], we consider those for Hankel transforms from the beginning; the results for cosine and sine transforms follow as special cases. As an application, we can give an answer to a problem of R. P. Boas on Fourier series.

We write R_0 for the class of slowly varying functions at infinity, that is, the class of positive measurable ℓ, defined on some neighbourhood of infinity, satisfying

$$\ell(\lambda x)/\ell(x) \to 1 \quad (x \to \infty) \quad \forall \lambda > 0.$$

For $\ell \in R_0$, the class Π_ℓ is the class of measurable f satisfying

$$\{f(\lambda x) - f(x)\}/\ell(x) \to c \log \lambda \quad (x \to \infty) \quad \forall \lambda > 0$$

for some constant c, called the *ℓ-index* of f. See [BGT] for background.

Let $\nu \geq -1/2$, $t^{\nu+\frac{1}{2}}h(t) \in L^1_{\text{loc}}[0,\infty)$, and h be ultimately decreasing to zero at infinity. We consider the *Hankel Transform*

$$H_\nu(x) := \int_0^{\infty-} h(t)(xt)^{1/2} J_\nu(xt)dt \quad (0 < x < \infty), \tag{1.1}$$

where $\int_0^{\infty-}$ denotes an improper integral $\lim_{M\to\infty}\int_0^M$ and J_ν is the Bessel function

$$J_\nu(x) = \sum_{j=0}^{\infty} c_{\nu,j} x^{\nu+2j} \qquad (0 \leq x < \infty)$$

with

$$c_{\nu,j} := \frac{(-1)^j}{2^{\nu+2j} \cdot j! \cdot \Gamma(\nu+j+1)} \qquad (\nu \geq -1/2,\ j = 0, 1, \cdots). \tag{1.2}$$

Since the improper integral on the right of (1.1) converges uniformly on each (a, ∞) with $a > 0$, H_ν is finite and continuous on $(0, \infty)$.

For $n \in \mathbb{N}$ and $x \in (0, \infty)$, we define $\bar{H}_{\nu,n}$ by

$$\bar{H}_{\nu,n}(x) := x^{\nu+\frac{1}{2}+2n} \left\{ H_\nu(1/x) - \sum_{j=0}^{n-1} c_{\nu,j} \int_0^\infty t^{\nu+\frac{1}{2}+2j} h(t)dt \cdot x^{-\nu-\frac{1}{2}-2j} \right\} \tag{1.3}$$

if $\int_0^\infty t^{\nu-\frac{3}{2}+2n} h(t)dt < \infty$.

Theorem 1. *Let $\ell \in R_0$ and $n \in \mathbb{N}$. Let $\nu \geq -1/2$, $t^{\nu+\frac{1}{2}} h(t) \in L^1_{\text{loc}}[0, \infty)$, and h be ultimately decreasing to zero at infinity, with Hankel transform H_ν. Then*

$$h(t) \sim t^{-\nu-\frac{3}{2}-2n} \ell(t) \qquad (t \to \infty) \tag{1.4}$$

if and only if

$$\int_0^\infty t^{\nu-\frac{3}{2}+2n} h(t)dt < \infty \quad \text{and} \quad \bar{H}_{\nu,n} \in \Pi_\ell \ \text{with } \ell\text{-index } c_{\nu,n}. \tag{1.5}$$

Note that Theorem 1 includes results for Fourier cosine and sine transforms, as

$$x^{1/2} J_{-1/2}(x) = \sqrt{\frac{2}{\pi}} \cos x, \qquad x^{1/2} J_{1/2}(x) = \sqrt{\frac{2}{\pi}} \sin x.$$

For $x \in (0, \infty)$, we define $\bar{H}_\nu$ by

$$\bar{H}_\nu(x) := x^{\nu+\frac{1}{2}} H_\nu(1/x). \tag{1.6}$$

We will prove Theorem 1 by reducing the problem to the following known result (which corresponds to the case $n = 0$ of (1.4)):

Theorem A ([BI], extending [I1], [I2]). *Let ν, h, H_ν and ℓ be as in Theorem 1. Then*

$$h(t) \sim t^{-\nu-\frac{3}{2}}\ell(t) \qquad (t \to \infty) \tag{1.7}$$

if and only if

$$\bar{H}_\nu \in \Pi_\ell \ \textit{with } \ell\textit{-index } c_{\nu,0}. \tag{1.8}$$

The cosine case $\nu = -\frac{1}{2}$ of Theorem A is due to [I1], the sine case $\nu = \frac{1}{2}$ to [I2], and the general case $\nu \geq -\frac{1}{2}$ to Bingham-Inoue [BI].

The theorems above treat the boundary cases to the following known Abel-Tauber theorem for Hankel transforms:

Theorem B ([RS], [SS], extending [P], [B]). *Let ν, h, H_ν and ℓ be as in Theorem 1.*

(1) *For $0 < \rho < \nu + \frac{3}{2}$,*

$$h(t) \sim t^{-\rho}\ell(t) \qquad (t \to \infty) \tag{1.9}$$

if and only if

$$H_\nu(x) \sim x^{\rho-1}\ell(1/x) \cdot 2^{\frac{1}{2}-\rho}\frac{\Gamma(\frac{3}{4}+\frac{\nu}{2}-\frac{\rho}{2})}{\Gamma(\frac{1}{4}+\frac{\nu}{2}+\frac{\rho}{2})} \qquad (x \to 0+). \tag{1.10}$$

(2) *Let $n \in \mathbb{N}$ and $\nu - \frac{1}{2} + 2n < \rho < \nu + \frac{3}{2} + 2n$. Then (1.9) holds if and only if $\int_0^\infty t^{\nu-\frac{3}{2}+2n}h(t)dt < \infty$ and*

$$\begin{aligned} H_\nu(x) - \sum_{j=0}^{n-1} c_{\nu,j}\int_0^\infty t^{\nu+\frac{1}{2}+2j}h(t)dt \cdot x^{\nu+\frac{1}{2}+2j} \\ \sim x^{\rho-1}\ell(1/x) \cdot 2^{\frac{1}{2}-\rho}\frac{\Gamma(\frac{3}{4}+\frac{\nu}{2}-\frac{\rho}{2})}{\Gamma(\frac{1}{4}+\frac{\nu}{2}+\frac{\rho}{2})} \qquad (x \to 0+). \end{aligned} \tag{1.11}$$

The part (1) of Theorem B is due to Pitman [P], Bingham [B], and Ridenhour-Soni [RS], while the part (2) to Soni-Soni [SS].

We focus on Fourier (cosine and sine) transforms. Let $f \in L^1_{\mathrm{loc}}[0,\infty)$ and f be ultimately decreasing to zero at infinity. We write F_{c} for the *Fourier cosine transform* of f:

$$F_{\mathrm{c}}(x) = \int_0^{\infty-} f(t)\cos(xt)dt \qquad (0 < x < \infty). \tag{1.12}$$

Similarly, let $g(t)t \in L^1_{\mathrm{loc}}[0,\infty)$, and g be ultimately decreasing to zero at infinity. We write G_{s} for the *Fourier sine transform* of g:

$$G_{\mathrm{s}}(x) = \int_0^{\infty-} g(t)\sin(xt)dt \qquad (0 < x < \infty). \tag{1.13}$$

Now, at least formally,

$$F_{\mathrm{c}}^{(2j)}(0) = (-1)^j \int_0^{\infty} t^{2j} f(t)dt, \qquad G_{\mathrm{s}}^{(2j+1)}(0) = (-1)^j \int_0^{\infty} t^{2j+1} g(t)dt.$$

So for $n \in \mathbb{N}$ we define $\bar{F}_{\mathrm{c},n}$ by

$$\bar{F}_{\mathrm{c},n}(X) := x^{2n} \left\{ F_{\mathrm{c}}(1/x) - \sum_{j=0}^{n-1} \frac{F_{\mathrm{c}}^{(2j)}(0)}{(2j)!} x^{-2j} \right\} \qquad (0 < x < \infty) \tag{1.14}$$

if $F_{\mathrm{c}} \in C^{2n-2}([0,\infty))$. Similarly, for $n \in \mathbb{N}$, we define $\bar{G}_{\mathrm{s},n}$ by

$$\bar{G}_{\mathrm{s},n}(x) := x^{2n+1} \left\{ G_{\mathrm{s}}(1/x) - \sum_{j=0}^{n-1} \frac{G_{\mathrm{s}}^{(2j+1)}(0)}{(2j+1)!} x^{-2j-1} \right\} \qquad (0 < x < \infty) \tag{1.15}$$

if $G_{\mathrm{s}} \in C^{2n-1}([0,\infty))$. Here as usual, $C^m([0,\infty))$ is the class of functions which are of $C^m(I)$-class for some open neighbourhood I of $[0,\infty)$.

Theorem 2. *Let $\ell \in R_0$ and $n \in \mathbb{N}$. Let $f \in L^1_{\mathrm{loc}}[0,\infty)$ and f be ultimately decreasing to zero at infinity, with Fourier cosine transform F_{c}. Then*

$$f(t) \sim t^{-2n-1}\ell(t) \qquad (t \to \infty) \tag{1.16}$$

if and only if

$$F_{\mathrm{c}} \in C^{2n-2}([0,\infty)) \quad \text{and} \quad \bar{F}_{\mathrm{c},n} \in \Pi_\ell \ \text{with } \ell\text{-index } \frac{(-1)^n}{(2n)!}. \tag{1.17}$$

Theorem 3. *Let $\ell \in R_0$ and $n \in \mathbb{N}$. Let $g(t)t \in L^1_{\mathrm{loc}}[0,\infty)$ and g be ultimately decreasing to zero at infinity, with Fourier sine transform G_{s}. Then*

$$g(t) \sim t^{-2n-2}\ell(t) \qquad (t \to \infty) \tag{1.18}$$

if and only if

$$G_{\mathrm{s}} \in C^{2n-1}([0,\infty)) \quad \text{and} \quad \bar{G}_{\mathrm{s},n} \in \Pi_\ell \ \text{with } \ell\text{-index } \frac{(-1)^n}{(2n+1)!}. \tag{1.19}$$

Remark. In Theorem 2, $F_{\mathrm{c}} \in C^{2n-2}([0,\infty))$ implies that the limit $F_{\mathrm{c}}(0+)$ exists and that F_{c}, with $F_{\mathrm{c}}(0) := F_{\mathrm{c}}(0+)$, is in $C^{2n-2}([0,\infty))$; similarly for the meaning of $G_{\mathrm{s}} \in C^{2n-1}([0,\infty))$ in Theorem 3.

The proofs of Theorems 2 and 3 are based on Theorem 1.

We give an application of Theorem 3 to probability theory. Let X be a real random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. The *tail-sum* of X is the function T defined by

$$T(x) := P(X \le -x) + P(X > x) \qquad (0 \le x < \infty).$$

Note that T is finite and decreases to zero at infinity. Now

$$\{1 - U(\xi)\}/\xi = \int_0^{\infty-} T(x)\sin(x\xi)dx \qquad (0 < \xi < \infty),$$

where U is the real part of the characteristic function of X:

$$U(\xi) := E[\cos(\xi X)] \qquad (\xi \in \mathbb{R})$$

(see [BGT, p. 336]). By Theorem 3, the asymptotic behavior

$$T(x) \sim x^{-2n-2}\ell(x) \qquad (x \to \infty)$$

with $n \in \mathbb{N}$ and $\ell \in R_0$ is characterized in terms of U.

We can apply Theorems 1 and A to Question 7.19 of Boas [Bo]. For $f \in L^1[0,\pi]$, we define its *Fourier cosine coefficients* a_n by

$$a_n := \frac{2}{\pi}\int_0^\pi f(t)\cos(nt)dt \quad (n = 1, 2, \cdots), \quad := \frac{1}{\pi}\int_0^\pi f(t)dt \quad (n = 0). \tag{1.20}$$

Similarly, for $g \in L^1[0,\pi]$, we define its *Fourier sine coefficients* b_n by

$$b_n := \frac{2}{\pi}\int_0^\pi g(t)\sin(nt)dt \qquad (n = 1, 2, \cdots). \tag{1.21}$$

Theorem 4. *Let $f \in L^1[0,\pi]$ with Fourier cosine coefficients (a_k). We assume that $a_k \geq 0$ for all $k \geq 0$. Let $n \in \mathbb{N}$ and $\ell \in R_0$. Then*

$$\sum_{k=m}^{\infty} a_k \sim \frac{\ell(m)}{m^{2n}} \cdot \frac{1}{2n} \qquad (m \to \infty) \tag{1.22}$$

if and only if

$$f \in C^{2n-2}([0,\pi]) \quad \text{and} \quad \bar{f}_n \in \Pi_\ell \ \text{with } \ell\text{-index } \frac{(-1)^n}{(2n)!}, \tag{1.23}$$

where

$$\bar{f}_n(x) := x^{2n}\left\{ f(1/x) - \sum_{j=0}^{n-1} \frac{f^{(2j)}(0)}{(2j)!}x^{-2j} \right\} \qquad (1/\pi \leq x < \infty). \tag{1.24}$$

Corollary. *In Theorem 4, we further assume that (a_k) is decreasing. Then* (1.23) *is equivalent to*

$$a_m \sim \frac{\ell(m)}{m^{2n+1}} \qquad (m \to \infty). \tag{1.25}$$

Theorem 5. *Let $g \in L^1[0,\pi]$ with Fourier sine coefficients (b_k). We assume that $b_k \geq 0$ for all $k \geq 1$. Let $n \in \mathbb{N}$ and $\ell \in R_0$. Then*

$$\sum_{k=m}^{\infty} b_k \sim \frac{\ell(m)}{m^{2n+1}} \cdot \frac{1}{2n+1} \qquad (m \to \infty) \tag{1.26}$$

if and only if

$$g \in C^{2n-1}([0,\pi]) \quad \text{and} \quad \bar{g}_n \in \Pi_\ell \ \text{ with } \ell\text{-index } \frac{(-1)^n}{(2n+1)!}, \tag{1.27}$$

where

$$\bar{g}_n(x) := x^{2n+1}\left\{ g(1/x) - \sum_{j=0}^{n-1} \frac{g^{(2j+1)}(0)}{(2j+1)!} x^{-2j-1} \right\} \qquad (1/\pi \leq x < \infty). \tag{1.28}$$

Corollary. *In Theorem 5, we further assume that (b_k) is decreasing. Then (1.27) is equivalent to*

$$b_m \sim \frac{\ell(m)}{m^{2n+2}} \qquad (m \to \infty). \tag{1.29}$$

Remark. We understand that $L^1[0,\pi]$ consists of equivalence classes with respect to the equivalence relation $f_1 \sim f_2 \Leftrightarrow f_1 = f_2$ a.e. So, e.g., in (1.23), $f \in C^{2n-2}([0,\pi])$ implies that there exists a function in $C^{2n-2}([0,\pi])$ which lies in the equivalence class of f and that we identify the function with f. In particular, if $\sum_{k=0}^{\infty} |a_k| < \infty$, then by [Z, Ch. III, Theorem 3.9] (Theorem of Lebesgue on Cesàro summability) $f \in C([0,\pi])$ and we may assume that $f(x) = \sum_{k=0}^{\infty} a_k \cos(kx)$ for $0 \leq x \leq \pi$. Similarly, if $\sum_{k=1}^{\infty} |b_k| < \infty$, then $g \in C([0,\pi])$ and we may assume that $g(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$ for $0 \leq x \leq \pi$.

For (1.26) with $n = 0$, we have the following:

Theorem 6. *Let g, (b_k) and ℓ be as in Theorem 5. We write $\bar{g}(x) := xg(1/x)$ for $x \geq 1/\pi$. Then*

$$\sum_{k=m}^{\infty} b_k \sim \frac{\ell(m)}{m} \qquad (m \to \infty) \tag{1.30}$$

if and only if

$$g \in C([0,\pi]) \quad \text{and} \quad \bar{g} \in \Pi_\ell \quad \text{with } \ell\text{-index } 1. \tag{1.31}$$

See also [I2, Theorem 1.2].

Theorems 4, 5 and 6 treat the boundary cases to the following known results due to Yong [Y]:

Theorem C ([Y]). *Let f, (a_k) and ℓ be as in Theorem 4. Let $n \in \mathbb{N}$ and $2n-1 < \rho < 2n+1$. Then*

$$\sum_{k=m}^{\infty} a_k \sim \frac{\ell(m)}{m^{\rho-1}} \cdot \frac{1}{\rho - 1} \qquad (m \to \infty) \tag{1.32}$$

if and only if $f \in C^{2n-2}([0,\pi])$ and

$$f(x) - \sum_{j=0}^{n-1} \frac{f^{(2j)}(0)}{(2j)!} x^{2j} \sim \frac{\pi}{2\Gamma(\rho)\cos(\rho\pi/2)} x^{\rho-1} \ell(1/x) \qquad (x \to 0+). \tag{1.33}$$

Theorem D ([Y]). *Let g, (b_k) and ℓ be as in Theorem 5. Let $n \in \mathbb{N}$ and $2n < \rho < 2n+2$. Then*

$$\sum_{k=m}^{\infty} b_k \sim \frac{\ell(m)}{m^{\rho-1}} \cdot \frac{1}{\rho - 1} \qquad (m \to \infty) \tag{1.34}$$

if and only if $g \in C^{2n-1}([0,\pi])$ and

$$g(x) - \sum_{j=0}^{n-1} \frac{g^{(2j+1)}(0)}{(2j+1)!} x^{2j+1} \sim \frac{\pi}{2\Gamma(\rho)\sin(\rho\pi/2)} x^{\rho-1} \ell(1/x) \qquad (x \to 0+). \tag{1.35}$$

Theorems 4, 5 and 6, together with Theorems C and D, give an answer to [Bo, Question 7.19].

In this paper, we give a proof only to Theorem 1 which is our main theorem. For the rest, we refer to [IK].

2. Proof of Theorem A

As we stated in §1, Theorem A is due to [BI]. Since our main theorem, that is Theorem 1, is proved using Theorem A, we give its proof following [BI] for completeness.

Step 1. Choose X so large that h is positive and non-increasing on $[X, \infty)$. We first show that we lose no generality by supposing that h vanishes on $[0, X)$.

Set $\tilde{h}(t) := I_{[X,\infty)}(t)h(t)$, and let $\tilde{H}_\nu$ be its Hankel transform:

$$\tilde{H}_\nu(x) := \int_0^{\infty-} \tilde{h}(t)(xt)^{1/2} J_\nu(xt)dt \qquad (x > 0).$$

By the mean-value theorem, there exists $c_1 \in (0, \infty)$ such that

$$|x^{-\nu} J_\nu(x) - y^{-\nu} J_\nu(y)| \leq c_1 |x - y| \qquad (0 \leq x, y \leq 1).$$

So for $\lambda > 1$,

$$\begin{aligned}
&\frac{1}{\ell(x)} |\{\bar{H}_\nu(\lambda x) - \bar{H}_\nu(x)\} - \{(\lambda x)^{\nu+\frac{1}{2}} \tilde{H}_\nu(1/\lambda x) - x^{\nu+\frac{1}{2}} \tilde{H}_\nu(1/x)\}| \\
=&\frac{1}{\ell(x)} \left| \int_0^X t^{n+\frac{1}{2}} h(t) \{(t/\lambda x)^{-\nu} J_\nu(t/\lambda x) - (t/x)^{-\nu} J_\nu(t/x)\} dt \right| \\
\leq& c_1 \frac{(1 - \lambda^{-1})}{x\ell(x)} X \int_0^X t^{\nu+\frac{1}{2}} |h(t)| dt \to 0 \qquad (x \to \infty).
\end{aligned}$$

So (1.8) holds if and only if $x^{n+\frac{1}{2}} \tilde{H}_\nu(1/x) \in \Pi_\ell$ with ℓ-index $c_{\nu,0}$. Hence we may replace h by $\tilde{h}$ — that is, we may assume that h vanishes on $[0, X)$.

Step 2: Abelian part. First we assume (1.7) and show (1.8). For $x > 0$,

$$\frac{1}{x^{\nu+\frac{3}{2}}h(x)}\left\{\bar{H}_\nu(x) - c_{\nu,0}\int_0^x h(t)t^{\nu+\frac{1}{2}}dt\right\} = I(x) + II(x), \tag{2.1}$$

where

$$I(x) := \int_0^1 \frac{(xu)^{\nu+2}h(xu)}{x^{\nu+2}h(x)}\left\{u^{1/2}J_\nu(u) - c_{\nu,0}u^{\nu+\frac{1}{2}}\right\}\frac{du}{u^{\nu+2}},$$

$$II(x) := \int_1^{\infty-}\frac{h(xu)}{h(x)}u^{1/2}J_\nu(u)du.$$

By the uniform convergence theorem for regularly varying functions ([BGT, Th. 1.5.2]), $(xu)^{\nu+2}h(xu)/x^{\nu+2}h(x)$ converges to $u^{1/2}$ as $x \to \infty$ uniformly in $u \in (0, 1]$. Hence we find that

$$I(x) \to \int_0^1 \frac{1}{u^{\nu+\frac{3}{2}}}\left\{u^{1/2}J_\nu(u) - c_{\nu,0}u^{\nu+\frac{1}{2}}\right\}du \qquad (x \to \infty).$$

We note that the integral above converges absolutely since there exists $c_2 \in (0, \infty)$ such that

$$\left|x^{-\nu}J_\nu(x) - c_{\nu,0}\right| \le c_2 x^2 \qquad (0 \le x \le 1). \tag{2.2}$$

In the same way, for any $Y > X$,

$$\int_1^Y \frac{h(xu)}{h(x)}u^{1/2}J_\nu(u)du \to \int_1^Y \frac{1}{u^{\nu+\frac{3}{2}}}u^{1/2}J_\nu(u)du \qquad (x \to \infty).$$

By the second integral mean-value theorem ([WW, §4.14]), for $x > 1$,

$$\int_Y^{\infty-}\frac{h(xu)}{h(x)}u^{1/2}J_\nu(u)du = \frac{h(xY+)}{h(x)}\int_Y^{Y'}u^{1/2}J_\nu(u)du$$

for some $Y' \in (Y, \infty)$. If we set

$$c_3 := \sup\left\{\left|\int_x^y u^{1/2}J_\nu(u)du\right| : 0 < x < y < \infty\right\},$$

then

$$\limsup_{x\to\infty}\left|\int_Y^{\infty-}\frac{h(xu)}{h(x)}u^{1/2}J_\nu(u)du\right| \le \limsup_{x\to\infty}\frac{h(xY)}{h(x)}c_3 = \frac{c_3}{Y^{\nu+\frac{3}{2}}},$$

which can be made arbitrarily small by choosing Y large enough. So

$$II(x) \to \int_1^\infty \frac{1}{u^{\nu+\frac{3}{2}}} u^{1/2} J_\nu(u)du \qquad (x \to \infty).$$

Therefore

$$\frac{1}{\ell(x)} \left\{ (\bar{H}_\nu(\lambda x) - c_{\nu,0} \int_0^{\lambda x} h(t)t^{\nu+\frac{1}{2}} dt) - (\bar{H}_\nu(x) - c_{\nu,0} \int_0^x h(t)t^{\nu+\frac{1}{2}} dt) \right\} \to 0$$

as $x \to \infty$. Hence, by the Uniform Convergence Theorem,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\bar{H}_\nu(\lambda x) - \bar{H}_\nu(x)}{\ell(x)} &= \lim_{x\to\infty} \frac{c_{\nu,0}}{\ell(x)} \int_x^{\lambda x} h(t)t^{\nu+\frac{1}{2}} dt \\
&= \lim_{x\to\infty} c_{\nu,0} \frac{x^{\nu+\frac{3}{2}} h(x)}{\ell(x)} \int_1^\lambda \frac{(xu)^{\nu+\frac{3}{2}} h(xu)}{x^{\nu+\frac{3}{2}} h(x)} \frac{du}{u} \\
&= c_{\nu,0} \int_1^\lambda \frac{du}{u} = c_{\nu,0} \log \lambda.
\end{aligned}$$

Thus (1.7) implies (1.8).

Step 3: Tauberian part. We now prove the implication from (1.8) to (1.7). By a formula of Gegenbauer,

$$\int_0^\infty t^{\nu+\frac{1}{2}} e^{-xt} (yt)^{1/2} J_\nu(yt)dt = d_\nu \frac{xy^{\nu+\frac{1}{2}}}{(x^2+y^2)^{\nu+\frac{3}{2}}} \qquad (x > 0),$$

where $d_\nu := \pi^{-1/2} 2^{\nu+1} \Gamma(\nu + \frac{3}{2})$ ([WW, §13.2 (5)]). So by Parseval's formula for Hankel transforms ([RS]), for $x > 0$

$$\int_0^\infty h(t)t^{\nu+\frac{1}{2}} e^{-xt} dt = d_\nu \int_{0+}^{\infty-} H_\nu(y) \frac{xy^{\nu+\frac{1}{2}}}{(x^2+y^2)^{\nu+\frac{3}{2}}} dy. \tag{2.5}$$

By the second integral mean-value theorem,

$$|H_\nu(y)| \leq c_3 h(X)/y \qquad (0 < y < \infty), \tag{2.6}$$

whence

$$\int_1^\infty |H_\nu(y)| \frac{y^{\nu+\frac{1}{2}}}{(x^2+y^2)^{\nu+\frac{3}{2}}} dy < \infty.$$

On the other hand, by [BGT, Th. 3.7.4], (1.8) implies $|\bar{H}_\nu| \in R_0$, whence

$$\int_0^1 |H_\nu(y)| \frac{y^{\nu+\frac{1}{2}}}{(x^2+y^2)^{\nu+\frac{3}{2}}} dy = \int_1^\infty \frac{|\bar{H}_\nu(u)|}{(x^2u^2+1)^{\nu+\frac{3}{2}}} du < \infty.$$

Thus the integral on the right of (2.5) converges absolutely — and so the results of [BGT Ch. 4] apply.

We use Laplace transforms. Write

$$U(t) := \int_0^t h(u) u^{\nu+\frac{1}{2}} du \qquad (t \geq 0),$$

$$\hat{U}(x) := \int_{[0,\infty)} e^{-xt} dU(t) = \int_0^\infty h(t) t^{\nu+\frac{1}{2}} e^{-xt} dt \qquad (x > 0).$$

Then by (2.5),

$$\hat{U}(1/x) = (k * \bar{H}_\nu)(x) \qquad (x > 0),$$

where

$$k(x) := d_\nu \frac{x^{2\nu+2}}{(1+x^2)^{\nu+\frac{3}{2}}} \qquad (x > 0)$$

and $k * \bar{H}_\nu$ denotes the Mellin convolution

$$(k * \bar{H}_\nu)(x) = \int_0^\infty k(x/t) \bar{H}_\nu(t) dt/t.$$

The absolute convergence strip of the Mellin transform

$$\check{k}(z) := \int_0^\infty t^{-z} k(t) dt/t$$

is $-1 < \Re z < 2\nu + 2$, and for z in the strip

$$\check{k}(z) = \frac{2^\nu}{\pi^{1/2}} \Gamma(\nu + 1 - \tfrac{1}{2}z) \Gamma(\tfrac{1}{2} + \tfrac{1}{2}z),$$

in particular,

$$\check{k}(0) = 2^{\nu}\Gamma(\nu + 1).$$

By (2.6), $\bar{H}_\nu$ is locally bounded on $[0, \infty)$. So by the argument of [BGT, p. 242] (Abelian theorem for differences), we find that (1.8) implies $k * \bar{H}_\nu \in \Pi_\ell$ with ℓ-index 1 — i.e., so is $\hat{U}(1/\cdot)$. So by a Tauberian theorem of de Haan (cf. [BGT, Th. 3.9.1]), we see that $U \in \Pi_\ell$ with ℓ-index 1. Finally, by [BGT, Th. 3.6.10] (with slow decrease replaced by slow increase), we obtain (1.7), completing the proof. □

3. Proof of Theorem 1

We note that (1.4) implies

$$\int_0^\infty t^{\nu - \frac{3}{2} + 2n} h(t) dt < \infty. \tag{3.1}$$

So, when proving the equivalence of (1.4) and (1.5), we may assume (3.1).

We define $h_0, \cdots, h_{n-1}$ by

$$\begin{aligned} h_0(t) &:= \int_t^\infty h(s) s^{\nu + \frac{1}{2}} ds \qquad (0 \le t < \infty), \\ h_j(t) &:= \int_t^\infty h_{j-1}(s) s ds \qquad (0 \le t < \infty, \ j = 1, \cdots, n-1). \end{aligned}$$

Since h is eventually non-negative, h_j are all eventually decreasing. By Fubini's theorem,

$$\begin{aligned} h_j(0) &= \int_0^\infty dt_j t_j \int_{t_j}^\infty dt_{j-1} t_{j-1} \cdots \int_{t_1}^\infty h(t_0) t_0^{\nu + \frac{1}{2}} dt_0 \\ &= \int_0^\infty dt_0 h(t_0) t_0^{\nu + \frac{1}{2}} \int_0^{t_0} dt_1 t_1 \cdots \int_0^{t_{j-1}} t_j dt_j \\ &= \frac{1}{2^j j!} \int_0^\infty h(t) t^{\nu + \frac{1}{2} + 2j} dt. \end{aligned}$$

Since

$$x^{-\mu}J_\mu(x) = O(1) \qquad (x \to \infty),$$

$$\frac{d}{dx}\left\{x^{-\mu}J_\mu(x)\right\} = -x^{-\mu}J_{\mu+1}(x),$$

$$x^{-\mu}J_\mu(x) \to c_{\mu,0} \qquad (x \to 0+)$$

for any $\mu \geq -1/2$ (see Watson [W], pages 199 and 45), we obtain, by integration by parts,

$$\begin{aligned} H_\nu(x) &= x^{\nu+\frac{1}{2}} \int_0^{\infty-} h(t)t^{\nu+\frac{1}{2}} \left\{(tx)^{-\nu}J_\nu(tx)\right\} dt \\ &= x^{\nu+\frac{1}{2}} h_0(0)c_{\nu,0} - x^{\nu+\frac{1}{2}+2} \int_0^{\infty-} h_0(t)t \left\{(tx)^{-\nu-1}J_{\nu+1}(tx)\right\} dt = \cdots \\ &= \sum_{j=0}^{n-1} (-1)^j x^{\nu+\frac{1}{2}+2j} h_j(0)c_{\nu+j,0} \\ &\qquad + (-1)^n x^{\nu+\frac{1}{2}+2n} \int_0^{\infty-} h_{n-1}(t)t\{(tx)^{-\nu-n}J_{\nu+n}(tx)\}dt \\ &= \sum_{j=0}^{n-1} (-1)^j x^{\nu+\frac{1}{2}+2j} h_j(0)c_{\nu+j,0} + (-1)^n x^n \int_0^{\infty-} g(t)(tx)^{1/2} J_{\nu+n}(tx)dt, \end{aligned}$$

where

$$g(t) := t^{-\nu+\frac{1}{2}-n} h_{n-1}(t) \qquad (0 < t < \infty).$$

Since

$$(-1)^j h_j(0)c_{\nu+j,0} = c_{\nu,j} \int_0^\infty t^{\nu+\frac{1}{2}+2j} h(t)dt,$$

we have

$$\tilde{H}_{\nu,n}(x) = (-1)^n x^{(\nu+n)+\frac{1}{2}} \int_0^{\infty-} g(t)(t/x)^{1/2} J_{\nu+n}(t/x)dt. \tag{3.2}$$

Now $t^{(\nu+n)+\frac{1}{2}} g(t) \in L^1_{\mathrm{loc}}[0,\infty)$ and g is eventually decreasing to zero, whence by Theorem A (with ν replaced by $\nu+n$) (1.5) is equivalent to

$$g(t) \sim t^{-\nu-n-\frac{3}{2}} \ell(t) \cdot \frac{(-1)^n c_{\nu,n}}{c_{\nu+n,0}} = t^{-\nu-n-\frac{3}{2}} \ell(t) \cdot \frac{1}{2^n n!} \qquad (t \to \infty)$$

or

$$h_{n-1}(t) \sim t^{-2}\ell(t) \cdot \frac{1}{2^n n!} \qquad (t \to \infty). \tag{3.3}$$

Since h_j is eventually decreasing, $\log\{h_j(t)t\}$ is slowly increasing, whence by the Monotone Density Theorem (see [BGT, §1.7]) (3.3) is equivalent to

$$h_0(t) = \int_t^\infty s^{\nu+\frac{1}{2}} h(s)ds \sim t^{-2n}\ell(t)\cdot\frac{1}{2n} \qquad (t\to\infty). \tag{3.4}$$

By assumption, h is eventually decreasing, whence $\log\left\{h(t)t^{\nu+\frac{1}{2}}\right\}$ is slowly increasing. Again by the Monotone Density Theorem, (3.4) is equivalent to (1.4). This completes the proof. □

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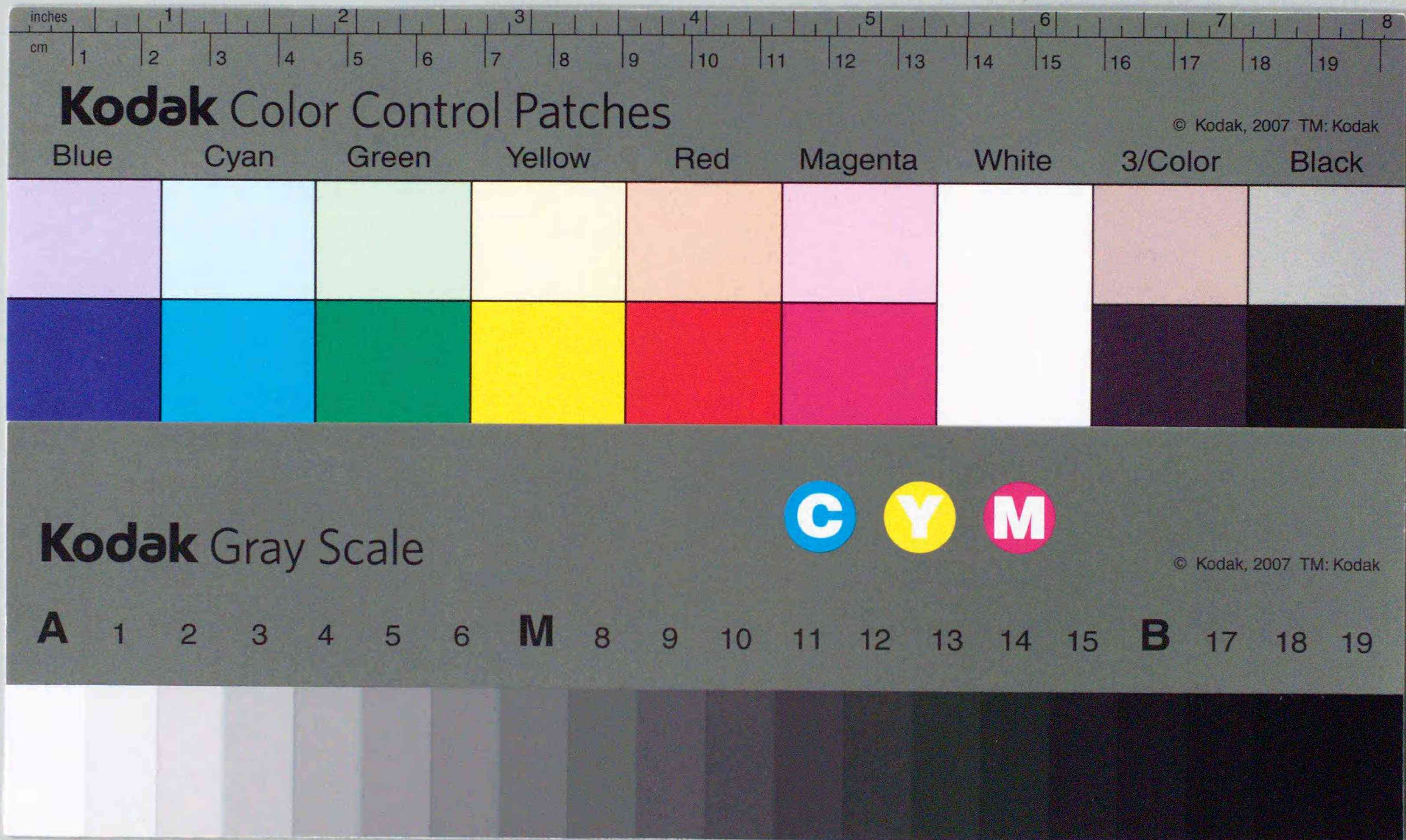
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