A study of closed hypersurfaces with constant mean curvature in a Riemannian manifold

by

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1 Introduction

We consider a hypersurface V^n imbedded in a Riemannian manifold R^{n+1} of $n+1$ dimensions. Let us denote by g_{ba} and h_{ba} the first fundamental tensor and the second fudamental tensor of V^n respectively.

Now, if we denote by $k_1, k_2, \cdots, k_n$ the principal curvatures of V^n, that is, the roots of the characteristic equation

$$det(h_{ba} - kg_{ba}) = 0,$$

then the *ν-th mean curvature* H_ν of V^n is defined to be the ν-th elementary symmetric function of $k_1, k_2, \cdots, k_n$ divided by the number of terms, that is,

$$\binom{n}{\nu} H_\nu = \sum_{c_1 < c_2 < \cdots < c_\nu} k_{c_1} k_{c_2} \cdots k_{c_\nu} \qquad (1 \leq \nu \leq n).$$

Particularly, for a surface V^2 in an Euclidean space E^3 of three dimensions, we put

$$H = \frac{1}{2}(k_1 + k_2), \qquad K = k_1 k_2,$$

and we call H and K the *mean curvature* and the *Gauss curvature* of V^2 respectively. Accordingly, H is equal to H_1 and K equal to H_2.

When h_{ba} are proportional to g_{ba} at a point P of V^n, the point P is called an *umbilic* point. Then it is easily seen that a point P is umbilic if and only if n principal curvatures be equal at P.

The spheres are characterized by various special properties of a closed surface in an Euclidean space E^3. In the present paper a *closed* surface means a compact connected surface without boundary. In particular, It has been proved that a closed surface in E^3 is a sphere if and only if the Gauss curvature K of the surface be constant. On the other hand, it is proved that a sphere is a closed surface with constant mean curvautre H. When $K > 0$ holds good at every point of a closed surface, the closed surface is called an *ovaloid*. It has been proved by H. Liebmann [15][2] that the only ovaloid with constant mean curvature H in E^3 is a sphere. This result was the starting point of various interesting investigations of closed surfaces. Afterward this result was generalized for a closed convex hypersurface in an n-dimensional Euclidean space E^n. Many investigators have studied characterizations of the sphere from various viewpoints, and there exist many interesting investigations, for instance, Chern's work [3] about generalizing the condition H=const. in the Liebmann's Theorem. In these investigations the integral formula of Minkowski type has played one of the important role.

Let V^2 be an ovaloid in E^3. Then, as the well-known integral formula of Minkowski, we have

$$\int_{V^2}(Kp + H)dA = 0,$$

where p denotes the oriented distance from a fixed point O in E^3 to the tangent space of V^2 at P and dA is the area element of V^2. This formula was generalized for a closed oreintable hypersurface V^n in E^{n+1} [6]. Afterward Y. Katsurada [8,9] derived the integral formulas of Minkowski type for a closed orientable hypersurface V^n in an $(n+1)$-dimensional Riemannian manifold R^{n+1} admitting a conformal Killing vector field ξ^i and gave certain characterizations of an umbilical hypersurface in R^{n+1}. The analogous problems for a closed orientable hypersurface V^n in a Riemannian manifold R^{n+1} have been discussed by many investigators.

It is one of the interesting problem to find certain conditions for an umbilical hypersurface in a Riemannian manifold to be isometric to a sphere. If every point of a closed orientable hypersurface in E^{n+1} is umbilic, then the hypersurfce is isometric to a sphere. However, in a general Riemannian manifold we can not expect the validity of the same result even if every point of a closed orientable hypersurface is umbilic. Some investigators [10,21] gave some results on this problem, making use of Obata's Theorem [16].

The *concircular* transformation is by definition a conformal transformation carrying any geodesic circle into another one. A vector field generates locally a one-parameter group of transformations. We shall say a vector field to be *isometric, homothetic, concircular* or *conformal*, if it generates a one-parameter group of corresponding transformations.

A proper conformal Killing vector field ξ with components ξ^i is characterized by the so-called conformal Killing equation

$$\mathcal{L}_\xi G_{ji} \equiv \nabla_j\xi_i + \nabla_i\xi_j = 2\Psi G_{ji}, \qquad (1.1)$$

for a non constant scalar field Ψ which is said to be *associated* with ξ^i, where G_{ji}, $\mathcal{L}_\xi G_{ji}$, and ∇_i denote the metric tensor of R^{n+1}, the Lie derivative of G_{ji} with respect to ξ, and the operator of covariant differentiation with respect to Christoffel symbols $\left\{ {k \atop ji} \right\}$ formed with G_{ji} respectively.

[2]Numbers in brackets refer to the references at the end of the paper.

It is known that in order that ξ^i is concircular, it is necessary and sufficient that the associated scalar field Ψ satisfies the equation

$$(1.2) \qquad \nabla_j \nabla_i \Psi = \phi G_{ji} \quad ([7]),$$

where ϕ is a scalar field.

We shall call a scalar field Ψ in R_{n+1} satisfying an equation of type (1.2) a *concircular* scalar field. If ϕ is linear in Ψ with constant coefficients, that is,

$$(1.3) \qquad \phi = \rho\Psi + \sigma,$$

where ρ is a non zero constant and σ a constant, then Ψ is called a *special* concircular scalar field [18].

For instance, let R^{n+1} be an Einstein manifold with non zero scalar curvature R which admits a proper conformal Killing vector field ξ^i, that is, ξ^i satisfies the equation (1.1). Then it is known that the associated scalar field Ψ satisfies the equation

$$(1.4) \qquad \nabla_j \nabla_i \Psi = -\frac{R}{n(n+1)} \Psi G_{ji} \quad (R = \textit{non zero const.}) \quad ([23, 21, 19]),$$

that is, Ψ is a special concircular scalar field such that $\rho = -(R/n(n+1))$ and $\sigma = 0$.

The author has studied closed hypersurfaces with constant mean curvature in a Riemannian manifold R^{n+1} admitting a conformal Killing vector field, or in R^{n+1} admitting a special concircular scalar field. If H_2 is constant and the principal curvatures $k_1, k_2, \cdots, k_n$ at each point of V^n are positive, then H_2 is positive constant, because

$$\binom{n}{2} H_2 = \sum_{d<c} k_d k_c.$$

This property has been the turning point of our study of a closed hypersurface V^n with H_2=constant. And the author obtained the following Theorems:

Theorem 1.1 ([12]) *Let R^{n+1} $(n \geq 3)$ be an orientable Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) *its second mean curvature H_2 is positive constant,*

(2) *$C^i \xi_i$ has fixed sign on V^n, where C^i denotes the normal vector to V^n.*

Then every point of V^n is umbilic.

Theorem 1.2 ([13, 14]) *Let R^{n+1} be an orientable Riemannian manifold which admits a special concircular scalar field Ψ satisfying the equation*

$$\nabla_j \Psi_i = (\rho\Psi + \sigma) G_{ji} \quad (\rho = \textit{const.} \neq 0,\ \sigma = \textit{const.}),$$

and V^n a closed orientable hypersurface in R^{n+1} such that

(1) *its first mean curvature* H_1 *is non zero constant,*

(2) Ω *has fixed sign on* V^n, *where* $\Psi_i = \nabla_i \Psi$, *and* $\Omega = C^i \Psi_i$.

Assume further that the ambient manifold R^{n+1} *satisfies one of the following conditions*

(a) R^{n+1} *has the harmonic curvature, that is,* $\nabla_k R_{ji} = \nabla_j R_{ki}$, *where* R_{ji} *denotes the Ricci tensor of* R^{n+1},

(b) R^{n+1} *has the parallel Ricci tensor, that is,* $\nabla_k R_{ji} = 0$,

(c) R^{n+1} *is a conformally flat Riemannian manifold and the scalar curvature* R *is constant,*

(d) R^{n+1} *satisfies the equation* $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$[3],

(e) $R^{ji} R_{ji}$ *is constant,*

(f) R^{n+1} *is a conformally flat Riemannian manifold and there exists a point* P_0 *on* V^n *such that* $S(P_0) = 0$, *where* S_{ji} *is the symmetric tensor defined by* $R_{ji} + n\rho G_{ji}$, *and* $S = S_{ji} G^{ji}$.

Then every point of V^n *is umbilic.*

Next, using these Theorems, the author proved the following ones respectively:

Theorem 1.3 ([12]) *Let* R^{n+1} $(n \geq 3)$ *be an orientable Riemannian manifold of constant curvature which admits a proper conformal Killing vector field* ξ^i *satisfying the equation*

$$\pounds_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

and V^n *a closed orientable hypersurface in* R^{n+1} *such that*

(1) H_2 *is positive constant,*

(2) $C^i \Psi_i$ *has fixed sign on* V^n,

(3) $C^i \Psi_i$ *is not constant along* V^n, *where* Ψ *satisfies* $\nabla_j \nabla_i \Psi = -(R/n(n+1))\Psi G_{ji}$.

Then V^n *is isometric to a sphere.*

Theorem 1.4 ([13, 14]) *Let* R^{n+1} *be an orientable Riemannian manifold which admits a special concircular scalar field* Ψ *satisfying the equation*

$$\nabla_j \Psi_i = (\rho\Psi + \sigma) G_{ji} \quad (\rho = const. \neq 0, \sigma = const.),$$

and V^n *a closed orientable hypersurface in* R^{n+1} *such that*

(1) H_1 *is non zero constant,*

(2) Ω *has fixed sign on* V^n,

(3) Ω *is not constant along* V^n.

Assume further that the ambient manifold R^{n+1} *satisfies one of the following conditions*

(a) R^{n+1} *has the harmonic curvature, that is,* $\nabla_k R_{ji} = \nabla_j R_{ki}$,

(b) R^{n+1} *has the parallel Ricci tensor, that is,* $\nabla_k R_{ji} = 0$,

(c) R^{n+1} *is a conformally flat Riemannian manifold and the scalar curvature* R *is constant,*

(d) R^{n+1} *satisfies the equation* $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$,

[3] See [4]

(e) $R^{ji}R_{ji}$ *is constant,*
(f) R^{n+1} *is a conformally flat Riemannian manifold and there exists a point* P_0 *on* V^n *such that* $S(P_0) = 0$.
Then V^n *is isometric to a sphere.*

Moreover, under the new assumption that Ψ is not constant along V^n, the author proved the following Theorems in a similar way:

Theorem 1.5 ([12]) *Let* R^{n+1} $(n \geq 3)$ *be an orientable Riemannian manifold of constant curvature which admits a conformal Killing vector field* ξ^i *satisfying the equation*

$$\pounds_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

and V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(1) H_2 *is positive constant,*
(2) $C^i\xi_i$ *has fixed sign on* V^n,
(3) Ψ *is not constant along* V^n.
Then V^n *is isometric to a sphere.*

Theorem 1.6 ([13, 14]) *Let* R^{n+1} *be an orientable Riemannian manifold which admits a special concircular scalar field* Ψ, *and* V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(1) H_1 *is non zero constant,*
(2) Ω *has fixed sign on* V^n,
(3) Ψ *is not constant along* V^n.
Assume further that the ambient manifold R^{n+1} *satisfies one of the following conditions*
(a) R^{n+1} *has the harmonic curvature, that is,* $\nabla_k R_{ji} = \nabla_j R_{ki}$,
(b) R^{n+1} *has the parallel Ricci tensor, that is,* $\nabla_k R_{ji} = 0$,
(c) R^{n+1} *is a conformally flat Riemannian manifold and the scalar curvature* R *is constant,*
(d) R^{n+1} *satisfies the equation* $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$,
(e) $R^{ji}R_{ji}$ *is constant,*
(f) R^{n+1} *is a conformally flat Riemannian manifold and there exists a point* P_0 *on* V^n *such that* $S(P_0) = 0$.
Then V^n *is isometric to a sphere.*

To prove Theorem 1.1, we need the following integral formulas of Minkowski type [4]

$$\int_{V^n} H_1 \theta \, dA + \int_{V^n} \Psi \, dA = 0 \tag{1.5}$$

and

$$\frac{2}{n(n-1)} \int_{V^n} \nabla_c p_a{}^c \beta^a \, dA + \int_{V^n} \{nH_1H_2 - (n-2)H_3\} \theta \, dA + 2\int_{V^n} \Psi H_2 \, dA = 0, \tag{1.6}$$

[4]See the equation (3.12) and (3.13) in § 3.

where the symmetric tensor field p_{ab} is defined by $h_c{}^c h_{ab} - h_a{}^c h_{cb}$, $p_a{}^c = p_{ab}g^{bc}$, ∇_c denotes the operator of the van der Waerden-Bortolotti covariant differentiation along V^n, $\theta = C^i \xi_i$ and dA denotes the area element of V^n. Multiplying (1.5) by $2H_2$ (=positive constant) and subtracting the result equation from (1.6), we find that

$$\frac{2}{n(n-1)} \int_{V^n} \nabla_c p_a{}^c \beta^a \, dA + (n-2) \int_{V^n} \Big\{ H_1 H_2 - H_3 \Big\} \theta \, dA = 0.$$

For a hypersurface V^n with positive constant second mean curvature H_2 in a Riemannian manifold of constant curvature, it is proved that $\nabla_c p_a{}^c = 0$ on V^n. Accordingly, from the above equation, we have

$$\int_{V^n} \Big\{ H_1 H_2 - H_3 \Big\} \theta \, dA = 0. \tag{1.7}$$

Moreover, since it is proved that H_1 has fixed sign on V^n for a hypersurface V^n with H_2=positive constant, the scalar field $H_1 H_2 - H_3$ is rewritten as follows:

$$H_1 H_2 - H_3 = \frac{1}{H_1} \Big\{ H_2(H_1^2 - H_2) + (H_2^2 - H_1 H_3) \Big\}. \tag{1.8}$$

On the other hand, it is known that

$$H_1^2 - H_2 = \frac{1}{n^2(n-1)} \sum_{a<b} (k_b - k_a)^2 \geq 0 \quad \text{and} \quad H_2^2 - H_1 H_3 \geq 0 \quad \text{on } V^n. \tag{1.9}$$

Accordingly, from (1.8), we have

$$H_1 H_2 - H_3 \geq 0 \quad (\text{or} \leq 0) \quad \text{on } V^n.$$

Hence, using the assumption (2), from (1.7), we find that

$$H_1 H_2 - H_3 = 0, \text{ that is, } H_2(H_1^2 - H_2) + (H_2^2 - H_1 H_3) = 0 \quad \text{on } V^n,$$

from which, using (1.9) and the assumption (1), we see that

$$H_1^2 - H_2 = 0 \quad \text{on } V^n.$$

Therefore, from the former of (1.9), we conclude that

$$k_1 = k_2 = \cdots = k_n$$

at each point of V^n. This means that each point of V^n is umbilic.

Now, in R^{n+1}, we assume the existence of a special concircular scalar field Ψ which satisfies the following partial differential equation

$$\nabla_j \Psi_i = (\rho \Psi + \sigma) G_{ji} \quad (\rho = \text{const.} \neq 0, \ \sigma = \text{const.}),$$

where $\Psi_i = \nabla_i \Psi$. If we put

$$\Phi = \rho \Psi + \sigma,$$

then it is proved that Φ satisfies the following partial differential equation

$$\nabla_j \Phi_i = \rho \Phi G_{ji},$$

where $\Phi_i = \nabla_i \Phi$. Accordingly, to prove Theorem 1.2, we need the following integral formulas of Minkowski type [5]

$$\int_{V^n} H_1 \Theta dA + \int_{V^n} \rho \Phi dA = 0 \tag{1.10}$$

and

$$\frac{1}{n}\int_{V^n} (\nabla_a h_c{}^c - R_{ij} B_a{}^i C^j)\phi^a dA + \int_{V^n} \left\{nH_1^2 - (n-1)H_2\right\}\Theta dA + \int_{V^n} \rho \Phi H_1 dA = 0, \tag{1.11}$$

where $B_a{}^i = \partial x^i / \partial u^a$ and $\Theta = C^i \Phi_i$. Since H_1 is non zero constant, taking account of $H_1 = (1/n) h_a{}^a$, we have, from (1.11),

$$-\frac{1}{n}\int_{V^n} R_{ij} B_a{}^i C^j \phi^a dA + \int_{V^n} \left\{nH_1^2 - (n-1)H_2\right\}\Theta dA + H_1 \int_{V^n} \rho \Phi dA = 0. \tag{1.12}$$

Eliminating $\int_{V^n} \rho \Phi dA$ from (1.10) and (1.12), we find that

$$-\frac{1}{n}\int_{V^n} R_{ij} B_a{}^i C^j \phi^a dA + (n-1)\int_{V^n} (H_1^2 - H_2)\Theta dA = 0. \tag{1.13}$$

On the other hand, making use of the special concircular scalar field Φ, the assumption (1) and (2), and the condition of the ambient manifold R^{n+1} under consideration, we will show in § 6 that $S_{ji} = 0$, say, $R_{ji} = -n\rho G_{ji}$ on V^n. Accordingly, from (1.13), we obtain

$$\int_{V^n} (H_1^2 - H_2)\Theta dA = 0. \tag{1.14}$$

Moreover, it is proved that, if Ω has fixed sign on V^n, then the same holds good also for Θ. Consequently, since

$$H_1^2 - H_2 = \frac{1}{n^2(n-1)} \sum_{a<b} (k_b - k_a)^2 \geq 0, \tag{1.15}$$

we see that $H_1^2 - H_2 = 0$ on V^n by virtue of (1.14). Therefore, from (1.15), we conclude that

$$k_1 = k_2 = \cdots = k_n$$

at each point of V^n, that is, every point of V^n is umbilic.

Finally, to prove that the umbilical hypersurface under consideration is isometric to a sphere, we use the Theorem [6] due to M. Obata [16].

[5] See the equation (3.19) and (3.20) in § 3.

[6] See Theorem A. in § 7.

For a closed hypersurface V^n with constant second mean curvature H_2 in a Riemannian manifold of constant curvature, Y. Katsurada gave Theorem 1.1 (cf., Theorem 3.2 in [8]) under the strong restriction that at each point of V^n the principal curvatures $k_1, k_2, \cdots, k_n$ of V^n are positive. The purpose of the present paper is to eliminate this restriction and prove Theorem 1.1. Y. Katsurada first gave the analogous Theorem (say, Theorem 3.1 in [9]) to Theorem 1.2 under the assumption that an ambient manifold R^{n+1} be an Einstein manifold. So, in a Riemannian manifold admitting a special concircular scalar field Ψ which satisfies the following partial differential equation

$$\nabla_j \Psi_i = (\rho\Psi + \sigma)G_{ji} \quad (\rho = const. \neq 0,\ \sigma = const.),$$

the another purpose of the present paper is to prove Theorems 1.2 under weaker assumptions by replacing, as an ambient manifold R^{n+1}, an Einstein manifold by a more general one. § 2 is devoted to give notations and general formulas in the theory of hypersurfaces in a general Riemannian manifold R^{n+1}. In § 3 we derive some integral formulas of Minkowski type which are valid for a closed orientable hypersurface V^n in R^{n+1} admitting a conformal Killing vector field ξ^i given by (1.1), or a special concircular scalar field Φ given by $\nabla_j\nabla_i\Phi = \rho\Phi G_{ji}$, which is a special case of (1.3). In § 4, we discuss some relations between Φ^i and $\nabla_k R_{ji}$, and others in R^{n+1} admitting the scalar field Φ, where $\Phi_j = \nabla_j\Phi$ and $\Phi^i = \Phi_j G^{ji}$. Our aim of § 5 is to give a generalization without the above mentioned restriction for a closed orientable hypersurface V^n whose second mean curvature H_2 is positive constant. In R^{n+1} admitting a conformal Killing vector field ξ^i, we apply the integral formulas obtained in § 3 to a closed orientable hypersurface V^n whose H_2 is positive constant, and give a generalization of Katsurada's Theorem. Moreover, in R^{n+1} admitting a special concircular scalar field Ψ given by $\nabla_j\nabla_i\Psi = (\rho\Psi + \sigma)G_{ji}$, in § 6 we similarly apply the integral formulas obtained in § 3 to a closed orientable hypersurface V^n whose first mean curvature H_1 is non zero constant, and prove Theorem 1.2. In the last section 7, making use of the results obtained in § 5 and § 6, we give certain conditions for a closed orientable hypersurface V^n in R^{n+1} to be isometric to a sphere.

2 Notation and general formulas

We consider an $(n+1)$-dimensional Riemannian manifold R^{n+1} $(n \geq 2)$ of class C^r $(r \geq 3)$ with local coordinates x^i, which admits a proper conformal Killing vector field ξ^i, that is, a vector field generating a local one-parameter group G of conformal transformations. In R^{n+1}, we consider a domain M, and if the domain M is simply covered by the orbits of transformations generated by ξ^i, then we call M a *regular* domain with respect to the vector field ξ^i.

The conformal Killing vector field ξ^i satisfies the conformal Killing equation (1.1), say,

$$\pounds_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji}, \tag{2.1}$$

for a non constant associated scalar field Ψ in R^{n+1}, where $\xi_i = \xi^j G_{ji}$ and G_{ji} is the metric tensor of R^{n+1}.

Let us denote by V^n an n-dimensional closed orientable hypersurface imbedded in the regular domain M and locally given by

$$x^i = x^i(u^a) \qquad i = 1, 2, \cdots, n+1;\ a = 1, 2, \cdots, n,$$

where u^a are local coordinates of V^n. Throughout the present paper, the indices $i, j, k, \cdots$ run from 1 to $n+1$ and the indices $a, b, c, \cdots$ from 1 to n.

If we put

$$B_a{}^i = \partial x^i / \partial u^a, \tag{2.2}$$

then $B_a{}^i$ $(a = 1, 2, \cdots, n)$ are n linearly independent vectors tangent to V^n and the first fundamental tensor g_{ba} of V^n is given by

$$g_{ba} = G_{ji} B_b{}^j B_a{}^i. \tag{2.3}$$

We assume that n vectors $B_1{}^i, B_2{}^i, \cdots, B_n{}^i$ give the positive orientation on V^n , and we denote by C^i the unit normal vector to V^n such that

$$B_1{}^i, B_2{}^i, \cdots, B_n{}^i, C^i$$

give the positive orientation in R^{n+1}.

Denoting by ∇_b the operator of the van der Waerden-Bortolotti covariant differentiation along V^n [17], we can write the equations of Gauss and Weingarten in the form

$$\nabla_b B_a{}^i = h_{ba} C^i, \tag{2.4}$$

$$\nabla_b C^i = -h_b{}^a B_a{}^i \tag{2.5}$$

respectively, where h_{ba} is the second fundamental tensor of V^n and $h_b{}^a = h_{bc}g^{ca}$. Also, the equations of Gauss and those of Codazzi are written as follows:

$$K_{dcba} - (h_{da}h_{cb} - h_{ca}h_{db}) = R_{kjih}B_d{}^k B_c{}^j B_b{}^i B_a{}^h, \tag{2.6}$$

$$\nabla_c h_{ba} - \nabla_b h_{ca} = R_{kjih}B_c{}^k B_b{}^j B_a{}^i C^h, \tag{2.7}$$

where K_{dcba} is the curvature tensor of V^n and R_{kjih} the curvature tensor of R^{n+1}. Transvecting g^{ba} to (2.7) and making use of $g^{ba}B_b{}^j B_a{}^i = G^{ji} - C^j C^i$, we find that

$$\nabla_c h_b{}^b - \nabla_b h_c{}^b = R_{kj}B_c{}^k C^j, \tag{2.8}$$

where $h_b{}^b = h_{ba}g^{ba}$ and $R_{kh} = R_{kjih}G^{ji}$.

Now, if we denote by $k_1, k_2, \cdots, k_n$ the principal curvatures of V^n, that is, the roots of the characteristic equation

$$det(h_{ba} - kg_{ba}) = 0,$$

the ν-th mean curvature H_ν of V^n is defined to be the ν-th elementary symmetric function of $k_1, k_2, \cdots, k_n$ divided by the number of terms, that is,

$$\binom{n}{\nu} H_\nu = \sum_{c_1<c_2<\cdots<c_\nu} k_{c_1}k_{c_2}\cdots k_{c_\nu} \qquad (1 \leq \nu \leq n).$$

Accordingly, H_1, H_2 and H_3 satisfy the following relations:

$$nH_1 = \sum_c k_c = h_a{}^a, \tag{2.9}$$

$$\binom{n}{2} H_2 = \sum_{d<c} k_d k_c = \frac{1}{2}\Big\{(h_a{}^a)^2 - h_a{}^b h_b{}^a\Big\} \tag{2.10}$$

and

$$\binom{n}{3} H_3 = \sum_{f<e<d} k_f k_e k_d = \frac{1}{3!}\Big\{(h_a{}^a)^3 + 2h_a{}^b h_b{}^c h_c{}^a - 3h_a{}^a(h_b{}^c h_c{}^b)\Big\}. \tag{2.11}$$

3 Integral formulas for a closed orientable hypersurface in R^{n+1}

We now assume that the hypersurface V^n under consideration is orientable and closed.

Let the group G be conformal, that is, ξ^i satisfy the equation

$$\pounds_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji}. \tag{3.1}$$

On the hypersurface V^n, we can put as follows

$$\xi^i = B_a{}^i \beta^a + \theta C^i$$

for a some vector field β^a and scalar field θ on V^n. Since $G_{ji} C^j B_a{}^i = 0$, it follows that

$$\theta = C^i \xi_i \tag{3.2}$$

and

$$\beta_a = B_a{}^i \xi_i, \tag{3.3}$$

where $\beta_a = \beta^b g_{ba}$ and $\xi_i = \xi^j G_{ji}$. We differentiate covariantly the above equation (3.3) along V^n, and, making use of (2.4) and (3.2), we get

$$\nabla_b \beta_a = h_{ba}\theta + \nabla_j \xi_i B_b{}^j B_a{}^i.$$

Transvecting g^{ba} to this eqation and using (2.9) and (3.1), we have

$$\nabla_b \beta^b = nH_1\theta + \frac{1}{2} g^{ba} B_b{}^j B_a{}^i \pounds_\xi G_{ji}, \tag{3.4}$$

where $\nabla_b \beta^b = \nabla_b \beta_a g^{ba}$.

If we now put

$$\gamma_b = 2p_b{}^a B_a{}^i \xi_i,$$

where the symbols p_{ba} are the components of the symmetric tensor of V^n defined by

$$p_{ba} = h_c{}^c h_{ba} - h_b{}^c h_{ca}, \tag{3.5}$$

and $p_b{}^a = p_{bc} g^{ca}$, then we have, by covariant differentiation along V^n and by virtue of (3.3), (2.4) and (3.2),

$$\nabla_c \gamma_b = 2\nabla_c p_b{}^a \beta_a + 2p_b{}^a h_{ac}\theta + 2p_b{}^a B_c{}^k B_a{}^i \nabla_k \xi_i.$$

And, transvecting g^{cb} to this equation and using (3.1), we get

$$\nabla_c \gamma^c = 2\nabla_c p_a{}^c \beta^a + 2p_c{}^a h_a{}^c \theta + p^{ca} B_c{}^k B_a{}^i \pounds_\xi G_{ki}. \tag{3.6}$$

On the other hand, from (2.9) and (2.10), the relation (2.11) is rewritten as follows:

$$(3.7) \qquad 2p_c{}^a h_a{}^c = n(n-1)\Big\{nH_1H_2 - (n-2)H_3\Big\}.$$

Also, from (2.10) and the definition of p_{ba}, we can see easily that

$$(3.8) \qquad p_a{}^a = n(n-1)H_2,$$

where $p_a{}^a = p_{ba}g^{ba}$. Consequently, by substituting (3.7) into (3.6), we find that

$$(3.9) \qquad \nabla_c\gamma^c = 2\nabla_c p_a{}^c\beta^a + n(n-1)\Big\{nH_1H_2 - (n-2)H_3\Big\}\theta + p^{ca}B_c{}^kB_a{}^i\mathcal{L}_\xi G_{ki}.$$

Therefore, since the vector field ξ^i is conformal Killing, that is, $\mathcal{L}_\xi G_{ji} = 2\Psi G_{ji}$, from (3.4) and (3.9), we find that

$$(3.10) \qquad \nabla_b\beta^b = n(H_1\theta + \Psi)$$

and

$$(3.11) \qquad \nabla_c\gamma^c = 2\nabla_c p_a{}^c\beta^a + n(n-1)\Big\{nH_1H_2 - (n-2)H_3\Big\}\theta + 2n(n-1)\Psi H_2,$$

by virtue of (2.3) and (3.8). And consequently, since the hypersurface V^n is orientable and closed, we apply Green's formula [22] to (3.10) and (3.11). We then obtain

$$(3.12) \qquad \int_{V^n} H_1\theta\, dA + \int_{V^n}\Psi\, dA = 0$$

and

$$(3.13) \qquad \frac{2}{n(n-1)}\int_{V^n}\nabla_c p_a{}^c\beta^a\, dA + \int_{V^n}\Big\{nH_1H_2 - (n-2)H_3\Big\}\theta\, dA + 2\int_{V^n}\Psi H_2\, dA = 0$$

respectively, where dA denotes the area element of V^n, and the integral formula (3.12) is due to Y. Katsurada [8, 9].

Next, we assume the existence of a special concircular scalar field, say, a non constant scalar one Φ which satisfies the partial differential equation defined by

$$(3.14) \qquad \nabla_j\Phi_i = \rho\Phi G_{ji} \qquad (\rho = \text{constant} \neq 0),$$

where $\Phi_i = \nabla_i\Phi$. For example, in an Einstein manifold with scalar curvature R $(\neq 0)$ admitting a conformal Killing vector field ξ^i, we know the fact that the associated scalar field Ψ satisfies the equation (1.4), say,

$$\nabla_j\nabla_i\Psi = -\frac{R}{n(n+1)}\Psi G_{ji} \quad (R = \textit{non zero const.}).$$

If $\Phi = 0$ on V^n, since the second covariant derivative of $\Phi = 0$ along V^n is given by

$$\nabla_j\Phi_i B_b{}^j B_a{}^i + \Phi_i\nabla_b B_a{}^i = 0,$$

substituting (3.14) and (2.4) into this equation and transvecting g^{ba} to the resulting equation, we can see that $H_1\Theta = 0$ on V^n, where $\Theta = C^i\Phi_i$. Hence we have the following

Lemma 3.1 *Let R^{n+1} be a Riemannian manifold which admits the special concircular scalar field Φ. If, on a hypersurface V^n in R^{n+1}, $H_1\Theta$ is not identically zero, then Φ is not identically zero on the V^n.*

Now, on the hypersurface V^n, we can put

$$\Phi^j = B_b{}^j\phi^b + \Theta C^j,$$

where $\Phi^j = \Phi_i G^{ji}$. Transvecting $G_{ji}B_a{}^i$ to this equation and making use of (2.3), we get

$$\phi_a = B_a{}^i\Phi_i, \tag{3.15}$$

from which, by covariant differentiation along V^n and by virtue of (2.4), (3.14) and (2.3), we obtain

$$\nabla_b\phi_a = \Theta h_{ba} + \rho\Phi g_{ba}.$$

And, transvecting g^{ba} to this equation and making use of (2.9), we get

$$\nabla_b\phi^b = n(H_1\Theta + \rho\Phi), \tag{3.16}$$

where $\nabla_b\phi^b = \nabla_b\phi_a g^{ba}$.

We now put

$$\omega_b = h_b{}^a B_a{}^i\Phi_i,$$

from which, by covariant differentiation along V^n, we obtain , by virtue of (2.4), (3.14) and (2.3),

$$\nabla_c\omega_b = \nabla_c h_b{}^a B_a{}^i\Phi_i + h_b{}^a h_{ca}\Theta + \rho\Phi h_{bc}.$$

And, transvecting g^{bc} to this equation, we get

$$\nabla_c\omega^c = \nabla_c h_a{}^c\phi^a + h_c{}^a h_a{}^c\Theta + \rho\Phi h_c{}^c, \tag{3.17}$$

by virtue of (3.15). On the other hand, since

$$h_c{}^c = nH_1 \quad \text{and} \quad h_c{}^a h_a{}^c = n^2H_1^2 - n(n-1)H_2,$$

by virtue of (2.9) and (2.10), we have, from (3.17),

$$\nabla_c\omega^c = \nabla_c h_a{}^c\phi^a + n\Big\{nH_1^2 - (n-1)H_2\Big\}\Theta + n\rho\Phi H_1,$$

and consequently, using (2.8), we obtain,

$$\nabla_c\omega^c = (\nabla_a h_c{}^c - R_{ij}B_a{}^iC^j)\phi^a + n\Big\{nH_1^2 - (n-1)H_2\Big\}\Theta + n\rho\Phi H_1, \tag{3.18}$$

And, since the hypersurface V^n under consideration is orientable and closed, we apply Green's formula [22] to (3.16) and (3.18). Then we obtain

$$\int_{V^n} H_1\Theta dA + \int_{V^n}\rho\Phi dA = 0 \tag{3.19}$$

and

$$\frac{1}{n}\int_{V^n}(\nabla_a h_c{}^c - R_{ij}B_a{}^iC^j)\phi^a dA + \int_{V^n}\Big\{nH_1^2 - (n-1)H_2\Big\}\Theta dA + \int_{V^n}\rho\Phi H_1 dA = 0 \tag{3.20}$$

respectively [9], where dA denotes the area element of V^n.

4 Relations in a Riemannian manifold admitting the special concircular scalar field Φ

Let R^{n+1} be a Riemannian manifold which admits a special concircular scalar field Φ defined by (3.14). Substituting (3.14) into the Ricci identity

$$\nabla_k \nabla_j \Phi_i - \nabla_j \nabla_k \Phi_i = -R_{kji}{}^l \Phi_l,$$

we find that

$$R_{kji}{}^l \Phi_l = \rho(\Phi_j G_{ki} - \Phi_k G_{ji}), \tag{4.1}$$

from which, by covariant differentiation and (3.14), we obtain

$$\nabla_h R_{kji}{}^l \Phi_l = -\rho\Phi\Big\{R_{kjih} - \rho(G_{ki}G_{jh} - G_{kh}G_{ji})\Big\}. \tag{4.2}$$

So, transvecting G^{ji} to this equation, we obtain

$$\nabla_h R_{kl} \Phi^l = -\rho\Phi(R_{kh} + n\rho G_{kh}), \tag{4.3}$$

where R_{kh} is the Ricci tensor of R^{n+1}, and if we put

$$S_{kh} = R_{kh} + n\rho G_{kh}, \tag{4.4}$$

then the tensor S_{kh} is symmetric in k and h, and consequently, (4.3) is rewritten as follows:

$$\nabla_h R_{kl} \Phi^l = -\rho\Phi S_{hk}. \tag{4.5}$$

Moreover, transvecting G^{hk} to this equation and making use of $\nabla_h R^h{}_l = (1/2)\nabla_l R$, we get

$$\nabla_l R \Phi^l = -2\rho\Phi S, \tag{4.6}$$

where $S = S_{hk}G^{hk}$, and $R = R_{hk}G^{hk}$, say, R is the scalar curvature of R^{n+1}. Also, transvecting G^{hk} to (4.4), we obtain

$$S = R + n(n+1)\rho. \tag{4.7}$$

Therefore, using (4.6), (4.7) and Lemma 3.1, we have

Lemma 4.1 *Let R^{n+1} be a Riemannian manifold which admits a special consircular scalar field Φ such that $\nabla_j \Phi_i = \rho\Phi G_{ji}$ (ρ = constant $\neq 0$). If its scalar curvature R is non zero constant and $H_1\Theta$ has fixed sign on V^n, we have*

$$\rho = -\frac{R}{n(n+1)}.$$

Next, transvecting G^{ji} to (4.1), we get

$$R_{kl}\Phi^l + n\rho\Phi_k = 0.$$

Thus, from (4.4), we have

(4.8) $$S_{kl}\Phi^l = 0.$$

Now, from $R_{kjil} = R_{lijk}$, the left-hand side of (4.2) is equal to $\nabla_h R_{lijk}\Phi^l$. Thus, transvecting G^{hk} to (4.2), we get, from (4.4),

(4.9) $$\nabla_h R_{lij}{}^h\Phi^l = -\rho\Phi S_{ij}.$$

On the other hand, transvecting G^{hk} to the Bianchi's identity, say, $\nabla_h R_{lijk} + \nabla_l R_{ihjk} + \nabla_i R_{hljk} = 0$, we find that

(4.10) $$\nabla_h R_{lij}{}^h = \nabla_l R_{ij} - \nabla_i R_{lj},$$

and consequently, transvecting Φ^l to this equation, we get, from (4.5) and (4.9),

(4.11) $$\nabla_l R_{ij}\Phi^l = -2\rho\Phi S_{ij}.$$

Lemma 4.2 *Let R^{n+1} be a Riemannian manifold which admits a special consircular scalar field Φ. If the scalar field Φ satisfies the following equation:*

(4.12) $$\Phi S_{kj} = 0,$$

then we have

$$(\Phi_l\Phi^l)S_{kj} = 0 \qquad \text{in } R^{n+1}.$$

Proof. Covariantly differentiating (4,12), we get, from (4.4),

$$\Phi_l S_{kj} + \Phi\nabla_l R_{kj} = 0.$$

Transvecting Φ^l to this equation and using of (4.10), we obtain

$$(\Phi_l\Phi^l)S_{kj} - 2\rho\Phi^2 S_{kj} = 0,$$

from which, taking account of the assumption (4.12), we conclude that Lemma 4.2 holds good.

5 A closed hypersurface with H_2=const.> 0 in R^{n+1} admitting a conformal Killing vector field ξ^i

From (2.9) and (2.10), it follows that

$$n^2 H_1^2 = n(n-1)H_2 + h_a{}^b h_b{}^a. \tag{5.1}$$

If we assume that the second mean curvature H_2 is positive constant, then the left hand menber of (5.1) is positive. Hence, there exists no point P on V^n such that $H_1(P) = 0$. Accordingly, H_1 must have fixed sign on V^n. Therefore we have

Lemma 5.1 *If the second mean curvature H_2 of V^n is positive constant, then the first mean curvature H_1 of V^n has fixed sign on V^n.*

We now assume that R^{n+1} is an Einstein manifold, say, $R_{kj} = (R/(n+1))G_{kj}$, where R_{kj} $(= R_{lkji}G^{li})$ and R $(= R_{kj}G^{kj})$ are the Ricci tensor and the scalar curvature of R^{n+1} respectively. Transvecting g^{da} to the equations of Gauss (2.6), we have

$$K_{cb} = (h_a{}^a h_{cb} - h_c{}^a h_{ab}) + R_{kjih} B_d{}^k B_c{}^j B_b{}^i B_a{}^h g^{da},$$

where K_{cb} is the Ricci tensor of V^n. Making use of (3.5) and $B_d{}^k B_a{}^h g^{da} = G^{kh} - C^k C^h$, we can write in the form

$$K_{cb} = p_{cb} + R_{ji} B_c{}^j B_b{}^i - R_{kjih} C^k B_c{}^j B_b{}^i C^h.$$

Accordingly, since R^{n+1} is Einstein, we obtain

$$K_{cb} = p_{cb} + \frac{R}{n+1} g_{cb} - R_{kjih} C^k B_c{}^j B_b{}^i C^h.$$

Moreover, transvecting g^{cb} to this equation, we have

$$K = p_b{}^b + \frac{nR}{n+1} - R_{kh} C^k C^h,$$

where K is the scalar curvature of V^n and $p_b{}^b = p_{cb} g^{cb}$. Consequently, since R^{n+1} is an Einstein manifold and $p_b{}^b = n(n-1)H_2$, we get

$$K = n(n-1)H_2 + \frac{n-1}{n+1} R.$$

Therefore, making use of the fact that the scalar curvature R in an Einstein manifold R^{n+1} is constant, we finally reach the following

Lemma 5.2 *Let V^n be a hypersurface in an Einstein manifold R^{n+1}. Then, necessary and sufficient condition that the second mean curvature H_2 be constant is that the scalar curvature K of V^n be constant.*

Next, as a special case of the Lemma, let R^{n+1} be a Riemannian manifold of constant curvature κ , say, $R_{kjih} = \kappa(G_{kh}G_{ji} - G_{ki}G_{jh})$, where $\kappa = R/n(n+1)$. Then, using (2.3), the equations of Gauss (2.6) is written in the form

$$K_{dcba} = h_{da}h_{cb} - h_{ca}h_{db} + \kappa(g_{da}g_{cb} - g_{ca}g_{db}).$$

Similarly, transvecting g^{da} to this equation, we have

$$K_{cb} = p_{cb} + (n-1)\kappa g_{cb},$$

from which, by covariant differentiation along V^n, we get

$$\nabla_a K_{cb} = \nabla_a p_{cb} + (n-1)\nabla_a \kappa g_{cb},$$

and consequently, since κ is constant, we obtain

$$\nabla_a K_{cb} = \nabla_a p_{cb}.$$

Moreover, transvecting g^{ab} to this equation, we have

(5.2) $$\nabla_b K_c{}^b = \nabla_b p_c{}^b.$$

On the other hand, as is well-known, the following equation is valid for any Riemannian manifolds:

(5.3) $$\nabla_b K_c{}^b = \frac{1}{2}\nabla_c K.$$

Accordingly, from (5.2) and (5.3), we get

$$\nabla_b p_c{}^b = \frac{1}{2}\nabla_c K.$$

Consequently, from Lemma 5.2, we have

Lemma 5.3 *Let V^n be a hypersurface in a Riemannian manifold of constant curvature. If the second mean curvature H_2 of V^n is constant, then $\nabla_b p_c{}^b = 0$ on V^n.*

Now we can prove the following

Theorem 5.4 *Let R^{n+1} $(n \geq 3)$ be an orientable Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i and V^n a closed orientable hypersurface in R^{n+1} such that*
(1) H_2 is positive constant,
(2) $\theta(= C^i\xi_i)$ has fixed sign on V^n,
then every point of V^n is umbilic.

PROOF. Multiplying (3.12) by $2H_2$ (=positive constant), we obtain

$$\int_{V^n} 2H_1H_2\theta\, dA + 2\int_{V^n} \Psi H_2\, dA = 0,$$

and subtracting the above formula from (3.13), we find that

$$\frac{2}{n(n-1)}\int_{V^n} \nabla_c p_a{}^c \beta^a\, dA + (n-2)\int_{V^n} \{H_1H_2 - H_3\}\theta\, dA = 0.$$

So, from Lemms 5.3 and $n \geq 3$, we have

$$\int_{V^n} \{H_1H_2 - H_3\}\theta\, dA = 0. \tag{5.4}$$

Moreover, from Lemma 5.1, that is, from the fact that H_1 has fixed sign on V^n, the scalar field on V^n defined by $H_1H_2 - H_3$ is rewritten as follows:

$$H_1H_2 - H_3 = \frac{1}{H_1}\{H_2(H_1^2 - H_2) + (H_2^2 - H_1H_3)\}. \tag{5.5}$$

On the other hand, we know the fact that

$$H_a^2 - H_{a-1}H_{a+1} \geq 0 \qquad (a = 1, 2, \cdots, n-1)\ ([5, 1]),$$

where $H_0 = 1$. In particular, we see that

$$H_1^2 - H_2 \geq 0 \quad \text{and} \quad H_2^2 - H_1H_3 \geq 0 \quad \text{on } V^n. \tag{5.6}$$

Accordingly, making use of the assumption (1) and Lemma 5.1, we have, from (5.5),

$$H_1H_2 - H_3 \geq 0 \quad (\text{or} \leq 0) \quad \text{on } V^n.$$

Hence, using the assumption (2), from (5.4), we find that

$$H_1H_2 - H_3 = 0 \quad \text{on } V^n.$$

Consequently, from (5.5), it follows that

$$H_2(H_1^2 - H_2) + (H_2^2 - H_1H_3) = 0 \quad \text{on } V^n,$$

from which, from (5.6) and the assumption (1), we see that

$$H_1^2 - H_2 = 0 \quad \text{on } V^n.$$

Therefore, since

$$H_1^2 - H_2 = \frac{1}{n^2(n-1)}\sum_{a<b}(k_b - k_a)^2,$$

we conclude that

$$k_1 = k_2 = \cdots = k_n$$

at each point of V^n. (Then we have $H_1 = k_1, H_2 = k_1^2$ and $H_3 = k_1^3$, from which we obtain $H_2^2 - H_1H_3 = 0$.) This means that each point of V^n is umbilic.

6 Closed hypersurfaces with H_1= const.$\neq 0$ in R^{n+1} admitting a special concircular scalar field Ψ

Now, in R^{n+1}, we assume the existence of a special concircular scalar field, say, a non constant scalar one Ψ which satisfies the following partial differential equation

$$\nabla_j \Psi_i = (\rho\Psi + \sigma)G_{ji} \quad (\rho = \text{const.} \neq 0,\ \sigma = \text{const.}), \tag{6.1}$$

where $\Psi_i = \nabla_i \Psi$.

If we put

$$\overline{\Phi} = \rho\Psi + \sigma,$$

then (6.1) becomes

$$\nabla_j \Psi_i = \overline{\Phi} G_{ji}, \tag{6.2}$$

and, by covariant differetiation, we have

$$\overline{\Phi}_i = \rho\Psi_i, \tag{6.3}$$

where $\overline{\Phi}_i = \nabla_i \overline{\Phi}$. Moreover, by covariant differentiation of (6.3), from (6.2), we find that

$$\nabla_j \overline{\Phi}_i = \rho\overline{\Phi} G_{ji},$$

that is, the scalar field $\overline{\Phi}$ satisfies the same partial differential equation as Φ given by (3.14). Therefore, if we put anew

$$\Phi = \rho\Psi + \sigma, \tag{6.4}$$

then, transvecting C^i to (6.3), we have

$$C^i\Phi_i = \rho C^i \Psi_i \qquad \text{on } V^n, \tag{6.5}$$

from which we get the following

Lemma 6.1 *If $C^i\Psi_i (= \Omega)$ has fixed sign on V^n, and is not constant along V^n, then the same holds good also for $C^i\Phi_i (= \Theta)$ respectively.*

Thus, without loss of generality we may discuss the properties of a hypersurface V^n in R^{n+1} admitting a concircular scalar field Φ defined by (3.14), that is,

$$\nabla_j \Phi_i = \rho\Phi G_{ji} \qquad (\rho = \text{const.} \neq 0).$$

Now, if we assume that the first mean curvature of V^n is non zero constant, say, $H_1 = \text{const.} \neq 0$, then , taking account of (2.9), we obtain, from (3.20),

$$-\frac{1}{n}\int_{V^n} R_{ij} B_a{}^i C^j \phi^a dA + \int_{V^n} \left\{ nH_1^2 - (n-1)H_2 \right\} \Theta dA + H_1 \int_{V^n} \rho\Phi dA = 0. \tag{6.6}$$

Eliminating $\int_{V^n} \rho\Phi dA$ from (3.19) and (6.6), we find that

$$-\frac{1}{n}\int_{V^n} R_{ij}B_a{}^iC^j\phi^a dA + (n-1)\int_{V^n}(H_1^2 - H_2)\Theta dA = 0. \tag{6.7}$$

First, we shall prove the following Theorem:

Theorem 6.2 *Let R^{n+1} be an orientable Riemannian manifold with harmonic curvature, say, $\nabla_k R_{ji} = \nabla_j R_{ki}$ which admits a special concircular scalar field Ψ satisfying the equation*

$$\nabla_j\Psi_i = (\rho\Psi + \sigma)G_{ji} \quad (\rho = const. \neq 0,\ \sigma = const.),$$

and V^n a closed orientable hypersurface in R^{n+1} such that

(1) $H_1 = const. \neq 0$,

(2) Ω has fixed sign on V^n, where $\Omega = C^i\Psi_i$.

Then every point of V^n is umbilic.

PROOF. Transvecting Φ^k to the assumption $\nabla_k R_{ji} = \nabla_j R_{ki}$, we get, from (4.11) and (4.5), $\Phi S_{ji} = 0$. Thus, using Lemma 4.2, we have $(\Phi_k\Phi^k)S_{ji} = 0$ in R^{n+1}. Accordingly, making use of the assumption (2) and Lemma 6.1, we can see that $S_{ji} = 0$ on V^n, say, $R_{ji} = -n\rho G_{ji}$ on V^n. Consequently, taking account of the assumption (1), we obtain, from (6.7),

$$\int_{V^n}(H_1^2 - H_2)\Theta dA = 0. \tag{6.8}$$

Also, since

$$H_1^2 - H_2 = \frac{1}{n^2(n-1)}\sum_{a<b}(k_b - k_a)^2, \tag{6.9}$$

we can see easily that $H_1^2 - H_2 \geq 0$. Thus, using the assumption (2) and Lemma 6.1, we conclude that $H_1^2 - H_2 = 0$ by virtue of (6.8), and therefore, because of (6.9), that

$$k_1 = k_2 = \cdots = k_n$$

at each point of V^n, that is, every point of V^n is umbilic.

So, we have the following

Corollary 6.3 *Let R^{n+1} be an orientable Riemannian manifold with parallel Ricci tensor, say, $\nabla_k R_{ji} = 0$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) Ω has fixed sign on V^n.

Then every point of V^n is umbilic.

Next, in a conformally flat Riemannian manifold R^{n+1},

$$R_{kji}{}^{h} = -\frac{1}{n-1}(R_{ki}\delta_j^{\,h} - R_{ji}\delta_k^{\,h} + G_{ki}R_j^{\,h} - G_{ji}R_k^{\,h}) + \frac{R}{n(n-1)}(G_{ki}\delta_j^{\,h} - G_{ji}\delta_k^{\,h}).$$

By covariant differentiation, we have

$$\nabla_l R_{kji}{}^{h} = -\frac{1}{n-1}(\nabla_l R_{ki}\delta_j^{\,h} - \nabla_l R_{ji}\delta_k^{\,h} + G_{ki}\nabla_l R_j^{\,h} - G_{ji}\nabla_l R_k^{\,h}) + \frac{\nabla_l R}{n(n-1)}(G_{ki}\delta_j^{\,h} - G_{ji}\delta_k^{\,h}),$$

from which, replacing l by h and summing for h, we have

$$\nabla_h R_{kji}{}^{h} = -\frac{1}{n-1}(\nabla_j R_{ki} - \nabla_k R_{ji} + G_{ki}\nabla_h R_j^{\,h} - G_{ji}\nabla_h R_k^{\,h}) + \frac{1}{n(n-1)}(G_{ki}\nabla_j R - G_{ji}\nabla_k R).$$

And, making use of (4.10) and $\nabla_h R_j^{\,h} = (1/2)\nabla_j R$, we find that

$$\nabla_j R_{ki} - \nabla_k R_{ji} - \frac{1}{2n}(G_{ki}\nabla_j R - G_{ji}\nabla_k R) = 0 \quad (n > 2). \tag{6.10}$$

Also, in case $n = 2$, a conformally flat Riemannian manifold is defined by (6.10). Therefore, assuming the scalar curvature R of R^{n+1} to be constant, we see easily, from (6.10), that $\nabla_j R_{ki} - \nabla_k R_{ji} = 0$. Thus, using Theorem 6.2, we have the following

Corollary 6.4 *Let R^{n+1} $(n \geq 2)$ be an orientable conformally flat Riemannian manifold with R=const. which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) *Ω has fixed sign on V^n.*

Then every point of V^n is umbilic,

Moreover, making use of (4.5) and (4.11), we can prove the following Theorem by an argument similar to that used in the proof of Theorem 6.2:

Theorem 6.5 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) *Ω has fixed sign on V^n.*

Then every point of V^n is umbilic.

We now assume that the length of the Ricci tensor R_{ji} is constant in R^{n+1}, that is,

$$R^{ji}R_{ji} = \lambda \qquad (\lambda = \text{const.}). \tag{6.11}$$

By covariant differentiation, we have

$$\nabla_k R^{ji}R_{ji} = 0. \tag{6.12}$$

On the other hand, from (4.4), (4.7) and (6.11), we can see that

$$S^{ji}S_{ji} = 2n\rho S - n^2(n+1)\rho^2 + \lambda. \tag{6.13}$$

Also, transvecting $\nabla_k R^{ji}$ to $S_{ji} = R_{ji} + n\rho G_{ji}$, say, (4.4) and using (6.12), we have

$$\nabla_k R^{ji} S_{ji} = n\rho \nabla_k R,$$

and, moreover, transvecting Φ^k to this equation, from (4.11) and (4.6), we find that

$$\Phi S^{ji} S_{ji} = n\rho \Phi S. \tag{6.14}$$

Consequently, substituting (6.13) into (6.14), we can see easily that

$$\Phi\Big\{n\rho S - n^2(n+1)\rho^2 + \lambda\Big\} = 0. \tag{6.15}$$

So, covariantly differentiating and using the fact that $\nabla_i S$ is equal to $\nabla_i R$, we have

$$\Phi_i\Big\{n\rho S - n^2(n+1)\rho^2 + \lambda\Big\} + n\rho\Phi\nabla_i R = 0,$$

from which, multiplying by Φ and using (6.15), we obtain

$$\Phi^2 \nabla_i R = 0. \tag{6.16}$$

Lemma 6.6 *Let R^{n+1} be a Riemannian manifold with $R^{ji}R_{ji}=$ const. which admits a special concircular scalar field Ψ, V^n a hypersurface in R^{n+1} and Ω fixed sign on V^n. If $\mu \neq 0$, then we have $\nabla_i R = 0$ in R^{n+1}, and if $\mu = 0$, then we have $\nabla_i R = 0$ on V^n, where μ is a constant number defined by*

$$\mu = \rho\Phi^2 - \Phi_j\Phi^j. \tag{6.17}$$

Proof. Since $\nabla_i(\rho\Phi^2 - \Phi_j\Phi^j) = 0$, we see that μ is constant in R^{n+1}.

Now, by covariant differentiation of (6.16), we get

$$2\Phi\Phi_j\nabla_i R + \Phi^2\nabla_j\nabla_i R = 0,$$

from which, using the fact that the tensor $\nabla_j\nabla_i R$ is symmetric, we obtain $\Phi(\Phi_j\nabla_i R - \Phi_i\nabla_j R) = 0$. Moreover, covariantly differentiating and using (3.14), we have

$$\Phi_k(\Phi_j\nabla_i R - \Phi_i\nabla_j R) + \rho\Phi^2(G_{kj}\nabla_i R - G_{ki}\nabla_j R) + \Phi(\Phi_j\nabla_k\nabla_i R - \Phi_i\nabla_k\nabla_j R) = 0,$$

and accordingly, using (6.16), we obtain

$$\Phi_k(\Phi_j\nabla_i R - \Phi_i\nabla_j R) + \Phi(\Phi_j\nabla_k\nabla_i R - \Phi_i\nabla_k\nabla_j R) = 0.$$

And consequently, transvecting G^{kj} to this equation, we have, from (4.6), (6.17) and (6.16),

$$-\mu\nabla_i R + \Phi\Big\{\Phi^j\nabla_j\nabla_i R + \Phi_i(2\rho S - \nabla^j\nabla_j R)\Big\} = 0, \tag{6.18}$$

where $\nabla^j\nabla_j R = G^{kj}\nabla_k\nabla_j R$. Thus, if $\mu \neq 0$, then we have, from (6.16) at a point P_1 such that $\Phi(P_1) \neq 0$, or from (6.18) at a point P_2 such that $\Phi(P_2) = 0$, $\nabla_i R = 0$. And if $\mu = 0$, then we have, from (6.17), $\rho\Phi^2 = \Phi_j\Phi^j$. On the other hand, using the assumption that Ω has fixed sign on V^n, we can see, from Lemma 6.1, that Θ has fixed sign on V^n, and accordingly, $\Phi_j\Phi^j > 0$ on V^n. Thus, it follows from (6.16) that $\nabla_i R = 0$ on V^n.

Lemma 6.7 *Let R^{n+1} be an orientable Riemannian manifold which admits a special concircular scalar field Ψ and V^n a closed orientable hypersurface in R^{n+1} such that*
(1) $H_1 = const. \neq 0$,
(2) Ω *has fixed sign on* V^n.
If $\nabla_i R = 0$ *on* V^n, *then we have* $S = 0$ *on* V^n.

PROOF. Transvecting $B_a{}^i$ to $\nabla_i R = 0$, we get $\nabla_a R = 0$, that is, R=const. on V^n. Consequently, because of (4.7), we have

$$S = \text{const.} \qquad \text{on } V^n. \tag{6.19}$$

Moreover, using the assumption that $\nabla_i R = 0$ on V^n, we get, from (4.6),

$$\Phi S = 0 \qquad \text{on } V^n.$$

Here Φ is not identically zero on V^n. Because, using the assumption (2) and Lemma 6.1, we can see that Θ has fixed sign on V^n, and accordingly, $H_1\Theta$ is not identically zero on V^n, which shows that this fact holds because of Lemma 3.1. Thus, taking account of (6.19), we can prove that $S = 0$ on V^n.

Next, making use of these Lemmas, we shall prove the following

Theorem 6.8 *Let R^{n+1} be an orientable Riemannian manifold with $R^{ji}R_{ji} = const.$ which admits a special concircular scalar field Ψ and V^n a closed orientable hypersurface in R^{n+1} such that*
(1) $H_1 = const. \neq 0$,
(2) Ω *has fixed sign on* V^n.
Then every point of V^n *is umbilic.*

PROOF. Transvectin Φ^k to (6.12), and using (4.11) and (4.4), we obtain

$$\Phi(S^{ji}S_{ji} - n\rho S) = 0.$$

Now, by covariant differentiation, we get, from (4.4) and (4.7),

$$\Phi_k(S^{ji}S_{ji} - n\rho S) + \Phi(2\nabla_k R^{ji}S_{ji} - n\rho\nabla_k R) = 0,$$

and, substituting (4.4) into the second term of the left-hand side of this equation and using (6.12), we can see that

$$\Phi_k(S^{ji}S_{ji} - n\rho S) + n\rho\Phi\nabla_k R = 0.$$

Thus, making use of Lemma 6.6 and Lemma 6.7, and transvecting Φ^k to the result equation, it follows that

$$(\Phi_k\Phi^k)(S^{ji}S_{ji}) = 0 \qquad \text{on } V^n,$$

and, since, taking account of the assumption that Ω has fixed sign on V^n, we can prove from Lemma 6.1 that $\Phi_k\Phi^k > 0$ on V^n, we conclude that

$$S_{ji} = 0, \quad \text{that is,} \quad R_{ji} = -n\rho G_{ji} \qquad \text{on } V^n.$$

Therefore, from (6.7), we obtain

$$\int_{V^n} (H_1^2 - H_2)\Theta dA = 0,$$

and consequently, by the argument similar to that used in the proof of Theorem 6.2, we can prove that Theorem 6.8 holds.

REMARK 1. Theorem 6.8 is a generalization of Corollary 6.3, because of the fact that $\nabla_k(R_{ji}R^{ji}) = 2\nabla_k R_{ji}R^{ji}$.

Moreover, as a generalization of Corollary 6.4, we have the following

Theorem 6.9 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(1) $H_1 = const. \neq 0$,
(2) *there exists a point P_0 on V^n such that $S(P_0) = 0$,*
(3) *Ω has fixed sign on V^n.*
Then every point of V^n is umbilic.

PROOF. In a conformally flat Riemannian manifold R^{n+1},

$$R_{kji}{}^h = -\frac{1}{n-1}(R_{ki}\delta_j{}^h - R_{ji}\delta_k{}^h + G_{ki}R_j{}^h - G_{ji}R_k{}^h) + \frac{R}{n(n-1)}(G_{ki}\delta_j{}^h - G_{ji}\delta_k{}^h).$$

Using the same calculations as the proof of Corollary 6.4, we obtain (6.10), that is,

$$\nabla_j R_{ki} - \nabla_k R_{ji} - \frac{1}{2n}(G_{ki}\nabla_j R - G_{ji}\nabla_k R) = 0. \quad (n \geq 2)$$

Now, transvecting $2n\Phi^k$ to this equation and making use of (4.5), (4.11) and (4.6), we get

$$2n\rho\Phi S_{ji} - (\nabla_j R\Phi_i + 2\rho\Phi S G_{ji}) = 0. \tag{6.20}$$

Moreover, transvecting Φ^i to this equation and making use of (4.8), we have

$$\nabla_j R\Phi_i\Phi^i + 2\rho\Phi S\Phi_j = 0. \tag{6.21}$$

And consequently, making use of (3.14) and (4.7), (6.21) is rewritten as follows:

$$\nabla_j(S\Phi_i\Phi^i) = 0, \tag{6.22}$$

from which, by the assumptions that there exists a point P_0 on V^n such that $S(P_0) = 0$, and the hypersurface V^n is closed, we find that

$$S\Phi_i\Phi^i = 0 \tag{6.23}$$

on V^n.

On the other hand, transvecting S^{ji} to (6.20) and making use of (4.8), we obtain

$$\Phi(nS_{ji}S^{ji} - S^2) = 0. \tag{6.24}$$

By covariant differentiation, we have, from (4.4) and (4.7),

$$\Phi_h(nS_{ji}S^{ji} - S^2) + 2\Phi(n\nabla_h R_{ji}S^{ji} - \nabla_h RS) = 0.$$

And, transvecting Φ^h to this equation, from (4.11), (4.6) and (6.24), we find that

$$\Phi_h\Phi^h(nS_{ji}S^{ji} - S^2) = 0. \tag{6.25}$$

Thus, making use of (6.23), we have $\Phi_h\Phi^h S_{ji}S^{ji} = 0$ on V^n. And, from the assumption (3), we find that $S_{ji} = 0$ on V^n, that is, $R_{ji} = -n\rho G_{ji}$ on V^n. Consequently, from (6.7), we obtain

$$\int_{V^n} \Big\{H_1^2 - H_2\Big\}\Theta dA = 0. \tag{6.26}$$

Also, we can see that $H_1^2 - H_2 \geq 0$, because

$$H_1^2 - H_2 = \frac{1}{n^2(n-1)}\sum_{b<a}(k_b - k_a)^2, \tag{6.27}$$

Thus, using the assumption (3) and Lemma 6.1, we find that $H_1^2 - H_2 = 0$ by virtue of (6.26), and consequently, because of (6.27), we conclude that $k_1 = k_2 = \cdots = k_n$ at each point of V^n. This means that every point of V^n is umbilic.

Remark 2. Theorem 6.9 is a generalization of Corollary 6.4. Because, using the assumption that R=const. and (4.6), we have $\Phi S = 0$, from which, taking account of Lemma 3.1, we can see that there exists a point P_0 on V^n such that $S(P_0) = 0$.

7 Some characterizations of a hypersurface to be isometric to a sphere

To prove that the hypersurface under consideration is isometric to a sphere, we need the following Theorem due to M. Obata [16].

Theorem A. (Obata) *Let V^n $(n \geq 2)$ be a complete Riemannian manifold which admits a non-null function ψ such that $\nabla_b \nabla_a \psi = -c^2 \psi\, g_{ba}$ ($c = const.$), where g_{ba} and ∇_a denote the metric tensor of V^n and the operator of covariant differentiation with respect to Christoffel symbols $\left\{ {c \atop ba} \right\}$ formed with g_{ba} respectively. Then V^n is isometric to a sphere of radius $1/c$.*

Now, using Theorem 6.2, we can prove the following

Theorem 7.1 *Let R^{n+1} be an orientable Riemannian manifold with harmonic curvature, say, $\nabla_k R_{ji} = \nabla_j R_{ki}$ which admits a special concircular scalar field Ψ satisfying the equation*

$$\nabla_j \Psi_i = (\rho \Psi + \sigma) G_{ji} \quad (\rho = const. \neq 0, \sigma = const.),$$

and V^n a closed orientable hypersurface in R^{n+1} such that

(1) $H_1 = const. \neq 0$,

(2) Ω has fixed sign on V^n, where $\Omega = C^i \Psi_i$,

(3) Ω is not constant along V^n.

Then V^n is isometric to a sphere.

Proof. By covariant differentiation of the definition $\Theta = C^i \Phi_i$ along V^n, we have, from (2.5) and (3.14),

$$\nabla_b \Theta = -h_b{}^a B_a{}^i \Phi_i. \tag{7.1}$$

Also, by virtue of Theorem 6.2, every point of V^n is umbilic, that is,

$$h_{bc} = H_1 g_{bc}. \tag{7.2}$$

Transvecting g^{ca} to this equation, we see that $h_b{}^a = H_1 \delta_b{}^a$. So, substituting this equation into (7.1), we have

$$\nabla_b \Theta = -H_1 B_b{}^i \Phi_i, \tag{7.3}$$

that is,

$$\nabla_b \Theta + H_1 \nabla_b \Phi = 0. \tag{7.4}$$

Accordingly, under the assumption that $H_1 = const.$, we can see that

$$\Theta + H_1 \Phi = C \quad (C = const.) \tag{7.5}$$

on V^n.

Now, by covariant differentiation of (7.3) along V^n, we get

$$\nabla_c\nabla_b\Theta = -H_1(\rho\Phi g_{cb} + \Theta h_{cb}), \tag{7.6}$$

by virtue of (2.4), (3.14) and (2.3). Thus, from (7.2) and (7.5), we find that

$$\nabla_c\nabla_b\Theta = -\Big\{(H_1^2 - \rho)\Theta + \rho C\Big\}g_{cb}. \tag{7.7}$$

Here we claim $H_1^2 - \rho \neq 0$. In fact, if $H_1^2 - \rho = 0$, then (7.7) becomes $\nabla_c\nabla_b\Theta = -\rho C g_{cb}$, from which $\Delta\Theta = -n\rho C$, that is, $\Delta\Theta = const.$, where $\Delta\Theta = g^{cb}\nabla_c\nabla_b\Theta$. However this is impossible, unless $\Theta = const.$ on V^n [22, 2]. But, using the assumption (3) and Lemma 6.1, we can see that Θ is not constant along V^n. Thus, $H_1^2 - \rho$ being different from zero, (7.7) is rewritten as follows:

$$\nabla_c\nabla_b\Big(\Theta + \frac{\rho C}{H_1^2 - \rho}\Big) = -(H_1^2 - \rho)\Big(\Theta + \frac{\rho C}{H_1^2 - \rho}\Big)g_{cb}, \tag{7.8}$$

from which we get

$$\Delta\Big(\Theta + \frac{\rho C}{H_1^2 - \rho}\Big) = -n(H_1^2 - \rho)\Big(\Theta + \frac{\rho C}{H_1^2 - \rho}\Big),$$

and consequently, it follows that $H_1^2 - \rho > 0$ [20]. Therefore, using Theorem A, the equation (7.8) shows that the hypersurface V^n under consideration is isometric to a sphere [10, 11].

And, from this Theorem 7.1, we have

Corollary 7.2 *Let R^{n+1} be an orientable Riemannian manifold with parallel Ricci tensor, say, $\nabla_k R_{ji} = 0$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,
(2) *Ω has fixed sign on V^n.*
(3) *Ω is not constant along V^n.*

Then V^n is isometric to a sphere.

Corollary 7.3 *Let R^{n+1} be an orientable conformally flat Riemannian manifold with R =const. which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,
(2) *Ω has fixed sign on V^n.*
(3) *Ω is not constant along V^n.*

Then V^n is isometric to a sphere.

Moreover, using Theorem 6.5, Theorem 6.8 and Theorem 6.9 respectively, we obtain

Theorem 7.4 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) Ω *has fixed sign on* V^n.

(3) Ω *is not constant along* V^n.

Then V^n is isometric to a sphere.

Theorem 7.5 *Let R^{n+1} be an orientable Riemannian manifold with $R^{ji}R_{ji} = const.$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) Ω *has fixed sign on* V^n.

(3) Ω *is not constant along* V^n.

Then V^n is isometric to a sphere.

Theorem 7.6 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) *there exists a point P_0 on V^n such that $S(P_0) = 0$,*

(3) Ω *has fixed sign on* V^n.

(4) Ω *is not constant along* V^n.

Then V^n is isometric to a sphere.

Lemma 7.7 *Let V^n be a hypersurface with H_2= positive const. in R^{n+1}. If V^n is an umbilical hypersurface, then H_1 is non zero constant.*

Proof. Since V^n is an umbilical hypersurface, we have

$$k_1 = k_2 = \cdots = k_n,$$

from which, taking account of (2.10), we obtain

$$H_2 = k_1^2 = H_1^2.$$

Accordingly, since H_2 is positive constant, we can see that H_1 is non zero constant.

Thus, using Theorem 5.4 and Lemma 7.7, we have the following

Theorem 7.8 *Let R^{n+1} $(n \geq 3)$ be an orientable Riemannian manifold of constant curvature which admits a proper conformal Killing vector field ξ^i satisfying the equation*

$$\pounds_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

and V^n a closed orientable hypersurface in R^{n+1} such that

(1) H_2 *is positive constant,*

(2) Ω $(= C^i\Psi_i)$ *has fixed sign on* V^n,

(3) Ω *is not constant along* V^n.

Then V^n *is isometric to a sphere, where* Ψ *satisfies* $\nabla_j\nabla_i\Psi = -(R/n(n+1))\Psi G_{ji}$ *and* $\Psi_i = \nabla_i\Psi$.

Next, under the new assumption that Ψ is not constant along V^n, instead of the condition of Ω, we prove the following Theorem in a similar way:

Theorem 7.9 *Let* R^{n+1} *be an orientable Riemannian manifold with harmonic curvature which admits a special concircular scalar field* Ψ, *and* V^n *a closed orientable hypersurface in* R^{n+1} *such that*

(1) $H_1 = const. \neq 0$,

(2) Ω *has fixed sign on* V^n,

(3) Ψ *is not constant along* V^n.

Then V^n *is isometric to a sphere.*

Proof. Since, $\nabla_b(\Phi_i B_a{}^i) = \nabla_j\Phi_i B_b{}^j B_a{}^i + \Phi_i\nabla_b B_a{}^i$, we see, from (3.14), (2.4) and $\Theta = C^i\Phi_i$, that

$$\nabla_b\nabla_a\Phi = \rho\Phi g_{ba} + \Theta h_{ba}. \tag{7.9}$$

Also, by virtue of Theorem 6.2, every point of V^n is umbilic, that is, $h_{ba} = H_1 g_{ba}$. Consequently, from (7.9), we have

$$\nabla_b\nabla_a\Phi = (\rho\Phi + H_1\Theta)g_{ba}.$$

So, substituting (7.5) into this equation, we find that

$$\nabla_b\nabla_a\Phi = \Big\{-(H_1^2 - \rho)\Phi + CH_1\Big\}g_{ba}. \tag{7.10}$$

Here, using the assumption (3) and (6.4), it follows that Φ is not constant along V^n, and accordingly, we can prove that $H_1^2 - \rho \neq 0$, by an argument similar to that used in the proof of Theorem 7.1. Thus, (7.10) is rewritten as follows:

$$\nabla_b\nabla_a\Big(\Phi - \frac{CH_1}{H_1^2 - \rho}\Big) = -(H_1^2 - \rho)\Big(\Phi - \frac{CH_1}{H_1^2 - \rho}\Big)g_{ba}, \tag{7.11}$$

from which we get

$$\Delta\Big(\Phi - \frac{CH_1}{H_1^2 - \rho}\Big) = -n(H_1^2 - \rho)\Big(\Phi - \frac{CH_1}{H_1^2 - \rho}\Big),$$

and consequently, it follows that $H_1^2 - \rho > 0$. Therefore, using Theorem A, the hypersurface V^n is isometric to a sphere [10, 21], by virtue of (7.11).

And, from this Theorem 7.9, we obtain

Corollary 7.10 *Let R^{n+1} be an orientable Riemannian manifold with parallel Ricci tensor which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) *Ω has fixed sign on V^n,*

(3) *Ψ is not constant along V^n.*

Then V^n is isometric to a sphere.

Corollary 7.11 *Let R^{n+1} be an orientable conformally flat Riemannian manifold with R =const. which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) *Ω has fixed sign on V^n,*

(3) *Ψ is not constant along V^n.*

Then V^n is isometric to a sphere.

Moreover, using Theorem 6.5, Theorem 6.8 and Theorem 6.9 respectively, we obtain similarly the following

Theorem 7.12 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) *Ω has fixed sign on V^n,*

(3) *Ψ is not constant along V^n.*

Then V^n is isometric to a sphere.

Theorem 7.13 *Let R^{n+1} be an orientable Riemannian manifold with $R^{ji}R_{ji} = const.$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) *Ω has fixed sign on V^n,*

(3) *Ψ is not constant along V^n.*

Then V^n is isometric to a sphere.

Theorem 7.14 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = const. \neq 0$,

(2) *there exists a point P_0 on V^n such that $S(P_0) = 0$,*

(3) *Ω has fixed sign on V^n,*

(4) *Ψ is not constant along V^n.*

Then V^n is isometric to a sphere.

Finally, using Theorem 5.4 and Lemma 7.7, we have the following

Theorem 7.15 *Let R^{n+1} $(n \geq 3)$ be an orientable Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i satisfying the equation*

$$\pounds_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

and V^n a closed orientable hypersurface in R^{n+1} such that

(1) *H_2 is positive constant,*

(2) *θ $(= C^i \xi_i)$ has fixed sign on V^n,*

(3) *Ψ is not constant along V^n.*

Then V^n is isometric to a sphere.

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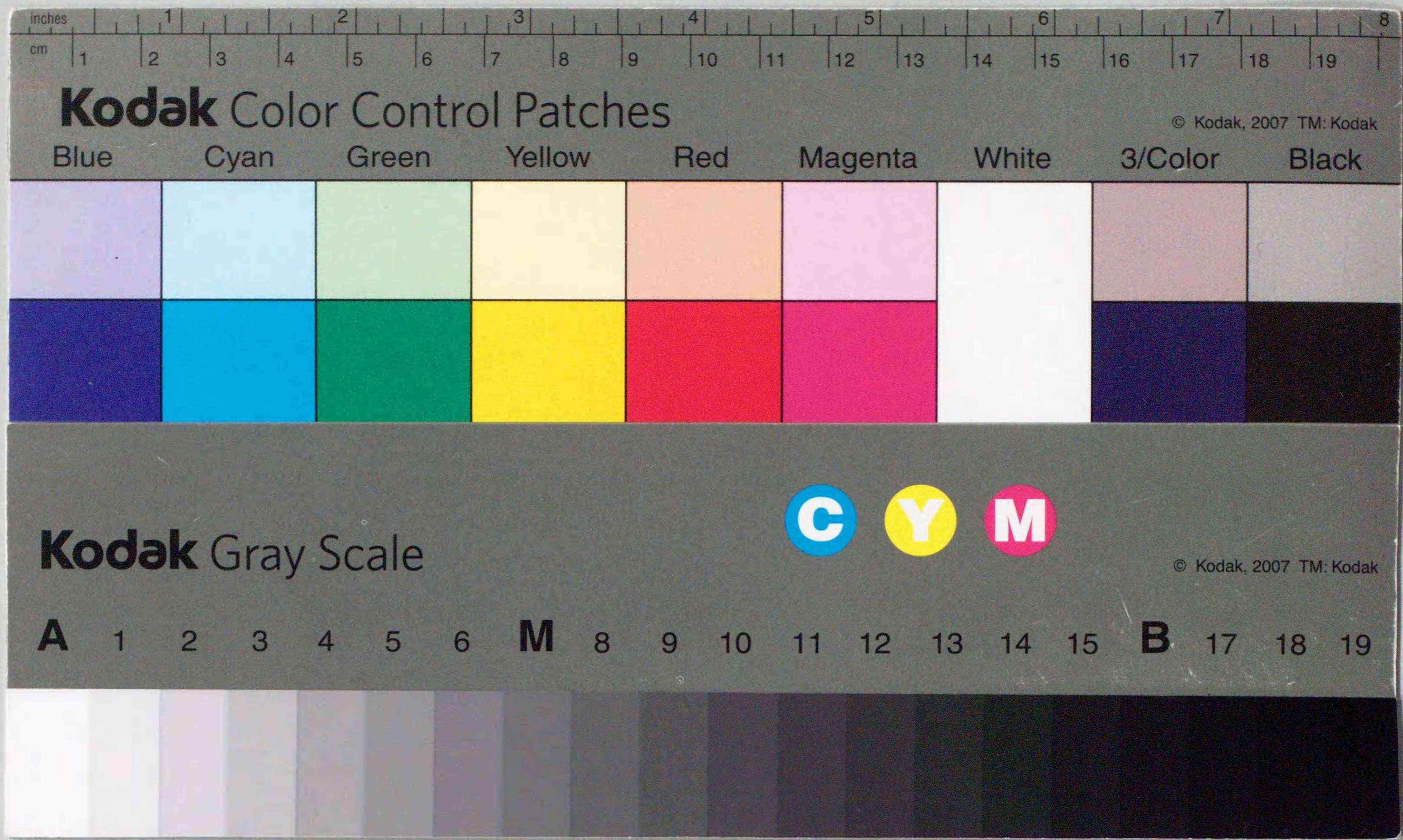
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Cyan
Green
Yellow
Red
Magenta
White
3/Color
Black
C
Y
M
Kodak Gray Scale
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A 1 2 3 4 5 6 M 8 9 10 11 12 13 14 15 B 17 18 19