Stability of Leptonic Self-complementarity

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We investigate a stability of leptonic self-complementarity such that sum of three mixing angles in lepton sector is 90 degrees. Current experimental data of neutrino oscillation indicates that the self-complementarity can be satisfied within 3\sigma ranges of each mixing angles. Thus the self-complementarity may be a key to study a flavor physics beyond the standard model, and important to discuss its stability. We analyze renormalization group equations in a context of minimal supersymmetric standard model for the self-complementarity. It is seen that one of Majorana phases plays an important role for the stability of self-complementarity. We find some stable solutions against quantum corrections at a low energy. An effective neutrino mass for neutrino-less double beta decay is also evaluated by the use of neutrino parameters giving rise to the stable solutions.

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Neutrino oscillation experiments established that there are two large mixing angles ($\theta_{12}$ and $\theta_{23}$) of Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix in lepton sector. Then a non-vanishing $\theta_{13}$ in the PMNS has been reported by recent long baseline and reactor neutrino experiments determine only two mass squared differences, $\Delta m^2_{21}$ and $\Delta m^2_{31}$. Therefore, two types of neutrino mass hierarchy are allowed, i.e. normal hierarchy (NH) $m_1 < m_2 < m_3$ and inverted hierarchy (IH) $m_3 < m_2 < m_1$. Further, neutrino experiments have not determined whether the neutrinos are Dirac or Majorana particles. Clearly, the nature of neutrinos would be a key to find physics beyond the standard model (SM).

In theoretical side of neutrino physics, various approaches have been discussed in order to investigate hidden flavor structure behind the SM, e.g. introductions of flavor symmetry, mass (matrix) texture analyses, and searches for exotic relations among flavor mixing angles etc. In this work, we focus on a leptonic self-complementarity [2] (see also [3] for related discussions)

\begin{equation}
\theta_{12} + \theta_{23} + \theta_{13} = \frac{\pi}{2} = 90^\circ. \tag{1}
\end{equation}

The current experimental data of neutrino oscillation indicates that the self-complementarity can be satisfied within 3\sigma ranges of each mixing angles. Therefore, the self-complementarity may be a key to investigate a flavor physics behind the SM, and important to discuss its stability.

We start with effective Yukawa interaction and Weinberg operator at a low energy scale such as electroweak (EW) scale $\Lambda_{\text{EW}}$ in a context of minimal supersymmetric standard model (MSSM),

\begin{equation}
\mathcal{L}_Y = -y_\sigma \overline{L}_H \sigma e_R + \kappa \frac{\mu}{2} (H_u L_L)(H_u L_L) + h.c., \tag{2}
\end{equation}

where $L_L$ are left-handed lepton doublets, $e_R$ are right-handed charged leptons, $H_u (H_d)$ is up(down)-type Higgs, $y_\sigma$ is Yukawa matrix of charged leptons, and $\kappa (H_u L_L)(H_u L_L)$ is the Weinberg operator, which can be effectively induced by integrating out a heavy particle(s). One of examples to obtain this operator is seesaw mechanism. Typical scale of the seesaw mechanism is $O(10^{14})$ GeV. Therefore, note that the effective coupling $\kappa$ is having mass dimension $-1$ and $\kappa^{-1} \sim O(10^{14})$ GeV. Such a heavy mass scale can realize tiny active neutrino mass scales through the seesaw mechanism. In this work, we utilize an useful parameterization for the PMNS matrix as [4]

\begin{equation}
V_{\text{PMNS}} \equiv V^\dagger_{\nu L} V_{\nu} D_\rho = \begin{pmatrix} c_{12} c_{13} & -s_{12} c_{23} e^{-i\delta} & s_{13} \\ -c_{12} s_{23} s_{13} - s_{12} c_{23} e^{-i\delta} & -c_{12} c_{23} s_{13} + s_{12} s_{23} e^{-i\delta} & 0 \\ -s_{12} s_{23} s_{13} + c_{12} c_{23} e^{-i\delta} & s_{23} c_{13} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ e^{i\sigma} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3}
\end{equation}

where $s_{ij} \equiv \sin \theta_{ij}$, $c \equiv \cos \theta_{ij}$ ($i, j = 1, 2, 3; \ i < j$), $\delta$ is a Dirac phase, and $D_\rho$ is a diagonal phase matrix including two Majorana phases, $\rho$ and $\sigma$. An neutrino mass matrix $M_\nu$ can be diagonalized as $V_{\nu} M_\nu V_{\nu}^\dagger = M_\nu^{\text{diag}} \equiv \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$ with $m_i \equiv \kappa_i v_u^2$ where $v_u$ is vacuum expectation value of up-type Higgs.

Next, we consider renormalization group equations (RGEs) in the MSSM. The RGE of $\kappa$ is given by

\begin{equation}
\text{Diag}(m_1, m_2, m_3) \text{ with } m_i \equiv \kappa_i v_u^2 \text{ where } v_u \text{ is vacuum expectation value of up-type Higgs.}
\end{equation}
\[16\pi^2(d\kappa/dt) = \alpha + [y_1 y_2]^T + \kappa(y_1 y_2)^T\]

with \(\alpha \equiv 6[-g_3^2/5 - g_2^2 + T_3(y_1^3 y_2)]\) where \(g_i\) are gauge coupling constants, \(t\) is a renormalization scale, \(\Lambda\) is a high energy scale such as the seesaw scale [5, 6]. One can also obtain RGEs of \(\theta_{ij}\) in a diagonal basis of \(y_i\) as \(d\theta_{ij}/dt = F_{ij}(\theta_{12}, \theta_{23}, \theta_{13}, \kappa_i, \delta, \rho, \sigma, y_r; t)\) where right-hand side (RHS) of this equation is given in [6] (see also [7] for other discussions of mixing angles under the RGEs). Now we turn to the self-complementarity relation (1) and investigate the following equation,

\[
\frac{d}{dt} \sum_{ij} \theta_{ij} = \sum_{ij} F_{ij}(\theta_{12}, \theta_{23}, \theta_{13}, \kappa_i, \delta, \rho, \sigma, y_r; t)
\]

\[
\equiv F(\theta_{12}, \theta_{23}, \theta_{13}, \kappa_i, \delta, \rho, \sigma, y_r; t),
\]

(4)

where \(ij\) is summed over 12, 23, and 13. The function \(F\) is described by 3 mixing angles, 3 effective couplings for the light neutrino masses (or equivalently light neutrino masses \(m_i\)), 3 CP-phases, a Yukawa coupling of \(\tau\), and renormalization scale. Then once we impose (1) on (4) at an energy scale \(t_0\), one of mixing angles in \(F\) is removed as e.g. \(F(\theta_{12}, \theta_{23}, \kappa_i, \delta, \rho, \sigma, y_r; t_0) = 0\). We now focus on an equation,

\[
\tilde{F}(\theta_{12}, \theta_{23}, \theta_{13}, \kappa_i, \delta, \rho, \sigma, y_r; t_0) = 0.
\]

(5)

This equation means that once the equation is satisfied at an energy scale \(t_0\), the self-complementarity is also satisfied at all other energy scales \(t\), i.e. the self-complementarity is stable against quantum corrections, if running effects of parameters except for mixing angles are tiny. In fact, we can find consistent solutions of (5) with experiments for both NH and IH cases. According to the latest experimental data of neutrino oscillation [8]

\[
31.3^\circ \leq \theta_{12} \leq 37.5^\circ,
\]

(6)

\[
38.6^\circ \leq \theta_{23} \leq 53.1^\circ,
\]

(7)

\[
7.0^\circ (7.3^\circ) \leq \theta_{13} \leq 10.9^\circ (11.1^\circ),
\]

(8)

at 3\(\sigma\) level for the NH(IH), the (1) can be satisfied.

Mass spectra of neutrinos at a low energy are defined by \(m_1 \equiv \sqrt{m_2^2 - |\Delta m^2_{31}|}\) and \(m_2 \equiv \sqrt{m_2^2 + |\Delta m^2_{31}|}\) with best fit values \(\Delta m^2_{31} = 7.62 \times 10^{-5}\) eV\(^2\) and \(\Delta m^2_{31} = 2.53 \times 10^{-3}\) eV\(^2\) for the NH, and \(m_1 \equiv \sqrt{m_2^2 - |\Delta m^2_{31}|}\) and \(m_3 \equiv \sqrt{m_2^2 + |\Delta m^2_{31}|}\) with \(\Delta m^2_{31} = 2.40 \times 10^{-3}\) eV\(^2\) for the IH. Therefore, the largest neutrino mass \(m_3(m_2)\), of NH(IH) case is a free parameter in our analyses. We analyze in range of \(\sqrt{|\Delta m^2_{31}|} \leq m_3 \leq 0.2\) eV \((\sqrt{|\Delta m^2_{31}|} + \Delta m^2_{31} \leq m_2 \leq 0.2\) eV\). The case of \(m_3(m_2) = 0.2\) eV corresponds to a degenerate mass spectrum. In such case, \(m_3(m_2)\) is bounded by a cosmological constraint on the sum of neutrino mass as \(\sum m_i \leq 0.6\) eV [9], and thus as \(m_3(m_2) \leq \sum m_i/3 \leq 0.2\) eV. Therefore, this must be implied as upper bound on the largest neutrino mass. Here note that since input values of neutrino parameters at low energy are used, the solutions of (5) are corresponding to ones at \(t_0 \simeq \ln(A_{EW}/\Lambda)\). The \(y_r\) has been also approximated at a low energy as \(y_r(-A_{EW}) = 10^{-2}\) in our analyses.

There are 7 parameters (3 mixing angles, 3 CP-phases, and 1 neutrino mass, \(m_3\) or \(m_2\) and 2 imposed equations ((1) and (5)). Therefore, number of free parameters is 5. Since it is however intricate to deal with all 5 parameters as completely free ones, we numerically analyze at some fixed neutrino masses as examples. According to our analyses, a CP-phase is important to give solutions of (5) with (1); we numerically found that there is no solution to satisfy (5) in cases of \((\delta = \rho = \sigma = 0)\) and \((\delta \neq 0, \rho = \sigma = 0)\) for both NH and IH, but \((\rho \neq 0, \delta = \sigma = 0)\) and \((\sigma 
eq 0, \delta = \rho = 0)\) can give solutions of (5) in some cases of NH. In a case of NH with a minimal \(m_3\) (i.e. \(m_3 = \sqrt{|\Delta m^2_{31}|}\)) and all cases of IH, neither \((\delta \neq 0, \delta = \rho = 0)\) nor \((\rho \neq 0, \delta = \sigma = 0)\) can give the solution. Therefore, in the following, we focus on the cases of \((\rho \neq 0, \delta = \sigma = 0)\) and \((\sigma \neq 0, \delta = \rho = 0)\) for other cases of NH in detail. Now we have 2 free parameters (one of Majorana phases and one of mixing angles) in order to look for solution, i.e. once we fix one of Majorana phases and one of mixing angles, all values of our parameters are uniquely determined as we will explain below.

We have scanned over \(0 \leq (|\rho| + |\sigma|) \leq \pi\). Some results of numerical analyses are shown in FIG. 1 as examples. In the figures, vertical and horizontal axes are \(\theta_{13}\) and \(\theta_{12}\), respectively, and shaded region mean that the self-complementarity (1) is correlatively satisfied within 3\(\sigma\) ranges of mixing angles (6)-(8). Lower(upper) slanting and right(left) sides are bounded by maximal(minimal) \(\theta_{13}\) and \(\theta_{12}\) at 3\(\sigma\) level, respectively. Note that an allowed range of \(\theta_{13}\) becomes narrow as \(41.6^\circ \leq \theta_{23} \leq 51.7^\circ\) compared to (7) due to (1). Both two lines in the figures are solutions of (5), which are contours of \(|\rho|\) or \(|\sigma|\) \((\rho \neq 0, \delta = \sigma = 0)\) for the NH, and there are 2 remaining parameters, \(\rho\) and one of mixing angles. In the case, 2 lines corresponding to \(|\rho| \simeq 124.0^\circ\) and 143.0\(^\circ\) are grazing shaded region at upper left described by \(A\) and lower right \(B\) points, respectively (there are 2 (different sign) solutions of \(\rho\) or \(\sigma\) in all cases, i.e. solutions are symmetric for reflection respect with \(\tau\)). Therefore, values of all 7 parameters in our analyses are uniquely determined at e.g. point \(A\) or \(B\). Further, the points \(A\) and \(B\) in all cases determine maximal and minimal values of parameters for the stability of self-complementarity. Therefore, the self-complementarity is stable in shaded regions of 124.0\(^\circ\) \(|\rho| \leq 143.0^\circ, 31.3^\circ \leq \theta_{12} \leq 37.5^\circ, 7.0^\circ \leq \theta_{13} \leq 10.9^\circ, \) and \(\min[\theta_{23}] \leq \theta_{23} \leq \max[\theta_{23}]\) where \(\max[\min[\theta_{ij}]\) is maximal(minimal) value of \(\theta_{ij}\). \(\max[\min[\theta_{23}]\) is evaluated by \(\max[\min[\theta_{23}] = 90^\circ\) - \(\min[\max[\theta_{23}]\) - \(\min[\max[\theta_{13}]]\) due to (1). Note that value of one of mixing angle (e.g. \(\theta_{23}\)) is not independently taken because of (1). In the case, we obtain \(\max[\min[\theta_{23}] \approx 51.7^\circ(41.6^\circ)\). The results are summarized in TAB. I.
TABLE I: Examples of solutions, and minimal and maximal values of neutrino parameters in the corresponding regions: Values of $|\sigma|$ in parentheses are maximal ones giving solutions of (5) but the solutions realized by these maximal values are not stable against running effects up to the seesaw scale. We have also numerically checked solutions given in T A B. I are stable, i.e. we can also obtain any solutions of (5) but the solutions realized by these maximal values are not stable against running effects from CP-phases. The values of $|\rho|$ and $|\sigma|$ without parentheses are complete stable ones against the running effects of CP-phases.

FIG. 1 (b) shows a case of $m_3 = \sqrt{\Delta m_{31}^2 + \Delta m_{21}^2}$ with $(\sigma \neq 0, \delta = \rho = 0)$. In the case, we obtain a region of $|\sigma|$ as $124.9^\circ \lesssim |\sigma| \lesssim 154.0^\circ$, where the self-complementarity is satisfied, in a similar analysis to the previous case. Regions of mixing angles for the realizability of self-complementarity is the same as the previous case. We have also analyzed a case of $m_3 = 0.2$ eV. Results are given in TAB. I. We cannot obtain any solutions of (5) in the case of $(\rho \neq 0, \delta = \sigma = 0)$ but can do in one of $(\sigma \neq 0, \delta = \rho = 0)$. In the case, values of $|\sigma|$ within $101.9^\circ \lesssim |\sigma| \lesssim 105.2^\circ$ can give the solutions. Allowed region of mixing angles are the same as ones in the $m_3 = \sqrt{\Delta m_{31}^2}$ case, i.e. contours of solutions can reach at the both points A and B like the case of $m_3 = \sqrt{\Delta m_{31}^2}$. For the Max(Min)$|\sigma|$, it is determined by Max(Min)$|\sigma|_{23}$ in contrast with the above two cases.

Of course, other parameters (CP-phases and neutrino masses) contributing to mixing angles evolve under the RGEs. First, we comment on running effects of CP-phases. It has been seen that the Majorana phases are important for the stability of self-complementarity. One may worry about running effects of phases on low energy solutions for the stability, i.e. whether such effects spoil the solutions at a high energy scale or not. We have approximated the running effects of $\delta$, $\rho$, and $\sigma$ from the seesaw scale on solutions by using leading-log estimation in the RGEs. These running effects $(\Delta \delta, \Delta \rho, \Delta \sigma)$ are $(\Delta \delta, \Delta \rho, \Delta \sigma) \sim (O(0.1^\circ), O(0.1^\circ), O(0.1^\circ))$ and $(O(1^\circ), O(1^\circ), O(1^\circ))$ for the cases of $m_3 = \sqrt{\Delta m_{31}^2}$, $0.2$ eV, respectively. Therefore, most regions of Majorana phases are stable against such small running effects up to the seesaw scale because of Max$|\sigma| - $Min$|\sigma| > \Delta \sigma$. In fact, we show complete stable values of $|\rho|$ and $|\sigma|$, which are described by values without parentheses in T A B. I, including the above running effects from CP-phases. Further, small (but non-vanishing) running effect of $\delta$ does not affect the stability of self-complementarity. Then one must remember that our analyses are only for the Majorana neutrino case.

Next, we consider running effects of neutrino masses. We have also evaluated the effects from neutrino masses by the use of leading-log approximation in corresponding RGEs for the neutrino masses, which are also given in [6]. These running effects from the seesaw scale to the electroweak scale are

$$(\Delta m_{1e}^{eff}, \Delta m_{2e}^{eff}, \Delta m_{3e}^{eff})$$

$$\sim \left\{ \begin{array}{l} (O(10^{-8}), O(10^{-7}), O(10^{-6})) \ eV \\
(O(10^{-6}), O(10^{-6}), O(10^{-6})) \ eV \end{array} \right.$$  

for the cases of $m_3 = \sqrt{\Delta m_{31}^2}$ and $0.2$ eV, respectively, where $\Delta m_{eff}$ affects only evolutions of mixing angles and CP-phases not absolute values of neutrino mass eigenvalues, i.e. overall (flavor mixing independent) contributions from running of Yukawa couplings (top Yukawa gives dominant contribution) are omitted. Even with these running effects of neutrino masses, the solutions given in the T A B. I are stable, i.e. we can also obtain solutions of (5) within the almost same range of $|\sigma|$ as ones in T A B. I. We have also numerically checked evolutions of mixing angles and their sum in order to make sure that the one of solutions in the NH is stable. In the calculation, we take

$$m_1 = 8.73 \times 10^{-3} \ eV,$$  

$$m_2 = 1.23 \times 10^{-2} \ eV,$$  

$$m_3 = \sqrt{\Delta m_{31}^2 + \Delta m_{21}^2},$$  

$$\delta = \sigma = 0, \rho = 133.5^\circ.$$  

<table>
<thead>
<tr>
<th>Phases</th>
<th>$\rho \neq 0, \delta = \sigma = 0$</th>
<th>$\rho \neq 0, \delta = \rho = 0$</th>
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<tbody>
<tr>
<td>$\min</td>
<td>\sigma</td>
<td><em>{12}(\sin^2 \theta</em>{12})$</td>
<td>31.3 (0.27)</td>
</tr>
<tr>
<td>$\max</td>
<td>\sigma</td>
<td><em>{12}(\sin^2 \theta</em>{12})$</td>
<td>51.7 (0.62)</td>
</tr>
<tr>
<td>$\min</td>
<td>\sigma</td>
<td><em>{23}(\sin^2 \theta</em>{23})$</td>
<td>143.0</td>
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<td>$\min</td>
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<td><em>{13}(\sin^2 \theta</em>{13})$</td>
<td>9.13</td>
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<tr>
<td>$\max</td>
<td>\sigma</td>
<td><em>{13}(\sin^2 \theta</em>{13})$</td>
<td>9.13</td>
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at low energy as an example and
\[ \theta_{12} = 33.5^\circ, \theta_{23} = 47.7^\circ, \theta_{13} = 8.8^\circ, \] (14)
as low energy boundary conditions for the RGEs. The running
effects of the mixing angles from the seesaw scale to the
electroweak one are
\[ (\Delta \theta_{12}, \Delta \theta_{23}, \Delta \theta_{13}) \sim (-O(10^{-4}), O(10^{-3}), -O(10^{-4})) \text{[degree].} \] (15)
Since the deviation of the leptonic self-complementarity
from 90° is \(O(10^{-4})\) degree, the self-complementarity relation
can be still stable.

Finally, we evaluate effective mass term of neutrino-less
double beta decay \((0\nu\beta\beta)\), \(\langle m \rangle_{ee} \equiv |\sum_{i=1}^{3}(V_{PMNS})^2_{ei}m_i|\),
in our parameter space. It is written down as
\[ \langle m \rangle_{ee} = |m_{14}^2s_{13}^2c_{13} + m_{24}^2s_{12}^2c_{12}s_{23} + m_{34}^2s_{13}^2|, \] (16)
in our notation. The phenomenon of \(0\nu\beta\beta\) can distinguish
whether neutrinos are Dirac or Majorana particles.

The results at benchmarks given in Tab. I are also presented
in the table.

The magnitude of \(\langle m \rangle_{ee}\) strongly depends on the scale
of \(m_1\) or \(m_2\) rather than mixing angles and CP-phases
in the cases. In the NH with \(m_3 = \sqrt{|\Delta m^2_{31}|}\) and
\(\sqrt{|\Delta m^2_{31}| + \Delta m^2_{21}}\), dominant contribution comes
from the second term of RHS of (16) because of the small \(s_{13}^2\)
and vanishing \(m_1\). We predict 6.08 meV \(\lesssim\) \(\langle m \rangle_{ee}\) \(\lesssim 97.3\) meV for the NH within the parameter space to make the self-complementarity stable. The Heidelberg-Moscow experiment [10] for \(0\nu\beta\beta\) is giving the most severe bound on \(\langle m \rangle_{ee}\), which is \(\langle m \rangle_{ee} \lesssim 210\) meV. The CUORE experiment [11] is expected to reach \(\langle m \rangle_{ee} = (24-93)\) meV
in the future. Therefore, a part of our predicting region
may be checked in future experiments.

We have investigated a stability of leptonic self-
complementarity relation in the PMNS sector against quantum corrections by considering RGEs in the MSSM. The current experimental data of neutrino oscillation indicates that that the self-complementarity can be satisfied at 3\(\sigma\) ranges of each mixing angle. This motivates us to study the self-complementarity and its stability as a key to find a physics behind the SM. As the results of analyses, we have found solutions stabilizing the self-complementarity by using low energy data of neutrino oscillation experiments. It has been seen that the Majorana play an important role to give the solutions. The self-complementarity relation can be satisfied up to an arbitrary high energy scale if neutrino parameters are correlatively within 31.3° \(\lesssim\) \(\theta_{12}\) \(\lesssim 37.5°, 7.0° \lesssim \theta_{13} \lesssim 10.9°, \text{and Max(\text{Min})}(\theta_{23}) \approx 51.7°(41.6°)\)
with 124.0° \(\lesssim\) \(|\Delta \theta_{23}| \lesssim 143.0° \text{or } 124.9° \lesssim\) \(|\sigma| \lesssim 153.9° \text{for } m_3 = \sqrt{|\Delta m^2_{31}| + \Delta m^2_{21}}, \text{and } 101.9° \lesssim\) \(|\sigma| \lesssim 104.6° \text{for } m_3 = 0.2\text{ eV of NH at a low energy. These solutions and leptonic self-complementarity relation are stable against running effects of CP-phases and neutrino masses. Regarding with the } 0\nu\beta\beta, \text{the effective neutrino mass can be predicted as } 6.08\text{ meV} \lesssim \langle m \rangle_{ee} \lesssim 97.3\text{ meV for the stable solution in NH case of } m_3 = \sqrt{|\Delta m^2_{31}|}.\)

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(a) \( m_3 = \sqrt{\Delta m_{31}^2 + \Delta m_{32}^2} \) and \((\rho \neq 0, \; \delta = \sigma = 0)\)

FIG. 1: Examples of solutions of (5)