Lightlike hypersurfaces along spacelike submanifolds in Minkowski space-time

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Abstract

We consider the singularities of lightlike hypersurfaces along spacelike submanifolds in Lorentz-Minkowski space with general codimension. As an application of the theory of Legendrian singularities, we investigate the geometric meanings of the singularities of lightlike hypersurfaces from the view point of the contact of spacelike submanifolds with lightcones.

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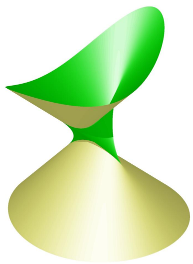
1 Introduction

The study of the extrinsic differential geometry of submanifolds in Lorentz manifolds is of special interest in Relativity theory. In particular, *lightlike hypersurfaces* are provided good models for the study of different horizon types ([5, 7, 24]). A lightlike hypersurface is also called a *light sheet* in Theoretical Physics (cf., [2]), which plays a principal role in the quantum theory of gravity. In this paper we investigate the singularities of lightlike hypersurfaces in Lorentz-Minkowski space. Although Lorentz-Minkowski space has no gravity, the singularities of lightlike hypersurfaces give a typical model of horizons and important information for the shape of horizons in general Lorentz manifolds. The lightlike hypersurfaces are constructed as ruled hypersurfaces along spacelike submanifolds with the lightlike rulings.

On the other hand, tools in the theory of singularities have proven to be useful description of geometrical properties of submanifolds immersed in different ambient spaces, from both the local and global viewpoint [9, 10, 12, 15, 16, 17, 21]. The natural connection between geometry and singularities relies on the basic fact that the contacts of a submanifold with the models of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions, or equivalently, of their associated Legendrian maps ([1, 25, 30]). When working in Lorentz-Minkowski space, the properties associated to the contacts of a given submanifold with lightlike hyperplanes or lightcones have a special relevance. In [11, 17], it was constructed a framework for the study of spacelike submanifolds with codimension two in Lorentz-Minkowski space and discovered a Lorentz invariant concerning their contacts with lightlike hyperplanes. The geometry related to this framework is called the *lightlike geometry* (or, the *lightlike flat geometry*) of spacelike submanifolds with codimension two. By using the invariants of lightlike geometry, the singularities of lightlike hypersurfaces along spacelike surfaces in Lorentz-Minkowski 4-space was investigated in [15]. It is not so difficult to generalize the result of [15] into the case for codimension two in general dimensional Lorentz-Minkowski space [18]. However, the situation is rather complicated for the general codimensional case. The main difference from the Euclidean space case is the fiber of the canal hypersurface of a spacelike submanifold is neither connected nor compact. In this paper we investigate singularities of lightlike hypersurfaces along general codimension spacelike submanifolds in Lorentz-Minkowski space. In order to avoid the above difficulty, we arbitrary choose a timelike future directed unit normal vector field along the spacelike submanifold which always exists for an orientable manifold (cf., [19]). Then we construct the unit spherical normal bundle relative to the above timeline unit normal vector field, which can be considered as a codimension two spacelike canal submanifold of the ambient Lorentz-Minkowski space. Therefore, we can apply the idea of the lightlike geometry of spacelike submanifolds of Lorentz-Minkowski space with codimension two. In this way we constructed the framework of the lightlike geometry of spacelike submanifolds with general codimension in [19] and investigated local and global properties. In this paper we apply this framework and the theory of Legendrian singularities to investigate the singularities of lightlike hypersurfaces along spacelike submanifolds with general codimension. Here, we draw the picture of the lightlike surface along an ellipse in the Euclidean plane canonically embedded in the Lorentz-Minkowski 3-space. We can observe that four swallowtail singularities (for the definition see §6) on the surface which correspond to the vertices of the ellipse. This means that there might be interesting geometric meanings of the singularities of lightlike surfaces.



Lightlike surface
Fig. 1.

In §3 we briefly review the framework of the lightlike geometry of spacelike submanifolds with general codimension which was constructed in [19]. The notion of lightlike hypersurfaces along spacelike submanifolds is introduced and the basic properties are investigated in §4. The notion of the distance squared functions families is useful for the study of lightlike hypersurfaces (cf., §4). The critical value set of the lightlike hypersurface along a spacelike submanifold is called the *lightlike focal set* of the submanifold. In §5 we show that the lightlike focal set of a spacelike submanifold is a point if and only if the lightlike hypersurface along the submanifold is a subset of a lightcone (Proposition 5.1). Therefore, a lightcone is a model hypersurface of lightlike hypersurfaces. The geometric meaning of the singularities of lightlike hypersurface is described by the theory of contact of submanifolds with model hypersurfaces. Moreover, as an application of the theory of Legendrian singularities, we show that two lightlike hypersurfaces are locally diffeomorphic if and only if the types of the contact of spacelike submanifolds with lightcones are the same in the sense of Montaldi[25] under some generic conditions (Theorem 5.5). In §6 we describe the case for codimension two as a special case. We also investigate spacelike curves in Lorentz-Minkowski 4-space as the simplest case of a higher codimension in §7. In §8 we consider the case that submanifolds are located in a spacelike hyperplane or in Hyperbolic space. In this case lightlike focal sets correspond to the focal sets in the Euclidean sense or the hyperbolic and de Sitter focal sets (cf., [13]).

2 Basic notations on Lorentz-Minkowski space

We introduce in this section some basic notions on Lorentz-Minkowski $n+1$-space. For basic concepts and properties, see [26].

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, 1, \dots, n)\ \}$ be an $n+1$-dimensional cartesian space. For any $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ *Lorentz-Minkowski* $n+1$*-space*. We write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. The *signature* of a vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is defined to be

$$\mathrm{sign}(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x}: \ \text{spacelike}, \\ 0 & \boldsymbol{x}: \ \text{lightlike}, \\ -1 & \boldsymbol{x}: \ \text{timelike}. \end{cases}$$

We have the canonical projection $\pi : \mathbb{R}^{n+1}_1 \to \mathbb{R}^n$ defined by $\pi(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n)$. Here we identify $\{\boldsymbol{0}\} \times \mathbb{R}^n$ with $\mathbb{R}^n$ and it is considered as Euclidean n-space whose scalar product is induced from the pseudo scalar product $\langle , \rangle$. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c \ \}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We remark that $\{\boldsymbol{0}\} \times \mathbb{R}^n$ is a spacelike hypersurface in $\mathbb{R}^{n+1}_1$.

We now define *Hyperbolic* n*-space* by

$$H^n(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}$$

and *de Sitter* n*-space* by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \ \}.$$

We define

$$LC_{\boldsymbol{a}} = \{\boldsymbol{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\},$$

which is called *the lightcone with the vertex* $\boldsymbol{a}$. We denote that $LC^* = LC_{\boldsymbol{0}} \setminus \{\boldsymbol{0}\}$. If $\boldsymbol{x} = (x_0, x_1, \dots, x_2)$ is a lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in S^{n-1}_+ = \{\boldsymbol{x} = (x_0, x_1, \dots, x_n) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0, \ x_0 = 1\}.$$

We call S^{n-1}_+ the *lightcone* (or, *spacelike*) *unit* $n-1$*-sphere.*

For any $\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ by

$$\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ x^2_0 & x^2_1 & \cdots & x^2_n \\ \vdots & \vdots & \cdots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}^i = (x^i_0, x^i_1, \dots, x^i_n)$. We can easily show that

$$\langle \boldsymbol{x}, \boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n \rangle = \det(\boldsymbol{x}, \boldsymbol{x}^1, \dots, \boldsymbol{x}^n),$$

so that $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ is pseudo orthogonal to any $\boldsymbol{x}^i$ $(i = 1, \dots, n)$.

3 Differential geometry on spacelike submanifolds

In this section we introduce the basic geometrical framework for the study of spacelike submanifolds in Minkowski $n+1$-space analogous to the case of codimension two in [17]. Let $\mathbb{R}^{n+1}_1$ be an oriented and time-oriented space. We choose $\boldsymbol{e}_0 = (1, 0, \ldots, 0)$ as the future timelike vector field. Let $\boldsymbol{X} : U \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike embedding of codimension k, where $U \subset \mathbb{R}^s$ $(s+k = n+1)$ is an open subset. We also write $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. We say that $\boldsymbol{X}$ is *spacelike* if the tangent space T_pM of M at p is a spacelike subspace (i.e., consists of spacelike vectors) for any point $p \in M$. For any $p = \boldsymbol{X}(u) \in M \subset \mathbb{R}^{n+1}_1$, we have

$$T_pM = \langle \boldsymbol{X}_{u_1}(u), \ldots, \boldsymbol{X}_{u_s}(u) \rangle_{\mathbb{R}}.$$

Let $N_p(M)$ be the pseudo-normal space of M at p in $\mathbb{R}^{n+1}_1$. Since T_pM is a spacelike subspace of $T_p\mathbb{R}^{n+1}_1$, $N_p(M)$ is a k-dimensional Lorentzian subspace of $T_p\mathbb{R}^{n+1}_1$ (cf.,[26]). On the pseudo-normal space $N_p(M)$, we have two kinds of pseudo spheres:

$$\begin{aligned} N_p(M; -1) &= \{ \boldsymbol{v} \in N_p(M) \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = -1 \} \\ N_p(M; 1) &= \{ \boldsymbol{v} \in N_p(M) \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1 \}, \end{aligned}$$

so that we have two unit spherical normal bundles over M:

$$N(M; -1) = \bigcup_{p \in M} N_p(M; -1) \text{ and } N(M; 1) = \bigcup_{p \in M} N_p(M; 1).$$

Then we have the Whitney sum decomposition

$$T\mathbb{R}^{n+1}_1 | M = TM \oplus N(M).$$

Since $M = \boldsymbol{X}(U)$ is spacelike, $\boldsymbol{e}_0$ is a transversal future directed timelike vector field along M. For any $\boldsymbol{v} \in T_p\mathbb{R}^{n+1}_1|M$, we have $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, where $\boldsymbol{v}_1 \in T_pM$ and $\boldsymbol{v}_2 \in N_p(M)$. If $\boldsymbol{v}$ is timelike, then $\boldsymbol{v}_2$ is timelike. Let $\pi_{N(M)} : T\mathbb{R}^{n+1}_1|M \longrightarrow N(M)$ be the canonical projection. Then $\pi_{N(M)}(\boldsymbol{e}_0)$ is a future directed timelike normal vector field along M. So we always have a future directed unit timelike normal vector field along M (even globally). We now arbitrarily choose a future directed unit timelike normal vector field $\boldsymbol{n}^T(u) \in N_p(M; -1)$, where $p = \boldsymbol{X}(u)$. Therefore we have the pseudo-orthonormal compliment $(\langle \boldsymbol{n}^T(u) \rangle_{\mathbb{R}})^{\perp}$ in $N_p(M)$ which is a $k-1$-dimensional subspace of $N_p(M)$. We can also choose a pseudo-normal section $\boldsymbol{n}^S(u) \in (\langle \boldsymbol{n}^T(u) \rangle_{\mathbb{R}})^{\perp} \cap N(M; 1)$ at least locally, then we have $\langle \boldsymbol{n}^S, \boldsymbol{n}^S \rangle = 1$ and $\langle \boldsymbol{n}^S, \boldsymbol{n}^T \rangle = 0$. We define a $k-1$-dimensional spacelike unit sphere in $N_p(M)$ by

$$N_1(M)_p[\boldsymbol{n}^T] = \{ \boldsymbol{\xi} \in N_p(M; 1) \mid \langle \boldsymbol{\xi}, \boldsymbol{n}^T(p) \rangle = 0 \}.$$

Then we have a *spacelike unit $k-2$-spherical bundle over M with respect to $\boldsymbol{n}^T$* defined by

$$N_1(M)[\boldsymbol{n}^T] = \bigcup_{p \in M} N_1(M)_p[\boldsymbol{n}^T].$$

Since we have $T_{(p,\xi)}N_1(M)[\boldsymbol{n}^T] = T_pM \times T_\xi N_1(M)_p[\boldsymbol{n}^T]$, we have the canonical Riemannian metric on $N_1(M)[\boldsymbol{n}^T]$. We denote the Riemannian metric on $N_1(M)[\boldsymbol{n}^T]$ by $(G_{ij}(p,\boldsymbol{\xi}))_{1\leqslant i,j\leqslant n-1}$.

For any future directed unit normal $\boldsymbol{n}^T$ along M, we arbitrary choose (at least locally) the unit spacelike normal vector field $\boldsymbol{n}^S$ with $\boldsymbol{n}^S(u) \in N_1(M)_p[\boldsymbol{n}^T]$, where $p = \boldsymbol{X}(u)$. We call $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ a *future directed pair* along M. Clearly, the vectors $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$ are lightlike. Here we choose $\boldsymbol{n}^T + \boldsymbol{n}^S$ as a lightlike normal vector field along M. We define a mapping

$$\mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{n}^S) : U \longrightarrow LC^*$$

by $\mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \boldsymbol{n}^T(u) + \boldsymbol{n}^S(u)$. We call it the *lightcone Gauss image* of $M = \boldsymbol{X}(U)$ with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$. Under the identification of M and U through $\boldsymbol{X}$, we have the linear mapping provided by the derivative of the lightcone Gauss image $\mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at each point $p \in M$,

$$d_p\mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{n}^S) : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_p(M).$$

Consider the orthogonal projections $\pi^t : T_pM \oplus N_p(M) \to T_p(M)$. We define

$$d_p\mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{n}^S)^t = \pi^t \circ d_p(\boldsymbol{n}^T + \boldsymbol{n}^S).$$

We call the linear transformations $S_p(\boldsymbol{n}^T, \boldsymbol{n}^S) = -d_p\mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{n}^S)^t$ the *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. Let $\{\kappa_i(\boldsymbol{n}^T, \boldsymbol{n}^S)(p)\}_{i=1}^s$ be the eigenvalues of $S_p(\boldsymbol{n}^T, \boldsymbol{n}^S)$, which are called the *lightcone principal curvatures with respect to* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at $p = \boldsymbol{X}(u)$. Then the *lightcone Gauss-Kronecker curvature with respect to* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at $p = \boldsymbol{X}(u)$ is defined by

$$K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = \mathrm{det} S_p(\boldsymbol{n}^T, \boldsymbol{n}^S).$$

We say that a point $p = \boldsymbol{X}(u)$ is an *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilical point* if

$$S_p(\boldsymbol{n}^T, \boldsymbol{n}^S) = \kappa(\boldsymbol{n}^T, \boldsymbol{n}^S)(p)1_{T_pM}.$$

We say that $M = \boldsymbol{X}(U)$ is *totally $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilical* if all points on M are $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilical. Moreover, $M = \boldsymbol{X}(U)$ is said to be *totally lightcone umbilical* if it is totally $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilical for any future directed pair $(\boldsymbol{n}^T, \boldsymbol{n}^S)$.

We deduce now the lightcone Weingarten formula. Since $\boldsymbol{X}_{u_i}$ $(i = 1, \dots s)$ are spacelike vectors, we have a Riemannian metric (the *lightcone first fundamental form*) on $M = \boldsymbol{X}(U)$ defined by $ds^2 = \sum_{i=1}^s g_{ij} du_i du_j$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. We also have a *lightcone second fundamental invariant with respect to the normal vector field* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ defined by $h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \langle -(\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. By the similar arguments to those in the proof of [17, Proposition 3.2], we have the following proposition.

Proposition 3.1 *We choose a pseudo-orthonormal frame* $\{\boldsymbol{n}^T, \boldsymbol{n}^S_1, \dots, \boldsymbol{n}^S_{k-1}\}$ *of* $N(M)$ *with* $\boldsymbol{n}^S_{k-1} = \boldsymbol{n}^S$. *Then we have the following lightcone Weingarten formula with respect to* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$:

(a) $\mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{n}^S)_{u_i} = \langle \boldsymbol{n}^S_{u_i}, \boldsymbol{n}^T\rangle(\boldsymbol{n}^T - \boldsymbol{n}^S) + \sum_{\ell=1}^{k-2}\langle(\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i}, \boldsymbol{n}^S_\ell\rangle \boldsymbol{n}^S_\ell - \sum_{j=1}^s h_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j}$

(b) $\pi^t \circ \mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^s h_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j}$.

Here $\left(h_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\right) = \left(h_{ik}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = (g_{kj})^{-1}$.

As a consequence of the above proposition, we have an explicit expression of the lightcone curvature by

$$K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) = \frac{\det\left(h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)}{\det\left(g_{\alpha\beta}\right)}.$$

Since $\langle -(\boldsymbol{n}^T + \boldsymbol{n}^S)(u), \boldsymbol{X}_{u_j}(u)\rangle = 0$, we have $h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \langle \boldsymbol{n}^T(u) + \boldsymbol{n}^S(u), \boldsymbol{X}_{u_i u_j}(u)\rangle$. Therefore the lightcone second fundamental invariant at a point $p_0 = \boldsymbol{X}(u_0)$ depends only on the values $\boldsymbol{n}^T(u_0) + \boldsymbol{n}^S(u_0)$ and $\boldsymbol{X}_{u_i u_j}(u_0)$, respectively Thus, the lightcone curvatures also depend only on $\boldsymbol{n}^T(u_0) + \boldsymbol{n}^S(u_0)$, $\boldsymbol{X}_{u_i}(u_0)$ and $\boldsymbol{X}_{u_i u_j}(u_0)$, independent of the derivation of the vector fields $\boldsymbol{n}^T$ and $\boldsymbol{n}^S$. We write $\kappa_i(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)(p_0)$ $(i = 1, \ldots, s)$ and $K_\ell(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)(u_0)$ as the lightcone curvatures at $p_0 = \boldsymbol{X}(u_0)$ with respect to $(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0) = (\boldsymbol{n}^T(u_0), \boldsymbol{n}^S(u_0))$. We might also say that a point $p_0 = \boldsymbol{X}(u_0)$ is $(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)$-*umbilical* because the lightcone $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-shape operator at p_0 depends only on the normal vectors $(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)$. So we denote that $h_{ij}(\boldsymbol{n}^T, \boldsymbol{\xi})(u_0) = h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u_0)$ and $K_\ell(\boldsymbol{n}^T, \boldsymbol{\xi})(p_0) = K_\ell(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)(p_0)$, where $\boldsymbol{\xi} = \boldsymbol{n}^S(u_0)$ for some local extension $\boldsymbol{n}^T(u)$ of $\boldsymbol{\xi}$. Analogously, we say that a point $p_0 = \boldsymbol{X}(u_0)$ is an $(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)$-*parabolic point* of $\boldsymbol{X} : U \longrightarrow \mathbb{R}^{n+1}_1$ if $K_\ell(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)(u_0) = 0$. And we say that a point $p_0 = \boldsymbol{X}(u_0)$ is a $(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)$-*flat point* if it is an $(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)$-umbilical point and $K_\ell(\boldsymbol{n}^T_0, \boldsymbol{n}^S_0)(u_0) = 0$.

On the other hand, we define a map $\mathbb{LG}(\boldsymbol{n}^T) : N_1(M)[\boldsymbol{n}^T] \longrightarrow LC^*$ by $\mathbb{LG}(\boldsymbol{n}^T)(u, \boldsymbol{\xi}) = \boldsymbol{n}^T(u) + \boldsymbol{\xi}$, which we call the *lightcone Gauss image* of $N_1(M)[\boldsymbol{n}^T]$. This map leads us to the notions of curvatures. Let $T_{(p,\xi)}N_1(M)[\boldsymbol{n}^T]$ be the tangent space of $N_1(M)[\boldsymbol{n}^T]$ at $(p, \boldsymbol{\xi})$. Under the canonical identification $(\mathbb{LG}(\boldsymbol{n}^T)^*T\mathbb{R}^{n+1}_1)_{(p,\boldsymbol{\xi})} = T_{(\boldsymbol{n}^T(p)+\boldsymbol{\xi})}\mathbb{R}^{n+1}_1 \equiv T_p\mathbb{R}^{n+1}_1$, we have

$$T_{(p,\boldsymbol{\xi})}N_1(M)[\boldsymbol{n}^T] = T_pM \oplus T_\xi S^{k-2} \subset T_pM \oplus N_p(M) = T_p\mathbb{R}^{n+1}_1,$$

where $T_\xi S^{k-2} \subset T_\xi N_p(M) \equiv N_p(M)$ and $p = \boldsymbol{X}(u)$. Let

$$\Pi^t : \mathbb{LG}(\boldsymbol{n}^T)^*T\mathbb{R}^{n+1}_1 = TN_1(M)[\boldsymbol{n}^T] \oplus \mathbb{R}^{k+1} \longrightarrow TN_1(M)[\boldsymbol{n}^T]$$

be the canonical projection. Then we have a linear transformation

$$S_\ell(\boldsymbol{n}^T)_{(p,\boldsymbol{\xi})} = -\Pi^t_{\mathbb{LG}(n^T)(p,\xi)} \circ d_{(p,\xi)}\mathbb{LG}(\boldsymbol{n}^T) : T_{(p,\xi)}N_1(M)[\boldsymbol{n}^T] \longrightarrow T_{(p,\xi)}N_1(M)[\boldsymbol{n}^T],$$

which is called the *lightcone shape operator* of $N_1(M)[\boldsymbol{n}^T]$ at $(p, \boldsymbol{\xi})$. Let $\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})$ be the eigenvalues of $S_\ell(\boldsymbol{n}^T)_{(p,\boldsymbol{\xi})}$, $(i = 1, \ldots, n-1)$. Here, we denote $\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})$, $(i = 1, \ldots, s)$ as the eigenvalues belonging to the eigenvectors on T_pM and $\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})$, $(i = s + 1, \ldots n-1)$ as the eigenvalues belonging to the eigenvectors on the tangent space of the fiber of $N_1(M)[\boldsymbol{n}^T]$. We have shown in [19] that $\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) = -1$, $(i = s+1, \ldots n-1)$. We call $\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})$, $(i = 1, \ldots, s)$ the *lightcone principal curvatures* of M with respect to $(\boldsymbol{n}^T, \boldsymbol{\xi})$ at $p \in M$. The *lightcone Lipschitz-Killing curvature* of $N_1(M)[\boldsymbol{n}^T]$ at $(p, \boldsymbol{\xi})$ is defined to be $K_\ell(\boldsymbol{n}^T)(p, \boldsymbol{\xi}) = \det S_\ell(\boldsymbol{n}^T)_{(p,\xi)}$.

We now define a mapping $\mathbb{DG}(\boldsymbol{n}^T) : N_1(M)[\boldsymbol{n}^t] \longrightarrow S^n_1$ by $\mathbb{DG}(\boldsymbol{n}^T)(p, \boldsymbol{\xi}) = \boldsymbol{\xi}$, which is called the *de Sitter Gauss image* of $N_1(M)[\boldsymbol{n}^T]$. By the similar way to the above case, we can define the *de Sitter shape operator* $S_d(\boldsymbol{n}^T)_{(p,\boldsymbol{\xi})}$. The *de Sitter principal curvatures* of M with respect to $(\boldsymbol{n}^T, \boldsymbol{\xi})$ at $p \in M$ are defined to be the eigenvalues of $S_d(\boldsymbol{n}^T)_{(p,\xi)}$ belonging to the eigenvectors on T_pM, which are denoted by $\kappa_d(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})$, $(i = 1, \ldots, s)$. We also define a mapping $\mathbb{G}(\boldsymbol{n}^T) : M \longrightarrow H^n(-1)$ by $\mathbb{G}(\boldsymbol{n}^T)(p) = \boldsymbol{n}^T(p)$. We call it the *hyperbolic Gauss image* of M with respect to $\boldsymbol{n}^T$. We define the *hyperbolic shape operator* $S_h(\boldsymbol{n}^T)_p$

with respect to $\boldsymbol{n}^T$ by $S_h(\boldsymbol{n}^T) = -\pi^t \circ d\mathbb{G}(\boldsymbol{n}^T)(p)$, where $\pi^t : T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_p(M) \longrightarrow TM$ is the orthogonal projection under the identification of $T_{\boldsymbol{n}^T(p)}\mathbb{R}^{n+1}_1 \equiv T_p\mathbb{R}^{n+1}_1$. We also define the *hyperbolic principal curvatures* $\kappa_h(\boldsymbol{n}^T)_i(p)$ $(i = 1, \dots, s)$ of M as the eigenvalues of $S_h(\boldsymbol{n}^T)$. By the assertion (b) of Proposition 3.1, we have the following corollary:

Corollary 3.2 *Under the above notations, we have the following assertions:*
(1) *The lightcone principal curvatures $\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})$, $(i = 1, \dots, s)$ are the eigenvalues of the matrix $(h^j_i(\boldsymbol{n}^T(p), \boldsymbol{n}^S(p)))$, where $\boldsymbol{n}^S$ is the local section of $N_1(M)[\boldsymbol{n}^T]$ with $\boldsymbol{n}^T(p) = \boldsymbol{\xi}$.*
(2) *We have the following relation:*

$$\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) = \kappa_h(\boldsymbol{n}^T)_i(p) + \kappa_d(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}), \ (i = 1, \dots, s).$$

4 Lightlike hypersurfaces

We define a hypersurface

$$\mathbb{LH}_M(\boldsymbol{n}^T) : N_1(M)[\boldsymbol{n}^T] \times \mathbb{R} \longrightarrow \mathbb{R}^{n+1}_1$$

by

$$\mathbb{LH}_M((p, \boldsymbol{\xi}), t) = \boldsymbol{X}(u) + t(\boldsymbol{n}^T + \boldsymbol{\xi})(u) = \boldsymbol{X}(u) + t\mathbb{LG}(\boldsymbol{n}^T)(u, \boldsymbol{\xi}),$$

where $p = \boldsymbol{X}(u)$, which is called the *lightlike hypersurface* along M relative to $\boldsymbol{n}^T$. In general, a hypersurface $H \subset \mathbb{R}^{n+1}_1$ is called a *lightlike hypersurface* if it is tangent to the lightcone at any regular point. We remark that $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R})$ is a lightlike hypersurface.

We introduce the notion of Lorentzian distance-squared functions on spacelike submanifold, which is useful for the study of singularities of lightlike hypersurfaces.

First we define a family of functions $G : M \times \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}$ on a spacelike submanifold $M = \boldsymbol{X}(U)$ by

$$G(p, \boldsymbol{\lambda}) = G(u, \boldsymbol{\lambda}) = \langle \boldsymbol{X}(u) - \boldsymbol{\lambda}, \boldsymbol{X}(u) - \boldsymbol{\lambda} \rangle,$$

where $p = \boldsymbol{X}(u)$. We call G the *Lorentzian distance-squared function* on the spacelike submanifold M. For any fixed $\boldsymbol{\lambda}_0 \in \mathbb{R}^{n+1}_1$, we write $g_{\boldsymbol{\lambda}_0}(p) = G(p, \boldsymbol{\lambda}_0)$ and have the following proposition.

Proposition 4.1 *Let M be a spacelike submanifold and $G : M \times (\mathbb{R}^{n+1}_1 \setminus M) \to \mathbb{R}$ the Lorentzian distance-squared function on M. Suppose that $p_0 \neq \boldsymbol{\lambda}_0$. Then we have the following*:

(1) $g_{\boldsymbol{\lambda}_0}(p_0) = \partial g_{\boldsymbol{\lambda}_0}/\partial u_i(p_0) = 0$ $(i = 1, \dots, s)$ *if and only if there exist $\boldsymbol{\xi}_0 \in N_1(M)_{p_0}[\boldsymbol{n}^T]$ and $\mu \in \mathbb{R} \setminus \{0\}$ such that*

$$p_0 - \boldsymbol{\lambda}_0 = \mu\mathbb{LG}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0).$$

(2) $g_{\boldsymbol{\lambda}_0}(p_0) = \partial g_{\boldsymbol{\lambda}_0}/\partial u_i(p_0) = \det\mathcal{H}(g_{\boldsymbol{\lambda}_0})(p_0) = 0$ $(i = 1, \dots, s)$ *if and only if there exist $\boldsymbol{\xi} \in N_1(M)_{p_0}[\boldsymbol{n}^T]$ and $\mu \in \mathbb{R} \setminus \{0\}$ such that*

$$p_0 - \boldsymbol{\lambda}_0 = \mu\mathbb{LG}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)$$

and $1/\mu$ is one of the non-zero lightcone principal curvatures $\kappa_i(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)$, $(i = 1, \dots, s)$.

Here, $\mathcal{H}(g_{\boldsymbol{\lambda}_0})(p_0)$ is the Hessian matrix of $g_{\boldsymbol{\lambda}_0}$ at p_0.

Proof. (1) We denote that $p = \boldsymbol{X}(u)$. The condition $g_{\boldsymbol{\lambda}_0}(p) = \langle \boldsymbol{X}(u) - \boldsymbol{\lambda}_0, \boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \rangle = 0$ means that $\boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \in LC^*$. We observe that $\partial g/\partial u_i((p) = 2\langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \rangle = 0$ if and only if $\boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \in N_pM$. Hence $g_{\boldsymbol{\lambda}_0}(p_0) = \partial g_{\boldsymbol{\lambda}_0}/\partial u_i((p_0) = 0$ $(i = 1, \dots, s)$ if and only if $p_0 - \boldsymbol{\lambda}_0 \in N_pM \cap LC^*$. Then we denote that $v = \boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \in LC^*$. If $\langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle = 0$, then $\boldsymbol{n}^T(u)$ have to be lightlike or spacelike. This contradiction to the fact that $\boldsymbol{n}^T(u)$ is a timelike unit vector, so that $\langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle \neq 0$. We set

$$\boldsymbol{\xi}_0 = \frac{-1}{\langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle} \boldsymbol{v} - \boldsymbol{n}^T(u).$$

Then we have

$$\begin{aligned} \langle \boldsymbol{\xi}_0, \boldsymbol{\xi}_0 \rangle &= -2 \frac{-1}{\langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle} \langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle - 1 = 1 \\ \langle \boldsymbol{\xi}_0, \boldsymbol{n}^T(u) \rangle &= \frac{-1}{\langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle} \langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle + 1 = 0. \end{aligned}$$

This means that $\boldsymbol{\xi}_0 \in N_1(M)_p(M)$. Since $-\boldsymbol{v} = \langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle (\boldsymbol{n}^T(u) + \boldsymbol{\xi}_0)$, we have $p_0 - \boldsymbol{\lambda}_0 = \mu \mathbb{LG}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)$, where $p_0 = \boldsymbol{X}(u)$ and $\mu = -\langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle$. The converse assertion is trivial by definition.

(2) By a straightforward calculation, we have

$$\frac{\partial^2 g}{\partial u_i \partial u_j} = 2 \left\{ \langle \boldsymbol{X}_{u_i u_j}, \boldsymbol{X} - \boldsymbol{\lambda}_0 \rangle + \langle \boldsymbol{X}_{u_i}, \boldsymbol{X}_{u_j} \rangle \right\}.$$

Under the conditions $p_0 - \boldsymbol{\lambda}_0 = \mu(\boldsymbol{n}^T(u) + \boldsymbol{\xi}_0)$ and $p_0 = \boldsymbol{X}(u)$, we have

$$\frac{\partial^2 g}{\partial u_i \partial u_j}(u) = 2 \left\{ \langle \boldsymbol{X}_{u_i u_j}(u), \mu(\boldsymbol{n}^T(u) + \boldsymbol{\xi}_0) \rangle + g_{ij}(u) \right\}.$$

Therefore, we have

$$\left(\frac{\partial^2 g}{\partial u_i \partial u_j}(u) \right) \left(g^{k\ell}(u) \right) = \left(2 \left\{ -\mu h^i_j(\boldsymbol{n}^T(u), \boldsymbol{n}^S(u)) + \delta^i_j \right\} \right),$$

where $\boldsymbol{n}^S$ is the local section of $N_1(M)[\boldsymbol{n}^T]$ with $\boldsymbol{n}^S(u) = \boldsymbol{\xi}_0$. It follows that $\det \mathcal{H}(g)(p_0) = 0$ if and only if $1/\mu$ is an eigenvalue of $(h^i_j(\boldsymbol{n}^T(u), \boldsymbol{n}^S(u)))$, which is equal to one of the lightcone principal curvatures $\kappa_i(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0), (i = 1, \dots, s)$ by Corollary 3.2. □

In order to understand the geometric meanings of the assertions of Proposition 4.1, we briefly review the theory of Legendrian singularities For detailed expressions, see [1, 30]. Let $\pi : PT^*(\mathbb{R}^{n+1}) \longrightarrow \mathbb{R}^{n+1}$ be the projective cotangent bundle with its canonical contact structure. We next review the geometric properties of this bundle. Consider the tangent bundle $\tau : TPT^*(\mathbb{R}^{n+}) \to PT^*(\mathbb{R}^{n+1})$ and the differential map $d\pi : TPT^*(\mathbb{R}^{n+1}) \to T\mathbb{R}^{n+1}$ of π. For any $X \in TPT^*(\mathbb{R}^{n+1})$, there exists an element $\alpha \in T^*(\mathbb{R}^{n+1}_1$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(\mathbb{R}^{n+1})$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(\mathbb{R}^{n+1})$ by

$$K = \{ X \in TPT^*(\mathbb{R}^{n+1}) \mid \tau(X)(d\pi(X)) = 0 \}.$$

We have the trivialization $PT^*(\mathbb{R}^{n+1}) \cong \mathbb{R}^{n+1} \times P^n(\mathbb{R})^*$, and call

$$((v_0, v_1, \ldots, v_n), [\xi_0 : \xi_1 : \cdots : \xi_n])$$

homogeneous coordinates of $PT^*(\mathbb{R}^{n+1})$, where $[\xi_0 : \xi_1 : \cdots : \xi_n]$ are the homogeneous coordinates of the dual projective space $P^n(\mathbb{R})^*$.

It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=0}^n \mu_i \xi_i = 0$, where $d\tilde{\pi}(X) = \sum_{i=0}^n \mu_i \partial/\partial v_i$. An immersion $i : L \to PT^*(\mathbb{R}^{n+1})$ is said to be *a Legendrian immersion* if $\dim L = n$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called *the Legendrian map* and the set $W(i) = \mathrm{image}\, \pi \circ i$, the *wave front set* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is *a Morse family of hypersurfaces* if the map germ

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is submersive, where $(q, x) = (q_1, \ldots, q_k, x_0, \ldots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0})$. In this case we have a smooth n-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0}) \;\middle|\; F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\mathscr{L}_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^{n+1}$ defined by

$$\mathscr{L}_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_0}(q, x) : \cdots : \frac{\partial F}{\partial x_n}(q, x) \right] \right)$$

is a Legendrian immersion. We call F *a generating family* of $\mathscr{L}_F$, and the wave front set is given by $W(\mathscr{L}_F) = \pi_n(\Sigma_*(F))$, where $\pi_n : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the canonical projection. In the theory of unfoldings of function germs, the wave front set $W(\mathscr{L}_F)$ is called a *discriminant set* of F, which we also denote $\mathcal{D}_F$. Therefore, Proposition 4.1 asserts that the discriminant set of the Lorentzian distance-squared function G is given by

$$\mathcal{D}_G = \left\{ \boldsymbol{\lambda} \;\middle|\; \boldsymbol{\lambda} = \boldsymbol{X}(p) + t(\boldsymbol{n}^T \pm \boldsymbol{\xi})(p),\ p \in M, \boldsymbol{\xi} \in N_1(M)_p[\boldsymbol{n}^T], t \in \mathbb{R} \right\},$$

which is the image of the lightlike hypersurface along M relative to $\boldsymbol{n}^T$.

By the assertion (2) of Proposition 4.1, a singular point of the lightlike hypersurface is a point $\boldsymbol{\lambda}_0 = p_0 + t_0(\boldsymbol{n}^T + \boldsymbol{\xi}_0)(p_0)$ for $t_0 = 1/\kappa_i(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)$, $i = 1, \ldots .s)$. Then we have the following corollary.

Corollary 4.2 *The critical value of* $\mathbb{LH}_M(\boldsymbol{n}^T)$ *is the point where* $\kappa_i(\boldsymbol{n}^T)(p, \boldsymbol{\xi}) \neq 0$ *and*

$$\boldsymbol{\lambda} = p + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})} \mathbb{LG}(\boldsymbol{n}^T)(p, \boldsymbol{\xi}).$$

We define a mapping $\mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i} : N_1(M)[\boldsymbol{n}^T] \longrightarrow \mathbb{R}^{n+1}_1$ by

$$\mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p, \boldsymbol{\xi}) = p + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})} \mathbb{LG}(\boldsymbol{n}^T)(p, \boldsymbol{\xi}).$$

We also define

$$\mathbb{LF}_M(\boldsymbol{n}^T) = \bigcup \left\{ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p,\boldsymbol{\xi}) \mid (p,\boldsymbol{\xi}) \in N_1(M)[\boldsymbol{n}^T] \text{ s.t. } \kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) \neq 0, i = 1,\dots,s \right\}.$$

We call $\mathbb{LF}_M(\boldsymbol{n}^T)$ the *lightlike focal set* of $M = \boldsymbol{X}(U)$ relative to $\boldsymbol{n}^T$. By definition, the lightlike focal set of $M = \boldsymbol{X}(U)$ relative to $\boldsymbol{n}^T$ is the critical values set of the lightlike hypersurface $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R})$ along M relative to $\boldsymbol{n}^T$.

We can show that the image of the lightlike hypersurface along M is independent of the choice of the future directed timelike normal vector field $\boldsymbol{n}^T$ as a corollary of Proposition 4.1.

Corollary 4.3 *Let $\boldsymbol{n}^T$ and $\overline{\boldsymbol{n}}^T$ be future directed timelike unit normal fields along M. Then we have*

$$\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R}) = \mathbb{LH}_M(\overline{\boldsymbol{n}}^T)(N_1(M)[\overline{\boldsymbol{n}}^T] \times \mathbb{R}) \text{ and } \mathbb{LF}_M(\boldsymbol{n}^T) = \mathbb{LF}_M(\overline{\boldsymbol{n}}^T).$$

Proof. By Proposition 4.1, the both images of the lightlike hypersurface along M relative to $\boldsymbol{n}^T$ and $\overline{\boldsymbol{n}}^T$ are the discriminant sets of the Lorentzian distance-squared function G on M. Moreover, the focal set of M is the critical value set of the lightlike hypersurface along M relative to $\boldsymbol{n}^T$. Since G is independent of the choice of $\boldsymbol{n}^T$, we have the assertion. □

We have the following proposition.

Proposition 4.4 *Let G be the Lorentzian distance-squared function on M. For any point $(u,\boldsymbol{\lambda}) \in G^{-1}(0)$, the germ of G at $(u,\boldsymbol{\lambda})$ is a Morse family of hypersurfaces.*

Proof. We denote that

$$\boldsymbol{X}(u) = (X_0(u), X_1(u), \dots, X_n(u)) \text{ and } \boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_n).$$

By definition, we have

$$G(u,\boldsymbol{\lambda}) = -(X_0(u)-\lambda_0)^2 + (X_1(u)-\lambda_1)^2 + \dots + (X_n(u)-\lambda_n)^2.$$

We now prove that the mapping

$$\Delta^* G = \left(G, \frac{\partial G}{\partial u_1}, \dots, \frac{\partial G}{\partial u_s} \right)$$

is non-singular at $(u,\boldsymbol{\lambda}) \in G^{-1}(0)$. Indeed, the Jacobian matrix of $\Delta^* G$ is given by

$$\begin{pmatrix} \mathrm{A} & \begin{matrix} 2(X_0-\lambda_0) & -2(X_1-\lambda_1) & \cdots & -2(X_n-\lambda_n) \\ 2X_{0u_1} & -2X_{1u_1} & \cdots & -2X_{nu_1} \\ \vdots & \vdots & \ddots & \vdots \\ 2X_{0u_s} & -2X_{1u_s} & \cdots & -2X_{nu_s} \end{matrix} \end{pmatrix},$$

where A is the following matrix:

$$\begin{pmatrix} 2\langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_1}\rangle & \cdots & 2\langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_s}\rangle \\ 2(\langle \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_1}\rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_1u_1}\rangle) & \cdots & 2(\langle \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_s}\rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_1u_s}\rangle) \\ \vdots & \ddots & \vdots \\ 2(\langle \boldsymbol{X}_{u_s}, \boldsymbol{X}_{u_1}\rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_su_1}\rangle) & \cdots & 2(\langle \boldsymbol{X}_{u_s}, \boldsymbol{X}_{u_s}\rangle + \langle \boldsymbol{X}-\boldsymbol{\lambda}, \boldsymbol{X}_{u_su_s}\rangle) \end{pmatrix}.$$

Since $\boldsymbol{X}$ is an immersion, the rank of the matrix

$$\begin{pmatrix} 2X_{0u_1} & -2X_{1u_1} & \cdots & -2X_{nu_1} \\ \vdots & \vdots & \ddots & \vdots \\ 2X_{0u_s} & -2X_{1u_s} & \cdots & -2X_{nu_s} \end{pmatrix}.$$

is equal to s. Moreover, $\boldsymbol{X} - \boldsymbol{\lambda}$ is lightlike, so that it is linearly independent with respect to tangent vectors $\boldsymbol{X}_{u_1}, \dots, \boldsymbol{X}_{u_s}$. This means that the rank of the matrix

$$\begin{pmatrix} 2(X_0 - \lambda_0) & -2(X_1 - \lambda_1) & \cdots & -2(X_n - \lambda_n) \\ 2X_{0u_1} & -2X_{1u_1} & \cdots & -2X_{nu_1} \\ \vdots & \vdots & \ddots & \vdots \\ 2X_{0u_s} & -2X_{1u_s} & \cdots & -2X_{nu_s} \end{pmatrix}$$

is equal to $s + 1$. Therefore the Jacobi matrix of $\Delta^* G$ is non-singular at $(u, \boldsymbol{\lambda}) \in G^{-1}(0)$. $\square$

Since G is a Morse family of hypersurfaces, we have a Legendrian immersion

$$\mathscr{L}_G : \Sigma_*(G) \longrightarrow PT^*(\mathbb{R}^{n+1}_1)$$

defined by

$$\mathscr{L}_G(u, \boldsymbol{\lambda}) = (\boldsymbol{\lambda}, [(X_0(u) - \lambda_0) : (\lambda_1 - X_1(u)) : \cdots : (\lambda_n - X_n(u))]),$$

where

$$\Sigma_*(G) = \{(u, \boldsymbol{\lambda}) \mid \boldsymbol{\lambda} = \mathbb{LH}_M(\boldsymbol{n}^T)(p, \boldsymbol{\xi}, t) \ ((p, \boldsymbol{\xi}), t) \in N_1(M)[\boldsymbol{n}^T] \times \mathbb{R}\}.$$

We observe that G is a generating family of the Legendrian immersion $\mathscr{L}_G$ whose wave front is $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R})$. Therefore we say that the Lorentzian distance-squared function G on M gives a Lorentz-Minkowski-canonical generating family for the Legendrian lift of $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R})$.

5 Contact with lightcones

In this section we consider the geometric meanings of the singularities of lightlike hypersurfaces from the view point of the theory of contact of submanifolds with model hypersurfaces in [25]. We begin with the following basic observations.

Proposition 5.1 *Let $\boldsymbol{\lambda}_0 \in \mathbb{R}^{n+1}_1$ and M a spacelike submanifold without points satisfying $K_\ell(\boldsymbol{n}^T)(p, \boldsymbol{\xi}) = 0$. Then $M \subset LC_{\boldsymbol{\lambda}_0}$ if and only if $\boldsymbol{\lambda}_0 = \mathbb{LF}_M(\boldsymbol{n}^T)$ is the lightcone focal set. In this case we have $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T]) \subset LC_{\boldsymbol{\lambda}_0}$ and $M = \boldsymbol{X}(U)$ is totally lightcone umbilical.*

Proof. By Proposition 3.1, $K_\ell(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0) \neq 0$ if and only if

$$\{(\boldsymbol{n}^T + \boldsymbol{n}^S), (\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_1}, \dots, (\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_s}\}$$

is linearly independent for $p_0 = \boldsymbol{X}(u_0) \in M$ and $\boldsymbol{\xi}_0 = \boldsymbol{n}^S(u_0)$, where $\boldsymbol{n}^S : U \longrightarrow N_1(M)[\boldsymbol{n}^T]$ is a local section. By definition, $M \subset LC_{\boldsymbol{\lambda}_0}$ if and only if $g_{\boldsymbol{\lambda}_0}(u) \equiv 0$ for any $u \in U$, where $g_{\boldsymbol{\lambda}_0}(u) = G(u, \boldsymbol{\lambda}_0)$ is the Lorentzian distance-squared function on M. It follows from Proposition 4.1 that there exists a smooth function $\mu : U \times N_1(M)[\boldsymbol{n}^T] \longrightarrow \mathbb{R}$ and section $\boldsymbol{n}^S : U \longrightarrow N_1(M)[\boldsymbol{n}^T]$ such that

$$\boldsymbol{X}(u) = \boldsymbol{\lambda}_0 + \mu(u, \boldsymbol{n}^S(u))(\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)).$$

In fact, we have $\mu(u, \boldsymbol{n}^S(u)) = -1/\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})$ $i = 1, \dots, s$, where $p = \boldsymbol{X}(u)$ and $\boldsymbol{\xi} = \boldsymbol{n}^S(u)$. It follows that $\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) = \widetilde{\kappa}_\ell(\boldsymbol{n}^T)_j(p, \boldsymbol{\xi})$, so that $M = \boldsymbol{X}(U)$ is totally lightcone umbilical. Therefore we have

$$\mathbb{LH}_M(\boldsymbol{n}^T)(u, \boldsymbol{n}^S(u), t) = \boldsymbol{\lambda}_0 + (t + \mu(u, \boldsymbol{n}^S(u))(\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)).$$

Hence we have $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T]) \subset LC_{\boldsymbol{\lambda}_0}$. By Corollary 5.2, the critical value set of $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T])$ is the lightcone focal set $\mathbb{LF}_M(\boldsymbol{n}^T)$. However, it is equal to $\boldsymbol{\lambda}_0$ by the previous arguments.

For the converse assertion, suppose that $\boldsymbol{\lambda}_0 = \mathbb{LF}_M(\boldsymbol{n}^T)$. Then we have

$$\boldsymbol{\lambda}_0 = \boldsymbol{X}(u) + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(\boldsymbol{X}(u), \boldsymbol{\xi})} \mathbb{LG}(\boldsymbol{n}^T)(u, \boldsymbol{\xi}),$$

for any $i = 1, \dots, s$ and $(p, \boldsymbol{\xi}) \in N_1(M)[\boldsymbol{n}^T]$, where $p = \boldsymbol{X}(u)$. Thus, we have

$$\kappa_\ell(\boldsymbol{n}^T)_i(\boldsymbol{X}(u), \boldsymbol{\xi}) = \kappa_\ell(\boldsymbol{n}^T)_j(\boldsymbol{X}(u), \boldsymbol{\xi})$$

for any $i, j = 1, \dots, s$, so that M is totally lightcone umbilical. Since $\mathbb{LG}(\boldsymbol{n}^T)(u, \boldsymbol{\xi})$ is lightlike, we have $\boldsymbol{X}(u) \in LC_{\boldsymbol{\lambda}_0}$. This completes the proof. $\square$

According to the above proposition, the lightcone is regarded as a model lightlike hypersurface in $\mathbb{R}^{n+1}_1$. We now consider the contact of spacelike submanifolds with lightcones in the view of Montaldi's theory. We review the theory of contact for submanifolds in [25]. Let X_i and Y_i, $i = 1, 2$, be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that *the contact of* X_1 and Y_1 at y_1 is same type as *the contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. Since this definition of contact is local, we can replace $\mathbb{R}^n$ by arbitrary n-manifold. Montaldi gives in [25] the following characterization of contact by using $\mathcal{K}$-equivalence.

Theorem 5.2 *Let X_i and Y_i, $i = 1, 2$, be submanifolds of $\mathbb{R}^n$ with* $\dim X_1 = \dim X_2$ *and* $\dim Y_1 = \dim Y_2$. *Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then*

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$$

if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

On the other hand, we return to the review on the theory of Legendrian singularities. We introduce a natural equivalence relation among Legendrian submanifold germs. Let

$F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then we say that $\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, z) \longrightarrow (PT^*\mathbb{R}^n, z')$ such that H preserves fibers of π and that $H(\mathscr{L}_F(\Sigma_*(F))) = \mathscr{L}_G(\Sigma_*(G))$, where $z = \mathscr{L}_F(0), z' = \mathscr{L}_G(0)$. By using the Legendrian equivalence, we can define the notion of Legendrian stability for Legendrian submanifold germs by the ordinary way (see, [1, Part III]). We can interpret the Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$. We say that F is an *infinitesimally $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \dots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [22].) The main result in the theory of Legendrian singularities ([1], §20.8 and [30], THEOREM 2) is the following:

Theorem 5.3 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then we have the following assertions:*
(1) $\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are Legendrian equivalent if and only if F and G are P-$\mathcal{K}$-equivalent,
(2) $\mathscr{L}_F(\Sigma_*(F))$ is Legendrian stable if and only if F is an infinitesimally $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$.

Since F and G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do not need the notion of stably P-$\mathcal{K}$-equivalences under this situation [30, page 27]. For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define *the local ring of f* by $Q_r(f) = \mathcal{E}_n/(f^*(\mathfrak{M}_p)\mathcal{E}_n + \mathfrak{M}_n^{r+1})$. We have the following classification result of Legendrian stable germs (cf. [15, Proposition A.4]) which is the key for the purpose in this section.

Proposition 5.4 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces and $f = F|\mathbb{R}^k \times \{\mathbf{0}\}, g = G|\mathbb{R}^k \times \{\mathbf{0}\}$. Suppose that $\mathscr{L}_F$ and $\mathscr{L}_G$ are Legendrian stable. The the following conditions are equivalent:*
 (1) $(W(\mathscr{L}_F), \mathbf{0})$ and $(W(\mathscr{L}_G), \mathbf{0})$ are diffeomorphic as set germs,
 (2) $\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are Legendrian equivalent,
 (3) $Q_{n+1}(f)$ and $Q_{n+1}(g)$ are isomorphic as $\mathbb{R}$-algebras.

Returning to lightlike hypersurfaces, we now consider the function

$$\mathcal{G} : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}$$

defined by $\mathcal{G}(\boldsymbol{x}, \boldsymbol{\lambda}) = \langle \boldsymbol{x} - \boldsymbol{\lambda}, \boldsymbol{x} - \boldsymbol{\lambda} \rangle$. Given $\boldsymbol{\lambda}_0 \in \mathbb{R}^{n+1}_1$, we denote $\mathfrak{g}_{\boldsymbol{\lambda}_0}(\boldsymbol{x}) = \mathcal{G}(\boldsymbol{x}, \boldsymbol{\lambda}_0)$, so that we have $\mathfrak{g}^{-1}_{\boldsymbol{\lambda}_0}(0) = LC_{\boldsymbol{\lambda}_0}$. For any $p_0 = \boldsymbol{X}(u_0) \in M$, $t_0 \in \mathbb{R}$ and $\boldsymbol{\xi}_0 \in N_1(M)_p[\boldsymbol{n}^T]$, we consider the point $\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + t_0(\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0)$. Then we have

$$\mathfrak{g}_{\boldsymbol{\lambda}_0} \circ \boldsymbol{X}(u_0)) = \mathcal{G} \circ (\boldsymbol{X} \times 1_{\mathbb{R}^{n+1}_1})(u_0, \boldsymbol{\lambda}_0) = G(p_0, \boldsymbol{\lambda}_0) = 0,$$

where $t_0 = 1/\kappa_\ell(\boldsymbol{n}^T)_i(p_0, \boldsymbol{\xi}_0)$, $i = 1, \dots, s$. We also have relations

$$\frac{\partial \mathfrak{g}_{\boldsymbol{\lambda}_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial G}{\partial u_i}(p_0, \boldsymbol{\lambda}_0) = 0, \ i = 1, \dots, s.$$

These imply that the lightcone $\mathfrak{g}^{-1}_{\boldsymbol{\lambda}_0}(0) = LC_{\boldsymbol{\lambda}_0}$ is tangent to $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$. In this case, we call $LC_{\boldsymbol{\lambda}_0}$ a *tangent lightcone* of $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$.

We now describe the contacts of spacelike submanifolds with lightcones. We denote by $Q^\sigma(\boldsymbol{X}, u_0)$ the local ring of the function germ $\widetilde{g}_{\boldsymbol{\lambda}^\sigma_0} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{\lambda}_0 = \mathbb{LC}_M(u_0, \boldsymbol{\xi}_0, t_0)$. We remark that we can explicitly write the local ring as follows:

$$Q_{n+1}(\boldsymbol{X}, u_0) = \frac{C^\infty_{u_0}(U)}{\langle\langle \boldsymbol{X}(u) - \boldsymbol{\lambda}_0, \boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \rangle\rangle_{C^\infty_{u_0}(U)} + \mathfrak{M}_{u_0}(U)^{n+2}},$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at u_0.

Let $\mathbb{LH}_{M_i}(\boldsymbol{n}^T_i) : (N_1(M_i)[\boldsymbol{n}^T_i] \times \mathbb{R}, (p_i, \boldsymbol{\xi}_i, t_i)) \longrightarrow (\mathbb{R}^{n+1}_1, \boldsymbol{\lambda}_i)$, $(i = 1, 2)$ be two lightlike hypersurface germs of spacelike submanifold germs $\boldsymbol{X}_i : (U, u^i) \longrightarrow (\mathbb{R}^{n+1}_1, p_i)$. Let $G_i : (U \times \mathbb{R}^{n+1}_1, (u^i, \boldsymbol{\lambda}^\sigma_i)) \longrightarrow \mathbb{R}$ be the Lorentzian distance-squared function germ of $\boldsymbol{X}_i$. Then we have the following theorem:

Theorem 5.5 *Let $\boldsymbol{X}_i : (U, u^i) \longrightarrow (\mathbb{R}^{n+1}_1, p_i)$, $i = 1, 2$, be spacelike surface germs such that the corresponding Legendrian submanifold germs $\mathscr{L}_{G_i}(\Sigma_*(G_i))$ are Legendrian stable. Then the following conditions are equivalent*:

(1) $(\mathbb{LH}_{M_1}(N_1(M_1)[\boldsymbol{n}^T_1] \times \mathbb{R}), \boldsymbol{\lambda}_1)$ *and* $(\mathbb{LH}_{M_2}(N_1(M_2)[\boldsymbol{n}^T_2] \times \mathbb{R}), \boldsymbol{\lambda}_2)$ *are diffeomorphic,*

(2) $(\mathscr{L}_{G_1}(\Sigma_*(G_1)), (u^1, \boldsymbol{\lambda}_1))$ *and* $(\mathscr{L}_{G_2}(\Sigma_*(G_2)), (u^2, \boldsymbol{\lambda}_2))$ *are Legendrian equivalent,*

(3) G_1 *and* G_2 *are* P-$\mathcal{K}$*-equivalent,*

(4) $g_{1,\boldsymbol{\lambda}_1}$ *and* $g_{2,\boldsymbol{\lambda}_2}$ *are* $\mathcal{K}$*-equivalent,*

(5) $K(\boldsymbol{X}_1(U), LC_{\boldsymbol{\lambda}_1}, p_1) = K(\boldsymbol{X}_2(U), LC_{\boldsymbol{\lambda}_2}, p_2)$.

(6) $Q_{n+1}(\boldsymbol{X}_1, u^1)$ *and* $Q_{n+1}(\boldsymbol{X}_2, u^2)$ *are isomorphic as* $\mathbb{R}$*-algebras.*

Proof. By Proposition 6.4, the conditions (1), (2) and (6) are equivalent. This condition is also equivalent to that two generating families G_1 and G_2 are P-$\mathcal{K}$-equivalent by Theorem 6.3. If we denote $g_{i,\boldsymbol{\lambda}_i}(u) = G_i(u, \boldsymbol{\lambda}_i)$, then we have $g_{i,\boldsymbol{\lambda}_i}(u) = \mathfrak{g}_{\boldsymbol{\lambda}_i} \circ \boldsymbol{X}_i(u)$. By Theorem 6.2, $K(\boldsymbol{X}_1(U), LC_{\boldsymbol{\lambda}_1}, p_1) = K(\boldsymbol{x}_2(U), LC\boldsymbol{\lambda}_2, p_2)$ if and only if $\widetilde{g}_{1,\boldsymbol{\lambda}_1}$ and $\widetilde{g}_{2,\boldsymbol{\lambda}_2}$ are $\mathcal{K}$-equivalent. This means that (4) and (5) are equivalent. By definition, (3) implies (4). The uniqueness of the infinitesimally $\mathcal{K}$-versal deformation of $g_{i,\boldsymbol{\lambda}_i}$ [22] leads that the condition (4) implies (3). This completes the proof. $\square$

For a spacelike embedding germ $\boldsymbol{X} : (U, u_0) \longrightarrow (\mathbb{R}^{n+1}_1, p_0)$, we consider a set germ $(\boldsymbol{X}^{-1}(LC_{\boldsymbol{\lambda}_0}), u_0)$, which is called the *tangent lightcone indicatrix germ* of $\boldsymbol{X}$, where $\boldsymbol{\lambda}_0 = \mathbb{LH}_M(p_0, \boldsymbol{\xi}_0, t_0)$ and $t_0 = -1/\kappa_\ell(\boldsymbol{n}^T)_i(p_0, \boldsymbol{\xi}_0)(i = 1, \dots s)$. We have the following corollary of Theorem 5.5.

Corollary 5.6 *Under the assumptions of Theorem* 5.5*, if the lightlike hypersurface germs* $(\mathbb{LH}_{M_1}(N_1(M_1)[\boldsymbol{n}_1^T]\times\mathbb{R}),\boldsymbol{\lambda}_1)$ *and* $(\mathbb{LH}_{M_2}(N_1(M_2)[\boldsymbol{n}_2^T]\times\mathbb{R}),\boldsymbol{\lambda}_2)$*, then tangent lightcone indicatrix germs*

$$(\boldsymbol{X}_1^{-1}(LC_{\boldsymbol{\lambda}_1}),u^1) \quad \text{and} \quad (\boldsymbol{X}_2^{-1}(LC_{\boldsymbol{\lambda}_2}),u^2)$$

are diffeomorphic as set germs.

Proof. Notice that the tangent lightcone indicatrix germ of $\boldsymbol{X}_i$ is the zero level set of $g_{i,\boldsymbol{\lambda}_i}$. Since $\mathcal{K}$-equivalence among function germs preserves the zero-level sets of function germs, the assertion follows from Theorem 5.5. $\square$

On the other hand, we consider generic properties of lightlike hypersurfaces along spacelike submanifolds. Let $\mathrm{Emb}_{\mathrm{sp}}(U,\mathbb{R}^{n+1}_1)$ be the space of spacelike embeddings with the Whitney C^∞-topology for an open set $U\subset\mathbb{R}^{n+1}_1$. We consider the function $\mathcal{G}:\mathbb{R}^{n+1}_1\times\mathbb{R}^{n+1}_1\longrightarrow\mathbb{R}$ again. We claim that $\mathfrak{g}_{\boldsymbol{\lambda}}$ is a submersion at $\boldsymbol{x}\neq\boldsymbol{\lambda}$ for any $\boldsymbol{\lambda}\in\mathbb{R}^{n+1}_1$. For any $\boldsymbol{X}\in\mathrm{Emb}_{\mathrm{sp}}(U,\mathbb{R}^{n+1}_1)$, we have $G=\mathcal{G}\circ(\boldsymbol{X}\times 1_{\mathbb{R}^{n+1}_1})$. We have the r-jet extension $j^r_1G:U\times\mathbb{R}^{n+1}_1\longrightarrow J^r(U,\mathbb{R})$ defined by $j^r_1G(u,\boldsymbol{\lambda})=j^rg_{\boldsymbol{\lambda}}(u)$, where $J^k(U,\mathbb{R})$ is the k-jet space of functions on U. We consider the trivialization $J^r(U,\mathbb{R})\equiv U\times\mathbb{R}\times J^r(s,1)$. For any submanifold $Q\subset J^r(s,1)$, we denote that $\widetilde{Q}=U\times\mathbb{R}\times Q$. As an application of [29, Lemma 6], the set

$$T_Q=\{\boldsymbol{X}\in\mathrm{Emb}_{\mathrm{sp}}(U,\mathbb{R}^{n+1}_1)\mid j^r_1G \text{ is transversal to } \widetilde{Q}\ \}$$

is a residual set of $\mathrm{Emb}_{\mathrm{sp}}(U,\mathbb{R}^{n+1}_1)$. Moreover, if Q is a closet subset , then T_Q is open. It is known [8] that there exists a semi-algebraic set $W^r(s,1)\subset J^k(s,1)$ and a stratification $\mathcal{A}^r(s,1)$ of $J^k(s,1)\setminus W^r(s,1)$ such that $\lim_{k\mapsto\infty}\operatorname{cod}W^r(s,1)=+\infty$. The stratification $\mathcal{A}^r(s,1)$ is called the *canonical stratification.* We define a stratification $\mathcal{A}^r(U,\mathbb{R})$ of $J^r(U,\mathbb{R})\setminus W^r(U,\mathbb{R})$ by

$$U\times(\mathbb{R}\setminus\{0\})\times(J^r(s,1)\setminus W^r(s,1)),\ U\times\{0\}\times\mathcal{A}^r(s,1),$$

where $W^r(U,\mathbb{R})=U\times\mathbb{R}\times W^r(s,1)$. In [28], it was shown that if $j^r_1G(U\times\mathbb{R}^{n+1}_1)\cap W^r(U,\mathbb{R})=\emptyset$ and j^r_1G is transversal to $\mathcal{A}^r(U,\mathbb{R})$, then the map $\pi|G^{-1}(0):G^{-1}(0)\longrightarrow\mathbb{R}$ is MT-stable map-germ at each point, where $\pi:U\times\mathbb{R}^{n+1}_1\longrightarrow\mathbb{R}^{n+1}_1$ is the canonical projection. Here, a map germ is said to be *MT-stable* if the jet extension is transversal to the canonical stratification of the jet space of sufficiently higher order (cf., [8, 23]). The main result of the theory of Topological stability of Mather is that MT-stability implies topological stability. By Proposition 4.1, the lightlike hypersurface $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T]\times\mathbb{R})$ is the discriminant set of G, which is equal to the critical value set of $\pi|G^{-1}(0)$. Since $\operatorname{cod}W^r(U,\mathbb{R})>s+n+1$ for sufficiently large k, the set

$$\mathcal{O}_1=\{\boldsymbol{X}\in\mathrm{Emb}_{\mathrm{sp}}(U,\mathbb{R}^{n+1}_1)\mid j^r_1G(U\times\mathbb{R}^{n+1}_1)\cap W^r(U,\mathbb{R})=\emptyset\ \}$$

is a residual set. It follows that the set

$$\mathcal{O}=\{\boldsymbol{X}\in\mathcal{O}_1\mid j^r_1G \text{ is transversal to } \mathcal{A}^r(U,\mathbb{R})\ \}$$

is a residual set. Therefore, we have the following theorem.

Theorem 5.7 *There exists a residual set $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}^{n+1}_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the lightlike hypersurface $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R})$ at any point is a germ of the critical value set of an MT-stable map germ.*

In the case when $n \leq 5$, by the classification results of the $\mathcal{K}$-equivalence among function germs, the canonical stratification $\mathcal{A}^k(s, 1)$ is given by the finite collection of the $\mathcal{K}$-orbits. Moreover, if $j^r_1 G$ is transversal to the $\mathcal{K}$-orbit of $j^r g_{\boldsymbol{\lambda}_0}(u_0)$ for sufficiently large r, then G is an infinitesimally $\mathcal{K}$-versal deformation of $g_{\boldsymbol{\lambda}}$ at $(u_0, \boldsymbol{\lambda}_0)$ [22]. By Theorem 5.3, we have the following theorem.

Theorem 5.8 *Suppose that $n \leq 5$. Then there exists a residual set $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}^{n+1}_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the lightlike hypersurface $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R})$ at any point is the germ of the wave front set of a stable Legendrian submanifold germ $\mathscr{L}_G(\Sigma_*(G))$.*

6 Spacelike submanifolds with codimension two

In the case when $s = n - 1$, $N_1(M)[\boldsymbol{n}^T]$ is a double covering of $M = \boldsymbol{X}(U)$. We can construct a spacelike unit normal section $\boldsymbol{n}^S(u) \in N_p(M)$ by

$$\boldsymbol{n}^S(u) = \frac{\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)}{\|\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\|}.$$

Then $\boldsymbol{\sigma}^{\pm}(u) = (\boldsymbol{X}(u), \pm \boldsymbol{n}^S(u))$ are sections of $N_1(M)[\boldsymbol{n}^T]$. Clearly, the vectors $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$ are lightlike. In [17], it was shown that $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u), \bar{\boldsymbol{n}}^T(u) \pm \bar{\boldsymbol{n}}^S(u)$ are respectively parallel for any two future directed unit timelike normal sections $\boldsymbol{n}^T(u), \bar{\boldsymbol{n}}^T(u) \in N_p(M)$. Therefore, $\widetilde{\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)} = \widetilde{\bar{\boldsymbol{n}}^T(u) \pm \bar{\boldsymbol{n}}^S(u)}$. It follows that we have a mapping $\widetilde{\mathbb{LG}}^{\pm} : U \longrightarrow S^{n-1}_+$ defined by $\widetilde{\mathbb{LG}}^{\pm}(u) = \widetilde{\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)}$. We call one of $\widetilde{\mathbb{LG}}^{\pm}$ a *normalized lightcone Gauss map* of $M = \boldsymbol{X}(U)$, which is independent of the choice of $\boldsymbol{n}^T$. Since $N_p(M)[\boldsymbol{n}^T]$ is a spacelike line in $N_p(M)$, we have $\boldsymbol{\xi} = \boldsymbol{n}^S(u)$ or $\boldsymbol{\xi} = -\boldsymbol{n}^S(u)$ for any $(\boldsymbol{X}(u), \boldsymbol{\xi}) \in N_1(M)[\boldsymbol{n}^T]$. In [17], the *normalized lightcone shape operator* $\widetilde{S}^{\pm}_{\ell,p} : T_pM \to T_pM$ was defined by taking the derivative of the normalized lightcone Gauss map $\widetilde{\mathbb{LG}}^{\pm}$. The *normalized principal curvatures* $\{\widetilde{\kappa}^{\pm}_{\ell,i}(p)\}_{i=1}^{n-1}$ are defined to be the eigenvalues of the normalized lightcone shape operator $\widetilde{S}^{\pm}_{\ell,p}$. It was shown that

$$\widetilde{\kappa}^{\pm}_{\ell,i}(p) = \frac{1}{\ell^{\pm}_0(p)} \kappa_i(\boldsymbol{n}^T, \pm \boldsymbol{n}^S)(p),$$

where $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u) = (\ell^{\pm}_0(p), \ell^{\pm}_1(p), \ldots, \ell^{\pm}_n(p))$ and $p = \boldsymbol{X}(u)$. We define the *normalized lightlike hypersurface* along $M = \boldsymbol{X}(U)$ as the images of the maps $\widetilde{\mathbb{LH}}^{\pm}_M : U \times \mathbb{R} \longrightarrow \mathbb{R}^{n+1}_1$ defined by $\widetilde{\mathbb{LH}}^{\pm}_M(u, t) = \boldsymbol{X}(u) + t\widetilde{\mathbb{LG}}^{\pm}(u)$. Since

$$\widetilde{\mathbb{LG}}^{\pm}(u) = \frac{1}{\ell^{\pm}_0(p)}(\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)) = \frac{1}{\ell^{\pm}_0(p)} \mathbb{LG}(\boldsymbol{n}^T)(p, \boldsymbol{n}^S(u)),$$

we have

$$\mathbb{LH}_M(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R}) = \widetilde{\mathbb{LH}}^+_M(U \times \mathbb{R}) \cup \widetilde{\mathbb{LH}}^-_M(U \times \mathbb{R}).$$

In this case the singular value of $\widetilde{\mathbb{LH}}^\pm_M(U \times \mathbb{R})$ is the point where $\widetilde{\kappa}^\pm_{\ell,i}(p) \neq 0$ and

$$\boldsymbol{\lambda}^\pm = \boldsymbol{X}(u) + \frac{1}{\widetilde{\kappa}^\pm_{\ell,i}(p)}\widetilde{\mathbb{LG}}^\pm(u) = \boldsymbol{X}(u) + \frac{1}{\kappa_i(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p)}\mathbb{LG}(\boldsymbol{n}^T)(p, \pm\boldsymbol{n}^S(u)).$$

Therefore we have a mapping $\widetilde{\mathbb{LE}}^\pm_{\widetilde{\kappa}^\pm_{\ell,i}} : U \longrightarrow \mathbb{R}^{n+1}_1$ defined by

$$\widetilde{\mathbb{LE}}^\pm_{\widetilde{\kappa}^\pm_{\ell,i}}(u) = \boldsymbol{X}(u) + \frac{1}{\widetilde{\kappa}^\pm_{\ell,i}(u)}\widetilde{\mathbb{LG}}^\pm(u).$$

Then we define

$$\widetilde{\mathbb{LE}}^\pm_M = \bigcup \left\{ \widetilde{\mathbb{LE}}^\pm_{\widetilde{\kappa}^\pm_{\ell,i}}(u) \mid u \in U \text{ such that } \widetilde{\kappa}^\pm_{\ell,i}(u) \neq 0, i = 1, \dots, n-1. \right\}.$$

By the above arguments, we know that $\widetilde{\mathbb{LE}}^\pm_M$ is nothing but the lightlike focal set of $M = \boldsymbol{X}(U)$. However, we call it the *lightlike evolute* of $M = \boldsymbol{X}(U)$ in the case when $\mathrm{codim} M = 2$.

The lightlike hypersurface $\widetilde{\mathbb{LH}}^\pm_M(U \times \mathbb{R})$ for a spacelike surface $M = \boldsymbol{X}(U)$ in $\mathbb{R}^4_1$ was investigated in [15] under a slightly different formulation. By a classification of stable Legendrian mappings in [30], we have the following proposition (cf., [15]).

Proposition 6.1 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the normalized lightlike hypersurfaces $\widetilde{\mathbb{LH}}^\pm_M(U \times \mathbb{R})$ at any point is diffeomorphic to the image of one of the map germs A_k $(1 \leq k \leq 4)$ or $D^\pm_4$: where, A_k, $D^\pm_4$-map germs $f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^4, 0)$ are given by*

(A_1) $f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0)$,
(A_2) $f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3)$,
(A_3) $f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3)$,
(A_4) $f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_1, u_2)$,
(D_4^+) $f(u_1, u_2, u_3) = (2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3)$,
(D_4^-) $f(u_1, u_2, u_3) = (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3)$.

By using the generic normal forms [30] of generating families (i.e. Lorentzian distance squared functions) of $\mathscr{L}_G(\Sigma_*(G))$ and Corollary 5.6, we have the following corollary.

Corollary 6.2 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the corresponding tangent lightcone indicatrix at any point $(x_0, y_0) \in U$ is diffeomorphic to one of the germs in the following list:*

(1) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 + y^2 = 0\}$ *(ordinary cusp)*
(2) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^4 \pm y^2 = 0\}$ *(tachnode or point)*
(3) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^5 + y^2 = 0\}$ *(rhamphoid cusp)*
(4) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 - xy^2 = 0\}$ *(three lines)*
(5) $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 + y^3 = 0\}$ *(line)*

In [9] the lightlike surface along a spacelike curve in $\mathbb{R}^3_1$ was investigated. Here, we give a brief review on the results. Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve with $\|\boldsymbol{\gamma}''(s)\| \neq 0$, where I is an open interval. Then we define $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and call $\boldsymbol{t}(s)$ a *unit tangent vector* of $\boldsymbol{\gamma}$ at s. The *curvature* of $\boldsymbol{\gamma}$ at s is defined to be $\kappa(s) = \sqrt{|\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s)\rangle|}$. If $\kappa(s) \neq 0$, then the *unit principal normal vector* $\boldsymbol{n}(s)$ of the curve $\boldsymbol{\gamma}$ at s is given by $\boldsymbol{\gamma}''(s) = \kappa(s)\boldsymbol{n}(s)$. We denote that $\delta(\boldsymbol{\gamma}(s)) = \mathrm{sign}(\boldsymbol{n}(s))$. The unit vector $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$ is called a *unit binormal vector* of the curve $\boldsymbol{\gamma}$ at s. Since $\boldsymbol{t}(s)$ is spacelike, we have $\langle \boldsymbol{b}(s), \boldsymbol{b}(s)\rangle = -\delta(\boldsymbol{\gamma}(s))$ and $\mathrm{sign}\,(\boldsymbol{\gamma}'(s)) = 1$ Then the following Frenet-Serret type formulae hold:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s), \\ \boldsymbol{n}'(s) = -\delta(\boldsymbol{\gamma}(s))\kappa(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) = \tau(s)\boldsymbol{n}(s), \end{cases}$$

where $\tau(s)$ is the torsion of the curve $\boldsymbol{\gamma}$ at s. In this case we distinguish two cases as follows:

Case 1) If $\delta(\boldsymbol{\gamma}) = -1$, then $\boldsymbol{n}$ is timelike, so the we choose $\boldsymbol{n}^T = \boldsymbol{n}$. We now consider the lightlike surface $\mathbb{LH}^{\pm}_C[\boldsymbol{n}](I \times \mathbb{R})$ along $C = \boldsymbol{\gamma}(I)$ defined by

$$\mathbb{LH}^{\pm}_C[\boldsymbol{n}](s,t) = \boldsymbol{\gamma}(s) + t(\boldsymbol{n} \pm \boldsymbol{b})(s).$$

By the Frenet-Serret type formulae, we have

$$\boldsymbol{n}'(s) \pm \boldsymbol{b}'(s) = -\delta(\boldsymbol{\gamma}(s))\kappa(s)\boldsymbol{t}(s) + \tau(s)(\boldsymbol{b}(s) \pm \boldsymbol{n}(s)),$$

so that $\kappa^{\pm}_{\ell}(s) = \delta(\boldsymbol{\gamma}(s))\kappa(s) = -\kappa(s)$.

Case 2) If $\delta(\boldsymbol{\gamma}) = 1$, then $\boldsymbol{n}$ is spacelike, so the we choose $\boldsymbol{n}^T = \boldsymbol{b}$. We now consider the lightlike surface $\mathbb{LH}^{\pm}_C[\boldsymbol{b}](I \times \mathbb{R})$ along $C = \boldsymbol{\gamma}(I)$ defined by

$$\mathbb{LH}^{\pm}_C[\boldsymbol{b}](s,t) = \boldsymbol{\gamma}(s) + t(\boldsymbol{b} \pm \boldsymbol{n})(s).$$

We also have

$$\boldsymbol{b}'(s) \pm \boldsymbol{n}'(s) = \mp\delta(\boldsymbol{\gamma}(s))\kappa(s)\boldsymbol{t}(s) + \tau(s)(\boldsymbol{b}(s) \pm \boldsymbol{n}(s)),$$

so that $\kappa^{\pm}_{\ell}(s) = \pm\delta(\boldsymbol{\gamma}(s))\kappa(s) = \pm\kappa(s)$.

On the other hand, we have

$$\boldsymbol{\gamma}(s) + t(\boldsymbol{b} \pm \boldsymbol{n})(s) = \boldsymbol{\gamma}(s) \pm t(\boldsymbol{n} \pm \boldsymbol{b})(s).$$

If we define a diffeomorphism $\Psi^{\pm} : I \times \mathbb{R} \longrightarrow I \times \mathbb{R}$ by $\Psi^{\pm}(s,t) = (s, \pm t)$, then we have $\mathbb{LH}^{\pm}_C[\boldsymbol{b}] = \mathbb{LH}^{\pm}_C[\boldsymbol{n}] \circ \Psi^{\pm}$. Therefore, we have $\mathbb{LH}^{\pm}_C[\boldsymbol{b}](I \times \mathbb{R}) = \mathbb{LH}^{\pm}_C[\boldsymbol{n}](I \times \mathbb{R})$. The assertions of Theorem B in [9] can be interpreted as the following theorem under the framework in this paper.

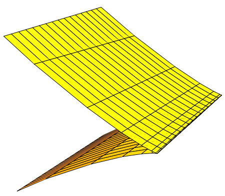
Theorem 6.3 1) *Suppose that $\delta(\boldsymbol{\gamma}) = -1$. Then we have the followings*:

(a) *The lightlike surface $\mathbb{LH}^{\pm}_C[\boldsymbol{n}](I \times \mathbb{R})$ is locally diffeomorphic to the cuspidal edge at $\mathbb{LH}^{\pm}_C[\boldsymbol{n}](s_0,t_0)$ if and only if $t_0 = \dfrac{-1}{\kappa(s_0)} = \dfrac{-1}{\kappa^{\pm}_{\ell}(s_0)}$. Moreover, the the lightlike evolute $\mathbb{LE}^{\pm}_C$ is the critical locus of the cuspidal edge.*

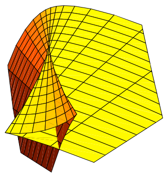
(b) *The lightlike surface* $\mathbb{LH}^{\pm}_C[\boldsymbol{n}](I\times\mathbb{R})$ *is locally diffeomorphic to the swallowtail at* $\mathbb{LH}^{\pm}_C[\boldsymbol{n}](s_0,t_0)$ *if and only if* $t_0=\dfrac{-1}{\kappa(s_0)}=\dfrac{-1}{\kappa^{\pm}_{\ell}(s_0)}$, $(\kappa'-\tau\kappa)(s)=0$ *and* $(\kappa'-\tau\kappa)'(s)\neq 0$.

2) *Suppose that* $\delta(\boldsymbol{\gamma})=1$. *Then we have the followings:*
(a) *The lightlike surface* $\mathbb{LH}^{\pm}_C[\boldsymbol{b}](I\times\mathbb{R})$ *is locally diffeomorphic to the cuspidal edge at* $\mathbb{LH}^{\pm}_C[\boldsymbol{b}](s_0,t_0)$ *if and only if* $t_0=\dfrac{\pm 1}{\kappa(s_0)}=\dfrac{1}{\kappa^{\pm}_{\ell}(s_0)}$. *Moreover, the the lightlike evolute* $\mathbb{LE}^{\pm}_C$ *is the critical locus of the cuspidal edge.*
(b) *The lightlike surface* $\mathbb{LH}^{\pm}_C[\boldsymbol{b}](I\times\mathbb{R})$ *is locally diffeomorphic to the swallow tail at* $\mathbb{LH}^{\pm}_C[\boldsymbol{b}](s_0,t_0)$ *if and only if* $t_0=\pm\dfrac{1}{\kappa(s_0)}=\dfrac{1}{\kappa^{\pm}_{\ell}(s_0)}$, $(\kappa'-\tau\kappa)(s)=0$ *and* $(\kappa'-\tau\kappa)'(s)\neq 0$.

Here, the cuspidaledge is a set germ $CE=\{(u_1,u_2^2,u_2^3)|(u_1,u_2)\in\mathbb{R}^2\}$ *and the swallowtail is a set germ* $SW=\{(3u_1^4+u_1^2u_2,4u_1^3+2u_1u_2,u_2)|(u_1,u_2)\in\mathbb{R}^2\}$ (*cf.*, Fig.2).



cuspidaledge



swallowtail

Fig. 2.

7 Spacelike curves in Minkowski 4-space

In §6 we investigated the spacelike submanifolds with codimension two, and we have a classification of the singularities of the lightlike hypersurfaces in $\mathbb{R}^4_1$. In this section we consider the higher codimensional case in $\mathbb{R}^4_1$, that is spacelike curves in Minkowski 4-space as a special case as the previous results. Let $\boldsymbol{\gamma}:I\longrightarrow\mathbb{R}^4_1$ be a spacelike curve with $\|\boldsymbol{\gamma}''(s)\|\neq 0$. In this case we write $C=\boldsymbol{\gamma}(I)$ instead of $M=\boldsymbol{\gamma}(I)$. Since $\|\boldsymbol{\gamma}'(s)\|>0$, we can reparameterize it by the arc-length s. So we have the unit tangent vector $\boldsymbol{t}(s)=\boldsymbol{\gamma}'(s)$ of $\boldsymbol{\gamma}(s)$. Moreover we have two unit normal vectors $\boldsymbol{n}_1(s)=\dfrac{\boldsymbol{\gamma}''(s)}{\kappa_1(s)}$, $\boldsymbol{n}_2(s)=\dfrac{\boldsymbol{n}_1'(s)+\delta\kappa_1(s)\boldsymbol{t}(s)}{\|\boldsymbol{n}_1'(s)+\delta\kappa_1(s)\boldsymbol{t}(s)\|}$ under the conditions that $\kappa_1(s)=\|\boldsymbol{\gamma}''(s)\|\neq 0$, $\kappa_2(s)=\|\boldsymbol{n}_1'(s)+\delta k_1(s)\boldsymbol{t}(s)\|\neq 0$, where $\delta_i=\mathrm{sign}(\boldsymbol{n}_i(s))$ and $\mathrm{sign}(\boldsymbol{n}_i(s))$ is the signature of $\boldsymbol{n}_i(s)$ $(i=1,2,3)$. Then we have another unit normal vector field $\boldsymbol{n}_3(s)$ defined by $\boldsymbol{n}_3(s)=\boldsymbol{t}(s)\wedge\boldsymbol{n}_1(s)\wedge\boldsymbol{n}_2(s)$. Therefore we can construct a pseudo-orthogonal frame $\{\boldsymbol{t}(s),\boldsymbol{n}_1(s),\boldsymbol{n}_2(s),\boldsymbol{n}_3(s)\}$, which satisfies the *Frenet-Serret type formulae*:

$$\begin{cases}\boldsymbol{t}'(s)=\kappa_1(s)\boldsymbol{n}_1(s),\\ \boldsymbol{n}_1'(s)=-\delta_1\kappa_1(s)\boldsymbol{t}(s)+\kappa_2(s)\boldsymbol{n}_2(s),\\ \boldsymbol{n}_2'(s)=\delta_3\kappa_2(s)\boldsymbol{n}_1(s)+\kappa_3(s)\boldsymbol{n}_3(s),\\ \boldsymbol{n}_3'(s)=\delta_1\kappa_3(s)n_2(s),\end{cases}$$

where $\kappa_2(s) = \delta_2\langle \boldsymbol{n}_1'(s), \boldsymbol{n}_2(s)\rangle$ and $\kappa_3(s) = \delta_3\langle \boldsymbol{n}_2'(s), \boldsymbol{n}_3(s)\rangle$. Since $\boldsymbol{t}(s)$ is spacelike, we distinguish the following three cases:

Case 1: $\boldsymbol{n}_1(s)$ is timelike, that is, $\delta_1 = -1$ and $\delta_2 = \delta_3 = 1$.
Case 2: $\boldsymbol{n}_2(s)$ is timelike, that is, $\delta_2 = -1$ and $\delta_1 = \delta_3 = 1$.
Case 3: $\boldsymbol{n}_3(s)$ is timelike, that is, $\delta_3 = -1$ and $\delta_1 = \delta_2 = 1$.

We consider the lightlike hypersurface along C, and calculate the Lorentzian distance-squared function on C which is useful for the study the singularities of lightlike hypersurfaces in the each case.

7.1 Case 1

Suppose that $\boldsymbol{n}_1(s)$ is timelike. In this case we adopt $\boldsymbol{n}^T(s) = \boldsymbol{n}_1(s)$ and denote that $\boldsymbol{b}_1(s) = \boldsymbol{n}_2(s), \boldsymbol{b}_2(s) = \boldsymbol{n}_3(s)$. Then we have the pseudo-orthogonal frame

$$\{\boldsymbol{t}(s), \boldsymbol{n}^T(s), \boldsymbol{b}_1(s), \boldsymbol{b}_2(s)\},$$

$\delta_1 = -1$ and $\delta_2 = \delta_3 = 1$, which satisfies the following Frenet-Serret type formulae:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa_1(s)\boldsymbol{n}^T(s), \\ \boldsymbol{n}^{T'}(s) = \kappa_1(s)\boldsymbol{t}(s) + \kappa_2(s)\boldsymbol{b}_1(s), \\ \boldsymbol{b}_1'(s) = \kappa_2(s)\boldsymbol{n}^T(s) + \kappa_3(s)\boldsymbol{b}_2(s), \\ \boldsymbol{b}_2'(s) = -\kappa_3(s)\boldsymbol{b}_1(s). \end{cases}$$

Since $N_1(C)[\boldsymbol{n}^T]$ is parametrized by

$$N_1(C)[\boldsymbol{n}^T] = \{(\boldsymbol{\gamma}(s), \boldsymbol{\xi}) \in \boldsymbol{\gamma}^*T\mathbb{R}^4_1 \mid \boldsymbol{\xi} = \cos\theta \boldsymbol{b}_1(s) + \sin\theta \boldsymbol{b}_2(s) \in N_{\boldsymbol{\gamma}(s)}(C),\ s \in I\},$$

the lightcone Gauss image of $N_1(C)_p[\boldsymbol{n}^T]$ is given by

$$\mathbb{LG}(\boldsymbol{n}^T)(s,\theta) = \boldsymbol{n}^T(s) + \cos\theta \boldsymbol{b}_1(s) + \sin\theta \boldsymbol{b}_2(s).$$

Then we have the lightlike hypersurface along C

$$\mathbb{LH}_C((s,\theta),t) = \boldsymbol{\gamma}(s) + t(\boldsymbol{n}^T(s) + \cos\theta \boldsymbol{b}_1(s) + \sin\theta \boldsymbol{b}_2(s)) = \boldsymbol{\gamma}(s) + t\mathbb{LG}(\boldsymbol{n}^T)(s,\theta).$$

We remark that the image of this lightlike hypersurface along C is independent of the choice of the future directed timelike normal vector field $\boldsymbol{n}^T$ by Corollary 5.3.
Now we investigate the Lorentzian distance-squared functions $G : I \times \mathbb{R}^4_1 \longrightarrow \mathbb{R}$ on a spacelike curve $C = \boldsymbol{\gamma}(I)$ defined by

$$G(p, \boldsymbol{\lambda}) = G(s, \boldsymbol{\lambda}) = \langle \boldsymbol{\gamma}(s) - \boldsymbol{\lambda}, \boldsymbol{\gamma}(s) - \boldsymbol{\lambda}\rangle,$$

where $p = \boldsymbol{\gamma}(s)$. For any fixed $\boldsymbol{\lambda}_0 \in \mathbb{R}^4_1$, we write $g(p) = g_{\boldsymbol{\lambda}_0}(p) = G(p, \boldsymbol{\lambda}_0)$.
By Proposition 4.1, the discriminant set of the Lorentzian distance-squared function G is given by

$$\mathcal{D}_G = \mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R}) = \left\{ \boldsymbol{\lambda} = \boldsymbol{\gamma}(s) + t\mathbb{LG}(s,\theta) \;\middle|\; \theta \in [0, 2\pi), s \in I,\ t \in \mathbb{R} \right\},$$

which is the image of the lightlike hypersurface along C. We also calculate that $g''(p) = 2\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}(s) - \boldsymbol{\lambda}_0\rangle + 2\langle \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'(s)\rangle = 2(-\mu\kappa_1 + 1)$. Then $g''(p) = 0$ if and only if $\mu = 1/\kappa_1(s)$. It means that a singular point of the lightlike hypersurface is a point $\boldsymbol{\lambda}_0 = \boldsymbol{\gamma}(s_0) + t_0\mathbb{LG}(\theta_0, s_0)$ for $t_0 = 1/\kappa_1(s_0)$. Therefore, the lightlike focal surface is

$$\mathbb{LF}_C = \left\{ \boldsymbol{\lambda} = \boldsymbol{\gamma}(s) - \frac{1}{\kappa_1(s)}\mathbb{LG}(\boldsymbol{n}^T)(s,\theta) \;\middle|\; s \in I,\ \theta \in [0, 2\pi) \right\}.$$

Moreover, if we calculate the third, 4th and 5th derivatives of $g(s)$, we have the following proposition.

Proposition 7.1 *Let C be a spacelike curve and $G : C \times (\mathbb{R}^4_1 \setminus C) \to \mathbb{R}$ the Lorentzian distance-squared function on C. Suppose that $p_0 = \boldsymbol{\gamma}(s_0) \neq \boldsymbol{\lambda}_0$. Then we have the followings*:
(1) $g(p_0) = g'(p_0) = 0$ *if and only if there exist $\theta_0 \in [0, 2\pi)$ and $\mu \in \mathbb{R} \setminus \{0\}$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = \mu\mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0).$$

(2) $g(p_0) = g'(p_0) = g''(p_0) = 0$ *if and only if there exists $\theta_0 \in [0, 2\pi)$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = \frac{1}{\kappa_1(s_0)}\mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0).$$

(3) $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = 0$ *if and only if there exists $\theta_0 \in [0, 2\pi)$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = \frac{1}{\kappa_1(s_0)}\mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0)$$

and $\kappa_1'(s_0) - \cos\theta_0\kappa_1(s_0)\kappa_2(s_0) = 0$, so that we can write $\theta_0 = \theta(s_0)$.
(4) $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = 0$ *if and only if there exists $\theta_0 = \theta(s_0) \in [0, 2\pi)$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = \frac{1}{\kappa_1(s_0)}\mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta(s_0)),$$

$\kappa_1'(s_0) - \cos\theta(s_0)\kappa_1(s_0)\kappa_2(s_0) = 0$ *and* $(2\kappa_1'(s_0)\kappa_2(s_0) + \kappa_1(s_0)\kappa_2'(s_0))\cos\theta(s_0) - \kappa_1''(s_0) - \kappa_1(s_0)\kappa_2^2(s_0) + \kappa_1(s_0)\kappa_2(s_0)\kappa_3(s_0)\sin\theta(s_0) = 0.$
(5) $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = g^{(5)}(p_0) = 0$ *if and only if there exists $\theta_0 = \theta(s_0) \in [0, 2\pi)$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = \frac{1}{\kappa_1(s_0)}\mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta(s_0)),$$

$\kappa_1'(s_0) - \cos\theta(s_0)\kappa_1(s_0)\kappa_2(s_0) = 0$, $(2\kappa_1'(s_0)\kappa_2(s_0) + \kappa_1(s_0)\kappa_2'(s_0))\cos\theta(s_0) - \kappa_1''(s_0) - \kappa_1(s_0)\kappa_2^2(s_0) + \kappa_1(s_0)\kappa_2(s_0)\kappa_3(s_0)\sin\theta(s_0) = 0$ *and* $((2\kappa_1'(s_0)\kappa_2(s_0) + \kappa_1(s_0)\kappa_2'(s_0))\cos\theta(s_0) - \kappa_1''(s_0) - \kappa_1(s_0)\kappa_2^2(s_0) + \kappa_1(s_0)\kappa_2(s_0)\kappa_3(s_0)\sin\theta(s_0))' = 0.$

Taking account of the above proposition, we denote that $\rho_1(s,\theta) = \kappa_1'(s) - \cos\theta\kappa_1(s)\kappa_2(s)$ and $\eta_1(s,\theta) = (2\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s))\cos\theta - \kappa_1''(s) - \kappa_1(s)\kappa_2^2(s) + \kappa_1(s)\kappa_2(s)\kappa_3(s)\sin\theta$, which might be important invariants of $C = \boldsymbol{\gamma}(I)$. Then we can show that $\rho_1(s,\theta) = \eta_1(s,\theta) = 0$ if and only if $\rho_1(s,\theta) = \sigma_1(s) = 0$, where

$$\sigma_1(s) = \left[\kappa_1\kappa_2(\kappa_1'' + \kappa_1\kappa_2^2) - \kappa_1'(2\kappa_1'\kappa_2 + \kappa_1\kappa_2') \mp \kappa_1\kappa_2\kappa_3\sqrt{(\kappa_1\kappa_2)^2 - (\kappa_1')^2}\right](s).$$

7.2 Case 2

Suppose that $\boldsymbol{n}_2(s)$ is timelike. Then we adopt $\boldsymbol{n}^T(s) = \boldsymbol{n}_2(s)$ and denote that $\boldsymbol{b}_1(s) = \boldsymbol{n}_1(s), \boldsymbol{b}_2(s) = \boldsymbol{n}_3(s)$. We have a pseudo-orthogonal frame $\{\boldsymbol{t}(s), \boldsymbol{n}^T(s), \boldsymbol{b}_1(s), \boldsymbol{b}_2(s)\}$, $\delta_2 = -1$ and $\delta_1 = \delta_3 = 1$, which satisfies the following Frenet-Serret type formulae:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa_1(s)\boldsymbol{b}_1(s), \\ \boldsymbol{b}_1'(s) = -\kappa_1(s)\boldsymbol{t}(s) + \kappa_2(s)\boldsymbol{n}^T(s), \\ \boldsymbol{n}^{T\prime}(s) = \kappa_2(s)\boldsymbol{b}_1(s) + \kappa_3(s)\boldsymbol{b}_2(s), \\ \boldsymbol{b}_2'(s) = \kappa_3(s)\boldsymbol{n}^T(s), \end{cases}$$

Here, $N_1(C)[\boldsymbol{n}^T]$ is parametrized by

$$N_1(C)[\boldsymbol{n}^T] = \{(\boldsymbol{\gamma}(s), \boldsymbol{\xi}) \in \boldsymbol{\gamma}^* T\mathbb{R}^4_1 \mid \boldsymbol{\xi} = \cos\theta \boldsymbol{b}_1(s) + \sin\theta \boldsymbol{b}_2(s) \in N_{\boldsymbol{\gamma}(s)}(C),\ s \in I\},$$

so that we have the lightlike hypersurface along $C = \boldsymbol{\gamma}(I)$:

$$\mathbb{LH}_C((s,\theta),t) = \boldsymbol{\gamma}(s) + t\mathbb{LG}(\boldsymbol{n}^T)(s,\theta).$$

We consider the Lorentzian distance-squared function $G : C \times \mathbb{R}^4_1 \longrightarrow \mathbb{R}$ on a spacelike curve $C = \boldsymbol{\gamma}(I)$. Under the similar notations to the case 1), we have the following proposition:

Proposition 7.2 *Let C be a spacelike curve and $G : C \times (\mathbb{R}^4_1 \setminus C) \to \mathbb{R}$ the Lorentzian distance-squared function on C. Suppose that $p_0 \neq \boldsymbol{\lambda}_0$. Then we have the following*:
(1) $g(p_0) = g'(p_0) = 0$ *if and only if there exist* $\theta_0 \in [0, 2\pi)$ *and* $\mu \in \mathbb{R} \setminus \{0\}$ *such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = \mu\mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0).$$

(2) $g(p_0) = g'(p_0) = g''(p_0) = 0$ *if and only if there exists* $\theta_0 \in [0, 2\pi)$ *such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta_0}\mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0).$$

(3) $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = 0$ *if and only if there exists* $\theta_0 \in [0, 2\pi)$ *such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta_0}\mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0)$$

and $\kappa_1'(s_0)\cos\theta_0 - \kappa_1(s_0)\kappa_2(s_0) = 0$, *so that we can write* $\theta_0 = \theta(s_0)$.
(4) $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = 0$ *if and only if there exists* $\theta_0 = \theta(s_0) \in [0, 2\pi)$ *such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta(s_0)}\mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta(s_0)),$$

$\kappa_1'(s_0)\cos\theta(s_0) - \kappa_1(s_0)\kappa_2(s_0) = 0$ *and* $(\kappa_1''(s_0) + \kappa_1(s_0)\kappa_2^2(s_0))\cos\theta(s_0) - 2\kappa_1'(s_0)\kappa_2(s_0) - \kappa_1(s_0)\kappa_2'(s_0) + \kappa_1(s_0)\kappa_2(s_0)\kappa_3(s_0)\sin\theta(s_0) = 0.$

(5) $g(p_0)=g'(p_0)=g''(p_0)=g'''(p_0)=g^{(4)}(p_0)=g^{(5)}(p_0)=0$ *if and only if there exists* $\theta_0=\theta(s_0)\in[0,2\pi)$ *such that*

$$\boldsymbol{\gamma}(s_0)-\boldsymbol{\lambda}_0=-\frac{1}{\kappa_1(s_0)\cos\theta(s_0)}\mathbb{LG}(\boldsymbol{n}^T)(s_0,\theta(s_0)),$$

$\kappa_1'(s_0)\cos\theta(s_0)-\kappa_1(s_0)\kappa_2(s_0)=0$, $(\kappa_1''(s_0)+\kappa_1(s_0)\kappa_2^2(s_0))\cos\theta(s_0)-2\kappa_1'(s_0)\kappa_2(s_0)-\kappa_1(s_0)\kappa_2'(s_0)+\kappa_1(s_0)\kappa_2(s_0)\kappa_3(s_0)\sin\theta(s_0)=0$ *and* $((\kappa_1''(s_0)+\kappa_1(s_0)\kappa_2^2(s_0))\cos\theta(s_0)-2\kappa_1'(s_0)\kappa_2(s_0)-\kappa_1(s_0)\kappa_2'(s_0)+\kappa_1(s_0)\kappa_2(s_0)\kappa_3(s_0)\sin\theta(s_0))'=0$.

The above proposition asserts that the discriminant set of the Lorentzian distance-squared function G is given by

$$\mathcal{D}_G=\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T]\times\mathbb{R})=\Big\{\boldsymbol{\lambda}=\boldsymbol{\gamma}(s)+t\mathbb{LG}(\boldsymbol{n}^T)(s,\theta)\ \Big|\ s\in I,\ \theta\in[0,2\pi),t\in\mathbb{R}\Big\}.$$

Moreover, the lightlike focal surface is

$$\mathbb{LF}_C=\Big\{\boldsymbol{\lambda}=\boldsymbol{\gamma}(s)-\frac{1}{\kappa_1(s)\cos\theta}\mathbb{LG}(\boldsymbol{n}^T)(s,\theta)\ \Big|\ s\in I,\ \theta\in[0,2\pi)\Big\}.$$

Here, we also denote that $\rho_2(s,\theta)=\kappa_1'(s)\cos\theta-\kappa_1(s)\kappa_2(s)$ and

$$\eta_2(s,\theta)=(\kappa_1''(s)+\kappa_1(s)\kappa_2^2(s))\cos\theta-2\kappa_1'(s)\kappa_2(s)-\kappa_1(s)\kappa_2'(s)+\kappa_1(s)\kappa_2(s)\kappa_3(s)\sin\theta.$$

We can also show that $\rho_2(s,\theta)=\eta_2(s,\theta)=0$ if and only if $\rho_2(s,\theta)=\sigma_2(s)=0$, where

$$\sigma_2(s)=\Big[\kappa_1\kappa_2(\kappa_1''+\kappa_1\kappa_2^2)-\kappa_1'(2\kappa_1'\kappa_2+\kappa_1\kappa_2')\pm\kappa_1\kappa_2\kappa_3\sqrt{-(\kappa_1\kappa_2)^2+(\kappa_1')^2}\Big](s).$$

7.3 Case 3

Suppose that $\boldsymbol{n}_3(s)$ is timelike. Then we adopt $\boldsymbol{n}^T(s)=\boldsymbol{n}_3(s)$ and denote that $\boldsymbol{b}_1(s)=\boldsymbol{n}_1(s),\boldsymbol{b}_2(s)=\boldsymbol{n}_2(s)$. We have a pseudo-orthogonal frame $\{\boldsymbol{t}(s),\boldsymbol{n}^T(s),\boldsymbol{b}_1(s),\boldsymbol{b}_2(s)\}$ and $\delta_3=-1$ and $\delta_1=\delta_2=1$,which satisfies the following Frenet-Serret type formulae:

$$\begin{cases}\boldsymbol{t}'(s)=\kappa_1(s)\boldsymbol{b}_1(s),\\ \boldsymbol{b}_1'(s)=-\kappa_1(s)\boldsymbol{t}(s)+\kappa_2(s)\boldsymbol{b}_2(s),\\ \boldsymbol{b}_2'(s)=-\kappa_2(s)\boldsymbol{b}_1(s)+\kappa_3(s)\boldsymbol{n}^T(s),\\ \boldsymbol{n}^{T\prime}(s)=\kappa_3(s)\boldsymbol{b}_2(s),\end{cases}$$

Here, $N_1(C)[\boldsymbol{n}^T]$ is parametrized by

$$N_1(C)[\boldsymbol{n}^T]=\{(\boldsymbol{\gamma}(s),\boldsymbol{\xi})\in\boldsymbol{\gamma}^*T\mathbb{R}^4_1\mid\boldsymbol{\xi}=\cos\theta\boldsymbol{b}_1(s)+\sin\theta\boldsymbol{b}_2(s)\in N_{\boldsymbol{\gamma}(s)}(C),\ s\in I\},$$

so that we have the lightlike hypersurface along C:

$$\mathbb{LH}_C((s,\theta),t)=\boldsymbol{\gamma}(s)+t\mathbb{LG}(\boldsymbol{n}^T)(s,\theta).$$

We investigate the Lorentzian distance-squared function on a spacelike curve $C=\boldsymbol{\gamma}(I)$ By the calculations similar to the cases 1 and 2, we have the following proposition:

Proposition 7.3 *Let C be a spacelike curve and $G : C \times (\mathbb{R}^4_1 \setminus C) \to \mathbb{R}$ the Lorentzian distance-squared function on $C = \boldsymbol{\gamma}(I)$. Suppose that $p_0 \neq \boldsymbol{\lambda}_0$. Then we have the following:*
(1) $g(p_0) = g'(p_0) = 0$ *if and only if there exist $\theta_0 \in [0, 2\pi)$ and $\mu \in \mathbb{R} \setminus \{0\}$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = \mu \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0).$$

(2) $g(p_0) = g'(p_0) = g''(p_0) = 0$ *if and only if there exists $\theta_0 \in [0, s\pi)$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta_0} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0).$$

(3) $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = 0$ *if and only if there exists $\theta_0 \in [0, s\pi)$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta_0} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0)$$

and $\kappa_1'(s_0)\cos\theta_0 + \kappa_1(s_0)\kappa_2(s_0)\sin\theta_0 = 0$, *so that we can write* $\theta_0 = \theta(s_0)$.
(4) $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = 0$ *if and only if there exists $\theta_0 = \theta(s_0) \in [0, 2\pi)$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta(s_0)} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta(s_0)),$$

$\kappa_1'(s_0)\cos\theta(s_0) + \kappa_1(s_0)\kappa_2(s_0)\sin\theta(s_0) = 0$. *and* $(2\kappa_1'(s_0)\kappa_2(s_0) + \kappa_1(s_0)\kappa_2'(s_0))\sin\theta(s_0) + (\kappa_1''(s_0) - \kappa_1(s_0)\kappa_2^2(s_0))\cos\theta(s_0) - \kappa_1(s_0)\kappa_2(s_0)\kappa_3(s_0) = 0$.
(5) $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = g^{(5)}(p_0) = 0$ *if and only if there exists $\theta_0 = \theta(s_0) \in [0, 2\pi)$ such that*

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta(s_0)} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta(s_0)),$$

$\kappa_1'(s_0)\cos\theta(s_0) + \kappa_1(s_0)\kappa_2(s_0)\sin\theta(s_0) = 0$, *and* $((2\kappa_1'(s_0)\kappa_2(s_0) + \kappa_1(s_0)\kappa_2'(s_0))\sin\theta(s_0)) + (\kappa_1''(s_0) - \kappa_1(s_0)\kappa_2^2(s_0))\cos\theta(s_0)) - \kappa_1(s_0)\kappa_2(s_0)\kappa_3(s_0))' = 0$.

The above proposition asserts that the discriminant set of the Lorentzian distance-squared function G is given by

$$\mathcal{D}_G = \mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R}) = \left\{ \boldsymbol{\lambda} = \boldsymbol{\gamma}(s) + t\mathbb{LG}(\boldsymbol{n}^T)(s, \theta) \middle| \ s \in I, \ \theta \in [0, 2\pi), t \in \mathbb{R} \right\}.$$

Moreover, the lightlike focal surface is

$$\mathbb{LF}_C = \left\{ \boldsymbol{\lambda} = \boldsymbol{\gamma}(s) - \frac{1}{\kappa_1(s)\cos\theta} \mathbb{LG}(\boldsymbol{n}^T)(s, \theta) \ \middle| \ s \in I, \ \theta \in [0, 2\pi) \right\}.$$

Here, we also denote that $\rho_3(s, \theta) = \kappa_1'(s)\cos\theta + \kappa_1(s)\kappa_2(s)\sin\theta$ and

$\eta_3(s, \theta) = (2\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s))\sin\theta + (\kappa_1''(s) - \kappa_1(s)\kappa_2^2(s))\cos\theta - \kappa_1(s)\kappa_2(s)\kappa_3(s)$,

We can also show that $\rho_3(s, \theta) = \eta_3(s, \theta) = 0$ if and only if $\rho_3(s, \theta) = \sigma_3(s) = 0$, where

$$\sigma_3(s) = \left[\kappa_1\kappa_2(\kappa_1'' - \kappa_1\kappa_2^2) - \kappa_1'(2\kappa_1'\kappa_2 + \kappa_1\kappa_2') \mp \kappa_1\kappa_2\kappa_3\sqrt{(\kappa_1\kappa_2)^2 + (\kappa_1')^2} \right](s).$$

We can unify the invariants $\sigma_i(s)$, $(i = 1, 2, 3)$ as follows:

$$\sigma(s) = \left[\kappa_1\kappa_2(\kappa_1'' - \kappa_1\kappa_2^2) - \kappa_1'(2\kappa_1'\kappa_2 + \kappa_1\kappa_2') \mp \delta_2\kappa_1\kappa_2\kappa_3\sqrt{\delta_1(\kappa_1\kappa_2)^2 + \delta_2(\kappa_1')^2} \right](s).$$

7.4 Classifications of singularities

By using the results of three cases, we classify the singularities of the lightlike hypersurface along $\boldsymbol{\gamma}$ as an application of the unfolding theory of functions. For a function $f(s)$, we say that f has *A_k-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$. Let F be an r-parameter unfolding of f and f has A_k-singularity $(k \geq 1)$ at s_0. We denote the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 as

$$j^{(k-1)} \left(\frac{\partial F}{\partial x_i}(s, \boldsymbol{x}_0) \right) (s_0) = \sum_{j=1}^{k-1} \alpha_{ji}(s - s_0)^j, \quad (i = 1, \cdots, r).$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k $(k \leq r)$, then F is called a *versal unfolding* of f, where $\alpha_{0i} = \partial F/\partial x_i(s_0, \boldsymbol{x}_0)$.

Inspired by the propositions in the previous subsections, we define the following set:

$$D_F^\ell = \left\{ \boldsymbol{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R},\ F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s}(s, \boldsymbol{x}) = \cdots = \frac{\partial^\ell F}{\partial s^\ell}(s, \boldsymbol{x}) = 0 \right\},$$

which is called a *discriminant set of order ℓ.* Of course, $D_F^1 = D_F$ and D_F^2 is the set of singular points of D_F. Therefore, we have the following proposition.

Proposition 7.4 *For all the cases, we have*

$$D_G = D_G^1 = \mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R}),\ D_G^2 = \mathbb{LF}_C \text{ and } D_G^3 \text{ is the critical value set of } \mathbb{LF}_C.$$

In order to understand the geometric properties of the discriminant set of order ℓ, we introduce an equivalence relation among the unfoldings of functions. Let F and G be r-parameter unfoldings of $f(s)$ and $g(s)$, respectively. We say that F and G are *P-$\mathcal{R}$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}\times\mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow (\mathbb{R}\times\mathbb{R}^r, (s_0', \boldsymbol{x}_0'))$ of the form $\Phi(s, \boldsymbol{x}) = (\Phi_1(s, \boldsymbol{x}), \phi(\boldsymbol{x}))$ such that $G \circ \Phi = F$. By straightforward calculations, we have the following proposition.

Proposition 7.5 *Let F and G be r-parameter unfoldings of $f(s)$ and $g(s)$, respectively. If F and G are P-$\mathcal{R}$-equivalent by a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow (\mathbb{R} \times \mathbb{R}^r, (s_0', \boldsymbol{x}_0'))$ of the form $\Phi(s, \boldsymbol{x}) = (\Phi_1(s, \boldsymbol{x}), \phi(\boldsymbol{x}))$, then $\phi(D_F^\ell) = D_G^\ell$ as set germs.*

We have the following classification theorem of versal unfoldings [3, Page 149, 6.6].

Theorem 7.6 *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of f which has A_k-singularity at s_0. Suppose F is a versal unfolding of f, then F is P-$\mathcal{R}$-equivalent to one of the following unfoldings:*
(a) $k = 1$; $\pm s^2 + x_1$,
(b) $k = 2$; $s^3 + x_1 + sx_2$,
(c) $k = 3$; $\pm s^4 + x_1 + sx_2 + s^2x_3$,
(d) $k = 4$; $s^5 + x_1 + sx_2 + s^2x_3 + s^3x_4$.

We have the following classification result as a corollary of the above theorem.

Corollary 7.7 *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of f which has A_k-singularity at s_0. Suppose F is a versal unfolding of f, then we have the following assertions:*
(a) *If $k = 1$, then D_F is diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$ and $D_F^2 = \emptyset$.*
(b) *If $k = 2$, then D_F is diffeomorphic to $C(2,3) \times \mathbb{R}^{r-2}$, D_F^2 is diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-2}$ and $D_F^3 = \emptyset$.*
(c) *If $k = 3$, then D_F is diffeomorphic to $SW \times \mathbb{R}^{r-3}$, D_F^2 is diffeomorphic to $C(2,3,4) \times \mathbb{R}^{r-3}$, D_F^3 is diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-3}$ and $D_F^4 = \emptyset$.*
(d) *If $k = 4$, then D_F is locally diffeomorphic to $BF \times \mathbb{R}^{r-4}$, D_F^2 is diffeomorphic to $C(BF) \times \mathbb{R}^{r-4}$, D_F^3 is diffeomorphic to $C(2,3,4,5) \times \mathbb{R}^{r-4}$, D_F^4 is diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-4}$ and $D_F^5 = \emptyset$.*
We remark that all of diffeomorphisms in the above assertions are diffeomorphism germs.

Here, we respectively call $C(2,3) = \{(x_1, x_2) \mid x_1 = u^2, x_2 = u^3\}$ a $(2,3)$-*cusp*, $C(2,3,4) = \{(x_1, x_2, x_3) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4\}$ a $(2,3,4)$-*cusp*, $C(2,3,4,5) = \{(x_1, x_2, x_3, x_4) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4, x_4 = u^5\}$ a $(2,3,4,5)$-*cusp*, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ a *swallow tail*, $BF = \{(x_1, x_2, x_3.x_4) \mid x_1 = 5u^4 + 3vu^2 + 2wu, x_2 = 4u^5 + 2vu^3 + wu^2, x_3 = u, x_4 = v\}$ a *butterfly*, and $C(BF) = \{(x_1, x_2, x_3, x_4) \mid x_1 = 6u^5 + u^3v, x_2 = 25u^4 + 9u^2v, x_3 = 10u^3 + 3uv, x_4 = v\}$ a *c-butterfly* (i.e., the critical value set of the butterfly). Here we have the following key proposition on G.

Proposition 7.8 *If $g(s)$ has A_k-singularity $(k = 1,2,3,4)$ at p_0, then G is a versal unfolding of g.*

Proof. We denote that $\boldsymbol{\gamma}(s) = (X_0(s), X_1(s), X_2(s), X_3(s))$ and $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$. By definition, we have

$$G(s, \boldsymbol{\lambda}) = -(X_0(s) - \lambda_0)^2 + (X_1(s) - \lambda_1)^2 + (X_2(s) - \lambda_2)^2 + (X_3(s) - \lambda_3)^2.$$

Thus we have

$$\frac{\partial G}{\partial \lambda_i}(s, \boldsymbol{\lambda}) = 2(X_i(s) - \lambda_i), \text{and } \frac{\partial^2 G}{\partial s \partial \lambda_i}(s, \boldsymbol{\lambda}) = 2X_i'(s), \text{ for}(i = 0,1,2,3)$$

For a fixed $\boldsymbol{\lambda}_0 = (\lambda_{00}, \lambda_{01}, \lambda_{02}, \lambda_{03})$, the 3-jet of $\partial G/\partial \lambda_i(s, \boldsymbol{\lambda}_0)(i = 0,,1,2,3)$ at s_0 is

$$j^{(3)} \frac{\partial G}{\partial \lambda_i}(s, \boldsymbol{\lambda}_0)(s_0) = 2X_i'(s_0)(s - s_0) - X_i''(s_0)(s - s_0)^2 - \frac{1}{3}X_i'''(s_0)(s - s_0)^3, \ (i = 0,1,2,3).$$

It is enough to show that the rank of the following matrix A is four,

$$B = \begin{pmatrix} 2(X_0(s) - \lambda_0) & 2(X_1(s) - \lambda_1) & 2(X_2(s) - \lambda_2) & 2(X_3(s) - \lambda_3) \\ 2X_0'(s_0) & 2X_1'(s_0) & 2X_2'(s_0) & 2X_3'(s_0) \\ 2X_0''(s_0) & 2X_1''(s_0) & 2X_2''(s_0) & 2X_3''(s_0) \\ 2X_0'''(s_0) & 2X_1'''(s_0) & 2X_2'''(s_0) & 2X_3'''(s_0) \end{pmatrix}.$$

In fact, $B = 2^t(\boldsymbol{\gamma}(s) - \boldsymbol{\lambda}, \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s)) = 2^t(\boldsymbol{\gamma}(s) - \boldsymbol{\lambda}, \boldsymbol{t}(s), \boldsymbol{t}'(s), \boldsymbol{t}''(s))$, and $\boldsymbol{\gamma}(s) - \boldsymbol{\lambda}, \boldsymbol{t}(s), \boldsymbol{t}'(s)$, and $\boldsymbol{t}''(s)$ are linearly independent each other in all Case 1,2,3, respectively. This completes the proof. $\square$

Finally, we can apply Corollary 8.5 to our condition. Then we have the following theorem:

Theorem 7.9 *Let $\boldsymbol{\gamma} : I \longrightarrow +\mathbb{R}^4_1$ be a spacelike curves with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$.*
(A) *For the lightlike hypersurfaces $\mathbb{LH}_C((s,\theta),t)$ of $C = \boldsymbol{\gamma}(I)$ in the Case 1, we have the following assertions:*
(1) *The lightlike hypersurface $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R})$ is locally diffeomorphic to $C(2,3) \times \mathbb{R}^2$ at $\boldsymbol{\lambda}_0$ if and only if there exist $\theta_0 \in [0, 2\pi)$ such that*

$$p_0 - \boldsymbol{\lambda}_0 = \frac{1}{\kappa_1(s_0)} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0),$$

and $\rho_1(s_0,\theta_0) \neq 0$. In this case, the lightlike focal set $\mathbb{LF}_C$ is non-singular.
(2) *The lightlike hypersurface $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R})$ is locally diffeomorphic to $SW \times \mathbb{R}$ at $\boldsymbol{\lambda}_0$ if and only if there exist $\theta_0 \in [0, 2\pi)$ such that*

$$p_0 - \boldsymbol{\lambda}_0 = \frac{1}{\kappa_1(s_0)} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0),$$

$\rho_1(s_0,\theta_0) = 0$ and $\sigma_1(s_0) \neq 0$. In this case, the lightlike focal set $\mathbb{LF}_C$ is locally diffeomorphic to $C(2,3,4) \times \mathbb{R}$ and the critical value set of $\mathbb{LF}_C$ is a regular curve.
(3) *The lightlike hypersurface $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R})$ is locally diffeomorphic to BF at $\boldsymbol{\lambda}_0$ if and only if there exist $\theta_0 \in [0, 2\pi)$ such that*

$$p_0 - \boldsymbol{\lambda}_0 = \frac{1}{\kappa_1(s_0)} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0),$$

$\rho_1(s_0,\theta_0) = 0$, $\sigma_1(s_0) = 0$ and $\sigma_1'(s_0) \neq 0$. In this case, the lightlike focal set $\mathbb{LF}_C$ is is locally diffeomorphic to $C(BF) \times \mathbb{R}$ and the critical value set is locally diffeomorphic to the $C(2,3,4,5)$-cusp.

(B) *For the lightlike hypersurfaces $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R})$ of $C = \boldsymbol{\gamma}(I)$ in the Case 2, we have the following assertions:*
(1) *The lightlike hypersurface $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R})$ is locally diffeomorphic to $C(2,3) \times \mathbb{R}^2$ at $\boldsymbol{\lambda}_0$ if and only if there exist $\theta_0 \in [0, 2\pi)$ such that*

$$p_0 - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta_0} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0)$$

and $\rho_2(s_0,\theta_0) \neq 0$. In this case, the lightlike focal set $\mathbb{LF}_C$ is non-singular.
(2) *The lightlike hypersurface $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R})$ is locally diffeomorphic to $SW \times \mathbb{R}$ at $\boldsymbol{\lambda}_0$ if and only if there exist $\theta_0 \in [0, 2\pi)$ such that*

$$p_0 - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta_0} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0),$$

$\rho_2(s_0,\theta_0) = 0$ and $\sigma(s_0) \neq 0$. In this case, the lightlike focal set $\mathbb{LF}_C$ is locally diffeomorphic to $C(2,3,4) \times \mathbb{R}$ and the critical value set of $\mathbb{LF}_C$ is a regular curve.
(3) *he lightlike hypersurface $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T] \times \mathbb{R})$ is locally diffeomorphic to BF at $\boldsymbol{\lambda}_0$ if and only if there exist $\theta_0 \in [0, 2\pi)$ such that*

$$p_0 - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0)\cos\theta_0} \mathbb{LG}(\boldsymbol{n}^T)(s_0, \theta_0),$$

$\rho_2(s_0,\theta_0)=0$, $\sigma_2(s_0)=0$ and $\sigma_2'(s_0)\neq 0$. In this case, the lightlike focal set $\mathbb{LF}_C$ is locally diffeomorphic to $C(BF)\times\mathbb{R}$ and the critical value set of $\mathbb{LF}_C$ is locally diffeomorphic to the $C(2,3,4,5)$-cusp.

(C) *For the lightlike hypersurfaces $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T]\times\mathbb{R})$ of $C=\boldsymbol{\gamma}(I)$ in the Case 3, we have the following assertions:*

(1) *The lightlike hypersurface $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T]\times\mathbb{R})$ is locally diffeomorphic to $C(2,3)\times\mathbb{R}^2$ at $\boldsymbol{\lambda}_0$ if and only if there exist $\theta_0\in[0,2\pi)$ such that*

$$p_0-\boldsymbol{\lambda}_0=-\frac{1}{\kappa_1(s_0)\cos\theta_0}\mathbb{LG}(\boldsymbol{n}^T)(s_0,\theta_0),$$

and $\rho_3(s_0,\theta_0)\neq 0$. In this case, the lightlike focal set $\mathbb{LF}_C$ is non-singular.

(2) *The lightlike hypersurface $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T]\times\mathbb{R})$ is locally diffeomorphic to $SW\times\mathbb{R}$ at $\boldsymbol{\lambda}_0$ if and only if there exist $\theta_0\in[0,2\pi)$ such that*

$$p_0-\boldsymbol{\lambda}_0=-\frac{1}{\kappa_1(s_0)\cos\theta_0}\mathbb{LG}(\boldsymbol{n}^T)(s_0,\theta_0),$$

$\rho_3(s_0,\theta_0)=0$ and $\sigma_3(s_0)\neq 0$. In this case, the lightlike focal set $\mathbb{LF}_C$ is locally diffeomorphic to $C(2,3,4)\times\mathbb{R}$ and the critical value set of $\mathbb{LF}_C$ is a regular curve.

(3) *The lightlike hypersurface $\mathbb{LH}_C(N_1(C)[\boldsymbol{n}^T]\times\mathbb{R})$ is locally diffeomorphic to BF at $\boldsymbol{\lambda}_0$ if and only if there exist $\theta_0\in[0,2\pi)$ such that*

$$p_0-\boldsymbol{\lambda}_0=-\frac{1}{\kappa_1(s_0)\cos\theta_0}\mathbb{LG}(\boldsymbol{n}^T)(s_0,\theta_0),$$

$\rho_3(s_0,\theta_0)=0$, $\sigma_3(s_0)=0$ and $\sigma_3'(s_0)\neq 0$. In this case, the lightlike focal set $\mathbb{LF}_C$ is locally diffeomorphic to $C(BF)\times\mathbb{R}$ and the critical value set of $\mathbb{LF}_C$ is locally diffeomorphic to the $C(2,3,4,5)$-cusp.

8 Submanifolds in Euclidean space or Hyperbolic space

In this section we consider submanifolds in Euclidean space and Hyperbolic space as special cases as the previous results.

8.1 Submanifolds in Euclidean space

Let $\mathbb{R}^n_0$ be the Euclidean space which is given by $x_0=0$ for $\boldsymbol{x}=(x_0,x_1,\dots,x_n)$. Consider an embedding $\boldsymbol{X}:U\longrightarrow\mathbb{R}^n_0$, where $U\subset\mathbb{R}^s$ is an open set. In this case we can adopt $\boldsymbol{n}^T=\boldsymbol{e}_0=(1,0,\dots,0)$ as a future directed timelike unit normal vector field along $M=\boldsymbol{X}(U)$ in $\mathbb{R}^{n+1}_1$. In this case $N_1(M)[\boldsymbol{n}^T]=N_1(M)[\boldsymbol{e}_0]$ is the unit normal bundle $N_1^e(M)$ of M in $\mathbb{R}^n_0$ in the Euclidean sense. Therefore, the lightcone Gauss map $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$ is given by $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p,\boldsymbol{\xi})=\boldsymbol{e}_0+\boldsymbol{\xi}=\boldsymbol{e}_0+\mathbb{G}(p,\boldsymbol{\xi})$, where $\mathbb{G}:N_1^e(M)\longrightarrow S^{n-1}$ is the Gauss map of the unit normal bundle $N_1^e(M)$ defined by $\mathbb{G}(p,\boldsymbol{\xi})=\boldsymbol{\xi}$[6]. Since $\boldsymbol{e}_0$ is a constant vector, we have $d_{(p,\boldsymbol{\xi})}\mathbb{LG}(\boldsymbol{n}^T)=d_{(p,\boldsymbol{\xi})}\mathbb{G}$, so that we have

$$\kappa_i(\boldsymbol{n}^T)(p,\boldsymbol{\xi})=\kappa_i(\boldsymbol{e}_0)(p,\boldsymbol{\xi})=\kappa_i(p,\boldsymbol{\xi}),$$

where $\kappa_i(p,\boldsymbol{\xi})$ $(i=1,\dots,s)$ are the eigenvalues of $-d_{(p,\boldsymbol{\xi})}\mathbb{G}$ belonging to the eigenvectors on T_pM, which are the principal curvatures of M with respect to $\boldsymbol{\xi}$ in the Euclidean sense.

On the other hand, the intersection of a lightcone with $\mathbb{R}^n_0$ is a hypersphere in $\mathbb{R}^n_0$, so that the contact of a submanifold in $\mathbb{R}^n_0$ with lightcones is equivalent to the contact with hyperspheres in $\mathbb{R}^n_0$. We define the projection $\pi : \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^n_0$ by $\pi(x_0,x_1,\dots,x_n) = (0,x_1,\dots,x_n)$. Then we have

$$\pi\circ\mathbb{LF}_{\kappa_i(\boldsymbol{n}^T)(p,\boldsymbol{\xi})}(p,\boldsymbol{\xi},t) = \boldsymbol{X}(u) + \frac{1}{\kappa_i(p,\boldsymbol{\xi})}\mathbb{G}(p,\boldsymbol{\xi}).$$

Therefore, $\pi\circ\mathbb{LF}_M$ is the focal set of $M=\boldsymbol{X}(U)$ in the Euclidean sense (cf., [27]). If $s=n-1$, $\pi\circ\mathbb{LF}_M$ is called the *evolute* of M in $\mathbb{R}^n_0$.

We remark that if $\boldsymbol{n}^T=\boldsymbol{v}$ is a constant timelike unit vector, the spacelike submanifold M is a submanifold in the spacelike hyperplane $HP(\boldsymbol{v},c)$. Since $HP(\boldsymbol{v},c)$ is isometric to the Euclidean space $\mathbb{R}^n_0$, all results for the case $\boldsymbol{n}=\boldsymbol{e}_0$ hold in this case.

8.2 Submanifolds in Hyperbolic space

Let $\boldsymbol{X} : U \longrightarrow H^n(-1)$ be an immersion into the hyperbolic space. Then we adopt $\boldsymbol{n}^T(u)=\boldsymbol{X}(u)$. In this case $N_1(M)[\boldsymbol{n}^T]$ is the unit normal bundle $N^h_1(M)$ of $M=\boldsymbol{X}(U)$ in $H^n(-1)$. Therefore, the lightcone Gauss image $\mathbb{LG}(\boldsymbol{n}^T)$ is given by $\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi}) = \boldsymbol{X}(u)+\boldsymbol{\xi} = \mathbb{L}(u,\boldsymbol{\xi})$, where $\mathbb{L} : N^h_1(M) \longrightarrow S^{n-1}_+$ is the hyperbolic Gauss indicatrix of the unit normal bundle $N^h_1(M)$(cf., [4]). Since we identify M with U through $\boldsymbol{X}$, $d\boldsymbol{X}(u)$ can be regarded as 1_{T_pM} for $p=\boldsymbol{X}(u)$. Therefore, we have $\kappa_h(\boldsymbol{n}^T)_i(p)=-1$ and we denote that $\kappa_d(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) = \kappa_d(\boldsymbol{\xi})_i(p)$, $(i=1,\dots,s)$, which we call the *de Sitter principal curvatures* of M at $p=\boldsymbol{X}(u)$ with respect to $\boldsymbol{\xi}$ (cf., [10, 14]). By Corollary 3.2, we have $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) = -1+\kappa_d(\boldsymbol{\xi})_i(p)$. The lightlike hypersurface along M is given by $\mathbb{LH}_M(p,\boldsymbol{\xi},t) = \boldsymbol{X}(u)+t(\boldsymbol{X}(u)+\boldsymbol{\xi})$, where $p=\boldsymbol{X}(u)$. Since $\langle\mathbb{LH}_M(p,\boldsymbol{\xi},t),\mathbb{LH}_M(p,\boldsymbol{\xi},t)\rangle = -1-2t$,

$$\mathbb{LH}_M(p,\boldsymbol{\xi},t) \ \text{ is } \begin{cases} \text{timeline if and only if } t>-\dfrac{1}{2}, \\ \text{lightlike if and only if } t=-\dfrac{1}{2}, \\ \text{spacelike if and only if } t<-\dfrac{1}{2}. \end{cases}$$

We now define a mapping

$$\Phi : \mathbb{R}^{n+1}_1 \setminus LC_{\boldsymbol{0}} \longrightarrow H^n(-1)\cup S^n_1$$

by $\Phi(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$. We have $\mathbb{R}^{n+1}_1 \setminus LC_{\boldsymbol{0}} = T\cup S$, where $S=\{\boldsymbol{x}\in\mathbb{R}^{n+1}_1 \mid \langle\boldsymbol{x},\boldsymbol{x}\rangle>0\ \}$ and $T=\{\boldsymbol{x}\in\mathbb{R}^{n+1}_1 \mid \langle\boldsymbol{x},\boldsymbol{x}\rangle<0\ \}$. We define $\Phi^S=\Phi|S : S\longrightarrow S^n_1$ and $\Phi^T=\Phi|T : T\longrightarrow H^n(-1)$.

We distinguish two cases as follows:

Case 1) $t>-\frac{1}{2}$, so that $\mathbb{LH}_M(p,\boldsymbol{\xi},t)$ is timelike. Thus we have $\mathbb{LH}_M(p,\boldsymbol{\xi},t)\in T$. It follows that we have the mapping $\mathbb{LH}^T_M : N_1(M)[\boldsymbol{n}^T]\times\{t\in\mathbb{R} \mid 2t+1>0\ \}\to T$ defined by

$$\mathbb{LH}^T_M(p,\boldsymbol{\xi},t) = \Phi^T\circ\mathbb{LH}_M(p,\boldsymbol{\xi},t) = \frac{1}{\sqrt{2t+1}}((t+1)\boldsymbol{X}(u)+t\boldsymbol{\xi}).$$

Case 2) $t < -\frac{1}{2}$, so that $\mathbb{LH}_M(p,\boldsymbol{\xi},t)$ is spacelike. Thus we have $\mathbb{LH}_M(p,\boldsymbol{\xi},t) \in S$. It follows that we have the mapping $\mathbb{LH}_M^S : N_1(M)[\boldsymbol{n}^T] \times \{t \in \mathbb{R} \mid 2t+1 < 0\ \} \to S$ defined by

$$\mathbb{LH}_M^S(p,\boldsymbol{\xi},t) = \Phi^S \circ \mathbb{LH}_M(p,\boldsymbol{\xi},t) = \frac{1}{\sqrt{-2t-1}}((t+1)\boldsymbol{X}(u) + t\boldsymbol{\xi}).$$

Then we have the following proposition.

Proposition 8.1 *Under the above notations, we have the followings:*
(1) *For* $t_0 > -\frac{1}{2}$, $(p_0,\boldsymbol{\xi}_0,t_0)$ *is a singular point of* $\mathbb{LH}_M$ *if and only if it is a singular point of* $\mathbb{LH}_M^T$*,*
(2) *For* $t_0 < -\frac{1}{2}$, $(p_0,\boldsymbol{\xi}_0,t_0)$ *is a singular point of* $\mathbb{LH}_M$ *if and only if it is a singular point of* $\mathbb{LH}_M^S$*.*

Proof. (1) Let $(p_0,\boldsymbol{\xi}_0,t_0)$ be a regular point of $\mathbb{LH}_M$. Then the tangent hyperplane at $(p_0,\boldsymbol{\xi}_0,t_0)$ of $\mathbb{LH}_M(N_1(M)[\boldsymbol{n}^T] \times \{t \in \mathbb{R} \mid 2t+1 > 0\ \})$ is a light like hyperplane, on where there are no timeline vectors. Since $\mathbb{LH}_M(t_0,\boldsymbol{\xi}_0,t_0)$ is timeline, it is transversal to the tangent hyperplane. Moreover, $\mathbb{LH}_M(t_0,\boldsymbol{\xi}_0,t_0)$ is directed to the fiber direction of the projection Φ^T, so that $\mathbb{LH}_M^T = \Phi^T \circ \mathbb{LH}_M$ is regular at $(p_0,\boldsymbol{\xi}_0,t_0)$. The converse assertion is trivial.

(2) For $t_0 > -\frac{1}{2}$, we assume that $(p_0,\boldsymbol{\xi}_0,t_0)$ be a regular point of $\mathbb{LH}_M$. For any $\boldsymbol{v} \in T_{(p_0,\boldsymbol{\xi}_0)}N_1(M)[\boldsymbol{n}^T]$, we take the directional derivative of the relation $\langle \mathbb{LH}_M, \mathbb{LH}_M\rangle = -1-2t$ with respect to $\boldsymbol{v}$ at $(p_0,\boldsymbol{\xi}_0,t_0)$. Then we have

$$0 = D_{\boldsymbol{v}}(-1-2t)|_{t=t_0} = 2\langle \mathbb{LH}_M, D_{\boldsymbol{v}}\mathbb{LH}_M\rangle|_{(p_0,\boldsymbol{\xi}_0,t_0)}.$$

Therefore, $\mathbb{LH}_M(p_0,\boldsymbol{\xi}_0,t_0)$ and $D_{\boldsymbol{v}}\mathbb{LH}_M(p_0,\boldsymbol{\xi}_0,t_0)$ are pseudo-orthogonal. Moreover,

$$\frac{\partial \mathbb{LH}_M}{\partial t}(p_0,\boldsymbol{\xi}_0,t_0) = p_0 + \boldsymbol{\xi}_0$$

is light like. Since $\mathbb{LH}_M(p_0,\boldsymbol{\xi}_0,t_0)$ is space like, it is transversal to the tangent hyperplane of $\mathbb{LH}_M(N_1(M)[\boldsymbol{n}^T] \times \{t \in \mathbb{R} \mid 2t+1 < 0\ \})$ at $(p_0,\boldsymbol{\xi}_0,t_0)$. Thus, $\mathbb{LH}_M^S = \Phi^S \circ \mathbb{LH}_M$ is regular at $(p_0,\boldsymbol{\xi}_0,t_0)$. The converse assertion is trivial. This completes the proof. □

On the other hand, by Corollary 4.2, the singular point of $\mathbb{LH}_M$ is $\left(p,\boldsymbol{\xi},\frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})}\right)$, $(i = 1,\dots,s)$, so that we have the following corollary.

Corollary 8.2 *We have the following assertions:*
(1) *If* $(\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}))^2 + 2\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) > 0$, *then the critical value of* $\mathbb{LH}_M^T$ *is*

$$\Phi^T \circ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p,\boldsymbol{\xi}) = \frac{|\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})+1|}{\sqrt{(\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}))^2 + 2\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})}}\left(p + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})+1}\boldsymbol{\xi}\right),$$

$(i = 1,\dots,s)$.
(2) *If* $(\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}))^2 + 2\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) < 0$, *then the critical value of* $\mathbb{LH}_M^S$ *is*

$$\Phi^S \circ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p,\boldsymbol{\xi}) = \frac{-(\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})+1)}{\sqrt{-(\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}))^2 - 2\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})}}\left(p + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})+1}\boldsymbol{\xi}\right),$$

$(i = 1,\dots,s)$.

Proof. Suppose that $(\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}))^2 + 2\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) > 0$. Since $t = 1/\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$, we have

$$\Phi^T \circ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})}(p,\boldsymbol{\xi}) = \sqrt{\frac{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})}{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})+2}}\frac{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})+1}{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})}\left(p + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})+1}\boldsymbol{\xi}\right).$$

For convenience we denote that $\kappa = \kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$. If $\kappa < 0$, then $\kappa + 2 < 0$, so that $\kappa + 1 < -1$. Therefore, we have

$$\sqrt{\frac{\kappa}{\kappa+2}}\frac{\kappa+1}{\kappa} = -\sqrt{\frac{\kappa}{(\kappa+2)\kappa^2}}(\kappa+1) = \frac{-(\kappa+1)}{\sqrt{\kappa^2+2\kappa}}.$$

If $\kappa > 0$, then $\kappa + 2 > 0$, so that we have

$$\sqrt{\frac{\kappa}{\kappa+2}}\frac{\kappa+1}{\kappa} = \sqrt{\frac{\kappa}{(\kappa+2)\kappa^2}}(\kappa+1) = \frac{(\kappa+1)}{\sqrt{\kappa^2+2\kappa}}.$$

Thus we have the formula (1).

Suppose that $(\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}))^2+2\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) < 0$. We also denote that $\kappa = \kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$. Then we have

$$\Phi^S \circ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})}(p,\boldsymbol{\xi}) = \sqrt{\frac{\kappa}{-\kappa-2}}\frac{\kappa+1}{\kappa}\left(p + \frac{1}{\kappa+1}\boldsymbol{\xi}\right).$$

Since $-2 < \kappa < 0$, we have

$$\sqrt{\frac{\kappa}{-\kappa-2}}\frac{\kappa+1}{\kappa} = -\sqrt{\frac{\kappa}{(-\kappa-2)\kappa^2}}(\kappa+1) = \frac{-(\kappa+1)}{\sqrt{-\kappa^2-2\kappa}}.$$

Thus we have the formula (2). This completes the proof. □

Since $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) = \kappa_d(\boldsymbol{\xi})_i(p) - 1$, we have

$$(\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}))^2 + 2\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) = (\kappa_d(\boldsymbol{\xi})_i(p))^2 - 1,$$

so that we have

$$\Phi^T \circ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p,\boldsymbol{\xi}) = \frac{|\kappa_d(\boldsymbol{\xi})_i(p)|}{\sqrt{(\kappa_d(\boldsymbol{\xi})_i(p))^2-1}}\left(p + \frac{1}{\kappa_d(\boldsymbol{\xi})_i(p)}\boldsymbol{\xi}\right)$$

and

$$\Phi^S \circ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p,\boldsymbol{\xi}) = \frac{-\kappa_d(\boldsymbol{\xi})_i(p)}{\sqrt{-(\kappa_d(\boldsymbol{\xi})_i(p))^2-1}}\left(p + \frac{1}{\kappa_d(\boldsymbol{\xi})_i(p)}\boldsymbol{\xi}\right).$$

We now introduce the notion of focal sets of submanifolds in the hyperbolic space. For a submanifold $M = \boldsymbol{X}(U) \subset H^n(-1)$, we define the *total focal set* of M by

$$\mathbb{TF}_M = \bigcup\left\{\frac{\pm\kappa_d(\boldsymbol{\xi})_i(p)}{\sqrt{|(\kappa_d(\boldsymbol{\xi})_i(p))^2-1|}}\left(p + \frac{1}{\kappa_d(\boldsymbol{\xi})_i(p)}\boldsymbol{\xi}\right)\ \middle|\ \kappa_d(\boldsymbol{\xi})_i(p) \neq \pm 1,\ i = 1,\dots,s\right\}.$$

We have the following decomposition of the total focal set:

$$\mathbb{TF}_M = \mathbb{HF}_M \cup \mathbb{SF}_M,$$

where

$$\mathbb{HF}_M = \bigcup \left\{ \frac{\pm\kappa_d(\boldsymbol{\xi})_i(p)}{\sqrt{(\kappa_d(\boldsymbol{\xi})_i(p))^2 - 1}} \left(p + \frac{1}{\kappa_d(\boldsymbol{\xi})_i(p)}\boldsymbol{\xi} \right) \;\middle|\; (\kappa_d(\boldsymbol{\xi})_i(p))^2 > 1,\ i = 1, \dots, s \right\}$$

and

$$\mathbb{SF}_M = \bigcup \left\{ \frac{\pm\kappa_d(\boldsymbol{\xi})_i(p)}{\sqrt{1 - (\kappa_d(\boldsymbol{\xi})_i(p))^2}} \left(p + \frac{1}{\kappa_d(\boldsymbol{\xi})_i(p)}\boldsymbol{\xi} \right) \;\middle|\; (\kappa_d(\boldsymbol{\xi})_i(p))^2 < 1,\ i = 1, \dots, s \right\}.$$

We call $\mathbb{HF}_M$ the *hyperbolic focal set* and $\mathbb{SF}_M$ the *de Sitter focal set.* We denote that

$$\mathbb{LF}^T_M = \bigcup \left\{ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p, \boldsymbol{\xi}) \;\middle|\; (\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}))^2 + 2\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) > 0 \right\}$$

and

$$\mathbb{LF}^S_M = \bigcup \left\{ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p, \boldsymbol{\xi}) \;\middle|\; (\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}))^2 + 2\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) < 0 \right\}.$$

We respectively call $\mathbb{LF}^T_M$ and $\mathbb{LF}^T_M$ the *timelike part of* and the *spacelike part of* the focal set of M. By the previous arguments, we have the following proposition.

Proposition 8.3 *Let $M = \boldsymbol{X}(U)$ be a submanifold in the hyperbolic space $H^n(-1)$. Then we have*

$$\Phi^T(\mathbb{LF}^T_M) \subset \mathbb{HF}_M \text{ and } \Phi^S(\mathbb{LF}^S_M) \subset \mathbb{SF}_M.$$

In [13] the notion of the evolutes of a hypersurface in the hyperbolic space was introduced and the singularities the evolutes are investigated. If M is a hypersurface of the hyperbolic space, then M is a spacelike submanifold in $\mathbb{R}^{n+1}_1$ with the codimension two and N^h_1 is a double covering of M. In this case, the above definition of the focal sets are the same as the definitions of the evolutes in [13]. Therefore, we denote that $\mathbb{LE}^T_M, \mathbb{LE}^S_M, \mathbb{HE}_M, \mathbb{SE}_M$ instead of $\mathbb{LF}^T_M, \mathbb{LF}^S_M, \mathbb{HF}_M, \mathbb{SF}_M$, respectively. Then we have the following corollary of Proposition 8.3.

Corollary 8.4 *Let $M = \boldsymbol{X}(U)$ be a hypersurface in the hyperbolic space $H^n(-1)$. Then we have*

$$\Phi^T(\mathbb{LE}^T_M) \subset \mathbb{HE}_M \text{ and } \Phi^S(\mathbb{LE}^S_M) \subset \mathbb{SE}_M.$$

References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol. I.* Birkhäuser, 1986.

[2] R. Bousso, *The holographic principle*, REVIEWS OF MODERN PHYSICS 74 (2002), 825–874.

[3] J. W. Bruce, P. J. Gibline, *Curves and singularities(2nd.ed.)*, Cambridge Univ. Press (1992).

[4] M. Buosi, S. Izumiya and M. A. Soares Ruas, *Total absolute horospherical curvature of sub manifolds in hyperbolic space*, Adv. Geom. 10 (2010), 603–620.

[5] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, International Series of Monographs on Physics. 69 Oxford Univeristy press, 1983.

[6] S. -S. Chern and R. K. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math., **79** (1957), 306–318

[7] S. Frittelli, E. T. Newman and G. Silva-Ortigoza, *The Eikonal equation in flat space: Null surfaces and their singularities I*, J. Math. Phys. 40 (1999), 383–407.

[8] C. G. Gibson, K. Wirthmuller, A. A. du Plessis and E. J. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Mathematics 552, Springer, Berlin, 1976.

[9] S. Izumiya, D. Pei and T. Sano, *The lightcone Gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space* , Glasgow. Math. J. 42 (2000), 75–89.

[10] S.Izumiya, D-H. Pei and T. Sano, *Singularities of hyperbolic Gauss maps.* Proc. London Math. Soc. 86 (2003), 485–512.

[11] S. Izumiya, D. Pei and M.C. Romero Fuster, The lightcone Gauss map of a spacelike surface in Minkowski 4-space, Asian Journal of Mathematics, 8 (2004), 511–530

[12] S. Izumiya, D. Pei and M.C. Romero Fuster, *Umbilicity of space-like submanifolds of Minkowski space*, Proc. Roy. Soc. Edinburh Sect. A 134 (2004), 375–387.

[13] S. Izumiya, D-H. Pei and M. Takahashi, *Singularities of evolutes of hypersurfaces in Hyperbolic space*, Proc. the Edinburgh Math. Soc. 47 (2004), 131–153.

[14] S. izumiya, D-H. Pei, M.C. Romero Fuster and M. Takahashi, The horospherical geometry of submanifolds in hyperbolic space, J. London Math. Soc. 71 (2005), 779–800.

[15] S. Izumiya, M. Kossowski, D-H. Pei and M. C. Romero Fuster, *Singularities of lightlike hypersurfaces in Minkowski four-space*, Tohoku Math. J. 58 (2006), 71–88

[16] S. Izumiya, D-H. Pei and M.C. Romero-Fuster, *The horospherical geometry of surfaces in Hyperbolic* 4*-space.* Israel J. Maths. 154 (2006), 361–379.

[17] S. Izumiya and M. C. Romero Fuster, *The lightlike flat geometry on spacelike submanifolds of codimension two in Minkowski space*, Sel. math., New ser. 13 (2007), 23–55

[18] S. Izumiya, *Lightlike hypersurfaces in Lorentz-Minkowski space*, The second Japanese-Australian Workshop on Real and Complex Singularities, RIMS Kokyuroku 1610 (2008), 51–65

[19] S. Izumiya, Total lightcone curvatures of spacelike submanifolds in Lorentz-Minkowski space, preprint (2012)

[20] M. Kossowski, *The intrinsic conformal structure and Gauss map of a light-like hypersurface in Minkowski space*, Trans. Amer. Math. Soc. 316 (1989), 369–383.

[21] J. A. Little, *On singularities of submanifolds of high dimensional Euclidean space*, Ann. Mat. Pura Appl.(4) 83 (1969), 261–335.

[22] J. Martinet, *Singularities of Smooth Functions and Maps*, London Math. Soc. Lecture Note Ser. 58, Cambridge Univ. Press, Cambridge-New York, 1982.

[23] J. N. Mather, How to stratify mappings and jet spaces, Lecture Notes in Mathematics 553 (1976)

[24] C. W. Misner, K. S. Thorpe and J. W. Wheeler, *Gravitation*, W. H. Freeman and Co., San Francisco, CA, 1973.

[25] J. A. Montaldi, *On contact between submanifolds*, Michigan Math. J. 33, (1986), 81–85.

[26] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[27] I. Porteous, *The normal singularities of submanifold.* J. Diff. Geom., vol 5, (1971), 543–564

[28] Y.-H. Wan, *Generic deformation of varieties*, Trans. Amer. Math. Soc., 259 (1980), 107–119.

[29] G. Wasserman, *Stability of Caustics*, Math. Ann. 210 (1975), 43–50.

[30] V. M. Zakalyukin, *Lagrangian and Legendrian singularities*, Funct. Anal. Appl. (1976), 23–31.

[31] V. M. Zakalyukin, *Reconstructions of fronts and caustics depending one parameter and versality of mappings*, J. Sov. Math. 27 (1984), 2713–2735.