HOKKAIDO UNIVERSITY

Title	Lightcone dualities for curves in the lightcone unit 3-sphere
Author(s)	Izumiya, Shyuichi; Jiang, Yang; Sato, Takami
Citation	Journal of Mathematical Physics, 54(6), 063511 https://doi.org/10.1063/1.4811451
Issue Date	2013-06
Doc URL	http://hdl.handle.net/2115/53140
Rights	Copyright 2013 American Institute of Physics. This article may be downloaded for personal use only. Any other use requires prior permission of the author and the American Institute of Physics. The following article appeared in J. Math. Phys. 54, 063511 (2013) and may be found at http://jmp.aip.org/resource/1/jmapaq/v54/i6/p063511_s1.
Type	article
File Information	JMathPhys_54_063511.pdf

Lightcone dualities for curves in the lightcone unit 3-sphere

Shyuichi Izumiya, Yang Jiang, and Takami Sato

Additional information on J. Math. Phys.

Lightcone dualities for curves in the lightcone unit 3-sphere

Shyuichi Izumiya,[1,a)] Yang Jiang,[1,2,b)] and Takami Sato[2,c)]
[1]*School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, People's Republic of China*
[2]*Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan*

In this paper, we consider the curves in the unit 3-sphere in the lightcone. The unit 3-sphere can be canonically embedded in the lightcone and de Sitter 4-space in Lorentz-Minkowski 5-space. We investigate these curves in the framework of the theory of Legendrian dualities between pseudo-spheres in Lorentz-Minkowski 5-space. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4811451]

I. INTRODUCTION

The evolutes of curves in the unit 2-sphere are studied in Ref. 7 from the view point of the Legendrian duality.[4,5] It is known that the evolute of a curve in the unit 2-sphere is the dual of the tangent indicatrix of the original curve.[11] The dual curve in the unit 2-sphere is defined to be equidistant by $\pi/2$ from the original one. For a curve in the unit 3-sphere, however, the dual is a surface. Therefore, the dual of the tangent indicatrix of a curve is a surface which is called the *focal surface* (or, the *focal set*) of the original curve. The critical locus of the focal surface is the evolute of the original curve (cf. Ref. 11). We remark that the focal set of a curve in the unit 2-sphere is a curve which is equal to the evolute.

On the other hand, the first author (S.I.) introduced the mandala of Legendrian dualities between pseudo-spheres in Lorentz-Minkowski 5-space.[4,5] There are three kinds of pseudo-spheres in Lorentz-Minkowski 5-space (i.e., the hyperbolic space, the de Sitter space, and the lightcone). Especially, if we investigate spacelike submanifolds in the lightcone, those Legendrian dualities are essentially useful (see also Ref. 9). For the de Sitter 4-space and the lightcone in Lorentz-Minkowski 5-space, there exist naturally embedded unit 3-spheres. The de Sitter 4-space corresponds to the cosmic model, and the lightcone also has its clear background in Physics.[15] In this paper, we investigate the curves in the unit 3-sphere in the framework of the theory of Legendrian dualities between pseudo-spheres in Lorentz-Minkowski 5-space (Refs. 1, 2, 5, 6, 10–13, etc.). If we have a regular curve in the unit 3-sphere, then we have the regular curve in the embedded unit 3-sphere in the lightcone or de Sitter space. Therefore, we naturally have the dual hypersurfaces in the lightcone as an application of the duality theorem in Ref. 5. There are two kinds of lightcone dual hypersurfaces of a curve in the unit 3-sphere. We will give the classifications of the singularities of these hypersurfaces. In physics, the singularities of the lightcone are also studied.[14] The critical value sets of these two hypersurfaces are called the lightcone focal surfaces, respectively. The projections of these focal surfaces to unit 3-sphere are different surfaces. In Ref. 7, we have shown that the projection images of the critical value sets of lightcone dual hypersurfaces for a curve in the unit 2-sphere coincide with the evolute of the original curve. Therefore, the situation of curves in the unit 3-sphere is quite different from that of curves in the unit 2-sphere. However, the projections of the critical sets of lightcone focal surfaces are equal to the evolute of the curve. In order to clarify such situation, we

---
a)Present address: Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan. Electronic mail: izumiya@math.sci.hokudai.ac.jp.
b)E-mail: xjiangyang@126.com
c)E-mail: taka_mi@ec.hokudai.ac.jp

introduce the notion of discriminant set of higher order for unfoldings of functions of one-variable (see Sec. VI).

A brief description of the organization for this paper is as follows: In Sec. II, we give basic concepts in this paper. In Sec. III, we formulate the Frenet-Serret type formulae for the curves in the unit 3-sphere. We also give the definition of the spherical focal surfaces of the curve in the unit 3-sphere. The spherical evolutes of the curve in the unit 3-sphere are given by the critical sets of the spherical focal surfaces. In Sec. IV, we define the lightcone dual hypersurfaces and the ligtcone focal surfaces of the curves in the unit 3-sphere. In Sec. V, we calculate the conditions for the $A_k(k = 1, 2, 3, 4)$ singularities for the lightcone height functions of the curves in the unit 3-sphere. In Sec. VI, we show that the projections of the critical sets of focal surfaces to the unit 3-sphere are the same, and they are equal to the spherical evolutes of the curve in the unit 3-sphere (Theorem 6.2). We also study the singularities of the lightcone dual hypersurfaces, the lightcone focal surfaces and the spherical evolutes of the curves in the unit 3-sphere which is one of the main results in this paper (cf. Theorem 6.9).

All maps considered here are of class C^∞ unless otherwise stated.

II. BASIC CONCEPTS

In this section, we give the basic concepts in this paper. Let $\mathbb{R}^5$ be a five-dimensional vector space. For any two vectors $\boldsymbol{x} = (x_0, x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_0, y_1, y_2, y_3, y_4)$ in $\mathbb{R}^5$, their pseudo-scalar product is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$. The pair $(\mathbb{R}^5, \langle, \rangle)$ is called *Lorentz-Minkowski 5-space*. We denote it as $\mathbb{R}_1^5$.

For any four vectors $\boldsymbol{x} = (x_0, x_1, x_2, x_3, x_4), \boldsymbol{y} = (y_0, y_1, y_2, y_3, y_4), \boldsymbol{z} = (z_0, z_1, z_2, z_3, z_4),$ $\boldsymbol{w} = (w_0, w_1, w_2, w_3, w_4) \in \mathbb{R}_1^5$, their pseudo-vector product is defined by

$$
\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} \wedge \boldsymbol{w} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \\ z_0 & z_1 & z_2 & z_3 & z_4 \\ w_0 & w_1 & w_2 & w_3 & w_4 \end{vmatrix},
$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical basis of $\mathbb{R}_1^5$. A non-zero vector $\boldsymbol{x} \in \mathbb{R}_1^5$ is called *spacelike*, *lightlike*, or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$, or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, respectively. The norm of $\boldsymbol{x} \in \mathbb{R}_1^5$ is defined by $\| \boldsymbol{x} \| = (\text{sign}(\boldsymbol{x})\langle \boldsymbol{x}, \boldsymbol{x} \rangle)^{1/2}$, where $\text{sign}(\boldsymbol{x})$ denotes the signature of $\boldsymbol{x}$ which is given by $\text{sign}(\boldsymbol{x}) = 1, 0,$ or -1 when $\boldsymbol{x}$ is a spacelike, lightlike, or timelike vector, respectively.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}_1^5$ be a regular curve in $\mathbb{R}_1^5$ (i.e., $\dot{\boldsymbol{\gamma}}(t) \neq \boldsymbol{0}$ for any $t \in I$), where I is an open interval. For any $t \in I$, the curve $\boldsymbol{\gamma}$ is called *spacelike*, *lightlike*, or *timelike* if $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle > 0$, $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle = 0$, or $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle < 0$, respectively. The arc-length of a spacelike or timelike curve $\boldsymbol{\gamma}$ measured from $\boldsymbol{\gamma}(t_0)(t_0 \in I)$ is $s(t) = \int_{t_0}^{t} \| \dot{\boldsymbol{\gamma}}(t) \| \, dt$.

The parameter s is determined such that $\| \boldsymbol{\gamma}'(s) \| = 1$ for the nonlightlike curve, where $\boldsymbol{\gamma}'(s) = d\boldsymbol{\gamma}/ds(s)$ is the unit tangent vector of $\boldsymbol{\gamma}$ at s. We define the *de Sitter 4-space* by

$$
S_1^4 = \{\boldsymbol{x} \in \mathbb{R}_1^5 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}.
$$

We define the *closed lightcone with the vertex $\boldsymbol{a}$* by

$$
LC_a = \{\boldsymbol{x} \in \mathbb{R}_1^5 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\}.
$$

We define the *open lightcone at the origin* by

$$
LC^* = \{\boldsymbol{x} \in \mathbb{R}_1^5 \backslash \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.
$$

Here we consider the unit sphere in the lightcone defined by

$$
S_+^3 = \{\boldsymbol{x} \in LC^* \mid x_0 = 1\} = \{\boldsymbol{x} \in \mathbb{R}_1^5 \mid x_0 = 1, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\},
$$

we call it the *lightcone unit sphere*. We define the *canonical lightcone projection* $\pi : LC^* \longrightarrow S_+^3$ by

$$\pi(x_0, x_1, x_2, x_3, x_4) = \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0}\right).$$

We also define the *three-dimensional Euclidean unit 3-sphere* in $\mathbb{R}_0^4$ by

$$S_0^3 = \{\boldsymbol{x} \in \mathbb{R}_1^5 \mid x_0 = 0, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\},$$

where $\mathbb{R}_0^4 = \{\boldsymbol{x} \in \mathbb{R}_1^5 \mid x_0 = 0\}$ is the Euclidean 4-space.

III. CURVES IN THE UNIT 3-SPHERE AND FOCAL SURFACES

Let $\boldsymbol{\gamma} : I \longrightarrow S_+^3$ be a regular curve. We have a map $\Phi : S_+^3 \to S_0^3$ defined by $\Phi(\boldsymbol{v}) = \boldsymbol{v} - \boldsymbol{e}_0$, which is an isometry. Then we have a regular curve $\overline{\boldsymbol{\gamma}} : I \to S_0^3$ defined by $\overline{\boldsymbol{\gamma}}(s) = \Phi(\boldsymbol{\gamma}(s))$ $= \boldsymbol{\gamma}(s) - \boldsymbol{e}_0$, so that $\boldsymbol{\gamma}$ and $\overline{\boldsymbol{\gamma}}$ have completely the same geometric properties as spherical curves. Since $\overline{\boldsymbol{\gamma}}$ is a spacelike curve, we can reparameterize it by the arc-length s. So we have the unit tangent vector $\boldsymbol{t}(s) = \overline{\boldsymbol{\gamma}}'(s)$ of $\overline{\boldsymbol{\gamma}}(s)$. Suppose that $\|\boldsymbol{t}'(s)\| \neq 1$. Then $\|\boldsymbol{t}'(s) + \overline{\boldsymbol{\gamma}}(s)\| \neq 0$, so that we have another unit vector $\boldsymbol{n}(s) = \frac{\boldsymbol{t}'(s) + \overline{\boldsymbol{\gamma}}(s)}{\|\boldsymbol{t}'(s) + \overline{\boldsymbol{\gamma}}(s)\|}$. We also define a unit vector by $\boldsymbol{b}(s) = \overline{\boldsymbol{\gamma}}(s) \wedge \boldsymbol{e}_0 \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$, then we have a pseudo-orthonormal frame field $\{\overline{\boldsymbol{\gamma}}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ of $\mathbb{R}_0^4$ along $\overline{\boldsymbol{\gamma}}(s)$. By standard arguments, we have the following *Frenet-Serret type formulae*:

$$\begin{cases} \overline{\boldsymbol{\gamma}}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = \kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s) \\ \boldsymbol{n}'(s) = -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{b}(s) \\ \boldsymbol{b}'(s) = -\tau_g(s)\boldsymbol{n}(s) \end{cases},$$

where $\kappa_g(s) = \|\boldsymbol{t}'(s) + \overline{\boldsymbol{\gamma}}(s)\|$ and $\tau_g(s) = -\det(\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}'(s), \overline{\boldsymbol{\gamma}}''(s), \overline{\boldsymbol{\gamma}}'''(s))/\kappa_g^2(s)$. We call $\{\overline{\boldsymbol{\gamma}}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ a *Sabban frame* of $\overline{\boldsymbol{\gamma}}$.[8] Here, κ_g is called a *geodesic curvature* and τ_g a *geodesic torsion* of $\overline{\boldsymbol{\gamma}}$ in S_0^3, respectively.

It is known that the evolute of the curve in the unit 2-sphere is the dual of the spherical curve given by the unit tangent vector $\boldsymbol{t}(s)$.[11] But for a curve in the unit 3-sphere, the dual is a surface. We call it the *focal surface* of the original curve. We now consider the focal surface of a curve $\overline{\boldsymbol{\gamma}} : I \to S_0^3$ analogous to the case for curves in Euclidean space. We define $F^\pm : I \times J \to S_0^3$ by

$$F^\pm(s, u) = u\overline{\boldsymbol{\gamma}}(s) + \frac{u}{\kappa_g(s)}\boldsymbol{n}(s) \pm \frac{\sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)}}{\kappa_g(s)}\boldsymbol{b}(s).$$

We call each image of $F^\pm$ the *spherical focal surface of* $\overline{\boldsymbol{\gamma}}$. We remark that the focal surfaces of $\overline{\boldsymbol{\gamma}}$ satisfy the equations $\langle \overline{\boldsymbol{\gamma}}'(s), F^\pm(s, u) \rangle = \langle \overline{\boldsymbol{\gamma}}''(s), F^\pm(s, u) \rangle = 0$. This means that each one of the focal surfaces $F^\pm(s, u)$ of $\boldsymbol{\gamma}$ is the spherical dual of $\boldsymbol{t}$ in the sense of Ref. 10. By straightforward calculations, we have

$$\frac{\partial F^\pm}{\partial u}(s, u) = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) \pm \frac{-u(\kappa_g^2(s) + 1)}{\kappa_g(s)\sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)}}\boldsymbol{b}(s),$$

$$\frac{\partial F^\pm}{\partial s}(s, u) = -\frac{u\kappa_g'(s) \pm \tau_g(s)\kappa_g(s)\sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)}}{\kappa_g^2(s)}\boldsymbol{n}(s)$$

$$+ \frac{u\tau_g(s)\kappa_g(s)\sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)} \pm u^2\kappa_g'(s)}{\kappa_g^2(s)\sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)}}\boldsymbol{b}(s).$$

It follows that $\{\partial F^{\pm}/\partial u, \partial F^{\pm}/\partial s\}$ are linearly dependent if and only if

$$\tau_g(s)\kappa_g(s)\sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)} \pm u\kappa_g'(s) = 0,$$

so that we have

$$u = \frac{\varepsilon_i \tau_g(s)\kappa_g^2(s)}{\sqrt{\kappa_g'^2(s) + \kappa_g^4(s)\tau_g^2(s) + \kappa_g^2(s)\tau_g^2(s)}}.(i = 1, 2; \varepsilon_1 = 1, \varepsilon_2 = -1).$$

Therefore, each critical value set of $F^{\pm}$ is given by

$$\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s) = \frac{\pm\tau_g(s)\kappa_g^2(s)}{\sqrt{\kappa_g'^2(s) + \kappa_g^4(s)\tau_g^2(s) + \kappa_g^2(s)\tau_g^2(s)}} \left\{ \overline{\gamma}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) + \left(\frac{1}{\kappa_g(s)}\right)' \frac{1}{\tau_g(s)}\boldsymbol{b}(s) \right\}.$$

We remark that each curve of $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}$ satisfies the equations

$$\langle \overline{\gamma}'(s), \boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s) \rangle = \langle \overline{\gamma}''(s), \boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s) \rangle = \langle \overline{\gamma}'''(s), \boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s) \rangle = 0.$$

In Ref. 11, Porteous introduced the notion of the evolute of $\overline{\gamma}$ in the unit 3-sphere. He defined it as the curve which satisfies the above equations, so that we call each image of $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}$ the *spherical evolute* of $\overline{\gamma}$ in the unit 3-sphere. We remark that $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{-}(s) = -\boldsymbol{\varepsilon}_{\overline{\gamma}}^{+}(s)$. For $s = s_0$, we fix $\boldsymbol{v}_0^{\pm} = \boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s_0)$ and $\langle \overline{\gamma}(s_0), \boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s_0) \rangle = c^{\pm}$. Since $\boldsymbol{v}_0^{-} = -\boldsymbol{v}_0^{+}$ and $c^{-} = -c^{+}$, we have a hyperplane

$$HP(\boldsymbol{v}_0^{+}, c^{+}) = \{\boldsymbol{x} \in \mathbb{R}_0^4 \mid \langle \boldsymbol{x}, \boldsymbol{v}_0^{+} \rangle = c^{+}\} = \{\boldsymbol{x} \in \mathbb{R}_0^4 \mid \langle \boldsymbol{x}, \boldsymbol{v}_0^{-} \rangle = c^{-}\} = HP(\boldsymbol{v}_0^{-}, c^{-}),$$

so that we have a sphere

$$S^2(\boldsymbol{v}_0^{\pm}, c^{\pm}) = HP(\boldsymbol{v}_0^{\pm}, c^{\pm}) \cap S_0^3.$$

We call $S^2(\boldsymbol{v}_0^{\pm}, c^{\pm})$ an *osculating sphere* of $\overline{\gamma}$ at s_0. Therefore, the spherical evolutes $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s)$ are the loci of the centers of osculating spheres of $\overline{\gamma}$.

Proposition 3.1. There exists a sphere $S^2(\boldsymbol{v}, c) \subset S_0^3$ such that $\overline{\gamma}(I) \subset S^2(\boldsymbol{v}, c)$ if and only if both of the spherical evolutes $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}$ of $\overline{\gamma}$ are single points.

Proof. If one of the spherical evolutes $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{+}$ of $\overline{\gamma}$ is a single point, we can set that $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{+}(s) = \boldsymbol{v}^{+}$. In this case another spherical evolute $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{-}$ is constant too. Then $\langle \overline{\gamma}(s), \boldsymbol{v}^{+} \rangle' = \langle \overline{\gamma}'(s), \boldsymbol{v}^{+} \rangle = \langle \boldsymbol{t}(s), \boldsymbol{\varepsilon}_{\overline{\gamma}}^{+}(s) \rangle = 0$, so we have $\langle \overline{\gamma}(s), \boldsymbol{v}^{+} \rangle = c^{+}$ and $\overline{\gamma}(I) \subset S^2(\boldsymbol{v}^{+}, c^{+})$. Conversely, if $\overline{\gamma}(I) \subset S^2(\boldsymbol{v}, c)$, then at any point on $\overline{\gamma}$, the osculating spheres is $S^2(\boldsymbol{v}, c)$ itself. So the locus of the centers of osculating spheres of $\overline{\gamma}$ is $\boldsymbol{v}$ and $-\boldsymbol{v}$. Therefore, both of the spherical evolutes $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}$ of $\overline{\gamma}$ are single points. This completes the proof. $\qquad\square$

IV. LIGHTCONE DUALS OF CURVES IN THE UNIT 3-SPHERE

In Ref. 5, the first author introduced the Legendrian dualities between pseudo-spheres in Lorentz-Minkowski 5-space which is a basic tool for the study of hypersurfaces in pseudo-spheres in Lorentz-Minkowski 5-space. We define one-forms $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^4 w_i dv_i$, $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^4 v_i dw_i$ in $\mathbb{R}_1^5 \times \mathbb{R}_1^5$ and consider the following two double fibrations:

(1) (a) $LC^* \times S_1^4 \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 1\}$,
     (b) $\pi_{31} : \Delta_3 \longrightarrow LC^*, \pi_{32} : \Delta_3 \longrightarrow S_1^4$,
     (c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle|\Delta_3, \theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle|\Delta_3$.
(2) (a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -2\}$,
     (b) $\pi_{41} : \Delta_4 \longrightarrow LC^*, \pi_{42} : \Delta_4 \longrightarrow LC^*$,
     (c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle|\Delta_4, \theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle|\Delta_4$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$. We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i, i=3,4. It has been shown in Ref. 5 that each

$(\Delta_i, K_i)(i = 3, 4)$ is a contact manifold and both of $\pi_{ij}(j = 1, 2)$ are Legendrian fibrations. Moreover, those contact manifolds are contact diffeomorphic to each other. In Ref. 5, we have defined four double fibrations (Δ_i, K_i), $i = 1, 2, 3, 4$, such that these are contact diffeomorphic to each other. Here, we only use (Δ_3, K_3) and (Δ_4, K_4). If we have an isotropic mapping $i: L \rightarrow \Delta_i$ (i.e., $i^*\theta_{i1} = 0$), we say that $\pi_{i1}(i(L))$ and $\pi_{i2}(i(L))$ are Δ_i-*dual* to each other, $i = 3, 4$. For detailed properties of Legendrian fibrations, see Ref. 1.

We now define hypersurfaces in LC^* associated with the curves in S_+^3 or S_0^3. Let $\boldsymbol{\gamma} : I \longrightarrow S_+^3$ be a unit speed curve. We define $\overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm} : I \times \mathbb{R}^2 \longrightarrow LC^*$ by

$$\overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s, u, v) = \overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) + v\boldsymbol{b}(s) \pm \sqrt{u^2 + v^2 + 1}\,\boldsymbol{e}_0.$$

We also define $LD_{\boldsymbol{\gamma}} : I \times \mathbb{R}^2 \longrightarrow LC^*$ by

$$LD_{\boldsymbol{\gamma}}(s, u, v) = \frac{u^2 + v^2 - 4}{4}\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) + v\boldsymbol{b}(s) + \frac{u^2 + v^2 + 4}{4}\boldsymbol{e}_0.$$

Then we have the following proposition.

Proposition 4.1. Under the above notation, we have the followings:

(1) $\overline{\boldsymbol{\gamma}}$ *and* $\overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}$ *are* Δ_3-*dual to each other.*
(2) $\boldsymbol{\gamma}$ *and* $LD_{\boldsymbol{\gamma}}$ *are* Δ_4-*dual to each other.*

Proof. Consider the mapping $\mathscr{L}_3(s, u, v) = (\overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s, u, v), \overline{\boldsymbol{\gamma}}(s))$. Then we have

$$\langle \overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s, u, v), \overline{\boldsymbol{\gamma}}(s) \rangle = \langle \overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}(s) \rangle = 1$$

and

$$\mathscr{L}_3^*\theta_{32} = \langle \overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s, u, v), \overline{\boldsymbol{\gamma}}'(s) \rangle ds = \langle \overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s, u, v), \boldsymbol{t}(s) \rangle ds = 0.$$

The assertion (1) holds.

We also consider the mapping $\mathscr{L}_4(s, u, v) = (LD_{\boldsymbol{\gamma}}(s, u, v), \boldsymbol{\gamma}(s))$. Since $\langle \boldsymbol{\gamma}(s), \boldsymbol{e}_0 \rangle = -1$ and $\langle \boldsymbol{\gamma}(s), \overline{\boldsymbol{\gamma}}(s) \rangle = 1$, we have $\langle LD_{\boldsymbol{\gamma}}(s, u, v), \boldsymbol{\gamma}(s) \rangle = (u^2 + v^2)/4 - 1 - ((u^2 + v^2)/4 + 1) = -2$. Moreover, we have

$$\mathscr{L}_4^*\theta_{42} = \langle LD_{\boldsymbol{\gamma}}(s, u, v)), \boldsymbol{\gamma}'(s) \rangle ds = \langle LD_{\boldsymbol{\gamma}}(s, u, v), \boldsymbol{t}(s) \rangle ds = 0.$$

This completes the proof. $\square$

We call each image of $\overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}$ the *Lightcone dual haypersurface of the de Sitter spherical curve* $\overline{\boldsymbol{\gamma}}$ and $LD_{\boldsymbol{\gamma}}$ the *Lightcone dual hypersurface of the lightlike spherical curve* $\boldsymbol{\gamma}$. Then we have two mappings $\pi \circ \overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm} : I \times \mathbb{R}^2 \rightarrow S_+^3$ and $\pi \circ LD_{\boldsymbol{\gamma}} : I \times \mathbb{R}^2 \rightarrow S_+^3$ defined by

$$\pi \circ \overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s, u, v) = \pm \left(\frac{1}{\sqrt{u^2 + v^2 + 1}}\overline{\boldsymbol{\gamma}}(s) + \frac{u}{\sqrt{u^2 + v^2 + 1}}\boldsymbol{n}(s) + \frac{v}{\sqrt{u^2 + v^2 + 1}}\boldsymbol{b}(s) \right) + \boldsymbol{e}_0,$$

$$\pi \circ LD_{\boldsymbol{\gamma}}(s, u, v) = \frac{u^2 + v^2 - 4}{u^2 + v^2 + 4}\overline{\boldsymbol{\gamma}}(s) + \frac{4u}{u^2 + v^2 + 4}\boldsymbol{n}(s) + \frac{4v}{u^2 + v^2 + 4}\boldsymbol{b}(s) + \boldsymbol{e}_0.$$

In this paper, we consider the singularities of these dual surfaces and mappings. By the Frenet-Serret type formulae, we have

$$\frac{\partial \overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}}{\partial u}(s, u, v) = \boldsymbol{n}(s) \pm \frac{u}{\sqrt{1 + u^2 + v^2}}\boldsymbol{e}_0,$$

$$\frac{\partial \overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}}{\partial v}(s, u, v) = \boldsymbol{b}(s) \pm \frac{v}{\sqrt{1 + u^2 + v^2}}\boldsymbol{e}_0,$$

$$\frac{\partial \overline{LD_{\overline{\gamma}}^{\pm}}}{\partial s}(s, u, v) = (1 - u\kappa_g(s))\boldsymbol{t}(s) - v\tau_g(s)\boldsymbol{n}(s) + u\tau_g(s)\boldsymbol{b}(s),$$

$$\frac{\partial LD_{\gamma}}{\partial u}(s, u, v) = \frac{u}{2}\overline{\gamma}(s) + \boldsymbol{n}(s) + \frac{u}{2}\boldsymbol{e}_0,$$

$$\frac{\partial LD_{\gamma}}{\partial v}(s, u, v) = \frac{v}{2}\overline{\gamma}(s) + \boldsymbol{b}(s) + \frac{v}{2}\boldsymbol{e}_0,$$

$$\frac{\partial LD_{\gamma}}{\partial s}(s, u, v) = \frac{u^2 + v^2 - 4u\kappa_g(s) - 4}{4}\boldsymbol{t}(s) - v\tau_g(s)\boldsymbol{n}(s) + u\tau_g(s)\boldsymbol{b}(s).$$

Then we have the following proposition.

Proposition 4.2. Let $\boldsymbol{\gamma} : I \longrightarrow S^3_+$ be a unit speed curve. Then we have the followings:

(1)   *(s, u, v) is a singular point of $\overline{LD_{\overline{\gamma}}^{\pm}}$ if and only if $u = 1/\kappa_g(s)$.*
(2)   *(s, u, v) is a singular point of LD_{γ} if and only if $v = \pm\sqrt{4 + 4u\kappa_g(s) - u^2}$.*

Proof. By the above calculations, $\partial \overline{LD_{\overline{\gamma}}^{\pm}}/\partial u(s, u, v), \partial \overline{LD_{\overline{\gamma}}^{\pm}}/\partial v(s, u, v), \partial \overline{LD_{\overline{\gamma}}^{\pm}}/\partial s(s, u, v)$ are linearly dependent if and only if $u = 1/\kappa_g(s)$. The assertion (1) follows. By the similar reason, we have the assertion (2). This completes the proof.   □

Therefore, the critical value sets of the above dual hypersurfaces are given by

$$C(\overline{LD_{\overline{\gamma}}^{\pm}}) = \left\{ \overline{\gamma}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) + v\boldsymbol{b}(s) \pm \sqrt{\frac{1 + \kappa_g^2(s) + v^2\kappa_g^2(s)}{\kappa_g^2(s)}}\boldsymbol{e}_0 \;\middle|\; v \in \mathbb{R}, s \in I, \kappa_g(s) \neq 0 \right\},$$

$$C(LD_{\gamma})^{\pm} = \{\kappa_g(s)u\overline{\gamma}(s) + u\boldsymbol{n}(s) \pm \sqrt{4 + 4u\kappa_g(s) - u^2}\boldsymbol{b}(s) + (\kappa_g(s)u + 2)\boldsymbol{e}_0 | u \in \mathbb{R}, s \in I \}.$$

We, respectively, denote that

$$LF_{\overline{\gamma}}^{\pm}(s, u) = \overline{\gamma}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) + v(u)\boldsymbol{b}(s) \pm \sqrt{\frac{1 + \kappa_g^2(s) + v^2(u)\kappa_g^2(s)}{\kappa_g^2(s)}}\boldsymbol{e}_0,$$

$$LF_{\gamma}^{\pm}(s, u) = \kappa_g(s)u\overline{\gamma}(s) + u\boldsymbol{n}(s) \pm \sqrt{4 + 4u\kappa_g(s) - u^2}\boldsymbol{b}(s) + (\kappa_g(s)u + 2)\boldsymbol{e}_0,$$

where we have the relation $v(u) = \pm\sqrt{4 + 4u\kappa_g(s) - u^2}$. We respectively call each one of $LF_{\overline{\gamma}}^{\pm}$ the *lightcone focal surface* of the de Sitter spherical curve $\overline{\gamma}$ and each one of $LF_{\gamma}^{\pm}$ the *ligtcone focal surface* of the lightcone spherical curve $\boldsymbol{\gamma}$. Then the projections of $LF_{\overline{\gamma}}^{\pm}$ and $LF_{\gamma}^{\pm}$ to S^3_+ are given as follows:

$$\pi(C(\overline{LD_{\overline{\gamma}}^{\pm}})) = \left\{ \frac{\pm(\kappa_g(s)\overline{\gamma}(s) + \boldsymbol{n}(s) + v\kappa_g(s)\boldsymbol{b}(s))}{\sqrt{1 + \kappa_g^2(s) + v^2\kappa_g^2(s)}} + \boldsymbol{e}_0 \;\middle|\; v \in \mathbb{R}, s \in I, \kappa_g(s) \neq 0 \right\},$$

$$\pi(C(LD_{\gamma})^{\pm}) = \left\{ \frac{u\kappa_g(s)\overline{\gamma}(s) + u\boldsymbol{n}(s) \pm \sqrt{4 + 4u\kappa_g(s) - u^2}\boldsymbol{b}(s)}{\kappa_g(s)u + 2} + \boldsymbol{e}_0 \;|u \in \mathbb{R}, s \in I \right\}.$$

On the other hand, we define $\widetilde{\pi} = \Phi \circ \pi : LC^* \to S^3_0$. By the previous calculations, $\widetilde{\pi}(C(\overline{LD_{\overline{\gamma}}^{\pm}}))$ is different from $\widetilde{\pi}(C(LD_{\gamma})^{\pm})$. In Ref. 7, it was shown that the projections of the critical value sets of the lightcone dual hypersurfaces of $\boldsymbol{\gamma}$ and $\overline{\gamma}$ are the same for a curve $\boldsymbol{\gamma} : I \to S^2_+$. Moreover, it is equal to the spherical evolute of $\overline{\gamma}$. Therefore, the situation for curves in S^3_+ is quite different from that for curves in S^2_+.

V. LIGHTCONE HEIGHT FUNCTIONS

In order to study the singularities of Lightcone dual surfaces of spherical curves, we introduce two families of functions and apply the theory of unfoldings. Let $\boldsymbol{\gamma} : I \longrightarrow S_+^3$ be a unit speed curve, according to the duality between lightcone and de Sitter 4-space in Sec. IV, we define the following family of functions:

$$\overline{H} : I \times LC^* \longrightarrow \mathbb{R}, \overline{H}(s, \boldsymbol{v}) = \langle \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v} \rangle - 1.$$

We call $\overline{H}$ a *lightcone height function* of the de Sitter spherical curve $\overline{\boldsymbol{\gamma}}$. For any fixed $\boldsymbol{v} \in LC^*$, we denote $\overline{h}_{\boldsymbol{v}}(s) = \overline{H}(s, \boldsymbol{v})$. For simplification, we denote

$$\rho(s) = \sqrt{(\kappa_g^4(s)\tau_g^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g'^2(s))/\kappa_g^4(s)\tau_g^2(s)}.$$

Then we have the following proposition for $\overline{h}_{\boldsymbol{v}}$.

Proposition 5.1. Let $\boldsymbol{\gamma} : I \longrightarrow S_+^3$ be a unit speed curve, then we have the followings:

(1) $\overline{h}_{\boldsymbol{v}}(s) = 0$ *if and only if there exist* $\lambda, \mu, \xi, \eta \in \mathbb{R}$ *with* $\eta^2 = 1 + \lambda^2 + \mu^2 + \xi^2$ *such that* $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \lambda \boldsymbol{t}(s) + \mu \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) + \eta \boldsymbol{e}_0.$

(2) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = 0$ *if and only if there exist* $\mu, \xi, \eta \in \mathbb{R}$ *with* $\eta^2 = 1 + \mu^2 + \xi^2$ *such that* $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \mu \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) + \eta \boldsymbol{e}_0 = \overline{\boldsymbol{\gamma}}(s) + \mu \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) \pm \sqrt{1 + \mu^2 + \xi^2} \boldsymbol{e}_0.$

(3) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = \overline{h}_{\boldsymbol{v}}''(s) = 0$ *if and only if* $\kappa_g(s) \neq 0$ *and*

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)} \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) \pm \sqrt{\frac{1 + \kappa_g^2(s) + \kappa_g^2(s)\xi^2}{\kappa_g^2(s)}} \boldsymbol{e}_0.$$

(4) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = \overline{h}_{\boldsymbol{v}}''(s) = \overline{h}_{\boldsymbol{v}}'''(s) = 0$ *if and only if* $\kappa_g(s) \neq 0$, $\tau_g(s) \neq 0$ *and*

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)} \boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)} \boldsymbol{b}(s) \pm \rho(s) \boldsymbol{e}_0.$$

(5) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = \overline{h}_{\boldsymbol{v}}''(s) = \overline{h}_{\boldsymbol{v}}'''(s) = \overline{h}_{\boldsymbol{v}}^{(4)}(s) = 0$ *if and only if* $\kappa_g(s) \neq 0$, $\tau_g(s) \neq 0$,

$$\left(\left(\frac{-1}{\kappa_g(s)} \right)' \frac{1}{\tau_g(s)} \right)' - \frac{\tau_g(s)}{\kappa_g(s)} = 0$$

and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)} \boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)} \boldsymbol{b}(s) \pm \rho(s) \boldsymbol{e}_0.$$

(6) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = \overline{h}_{\boldsymbol{v}}''(s) = \overline{h}_{\boldsymbol{v}}'''(s) = \overline{h}_{\boldsymbol{v}}^{(4)}(s) = \overline{h}_{\boldsymbol{v}}^{(5)}(s) = 0$ *if and only if* $\kappa_g(s) \neq 0$, $\tau_g(s) \neq 0$,

$$\left(\left(\frac{-1}{\kappa_g(s)} \right)' \frac{1}{\tau_g(s)} \right)' - \frac{\tau_g(s)}{\kappa_g(s)} = \left\{ \left(\left(\frac{-1}{\kappa_g(s)} \right)' \frac{1}{\tau_g(s)} \right)' - \frac{\tau_g(s)}{\kappa_g(s)} \right\}' = 0$$

and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)} \boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)} \boldsymbol{b}(s) \pm \rho(s) \boldsymbol{e}_0.$$

Proof. (1) Since $\boldsymbol{v} \in LC^*$, there exist $\omega, \lambda, \mu, \xi, \eta \in \mathbb{R}$ with $\omega^2 + \lambda^2 + \mu^2 + \xi^2 - \eta^2 = 0$ such that $\boldsymbol{v} = \omega \overline{\boldsymbol{\gamma}}(s) + \lambda \boldsymbol{t}(s) + \mu \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) + \eta \boldsymbol{e}_0$. From $\overline{h}_{\boldsymbol{v}}(s) = \langle \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v} \rangle - 1 = 0$, we have $\omega = 1$. So $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \lambda \boldsymbol{t}(s) + \mu \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) + \eta \boldsymbol{e}_0$ and $\eta^2 = 1 + \lambda^2 + \mu^2 + \xi^2$. The converse direction also holds.

(2) Since $\overline{h}_{\boldsymbol{v}}'(s) = \langle \boldsymbol{t}(s), \boldsymbol{v} \rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = 0$ if and only if

$$\overline{h}_{\boldsymbol{v}}'(s) = \langle \boldsymbol{t}(s), \boldsymbol{v} \rangle = \langle \boldsymbol{t}(s), \overline{\boldsymbol{\gamma}}(s) + \lambda \boldsymbol{t}(s) + \mu \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) + \eta \boldsymbol{e}_0 \rangle = \lambda = 0.$$

It follows from the fact $\eta^2 = 1 + \mu^2 + \xi^2$ that $\eta = \pm\sqrt{1 + \mu^2 + \xi^2}$. Then we have $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s)$ $+ \mu\boldsymbol{n}(s) + \xi\boldsymbol{b}(s) + \eta\boldsymbol{e}_0 = \overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{n}(s) + \xi\boldsymbol{b}(s) \pm \sqrt{1 + \mu^2 + \xi^2}\boldsymbol{e}_0$.

(3) Since $\overline{h}_{\boldsymbol{v}}''(s) = \langle \kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = \overline{h}_{\boldsymbol{v}}''(s) = 0$ if and only if

$$\overline{h}_{\boldsymbol{v}}''(s) = \langle \kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{n}(s) + \xi\boldsymbol{b}(s) \pm \sqrt{1 + \mu^2 + \xi^2}\boldsymbol{e}_0\rangle = \kappa_g(s)\mu - 1 = 0.$$

Then we have $\kappa_g(s) \neq 0$, $\mu = 1/\kappa_g(s)$ and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) + \xi\boldsymbol{b}(s) \pm \sqrt{(1 + \kappa_g^2(s) + \kappa_g^2(s)\xi^2)/\kappa_g^2(s)}\boldsymbol{e}_0.$$

(4) Since $\overline{h}_{\boldsymbol{v}}'''(s) = \langle \kappa_g'(s)\boldsymbol{n}(s) - (\kappa_g^2(s) + 1)\boldsymbol{t}(s) + \kappa_g(s)\tau_g(s)\boldsymbol{b}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = \overline{h}_{\boldsymbol{v}}''(s)$ $= \overline{h}_{\boldsymbol{v}}'''(s) = 0$ if and only if

$$\overline{h}_{\boldsymbol{v}}'''(s) = \langle \kappa_g'(s)\boldsymbol{n}(s) - (\kappa_g^2(s) + 1)\boldsymbol{t}(s) + \kappa_g(s)\tau_g(s)\boldsymbol{b}(s),$$

$$\overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) + \xi\boldsymbol{b}(s) \pm \sqrt{(1 + \kappa_g^2(s) + \kappa_g^2(s)\xi^2)/\kappa_g^2(s)}\boldsymbol{e}_0\rangle$$

$$= \kappa_g'(s)/\kappa_g(s) + \kappa_g(s)\tau_g(s)\xi = 0.$$

Then we have $\kappa_g(s) \neq 0$, $\tau_g(s) \neq 0$, $\xi = -\kappa_g'(s)/\kappa_g^2(s)\tau_g(s)$, and $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s)$ $- \kappa_g'(s)\boldsymbol{b}(s)/\kappa_g^2(s)\tau_g(s) \pm \rho(s)\boldsymbol{e}_0$.

The proof of (5) and (6) follows by computing the 4th and 5th derivatives of $\overline{h}_{\boldsymbol{v}}(s)$ with respect to s. The calculations are lengthy and are omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

For a function $f(s)$, we say that f has A_k-*singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$. By Proposition 5.1 (6), we know that for $\overline{h}_{\boldsymbol{v}}$, only A_k-singularity, $k = 1, 2, 3, 4$, are generic singularities.

Let $\boldsymbol{\gamma} : I \longrightarrow S_+^3$ be a unit speed curve, according to the duality in the lightcone in Sec. IV, we define the following family of functions:

$$H : I \times LC^* \longrightarrow \mathbb{R}, \quad H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v}\rangle + 2.$$

We call H a *lightcone height function* of the lightlike spherical curve $\boldsymbol{\gamma}$. For any fixed $\boldsymbol{v} \in LC^*$, we denote $h_{\boldsymbol{v}}(s) = H(s, \boldsymbol{v})$. For simplification, we denote

$$\sigma^{\pm}(s) = (\kappa_g^2(s)\tau_g(s) \pm \sqrt{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s)})/(\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s)).$$

The we have the following proposition for $h_{\boldsymbol{v}}$.

Proposition 5.2. Let $\boldsymbol{\gamma} : I \longrightarrow S_+^3$ be a unit speed curve, then we have the followings:

(1) $h_{\boldsymbol{v}}(s) = 0$ if and only if $\boldsymbol{v} = \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + (\lambda + 2)\boldsymbol{e}_0$, where λ, μ, ξ, η $\in \mathbb{R}$ and $\mu^2 + \xi^2 + \eta^2 - 4\lambda - 4 = 0$.

(2) $h_{\boldsymbol{v}}(s) = h_{\boldsymbol{v}}'(s) = 0$ if and only if $\boldsymbol{v} = ((\xi^2 + \eta^2)/4 - 1)\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n} + \eta\boldsymbol{b}(s) + ((\xi^2 + \eta^2)/4$ $+ 1)\boldsymbol{e}_0$.

(3) $h_{\boldsymbol{v}}(s) = h_{\boldsymbol{v}}'(s) = h_{\boldsymbol{v}}''(s) = 0$ if and only if

$$\boldsymbol{v} = \kappa_g(s)\xi\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) \pm \sqrt{4 + 4\kappa_g(s)\xi - \xi^2}\boldsymbol{b}(s) + (\kappa_g(s)\xi + 2)\boldsymbol{e}_0.$$

(4) $h_{\boldsymbol{v}}(s) = h_{\boldsymbol{v}}'(s) = h_{\boldsymbol{v}}''(s) = h_{\boldsymbol{v}}'''(s) = 0$ if and only if $\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0$ and

$$\boldsymbol{v} = 2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s) - 2\kappa_g'(s)\sigma^{\pm}(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 2)\boldsymbol{e}_0.$$

(5) $h_{\boldsymbol{v}}(s) = h_{\boldsymbol{v}}'(s) = h_{\boldsymbol{v}}''(s) = h_{\boldsymbol{v}}'''(s) = h_{\boldsymbol{v}}^{(4)}(s) = 0$ if and only if $\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0$,

$$\left(\left(\frac{-1}{\kappa_g(s)}\right)' \frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)} = 0$$

and

$$\boldsymbol{v} = 2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^\pm(s)\boldsymbol{n}(s) - 2\kappa_g'(s)\sigma^\pm(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s) + 2)\boldsymbol{e}_0.$$

(6) $h_{\boldsymbol{v}}(s) = h_{\boldsymbol{v}}'(s) = h_{\boldsymbol{v}}''(s) = h_{\boldsymbol{v}}'''(s) = h_{\boldsymbol{v}}^{(4)}(s) = h_{\boldsymbol{v}}^{(5)}(s) = 0$ *if and only if* $\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s)$
$\neq 0$,

$$\left(\left(\frac{-1}{\kappa_g(s)}\right)' \frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)} = \left\{\left(\left(\frac{-1}{\kappa_g(s)}\right)' \frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)}\right\}' = 0$$

and

$$\boldsymbol{v} = 2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^\pm(s)\boldsymbol{n}(s) - 2\kappa_g'(s)\sigma^\pm(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s) + 2)\boldsymbol{e}_0.$$

The proof of this proposition is similar to the proof of Proposition 5.1, we omit it. According to the assertions of Propositions 5.1 and 5.2, we define an invariant

$$\kappa_S(s) = \left(\left(\frac{-1}{\kappa_g(s)}\right)' \frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)}.$$

We call $\kappa_S(s)$ *spherical curvature* of $\overline{\boldsymbol{\gamma}}$. We have the following proposition.

Proposition 5.3. For a unit speed curve $\boldsymbol{\gamma} : I \to S_+^3$, *both of the spherical evolutes* $\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}^\pm$ *are single points if and only if* $\kappa_S \equiv 0$.

Proof.

$$\kappa_S(s) = \left(\frac{(-1/\kappa_g(s))'}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)} = \frac{\kappa_g(s)\kappa_g''(s)\tau_g(s) - 2\kappa_g'^2(s)\tau_g(s) - \kappa_g(s)\kappa_g'(s)\tau_g'(s) - \kappa_g^2(s)\tau_g^3(s)}{\kappa_g^3(s)\tau_g^2(s)},$$

$$\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}'^\pm(s) = \frac{-\kappa_S(s)\kappa_g^4(s)\tau_g^2(s)\kappa_g'(s)}{(\kappa_g'^2(s) + \kappa_g^4(s)\tau_g^2(s) + \kappa_g^2(s)\tau_g^2(s))^{3/2}} \left\{\overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) + \frac{(1/\kappa_g(s))'}{\tau_g(s)}\boldsymbol{b}(s)\right\}$$

$$- \frac{\kappa_S(s)\kappa_g^3(s)\tau_g^2(s)}{\kappa_g(s)\tau_g(s)(\kappa_g'^2(s) + \kappa_g^4(s)\tau_g^2(s) + \kappa_g^2(s)\tau_g^2(s))^{1/2}}\boldsymbol{b}(s).$$

So $\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}'^\pm \equiv 0$ if and only if $\kappa_S \equiv 0$. This completes the proof. $\square$

VI. SINGULARITIES OF LIGHTCONE DUALS OF SPHERICAL CURVES

In this section, we classify the singularities of $\overline{LD_{\overline{\boldsymbol{\gamma}}}^\pm}$ and $LD_{\boldsymbol{\gamma}}$ as an application of the unfolding theory of functions. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be a function germ, we call F an *r-parameter unfolding* of f, where $f(s) = F_{\boldsymbol{x}_0}(s, \boldsymbol{x}_0)$. The *discriminant set* of F is defined by

$$D_F = \left\{\boldsymbol{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, \ F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s}(s, \boldsymbol{x}) = 0\right\}.$$

By Propositions 5.1 (2) and 5.2 (2), the discriminant set of $\overline{H}$ and H are

$$D_{\overline{H}} = \{\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) + v\boldsymbol{b}(s) \pm \sqrt{u^2 + v^2 + 1}\,\boldsymbol{e}_0 \mid s \in I, u, v \in \mathbb{R}\},$$

$$D_H = \{(u^2 + v^2 - 4)\overline{\boldsymbol{\gamma}}(s)/4 + u\boldsymbol{n}(s) + v\boldsymbol{b}(s) + (u^2 + v^2 + 4)\boldsymbol{e}_0/4 \mid s \in I, u, v \in \mathbb{R}\}.$$

These are the lightcone dual hypersurfaces of $\overline{\gamma}$ and the lightcone dual surface of γ, respectively. Moreover, the both assertions (4) of Propositions 5.1 and 5.2 describe the singularities of the lightcone focal surfaces of γ and $\overline{\gamma}$, respectively.

Proposition 6.1. The critical value sets of $LF_{\overline{\gamma}}^{\pm}$ and $LF_{\gamma}^{\pm}$ are give as follows:

$$C(LF_{\overline{\gamma}}^{\pm}) = \left\{ \overline{\gamma}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)}\boldsymbol{b}(s) \pm \rho(s)\boldsymbol{e}_0 \;\Big|\; s \in I \right\},$$

$$C(LF_{\gamma}^{\pm}) = \{ 2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\gamma}(s) + 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s)$$

$$-2\kappa_g'(s)\sigma^{\pm}(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 2)\boldsymbol{e}_0 \mid s \in I \}.$$

Then we have the following theorem as a corollary.

Theorem 6.2. *Both of the projections of the critical value sets $C(LF_{\overline{\gamma}}^{\pm})$ and $C(LF_{\gamma}^{\pm})$ in the unit 3-sphere S_0^3 are the images of the spherical evolutes of $\overline{\gamma}$, that is,*

$$\widetilde{\pi}(C(LF_{\overline{\gamma}}^{\pm})) = \widetilde{\pi}(C(LF_{\gamma}^{\pm})) = \{\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s) \mid s \in I \}.$$

Proof. We know that

$$\widetilde{\pi}(C(LF_{\overline{\gamma}}^{\pm})) = \left\{ \pm\frac{1}{\rho(s)}\left(\overline{\gamma}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)}\boldsymbol{b}(s) \right) \;\Big|\; s \in I \right\}$$

and

$$\widetilde{\pi}(C(LF_{\gamma}^{\pm})) = \left\{ \sigma^{\pm}(s)\frac{\kappa_g^2(s)\tau_g(s)\overline{\gamma}(s) + \kappa_g(s)\tau_g(s)\boldsymbol{n}(s) - \kappa_g'(s)\boldsymbol{b}(s)}{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 1} \;\Big|\; s \in I \right\}.$$

By straightforward calculations, we have

$$\frac{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)}{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 1}$$

$$= \frac{\kappa_g^2(s)\tau_g(s)(\kappa_g^2(s)\tau_g(s) \pm \sqrt{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s)})}{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s) \pm \kappa_g^2(s)\tau_g(s)\sqrt{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s)}}$$

$$= \frac{\pm\kappa_g^2(s)\tau_g(s)}{\sqrt{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s)}} = \frac{\pm 1}{\rho(s)}.$$

Similarly, we can calculate that

$$\frac{\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)}{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 1} = \frac{\pm 1}{\rho(s)\kappa_g(s)},$$

$$\frac{\kappa_g'(s)\sigma^{\pm}(s)}{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 1} = \frac{\pm\kappa_g'(s)}{\rho(s)\kappa_g^2(s)\tau_g(s)}.$$

Thus we have

$$\widetilde{\pi}(C(LF_{\overline{\gamma}}^{\pm})) = \widetilde{\pi}(C(LF_{\gamma}^{\pm})) = \{\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s) \mid s \in I \}.$$

This completes the proof.   □

Inspired by Propositons 5.1, 5.2, and Theorem 6.2, we define the following set:

$$D_F^\ell = \left\{ \boldsymbol{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, \ F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s}(s, \boldsymbol{x}) = \cdots = \frac{\partial^\ell F}{\partial s^\ell}(s, \boldsymbol{x}) = 0 \right\},$$

which we call the *discriminant set of order* ℓ. Of course, $D_F^1 = D_F$. In order to understand the geometric properties of the discriminant set of order ℓ, we introduce an equivalence relation among the unfoldings of functions. Let F and G be r-parameter unfoldings of $f(s)$ and $g(s)$, respectively. We say that F and G are *P-$\mathcal{R}$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow (\mathbb{R} \times \mathbb{R}^r, (s_0', \boldsymbol{x}_0'))$ of the form $\Phi(s, \boldsymbol{x}) = (\Phi_1(s, \boldsymbol{x}), \phi(\boldsymbol{x}))$ such that $G \circ \Phi = F$. By straightforward calculations, we have the following proposition.

Proposition 6.3. Let F and G be r-parameter unfoldings of f(s) and g(s), respectively. If F and G are P-$\mathcal{R}$-equivalent by a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow (\mathbb{R} \times \mathbb{R}^r, (s_0', \boldsymbol{x}_0'))$ of the form $\Phi(s, \boldsymbol{x}) = (\Phi_1(s, \boldsymbol{x}), \phi(\boldsymbol{x}))$, then $\phi(D_F^\ell) = D_G^\ell$ as set germs.

By Propositions 5.1 and 5.2, we have the following proposition.

Proposition 6.4. Under the same notations as in the previous paragraphs, we have

$$D_{\overline{H}} = D_{\overline{H}}^1 = \text{Images } \overline{LD_{\overline{\gamma}}^\pm}, \ D_{\overline{H}}^2 = \text{Images } LF_{\overline{\gamma}}^\pm, \ \widetilde{\pi}(D_{\overline{H}}^3) = \text{Images } \boldsymbol{\varepsilon}_{\overline{\gamma}}^\pm,$$

$$D_H = D_H^1 = \text{Image } LD_{\boldsymbol{\gamma}}, \ D_H^2 = \text{Images } LF_{\boldsymbol{\gamma}}^\pm, \ \widetilde{\pi}(D_H^3) = \text{Images } \boldsymbol{\varepsilon}_{\overline{\gamma}}^\pm.$$

Let F be an r-parameter unfolding of f and f has A_k-singularity ($k \geq 1$) at s_0. We denote the $(k - 1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 as

$$j^{(k-1)} \left(\frac{\partial F}{\partial x_i}(s, \boldsymbol{x}_0) \right) (s_0) = \sum_{j=1}^{k-1} \alpha_{ji}(s - s_0)^j, i = 1, \cdots, r.$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is $k(k \leq r)$, then F is called a *versal unfolding* of f, where $\alpha_{0i} = \partial F/\partial x_i(s_0, \boldsymbol{x}_0)$. We have the following classification theorem of versal unfoldings (see page 149, Sec. 6.6 of Ref. 3).

Theorem 6.5. *Let* $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ *be an r-parameter unfolding of f which has A_k-singularity at s_0. Suppose F is a versal unfolding of f, then F is P-$\mathcal{R}$-equivalent to one of the following unfoldings:*

(a) $k = 1 : \pm s^2 + x_1$,
(b) $k = 2 : s^3 + x_1 + sx_2$,
(c) $k = 3 : \pm s^4 + x_1 + sx_2 + s^2 x_3$,
(d) $k = 4 : s^5 + x_1 + sx_2 + s^2 x_3 + s^3 x_4$.

We have the following classification result as a corollary of the above theorem.

Corollary 6.6. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ *be an r-parameter unfolding of f which has A_k-singularity at s_0. Suppose F is a versal unfolding of f, then we have the following assertions:*

(a) *For $k = 1$, D_F is diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$ and $D_F^2 = \emptyset$.*

(b) *For $k = 2$, D_F is diffeomorphic to $C(2, 3) \times \mathbb{R}^{r-2}$, D_F^2 is diffeomorphic to $\{\boldsymbol{0}\} \times \mathbb{R}^{r-2}$ and $D_F^3 = \emptyset$.*

(c) *For $k = 3$, D_F is diffeomorphic to $SW \times \mathbb{R}^{r-3}$, D_F^2 is diffeomorphic to $C(2, 3, 4) \times \mathbb{R}^{r-3}$, D_F^3 is diffeomorphic to $\{\boldsymbol{0}\} \times \mathbb{R}^{r-3}$ and $D_F^4 = \emptyset$.*

(d) *For $k = 4$, D_F is locally diffeomorphic to $BF \times \mathbb{R}^{r-4}$, D_F^2 is diffeomorphic to $C(BF) \times \mathbb{R}^{r-4}$, D_F^3 is diffeomorphic to $C(2, 3, 4, 5) \times \mathbb{R}^{r-4}$, D_F^4 is diffeomorphic to $\{\boldsymbol{0}\} \times \mathbb{R}^{r-4}$ and $D_F^5 = \emptyset$.*

We remark that all of diffeomorphisms in the above assertions are diffeomorphism germs.

Here, we respectively call

$$C(2, 3) = \{(x_1, x_2) \mid x_1 = u^2, x_2 = u^3\}$$

a *(2, 3)-cusp*,

$$C(2, 3, 4) = \{(x_1, x_2, x_3) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4\}$$

a *(2, 3, 4)-cusp*,

$$C(2, 3, 4, 5) = \{(x_1, x_2, x_3, x_4) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4, x_4 = u^5\}$$

a *(2, 3, 4, 5)-cusp*,

$$SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$$

a *swallow tail*,

$$BF = \{(x_1, x_2, x_3.x_4) \mid x_1 = 5u^4 + 3vu^2 + 2wu, x_2 = 4u^5 + 2vu^3 + wu^2, x_3 = u, x_4 = v\}$$

a *butterfly*,

$$C(BF) = \{(x_1, x_2, x_3, x_4) \mid x_1 = 6u^5 + u^3 v, x_2 = 25u^4 + 9u^2 v, x_3 = 10u^3 + 3uv, x_4 = v\}$$

a *c-butterfly* (i.e., the critical value set of the butterfly).

We have the following key propositions for H and $\overline{H}$.

Proposition 6.7. If $\overline{h}_{v_0}$ has an A_k-singularity, $k = 1, 2, 3, 4$, at s_0, then $\overline{H}$ is a versal unfolding of $\overline{h}_{v_0}$.

Proof. For $v \in LC^*$, we have $v = (\pm(v_1^2 + v_2^2 + v_3^2 + v_4^2)^{1/2}, v_1, v_2, v_3, v_4)$. We denote that $\overline{\gamma}(s) = (0, x_1(s), x_2(s), x_3, (s), x_4(s))$. Then

$$\overline{H}(s, v) = \langle \overline{\gamma}(s), v \rangle - 1 = x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 + x_4(s)v_4 - 1.$$

Thus we have

$$\frac{\partial \overline{H}}{\partial v_1}(s, v) = x_1(s), \quad \frac{\partial \overline{H}}{\partial v_2}(s, v) = x_2(s), \quad \frac{\partial \overline{H}}{\partial v_3}(s, v) = x_3(s), \quad \frac{\partial \overline{H}}{\partial v_4}(s, v) = x_4(s),$$

$$\frac{\partial^2 \overline{H}}{\partial s \partial v_1}(s, v) = x_1'(s), \quad \frac{\partial^2 \overline{H}}{\partial s \partial v_2}(s, v) = x_2'(s), \quad \frac{\partial^2 \overline{H}}{\partial s \partial v_3}(s, v) = x_3'(s), \quad \frac{\partial^2 \overline{H}}{\partial s \partial v_4}(s, v) = x_4'(s),$$

$$\frac{\partial^3 \overline{H}}{\partial s^2 \partial v_1}(s, v) = x_1''(s), \quad \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_2}(s, v) = x_2''(s), \quad \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_3}(s, v) = x_3''(s), \quad \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_4}(s, v) = x_4''(s),$$

$$\frac{\partial^4 \overline{H}}{\partial s^3 \partial v_1}(s, v) = x_1'''(s), \quad \frac{\partial^4 \overline{H}}{\partial s^3 \partial v_2}(s, v) = x_2'''(s), \quad \frac{\partial^4 \overline{H}}{\partial s^3 \partial v_3}(s, v) = x_3'''(s), \quad \frac{\partial^4 \overline{H}}{\partial s^3 \partial v_4}(s, v) = x_4'''(s).$$

For a fixed point $v_0 = (v_{00}, v_{01}, v_{02}, v_{03}, v_{04})$, the 3-jet of $\partial \overline{H}/\partial v_i(s, v_0)$ at s_0 is

$$j^{(3)} \frac{\partial \overline{H}}{\partial v_i}(s, v_0)(s_0) = x_i'(s_0)(s - s_0) + x_i''(s_0)(s - s_0)^2/2 + x_i'''(s_0)(s - s_0)^3/6, \ i = 1, 2, 3, 4.$$

For $k = 1, 2, 3$, it is obviously that $\overline{H}$ is a versal unfolding of $\overline{h}_{v_0}$. So we only consider the case $k = 4$. We need to show that the rank of the matrix A is 4, where

$$A = \begin{pmatrix} x_1(s_0) & x_2(s_0) & x_3(s_0) & x_4(s_0) \\ x_1'(s_0) & x_2'(s_0) & x_3'(s_0) & x_4'(s_0) \\ x_1''(s_0) & x_2''(s_0) & x_3''(s_0) & x_4''(s_0) \\ x_1'''(s_0) & x_2'''(s_0) & x_3'''(s_0) & x_4'''(s_0) \end{pmatrix}.$$

We have

$$\det A = \langle \boldsymbol{e}_0 \wedge \overline{\boldsymbol{\gamma}}(s_0) \wedge \overline{\boldsymbol{\gamma}}'(s_0) \wedge \overline{\boldsymbol{\gamma}}''(s_0), \overline{\boldsymbol{\gamma}}'''(s_0) \rangle = -\kappa_g^2(s_0)\tau_g(s_0) \neq 0,$$

which completes the proof.   □

Proposition 6.8. If $h_{\boldsymbol{v}_0}$ has an A_k-singularity, $k = 1, 2, 3, 4$, at s_0, then H is a versal unfolding of $h_{\boldsymbol{v}_0}$.

Proof.   For $\boldsymbol{v} \in LC^*$,   we   have $\boldsymbol{v} = (v_0, v_1, v_2, v_3, v_4) = (\pm(v_1^2 + v_2^2 + v_3^2 + v_4^2)^{1/2}, v_1, v_2, v_3, v_4)$. We write $\overline{\boldsymbol{\gamma}}(s) = (1, x_1(s), x_2(s), x_3(s), x_4(s))$. Then we have

$$H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle + 2 = \mp(v_1^2 + v_2^2 + v_3^2 + v_4^2)^{1/2} + x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 + x_4(s)v_4 + 2.$$

Thus we have

$$\frac{\partial H}{\partial v_1}(s, \boldsymbol{v}) = -v_1/v_0 + x_1(s), \ \frac{\partial H}{\partial v_2}(s, \boldsymbol{v}) = -v_2/v_0 + x_2(s),$$

$$\frac{\partial H}{\partial v_3}(s, \boldsymbol{v}) = -v_3/v_0 + x_3(s), \ \frac{\partial H}{\partial v_4}(s, \boldsymbol{v}) = -v_4/v_0 + x_4(s),$$

$$\frac{\partial^2 H}{\partial s \partial v_1}(s, \boldsymbol{v}) = x_1'(s), \ \frac{\partial^2 H}{\partial s \partial v_2}(s, \boldsymbol{v}) = x_2'(s), \ \frac{\partial^2 H}{\partial s \partial v_3}(s, \boldsymbol{v}) = x_3'(s), \ \frac{\partial^2 H}{\partial s \partial v_4}(s, \boldsymbol{v}) = x_4'(s),$$

$$\frac{\partial^3 H}{\partial s^2 \partial v_1}(s, \boldsymbol{v}) = x_1''(s), \ \frac{\partial^3 H}{\partial s^2 \partial v_2}(s, \boldsymbol{v}) = x_2''(s), \ \frac{\partial^3 H}{\partial s^2 \partial v_3}(s, \boldsymbol{v}) = x_3''(s), \ \frac{\partial^3 H}{\partial s^2 \partial v_4}(s, \boldsymbol{v}) = x_4''(s),$$

$$\frac{\partial^4 H}{\partial s^3 \partial v_1}(s, \boldsymbol{v}) = x_1'''(s), \ \frac{\partial^4 H}{\partial s^3 \partial v_2}(s, \boldsymbol{v}) = x_2'''(s), \ \frac{\partial^4 H}{\partial s^3 \partial v_3}(s, \boldsymbol{v}) = x_3'''(s), \ \frac{\partial^4 H}{\partial s^3 \partial v_4}(s, \boldsymbol{v}) = x_4'''(s).$$

For a fixed $\boldsymbol{v}_0 = (v_{00}, v_{01}, v_{02}, v_{03}, v_{04})$, the 3-jet of $\partial H/\partial v_i(s, \boldsymbol{v}_0)$ at s_0 is

$$j^{(3)}\frac{\partial H}{\partial v_i}(s, \boldsymbol{v}_0)(s_0) = x_i'(s_0)(s - s_0) + x_i''(s_0)(s - s_0)^2/2 + x_i'''(s_0)(s - s_0)^3/6, i = 1, 2, 3, 4.$$

For $k = 1, 2, 3$, it is obviously that H is a versal unfolding of $h_{\boldsymbol{v}_0}$. So we only consider the case $k = 4$. We need to show that the rank of the matrix B is 4, where

$$B = \begin{pmatrix} -v_{01}/v_{00} + x_1(s_0) & -v_{02}/v_{00} + x_2(s_0) & -v_{03}/v_{00} + x_3(s_0) & -v_{04}/v_{00} + x_4(s_0) \\ x_1'(s_0) & x_2'(s_0) & x_3'(s_0) & x_4'(s_0) \\ x_1''(s_0) & x_2''(s_0) & x_3''(s_0) & x_4''(s_0) \\ x_1'''(s_0) & x_2'''(s_0) & x_3'''(s_0) & x_4'''(s_0) \end{pmatrix}.$$

By straightforward calculations, we have

$$\det B = \langle \boldsymbol{e}_0 \wedge \overline{\boldsymbol{\gamma}}'(s_0) \wedge \overline{\boldsymbol{\gamma}}''(s_0) \wedge \overline{\boldsymbol{\gamma}}'''(s_0), \boldsymbol{v}_0 \rangle/v_{00} + \langle \boldsymbol{e}_0 \wedge \overline{\boldsymbol{\gamma}}(s_0) \wedge \overline{\boldsymbol{\gamma}}'(s_0) \wedge \overline{\boldsymbol{\gamma}}''(s_0), \overline{\boldsymbol{\gamma}}'''(s_0) \rangle$$

$$= \langle \kappa_g^2(s_0)\tau_g(s_0)\overline{\boldsymbol{\gamma}}(s_0), \boldsymbol{v}_0 \rangle/v_{00} - \langle \kappa_g'(s_0)\boldsymbol{b}(s_0), \boldsymbol{v}_0 \rangle/v_{00} + \langle \kappa_g(s_0)\tau_g(s_0)\boldsymbol{n}(s_0), \boldsymbol{v}_0 \rangle/v_{00} - \kappa_g^2(s_0)\tau_g(s_0).$$

In this case, $h_{\boldsymbol{v}_0}(s)$ has an A_4-singularity, then we have

$$\boldsymbol{v}_0 = 2\kappa_g^2(s_0)\tau_g(s_0)\sigma^\pm(s_0)\overline{\boldsymbol{\gamma}}(s_0) + 2\kappa_g(s_0)\tau_g(s_0)\sigma^\pm(s_0)\boldsymbol{n}(s_0) - 2\kappa_g'(s_0)\sigma^\pm(s_0)\boldsymbol{b}(s_0)$$

$$+(2\kappa_g^2(s_0)\tau_g(s_0)\sigma^\pm(s_0) + 2)\boldsymbol{e}_0.$$

Moreover, we have

$$v_{00} = 2\kappa_g^2(s_0)\tau_g(s_0)\sigma^{\pm}(s_0) + 2.$$

Therefore by calculation, we have

$$\det B = \pm \frac{\kappa_g^2(s_0)\tau_g^2(s_0) + \kappa_g'^2(s_0)}{\sqrt{\kappa_g'^2(s_0) + \kappa_g^2(s_0)\tau_g^2(s_0) + \kappa_g^4(s_0)\tau_g^2(s_0)} \pm \kappa_g^2(s_0)\tau_g(s_0)} \neq 0,$$

which completes the proof.   □

We have the following theorem.

Theorem 6.9. *Let $\gamma : I \longrightarrow S_+^3$ be a unit speed curve.*

(A) For the lightcone duals $\overline{LD}_{\overline{\gamma}}^{\pm}$ of $\overline{\gamma}$, we have the following assertions:

(1) Each one of the lightcone duals $\overline{LD}_{\overline{\gamma}}^{\pm}$ of $\overline{\gamma}$ is locally diffeomorphic to $C(2, 3) \times \mathbb{R}^2$ at (s_0, u_0, v_0) if and only if

$$\kappa_g(s_0) \neq 0, \ u_0 = \frac{1}{\kappa_g(s_0)} \text{ and } v_0 \neq \left(\frac{1}{\kappa_g(s_0)}\right)' \frac{1}{\tau_g(s_0)}.$$

In this case, each one of $LF_{\overline{\gamma}}^{\pm}$ is non-singular and each one of Images $\varepsilon_{\overline{\gamma}}^{\pm}$ is empty.

(2) Each one of the lightcone duals $\overline{LD}_{\overline{\gamma}}^{\pm}$ of $\overline{\gamma}$ is locally diffeomorphic to $SW \times \mathbb{R}$ at (s_0, u_0, v_0) if and only if

$$\kappa_g(s_0) \neq 0, \ \tau_g(s_0) \neq 0, \ u_0 = \frac{1}{\kappa_g(s_0)}, \ v_0 = \left(\frac{1}{\kappa_g(s_0)}\right)' \frac{1}{\tau_g(s_0)}, \text{ and } \kappa_S(s_0) \neq 0.$$

In this case, each one of $LF_{\overline{\gamma}}^{\pm}$ is locally diffeomorphic to $C(2, 3, 4) \times \mathbb{R}$ and each one of Images $\varepsilon_{\overline{\gamma}}^{\pm}$ is a regular curve.

(3) Each one of the lightcone duals $\overline{LD}_{\overline{\gamma}}^{\pm}$ of $\overline{\gamma}$ is locally diffeomorphic to BF at (s_0, u_0, v_0) if and only if

$$\kappa_g(s_0) \neq 0, \ \tau_g(s_0) \neq 0, \ u_0 = \frac{1}{\kappa_g(s_0)}, \ v_0 = \left(\frac{1}{\kappa_g(s_0)}\right)' \frac{1}{\tau_g(s_0)}, \ \kappa_S(s_0) = 0, \text{ and } \kappa_S'(s_0) \neq 0.$$

In this case, each one of $LF_{\overline{\gamma}}^{\pm}$ is locally diffeomorphic to $C(BF) \times \mathbb{R}$ and each one of Images $\varepsilon_{\overline{\gamma}}^{\pm}$ is locally diffeomorphic to the projection of the $C(2, 3, 4, 5)$ to S_0^3.

(B) For the lightcone dual LD_{γ} of γ, we have the following assertions:

(1) The lightcone dual LD_{γ} of γ is locally diffeomorphic to $C(2, 3) \times \mathbb{R}^2$ at (s_0, u_0, v_0) if and only if

$$u_0 \neq 2\kappa_g(s_0)\tau_g(s_0)\sigma^{\pm}(s_0) \text{ and } v_0 = \pm\sqrt{4 + 4\kappa_g(s_0)u_0 - u_0^2}.$$

In this case, each one of $LF_{\gamma}^{\pm}$ is non-singular and each one of Images $\varepsilon_{\gamma}^{\pm}$ is empty.

(2) The lightcone dual LD_{γ} of γ is locally diffeomorphic to $SW \times \mathbb{R}$ at (s_0, u_0, v_0) if and only if

$$\kappa_g'^2(s_0) + \kappa_g^2(s_0)\tau_g^2(s_0) \neq 0, \ u_0 = 2\kappa_g(s_0)\tau_g(s_0)\sigma^{\pm}(s_0), \ v_0 = -2\kappa_g'(s_0)\sigma^{\pm}(s_0), \text{ and } \kappa_S(s_0) \neq 0.$$

In this case, each one of $LF_{\gamma}^{\pm}$ is locally diffeomorphic to $C(2, 3, 4) \times \mathbb{R}$ and each one of Images $\varepsilon_{\overline{\gamma}}^{\pm}$ is a regular curve.

(3) The lightcone dual LD_{γ} of γ is locally diffeomorphic to BF at (s_0, u_0, v_0) if and only if

$$\kappa_g'^2(s_0) + \kappa_g^2(s_0)\tau_g^2(s_0) \neq 0, \ u_0 = 2\kappa_g(s_0)\tau_g(s_0)\sigma^{\pm}(s_0), \ v_0 = -2\kappa_g'(s_0)\sigma^{\pm}(s_0),$$

$$\kappa_S(s_0) = 0 \text{ and } \kappa_S'(s_0) \neq 0.$$

In this case, each one of $LF_{\gamma}^{\pm}$ is locally diffeomorphic to $C(BF) \times \mathbb{R}$ and each one of Images $\varepsilon_{\overline{\gamma}}^{\pm}$ *is locally diffeomorphic to the projection of the $C(2, 3, 4, 5)$-cusp to S_0^3.*

Proof. By Propositions 5.1 and 5.2, the discriminant sets of $\overline{H}$ and H are the lightcone duals of $\overline{\gamma}$ and γ, respectively. By Propositions 5.1 and 5.2, both of $\overline{h}_{v_0}$ and h_{v_0} have A_k singularities ($k = 1, 2, 3, 4$), respectively, if and only if the above conditions on the geodesic curvatures and geodesic torsions hold. By Propositions 6.7 and 6.8, $\overline{H}$ and H are versal unfoldings of $\overline{h}_{v_0}$ and h_{v_0} at any point $s_0 \in I$, respectively. We apply Corollary 6.6, so that we have the above assertions. This completes the proof. $\square$

ACKNOWLEDGMENTS

[1] V. I. Arnol'd, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of Differentiable Maps* (Birkhäuser, 1986), Vol. I.

[2] J. W. Bruce and M. C. Romero-Fuster, "Duality and projections of curves and surfaces in 3-space," Q. J. Math. Oxford Ser. **42**, 433–441 (1991).

[3] J. W. Bruce and P. J. Giblin, *Curves and Singularities*, 2nd ed. (Cambridge University Press, 1992).

[4] L. Chen and S. Izumiya, "A mandala of Legendrian dualities for pseudo-spheres in semi-Euclidean space," Proc. Jpn. Acad., Ser. A: Math. Sci. **85**, 49–54 (2009).

[5] S. Izumiya, "Lengendrian dualities and spacelike hypersurfaces in the lightcone," Mosc. Math. J. **9**, 325–357 (2009).

[6] S. Izumiya, "Differential geometry from the viewpoint of Lagrangian or Legendrian singularity theory," in *Singularity Theory*, edited by D. Chéniot *et al.* (World Scientific, 2007), pp. 241–275.

[7] S. Izumiya, Y. Jiang, and D. H. Pei, "Lightcone dualities for curves in the sphere," Q. J. Math. Oxford Ser. **64**, 221–234 (2013).

[8] J. J. Koenderink, *Solid Shape* (The MIT Press, 1990).

[9] H. L. Liu and S. D. Jung, "Hypersurfaces in lightlike cone," J. Geom. Phys. **58**, 913–922 (2008).

[10] T. Nagai, "The Gauss map of a hypersurface in Euclidean sphere and the spherical Legendrian duality," Topol. Appl. **159**, 545–554 (2012).

[11] I. R. Porteous, "Some remarks on duality in S^3," *Geometry and Topology of Caustics*, Banach Center Publications Vol. 50 (Polish Academy of Sciences, Warsaw, 2004), pp. 217–226.

[12] M. C. Romero Fuster, "Sphere stratifications and the Gauss map," Proc. - R. Soc. Edinburgh, Sect. A: Math. **95**, 115–136 (1983).

[13] O. P. Shcherbak, "Projectively dual space curves and Legendre singularities," Sel. Math. Sov. **5**, 391–421 (1986).

[14] F. Schwarz, "Asymptotic expansions of Fourier integrals with light cone singularities," J. Math. Phys. **13**, 1621 (1972).

[15] F. Finster, "Light-cone expansion of the Dirac sea in the presence of chiral and scalar potentials," J. Math. Phys. **41**, 6689 (2000).