Functional Integral Representations and Golden–Thompson Inequalities in Boson–Fermion Systems

Asao Arai
Department of Mathematics, Hokkaido University
Sapporo 060-0810
Japan
E-mail: arai@math.sci.hokudai.ac.jp

August 29, 2013

Abstract

For a general class of boson–fermion Hamiltonians H acting in the tensor product Hilbert space $L^2(\mathbb{R}^n) \otimes \wedge(\mathbb{C}^r)$ of $L^2(\mathbb{R}^n)$ and the fermion Fock space $\wedge(\mathbb{C}^r)$ over $\mathbb{C}^r$ $(n, r \in \mathbb{N})$, we establish, in terms of an n-dimensional conditional oscillator measure, a functional integral representation for the trace $\mathrm{Tr}\,(F \otimes z^{N_{\mathrm{f}}} e^{-tH})$ $(F \in L^\infty(\mathbb{R}^n), z \in \mathbb{C}\backslash\{0\}, t > 0)$, where N_{f} is the fermion number operator on $\wedge(\mathbb{C}^r)$. We prove a Golden–Thompson type inequality for $|\mathrm{Tr}\,(F \otimes z^{N_{\mathrm{f}}} e^{-tH})|$. Also we discuss applications to a model in supersymmetric quantum mechanics and present an improved version of the Golden–Thompson inequality in supersymmetric quantum mechanics given by Klimek and Lesniewski (Lett. Math. Phys. **21** (1991), 237–244). An upper bound for the number of the supersymmetric states is given as well as a sufficient condition for the spontaneous supersymmetry breaking. Moreover, we derive a functional integral representation for the analytical index of a Dirac type operator on $\mathbb{R}^n$ (Witten index) associated with the supersymmetric quantum mechanical model.

Keywords: boson–fermion system, conditional oscillator measure, Dirac operator, functional integral, Golden–Thompson inequality, ground state energy, supersymmetric quantum mechanics.
2010 Mathematics Subject Classification: 81S40, 81Q60, 81Q10.

Contents

1 Introduction

In their interesting paper [9], Klimek and Lesniewski derived a Golden–Thompson inequality for a model in supersymmetric quantum mechanics on the n-dimensional Euclidean vector space $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) | x_j \in \mathbb{R}, j = 1, \ldots, n\}$ with $n \in \mathbb{N}$ (the original versions of Golden–Thompson inequality are given in [7, 18]; for abstract generalizations, see, e.g., [13, p.320]). In the present paper, we improve their result and extend it to a general class of boson–fermion systems with finite degrees of freedom which, as special cases, includes supersymmetric quantum mechanical ones (e.g., [17]). We also discuss some consequences of the main results and application to supersymmetric quantum mechanics.

One of the motivations for the present work comes from trying to improve the Golden–Thompson inequality by Klimek and Lesniewski [9] in two aspects: The one is that it is not best possible in a sense as is explained in Example 1.1 below. The other one, which is related to the first one, lies in that the inequality does not have a form suitable for infinite dimensional extensions. In fact, we originally have been interested in infinite dimensional versions of Golden–Thompson type inequalities which may play important roles in statistical mechanics of quantum fields. As for a general class of Bose fields, Golden–Thompson type inequalities have been established in a previous paper [3]. But it seems to be still left open to establish such inequalities for quantum field models which describe boson–fermion interactions (partial results related to the subject have been obtained in [1, 2] where general mathematical frameworks for supersymmetric quantum field models are formulated and a functional integral representation for the analytical index of an infinite dimensional Dirac type operator is given).

With the motivation mentioned above and from a view-point aiming at generality and unification, it would be more natural to study a general (not necessarily supersymmetric) boson-fermion system with finite degrees of freedom. This is the idea underlying the present work. A boson-fermion system with finite degrees of freedom is interesting not only as a generalization of ordinary quantum mechanical systems with spin, but also as a finite mode approximation of a quantum field model with a Bose field and a Fermi field (e.g., the Witten model in supersymmetric quantum mechanics [20, 21], which is a boson-fermion model with finite degrees of freedom, is heuristically obtained as the so-called dimensional reduction of a supersymmetric quantum field model).

The Hilbert space of the boson–fermion system we consider in this paper is the tensor product Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^n) \otimes \wedge(\mathbb{C}^r) \tag{1.1}$$

of $L^2(\mathbb{R}^n)$ and the fermion Fock space

$$\wedge(\mathbb{C}^r) := \oplus_{p=0}^r \wedge^p (\mathbb{C}^r) = \{\psi = (\psi^{(p)})_{p=0}^r | \psi^{(p)} \in \wedge^p(\mathbb{C}^r), p = 0, 1, \ldots, r\} \tag{1.2}$$

over the r-dimensional complex Hilbert space $\mathbb{C}^r$, where $\wedge^p(\mathbb{C}^r)$ is the p-fold anti-symmetric tensor product of $\mathbb{C}^r$. In what follows, we mainly use the following natural isomorphism:

$$\mathcal{H} \cong L^2(\mathbb{R}^n; \wedge(\mathbb{C}^r)), \tag{1.3}$$

the Hilbert space of $\wedge(\mathbb{C}^r)$-valued square integrable functions on $\mathbb{R}^n$.

To explain in what sense the Golden–Thompson inequality in [9] is unsatisfactory, we first review it briefly. We consider the case $r = n$ as in [9]. We denote the multiplication operator by a Borel measurable function F on $L^2(\mathbb{R}^n)$ by the same symbol F. Let P be a real polynomial of n variables $x_1, \ldots, x_n \in \mathbb{R}$, Δ be the generalized Laplacian acting in $L^2(\mathbb{R}^n)$ and b_j $(j = 1, \ldots, n)$ be the fermion annihilation operator on the fermion Fock space $\wedge(\mathbb{C}^n)$, i.e., it is a linear operator on $\wedge(\mathbb{C}^n)$ whose adjoint b_j^* satisfies

$$(b_j^*\psi)^{(0)} = 0, \quad (b_j^*\psi)^{(p)} = \sqrt{p}A_p(e_j \otimes \psi^{(p-1)}), \quad 1 \le p \le n,\ j = 1, \ldots, n,$$

where A_p is the antisymmetrization operator on the p-fold tensor product $\otimes^p\mathbb{C}^n$ of $\mathbb{C}^n$ and $\{e_j\}_{j=1}^n$ is the standard orthonormal basis of $\mathbb{C}^n$. The operators b_j's satisfy the canonical anti-commutation relations

$$\{b_j, b_k\} = 0, \quad \{b_j, b_k^*\} = \delta_{jk} \quad (j, k = 1, \ldots, n),$$

where $\{A, B\} := AB + BA$ and δ_{jk} is the Kronecker delta. The Hamiltonian H_{KL} of the supersymmetric quantum mechanical model considered in [9] has the following form:

$$H_{\mathrm{KL}} = -\frac{\hbar^2}{2}\Delta - \frac{\hbar}{2}\Delta P + \frac{1}{2}|\nabla P|^2 + \sum_{j,k=1}^n \hbar(\partial_j\partial_k P)b_j^* b_k, \tag{1.4}$$

acting in $\mathcal{H}$ with $r = n$, where $\hbar > 0$ is a parameter denoting physically the Planck constant divided by 2π, $\partial_j := \partial/\partial x_j$ $(j = 1, \ldots, n)$ and $\nabla := (\partial_1, \ldots, \partial_n)$ (in the paper [9], a physical unit system is taken such that $\hbar = 1$, but we make explicit the dependence on $\hbar$ so that the behavior of the classical limit $\hbar \to 0$ may be visible). The Golden–Thompson inequality proved in [9] is as follows:

$$\operatorname{Tr} e^{-tH_{\mathrm{KL}}} \le \frac{1}{(2\pi t)^{n/2}\hbar^n} \int_{\mathbb{R}^n} \det(I + e^{-t\hbar\nabla\otimes\nabla P(x)}) e^{-\frac{t}{2}\left(|\nabla P(x)|^2 - \hbar\Delta P(x)\right)} \mathrm{d}x \tag{1.5}$$

for all $t > 0$ such that the integral on the right hand side is finite, where Tr (resp. det) denotes trace (resp. determinant), I denotes identity, and $\nabla \otimes \nabla P(x)$ ($x \in \mathbb{R}^n$) is the $n \times n$ matrix whose (j, k) component is equal to $\partial_j \partial_k P(x)$ ($j, k = 1, \ldots, n$).

We note that inequality (1.5) is not best possible in the following sense. Namely, in the case where H_{KL} is the Hamiltonian of a supersymmetric quantum harmonic oscillator with n degrees of freedom (Example 1.1 just below), the equality in (1.5) does not hold, i.e., the left hand side is less than the right hand side. For the reader's convenience and for a preparation for later sections, we demonstrate this fact as an example:

Example 1.1 Consider the case where

$$P(x) = \frac{1}{2}\sum_{i=1}^{n} \omega_i x_i^2, \quad x \in \mathbb{R}^n$$

with constants $\omega_i > 0, i = 1, \cdots, n$. Then H_{KL} takes the form

$$H_\omega := H_{\mathrm{b}} + H_{\mathrm{f}} \tag{1.6}$$

with

$$H_{\mathrm{b}} := -\frac{\hbar^2}{2}\Delta - \frac{\hbar}{2}\sum_{j=1}^{n} \omega_j + \frac{1}{2}\sum_{j=1}^{n} \omega_j^2 x_j^2, \tag{1.7}$$

$$H_{\mathrm{f}} := \sum_{j=1}^{n} \hbar\omega_j b_j^* b_j. \tag{1.8}$$

The operator H_{b} (resp. H_{f}) is the Hamiltonian of an n-dimensional quantum harmonic oscillator (resp. a Hamiltonian of n free fermions). The operator H_ω is called the Hamiltonian of a supersymmetric quantum harmonic oscillator with n degrees of freedom.

For a Hilbert space $\mathcal{K}$, we denote by $\mathcal{I}_1(\mathcal{K})$ the set of trace class operators on $\mathcal{K}$.

It follows from the well known spectral property of H_{b} and H_{f} that, for all $t > 0$, $e^{-tH_{\mathrm{b}}} \in \mathcal{I}_1(L^2(\mathbb{R}^n))$ with

$$\mathrm{Tr}\, e^{-tH_{\mathrm{b}}} = \frac{1}{\prod_{j=1}^{n}\left(1 - e^{-t\hbar\omega_j}\right)} \tag{1.9}$$

and

$$\mathrm{Tr}\, e^{-tH_{\mathrm{f}}} = \prod_{j=1}^{n}\left(1 + e^{-t\hbar\omega_j}\right). \tag{1.10}$$

Hence, for all $t > 0$, e^{-tH_ω} is in $\mathcal{I}_1(\mathcal{H})$ and

$$\mathrm{Tr}\, e^{-tH_\omega} = \left(\mathrm{Tr}\, e^{-tH_{\mathrm{b}}}\right)\left(\mathrm{Tr}\, e^{-tH_{\mathrm{f}}}\right) = \prod_{j=1}^{n} \frac{1 + e^{-t\hbar\omega_j}}{1 - e^{-t\hbar\omega_j}} = \prod_{j=1}^{n} \coth\frac{t\hbar\omega_j}{2}.$$

On the other hand, denoting the right hand side of (1.5) by $I_P(t)$, one has

$$I_P(t) \;=\; \frac{1}{(2\pi t)^{n/2}\hbar^n}\int_{\mathbb{R}^n}\left\{\prod_{j=1}^{n}(I + e^{-t\hbar\omega_j})\right\} e^{-t\sum_{j=1}^{n}\omega_j^2 x_j^2/2 + t\hbar\sum_{j=1}^{n}\omega_j/2}\mathrm{d}x$$

$$= \prod_{j=1}^{n} \frac{\cosh \dfrac{t\hbar\omega_j}{2}}{\dfrac{t\hbar\omega_j}{2}}.$$

But $\sinh \chi > \chi$ for all $\chi > 0$, which implies that $\operatorname{Tr} e^{-tH_\omega} < I_P(t)$. Thus, in the present case, the equality in (1.5) does not hold.

As is well known, a free supersymmetric quantum field model (a model describing a quantum system with supersymmetry consisting of a free Bose field and a free Fermi field) may be viewed as an infinite system of a supersymmetric quantum harmonic oscillator with one degree of freedom. An example of the Hamiltonian of such a model is symbolically expressed as H_ω with $n = \infty$. Hence, from a view point of supersymmetric quantum field theory, it would be natural to find a Golden–Thompson type inequality which attains the equality in the case where the Hamiltonian is that of a supersymmetric quantum harmonic oscillator.

To derive (1.5), Klimek and Lesniewski [9] employed, for the boson system, a functional integral representation based on the n-dimensional conditional Wiener measure . But, for our purpose mentioned in the preceding paragraph, it turns out that a conditional oscillator measure is suitable. Thus we use an n-dimensional conditional oscillator measure to represent quantities of the boson system in terms of functional integrals.

The outline of the present paper is as follows. In Section 2 we introduce a general class of boson–fermion systems with finite degrees of freedom with the Hilbert space of each system being $\mathcal{H}$ given by (1.1). The boson system is given by a perturbation of an n-dimensional quantum harmonic oscillator by a potential $V : \mathbb{R}^n \to \mathbb{R}$ so that the Hamiltonian of the boson system takes the form

$$H_{\mathrm{b},V} := H_{\mathrm{b}} + V \tag{1.11}$$

and the interaction of the boson system with a fermion system is induced by an $r \times r$ matrix-valued function $\mathbb{U}$ on $\mathbb{R}^n$. We prove some basic facts on the Hamiltonian $H(V, \mathbb{U})$ of the boson–fermion system, including its self-adjointness and the nuclearity of the heat semi-group $e^{-tH(V,\mathbb{U})}$ with $t > 0$ (Lemma 2.3).

In Section 3 we introduce an n-dimensional conditional oscillator process which allows one to derive a functional integral representation for the trace of quantities formed out of $e^{-sH_{\mathrm{b}}}$ $(s > 0)$ and bounded multiplication operators (Lemma 3.1). As an application of such functional integral representations, we prove a Golden–Thompson type inequality for $\operatorname{Tr} e^{-tH_{\mathrm{b},V}}$ (Theorem 3.5). This inequality attains the equality in the case $V = 0$ (the quantum harmonic oscillator case). In this sense, it is better than the standard Golden–Thompson inequality for $\operatorname{Tr} e^{-t(-\hbar^2\Delta/2+V)}$ (e.g., [15, Theorem 9.2]), which does not attain the equality for the quantum harmonic oscillator case. Moreover, the limit $\omega_j \to 0$ $(j = 1, \ldots, n)$ of the new Golden–Thompson type inequality yields the standard Golden–Thompson inequality (Corollary 3.6). We also derive functional integral representations of traces of quantities formed out of $e^{-sH_{\mathrm{b},V}}$ $(s > 0)$ and bounded multiplication operators on $L^2(\mathbb{R}^n)$ (Theorem 3.8). Also some consequences of the derived Golden–Thompson inequality are presented: Corollary 3.9 (A lower bound for the Helmholtz free-energy function of $H_{\mathrm{b},V}$), Corollary 3.11 (an upper bound for the number of eigenvalues of $H_{\mathrm{b},V}$ less than or equal to a given number) and Corollary 3.12 (an estimate for the lowest eigenvalue of $H_{\mathrm{b},V}$ from below).

In Section 4 we establish a functional integral representation for $\mathrm{Tr}\, F z^{N_\mathrm{f}} e^{-tH(V,\mathbb{U})}$ ($t > 0$) in terms of the conditional oscillator measure (Theorems 4.2 and 4.6), where F is a bounded multiplication operator on $L^2(\mathbb{R}^n)$, $z \in \mathbb{C} \setminus \{0\}$ and $N_\mathrm{f} = \sum_{k=1}^r b_k^* b_k$ is the fermion number operator.

As a corollary to the result in Section 4, we derive, in Section 5, a Golden–Thompson type inequality for the boson–fermion system under consideration (Theorem 5.1). As desired, this inequality attains the equality in the case where the quantum system consists of an n-dimensional quantum harmonic oscillator and r free fermions (see Remark 5.2)). As in the case of the boson system, we obtain results on the following aspects (Corollary 5.3): A lower bound for the Helmholtz free-energy function of $H(V,\mathbb{U})$, an upper bound for the number of eigenvalues of $H(V,\mathbb{U})$ less than or equal to a given number and an estimate of the lowest eigenvalue of $H(V,\mathbb{U})$ from below.

Section 6 is devoted to applications of the results obtained in the preceding sections to a model of supersymmetric quantum mechanics with Hamiltonian H_{SS}. We derive a Golden–Thompson inequality and a functional integral representation for $|\mathrm{Tr}\, F z^{N_\mathrm{f}} e^{-tH_{\mathrm{SS}}}|$ with $z \in \mathbb{C} \setminus \{0\}$ and F being a bounded multiplication operator on $L^2(\mathbb{R}^n)$ (Theorem 6.3). The Golden–Thompson inequality for $\mathrm{Tr}\, e^{-tH_{\mathrm{SS}}}$ improves (1.5). As applications, we prove an inequality for the number of supersymmetric states (Corollary 6.5) and give a sufficient condition for the spontaneous supersymmetry breaking (Corollary 6.6).

In the last section, we consider a Dirac operator associated with the model in Section 6 and prove a formula for the analytical index of it (Witten index) in terms of a functional integral (Theorem 7.2).

In Appendix, we collect and prove some facts in operator theory used in the text of the present paper. They may have independent interests.

2 A General Class of Boson–Fermion Hamiltonians

For a linear operator T, we denote its domain by $D(T)$. If T is closable, we denote its closure by $\overline{T}$. For a subspace $\mathcal{D} \subset D(T)$, $T \upharpoonright \mathcal{D}$ denotes the restriction of T to $\mathcal{D}$.

We denote by D_j the generalized partial differential operator in the variable x_j and set

$$p_j := -i\hbar D_j \tag{2.1}$$

as an operator on $L^2(\mathbb{R}^n)$ (i is the imaginary unit) with $D(p_j) = D(D_j) = \{\Psi \in L^2(\mathbb{R}^n) | D_j\Psi \in L^2(\mathbb{R}^n)\}$, which physically denotes the j-th momentum operator on $L^2(\mathbb{R}^n)$. As is well known, p_j is self-adjoint and essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, the set of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. Let $\omega_j > 0$ ($j = 1, \ldots, n$). Then one can define the following closed operators:

$$a_j := \frac{i}{\sqrt{2\hbar\omega_j}} \overline{(p_j - i\omega_j x_j) \upharpoonright C_0^\infty(\mathbb{R}^n)}, \quad j = 1, \ldots, n, \tag{2.2}$$

which is called the annihilation operator of the j-th boson (the adjoint a_j^* of a_j is called the creation operator of the j-th boson). The operators $\{a_1, \ldots, a_n\}$ obey the canonical commutation relations

$$[a_j, a_k^*] = \delta_{jk}, \quad [a_j, a_k] = 0 \quad (j, k = 1, \ldots, n)$$

on $C_0^\infty(\mathbb{R}^n)$, where $[A, B] := AB - BA$.

In terms of a_j ($j = 1, \ldots, n$), the Hamiltonian $H_{\rm b}$ defined by (1.7) is written as

$$H_{\rm b} = \sum_{j=1}^{n} \hbar\omega_j a_j^* a_j. \tag{2.3}$$

We fix a real-valued Borel measurable function V on $\mathbb{R}^n$ which is finite a.e (almost everywhere) with respect to the Lebesgue measure on $\mathbb{R}^n$. We take $H_{{\rm b},V}$ defined by (1.11) as a bosonic Hamiltonian. We assume the following:

(A.1) $H_{{\rm b},V}$ is self-adjoint and bounded below.

(A.1′) For some $t_0 > 0$, $e^{-t_0 H_{{\rm b},V}}$ is in $\mathcal{J}_1(L^2(\mathbb{R}^n))$.

Remark 2.1 (i) If one treats only a bosonic theory, then the self-adjointness of $H_{{\rm b},V}$ in (A.1) may be weakened to the essential self-adjointness of $H_{{\rm b},V}$ on $C_0^\infty(\mathbb{R}^n)$ (see Section 3). But, in the boson-fermion theory we consider below, we use the self-adjointness of $H_{{\rm b},V}$.

(ii) In (A.1), V is not necessarily bounded below (if V is bounded below, then it is obvious that $H_{{\rm b},V}$ is bounded below). Also the following fact should be kept in mind. Suppose that V is bounded below and satisfies $\int_{|x|\le R} |V(x)|^2 {\rm d}x < \infty$ for all $R > 0$ (i.e., $V \in L^2_{\rm loc}(\mathbb{R}^n)$) . Then the function: $x \mapsto \sum_{i=1}^n \omega_i^2 x_i^2/2 + V(x)$ is in $L^2_{\rm loc}(\mathbb{R}^n)$ and bounded below. Hence, by a general theorem [11, Theorem X.28], $H_{{\rm b},V}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and bounded below. As already mentioned in Introduction, $e^{-tH_{\rm b}}$ is in $\mathcal{J}_1(L^2(\mathbb{R}^n))$ for all $t > 0$. In the present case, e^{-tV} is bounded. Hence $e^{-tH_{\rm b}/2}e^{-tV}e^{-tH_{\rm b}/2}$ is in $\mathcal{J}_1(L^2(\mathbb{R}^n))$. Therefore, by a general theorem [13, p.320, Corollary], $e^{-t\overline{H}_{{\rm b},V}}$ is trace class on $L^2(\mathbb{R}^n)$. Hence (A.1′) holds with $H_{{\rm b},V}$ replaced by $\overline{H}_{{\rm b},V}$ and $t_0 > 0$ arbitrary. But it may depend on V if $\overline{H}_{{\rm b},V} = H_{{\rm b},V}$, i.e., $H_{{\rm b},V}$ is self-adjoint (we shall use the closedness of $H_{{\rm b},V}$ below).

(iii) In general, if a self-adjoint operator A on a Hilbert space $\mathcal{K}$ satisfies $e^{-t_0 A} \in \mathcal{J}_1(\mathcal{K})$ for some $t_0 > 0$, then A is bounded below and, for all $t \ge t_0$, $e^{-tA} \in \mathcal{J}_1(\mathcal{K})$. Hence (A.1′) implies that $H_{{\rm b},V}$ is bounded below and, for all $t \ge t_0$, $e^{-tH_{{\rm b},V}} \in \mathcal{J}_1(L^2(\mathbb{R}^n))$ (hence, if (A.1′) is assumed, then the condition of the lower-boundedness of $H_{{\rm b},V}$ in (A.1) is not needed).

For $j, k = 1, \ldots, r$, let $U_{jk} \in L^2_{\rm loc}(\mathbb{R}^n)$ such that $U_{jk}(x)^* = U_{kj}(x)$ for all $j, k = 1, \ldots, r$ and a.e.$x \in \mathbb{R}^n$ (for a complex number $z \in \mathbb{C}$, z^* denotes the complex conjugate of $z \in \mathbb{C}$) and define an $r \times r$ Hermitian matrix-valued function:

$$\mathbb{U} := (U_{jk})_{j,k=1,\ldots,r}. \tag{2.4}$$

Then the operator

$$H_{\mathrm{f},\mathbb{U}} := \sum_{j,k=1}^{r} U_{jk} b_j^* b_k = \int_{\mathbb{R}^n}^{\oplus} \sum_{j,k=1}^{r} U_{jk}(x) b_j^* b_k \mathrm{d}x, \tag{2.5}$$

acting in $\mathcal{H}$ is symmetric with $D(H_{\mathrm{f},\mathbb{U}}) \supset C_0^\infty(\mathbb{R}^n)\hat{\otimes} \wedge (\mathbb{C}^r)$ ($\hat{\otimes}$ means algebraic tensor product), where, in the second equality of (2.5), we have used the natural identification $\mathcal{H} \cong \int_{\mathbb{R}^n}^{\oplus} \wedge(\mathbb{C}^r)\mathrm{d}x$, the constant fibre direct integral with fiber $\wedge(\mathbb{C}^r)$ and base space $(\mathbb{R}^n, \mathrm{d}x)$ (e.g., [13, pp.280–286]). We take this operator as an operator describing an interaction of n bosons with r fermions.

Remark 2.2 If each U_{jk} is essentially bounded on $\mathbb{R}^n$, then $H_{\mathrm{f},\mathbb{U}}$ is a bounded self-adjoint operator on $\mathcal{H}$. In general, if the operator-valued function $(\sum_{j,k=1}^r U_{jk}(\cdot)b_j^*b_k + i)^{-1}$ is measurable (e.g., the case where each U_{jk} is continuous on $\mathbb{R}^n$), then it follows from a general theorem [13, Theorem XIII.85] that $H_{\mathrm{f},\mathbb{U}}$ is self-adjoint.

The total Hamiltonian of the boson–fermion system we consider is given by

$$H(V, \mathbb{U}) := H_{\mathrm{b},V} + H_{\mathrm{f},\mathbb{U}}. \tag{2.6}$$

We call it a boson–fermion Hamiltonian.

In addition to (A.1) and (A.1′), we assume the following too:

(A.2) There exist constants $\alpha \in [0, 1)$ and $a, b > 0$ such that

$$|U_{jk}(x)|^2 \le a|V(x)|^{2\alpha} + b, \quad x \in \mathbb{R}^n, j, k = 1, \dots, r. \tag{2.7}$$

Lemma 2.3

(i) *Assume (A.1) and (A.2). Then $H(V, \mathbb{U})$ is self-adjoint and bounded below.*

(ii) *Assume (A.1), (A.1′) and (A.2). Then, for all $t > t_0$, $e^{-tH(V,\mathbb{U})}$ is in $\mathcal{I}_1(\mathcal{H})$.*

Proof. (i) For a bounded linear operator T on a Hilbert space, we denote by $\|T\|$ the operator norm of T. It is easy to see that

$$\|b_j\| = 1, \quad \|b_j^*\| = 1, \quad j = 1, \dots, r. \tag{2.8}$$

By this fact and (2.7), we have for all $\Psi \in D(V)$

$$\|H_{\mathrm{f},\mathbb{U}}\Psi\| \le \sum_{j,k=1}^{r} \|U_{jk}\Psi\| \le r^2(\sqrt{a}\||V|^\alpha\Psi\| + \sqrt{b}\|\Psi\|).$$

Since $0 \le \alpha < 1$, for every $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that

$$\||V|^\alpha\Psi\| \le \varepsilon\|V\Psi\| + b_\varepsilon\|\Psi\|.$$

Let ε_0 be the infimum of the spectrum of $H_{\mathrm{b},V}$. Then $H_{\mathrm{b},V} - \varepsilon_0 \geq 0$. By the closedness of $H_{\mathrm{b},V}$, there exists a constant $c > 0$ such that

$$\|V\Psi\| \leq c\|(H_{\mathrm{b},V} - \varepsilon_0 + 1)\Psi\|, \quad \Psi \in D(H_{\mathrm{b},V}). \tag{2.9}$$

Thus $H_{\mathrm{f},\mathbb{U}}$ is infinitesimally small with respect to $H_{\mathrm{b},V}$. Hence, by the Kato–Rellich theorem (e.g., [11, Theorem X.12]), $H(V,\mathbb{U})$ is self-adjoint and bounded below.

(ii) Let $0 < \kappa < 1$. Then we have

$$H(V,\mathbb{U}) = (1-\kappa)H_{\mathrm{b},V} + \kappa H_{\mathrm{b},V} + H_{\mathrm{f},\mathbb{U}}.$$

Since $H_{\mathrm{f},\mathbb{U}}$ is infinitesimally small with respect to $H_{\mathrm{b},V}$ as is shown in part (i), it follows that $H_{\mathrm{f},\mathbb{U}}$ is infinitesimally small with respect to $\kappa H_{\mathrm{b},V}$. Hence, by the Kato–Rellich theorem again, $\kappa H_{\mathrm{b},V} + H_{\mathrm{f},\mathbb{U}}$ is self-adjoint and bounded below. By Remark 2.1-(iii), $e^{-t(1-\kappa)H_{\mathrm{b},V}}$ is in $\mathcal{I}_1(\mathcal{H})$ for all $t \geq t_0/(1-\kappa)$. Hence, by a general theorem [13, p.320, Corollary], $e^{-tH(V,\mathbb{U})}$ is in $\mathcal{I}_1(\mathcal{H})$ for all $t \geq t_0/(1-\kappa)$. Since $\kappa \in (0,1)$ is arbitrary, it follows that $e^{-tH(V,\mathbb{U})} \in \mathcal{I}_1(\mathcal{H})$ for all $t > t_0$. ∎

3 Functional Integral Representations for the Boson System

In this section, for a class of V such that $H_{\mathrm{b},V}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and bounded below, we shall prove that, for all $t > 0$, $e^{-t\overline{H}_{\mathrm{b},V}}$ is in $\mathcal{I}_1(\mathbb{R}^n)$ and derive, in terms of functional integrals, trace formulae for quantities formed out of $e^{-s\overline{H}_{\mathrm{b},V}}$ $(s > 0)$ and bounded multiplication operators on $L^2(\mathbb{R}^n)$ as well as a Golden–Thompson inequality. Also we discuss some consequences of the Golden–Thompson inequality.

As mentioned in Introduction, we use a functional integration based on an n-dimensional conditional oscillator process, which is constructed from an n-dimensional oscillator process [15, pp.34–38]. This is a point different from the methods in [9] where the n-dimensional conditional Wiener process is used.

3.1 Trace formulae based on a conditional oscillator measure

For $t > 0$ and $j = 1, \ldots, n$, we define a function $K_t^{(j)}$ on $\mathbb{R}\times\mathbb{R}$ by

$$K_t^{(j)}(x_j, y_j) := \sqrt{\frac{\omega_j}{2\pi\hbar}}\frac{e^{\hbar\omega_j t/2}}{\sqrt{\sinh\hbar\omega_j t}}\exp\left(-\frac{\omega_j}{2\hbar}(x_j^2 + y_j^2)\coth\hbar\omega_j t + \frac{\omega_j}{\hbar\sinh\hbar\omega_j t}x_j y_j\right),$$
$$(x_j, y_j) \in \mathbb{R}\times\mathbb{R}. \tag{3.1}$$

For convenience, we set

$$K_0^{(j)}(x_j, y_j) := \delta(x_j - y_j), \quad j = 1, \ldots, n, \tag{3.2}$$

the delta distribution on $\mathbb{R}\times\mathbb{R}$.

We define

$$K_t(x,y) = \prod_{j=1}^{n} K_t^{(j)}(x_j, y_j), \quad t \geq 0, x, y \in \mathbb{R}^n. \tag{3.3}$$

It is obvious that $K_t(\cdot,\cdot)$ is a symmetric function on $\mathbb{R}^n \times \mathbb{R}^n$:

$$K_t(x,y) = K_t(y,x), \quad x, y \in \mathbb{R}^n, t > 0. \tag{3.4}$$

It follows from a well known formula for the integral kernel of e^{-tH_b} with $n = 1$ (e.g., [15, pp.37–38], [6, Theorem 1.5.10]) that e^{-tH_b} is an integral operator with an integral kernel equal to $K_t(x,y)$:

$$(e^{-tH_\mathrm{b}} f)(x) = \int_{\mathbb{R}^n} K_t(x,y) f(y) dy, \quad f \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n, t > 0. \tag{3.5}$$

We already know that, for all $t > 0$, e^{-tH_b} is a positive trace class operator on $L^2(\mathbb{R}^n)$. It is easy to see that, for all $t > 0$, $K_t(x,y)$ is continuous in $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Hence, by a general fact (e.g., [12, p.65, Lemma]), we have

$$\mathrm{Tr}\, e^{-tH_\mathrm{b}} = \int_{\mathbb{R}^n} K_t(x,x) \mathrm{d}x, \quad t > 0. \tag{3.6}$$

This can be shown also by computing the right hand side explicitly and using (1.9).

For each $a, c \in \mathbb{R}^n$ and $t > 0$, there exist random variables $\{q(s) | s \in [0,t]\}$ such that, for all $m \in \mathbb{N}$, the joint distribution for $(q(s_1), \cdots, q(s_m))$ $(0 \leq s_1 \leq s_2 \leq \cdots \leq s_m \leq t)$ is

$$K_t(a,c)^{-1} K_{s_1}(a, x_1) K_{s_2 - s_1}(x_1, x_2) \cdots K_{s_m - s_{m-1}}(x_{m-1}, x_m) K_{t - s_m}(x_m, c) \mathrm{d}x_1 \cdots \mathrm{d}x_m.$$

We denote by $P_{a,c;t}$ the corresponding probability measure and define a finite measure $\mu_{a,c;t}$ by

$$\mathrm{d}\mu_{a,c;t} := K_t(a,c) \mathrm{d}P_{a,c;t}. \tag{3.7}$$

We call $\{q(s) | s \in [0,t]\}$ a conditional oscillator process associated with H_b and $\mu_{a,c;t}$ its conditional oscillator measure. Note that

$$\int 1 \, \mathrm{d}\mu_{x,y;t} = K_t(x,y), \quad x, y \in \mathbb{R}^n.$$

For a complex Hilbert space $\mathcal{K}$, we denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product (linear in the second variable) and norm of $\mathcal{K}$ respectively. We denote by $L^\infty(\mathbb{R}^n)$ the set of essentially bounded Borel measurable functions on $\mathbb{R}^n$ and by $\|f\|_\infty$ the essential supremum of $|f|$.

Lemma 3.1 *Let* $0 < s_1 < \cdots < s_m < t$ *and* $f_j \in L^\infty(\mathbb{R}^n)$ $(j = 1, \ldots, m)$. *Then* $e^{-s_1 H_\mathrm{b}} f_1 e^{-(s_2 - s_1) H_\mathrm{b}} f_2 \cdots f_m e^{-(t - s_m) H_\mathrm{b}}$ *is in* $\mathcal{I}_1(L^2(\mathbb{R}^n))$ *and*

$$\mathrm{Tr}\left(e^{-s_1 H_\mathrm{b}} f_1 e^{-(s_2 - s_1) H_\mathrm{b}} f_2 \cdots f_m e^{-(t - s_m) H_\mathrm{b}}\right) = \int_{\mathbb{R}^n} \mathrm{d}x \left(\int f_1(q(s_1)) \cdots f_m(q(s_m)) \mathrm{d}\mu_{x,x;t} \right). \tag{3.8}$$

Proof. Since $e^{-sH_{\mathrm{b}}}$ is trace class for all $s>0$ and each f_j is bounded as a multiplication operator on $L^2(\mathbb{R}^n)$, the operator

$$M := e^{-s_1 H_{\mathrm{b}}} f_1 e^{-(s_2-s_1)H_{\mathrm{b}}} f_2 \cdots f_m e^{-(t-s_m)H_{\mathrm{b}}}$$

is trace class on $L^2(\mathbb{R}^n)$. By (3.5), M is an integral operator with integral kernel

$$M(x,y) = \int_{\mathbb{R}^{nm}} K_{s_1}(x,x_1)K_{s_2-s_1}(x_1,x_2)\cdots K_{t-s_m}(x_m,y) f_1(x_1)\cdots f_m(x_m)\mathrm{d}x_1\cdots \mathrm{d}x_m.$$

We note that M is not necessarily a nonnegative operator. Hence one can not immediately conclude that the heuristic form "$\mathrm{Tr}\, M = \int_{\mathbb{R}^n} M(x,x)\mathrm{d}x$" is true (cf. [12, pp.65–66, Lemma]). Thus we take another route. Let $\{g_\ell\}_{\ell=1}^\infty$ be a complete orthonormal system of $L^2(\mathbb{R}^n)$. Then

$$\mathrm{Tr}\, M = \sum_{\ell=1}^\infty \langle g_\ell, M g_\ell\rangle = \sum_{\ell=1}^\infty \int_{\mathbb{R}^n} \mathrm{d}x g_\ell(x)^* \left(\int_{\mathbb{R}^n} M(x,y) g_\ell(y)\mathrm{d}y\right).$$

We have

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} |g_\ell(x)^* M(x,y) g_\ell(y)|\mathrm{d}x\mathrm{d}y\\
&\le \int_{(\mathbb{R}^n)^{m+2}} |g_\ell(x)^* K_{s_1}(x,x_1)K_{s_2-s_1}(x_1,x_2)\cdots K_{t-s_m}(x_m,y) f_1(x_1)\cdots f_m(x_m) g_\ell(y)|\\
&\qquad\qquad \times \mathrm{d}x\mathrm{d}x_1\cdots\mathrm{d}x_m\mathrm{d}y\\
&\le \left(\prod_{j=1}^m \|f_j\|_\infty\right)\left\langle |g_\ell|, e^{-tH_{\mathrm{b}}}|g_\ell|\right\rangle < \infty.
\end{aligned}$$

Hence, by Fubini's theorem, we have

$$\begin{aligned}
\mathrm{Tr}\, M &= \lim_{N\to\infty}\int_{\mathbb{R}^{nm}} F_N(x_1,x_m) K_{s_2-s_1}(x_1,x_2)\cdots K_{s_m-s_{m-1}}(x_{m-1},x_m)\\
&\qquad \times f_1(x_1)\cdots f_m(x_m)\mathrm{d}x_1\cdots\mathrm{d}x_m,
\end{aligned}$$

where

$$F_N(x_1,x_m) := \sum_{\ell=1}^N \langle g_\ell, K_{s_1}(\cdot,x_1)\rangle \langle K_{t-s_m}(x_m,\cdot), g_\ell\rangle.$$

By the Schwarz inequality with respect to the sum $\sum_{\ell=1}^N$ and the Bessel inequality, we have

$$|F_N(x_1,x_m)| \le \|K_{s_1}(\cdot,x_1)\|\|K_{t-s_m}(x_m,\cdot)\| = \sqrt{K_{2s_1}(x_1,x_1)}\sqrt{K_{2(t-s_m)}(x_m,x_m)},$$

where we have used (3.4) and

$$\int_{\mathbb{R}^n} K_s(x,y)K_{s'}(y,z)\mathrm{d}y = K_{s+s'}(x,z) \quad (s,s'>0,\ x,z\in\mathbb{R}^n). \tag{3.9}$$

Hence

$$
\begin{aligned}
&|F_N(x_1,x_m)K_{s_2-s_1}(x_1,x_2)\cdots K_{s_m-s_{m-1}}(x_{m-1},x_m)f_1(x_1)\cdots f_m(x_m)| \\
&\le \left(\prod_{j=1}^m \|f_j\|_\infty\right)\sqrt{K_{2s_1}(x_1,x_1)}\sqrt{K_{2(t-s_m)}(x_m,x_m)} \\
&\qquad\qquad\qquad \times K_{s_2-s_1}(x_1,x_2)\cdots K_{s_m-s_{m-1}}(x_{m-1},x_m).
\end{aligned}
$$

Using (3.9), we have

$$
\begin{aligned}
&\int_{(\mathbb{R}^n)^m}\sqrt{K_{2s_1}(x_1,x_1)}\sqrt{K_{2(t-s_m)}(x_m,x_m)} \\
&\qquad\qquad \times K_{s_2-s_1}(x_1,x_2)\cdots K_{s_m-s_{m-1}}(x_{m-1},x_m)\mathrm{d}x_1\cdots\mathrm{d}x_m \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n}\sqrt{K_{2s_1}(x_1,x_1)}\sqrt{K_{2(t-s_m)}(x_m,x_m)}K_{s_m-s_1}(x_1,x_m)\mathrm{d}x_1\mathrm{d}x_m,
\end{aligned}
$$

which is finite, because the function $h_s : x \mapsto \sqrt{K_s(x,x)}$ is in $L^2(\mathbb{R}^n)$ (note that the right hand side is written $\left\langle h_{2s_1}, e^{-(s_m-s_1)H_\mathrm{b}}h_{2(t-s_m)}\right\rangle$). Moreover, we have

$$\lim_{N\to\infty}F_N(x_1,x_m) = \langle K_{t-s_m}(x_m,\cdot),K_{s_1}(\cdot,x_1)\rangle = \int_{\mathbb{R}^n}K_{s_1}(x,x_1)K_{t-s_m}(x_m,x)\mathrm{d}x.$$

Thus, by the Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
\operatorname{Tr} M &= \int_{\mathbb{R}^{nm}}\left(\int_{\mathbb{R}^n}K_{s_1}(x,x_1)K_{t-s_m}(x_m,x)\mathrm{d}x\right)K_{s_2-s_1}(x_1,x_2)\cdots K_{s_m-s_{m-1}}(x_{m-1},x_m) \\
&\qquad\qquad \times f_1(x_1)\cdots f_m(x_m)\mathrm{d}x_1\cdots\mathrm{d}x_m.
\end{aligned}
$$

Using Fubini's theorem again, we have

$$
\begin{aligned}
\operatorname{Tr} M &= \int_{\mathbb{R}^n}\mathrm{d}x\int_{\mathbb{R}^{nm}}K_{s_1}(x,x_1)f_1(x_1)K_{s_2-s_1}(x_1,x_2)f_2(x_2)\cdots K_{s_m-s_{m-1}}(x_{m-1},x_m) \\
&\qquad\qquad \times f_m(x_m)K_{t-s_m}(x_m,x)\mathrm{d}x_1\cdots\mathrm{d}x_m.
\end{aligned}
$$

The right hand side is equal to that of (3.8). Thus (3.8) holds. ∎

Lemma 3.2 *Suppose that V is in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ with*

$$V_0 := \operatorname{ess.inf}_{x\in\mathbb{R}^n}V(x) > -\infty. \tag{3.10}$$

Then $H_{\mathrm{b},V}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and bounded below. Moreover, for all $t>0$, $e^{-t\overline{H}_{\mathrm{b},V}}$ is in $\mathcal{I}_1(L^2(\mathbb{R}^n))$ and

$$\operatorname{Tr} e^{-t\overline{H}_{\mathrm{b},V}} \le e^{-tV_0}\operatorname{Tr} e^{-tH_\mathrm{b}}. \tag{3.11}$$

Proof. The statements except (3.11) have already been shown in Remark 2.1-(ii). By the generalized Golden-Thompson inequality [13, p.320, Corollary], we have for all $t>0$

$$\operatorname{Tr} e^{-t\overline{H}_{\mathrm{b},V}} \le \operatorname{Tr}\left(e^{-tH_\mathrm{b}/2}e^{-tV}e^{-tH_\mathrm{b}/2}\right),$$

which, together with the fact that $e^{-tV}\le e^{-tV_0}$, implies (3.11). ∎

Lemma 3.3 *Suppose that $V_1, \dots, V_{m+1}$ ($m \in \mathbb{N}$) are in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and bounded below. Let $0 < s_1 < \cdots < s_m < t$ and $f_j \in L^\infty(\mathbb{R}^n)$ ($j = 1, \dots, m$). Then*

$$e^{-s_1 \overline{H}_{\mathrm{b},V_1}} f_1 e^{-(s_2-s_1)\overline{H}_{\mathrm{b},V_2}} f_2 \cdots f_m e^{-(t-s_m)\overline{H}_{\mathrm{b},V_{m+1}}}$$

is in $\mathfrak{I}_1(L^2(\mathbb{R}^n))$ and

$$\begin{aligned}&\mathrm{Tr}\left(e^{-s_1 \overline{H}_{\mathrm{b},V_1}} f_1 e^{-(s_2-s_1)\overline{H}_{\mathrm{b},V_2}} f_2 \cdots f_m e^{-(t-s_m)\overline{H}_{\mathrm{b},V_m}}\right)\\ &= \int_{\mathbb{R}^n} \mathrm{d}x \left(\int f_1(q(s_1)) \cdots f_m(q(s_m)) e^{-\sum_{j=1}^{m+1} \int_{s_{j-1}}^{s_j} V_j(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t} \right), \qquad (3.12)\end{aligned}$$

where $s_0 = 0, s_{m+1} = t$.

Proof. We prove the lemma only for the case where $V_1, \dots, V_m$ are continuous. The extension to the case where $V_j \in L^2_{\mathrm{loc}}(\mathbb{R}^)$ ($j = 1, \dots, m$) is routine work (e.g., see [15, p.51]).

By Lemma 3.2, $e^{-s_1 \overline{H}_{\mathrm{b},V_1}}$ is in $\mathfrak{I}_1(L^2(\mathbb{R}^n))$ and $f_1 e^{-(s_2-s_1)\overline{H}_{\mathrm{b},V_2}} f_2 \cdots f_m e^{-(t-s_m)\overline{H}_{\mathrm{b},V_m}}$ is bounded. Hence the first statement of the lemma follows.

By the Trotter product formula (e.g., [10, Theorem VIII.31]), we have

$$e^{-t\overline{H}_{\mathrm{b},V_j}} = \text{s-}\lim_{N\to\infty} A_N^j(t), \quad t > 0$$

with

$$A_N^j(t) := \left(e^{-tH_{\mathrm{b}}/N} e^{-tV_j/N}\right)^N,$$

where s- lim means strong limit. Let L be the left hand side of (3.12). Then, by Proposition B.1 in Appendix, we have

$$L = \lim_{N_1\to\infty} \cdots \lim_{N_{m+1}\to\infty} \mathrm{Tr}\left(A^1_{N_1}(s_1) f_1 A^2_{N_2}(s_2 - s_1) f_2 \cdots A^m_{N_m}(s_m - s_{m-1}) f_m A^{m+1}_{N_{m+1}}(t - s_m)\right).$$

By Lemma 3.1, we have

$$\begin{aligned}&\mathrm{Tr}\left(A^1_{N_1}(s_1) f_1 A^2_{N_2}(s_2 - s_1) f_2 \cdots A^m_{N_m}(s_m - s_{m-1}) f_m A^{m+1}_{N_{m+1}}(t - s_m)\right)\\ &= \int_{\mathbb{R}^n} \mathrm{d}x \int f_1(q(s_1)) \cdots f_m(q(s_m)) \exp\left(-\frac{s_1}{N_1} \sum_{k=1}^{N_1} V_1(q(ks_1/N_1))\right)\\ &\exp\left(-\frac{(s_2-s_1)}{N_2} \sum_{k=1}^{N_2} V_2(q(s_1 + k(s_2-s_1)/N_2))\right)\\ &\cdots \exp\left(-\frac{(t-s_m)}{N_{m+1}} \sum_{k=1}^{N_{m+1}} V_{m+1}(q(s_m + k(t-s_m)/N_{m+1}))\right) \mathrm{d}\mu_{x,x;t} \qquad (3.13)\end{aligned}$$

The integrand on the right hand side is bounded by $\|f_1\|_\infty \cdots \|f_m\|_\infty e^{-\sum_{j=1}^{m+1}(s_j - s_{j-1})c_j}$ with $c_j := \inf_{x\in\mathbb{R}^n} V_j(x)$. Hence, by the Lebesgue dominated convergence theorem, we obtain

$$L = \int_{\mathbb{R}^n} \mathrm{d}x \int f_1(q(s_1)) \cdots f_m(q(s_m)) e^{-\int_0^{s_1} V_1(q(s))\mathrm{d}s} e^{-\int_{s_1}^{s_2} V_2(q(s))\mathrm{d}s} \cdots e^{-\int_{s_m}^{t} V_{m+1}(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t}.$$

Thus (3.12) follows. ∎

3.2 Golden–Thompson inequality and generalization of trace formulae

In what follows, we take an additional assumption:

(A.3) The function V is in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and, for all $t > 0$,

$$\int_{\mathbb{R}^n} K_t(x,x)e^{-tV(x)}\mathrm{d}x < \infty. \tag{3.14}$$

We remark that, in (A.3), V is not necessarily bounded below. If V is bounded below, then (3.14) holds for all $t > 0$.

Lemma 3.4 *Under Assumption (A.3), the following inequality holds:*

$$\int_{\mathbb{R}^n} \mathrm{d}x \int e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t} \le \int_{\mathbb{R}^n} K_t(x,x)e^{-tV(x)}\mathrm{d}x. \tag{3.15}$$

Proof. By Jensen's inequality, we have

$$e^{-\int_0^t V(q(s))\mathrm{d}s} \le \frac{1}{t}\int_0^t e^{-tV(q(s))}\mathrm{d}s.$$

Since $e^{-tV(q(s))}$ is a positive function, one can use Fubini's theorem to obtain

$$\begin{aligned} \int e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t} &\le \frac{1}{t}\int_0^t \mathrm{d}s \int e^{-tV(q(s))}d\mu_{x,x;t} \\ &= \frac{1}{t}\int_0^t \mathrm{d}s \int_{\mathbb{R}^n} e^{-tV(x_1)}K_s(x,x_1)K_{t-s}(x_1,x)\mathrm{d}x_1. \end{aligned}$$

Hence, using (3.9) and Fubini's theorem again, we obtain

$$\int_{\mathbb{R}^n} \mathrm{d}x \int e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t} \le \int_{\mathbb{R}^n} e^{-tV(x_1)}K_t(x_1,x_1)\mathrm{d}x_1.$$

Thus (3.15) holds. ∎

As we shall show below, as far as we treat only a bosonic theory, we do not need (A.1). The following weakned one is sufficient:

(A.4) The bosonic Hamiltonian $H_{\mathrm{b},V}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and bounded below.

For a self-adjoint operator A on a Hilbert space $\mathcal{K}$, we denote its form domain by $Q(A)$:

$$Q(A) := \left\{\psi \in \mathcal{K} | \int_{\mathbb{R}} |\lambda|\mathrm{d}\|E_A(\lambda)\psi\|^2 < \infty\right\},$$

where E_A is the spectral measure of A. If two self-adjoint operators A and B on $\mathcal{K}$ satisfy that $Q(B) \subset Q(A)$ and $\int_{\mathbb{R}} \lambda \mathrm{d}\langle\psi, E_A(\lambda)\psi\rangle \le \int_{\mathbb{R}} \lambda\mathrm{d}\langle\psi, E_B(\lambda)\psi\rangle$ for all $\psi \in Q(B)$, then we write $A \preceq B$.

Theorem 3.5 (Golden–Thompson inequality for the boson system) *Assume (A.3) and (A.4). Then, for all $t > 0$, $e^{-t\overline{H}_{\mathrm{b},V}}$ is in $\mathcal{J}_1(L^2(\mathbb{R}^n))$ and*

$$\operatorname{Tr} e^{-t\overline{H}_{\mathrm{b},V}} \leq \int_{\mathbb{R}^n} K_t(x,x)e^{-tV(x)}\mathrm{d}x \tag{3.16}$$

and

$$\operatorname{Tr} e^{-t\overline{H}_{\mathrm{b},V}} = \int_{\mathbb{R}^n} \mathrm{d}x \int e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t}. \tag{3.17}$$

Proof. We prove the theorem only for the case where V is continuous on $\mathbb{R}^n$. The extension to V satisfying the original assumption of the theorem is routine work (see, e.g., [15, p.51]).

For $N \in \mathbb{N}$, let $V_N := V + V^2/N$ and

$$H_N := H_{\mathrm{b},V_N} = H_{\mathrm{b},V} + \frac{1}{N}V^2. \tag{3.18}$$

Since V_N is in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and bounded below, we can apply Lemm 3.2 with V replaced by V_N to conclude that H_N is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, bounded below, and $e^{-t\overline{H}_N}$ is in $\mathcal{J}_1(L^2(\mathbb{R}^n))$. It is easy to see that $\overline{H}_{\mathrm{b},V} \preceq \overline{H}_N$ for all $N \in \mathbb{N}$. By Lemma 3.3, we have

$$\operatorname{Tr} e^{-t\overline{H}_N} = \int_{\mathbb{R}^n} \mathrm{d}x \int e^{-\int_0^t V(q(s))\mathrm{d}s} e^{-\int_0^t V(q(s))^2\mathrm{d}s/N}\mathrm{d}\mu_{x,x;t}.$$

One can estimate the right hand side by Lemma 3.4 with $e^{-\int_0^t V(q(s))^2\mathrm{d}s/N} < 1$ to obtain

$$\operatorname{Tr} e^{-t\overline{H}_N} \quad \leq \quad \int_{\mathbb{R}^n} \mathrm{d}x \int e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t} \leq \int_{\mathbb{R}^n} K_t(x,x)e^{-tV(x)}\mathrm{d}x < \infty.$$

Hence, by the monotone convergence theorem, we obtain

$$\lim_{N\to\infty} \operatorname{Tr} e^{-t\overline{H}_N} = \int_{\mathbb{R}^n} \mathrm{d}x \int e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t}.$$

By these facts, we can apply Lemma B.2 in Appendix with $A_N = \overline{H}_N, A = \overline{H}_{\mathrm{b},V}$ and $\mathcal{D} = C_0^\infty(\mathbb{R}^n)$ ($\mathcal{H} = L^2(\mathbb{R}^n)$) to conclude that $e^{-t\overline{H}_{\mathrm{b},V}} \in \mathcal{J}_1(L^2(\mathbb{R}^n))$ and

$$\operatorname{Tr} e^{-t\overline{H}_{\mathrm{b},V}} = \lim_{N\to\infty} \operatorname{Tr} e^{-t\overline{H}_N}.$$

Hence (3.17) and (3.16) hold. ∎

Theorem 3.5 implies the standard Golden–Thompson inequality for the Schrödinger operator with potential V (e.g., [15, Theorem 9.2]):

Corollary 3.6 *Assume (A.4) and that*

$$H_V := -\frac{\hbar^2}{2}\Delta + V \tag{3.19}$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and bounded below. Suppose that V is in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and, for some $t>0$,

$$\int_{\mathbb{R}^n} e^{-tV(x)}\mathrm{d}x < \infty \tag{3.20}$$

Then $e^{-t\overline{H}_V}$ is in $\mathfrak{I}_1(L^2(\mathbb{R}^n))$ and

$$\mathrm{Tr}\, e^{-t\overline{H}_V} \le \frac{1}{(2\pi t)^{n/2}\hbar^n}\int_{\mathbb{R}^n} e^{-tV(x)}\mathrm{d}x \tag{3.21}$$

Proof. It is easy to see that

$$K_t(x,x) = \prod_{j=1}^n \sqrt{\frac{\omega_j}{2\pi\hbar}}\,\frac{e^{\hbar\omega_j t/2}}{\sqrt{\sinh\hbar\omega_j t}}\exp\left(-\frac{\omega_j}{\hbar}x_j^2\tanh\frac{\hbar\omega_j t}{2}\right). \tag{3.22}$$

Hence, for each constant $\chi_0>0$,

$$0<K_t(x,x)\le\frac{1}{(2\pi t)^{n/2}\hbar^n}\left(\sup_{0<\chi<\chi_0}\sqrt{\frac{\chi}{\sinh\chi}}e^{\chi/2}\right)^n<\infty,\quad 0<\omega_j\le\frac{\chi_0}{\hbar t}\ \ (j=1,\dots,n) \tag{3.23}$$

and

$$\lim_{\omega\to 0}K_t(x,x)=\frac{1}{(2\pi t)^{n/2}\hbar^n}, \tag{3.24}$$

where $\omega=(\omega_1,\dots,\omega_n)\in\mathbb{R}^n$. By (3.23) and (3.20), (A.3) holds. Hence, by Theorem 3.5, (3.16) holds. By (3.23), (3.24) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\omega\to0}\int_{\mathbb{R}^n}K_t(x,x)e^{-tV(x)}\mathrm{d}x=\frac{1}{(2\pi t)^{n/2}\hbar^n}\int_{\mathbb{R}^n}e^{-tV(x)}\mathrm{d}x.$$

It is easy to see that, for all $f\in C_0^\infty(\mathbb{R}^n)$, $\text{s-}\lim_{\omega\to0}\overline{H}_{\mathrm{b},V}f=\overline{H}_Vf$. Hence, it follows from an application of Lemma B.2 in Appendix that $e^{-t\overline{H}_V}$ is in $\mathfrak{I}_1(L^2(\mathbb{R}^n))$ and

$$\mathrm{Tr}\,e^{-\overline{H}_V}\le\liminf_{\omega\to0}\mathrm{Tr}\,e^{-t\overline{H}_{\mathrm{b},V}}.$$

Thus (3.21) follows. ∎

Remark 3.7 Suppose that V is in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n}e^{-t\sum_{i=1}^n\omega_i^2x_i^2/2}e^{-tV(x)}\mathrm{d}x<\infty$$

for some $t>0$ and $H_{\mathrm{b},V}$ is bounded below. Then, applying (3.21) with V replaced by $\sum_{i=1}^n\omega_i^2x_i^2/2-\sum_{i=1}^n\hbar\omega_i/2+V(x)$, we have

$$\mathrm{Tr}\,e^{-t\overline{H}_{\mathrm{b},V}}\le\frac{1}{(2\pi t)^{n/2}\hbar^n}e^{t\sum_{i=1}^n\hbar\omega_i/2}\int_{\mathbb{R}^n}e^{-t\sum_{i=1}^n\omega_i^2x_i^2/2}e^{-tV(x)}\mathrm{d}x.$$

But this does not imply (3.16), because the right hand side is not necessarily less than or equal to $\int_{\mathbb{R}^n} K_t(x,x)e^{-tV(x)}\mathrm{d}x$. For example, for $V=0$, we have

$$\operatorname{Tr} e^{-tH_{\mathrm{b}}} = \int_{\mathbb{R}^n} K_t(x,x)\mathrm{d}x < \frac{1}{(2\pi t)^{n/2}\hbar^n} e^{t\sum_{i=1}^n \hbar\omega_i/2} \int_{\mathbb{R}^n} e^{-t\sum_{i=1}^n \omega_i^2 x_i^2/2}\mathrm{d}x.$$

Thus the Golden–Thompson type inequality (3.16) is more general than the standard one (3.21) in the sense of Corollary 3.6.

The next theorem gives functional integral representations for the trace of quantities formed out of $e^{-s\overline{H}_{\mathrm{b},V}}$ $(s>0)$ with V not necessarily bounded below and bounded multiplication operators:

Theorem 3.8 *Let $V_1,\dots,V_m \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ be such that, for all $t>0$ and $j=1,\dots,m$,*

$$\int_{\mathbb{R}^n} K_t(x,x)e^{-tV_j(x)}\mathrm{d}x < \infty,$$

H_{b,V_j} is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and bounded below. Let $0<s_1<\dots<s_m<t$ and $f_j \in L^\infty(\mathbb{R}^n)$ $(j=1,\dots,m)$. Then

$$e^{-s_1\overline{H}_{\mathrm{b},V_1}} f_1 e^{-(s_2-s_1)\overline{H}_{\mathrm{b},V_2}} f_2 \cdots f_m e^{-(t-s_m)\overline{H}_{\mathrm{b},V_m}}$$

is in $\mathcal{I}_1(L^2(\mathbb{R}^n))$ and

$$\begin{aligned} &\operatorname{Tr}\left(e^{-s_1\overline{H}_{\mathrm{b},V_1}} f_1 e^{-(s_2-s_1)\overline{H}_{\mathrm{b},V_2}} f_2 \cdots f_m e^{-(t-s_m)\overline{H}_{\mathrm{b},V_m}}\right) \\ &= \int_{\mathbb{R}^n} \mathrm{d}x \left(\int f_1(q(s_1))\cdots f_m(q(s_m)) e^{-\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j} V_j(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t}\right), \end{aligned} \tag{3.25}$$

where $s_0=0$, $s_{m+1}=t$.

Proof. By Theorem 3.5, $e^{-(s_j-s_{j-1})\overline{H}_{\mathrm{b},V_j}}$ is in $\mathcal{I}_1(L^2(\mathbb{R}^n))$, from which the first half of the theorem follows.

To prove (3.25), we first consider the case where each V_j is continuous on $\mathbb{R}^n$. Let H_{N_j} $(N_j \in \mathbb{N})$ be the H_N given by (3.18) with V (resp. N) replaced by V_j (resp. N_j). Then, by Lemma 3.3,

$$\begin{aligned} &\operatorname{Tr}\left(e^{-s_1\overline{H}_{N_1}} f_1 e^{-(s_2-s_1)\overline{H}_{N_2}} f_2 \cdots f_m e^{-(t-s_m)\overline{H}_{N_{m+1}}}\right) \\ &= \int_{\mathbb{R}^n} \mathrm{d}x \int f_1(q(s_1))\cdots f_m(q(s_m)) e^{-\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j} V_j(q(s))\mathrm{d}s} e^{-\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j} V_j(q(s))^2\mathrm{d}s/N_j}\mathrm{d}\mu_{x,x;t}. \end{aligned} \tag{3.26}$$

We have

$$\begin{aligned} &\left| f_1(q(s_1))\cdots f_m(q(s_m)) e^{-\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j} V_j(q(s))\mathrm{d}s} e^{-\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j} V_j(q(s))^2\mathrm{d}s/N_j} \right| \\ &\le \left(\prod_{j=1}^m \|f_j\|_\infty\right) e^{-\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j} V_j(q(s))\mathrm{d}s}. \end{aligned}$$

Noting that

$$\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j} V_j(q(s))\mathrm{d}s = \int_0^t \left(\sum_{j=1}^{m+1}\chi_{[s_{j-1},s_j]}(s)V_j(q(s))\right)\mathrm{d}s,$$

where $\chi_{[s,s']}$ ($s \le s'$) is the characteristic function of the interval $[s,s'] \subset \mathbb{R}$, one can apply the method of proof in Lemma 3.4 to obtain

$$\int_{\mathbb{R}^n}\mathrm{d}x\int e^{-\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j}V_j(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t} \le \frac{1}{t}\sum_{j=1}^{m+1}(s_j - s_{j-1})\int_{\mathbb{R}^n} e^{-tV_j(x)}K_t(x,x)\mathrm{d}x. \quad (3.27)$$

Hence, by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{N_1\to\infty}\mathrm{Tr}\left(e^{-s_1\overline{H}_{N_1}}f_1e^{-(s_2-s_1)\overline{H}_{N_2}}f_2\cdots f_me^{-(t-s_m)\overline{H}_{N_{m+1}}}\right)\\ &= \int_{\mathbb{R}^n}\mathrm{d}x\int f_1(q(s_1))\cdots f_m(q(s_m))e^{-\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j}V_j(q(s))\mathrm{d}s}e^{-\sum_{j=2}^{m+1}\int_{s_{j-1}}^{s_j}V(q(s))^2\mathrm{d}s/N_j}\mathrm{d}\mu_{x,x;t}.\end{aligned}$$

On the other hand, by Proposition B.1 in Appendix, the left hand side is equal to $\mathrm{Tr}\left(e^{-s_1\overline{H}_{\mathrm{b},V_1}}f_1e^{-(s_2-s_1)\overline{H}_{N_2}}f_2\cdots f_me^{-(t-s_m)\overline{H}_{N_{m+1}}}\right)$. Hence

$$\begin{aligned}&\mathrm{Tr}\left(e^{-s_1\overline{H}_{\mathrm{b},V_1}}f_1e^{-(s_2-s_1)\overline{H}_{N_2}}f_2\cdots f_me^{-(t-s_m)\overline{H}_{N_{m+1}}}\right)\\ &= \int_{\mathbb{R}^n}\mathrm{d}x\int f_1(q(s_1))\cdots f_m(q(s_m))e^{-\sum_{j=1}^{m+1}\int_{s_{j-1}}^{s_j}V_j(q(s))\mathrm{d}s}e^{-\sum_{j=2}^{m+1}\int_{s_{j-1}}^{s_j}V_j(q(s))^2\mathrm{d}s/N_j}\mathrm{d}\mu_{x,x;t}.\end{aligned}$$

In the same manner, taking the limit $N_2\to\infty, N_3\to\infty,\dots N_{m+1}\to\infty$ successively, we obtain (3.25).

We next consider the case where V_1 is in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $V_2,\dots,V_{m+1}$ are continuous on $\mathbb{R}^n$. It is enough to prove (3.25) in the case where each f_j is non-negative. Then one easily sees that, with (3.27), the standard argument described in [15, p.51] works. Similarly one can extend (3.25) successively to the case where $V_2,\dots,V_{m+1}$ obey the assumption of the theorem. ∎

3.3 Some consequences

In this subsection, we discuss some implications of the Golden–Thompson inequality (3.16). In general, for a self-adjoint operator H on a Hilbert space $\mathcal{K}$ such that $e^{-\beta H}$ is trace class for all $\beta > 0$, the quantity

$$F_\beta(H) := -\frac{1}{\beta}\log\mathrm{Tr}\, e^{-\beta H} \quad (3.28)$$

is called, in the context of quantum statistical mechanics, the Helmholtz free-energy function of the quantum system [7] whose Hamiltonian is H, where $\beta > 0$ is a parameter denoting physically the inverse temperature.

Corollary 3.9 *Under the same assumption as in Theorem 3.5,*

$$F_\beta(\overline{H}_{\mathrm{b},V}) \geq -\frac{1}{\beta}\log\left(\int_{\mathbb{R}^n} K_\beta(x,x)e^{-\beta V(x)}\mathrm{d}x\right), \quad \beta > 0. \tag{3.29}$$

Proof. This follows from (3.16). ∎

Remark 3.10 Under the same assumption as in Corollary 3.6, taking the limit $\omega \to 0$ in (3.29) yields the well known inequality for the Helmholtz free-energy function given by Golden [7, Eq.(17)]. In this sense, (3.29) is more general than the standard one.

For a linear operator A on a Hilbert space, we denote its spectrum by $\sigma(A)$. One says that the spectrum of A is purely discrete if $\sigma(A)$ consists of isolated eigenvalues of A with finite multiplicity.

If A is self-adjoint and bounded below, then $\inf\sigma(A)$ is called (by abuse of words) the ground state energy of A.

For each $E \in \mathbb{R}$, we denote by $N_E(A) \geq 0$ the number of eigenvalues of A less than or equal to E, counting multiplicities (if there exist no such eigenvalues, then $N_E(A) := 0$).

Corollary 3.11 *Under the same assumption as in Theorem 3.5, the spectrum of $\overline{H}_{\mathrm{b},V}$ is purely discrete and, for each $E \in \mathbb{R}$,*

$$N_E(\overline{H}_{\mathrm{b},V}) \leq \inf_{t>0}\int_{\mathbb{R}^n} K_t(x,x)e^{-t(V(x)-E)}\mathrm{d}x. \tag{3.30}$$

Proof. The pure discreteness of $\sigma(\overline{H}_{\mathrm{b},V})$ follows from the compactness of $e^{-t\overline{H}_{\mathrm{b},V}}$ $(t > 0)$. It is sufficient to consider the case where $N_E(\overline{H}_{\mathrm{b},V}) > 0$. Let $E_1, \cdots, E_N$ be distinct eigenvalues of $\overline{H}_{\mathrm{b},V}$ with $E_1 < \ldots < E_N \leq E$. We denote the multiplicity of E_j by m_j $(j = 1, \ldots, N)$. Then, for all $t > 0$,

$$N_E(\overline{H}_{\mathrm{b},V}) \;=\; \sum_{j=1}^N m_j \leq \sum_{j=1}^N m_j e^{t(E-E_j)} \leq e^{tE}\mathrm{Tr}\, e^{-t\overline{H}_{\mathrm{b},V}}.$$

By this fact and (3.16), we obtain (3.30). ∎

Corollary 3.12 *In addition to the assumption of Theorem 3.5, suppose that, for some $E \in \mathbb{R}$, $V(x) > E$ a.e. $x \in \mathbb{R}^n$ and, for some $t_0 > 0$, $\int_{\mathbb{R}^n} e^{-t_0V(x)}\mathrm{d}x < \infty$. Then*

$$\inf\sigma(\overline{H}_{\mathrm{b},V}) > E. \tag{3.31}$$

Proof. We have $\lim_{t\to\infty} K_t(x,x)e^{-t(V(x)-E)} = 0$ for a.e.$x \in \mathbb{R}^n$ and

$$K_t(x,x)e^{-t(V(x)-E)} \leq e^{-t_0(V(x)-E)}\prod_{j=1}^n \sqrt{\frac{\omega_j}{\pi\hbar}}\frac{1}{\sqrt{1-e^{-2\hbar\omega_j t_0}}}, \quad t \geq t_0, \quad \text{a.e.}x \in \mathbb{R}^n.$$

Hence, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{t\to\infty}\int_{\mathbb{R}^n} K_t(x,x)e^{-t(V(x)-E)}\mathrm{d}x = 0.$$

Therefore $\inf_{t>0}\int_{\mathbb{R}^n} K_t(x,x)e^{-t(V(x)-E)}\mathrm{d}x = 0$. By this result and (3.30), we have $N_E(\overline{H}_{\mathrm{b},V}) = 0$. This and the pure discreteness of $\sigma(\overline{H}_{\mathrm{b},V})$ imply (3.31). ∎

Inequality (3.31) shows that the ground state energy of $\overline{H}_{\mathrm{b},V}$ is strictly more than E.

Remark 3.13 In the case $V(x) > E$ for a.e.$x \in \mathbb{R}^n$, it is obvious that $\overline{H}_{\mathrm{b},V} \geq E$. The classical Hamiltonian

$$H_{\mathrm{cl}}(x,p) = \frac{p^2}{2} + \sum_{i=1}^{n}\frac{\omega_i^2 x_i^2}{2} - \sum_{i=1}^{n}\frac{\hbar\omega_i}{2} + V(x), \quad (x,p)\in\mathbb{R}^n\times\mathbb{R}^n,$$

corresponding to $\overline{H}_{\mathrm{b},V}$ satisfies $H_{\mathrm{cl}}(x,p) \geq E_{\mathrm{cl}} := E - \sum_{i=1}^{n}\hbar\omega_i/2$ and its infimum may be E_{cl} (e.g., if $V(0) = E$, then $H_{\mathrm{cl}}(0,0) = E_{\mathrm{cl}}$). Hence (3.31) means an enhancement of the ground state energy due to quantization. This kind of phenomenon has been discussed in [4] (cf. Theorem 1.3 therein) for a more general class of V. The class of V in Corollary 3.12 is more restricted than the one in [4, Theorem 1.3], but it gives a stronger result in the sense that, in the case $V(0) = E$, the amount of the enhancement is strictly more than $\sum_{i=1}^{n}\hbar\omega_i/2$.

4 A Trace Formula for the Boson–Fermion System

We next consider the heat semi-group $\{e^{-tH(V,\mathbb{U})}\}_{t\geq 0}$ generated by the boson-fermion Hamiltonian $H(V,\mathbb{U})$) defined by (2.6).

The fermion number operator is defined by

$$N_{\mathrm{f}} := \sum_{j=1}^{r} b_j^* b_j, \tag{4.1}$$

which is a bounded nonnegative self-adjoint operator satisfying $N_{\mathrm{f}} \restriction \wedge^p(\mathbb{C}^r) = p$ for all $p = 0,\dots,r$. Hence $\sigma(N_{\mathrm{f}}) = \{0,1,\dots,r\}$. Therefore, for all $z \in \mathbb{C}\setminus\{0\}$, one can define a linear operator $z^{N_{\mathrm{f}}}$ on $\wedge(\mathbb{C}^r)$ by functional calculus and one has $z^{N_{\mathrm{f}}} \restriction \wedge^p(\mathbb{C}^r) = z^p$ $(p = 0,1,\dots,r)$.

Lemma 4.1 *Let $T = (T_{jk})_{j,k=1,\dots,r}$ be an $r\times r$ complex matrix and $z \in \mathbb{C}\setminus\{0\}$. Then*

$$\mathrm{Tr}\, z^{N_{\mathrm{f}}} e^{\sum_{j,k=1}^{r} T_{jk} b_j^* b_k} = \det(I + z e^T). \tag{4.2}$$

Proof. We first prove (4.2) with $z = 1$:

$$\mathrm{Tr}\, e^{\sum_{j,k=1}^{r} T_{jk} b_j^* b_k} = \det(I + e^T). \tag{4.3}$$

We define a linear operator $\hat{T} : \mathbb{C}^r \to \mathbb{C}^r$ by

$$\hat{T}\psi := \sum_{j,k=1}^{r} T_{jk} \langle e_k, \psi \rangle e_j, \quad \psi \in \mathbb{C}^r,$$

where $\{e_j\}_{j=1}^r$ is the standard orthonormal basis of $\mathbb{C}^r$. Let $\hat{T}^{(0)} := 0$ as a linear operator on $\wedge^0(\mathbb{C}^r) = \mathbb{C}$ and, for $p = 1, \ldots, r$,

$$\hat{T}^{(p)} := \sum_{j=1}^{p} I \otimes \cdots \otimes \overset{j\text{th}}{\hat{T}} \otimes \cdots \otimes I$$

acting on $\wedge^p(\mathbb{C}^r)$. Then it is easy to see that

$$\sum_{j,k=1}^{r} T_{jk} b_j^* b_k = \oplus_{p=0}^{r} \hat{T}^{(p)}.$$

Hence

$$\operatorname{Tr} e^{\sum_{j,k=1}^r T_{jk} b_j^* b_k} = \sum_{p=0}^{r} \operatorname{Tr} e^{\hat{T}^{(p)}} = 1 + \sum_{p=1}^{r} \operatorname{Tr} \left(\wedge^p e^{\hat{T}} \right) = \det(I + e^{\hat{T}}),$$

where, in the last equality, we have used a well known formula (e.g., [13, p.322, (188)]) on the determinant of a finite dimensional linear operator. Since the matrix representation of $\hat{T}$ with respect to the basis $\{e_j\}_{j=1}^r$ of $\mathbb{C}^r$ is given by the matrix T, we have $\det(I + e^{\hat{T}}) = \det(I + e^T)$. Thus (4.3) holds.

Letting $S_{jk} := \delta_{jk} \log z + T_{jk}$, we have

$$z^{N_{\mathrm{f}}} e^{\sum_{j,k=1}^r T_{jk} b_j^* b_k} = e^{\sum_{j,k=1}^r S_{jk} b_j^* b_k}.$$

Hence, we can apply (4.3) with T replaced by $S = (S_{jk})$ to obtain (4.2). ∎

Theorem 4.2 *Assume (A.1), (A.2) and (A.3). Suppose that, for all $t > 0$,*

$$\int_{V(x)<0} e^{t \sum_{j,k=1}^r |U_{jk}(x)|} e^{-tV(x)} K_t(x,x) \mathrm{d}x < \infty. \tag{4.4}$$

Let $z \in \mathbb{C} \setminus \{0\}$ and $F \in L^\infty(\mathbb{R}^n)$. Then, for all $t > 0$, $e^{-tH(V,\mathbb{U})}$ is in $\mathcal{I}_1(\mathcal{H})$ and

$$\operatorname{Tr} \left(F z^{N_{\mathrm{f}}} e^{-tH(V,\mathbb{U})} \right) = \int_{\mathbb{R}^n} \mathrm{d}x F(x) \int \det \left(I + z e^{-\int_0^t \mathbb{U}(q(s))\mathrm{d}s} \right) e^{-\int_0^t V(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t} \tag{4.5}$$

Proof. By the proof of Lemma 2.3, we have

$$\|H_{\mathrm{f},\mathbb{U}}\Psi\| \le d_1 \| |V|^\alpha \Psi\| + d_2 \|\Psi\|, \quad \Psi \in D(|V|^\alpha)$$

with some constants $d_j > 0$ $(j = 1, 2)$. By (2.9) and the Heinz inequality [14, Proposition 10.14], we have

$$\| |V|^\alpha \Psi\| \le c^\alpha \|(H_{\mathrm{b},V} - \varepsilon_0 + 1)^\alpha \Psi\|, \quad \Psi \in D((H_{\mathrm{b},V} - \varepsilon_0 + 1)^\alpha).$$

Therefore, $H_{\mathrm{f},\mathbb{U}}$ is $(H_{\mathrm{b},V} - \varepsilon_0)^\alpha$-bounded. Thus we can apply Theorem B.5 in Appendix to conclude that, for all $t > 0$, $e^{-tH(V,\mathbb{U})}$ is in $\mathcal{I}_1(\mathcal{H})$ and

$$
\begin{aligned}
&\mathrm{Tr}\,\left(Fz^{N_\mathrm{f}}e^{-tH(V,\mathbb{U})}\right)\\
&= \mathrm{Tr}\,\left(Fz^{N_\mathrm{f}}e^{-tH_{\mathrm{b},V}}\right) + \sum_{\ell=1}^{\infty}(-1)^\ell \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{\ell-1}}^t \mathrm{d}s_\ell\\
&\qquad \times \mathrm{Tr}\,\left(Fz^{N_\mathrm{f}}e^{-s_1 H_{\mathrm{b},V}}H_{\mathrm{f},\mathbb{U}}e^{-(s_2-s_1)H_{\mathrm{b},V}}H_{\mathrm{f},\mathbb{U}}\cdots e^{-(s_\ell-s_{\ell-1})H_{\mathrm{b},V}}H_{\mathrm{f},\mathbb{U}}e^{-(t-s_\ell)H_{\mathrm{b},V}}\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
&\mathrm{Tr}\,\left(Fz^{N_\mathrm{f}}e^{-s_1 H_{\mathrm{b},V}}H_{\mathrm{f},\mathbb{U}}e^{-(s_2-s_1)H_{\mathrm{b},V}}H_{\mathrm{f},\mathbb{U}}\cdots e^{-(s_\ell-s_{\ell-1})H_{\mathrm{b},V}}H_{\mathrm{f},\mathbb{U}}e^{-(t-s_\ell)H_{\mathrm{b},V}}\right)\\
&= \sum_{j_1,k_1,\dots,j_\ell,k_\ell=1}^{r} \mathrm{Tr}\,(z^{N_\mathrm{f}}b_{j_1}^* b_{k_1}\cdots b_{j_\ell}^* b_{k_\ell})\\
&\times \mathrm{Tr}\,(Fe^{-s_1 H_{\mathrm{b},V}}U_{j_1k_1}e^{-(s_2-s_1)H_{\mathrm{b},V}}U_{j_2k_2}\cdots e^{-(s_\ell-s_{\ell-1})}U_{j_\ell k_\ell}e^{-(t-s_\ell)H_{\mathrm{b},V}})\\
&= \sum_{j_1,k_1,\dots,j_\ell,k_\ell=1}^{r} \mathrm{Tr}\,(z^{N_\mathrm{f}}b_{j_1}^* b_{k_1}\cdots b_{j_\ell}^* b_{k_\ell})\\
&\times \int_{\mathbb{R}^n}\mathrm{d}xF(x)\int U_{j_1k_1}(q(s_1))U_{j_2k_2}(q(s_2))\cdots U_{j_\ell k_\ell}(q(s_\ell))e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t}\\
&= \int_{\mathbb{R}^n}\mathrm{d}xF(x)\int \mathrm{Tr}\,(z^{N_\mathrm{f}}H_\mathrm{f}(s_1)\cdots H_\mathrm{f}(s_\ell))e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t},
\end{aligned}
$$

where

$$
H_\mathrm{f}(s) := \sum_{j,k=1}^{r} U_{jk}(q(s))b_j^* b_k, \quad s \in [0,t].
$$

By a well known formula (e.g., [1, Lemma 3.2]), one can show that $\mathrm{Tr}\,(z^{N_\mathrm{f}}b_{j_1}^* b_{k_1}\cdots b_{j_\ell}^* b_{k_\ell})$ is symmetric for every permutation of $(j_1,k_1),\dots,(j_\ell,k_\ell)$. Hence

$$
\int_{\mathbb{R}^n}\mathrm{d}xF(x)\int \mathrm{Tr}\,(z^{N_\mathrm{f}}H_\mathrm{f}(s_1)\cdots H_\mathrm{f}(s_\ell))\mathrm{d}e^{-\int_0^t V(q(s))\mathrm{d}s}\mu_{x,x;t}
$$

is symmetric in $s_1,\dots,s_\ell$. Therefore

$$
\begin{aligned}
&\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{\ell-1}}^t \mathrm{d}s_\ell\\
&\qquad \times \mathrm{Tr}\,\left(Fz^{N_\mathrm{f}}e^{-s_1 H_{\mathrm{b},V}}H_{\mathrm{f},\mathbb{U}}e^{-(s_2-s_1)H_{\mathrm{b},V}}H_{\mathrm{f},\mathbb{U}}\cdots e^{-(s_\ell-s_{\ell-1})H_{\mathrm{b},V}}H_{\mathrm{f},\mathbb{U}}e^{-(t-s_\ell)H_{\mathrm{b},V}}\right).\\
&= \frac{1}{\ell!}\int_0^t \mathrm{d}s_1\cdots\int_0^t \mathrm{d}s_\ell\int_{\mathbb{R}^n}\mathrm{d}xF(x)\int \mathrm{Tr}\,(z^{N_\mathrm{f}}H_\mathrm{f}(s_1)\cdots H_\mathrm{f}(s_\ell))e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t}\\
&= \frac{1}{\ell!}\int_{\mathbb{R}^n}\mathrm{d}xF(x)\int \mathrm{Tr}\,(z^{N_\mathrm{f}}Z(t)^\ell)e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t},
\end{aligned}
$$

where $Z(t) := \int_0^t H_\mathrm{f}(s)\mathrm{d}s$. Hence

$$
\mathrm{Tr}\,(Fz^{N_\mathrm{f}}e^{-tH(V,\mathbb{U})}) = \lim_{N\to\infty}\int_{\mathbb{R}^n}\mathrm{d}xF(x)\int\sum_{\ell=0}^{N}\frac{(-1)^\ell}{\ell!}\mathrm{Tr}\,(z^{N_\mathrm{f}}Z(t)^\ell)e^{-\int_0^t V(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t}.
$$

We have

$$\|Z(t)\| \le \sum_{j,k=1}^{r} \int_0^t |U_{jk}(q(s)|\mathrm{d}s.$$

Hence

$$\left|\sum_{\ell=0}^{N} \frac{(-1)^\ell}{\ell!} \mathrm{Tr}\,(z^{N_\mathrm{f}} Z(t)^\ell)\right| \le (1+|z|)^r e^{\sum_{j,k=1}^r \int_0^t |U_{jk}(q(s))|\mathrm{d}s}.$$

To proceed further, we show that the integral

$$I_t := \int_{\mathbb{R}^n} K_t(x,x) e^{t\sum_{j,k=1}^r |U_{jk}(x)| - tV(x)} \mathrm{d}x$$

is finite. This is done as follows. We write

$$I_t = I_t^{(1)} + I_t^{(2)}$$

with

$$I_t^{(1)} := \int_{V(x)<0} K_t(x,x) e^{t\sum_{j,k=1}^r |U_{jk}(x)| - tV(x)} \mathrm{d}x,$$
$$I_t^{(2)} := \int_{V(x)\ge 0} K_t(x,x) e^{t\sum_{j,k=1}^r |U_{jk}(x)| - tV(x)} \mathrm{d}x.$$

Then $I_t^{(1)}$ is finite by (4.4). As for $I_t^{(2)}$, we note that, in the case where $V(x) \ge 0$,

$$|U_{jk}(x)| \le \sqrt{a} V(x)^\alpha + \sqrt{b} \le \frac{1}{r^2} V(x) + c_r,$$

where $c_r > 0$ is a constant. Hence $\sum_{j,k=1}^r |U_{jk}(x)| - V(x) \le r^2 c_r$, which implies that $I_t^{(2)} \le \int_{\mathbb{R}^n} K_t(x,x) e^{tr^2 c_r} \mathrm{d}x < \infty$.

As in Lemma 3.4, we have

$$\int_{\mathbb{R}^n} \mathrm{d}x |F(x)| \int e^{\sum_{j,k=1}^r \int_0^t |U_{jk}(q(s))|\mathrm{d}s - V(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t} \le \|F\|_\infty I_t.$$

Hence, by the Lebesgue dominated convergence theorem, we obtain

$$\mathrm{Tr}\,(F z^{N_\mathrm{f}} e^{-tH(V,\mathbb{U})}) = \int_{\mathbb{R}^n} \mathrm{d}x F(x) \int \mathrm{Tr}\,(z^{N_\mathrm{f}} e^{-Z(t)}) e^{-\int_0^t V(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t}.$$

Since

$$Z(t) = \sum_{j,k=1}^r \left(\int_0^t U_{jk}(q(s))\mathrm{d}s\right) b_j^* b_k,$$

it follows from Lemma 4.1 that

$$\mathrm{Tr}\, z^{N_\mathrm{f}} e^{-Z(t)} = \det\left(I + z e^{-\int_0^t \mathbb{U}(q(s))\mathrm{d}s}\right).$$

Thus (4.5) follows. ∎

In applications, it may be useful to find a general condition which does not require condition (4.4). For this purpose, we consider the following condition:

(A.5) There exists a nonnegative function $Y \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying:

(i) For all $\varepsilon \in (0, \delta)$ with some constant $\delta > 0$, $H_{\mathrm{b},\varepsilon Y}$ is self-adjoint.

(ii) For each $\eta > 0$, there exists a constant $c_\eta > 0$ such that $|V(x)|^2 \le \eta^2 Y(x)^2 + c_\eta^2$, a.e.$x \in \mathbb{R}^n$.

(iii) There exist constants $\alpha \in [0, 1)$ and $a, b > 0$ such that

$$|U_{jk}(x)|^2 \le a|Y(x)|^{2\alpha} + b, \quad \text{a.e.} x \in \mathbb{R}^n,\ j, k = 1, \dots, r.$$

Lemma 4.3 *Assume (A.5). Let $\varepsilon \in (0, \delta)$. Then: $H_{\mathrm{b},V+\varepsilon Y}$ and $H(V + \varepsilon Y, \mathbb{U})$ are self-adjoint and bounded below.*

Proof. By the closedness of $H_{\mathrm{b},\varepsilon Y}$, there exists a constant $d_\varepsilon > 0$ such that

$$\varepsilon\|Y\Psi\| \le d_\varepsilon\|(H_{\mathrm{b},\varepsilon Y} + 1)\Psi\|, \quad \Psi \in D(H_{\mathrm{b},\varepsilon Y}) = D(H_{\mathrm{b}}) \cap D(Y).$$

By (A.5)-(ii), we have for all $\Psi \in D(Y)$

$$\|V\Psi\| \le \eta\|Y\Psi\| + c_\eta\|\Psi\|,$$

where $\eta > 0$ is arbitrary. Hence V is infinitesimally small with respect to $H_{\mathrm{b},\varepsilon Y}$. Hence, by the Kato–Rellich theorem (e.g., [11, Theorem X.12]), $H_{\mathrm{b},V+\varepsilon Y} = H_{\mathrm{b},\varepsilon Y} + V$ is self-adjoint and bounded below.

Let $V_\varepsilon(x) := V(x) + \varepsilon Y(x)$. Then, for a.e.$x \in \mathbb{R}^n$,

$$|Y(x)|^2 \le \frac{2}{\varepsilon^2}(|V_\varepsilon(x)|^2 + |V(x)|^2) \le \frac{2}{\varepsilon^2}(|V_\varepsilon(x)|^2 + \eta^2|Y(x)|^2 + c_\eta^2).$$

Hence

$$\left(1 - \frac{2\eta^2}{\varepsilon^2}\right)|Y(x)|^2 \le \frac{2}{\varepsilon^2}(|V_\varepsilon(x)|^2 + c_\eta^2).$$

Take $\eta > 0$ such that $2\eta^2/\varepsilon^2 < 1$. Then, by (A.5)-(iii), we obtain

$$|U_{jk}(x)|^2 \le a'|V_\varepsilon(x)|^{2\alpha} + b'$$

with constants $a', b' > 0$. Thus we can apply Lemma 2.3 with V replaced by V_ε to conclude that $H(V_\varepsilon, \mathbb{U})$ is self-adjoint and bounded below. ∎

Lemma 4.4 *Assume (A.5). Let $\varepsilon \in (0, \delta)$, $z \in \mathbb{C} \setminus \{0\}$ and $F \in L^\infty(\mathbb{R}^n)$. Suppose that, for all $t > 0$,*

$$\int_{\mathbb{R}^n} K_t(x, x) \det(I + |z|e^{-t\mathbb{U}(x)})e^{-tV(x)}e^{-t\varepsilon Y(x)}\mathrm{d}x < \infty. \tag{4.6}$$

Then, for all $t > 0$, $e^{-tH(V+\varepsilon Y,\mathbb{U})}$ is in $\mathfrak{I}_1(\mathcal{H})$ and

$$\begin{aligned}&\mathrm{Tr}\,(Fz^{N_\mathrm{f}}e^{-tH(V+\varepsilon Y,\mathbb{U})})\\ &= \int_{\mathbb{R}^n} \mathrm{d}xF(x) \int \det(I + ze^{-\int_0^t \mathbb{U}(q(s))\mathrm{d}s})e^{-\int_0^t V(q(s))\mathrm{d}s}e^{-\varepsilon\int_0^t Y(q(s))\mathrm{d}s}\mathrm{d}\mu_{x,x;t}.\end{aligned} \tag{4.7}$$

Proof. We have already seen that (A.1) and (A.2) holds with V replaced by $V_\varepsilon = V + \varepsilon Y$. Taking $\eta > 0$ such that $\varepsilon > \eta$, we have $V_\varepsilon + c_\eta \geq 0$. Hence (4.4) with V replaced by $V_\varepsilon + c_\eta$ is trivially satisfied. Since $\det(I + |z|e^{-t\mathbb{U}(x)}) \geq 1$, we have

$$\int_{\mathbb{R}^n} K_t(x,x) e^{-t(V_\varepsilon(x)+c_\eta)} \mathrm{d}x \leq \int_{\mathbb{R}^n} K_t(x,x) \det(I + |z|e^{-t\mathbb{U}(x)}) e^{-tV(x)} e^{-t\varepsilon Y(x)} e^{-tc_\eta} \mathrm{d}x < \infty.$$

Therefore we can apply Theorem 4.2 to obtain the desired result. ∎

Lemma 4.5 *Let $z \in \mathbb{C} \setminus \{0\}$ and $t > 0$ be fixed. Let Φ be a Borel measurable function on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} K_t(x,x) \det(I + |z|e^{-t\mathbb{U}(x)}) e^{-t\Phi(x)} \mathrm{d}x < \infty. \tag{4.8}$$

Then

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \mathrm{d}x \int \det(I + z e^{-\int_0^t \mathbb{U}(q(s))\mathrm{d}s}) e^{-\int_0^t \Phi(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t} \right| \\ &\leq \int_{\mathbb{R}^n} K_t(x,x) \det(I + |z|e^{-t\mathbb{U}(x)}) e^{-t\Phi(x)} \mathrm{d}x. \end{aligned} \tag{4.9}$$

Proof. Let $\mathfrak{L}_\mathrm{h}(\mathbb{C}^n)$ be the real vector space of $n \times n$ Hermitian matrices. Then the function: $\mathfrak{L}_\mathrm{h}(\mathbb{C}^n) \times \mathbb{R} \ni (A, \lambda) \mapsto e^\lambda \det(I + e^A)$ is convex [9, Proposition 2.2]. Noting that $|z|e^A = e^{A + \log|z|}$, one sees that $e^\lambda \det(I + |z|e^A)$ also is convex in $(A, \lambda) \in \mathfrak{L}_\mathrm{h}(\mathbb{C}^n) \times \mathbb{R}$. Hence, by Jensen's inequality, we have

$$\det(I + |z|e^{-\int_0^t \mathbb{U}(q(s))\mathrm{d}s}) e^{-\int_0^t \Phi(q(s))\mathrm{d}s} \leq \frac{1}{t} \int_0^t \det(I + |z|e^{-t\mathbb{U}(q(s))}) e^{-t\Phi(q(s))} \mathrm{d}s. \tag{4.10}$$

We also note that, for every positive $n \times n$ matrix A

$$|\det(I + zA)| \leq \det(I + |z|A).$$

Then, in the same manner as in the proof of Lemma 3.4, we obtain (4.9). ∎

For generality, we introduce the following condition:

(A.6) The boson-fermion Hamiltonian $H(V, \mathbb{U})$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n) \hat{\otimes} \wedge (\mathbb{C}^r)$ and bounded below.

This condition is satisfied, e.g., if (A.1) and (A.2) hold with the property that $C_0^\infty(\mathbb{R}^n)$ is a core of $H_{\mathrm{b},V}$.

Theorem 4.6 *Let $z \in \mathbb{C} \setminus \{0\}$. Assume (A.5) and (A.6) with*

$$\int_{\mathbb{R}^n} K_t(x,x) \det(I + e^{-t\mathbb{U}(x)}) e^{-tV(x)} \mathrm{d}x < \infty, \quad \forall t > 0. \tag{4.11}$$

Then, for all $t > 0$, $e^{-t\overline{H(V,\mathbb{U})}}$ is in $\mathfrak{I}_1(\mathcal{H})$. Moreover, if

$$\int_{\mathbb{R}^n} K_t(x,x) \det(I + |z|e^{-t\mathbb{U}(x)}) e^{-tV(x)} \mathrm{d}x < \infty, \quad \forall t > 0, \tag{4.12}$$

then, for all $t > 0$ and $F \in L^\infty(\mathbb{R}^n)$, (4.5) holds with $H(V, \mathbb{U})$ replaced by $\overline{H(V, \mathbb{U})}$.

Proof. Let $A := \overline{H(V,\mathbb{U})}$ and $A_N := H(V + Y/N, \mathbb{U})$ $(N \in \mathbb{N})$. Then $A \preceq A_N$. Hence $A_N \geq \inf \sigma(A)$. It is easy to see that $\lim_{N\to\infty} A_N\Psi = A\Psi$ for all $\Psi \in C_0^\infty(\mathbb{R}^n)\hat{\otimes} \wedge (\mathbb{C}^r)$. By (4.7) and (4.9) we have

$$\operatorname{Tr} e^{-tA_N} \leq \int_{\mathbb{R}^n} \mathrm{d}x K_t(x,x) \det(I + e^{-t\mathbb{U}(x)})e^{-tV(x)}, \tag{4.13}$$

which, by (4.11), is finite independently of N. Hence we can apply Lemma B.2 in Appendix to conclude that $e^{-tA} \in \mathfrak{I}_1(L^2(\mathbb{R}^n))$ and $\lim_{N\to\infty} \|e^{-tA_N} - e^{-tA}\|_1 = 0$, where $\|\cdot\|_1$ denotes trace norm. Then, taking the limit $N \to \infty$ in (4.7) with $\varepsilon = 1/N$, we obtain

$$\begin{aligned}
&\operatorname{Tr}(F z^{N_{\mathrm{f}}} e^{-t\overline{H(V,\mathbb{U})}}) \\
&= \lim_{N\to\infty} \int_{\mathbb{R}^n} \mathrm{d}x F(x) \int \det(I + z e^{-\int_0^t \mathbb{U}(q(s))\mathrm{d}s}) e^{-\int_0^t V(q(s))\mathrm{d}s} e^{-\int_0^t Y(q(s))\mathrm{d}s/N} \mathrm{d}\mu_{x,x;t}.
\end{aligned}$$

We have

$$\begin{aligned}
&\left| F(x) \int \det(I + z e^{-\int_0^t \mathbb{U}(q(s))\mathrm{d}s}) e^{-\int_0^t V(q(s))\mathrm{d}s} e^{-\int_0^t Y(q(s))\mathrm{d}s/N} \mathrm{d}\mu_{x,x;t} \right| \\
&\leq \|F\|_\infty \int \det(I + |z| e^{-\int_0^t \mathbb{U}(q(s))\mathrm{d}s}) e^{-\int_0^t V(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t},
\end{aligned}$$

which is integrable by (4.12) and Lemma 4.5. Hence, using the Lebesgue dominated convergence theorem, we obtain (4.5). ∎

5 Golden–Thompson Type Inequalities

Now we can derive Golden–Thompson type inequalities for $e^{-tH(V,\mathbb{U})}$. It is easy to see that, for all $(x,y) \in \mathbb{R}^{2n}$ with $x_j \neq y_j$ $(j = 1,\ldots,n)$ and $t > 0$, the integral

$$L_t(x,y) := \frac{1}{t} \int_0^t K_s(x,y) K_{t-s}(y,x) \mathrm{d}s \tag{5.1}$$

is finite.

We consider the following two cases:

(C.1) Conditions (A.1), (A.2), (A.3) and (4.4) hold.

(C.2) Conditions (A.5), (A.6) and (4.12) hold.

In the rest of this section, for notational simplicity, we denote $\overline{H(V,\mathbb{U})}$ in case (C.2) simply by $H(V,\mathbb{U})$.

Theorem 5.1 *Assume (C.1) or (C.2). Let $z \in \mathbb{C}\setminus\{0\}$, $t > 0$ and $F \in L^\infty(\mathbb{R}^n)$. Then*

$$\left|\operatorname{Tr}(F z^{N_{\mathrm{f}}} e^{-tH(V,\mathbb{U})})\right| \leq \int_{\mathbb{R}^n} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y |F(x)| L_t(x,y) \det(I + |z| e^{-t\mathbb{U}(y)}) e^{-tV(y)} \tag{5.2}$$

In particular,

$$\operatorname{Tr} e^{-tH(V,\mathbb{U})} \leq \int_{\mathbb{R}^n} \mathrm{d}x K_t(x,x) \det(I + e^{-t\mathbb{U}(x)}) e^{-tV(x)}. \tag{5.3}$$

Proof. By the present assumption, (4.5) holds. Hence, by applying (4.10) with $\Phi = V$, we have

$$\left|\mathrm{Tr}\left(Fz^{N_{\mathrm{f}}}e^{-tH(V,\mathbb{U})}\right)\right| \le \int_{\mathbb{R}^n} \mathrm{d}x|F(x)| \int \frac{1}{t}\int_0^t \mathrm{d}s \det\left(I+|z|e^{-t\mathbb{U}(q(s))}\right)e^{-tV(q(s))}\mathrm{d}\mu_{x,x;t}.$$

We note that, for each $s \in [0,t]$,

$$\begin{aligned}&\int \det\left(I+|z|e^{-t\mathbb{U}(q(s))}\right)e^{-tV(q(s))}\mathrm{d}\mu_{x,x;t}\\ &= \int_{\mathbb{R}^n}\mathrm{d}y \det\left(I+|z|e^{-t\mathbb{U}(y)}\right)e^{-tV(y)}K_s(x,y)K_{t-s}(y,x).\end{aligned}$$

Hence (5.2) follows. Inequality (5.3) follows from putting $F = 1$ and $z = 1$ in (5.2) (or use (4.13)). ∎

Remark 5.2 Consider the case where $U_{jk} = \delta_{jk}\lambda_j$ ($\lambda_j > 0, j = 1, \ldots, r$) and $V = 0$. Then

$$H(V,\mathbb{U}) = H_{\mathrm{b}} + \sum_{j=1}^r \lambda_j b_j^* b_j,$$

i.e., it is the Hamiltonian of an n-dimensional quantum harmonic oscillator and r free fermions. In this case, we have

$$\mathrm{Tr}\left(Fe^{-tH(V,\mathbb{U})}\right) = \left(\mathrm{Tr}\, Fe^{-tH_{\mathrm{b}}}\right)\left(\mathrm{Tr}\, e^{-t\sum_{j=1}^r \lambda_j b_j^* b_j}\right) = \int_{\mathbb{R}^n} F(x)K_t(x,x)\mathrm{d}x \cdot \prod_{j=1}^r (1+e^{-t\lambda_j}). \tag{5.4}$$

On the other hand, in the present case, we have $\det(I+e^{-t\mathbb{U}(y)}) = \prod_{j=1}^r(1+e^{-t\lambda_j})$. Hence, using the easily proved equality

$$\int_{\mathbb{R}^n} L_t(x,y)\mathrm{d}y = K_t(x,x),$$

we see that the right hand side of (5.2) is equal to that of (5.4). Thus, in the present case, equality in (5.2) holds.

Corollary 5.3 *Assume (C.1) or (C.2). Then:*

(i) *For all $\beta > 0$,*

$$F_\beta(H(V,\mathbb{U})) \ge -\frac{1}{\beta}\log\left(\int_{\mathbb{R}^n}\mathrm{d}xK_\beta(x,x)\det(I+e^{-\beta\mathbb{U}(x)})e^{-\beta V(x)}\right).$$

(ii) *The spectrum of $H(V,\mathbb{U})$ is purely discrete and, for each $E \in \mathbb{R}$,*

$$N_E(H(V,\mathbb{U})) \le \inf_{t>0}\int_{\mathbb{R}^n}\mathrm{d}xK_t(x,x)\det(I+e^{-t\mathbb{U}(x)})e^{-t(V(x)-E)}.$$

(iii) *Suppose, in addition, that, for some* $E \in \mathbb{R}$ *and a.e.*$x \in \mathbb{R}^n$,

$$V(x) - r^2\sqrt{a}|V(x)|^\alpha - r^2\sqrt{b} > E, \tag{5.5}$$

where a, b *and* α *are constants in (A.2), and* $\int_{\mathbb{R}^n} e^{-t_0 V(x)} \mathrm{d}x < \infty$ *for some* $t_0 > 0$. *Then* $\inf \sigma(H(V, \mathbb{U})) > E$.

Proof. The proof of (i) (resp. (ii)) is similar to that of Corollary 3.9 (redp. Corollary 3.11).

As for (iii), we first note that

$$\det(I + e^{-t\mathbb{U}(x)}) \leq (I + e^{t\|\mathbb{U}(x)\|})^r \leq 2^r e^{rt\|\mathbb{U}(x)\|}.$$

On the other hand, we have

$$\|\mathbb{U}(x)\| \leq \sqrt{\sum_{j,k=1}^{r} |U_{jk}(x)|^2} \leq \sqrt{r^2(a|V(x)|^{2\alpha} + b)} \leq r\sqrt{a}|V(x)|^\alpha + r\sqrt{b}.$$

Hence

$$\det(I + e^{-t\mathbb{U}(x)})e^{-tV(x)} \leq 2^r e^{-t(V(x) - r^2\sqrt{a}|V(x)|^\alpha - r^2\sqrt{b})}.$$

Therefore

$$N_E(H(V, \mathbb{U})) \leq 2^r \int_{\mathbb{R}^n} K_t(x, x) e^{-t(V(x) - E - r^2\sqrt{a}|V(x)|^\alpha - r^2\sqrt{b})} \mathrm{d}x.$$

Condition (5.5) implies that $V(x) - E > 0$ a.e.$x \in \mathbb{R}^n$. Hence, for every $\varepsilon \in (0, 1)$, there exists a constant $d_\varepsilon > 0$ such that $\varepsilon(V(x) - E) - r^2\sqrt{a}|V(x)|^\alpha - r^2\sqrt{b} \geq -d_\varepsilon$ for a.e.$x \in \mathbb{R}^n$. Hence

$$V(x) - E - r^2\sqrt{a}|V(x)|^\alpha - r^2\sqrt{b} \geq (1 - \varepsilon)(V(x) - E) - d_\varepsilon, \quad \text{a.e.} x \in \mathbb{R}^n.$$

Therefore, taking $t_0' = t_0/(1 - \varepsilon)$, one sees that $\int_{\mathbb{R}^n} e^{-t_0'(V(x) - E - r^2\sqrt{a}|V(x)|^\alpha - r^2\sqrt{b})} \mathrm{d}x < \infty$. Then, in the same manner as in the proof of Corollary 3.12, one can show that $N_E(H(V, \mathbb{U})) = 0$. ∎

Remark 5.4 It may be interesting to note that, in Corollary 5.3-(iii), $\mathbb{U}(x)$ is not necessarily bounded below.

6 Application to Supersymmetric Quantum Mechanics

In this section we apply the results established in the preceding sections to a model in supersymmetric quantum mechanics.

We consider the case $r = n$. Let H_b and H_f be given by (1.7) and (1.8) respectively. Then the operator

$$H_0 := H_\mathrm{b} + H_\mathrm{f} \tag{6.1}$$

is a nonnegative self-adjoint operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \otimes \wedge(\mathbb{C}^n)$. It is easy to see that the operator

$$Q_0 := i \sum_{j=1}^{n} \sqrt{\hbar\omega_j}(a_j b_j^* - a_j^* b_j), \tag{6.2}$$

is symmetric. By direct computations, we have

$$H_0 = Q_0^2 \tag{6.3}$$

on $C_0^\infty(\mathbb{R}^n)\hat{\otimes} \wedge (\mathbb{C}^n)$. Applying a general theorem in the theory of tensor products of self-adjoint operators, H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)\hat{\otimes} \wedge (\mathbb{C}^n)$. Hence, it follows from a well known fact (e.g., [11, p.341, Problem 28]) that Q_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)\hat{\otimes} \wedge (\mathbb{C}^n)$ with operator equality

$$H_0 = \overline{Q}_0^2. \tag{6.4}$$

Let

$$\tau := (-1)^{N_{\mathrm{f}}}. \tag{6.5}$$

Then τ is a self-adjoint involution: $\tau^* = \tau, \tau^2 = I$. It is easy to see that

$$\{\tau, b_j^*\} = 0, \quad \{\tau, b_j\} = 0, \quad j = 1, \ldots, n.$$

It follows that

$$\tau\overline{Q}_0 \subset -\overline{Q}_0\tau. \tag{6.6}$$

Hence $\overline{Q}_0$ is a supercharge with respect to τ and H_0 is the supersymmetric Hamiltonian with supercharge $\overline{Q}_0$ (e.g., [17, p.140]). Mathematically Q_0 is a Dirac type operator on $\mathbb{R}^n$.

We now consider a perturbation of Q_0. Let W be a real distribution on $\mathbb{R}^n$ such that

$$W_j := D_j W \in L^4_{\mathrm{loc}}(\mathbb{R}^n), \quad W_{jk} := D_j D_k W \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \quad (j, k = 1, \ldots, n).$$

Then one can define a symmetric operator

$$Q_1 := \frac{i}{\sqrt{2}} \sum_{j=1}^{n} \left(b_j^* W_j - b_j W_j\right) \tag{6.7}$$

with $D(Q_1) \supset C_0^\infty(\mathbb{R}^n)\hat{\otimes} \wedge (\mathbb{C}^n)$. Hence

$$Q := Q_0 + Q_1 \tag{6.8}$$

is a symmetric operator with $D(Q) \supset C_0^\infty(\mathbb{R}^n)\hat{\otimes} \wedge (\mathbb{C}^n)$. We have

$$\tau\overline{Q} \subset -\overline{Q}\tau. \tag{6.9}$$

Hence Q also is a Dirac type operator on $\mathbb{R}^n$. If $\overline{Q}$ is self-adjoint, then $\overline{Q}$ is a supercharge with respect to τ.

For the moment, we do not discuss the essential self-adjointness of Q. By von Neumann's theorem, the operator

$$H_{\mathrm{SS}} := \overline{Q}^{*}\overline{Q} \tag{6.10}$$

is a nonnegative self-adjoint operator on $\mathcal{H}$. If $\overline{Q}$ is self-adjoint, then H_{SS} is the supersymmetric Hamiltonian with respect to the supercharge $\overline{Q}$.

To see a concrete form of H_{SS} restricted to a subspace of $D(H_{\mathrm{SS}})$, let

$$\omega x := (\omega_1 x_1, \cdots, \omega_n x_n) \in \mathbb{R}^n, \tag{6.11}$$

$$\Phi_W(x) := \omega x \cdot DW(x) + \frac{1}{2}|DW(x)|^2 - \frac{\hbar}{2}\Delta W(x), \quad x \in \mathbb{R}^n, \tag{6.12}$$

where $DW := (D_1 W, \ldots, D_n W)$, and

$$H_{\mathrm{b}}(W) := H_{\mathrm{b}} + \Phi_W, \tag{6.13}$$

$$H_{\mathrm{f}}(W) = H_{\mathrm{f}} + \hbar \sum_{j,k=1}^{n} W_{jk} b_j^* b_k. \tag{6.14}$$

Then we have

$$H_{\mathrm{SS}} = H_{\mathrm{b}}(W) + H_{\mathrm{f}}(W) \tag{6.15}$$

on $C_0^\infty(\mathbb{R}^n)\hat{\otimes} \wedge (\mathbb{C}^n)$.

As for W, we assume the following too:

(W.1) There exists a nonnegative continuous function $U \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying the following conditions:

(a) For all $\varepsilon \in (0,\delta)$ with a constant $\delta > 0$, $H_{\mathrm{b}} + \varepsilon U$ is self-adjoint.

(b) For all $\eta > 0$, there exists a constant $c_\eta > 0$ such that

$$|\Phi_W(x)|^2 \le \eta^2 U(x)^2 + c_\eta^2, \quad \text{a.e.} x \in \mathbb{R}^n.$$

(c) There exist constants $\alpha \in [0,1)$ and $a, b > 0$ such that

$$|W_{jk}(x)|^2 \le a U(x)^{2\alpha} + b, \quad \text{a.e.} x \in \mathbb{R}^n, \quad j,k = 1,\ldots,n.$$

(d) $D(\overline{Q}) \cap D(U^{1/2})$ is a core of $\overline{Q}$.

We define

$$H_\varepsilon := H_{\mathrm{b}}(W) + \varepsilon U + H_{\mathrm{f}}(W), \quad \varepsilon \in (0,\delta). \tag{6.16}$$

We set

$$\mathbb{W} := (W_{jk})_{j,k=1,\ldots,n}. \tag{6.17}$$

We denote by Ω the $n \times n$ matrix such that

$$\Omega_{jk} = \omega_j \delta_{jk}, \quad j,k = 1,\ldots,n. \tag{6.18}$$

Lemma 6.1 *Assume (W.1) and let $\varepsilon \in (0,\delta)$. Then:*

(i) *H_ε is self-adjoint and bounded below.*

(ii) *For all $t>0$, e^{-tH_ε} is trace class.*

Proof. (i) Assumption (W.1)-(b) implies that Φ_W is infinitesimally small with respect to U. Since $H_{\rm b}+\varepsilon U$ is closed, it follows that Φ_W is infinitesimally small with respect to $H_{\rm b}+\varepsilon U$. Hence, by the Kato–Rellich theorem, $H_{\rm b}(W)+\varepsilon U$ is self-adjoint and bounded below. Assumption (W.1)-(b) and (W.1)-(c) imply that there exist constants $\alpha\in[0,1)$ and $a',b'>0$ such that

$$|W_{jk}(x)|^2\le a'|\varepsilon U(x)+\Phi_W(x)|^{2\alpha}+b',\quad \text{a.e.}\mathbb{R}^n,\quad j,k=1,\dots,n.$$

Hence, we can apply Lemma 2.3-(i) with V and $\mathbb{U}$ replaced by $\varepsilon U+\Phi_W$ and $\hbar(\Omega+\mathbb{W})$ respectively to conclude that H_ε is self-adjoint and bounded below.

(ii) It is obvious that, for all $t>0$, $e^{-t(H_{\rm b}+\varepsilon U)}\in\mathcal{J}_1(L^2(\mathbb{R}^n))$. For each $\kappa\in(0,1)$, we have

$$H_{\rm b}(W)+\varepsilon U=(1-\kappa)(H_{\rm b}+\varepsilon U)+\kappa(H_{\rm b}+\varepsilon U)+\Phi_W.$$

Then, in the same way as in the proof of Lemma 2.3-(ii), one sees that, for all $t>0$, e^{-tH_ε} is in $\mathcal{J}_1(\mathcal{H})$ (note that, in this reasoning, the finite dimensionality of $\wedge(\mathbb{C}^n)$ is used). ∎

Lemma 6.2 *Assume (W.1). Let $\varepsilon\in(0,\delta), z\in\mathbb{C}\setminus\{0\}$ and $F\in L^\infty(\mathbb{R}^n)$. Suppose that*

$$\int_{\mathbb{R}^n}K_t(x,x)\det\left(I+|z|e^{-t\hbar(\Omega+\mathbb{W}(x))}\right)e^{-t\Phi_W(x)-t\varepsilon U(x)}{\rm d}x<\infty,\quad \forall t>0. \tag{6.19}$$

Then:

(i) *For all $t>0$,*

$$\begin{aligned}\operatorname{Tr}(Fz^{N_{\rm f}}e^{-tH_\varepsilon})=\int_{\mathbb{R}^n}F(x)\quad&\int\det\left(I+ze^{-t\hbar\Omega-\hbar\int_0^t\mathbb{W}(q(s)){\rm d}s}\right)\\&\times e^{-\int_0^t\Phi_W(q(s)){\rm d}s}e^{-\varepsilon\int_0^tU(q(s)){\rm d}s}{\rm d}\mu_{x,x;t}\end{aligned}\tag{6.20}$$

(ii) *For all $t>0$,*

$$\begin{aligned}\left|\operatorname{Tr}(Fz^{N_{\rm f}}e^{-tH_\varepsilon})\right|\ \le\ &\int_{\mathbb{R}^n}{\rm d}x|F(x)|\int_{\mathbb{R}^n}{\rm d}yL_t(x,y)\\&\times\det\left(I+|z|e^{-t\hbar(\Omega+\mathbb{W}(y))}\right)e^{-t\Phi_W(y)-t\varepsilon U(y)}.\end{aligned}\tag{6.21}$$

In particular, if (6.19) holds for $z=1$, then

$$\operatorname{Tr}e^{-tH_\varepsilon}\le\int_{\mathbb{R}^n}{\rm d}xK_t(x,x)\det\left(I+e^{-t\hbar(\Omega+\mathbb{W}(x)}\right)e^{-tV(x)-t\varepsilon U(x)}. \tag{6.22}$$

Proof. (i) Similar to the proof of Theorem 4.6.
(ii) Similar to the proof of Theorem 5.1. ∎

Theorem 6.3 *Assume (W.1). Let $z \in \mathbb{C} \setminus \{0\}$ and $F \in L^\infty(\mathbb{R}^n)$.*

(i) *Suppose that*

$$\int_{\mathbb{R}^n} K_t(x,x) \det\left(I + e^{-t\hbar(\Omega+\mathbb{W}(x))}\right) e^{-t\Phi_W(x)} \mathrm{d}x < \infty, \quad \forall t > 0. \tag{6.23}$$

Then, for all $t > 0$, $e^{-tH_{\mathrm{SS}}}$ is trace class and the spectrum of H_{SS} is purely discrete. Moreover,

$$\mathrm{Tr}\, e^{-tH_{\mathrm{SS}}} \leq \int_{\mathbb{R}^n} K_t(x,x) \det(I + e^{-t\hbar(\Omega+\mathbb{W}(x))}) e^{-t\Phi_W(x)}, \quad \forall t > 0. \tag{6.24}$$

(ii) *Suppose that*

$$\int_{\mathbb{R}^n} K_t(x,x) \det\left(I + |z| e^{-t\hbar(\Omega+\mathbb{W}(x))}\right) e^{-t\Phi_W(x)} \mathrm{d}x < \infty, \quad \forall t > 0. \tag{6.25}$$

Then, for all $t > 0$,

$$\begin{aligned}|\mathrm{Tr}\,(F z^{N_{\mathrm{f}}} e^{-tH_{\mathrm{SS}}})| \;\leq\; & \int_{\mathbb{R}^n} \mathrm{d}x |F(x)| \int_{\mathbb{R}^n} \mathrm{d}y L_t(x,y) \\ & \times \det\left(I + |z| e^{-t\hbar(\Omega+\mathbb{W}(y))}\right) e^{-t\Phi_W(y)}\end{aligned} \tag{6.26}$$

and

$$\begin{aligned}\mathrm{Tr}\,(F z^{N_{\mathrm{f}}} e^{-tH_{\mathrm{SS}}}) \;=\; & \int_{\mathbb{R}^n} F(x) \int \det\left(I + z e^{-t\hbar\Omega - \hbar\int_0^t \mathbb{W}(q(s))\mathrm{d}s}\right) \\ & \times e^{-\int_0^t \Phi_W(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t}.\end{aligned} \tag{6.27}$$

Proof. (i) By (6.22) and the non-negativity of U, we have

$$\mathrm{Tr}\, e^{-tH_\varepsilon} \leq \int_{\mathbb{R}^n} \mathrm{d}x K_t(x,x) \det\left(I + e^{-t\hbar(\Omega+\mathbb{W}(x))}\right) e^{-t\Phi_W(x)}.$$

The right hand side is finite by assumption (6.23) and independent of ε. Since H_ε is self-adjoint, it coincides with the self-adjoint operator $H_{\mathrm{SS}} \dot{+} \varepsilon U$ (the form sum). Note that $D(H_{\mathrm{SS}}^{1/2}) = D(\overline{Q})$. Hence, by Assumption (W.1)-(d), $D(H_{\mathrm{SS}}^{1/2}) \cap D(U^{1/2}) = D(\overline{Q}) \cap D(U^{1/2})$ is a core of $H_{\mathrm{SS}}^{1/2}$. Therefore, we can apply a general fact [9, Proposition 2.3] to conclude that $e^{-tH_{\mathrm{SS}}}$ is trace class and

$$\lim_{\varepsilon \downarrow 0} \mathrm{Tr}\, e^{-tH_\varepsilon} = \mathrm{Tr}\, e^{-tH_{\mathrm{SS}}}. \tag{6.28}$$

Thus (6.24) holds.

(ii) By (6.21) and $U \geq 0$, we have

$$\begin{aligned}|\mathrm{Tr}\,(F z^{N_{\mathrm{f}}} e^{-tH_\varepsilon})| \;\leq\; & \int_{\mathbb{R}^n} \mathrm{d}x |F(x)| \int_{\mathbb{R}^n} \mathrm{d}y L_t(x,y) \\ & \times \det\left(I + |z| e^{-t\hbar(\Omega+\mathbb{W}(y))}\right) e^{-t\Phi_W(y)}.\end{aligned} \tag{6.29}$$

If $\delta > \varepsilon_1 > \varepsilon_2 > 0$, then

$$\langle H_{\varepsilon_1}^{1/2}\Psi, H_{\varepsilon_1}^{1/2}\Psi\rangle \geq \langle H_{\varepsilon_2}^{1/2}\Psi, H_{\varepsilon_2}^{1/2}\Psi\rangle \geq 0, \forall\Psi \in D(H_{\varepsilon_1}^{1/2}) = D(H_{\rm SS}^{1/2}) \cap D(U^{1/2})$$

and

$$\lim_{\varepsilon\downarrow 0} \|H_\varepsilon^{1/2}\Psi\|^2 = \|H_{\rm SS}^{1/2}\Psi\|^2, \quad \Psi \in \cap_{\varepsilon\in(0,\delta)} D(H_\varepsilon^{1/2}) = D(H_{\rm SS}^{1/2}) \cap D(U^{1/2}).$$

By Assumption (W.1)-(d), $D(H_{\rm SS}^{1/2}) \cap D(U^{1/2})$ is a core of $H_{\rm SS}^{1/2}$. Hence, by a general convergence theorem [8, Theorem 3.11], H_ε converges to $H_{\rm SS}$ in the strong resolvent sense. This implies that $e^{-tH_\varepsilon} \to e^{-tH_{\rm SS}}$ strongly as $\varepsilon \downarrow 0$. By this fact and (6.28), we can apply Grümm's convergence theorem [16, Theorem 2.19] to conclude that $\lim_{\varepsilon\downarrow 0} \|e^{-tH_\varepsilon} - e^{-tH_{\rm SS}}\|_1 = 0$, where $\|\cdot\|_1$ denotes trace norm. Hence, for all bounded linear operators B on $\mathcal{H}$,

$$\lim_{\varepsilon\downarrow 0} \mathrm{Tr}\,(Be^{-tH_\varepsilon}) = \mathrm{Tr}\,(Be^{-tH_{\rm SS}}). \tag{6.30}$$

This result and (6.29) imply (6.26).

(iii) By (6.30) and (6.20), we have

$$\begin{aligned}\mathrm{Tr}\,(Fz^{N_{\rm f}}e^{-tH_{\rm SS}}) &= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} F(x)\int \det\left(I + ze^{-t\hbar\Omega - \hbar\int_0^t \mathbb{W}(q(s))\mathrm{d}s}\right) \\ &\qquad \times e^{-\int_0^t \Phi_W(q(s))\mathrm{d}s} e^{-\varepsilon\int_0^t U(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t}.\end{aligned}$$

Using the Lebesgure dominated convergence theorem, one sees that the right hand side is equal to that of (6.27). ∎

Remark 6.4 The supersymmetric Golden–Thompson inequality (6.24) includes (1.5) as a limiting case in the following sense. Assume that the supersymmetric Hamiltonian

$$H_{\rm SS}^0(W) := -\frac{\hbar^2}{2}\Delta - \frac{\hbar}{2}\Delta W + \frac{1}{2}|DW|^2 + \hbar\sum_{j,k=1}^n W_{jk}b_j^* b_k,$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \det(I + e^{-t\hbar\mathbb{W}(x)}) e^{tc\sum_{j=1}^n |x_j W_j(x)|} e^{-\frac{t}{2}\left(|DW(x)|^2 - \hbar\Delta W(x)\right)} \mathrm{d}x < \infty$$

for some constant $c > 0$ (note that $H_{\rm SS}^0(W)$ is just H_{KL} defined by (1.4) with P replaced by W and obtained as a limit of $H_{\rm SS}$ as $\omega \to 0$ in a suitable sense). Then, considering the limit $\omega \to 0$ in (6.24), one has

$$\mathrm{Tr}\,(e^{-t\overline{H}_{\rm SS}^0(W)}) \leq \frac{1}{(2\pi t)^{n/2}\hbar^n}\int_{\mathbb{R}^n} \det\left(I + e^{-t\hbar\mathbb{W}(x)}\right) e^{-\frac{t}{2}\left(|DW(x)|^2 - \hbar\Delta W(x)\right)} \mathrm{d}x, \tag{6.31}$$

which is just (1.5) with P replaced by W. The proof of this fact is similar to that of Corollary 3.6 (note that $\det(I + e^{-t\hbar(\Omega+\mathbb{W}(x)}) \leq \det(I + e^{-t\hbar\mathbb{W}(x)})$, since Ω is positive). Thus, in the sense just described, (6.24) is more general than (1.5).

A non-zero vector in $\ker \overline{Q}$ is called a supersymmetric state. Hence, the number of supersymmetric states is given by $\dim \ker \overline{Q}$. One says that the supersymmetry is spontaneously broken if there exist no supersymmetric states, i.e., $\ker \overline{Q} = \{0\}$.

Using the supersymmetric Golden–Thompson inequality (6.24), one can derive an upper bound for the number of supersymmetric states:

Corollary 6.5 *Assume (W.1) and (6.23). Then*

$$\dim \ker \overline{Q} \leq \inf_{t>0} \int_{\mathbb{R}^n} K_t(x,x) \det \left(I + e^{-t\hbar(\Omega+\mathbb{W}(x))}\right) e^{-t\Phi_W(x)} \mathrm{d}x. \tag{6.32}$$

Proof. We have $\ker \overline{Q} = \ker H_{\mathrm{SS}}$ and $\dim \ker H_{\mathrm{SS}} \leq \operatorname{Tr} e^{-tH_{\mathrm{SS}}}$ for all $t > 0$. These facts and (6.24) imply (6.32). ∎

The next corollary gives a sufficient condition for the supersymmetry to be broken spontaneously:

Corollary 6.6 *Assume (W.1) and (6.23). Suppose that*

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} K_t(x,x) \det \left(I + e^{-t\hbar(\Omega+\mathbb{W}(x))}\right) e^{-t\Phi_W(x)} \mathrm{d}x = 0. \tag{6.33}$$

Then $\dim \ker \overline{Q} = \{0\}$*, i.e., the supersymmetry is spontaneously broken.*

Proof. This is just a simple application of (6.32). ∎

7 Functional Integral Representation for the Witten Index

We continue to consider the case $r = n$. Then the Hilbert space $\mathcal{H}$ is decomposed as

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = \left\{ \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} | \Psi \in \mathcal{H}_+, \Phi \in \mathcal{H}_- \right\} \tag{7.1}$$

with

$$\mathcal{H}_+ := \oplus_{p=0}^{[n/2]} L^2(\mathbb{R}^n; \wedge^{2p}(\mathbb{C}^n)), \quad \mathcal{H}_- := \oplus_{p=0}^{[(n-1)/2]} L^2(\mathbb{R}^n; \wedge^{2p+1}(\mathbb{C}^n)),$$

where, for $w > 0$, $[w]$ denotes the largest integer less than or equal to w.

Let Q be as in Section 6. Then, by (6.9), there exist densely defined closed linear operators $Q_\pm$ from $\mathcal{H}_\pm$ to $\mathcal{H}_\mp$ such that

$$\overline{Q} = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix}, \tag{7.2}$$

where the operator matrix is relative to the orthogonal decomposition (7.1).

We assume the following in addition to (W.1):

(W.2) The operator Q is essentially self-adjoint.

Then $\overline{Q}$ is self-adjoint. Hence $Q_+^* = Q_-$ and

$$H_{\mathrm{SS}} = \overline{Q}^2 = \begin{pmatrix} Q_+^* Q_+ & 0 \\ 0 & Q_+ Q_+^* \end{pmatrix}. \tag{7.3}$$

For a densely defined closed linear operator T, the integer

$$\mathrm{ind}(T) := \dim\ker T - \dim\ker T^*$$

is called the analytical index of T, provided that at least one of $\dim\ker T$ and $\dim\ker T^*$ is finite.

In the context of supersymmetric quantum mechanics, $\mathrm{ind}(Q_+)$ is called the Witten index.

Lemma 7.1 *Assume (W.1) and (W.2). Then Q_+ is a Fredholm operator and*

$$\mathrm{ind}(Q_+) = \mathrm{Tr}\,\tau e^{-tH_{\mathrm{SS}}}. \tag{7.4}$$

Proof. Under the present assumption, $e^{-tH_{\mathrm{SS}}} \in \mathfrak{I}_1(\mathcal{H})$ (Theorem 6.3-(i)). Hence we can apply a general theorem [17, Theorem 5.19] to conclude that $\overline{Q}$ is Fredholm, which, together with (7.2), imply that Q_+ is Fredholm. Formula (7.4) follows from [17, Theorem 5.19]. ∎

Theorem 7.2 *Assume (W.1), (W.2) and (6.23). Then, for all $t > 0$,*

$$\mathrm{ind}(Q_+) = \int_{\mathbb{R}^n} \mathrm{d}x \int \det\left(I - e^{-t\hbar\Omega - \hbar\int_0^t \mathbb{W}(q(s))\mathrm{d}s}\right) e^{-\int_0^t \Phi_W(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t}, \tag{7.5}$$

independently of t.

Proof. This follows from (7.4) and (6.27) with $F = 1$ and $z = -1$. ∎

Remark 7.3 If $W = 0$ (the case of the supersymmetric quantum harmonic oscillator), then (7.5) implies that

$$\mathrm{ind}(Q_+) = \int_{\mathbb{R}^n} \mathrm{d}x \int \det\left(I - e^{-t\hbar\Omega}\right) \mathrm{d}\mu_{x,x;t} = \left(\mathrm{Tr}\, e^{-tH_{\mathrm{b}}}\right) \det\left(I - e^{-t\hbar\Omega}\right) = 1,$$

coinciding with the calculation by an operator theoretical method (it is easy to see that, in the case $W = 0$, $\dim\ker Q_+ = 1, \dim\ker Q_- = 0$). But, for a general W, it is difficult in general to calculate $\mathrm{ind}(Q_+)$ explicitly (some examples of Q_+ whose Witten index is explicitly calculated are given, e.g., in [17, Section 5.11]; heuristic arguments to calculate the Witten index of various concrete models in supersymmetric quantum mechanics are found in the physics literature (e.g., [20, 21], [5] and references therein). Theorem 7.2 implies that

$$\mathrm{ind}(Q_+) = \lim_{t\to\infty} \int_{\mathbb{R}^n} \mathrm{d}x \int \det\left(I - e^{-t\hbar\Omega - \hbar\int_0^t \mathbb{W}(q(s))\mathrm{d}s}\right) e^{-\int_0^t \Phi_W(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t}$$

and

$$\mathrm{ind}(Q_+) = \lim_{t\downarrow 0} \int_{\mathbb{R}^n} \mathrm{d}x \int \det\left(I - e^{-t\hbar\Omega - \hbar\int_0^t \mathbb{W}(q(s))\mathrm{d}s}\right) e^{-\int_0^t \Phi_W(q(s))\mathrm{d}s} \mathrm{d}\mu_{x,x;t}.$$

By computing the right hand sides, one may obtain more explicit forms of $\mathrm{ind}(Q_+)$. But this would require careful and elaborate mathematical analysis which may be the subject of an independent article. Thus, in the present paper, we do not discuss this aspect and leave it for future study.

Appendix

In this appendix, we denote by $\mathcal{H}$ an abstract complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ (linear in the second variable) and norm $\|\cdot\|$. We denote by $\mathfrak{B}(\mathcal{H})$ the set of bounded linear operators B on $\mathcal{H}$ with $D(B) = \mathcal{H}$.

A Perturbation of a Self-adjoint Operator

Lemma A.1 *Let A be a non-negative self-adjoint operator on $\mathcal{H}$ and B be a symmetric operator on $\mathcal{H}$ which is relatively bounded with respect to A^α for some $\alpha \in [0,1)$, i.e., $D(A^\alpha) \subset D(B)$ and there exist constants $a, b \geq 0$ such that*

$$\|B\Psi\| \leq a\|A^\alpha\Psi\| + b\|\Psi\|, \quad \Psi \in D(A^\alpha). \tag{A.1}$$

Then $A + B$ is self-adjoint with $D(A+B) = D(A)$ and bounded below.

Proof. We denote by E_A the spectral measure of A. For all $\Psi \in D(A)$, we have $\|A^\alpha\Psi\|^2 = \int_{[0,\infty)} \lambda^{2\alpha}\mathrm{d}\|E_A(\lambda)\Psi\|^2$. Since $0 \leq \alpha < 1$, for each $\varepsilon > 0$, there exists a constant $c_\varepsilon \geq 0$ such that $\lambda^{2\alpha} \leq \varepsilon^2\lambda^2 + c_\varepsilon^2$ for all $\lambda \geq 0$. Hence $\|A^\alpha\Psi\|^2 \leq \varepsilon^2\|A\Psi\|^2 + c_\varepsilon^2\|\Psi\|^2$. Therefore, by (A.1), $\|B\Psi\| \leq a\varepsilon\|A\Psi\| + (ac_\varepsilon + b)\|\Psi\|$, $\Psi \in D(A)$. Hence B is infinitesimally small with respect to A. Thus, by the Kato-Rellich theorem (e.g., [11, Theorem X.12]), $A + B$ is self-adjoint with $D(A+B) = D(A)$ and bounded below. ∎

B Trace of Operators

In this section, we assume that $\mathcal{H}$ is separable. We denote the set of trace class operators on $\mathcal{H}$ by $\mathcal{I}_1(\mathcal{H})$ and the trace of $T \in \mathcal{I}_1(\mathcal{H})$ by $\mathrm{Tr}\, T$.

Proposition B.1 *Let $A_n, A \in \mathfrak{B}(\mathcal{H})$ $(n = 1, 2, \ldots)$ and* $\operatorname{s-lim}_{n\to\infty} A_n = A$, *where* s-lim *means strong limit. Then, for all $T \in \mathcal{I}_1(\mathcal{H})$,*

$$\lim_{n\to\infty} \mathrm{Tr}\,(A_nT) = \mathrm{Tr}\,(AT). \tag{B.1}$$

Proof. By a general theorem (e.g., [10, Theorem VI.22-(h)]), there exist Hilbert–Schmidt operators S and R such that $T = SR$. Let $\{e_k\}_{k=1}^{\infty}$ be a complete orthonormal system of $\mathcal{H}$ (we give a proof only for the case where $\mathcal{H}$ is infinite dimensional). Then we have

$$\mathrm{Tr}\,(A_n T) = \mathrm{Tr}\,(RA_n S) = \sum_{k=1}^{\infty} \langle R^* e_k, A_n S e_k \rangle \,.$$

It follows from the strong convergence of $\{A_n\}_n$ to A and the principle of uniform boundedness that $C := \sup_{n\geq 1} \|A_n\| < \infty$. Hence $|\langle R^* e_k, A_n S e_k \rangle| \leq C \|R^* e_k\| \|S e_k\|$. Since R^* and S are Hilbert–Schmidt, it follows that $\sum_{k=1}^{\infty} \|R^* e_k\| \|S e_k\| < \infty$. Hence, by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \mathrm{Tr}\,(A_n T) = \sum_{k=1}^{\infty} \langle R^* e_k, A S e_k \rangle = \mathrm{Tr}\,(RAS) = \mathrm{Tr}\,(AT).$$

∎

We denote the trace norm of $T \in \mathcal{I}_1(\mathcal{H})$ by $\|T\|_1$.

Lemma B.2 *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of self-adjoint operators on $\mathcal{H}$ satisfying the following:*

(a) *There exists a constant $c \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, $c \leq A_n$.*

(b) *For some $t > 0$ and all $n \in \mathbb{N}$, $e^{-tA_n} \in \mathcal{I}_1(\mathcal{H})$ with $T_0 := \sup_{n\in\mathbb{N}} \mathrm{Tr}\, e^{-tA_n} < \infty$.*

Suppose that there exist a self-adjoint operator A on $\mathcal{H}$ and a core $\mathcal{D}$ of A with the following property: $\mathcal{D} \subset \cap_{n\in\mathbb{N}} D(A_n)$ and, for all $\psi \in \mathcal{D}$. $\lim_{n\to\infty} A_n\psi = A\psi$. Then:

(i) *$e^{-tA} \in \mathcal{I}_1(\mathcal{H})$ and*

$$\mathrm{Tr}\, e^{-tA} \leq \liminf_{n\to\infty} \mathrm{Tr}\, e^{-tA_n}. \tag{B.2}$$

(ii) *If $A \preceq A_n$ ($\forall n \in \mathbb{N}$) in addition (for the symbol $\preceq$, see the paragraph just before Theorem 3.5), then*

$$\lim_{n\to\infty} \mathrm{Tr}\, e^{-tA_n} = \mathrm{Tr}\, e^{-tA} \tag{B.3}$$

and

$$\lim_{n\to\infty} \|e^{-tA_n} - e^{-tA}\|_1 = 0. \tag{B.4}$$

In particular, for all $B \in \mathfrak{B}(\mathcal{H})$,

$$\lim_{n\to\infty} \mathrm{Tr}\,(Be^{-tA_n}) = \mathrm{Tr}\,(Be^{-tA}). \tag{B.5}$$

Proof. (i) The assumption for $\{A_n\}_n$ implies that A is bounded below with $A \geq c$. Hence, by a general convergence theorem [10, Theorem VIII.25(a), Theorem VIII.20(b)],

$$\text{s-}\lim_{n\to\infty} e^{-tA_n} = e^{-tA}. \tag{B.6}$$

Let $\{e_k\}_{k=1}^{\infty}$ be a complete orthonormal system of $\mathcal{H}$. Then , for all $N \in \mathbb{N}$, $\mathrm{Tr}\, e^{-tA_n} \geq \sum_{k=1}^{N} \left\langle e_k, e^{-tA_n} e_k \right\rangle$. Hence

$$T_0 \geq \liminf_{n\to\infty} \mathrm{Tr}\, e^{-tA_n} \geq \sum_{k=1}^{N} \left\langle e_k, e^{-tA} e_k \right\rangle .$$

Taking the limit $N \to \infty$, we obtain

$$\sum_{k=1}^{\infty} \left\langle e_k, e^{-tA} e_k \right\rangle \leq \liminf_{n\to\infty} \mathrm{Tr}\, e^{-tA_n} \leq T_0.$$

Thus $e^{-tA} \in \mathcal{I}_1(\mathcal{H})$ and (B.2) holds.

(ii) It follows from $A \preceq A_n$ ($\forall n \in \mathbb{N}$) and the min-max principle (cf. [13, p.364, Problem 1]) that $\mathrm{Tr}\, e^{-tA_n} \leq \mathrm{Tr}\, e^{-tA}$ for all $n \in \mathbb{N}$. Hence $\limsup_{n\to\infty} \mathrm{Tr}\, e^{-tA_n} \leq \mathrm{Tr}\, e^{-tA}$. By this fact and (B.2), we obtain (B.3). Since e^{-tA_n} and e^{-tA} are positive self-adjoint, (B.4) follows from (B.6), (B.2) and Grümm's convergence theorem [16, Theorem 2.19]. Formula (B.5) follows from the well known inequality $|\mathrm{Tr}\,(BT)| \leq \|B\| \|T\|_1$ ($\forall B \in \mathfrak{B}(\mathcal{H}), \forall T \in \mathcal{I}_1(\mathcal{H})$). ∎

Lemma B.3 *Let A be a nonnegative self-adjoint operator on $\mathcal{H}$ such that, for all $t > 0$, $e^{-tA} \in \mathcal{I}_1(\mathcal{H})$. Let B_j ($j = 1, \ldots, n$) be a linear operator on $\mathcal{H}$ which is relatively bounded with respect to A^{α_j} with some $\alpha_j \in [0, 1)$. Let*

$$c_j := \|B_j(A+1)^{-\alpha_j}\|, \quad d_j := \max_{\lambda \geq 1} \lambda^{\alpha_j} e^{-\lambda}, \tag{B.7}$$

and $0 < s_1 < s_2 < \cdots < s_n < t$. Then the operator

$$K(s_1, \ldots, s_n) := e^{-s_1 A} B_1 e^{-(s_2 - s_1)A} B_2 \cdots e^{-(s_n - s_{n-1})A} B_n e^{-(t - s_n)A} \tag{B.8}$$

is in $\mathcal{I}_1(\mathcal{H})$ and

$$\|K(s_1, \ldots, s_n)\|_1 \leq e^{t/4} \|e^{-tA/4}\|_1 \prod_{j=1}^{n} \frac{4^{\alpha_j} c_j d_j}{(s_{j+1} - s_j)^{\alpha_j}}, \tag{B.9}$$

where $s_{n+1} := t$.

Proof. Throughout the proof, we set $K := K(s_1, \ldots, s_n)$. Let

$$K_j := e^{-(s_j - s_{j-1})A/2} B_j e^{-(s_{j+1} - s_j)A/2} \quad (j = 1, \ldots, n)$$

with $s_0 := 0$. Then we have

$$K = e^{-s_1 A/2} K_1 K_2 \cdots K_n e^{-(t - s_n)A/2}.$$

On the other hand, $B_j e^{-(s_{j+1} - s_j)A/2}$ is bounded, because it is the product of two bounded operators:

$$B_j e^{-(s_{j+1} - s_j)A/2} = B_j (A+1)^{-\alpha_j} \cdot (A+1)^{\alpha_j} e^{-(s_{j+1} - s_j)A/2}.$$

Hence K_j is trace class. Therefore K is trace class.

As usual, for a bounded linear opeartor T and $p > 0$, we set $\|T\|_p := (\mathrm{Tr}\,|T|^p)^{1/p}$ (the Schatten p-norm). By Hölder's inequality for the trace norm [16, Theorem 2.8], we have

$$\|K\|_1 \le \|e^{tA/2}\|_1^{s_1/t} \prod_{j=1}^n \|K_j\|_{t/(s_{j+1}-s_j)}.$$

We note that

$$\begin{aligned}\|K_j\|_{t/(s_{j+1}-s_j)} &\le \|e^{-(s_{j+1}-s_j)A/4}\|_{t/(s_{j+1}-s_j)}\|e^{-(s_j-s_{j-1})A/2}B_j e^{-(s_{j+1}-s_j)A/4}\| \\ &= \|e^{-tA/4}\|_1^{(s_{j+1}-s_j)/t}\|e^{-(s_j-s_{j-1})A/2}B_j e^{-(s_{j+1}-s_j)A/4}\|.\end{aligned}$$

Hence

$$\|K\|_1 \le \|e^{-tA/4}\|_1 \prod_{j=1}^n \|e^{-(s_j-s_{j-1})A/2}B_j e^{-(s_{j+1}-s_j)A/4}\|.$$

For all $r, s > 0$, we have

$$\begin{aligned}\|e^{-sA}B_j e^{-rA}\| &= \|e^{-sA}B_j(A+1)^{-\alpha_j}(A+1)^{\alpha_j}e^{-r(A+1)}\|e^r \\ &\le c_j d_j \frac{e^r}{r^{\alpha_j}}.\end{aligned}$$

Thus (B.9) follows. ∎

Lemma B.4 *Let A be a nonnegative self-adjoint operator on $\mathcal{H}$ and B be a symmetric operator on $\mathcal{H}$ which is relatively bounded with respect to A. Suppose that $A + B$ is self-adjoint and bounded below. Then, for all $t > 0$,*

$$e^{-t(A+B)} = e^{-tA} - \int_0^t e^{-sA}Be^{-(t-s)(A+B)}\mathrm{d}s, \tag{B.10}$$

where the integral on the right hand side is taken in the sense of strong Riemann integral.

Proof. A simple application of a general formla in perturbation theory of semi-groups (e.g., [8, p.502, (2.22)]). ∎

The following theorem gives a mathematically rigorous basis for the heuristic perturbation expansion for $e^{-t(A+B)}$ *in the trace norm*:

Theorem B.5 *Let A and B as in Lemma A.1. Suppose that, for all $t > 0$, $e^{-tA} \in \mathfrak{I}_1(\mathcal{H})$. Then, for all $t > 0$, $e^{-t(A+B)}$ is in $\mathfrak{I}_1(\mathcal{H})$ and*

$$\begin{aligned}e^{-t(A+B)} &= e^{-tA} + \sum_{n=1}^\infty (-1)^n \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{n-1}}^t \mathrm{d}s_n \\ &\qquad \times e^{-s_1A}Be^{-(s_2-s_1)A}B\cdots e^{-(s_n-s_{n-1})A}Be^{-(t-s_n)A}\end{aligned} \tag{B.11}$$

in the trace norm. Moreover, for all bounded linear operators S on $\mathcal{H}$,

$$\begin{aligned}\mathrm{Tr}\left(Se^{-t(A+B)}\right) &= \mathrm{Tr}\left(Se^{-tA}\right) + \sum_{n=1}^\infty (-1)^n \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{n-1}}^t \mathrm{d}s_n \\ &\qquad \times \mathrm{Tr}\left(Se^{-s_1A}Be^{-(s_2-s_1)A}B\cdots e^{-(s_n-s_{n-1})A}Be^{-(t-s_n)A}\right)\end{aligned} \tag{B.12}$$

Proof. By Lemma A.1, $A+B$ is self-adjoint and bounded below. As is shown in the proof of Lemma A.1, B is infinitesimally small with respect to A. Hence, for all $\kappa>0$, $\kappa A+B$ is self-adjoint and bounded below. Let $0<\kappa<1$ and write

$$A+B=(1-\kappa)A+(\kappa A+B).$$

For all $t>0$, $e^{-t(1-\kappa)A/2}e^{-t(\kappa A+B)}e^{-t(1-\kappa)A/2}$ is trace class. Hence, by a general theorem [13, p.320, Corollary], $e^{-t(A+B)}$ is trace class. Iterating (B.10), we have

$$\begin{aligned} e^{-t(A+B)} &= e^{-tA}+\sum_{n=1}^{N}(-1)^n\int_0^t \mathrm{d}s_1\int_{s_1}^t \mathrm{d}s_2\cdots\int_{s_{n-1}}^t \mathrm{d}s_n \\ &\quad \times e^{-s_1A}Be^{-(s_2-s_1)A}B\cdots e^{-(s_n-s_{n-1})A}Be^{-(t-s_n)A}+R_N, \end{aligned} \tag{B.13}$$

where

$$\begin{aligned} R_N &:= (-1)^{N+1}\int_0^t \mathrm{d}s_1\int_{s_1}^t \mathrm{d}s_2\cdots\int_{s_N}^t \mathrm{d}s_{N+1} \\ &\quad \times e^{-s_1A}Be^{-(s_2-s_1)A}B\cdots e^{-(s_{N+1}-s_N)A}Be^{-(t-s_{N+1})(A+B)}. \end{aligned}$$

By Lemma B.3, $e^{-s_1A}Be^{-(s_2-s_1)A}B\cdots e^{-(s_n-s_{n-1})A}Be^{-(t-s_n)A}$ $(0<s_1<\cdots<s_n<t)$ is trace class and

$$\|e^{-s_1A}Be^{-(s_2-s_1)A}B\cdots e^{-(s_n-s_{n-1})A}Be^{-(t-s_n)A}\|_1\le e^tC^n\|e^{-tA/4}\|_1\prod_{j=1}^n\frac{1}{(s_{j+1}-s_j)^\alpha}, \tag{B.14}$$

where $C:=\|B(A+1)^{-\alpha}\|\cdot 4^\alpha\sup_{\lambda\ge1}\lambda^\alpha e^{-\lambda}$.

It is easy to prove the following integral formula:

$$\int_a^t\frac{1}{(s-a)^p(t-s)^q}\mathrm{d}s=\frac{B(1-p,1-q)}{(t-a)^{p+q-1}},\quad 0<a<t, 0\le p,q<1, \tag{B.15}$$

where $B(\cdot,\cdot)$ is the Beta function. Using this formula, we obtain

$$\int_0^t \mathrm{d}s_1\int_{s_1}^t \mathrm{d}s_2\cdots\int_{s_{n-1}}^t \mathrm{d}s_n\prod_{j=1}^n\frac{1}{(s_{j+1}-s_j)^\alpha}=\frac{\Gamma(1-\alpha)^n}{n(1-\alpha)\Gamma(n(1-\alpha))}t^{n(1-\alpha)},$$

where $\Gamma(\cdot)$ is the Gamma function. Hence

$$\begin{aligned} &\int_0^t \mathrm{d}s_1\int_{s_1}^t \mathrm{d}s_2\cdots\int_{s_{n-1}}^t \mathrm{d}s_n\|e^{-s_1A}Be^{-(s_2-s_1)A}B\cdots e^{-(s_n-s_{n-1})A}Be^{-(t-s_n)A}\|_1 \\ &\le e^tC^n\|e^{-tA/4}\|_1\frac{\Gamma(1-\alpha)^n}{n(1-\alpha)\Gamma(n(1-\alpha))}t^{n(1-\alpha)}. \end{aligned}$$

Thereore, each term in the sum $\sum_{n=1}^N$ on the right hand side of (B.13) is trace class. ¿From the proof of Lemma B.3, we see that (B.14) with $(t-s_n)A$ replaced by $(t-s_n)(A+B)$ holds. Hence R_N is trace class and

$$\|R_N\|_1\le e^tC^{N+1}\|e^{-tA/4}\|_1\frac{\Gamma(1-\alpha)^{N+1}}{(N+1)(1-\alpha)\Gamma((N+1)(1-\alpha))}t^{(N+1)(1-\alpha)}.$$

As is well known (e.g., [19, p.279]),

$$\Gamma((N+1)(1-\alpha)) \sim \sqrt{2\pi}e^{-(N+1)(1-\alpha)}((N+1)(1-\alpha))^{(N+1)(1-\alpha)-1/2} \quad (N \to \infty).$$

Hence it follows that $\lim_{N\to\infty} \|R_N\|_1 = 0$. Thus (B.11) holds in the trace norm. Formula (B.12) follows from the continuity of the mapping $\mathrm{Tr}\,(S\cdot)$ on $\mathcal{I}_1(\mathcal{H})$ and estimate (B.14) which allows one to interchange between the trace operation Tr and the iterated integral $\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{n-1}}^t \mathrm{d}s_n$. ∎

Acknowledgment

This work is supported by the Grant-In-Aid No.24540154 for Scientific Research from JSPS.

References

[1] A. Arai, Path integral representation of the index of Kähler–Dirac operators on an infinite dimensional manifold, *J. Funct. Anal.* **82** (1989), 330–369.

[2] A. Arai, A general class of infinite dimensional Dirac operators and path integral representation of their index, *J. Funct. Anal.* **105** (1992), 342–408.

[3] A. Arai, Trace formulas, a Golden–Thompson inequality and classical limit in boson Fock space, *J. Funct. Anal.* **136** (1996), 510–547.

[4] A. Arai, A new estimate for the groundstate energy of Schrödinger operators, *Lett. Math. Phys.* **42** (1997), 215–227.

[5] B. K. Bagchi, Supersymmetry in Quantum and Classical Mechanics, Chapman & Hall/CRC, 2001.

[6] J. Glimm and A. Jaffe, *Quantum Physics*, Springer-Verlag, New York, 1981.

[7] S. Golden, Lower bounds for the Helmholtz function, *Phy. Rev.* **137** no. 4B (1965), B1127–B1128.

[8] T. Kato, *Perturbation Theory for Linear Operators*, 2nd Edition, Springer-Verlag, Berlin Heidelberg, 1976.

[9] S. Klimek and A. Lesniewski, A Golden–Thompson inequality in supersymmetric quantum mechanics, *Lett. Math. Phys.* **21** (1991), 237–244.

[10] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol.I: Functional Analysis*, Academic Press, New York, 1972.

[11] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol.II: Fourier Analysis and Self-adjointness*, Academic Press, New York, 1975.

[12] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol.III: Scattering Theory*, Academic Press, New York, 1979.

[13] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol.IV: Analysis of Operators*, Academic Press, New York, 1978.

[14] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Springer, Dordrecht, 2012.

[15] B. Simon, *Functional Integration and Quantum Physics*, Academic Press, New York, 1979.

[16] B. Simon, *Trace Ideals and Their Applications*, American Mathematical Society, 2005.

[17] B. Thaller, *The Dirac Equation*, Springer-Verlag, Berlin Heidelberg, 1992.

[18] C. J. Thompson, Inequality with applications in statistical mechanics, *J. Math. Phys.* **6** (1965), 1812–1813.

[19] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Fourth Edition (reprinted), 1969.

[20] E. Witten, Dynamical breaking of supersymmetry, *Nucl. Phys.* **B185** (1981), 513–554.

[21] E. Witten, Constraints on supersymmetry breaking, *Nucl. Phys.* **B202** (1982), 253–316.