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On the Existence of Competitive Equilibrium in Classical Finite Economies without Survival Assumption : Resource Relatedness Case

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This paper shows that the theorem on the existence of competitive equilibrium in classical finite economies with ordered preferences but without individual survival is proved with using generalized notion of resource relatedeness condition of Arrow-Hahn (1971) as well as with using irreducibility of McKenzie (1959, 1981).

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1. Introduction

One of the crucial aspects in the proof of the existence of competitive equilibrium in market economies is to assure that every consumer has cheaper points at a quasi-equilibrium or compensated equilibrium. The simplest condition for this situation to hold is the strong individual survival assumption used in Arrow-Debreu (1954, Theorem 1) where every consumer can provide some of all goods from his initial endowment. This condition is also used in Debreu (1959). This condition is, however, inappropriate in world trade economies with non-traded goods since each country is unable to be endowed with the non-traded goods of other countries by the definition. Then Arrow-Debreu (1954, Theorem 2) weakened, besides the individual survival assumption, this strong individual survival assumption to the condition that every consumer has a productive goods. This goods can produce more of goods which are commonly desirable to every consumer. This situation to hold corresponds to the fact that labor of consumers is useful in the economy. Moreover this condition is generalized as the resource relatedness condition in Arrow-Hahn (1971). From the same reason, McKenzie (1959) uses irreducibility, besides the individual survival assumption, which is based on the situation where each consumer has something, not necessarily labor, which is desirable to the rest of consumers. In any case, all of these suppose the individual survival assumption where the initial endowments of consumers are in their consumption sets, so

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that they can survive even without trading other consumers though markets.

Moore (1975), then, shows that irreducibility assures that every consumer has cheaper points at a compensated equilibrium, which is a variant of quasi-equilibrium, even without the individual survival assumption. Thus, as long as consumers have something which is desirable to others, they can survive with trading each other through markets even they can not survive alone. In other words, the existence of others are beneficial and helpful in order for economies to go well without causing people who can not survive. Notice, however, that although this result seems to imply the importance of markets trading for the survival of consumers, it is irreducibility that is fundamental for market to make consumers to survive.[1)]

In the world economy there are so many goods and consumers in each country and they can have goods imported from the outside of the country even they can not have them as a part of their initial endowments. Thus the individual survival assumption is extremely difficult to hold to them in this world economy. Then, it is important, for the world trade to make people survival, to assure that resource relatedness or irreducibility holds in the entire world.[2)]

Although Moore (1975) uses transitive preferences, McKenzie (1981, Theorem 3) establishes, with using irreducibility, the existence of competitive equilibrium in finite economies without individual survival and transitive preferences, based on the existence of competitive equilibrium without transitive preferences established by Gale-MasColell (1975) and Shafer-Sonnenschein (1975).

The irreducibility condition of McKenzie (1959, 1981) as well as the resource relatedness condition of Arrow-Hahn (1971) are used to assure that every consumer has cheaper points at a quasie-quilibrium (compensated equilibrim). Although these conditions play the similar role for the similar purpose, resource relatedness, however, supposes the individual survival assumption of consumers in the definition, and hence it can not applies to the case without the individual survival assumption. Thus, the irreducibility condition has much wider applicability to competitive economies than that of re-

1) Figure1, 2, and 3 below illustrate the role of resource relatedenss as well as irreducibility in the existence of competitive equilibrium in an exchange economy without the individual survival assumption.

2) Recently globalization of the world economy is often mentioned at many occasions. For the result of the globalization to go well without making any of people in the world unable to survive, it is necessary to guarantee that the world economy after globalized satisfies the irreducibility condition or resource relatedeness condition so that people are beneficial each other. From this viewpoint, the reason why so many anti-globalization protects happen is that the world economy after globalization may not satisfy the irreducibility condition or resource relatedeness condition. This argument also relates closely to that of gains from free trade. For the relation between glozalization and gains from free trade, see Bhagwati (2002)

source relatedness since irreducibility assures that every consumer has cheaper points at a compensated equilibrium even without the individual survival assumption.

Then it is an interesting question whether resource relatedness is employed even in the case without the individual survival assumption to make every consumer have cheaper points at a quasi-equilibrium. This is not addressed before in the literature on the existence of competitive equilibrium. Of course, since the original definition of resource relatedness based on the individual survival assumption, it is necessary to generalize the notion of resource relatedness so that it is applicable to the case without the individual survival assumption. Thus it is interesting to generalize the notion of resource relatedness in economies without the individual survival assumption and to show that the generalized resource relatedness dispenses with the individual survival assumption and still makes possible to prove the existence of competitive equilibrium in such economies even without the individual survival assumption.

Therefore, this paper considers the existence of competitive equilibrium in market economies where the notion of resource relatedness of Arrow-Hahn (1971) is generalized but the individual survival assumption is not assumed. Transitivity of preferences and non-empty interior of the production set is, however, used to make the argument simple. The result obtained is close to that of McKenzie (1981, Theorem 2) where irreducibility of McKenzie (1958, 81) is used in the case without the individual survival assumption instead of resource relatedness. This paper also discusses the comparison between the generalized resource relatedness and irreducibility from the viewpoint of existence of competitive equilibrium in economies without the individual survival assumption.

2. Model

This paper employs a classical finite economy so that the number of commodities and consumers are finite. Also technology is assumed to be constant returns to scale. Let I be the set of consumers in the economy, $\{1, \cdot\cdot, H\}$, J be the set of firms exist in the economy, $\{1, \cdot\cdot, F\}$, and L be the set of the the commodities exist in the economy, $\{1, \cdot\cdot, N\}$. Then the commodity space of the economy is R^N. Let Y_k and C_i be the production set of the k-th firm and the consumption set of the i-th consumer. Since this paper employs net trading sets as consumption sets, consumption set C_i is indeed removed by the initial endowment of the i-th consumer.[3] Then C_i is bounded below by $\mathrm{a}_\mathrm{i} \in -(\mathrm{R}^N_+)$ and hence, $\mathrm{C}_\mathrm{i} \subset (\mathrm{a}_\mathrm{i} + \mathrm{R}^\mathrm{N}_+)(= \mathrm{C}'_\mathrm{i})$ holds for $\mathrm{i} \in \mathrm{I}$. This paper, however, does not use the individual survival assumption, $0 \in \mathrm{C}_\mathrm{i}$, so that $0 \notin \mathrm{C}_{\mathrm{i}'}$ may occur for some

$i' \in I$. Let R_i be a (weak) preference relation of the i-th consumer defined over C_i. $x' R_i x$ expresses that x' is at least as good as x.[4] P_i is, then, a strict preference relation (defined over C_i) derived from R_i, and $x' P_i x$ expresses that x' is preferred to x.[5] Also let Y, C and C' be $\sum_{k \in J} Y_k$, $\sum_{i \in I} C_i$ and $\sum_{i \in I} C'_i$, i.e., the aggregate production set, the aggregate consumption set, and the aggregate extended consumption set, respectively. Then $x \in C$ and $y \in Y$ are equivalent to $x_i \in C_i$ for $i \in I$ and $y_k \in Y_k$ for $k \in J$. This paper uses the aggregate production set as a primitive concept. Since C_i is bounded by a_i ($\in -(R^N_+)$), C is bounded by $a = \sum_{i \in I} a_i$ ($\leq a_i$)($\in -(R^N_+)$) and hence $C \subset (a + R^N_+)$ holds. Note that $C_i \subset (a + R^N_+)$ as well as $(\sum_{h \neq i} C_h) \subset (\sum_{h \neq i} C'_h)$ $(= \sum_{h \neq i} (a_h + R^N_+)) \subset (a + R^N_+)$ holds from the definition of a.

A vector $(x_i, y)_{i \in I}$ is called an *allocation* in the economy if $x_i \in C_i$ for $i \in I$ and $y \in Y$ hold. Also an allocation $(x_i, y)_{i \in I}$ is called *feasible* if $\sum_{i \in I} x_i$ $(\equiv x) = y (\equiv \sum_{k \in J} y_k)$ also holds. We define $\widehat{C_i}$ and $\widehat{Y}$ by $\widehat{C_i} = C_i \cap (Y - \sum_{h \neq i} C_h)$ and $Y \cap C$. Then the feasibility of $(x_i, y)_{i \in I}$ implies $x_i \in \widehat{C_i}$ for $i \in I$ and $y \in \widehat{Y}$. We also define $\widehat{C'_i}$ and $\widehat{Y'}$ by $C'_i \cap (Y - (\sum_{h \neq i} C'_h))$ and $Y \cap C'$. Since $(\sum_{h \neq i} C_h) \subset (\sum_{h \neq i} C'_h)$ and $C \subset C'$ hold, $\widehat{C_i} (= [C_i \cap (Y - \sum_{h \neq i} C_h)]) \subset \widehat{C'_i}$ $(= [C'_i \cap (Y - (\sum_{h \neq i} C'_h))])$ and $\widehat{Y}$ $(= (Y \cap C)) \subset \widehat{Y'}$ $(= [Y \cap C'])$ hold. Then if $\widehat{C'_i}$ is bounded, then so is $\widehat{C_i}$. In fact, we show below that $\widehat{C'_i}$ is bounded. Then there is a sufficiently large hypercube K around 0 satisfying $\widehat{C'_i} \subset$ *int* (K), and we truncate the relevant part of the consumption set C_i and define $\widetilde{C_i}$ by $C_i \cap K$[6]. We further define $\widehat{C''_i}$ by $C_i \cap (Y - (\sum_{h \neq i} C'_h))$. Then $\widehat{C''_i} = \widehat{C'_i} \cap C_i$ holds. $\widehat{C''_i}$ is a part of i-th consumer's consumption set which is consistent with feasibility condition when the other consumers' consumption sets are the extended C'_i instead of the original C_i. Since $(\sum_{h \neq i} C_h) \subset (\sum_{h \neq i} C'_h)$ holds, $\widehat{C_i} (= [C_i \cap (Y - \sum_{h \neq i} C_h)]) \subset \widehat{C''_i}$ $(= [C_i \cap (Y - \sum_{h \neq i} C'_h))]) \subset \widehat{C'_i}$ $(= [C'_i \cap (Y - \sum_{h \neq i} C'_h)])(\subset$ *int* (K)) holds. Then, $\widehat{C''_i} \subset \widetilde{C'_i} = (C_i \cap K)$ also holds.

3) When C_i is a consumption set and w_i is his initial endowment, the trading set is defined as $C_i - w_i$ and is used as the consumption set in the text. Note that C_i as well as w_i is assumed to be in R^N_+ owing to the device by Arrow-Hahn (1971).

4) R_i is also considered as a correspondence from C_i into C_i by defining R_i $(x) = \{x' C_i | x' R_i x\}$. Note also that we can define a correspondence R_i^{-1} from C_i into C_i by defining R_i^{-1} $(x) = \{x' \in C_i | x R_i x'\}$. This R_i^{-1} is usually called a lower section of the correspondence R_i. In the case with two goods, R_i (x) corresponds to the region above the indifference curve I(x) through x and contains I(x) as its boundary, and R_i^{-1} (x), on the other hand, corresponds to the region below the indifference curve I(x) through x and contains I(x) as its boundary. Moreover, the graph of R_i, $Gr(R_i)$ is defined as $\{(x, x') \in C_i \times C_i | x' \in R_i(x)\}$.

5) $x' P_i x$ is defined by $x' R_i x$ but not $x R_i x'$. Then P_i is also considered as a correspondence from C_i into C_i by defining $P_i(x) = \{x' \in C_i | x' P_i x\}$. As in the case of R_i, we can define from the correspondence P_i its lower section P_i^{-1} from C_i into C_i by P_i^{-1} $(x) = \{x' \in C_i | x P_i x'\}$. In the case with two goods, P_i(x) corresponds to the region above the indifference curve I(x) through x but does not contain I(x), and P_i^{-1} (x) corresponds to the region below the indifference curve I(x) through x but does not contain I(x) as its boundary. Note that since transitivity of preference relation R_i is assumed, $R_i(x) = (P_i^{-1}(x))^c$ and $R_i^{-1}(x) = (P_i(x))^c$ as well as $P_i(x) = (R_i^{-1}(x))^c$ and $P_i^{-1}(x) = (R_i(x))^c$ hold. The graph of P_i, $Gr(P_i)$ is defined as $\{(x, x') \in C_i \times C_i | x' \in P_i(x)\}$.

6) The existence of such a K is possible by lemma 2 below.

Since we do not employ $0 \in \widetilde{C}_i$, the individual survival assumption, $0 \notin \widetilde{C}_{i'}$ may occur for some $i' \in I$. In this situation, the budget set of consumer i' may be empty and hence demand set may be empty. Thus, we define $\overline{C}_i$ as the convex hull of 0 and $\widetilde{C}_i$ and extend $\widetilde{C}_i$ to $\overline{C}_i$ for each $i \in I$. Then since $0 \in \overline{C}_i$ holds for each $i \in I$, and hence each consumer can purchase 0, his budget set becomes non-empty for any prices. This is a device used in Debreu (1962) to exclude the emptiness of budget sets, and used also in McKenzie (1981). Note that $\overline{C}_i \subset C'_i$ holds from the defintition of C'_i and $\widehat{C}''_i (= \widehat{C}'_i \cap C_i) \subset \widetilde{C}_i = (C_i \cap K) = (\overline{C}_i \cap C_i)$ also holds from the definition of $\widehat{C}''_i$. Note also that since $\overline{C}_i$ is the convex hull of 0 and $\widetilde{C}_i$, $x_i \in \overline{C}_i$ implies that $x_i = \alpha x'_i$ for some $x'_i \in C_i$ and $\alpha \in (0, 1]$. Thus, when $p \cdot x_i < 0$ for $x_i \in \overline{C}_i$, there is $x'_i \in C_i$, satisfying $p \cdot x'_i < 0$. When a consumer has a cheaper point over $\overline{C}_i$, then such a consumer also has a cheaper pint over the original consumption set C_i.

We call a vector $(x_i, y)_{i \in I}$ a *pseudo-allocation* if $x_i \in \overline{C}_i$ for $i \in I$, $y \in Y$ hold, and a *feasible pseudo-allocation* if $\sum_{i \in I} x_i = y$ further hold.[7] Note that even if $x_i \in \overline{C}_i$ holds for any $i \in I$, $x_{i'} \notin \widetilde{C}_{i'}$ and hence $x_{i'} \notin C_{i'}$ may occur for some $i' \in I$. This is a reason why we call such a vector $(x_i, y)_{i \in I}$ a pseudo-allocation. Note also that the lack of the individual survival assumption implies that $x_{i'} \in \overline{C}_{i'}$ but $x_{i'} \notin \widetilde{C}_{i'}$ and hence $x_{i'} \notin C_{i'}$ holds for some $i' \in I$ even if $y \in Y$ and $\sum_{i \in I} x_i = y$ hold. Then although $x_i \notin C_i$ occurs for some $i \in I$, $x_i \in \widehat{C}'_i$ still holds since $(x_i \in)$ $\overline{C}_i \subset C'_i$ and $\sum_{i \in I} x_i = y$ imply $x_i \in \widehat{C}'_i$ $(\subset \mathit{int}(K))$ holds. This is indeed a reason why we also use $\widehat{C}'_i$ besides $\widehat{C}_i$. In the argument later of proving a competitive equilibrium in the economy, we first show that there is a candidate $(x_i, y)_{i \in I}$ for a competitive equilibrium such that $x_i \in \overline{C}_i$ for $i \in I$, $y \in Y$, and $\sum_{i \in I} x_i = y$ hold. Thus, $(x_i, y)_{i \in I}$ is shown to be only a feasible pseudo-allocation. We then establish that indeed $x_i \in \widehat{C}_i$ holds for $i \in I$ owing to resource relatedness of the economy and aggregate adequacy of the economy. This basic argument is similar to that of the case with irreducibility of the economy.

We call a vector $(x^*_i, y^*, p^*)_{i \in I}$ with $p^* \in R^N \setminus \{0\}$ a *competitive equilibrium* when it satisfies the following three conditions:

7) In the case without ordered preferences of Kubota (1997), as the extended consumption set of the i-th consumer C'_i, the closed convex hull of 0 and C_i is used for each $i \in I$. Trivially the new consumption sets C'_i contain 0. According to this expansion of net trading set, the preference relation P_i is also extended properly. Let C' be $\sum_{i \in I} C'_i$, the aggregate extended consumption set. Then a vector $(x_i, y)_{i \in I}$ is called a *pseudo-allocation* if $x_i \in C'_i$ and $y \in Y$ hold for $i \in I$, and called *feasible* if $\sum_{i \in I} x_i = y$ also holds. Similarly, define $\widehat{C}'_i = (C'_i \cap (Y - \sum_{h \neq i} C'_h))$ and $\widehat{Y}' = (Y \cap C')$. The feasibility of pseudo-allocation $(x_i, y)_{i \in I}$ means $x_i \in \widehat{C}'_i$ for $i \in I$ and $y \in \widehat{Y}'$. Then, even when $(x_i, y)_{i \in I}$ is *feasible pseudo-allocation*, it is, however, not necessarily an *allocation* since some x_i may not be in C_i. In the case without ordered preferences, it is shown first that a candidate for a competitive equilibrium is a feasible pseudo-allocation, and then established that it is indeed an allocation owing to irreducibility of economy. It is also an interesting question to establish the similar result in economies without individual survival and transitive preferences when the irreducibility assumption is replaced with the generalized notion of resource relatedness in this paper.

(i) The allocation $(x_i^*, y^*)_{i \in I}$ is feasible.
(ii) $p^* \cdot x_i^* \leqq 0$ holds and $x_i' \in P_i(x_i^*)$ implies $p^* \cdot x_i' > 0$.
(iii) $p^* \cdot y^* = 0$ holds and $y \in Y$ implies $0 \geqq p \cdot y$.

Note that when we use the production set Y_k of each firm as a primitive concept instead of the aggregate production set, the aggregate profit maximization condition (iii) is equivalent to the profit maximization condition of each firm. [8)]

We also call a vector $(x_i^*, y^*, p^*)_{i \in I}$ with $p^* \in R^N \setminus \{0\}$ a *pseudo quasi-equilibrium* when it satisfies the following three conditions :

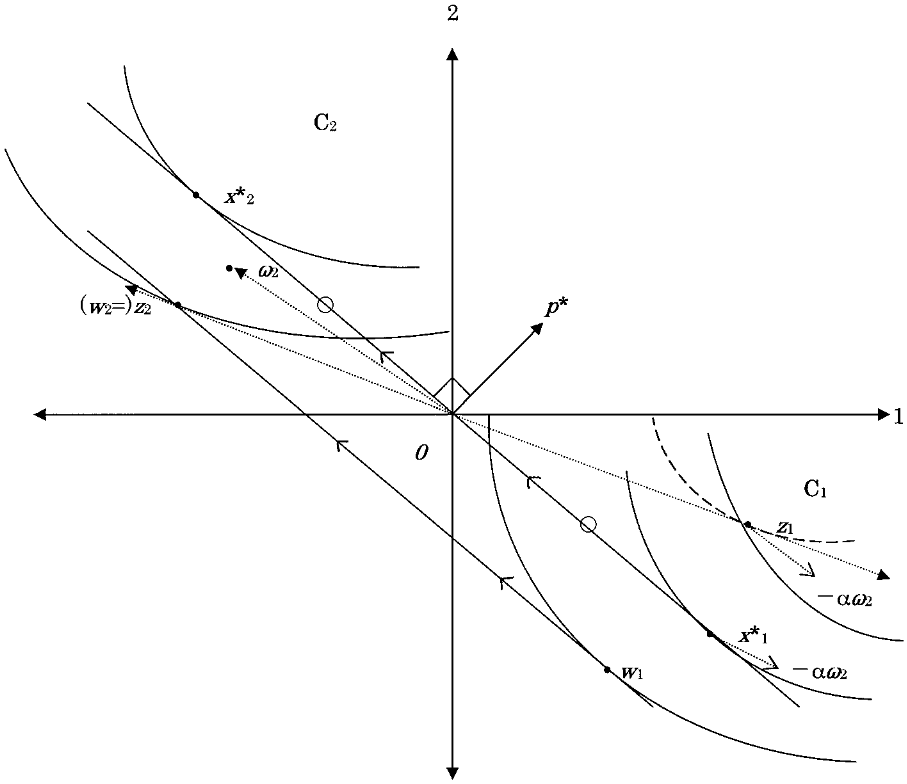
(i′) The allocation $(x_i^*, y^*)_{i \in I}$ is a pseudo feasible allocation.
(ii′) $p^* \cdot x_i^* \leqq 0$ holds and $x_i \in P_i(x_i^*)$ implies $p^* \cdot x_i > 0$ for $i \in \{i \in I | p^* \cdot x_i' < 0$ for some $x_i' \in C_i\}$.
(iii′) $p^* \cdot y^* = 0$ holds and $y \in Y$ implies $0 \geqq p^* \cdot y$.

There are two differences in this definition with that of competitive equilibrium. Since even when $y \in Y$ and $\sum_{i \in I} x_i = y$ hold besides $x_i \in \overline{C_i}$, $x_i \notin C_i$ may occur for some other $i \in I$ due to the lack of the individual survival assumption, we use $(x_i, y)_{i \in I}$ as a feasible pseudo allocation. Moreover, owing to the definition of Debreu (1962), we apply preference maximization under budget constraint only to the consumers with cheaper points (ii′). This modification is necessary since every consumer is unable to be shown to have cheaper points generally without the resource relatedness condition even with the aggregate adequacy condition. Thus, we first show the existence of a pseudo quasi-equilibrium in the economy, and then show that it is indeed a feasible allocation owing to the resource relatedness condition and the aggregate adequacy condition.

The following figure illustrates the existence of competitive equilibrium in a pure exchange economy with two consumers but without the individual survival assumption.[9)]
Since each C_1 and C_2 does not contains the origin, the individual survival assumption does not hold, and hence each consumer can not survive alone. $x_1^* + x_2^* = 0$ at (x_1^*, x_2^*) implies that (x_1^*, x_2^*) is feasible. Also $p^* \cdot x_1^* = p^* \cdot x_2^* = 0$ for p^* implies that income is exhausted at (x_1^*, x_2^*) with p^*. Moreover, since $p^* \cdot w_1 < 0$ and $p^* \cdot w_2 < 0$ hold, the cheaper point condition holds at (x_1^*, x_2^*) with

8) See, for example, Debreu (1959, 3.4. (1), p.45). Note also that in this case the income of every consumer is composed from only his initial endowment when $0 \in Y_k$ is assumed and hence zero profit to Y_k as well as Y holds because of the choice of price systems in the dual cone of a convex cone Y with the vertex at the origin. Then it is irrelevant whether consumers to possess the shares of firms or not.
9) This figure is a variant of McKengie (1999, Figure 2).

Figure 1 . Existence of Competitive Equilibrium without the Individual Survival : Pure Exchange Economy Case



p^*. Also, $x_i' \in P_1$ $(x_i^*) \rightarrow p^* \cdot x_i' > 0$ and $x_2' \in P_2$ $(x_2^*) \rightarrow p^* \cdot x_2' > 0$ hold, and hence, each consumer achieves the preference maximization under budget constraint. Thus, (x_1^*, x_2^*, p^*) is a competitive equilibrium. Note that an offer of (α times) ω_2 by consumer 2 makes consumer 1 a better position over z_1. This indeed implies that the resource relatedness condition holds in this case, as explained later. The basic point is that the indifference curve passing through z_1 intersects the budget line through z_1 from above. When the indfference curve (as a dotted line) passing through z_1 tangents the budget line at z_1, an offer of (α times) ω_2 by consumer 2 make consumer 1 move below the budget line through z_1. It is impossible for consumer 1 to be better off than at z_1 by an offer of (α times) w_2 from consumer 2 . In this situation, the resource relatedness condition does not hold. It is violated at (z_1, z_2).

We first define resource relatedness of an economy without individual survival before proceeding to the proof of the existence of competitive equilibrium in the economy below. We first consider the original definition of resource relatedness by Arrow-Hahn (1971) where individual survival is assumed. The i-th consumer is called *resource related* to the j-th consumer when the following condition holds: There is $-\delta e_i \in C_i$ for $\delta > 0$ such that for any feasible allocation $(x_h, y)_{h \in I}$ there is $(x'_h)_{h \in I}$ satisfying $\sum_{h \in I} x'_h \in \sum_{h \in I}[C_h \cap (Y + \delta e_i)]$ with $x'_h \in R_h(x_h)$ holds for $h \in I$ and $x'_j \in P_j\ (x_j)$ holds. The basic idea of this definition is that when consumer i offers one goods that he can such as his labor to the production process in the entire economy, then the outcome is to make every consumer not worse off and the j-th consumer better off than before. This definition owes to the existence of productive goods and commonly desirable consumption goods used in Arrow-Debreu (1954). It also weakens the assumption that every consumer can offer any goods to the one that every consumer can offer at least one goods. When the i-th consumer has a sufficient income at a quasi-equilibrium, he is willing to pay implicitly for the goods that j-th consumer can offer, and as a result, j-th consumer turns out to have a sufficient income as well. This is a main object to employ resource relatedness.

Furthermore, the i-th consumer is called *indirectly resource related* to the j-th consumer when there are a series of consumers $\{h_i\}_{i=1}^{m}$ such that the i-th consumer is resource related to the h_1-th consumer, the h_1-th consumer is resource related to the h_2-th consumer, ……, the h_m-th consumer is resource related to the j-th consumer. That is, the i-th consumer is linked with the j-th consumer through resource related consumers between them.

In the case without individual survival, however, some of candidates for competitive equilibrium are only pseudo quasi-equilibria so that they may be only feasible pseudo-allocations. Thus, we need to have the similar condition applicable even to feasible pseudo-allocations not only to feasible allocations. Moreover, although individual survival is assumed in the original definition and every consumer has at least some goods to offer, in the case without individual survival he may have only a bundle of goods to offer instead of some certain goods. Thus, the definition of resource relatedness without induvidual survival have to take account of these two aspects. It is defined as following. The i-th consumer is called *resource related* to the j-th consumer when the following condition holds:

There is $z_i \in \overline{C_i}$ such that for any pseudo feasible allocation $(x_h, y)_{h \in I}$ with $x_j \in C_j$ there is $(x'_h)_{h \in I}$ satisfying $\sum_{h \in I} x'_h \in [((\sum_{h \in I^1} C_h) + (\sum_{h \in I^2} \overline{C_h})) \cap (Y - \delta z_i)]$ $(\exists \delta > 0)$ such that $x'_h \in R_h(x_h)$ holds for $h \in I^1$, $x'_h \in \overline{C_h}$ holds for $h \in I^2$, and $x'_j \in P_j(x_j)$ holds, where $I^1 = \{h \in I : x_h \in C_h\}$ and $I^2 = \{h \in I : x_h \in \overline{C_h} \backslash C_h\}$.

The basic idea of this definition is analogous to the original. When the i-th consumer offers some bundle of goods that he can, instead of some certain goods in the original definition, to the production process in the entire economy, then an outcome is beneficial for the j-th consumer. Here x'_h as well as x_h is in $\overline{C_h}$ for $h \in I^2$ instead of $x'_h \in R_h\ (x_h)$ in the original definition. The consumers in I^2 may gain small enough so that thay may stay within $\overline{C_h}\setminus C_h$ from the i-th consumer's offer of some bundle of goods from $\overline{C_i}$ to the production process in the entire economy.[10)] When, however, $0 \in C_h$ and hence $\overline{C_h} = \widetilde{C_h}(\subset C_h)$ holds for $h \in I$, then feasible pseudo allocations are feasible allocations so that $I^2 = \{h \in I : x_h \in \overline{C_h}\setminus C_h\} = \emptyset$ or $I = I^1$ holds, and $-\delta e_i$ is used as z_i, the original definition satisfies the general definition as a special case. Thus this definition of resource relatedness is considered as a generalization of the original resource relatedness in the case without the individual survival assumption. Note that when the i-th consumer is resource related to the j-th consumer in this generalized sense there is $x_j' \in C_j$ with $x_j' \in P_j(x_j)$ so that non-satiation over $\widehat{C_j''}(\subset \widetilde{C_j}(=(C_j \cap K) = (\overline{C_i} \cap C_j)))$ holds for the j-th consumer.

The indirectly resource relatedness between the i-th consumer and the j-th consumer is defined analogously as the original. The i-th consumer is called *indirectly resource related* to the j-th consumer when there is a series of consumers $\{h_i\}_{i=1}^{m}$ such that the i-th consumer is resource related to the h_1-th consumer, the h_1-th consumer is resource related to the h_2-th consumer, ……, and the h_m-th consumer is resource related to the j-th consumer. That is, the i-th consumer is linked with the j-th consumer through resource related consumers between them. We discuss this definition of generalized resource relatedness in detail and compare to the original one of Arrow-Hahn (1971) and irreducibility of McKenzie (1959, 1981) in section 5. Note that as in the generalized resource relatedness case, when the i-th consumer is indirectly resource related to the j-th consumer in this generalized sense, there is $x_j' \in C_j$ with $x_j' \in P_j(x_j)$ so that non-satiation over $\widehat{C_j''}(\subset \widetilde{C_j})$ still holds for the j-th consume as well.

We first list the assumptions necessary to prove the existence of competitive equilibrium in the economy without individual survival. They are as following:

(1) Y is a closed convex cone with the vertex at the origin 0 .
(2) $Y \cap R^N_+ = \{0\}$.
(3) C_i is non-empty $(=\emptyset)$, convex, closed, and bounded below by $a_i\ (\in -(R^N_+))$.
(4) R_i is a complete preorder, and $Gr(R_i) = \{(x, x') \in C_i \times C_i \mid x' \in R_i(x)\}$ is closed (in $C_i \times C_i$).

10) Of course, the simplest generalization of the definition is to use $I^2 = \{h \in I : x_h \in \overline{C_h}\setminus C_h\} = \emptyset$ or $I = I^1$ so that the definition becomes as following: There is $z_i \in \overline{C_i}$ such that for any pseudo feasible allocation $(x_h, y)_{h \in I}$ with $x_j \in C_j$ there is $(x_h')_{h \in I}$ satisfying $\sum_{h \in I} x_h' \in [(\sum_{h \in I} C_h) \cap (Y - \delta z_i)] (\exists \delta > 0)$such that $x_h' \in R_h(x_h)$ holds for $h \in I$ and $x_j' \in P_j(x_j)$ holds. This definition supposes that the h-th consumer with $x_h \in \overline{C_h}\setminus C_h$ ends up with $x_h' \in C_h$ so that this consumer gains large enough to move into C_h from the i-th consumer's offer of some bundle of goods from $\overline{C_i}$ to the production process in the entire economy. There is, however, a problem with this definiton. We discuss this point in section 5 .

(5) $x' \in P_i(x)$ implies $\alpha x' + (1-\alpha)x \in P_i(x)\ \forall \alpha \in (0, 1)$.
(6) $int(Y) \cap C \neq \emptyset$.
(7) The economy is indirectly resource related.

Since (1),……, (6) are standard and we already discussed (7) above, we discuss them briefly. In (1), the aggregate convexity of the production set Y is used instead of the convexity of the production set of each firm Y_k.[11] When the production set of each firm is convex (and cone), then the aggregate production set is convex (and cone) as well.[12] Also when Y_k contains the origin 0, owing to the choice of price space, the income from holding the shares of firms is always zero. Thus, the source of a consumers attributes to his initial endowments although initial endowments of consumers do not appear since net trading sets are used for consumption sets explicitly in this paper. (2) is necessary to exclude the set of feasible allocations to be unbounded.

Since C_i is bounded by a_i in (3), C is also bounded by $a = \sum_{i \in I} a_i (\leq a_i)$ and hence $C \subset (a + R^N_+)$ as well as $C_i \subset (a + R^N_+)$ holds. These facts are already used to define $\widehat{C_i}$, $\widehat{C'_i}$ and $\overline{C_i}$. Since preference relation R_i is assumed to satisfy completeness, the continuity condition in (4) is equivalent to the other three alternative conditions: $(i)\ Gr(P_i) = \{(x, x') \in C_i \times C_i \mid x' \in P(x)\}$ is open in $C_i \times C_i$, (ii) $\{x' \in C_i \mid x' \in P_i(x)\}$ and $\{x' \in C_i \mid x' \in P_i^{-1}(x)\}$ is open in $C_i\ \forall x \in C_i$. (iii) $\{x' \in C_i \mid x' \in R_i(x)\}$ and $\{x' \in C_i \mid x' \in R_i^{-1}(x)\}$ is closed (in C_i) $\forall x \in C_i$.[13] Note also that transitivity of preferences with resect to R_i gives rise to that with resect to P_i and R_i.[14] (5) is a version of convexity of preferences and gives rise to that $x' \in R_i(x)$ implies $\alpha x' + (1-\alpha)x \in R_i(x)\ \forall \alpha \in (0, 1)$, weak convexity of preferences under the continuity of preferences (4).[15] Note also that the convexity condition (5) implies the local non-satiation of preferences as long as non-satiation $P_i(x) \neq \emptyset$ holds at $x \in C_i$.

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11) This aggregate convexity condition is used in Debreu (1959, 5.7 (1) (d.2), p.84-5. Notes 1., p.88) and Uzawa (1962).

12) As shown in McKenzie (1959), an economy with general convex production sets is transformed into that with convex cone prodction set once entrepreneurial factors of firms are intoduced. Thus, we employ the aggregate convex cone production set instead of general convex production sets.

13) For this result, see, for example, Nikaido (1968, lemma15.4 p.239) or Hildenbrand-Kirman (1988, prop 2.2, p.63).

14) Suppose $x' \in R_i(x)$ and $x'' \in P_i(x')$ but $x'' \notin P_i(x)$. Since $x'' \notin P_i(x)$ is equivalento to $x \in R_i(x'')$ by the definition, $x' \in R_i(x)$ and $x \in R_i(x'')$ give rise to $x' \in R_i(x'')$ from transitivity of preference R_i. Since $x' \in R_i(x'')$ is equivalent to $x'' \notin P_i(x')$, this is a contradition to $x'' \in P_i(x')$, and hence $x'' \in P_i(x)$ follows. Similar argument gives rise to $x' \in P_i(x)$ and $x'' \in R_i(x')$ implies $x'' \in P_i(x)$.

15) Since (5) implies that $x' \in P_i(x) \rightarrow \alpha x' + (1-\alpha)x \in R_i(x)\ \forall \alpha \in (0, 1)$ it is enough to show $x' \in I_i(x)$ and $x' \neq x \rightarrow \alpha x' + (1-\alpha)x \in R_i(x)\ \forall \alpha \in (0, 1)$. Suppose that there is $\alpha' \in (0, 1)$ satisfying $x'(\alpha')x \equiv \alpha' x' + (1-\alpha')x \in P_i^{-1}(x(x'))$. Then $x'(\alpha'')x \in [P_i(x'(\alpha')x) \cap P_i^{-1}(x(x'))]$ holds $\exists \alpha'' \in (\alpha', 1)$ by (4) and (5) and $x'(\alpha'')x \in [x'(\alpha')x, x']$. But $x'(\alpha')x \in P_i^{-1}(x'(\alpha'')x)$ holds again by (5) and $x'(\alpha')x \in [x'(\alpha'')x, x]$. Thus a contradiction occurs, and hence $x' \neq x \rightarrow \alpha x' + (1-\alpha)x \in R_i(x)\ \forall \alpha \in (0, 1)$. Thus, (5) implies $x' \in R_i(x) \rightarrow \alpha x' + (1-\alpha)x \in R_i(x)\ \forall \alpha \in (0, 1)$. This proof follows Debreu (1959, 4.7 (1), p.60)

(6) implies $int\,(\mathrm{Y}) \neq \emptyset$. Free disposal of production, $\mathrm{Y} \supset -\mathrm{R}^N_+$, is not assumed. We use $\mathrm{S}' \equiv \{\mathrm{p} \in (\mathrm{Y})^* : \mathrm{p} \cdot \bar{\mathrm{y}} = -1\}$ for some $\bar{\mathrm{y}} \in int(\mathrm{Y})$ as the set of prices.[16] Then since $\mathrm{p} \in (\mathrm{Y})^*$ and $\bar{\mathrm{y}} \in int(\mathrm{Y})$ imply $\mathrm{p} \cdot \bar{\mathrm{y}} < 0$, $\mathrm{p}' \equiv -(\mathrm{p} / \mathrm{p} \cdot \bar{\mathrm{y}}) \in \mathrm{S}'$ follows. Moreover, $\mathrm{p}' \cdot \mathrm{z} \leqq 0 \quad \forall\, \mathrm{p}' \in \mathrm{S}'$ implies $\mathrm{z} \in (\mathrm{Y})^{**} (= \mathrm{Y})$ since $\mathrm{p} \cdot \mathrm{z} = (\mathrm{p} \cdot \bar{\mathrm{y}})\, \mathrm{p} / (\mathrm{p} \cdot \bar{\mathrm{y}}) \cdot \mathrm{z} = -(\mathrm{p} \cdot \bar{\mathrm{y}})\, \mathrm{p}' \cdot \mathrm{z} \leqq 0$ holds $\forall\, \mathrm{p} \in (\mathrm{Y})^*$.[17] This fact is used later in the proof of Theorem 1. (6) also implies that there are some $\bar{\mathrm{x}}_\mathrm{i} \in \widetilde{\mathrm{C}}_\mathrm{i}\ \forall\, \mathrm{i} \in \mathrm{I}$ satisfying $\bar{\mathrm{x}} (\equiv \sum_{\mathrm{i} \in \mathrm{I}} \bar{\mathrm{x}}_\mathrm{i}) \in int\,(\mathrm{Y})$. Then profit maximization (iii) implies $\mathrm{p} \cdot \bar{\mathrm{x}} < 0$ $(= \mathrm{p} \cdot \mathrm{y}^*)$, and $\mathrm{p} \cdot \bar{\mathrm{x}}_\mathrm{i} < 0$ holds for some consumers since $0 \leqq \mathrm{p} \cdot \bar{\mathrm{x}}_\mathrm{i}\ \forall\, \mathrm{i} \in \mathrm{I}$ gives rise to $\mathrm{p} \cdot \bar{\mathrm{x}} \geqq 0$. That is, these consumers have cheaper points at price p.

(7), then, as seen later, implies that every consumer who is resource related with the consumers with cheaper points also turns up to have a cheaper point as well. Thus, as long as there is at least one consumer with a cheaper point the other consumers also end up with having cheaper points under indirect resource relatedness. This corresponds to that in the case with irreducibility.[18] (7) also implies non-satiation over $\widehat{\mathrm{C}}''_\mathrm{j}\ (= \widehat{\mathrm{C}}'_\mathrm{i} \cap \mathrm{C}_\mathrm{i})(\subset \widetilde{\mathrm{C}}_\mathrm{j} (= (\overline{\mathrm{C}}_\mathrm{j} \cap \mathrm{C}_\mathrm{j}))$, there is $\mathrm{x}''_\mathrm{i} \in \mathrm{C}_\mathrm{i}$ with $\mathrm{x}''_\mathrm{i} \in P_\mathrm{i}(\mathrm{x}_\mathrm{i})$. Thus, when a consumer has a cheaper point at a pseudo quasi-equilibrium $(\mathrm{x}^*_\mathrm{i}, \mathrm{y}^*, \mathrm{p}^*)_{\mathrm{i} \in \mathrm{I}}$, his demand point turns out to belong to $\widehat{\mathrm{C}}''_\mathrm{j}\ (\subset \widetilde{\mathrm{C}}_\mathrm{j} (= (\overline{\mathrm{C}}_\mathrm{j} \cap \mathrm{C}_\mathrm{j}))$ and hence non-satiation $P_\mathrm{i}(\mathrm{x}^*_\mathrm{i}) \neq \emptyset$ holds from the generalized indirect resource relatedness.

The following figure illustrates the situation corresponding to the famous Allow's corner example in a pure exchange economy without the individual survival assumption. In this figure competitive equilibrium does not exist. It turns out that the resource relatedness condition does not hold in this case.

Notice first the consumption set of consumer 1 does not contain the origin. Thus, imdividual survival does not hold for consumer 1. Note also that the lower boundary of the consumer set of consumer 1 contains straight line segment unlike in original Arrow's corner case. Then the same reason used in Arrow's corner implies that $(-\mathrm{x}_2, \mathrm{x}_2, \mathrm{p})$ is a quasi-equilibrium in the case without the individual survive assumption. That is, $(-\mathrm{x}_2, \mathrm{x}_2)$ is feasible, $\mathrm{p} \cdot \mathrm{x}_2 = 0$, and, $(\mathrm{x}'_1 \in P_1(-\mathrm{x}_2) \rightarrow \mathrm{p} \cdot \mathrm{x}'_1 \geqq 0$ and) $\mathrm{x}'_2 \in P_2(\mathrm{x}_2) \rightarrow \mathrm{p} \cdot \mathrm{x}'_2 \geqq 0$ hold. However, $\mathrm{x}'_1 \in P_1(-\mathrm{x}_2) \rightarrow \mathrm{p} \cdot \mathrm{x}'_1 > 0$, does not hold for consumer 1 since the indifference curve passing though $-\mathrm{x}_2$ intersects to the budget line at $-\mathrm{x}_2$. Indeed, the demand point of consumer 1 under p is x_1, not $-\mathrm{x}_2$. On the other hand, $\mathrm{x}'_2 \in P_2(\mathrm{x}_2) \rightarrow \mathrm{p} \cdot \mathrm{x}'_2 > 0$ holds for consumer 2 since the indifference curve passing though x_2 is tangent to the budget line at x_2. Thus, $(-\mathrm{x}_2, \mathrm{x}_2, \mathrm{p})$ is not a competitive equilibrium but only a quasi-equilibrium.

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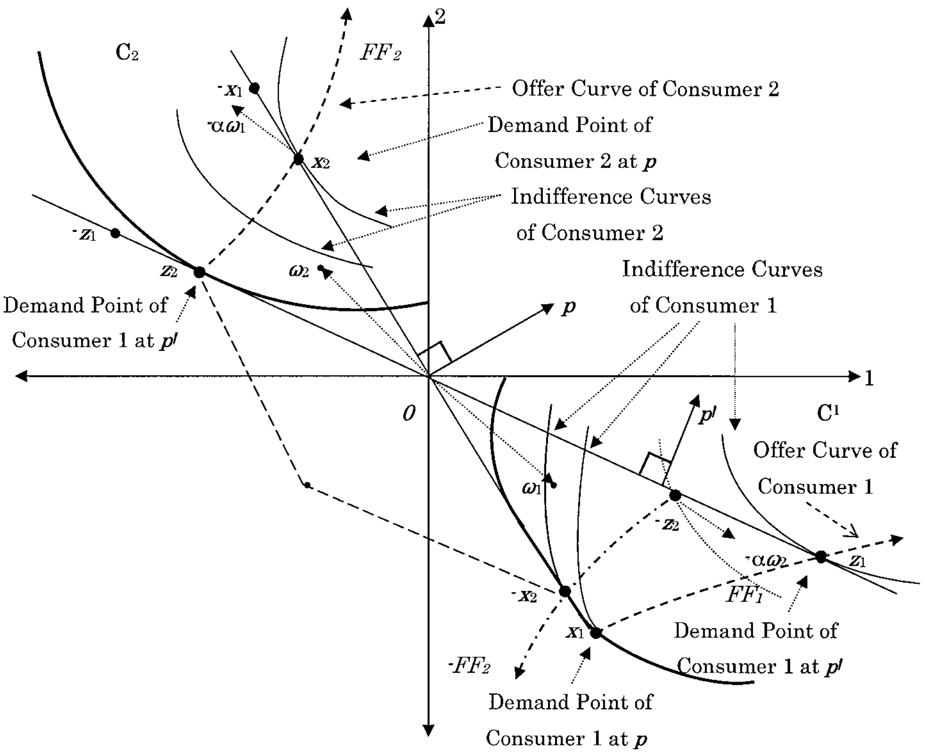
16) As seen below, S' is non-empty, convex, and compact from lemm 1.

17) Since Y is a closed convex cone, $\mathrm{Y} = (\mathrm{Y})^{**}$ holds. See, for example, Nikaido (1968, Theorem 3.4, p.34) for this result.

18) See section 5 for the discussion about resource relatedness and irreducibility.

Figure 2 . Arrow's Corner without Individual Survival



The reason for $(-x_2, x_2, p)$ not to be a competitive equilibrium is that the resource relatedness condition is violated at $(-x_2, x_2)$. Since the indifference curve passing through x_2 is tangent to the budget line at x_2, an offer of (α times) ω_1 by consumer 1 makes consumer 2 move below the budget line through x_2. That is, an offer of (α times) ω_1 from consumer 1 is unable to make consumer 2 better off than at x_2. Thus, the resource relatedness condition does not hold in this situation. Similarly, although z_1 and z_2 are demand points of both consumers, (z_1, z_2, p') is not a competitive equilibrium since (z_1, z_2) is not feasible, i.e., $z_1+z_2 \neq 0$. Indeed, when we consider FF_i, the offer curves of both consumers, $i=1, 2$, there is no competitive equilibrium in this case since there is no intersection between FF_1 and $-FF_2$. Notice that $-x_2+z_2<0$ implies that this economy satisfies the aggregate adequacy condition, i.e., this economy is sufficiently productive. Note that the condition required by resource relatedness holds at $(-z_2, z_2)$. That is, since the indifference curve passing through $-z_2$ cuts the budget line at $-z_2$, an offer of (α times) ω_2 by consumer 2 makes consumer 1 better off than at $-z_2$. It is possible for consumer 1 to be better off than at $-z_2$ by an offer of (α times) ω_2 from consumer 2 .

3. Some Lemmas

We need to show that S' and $\overline{C}(\equiv \sum_{i\in I}\overline{C'})$ are non-empty, convex, and compact. We first show that S' is non-empty, convex, and compact.

Lemma 1: *Under (1), (2), and (6),* S' *is non-empty, convex, and compact.*

Proof. See Appendix ■

We next show that $\overline{C}(\equiv \sum_{i\in I}\overline{C_i})$ is non-empty, convex, and compact. By the definition of $\overline{C_i}$, it is enough to show that $\widehat{C'_i}$ is non-empty, convex, and compact for this purpose.

Lemma 2: *Under (1), (2), (3),and (6),* $\widehat{C'_i}$ *is non-empty, convex, and compact.*

Proof. See Appendix ■

Note first that $\widehat{C_i}$ is also (non-empty, convex, and) compact since $\widehat{C_i}\subset\widehat{C'_i}$. Let $K\subset R^N$ be a hypercube around 0 satisfying $int(K)\supset\widehat{C'_i}$ and $\widehat{Y'}$ for $i\in I$. Then $\widetilde{C_i}$ is defined as $C_i\cap K$. $\widetilde{C_i}$ is non-empty, convex, and compact. Define $m_i : S'\rightarrow R$ by $m_i(p)=min\{p\cdot x'|x'\in\widetilde{C_i}\}$. $m_i(p)$ is his subsistence level of income over $\widetilde{C_i}$ at a price level p. Note that since $\widetilde{C_i}$ is non-empty and compact, there is $x'\in\widetilde{C_i}$ satisfying $m_i(p)=p\cdot x'$ for any $p\in S'$. Also, $\overline{C_i}$ is defined as the convex hull of $\widetilde{C_i}$ and $\{0\}$. Then, we define the modified demand correspondence of the i-th consumer in the economy, $D_i : S'\rightarrow\overline{C_i}$, by

$$
\begin{aligned}
D_i(p) &= \{x\in\widetilde{C_i}\,|p\cdot x\leqq 0, x'\in\widetilde{C_i} \text{ and } x'\in P_i(x)\rightarrow p\cdot x'>0\} \text{ if } 0>m_i(p)\\
&= \{x\in\overline{C_i}\,|p\cdot x\leqq 0\} \text{ if } 0\leqq m_i(p),
\end{aligned}
$$

Here we use $\widetilde{C_i}$ and $\overline{C_i}$ instead of C_i. This is necessary to get the non-emptiness of budget sets in the consumption sets truncated by K. In the case of $0>m_i(p)$, $D_i(p)$ is also considered to be mapped into $\overline{C_i}$ as well since $D_i(p)\subset\widetilde{C_i}\subset\overline{C_i}$ holds from the definition.

Note that $\{x'\in\overline{C_i}\,|p\cdot x'\leqq 0\}$ is non-empty for $p\in S'$ since $0\in\overline{C_i}$ holds owing to the definition of $\overline{C_i}$ regardless of whether $0\leqq m_i(p)$ or $0>m_i(p)$ holds for $p\in S'$. In fact, due to the lack of the individual survival assumption, $0<m_i(p)$ is possible to occur for some $p\in S'$. Then, $\{x'\in\widetilde{C_i}\,|p\cdot x'\leqq 0\}=\emptyset$ may occur, and hence his budget set in $\widetilde{C_i}$ is empty.[19] This is the reason to use $\overline{C_i}$ as the convex combination of $\widetilde{C_i}$ and $\{0\}$ instead of $\widetilde{C_i}$, particularly, in the case of $0<m_i(p)$. Then even when $0<m_i(p)$ occurs for $p\in S'$, $0\in\{x'\in\overline{C_i}\,|p\cdot x'\leqq 0\}\neq\emptyset$ still holds. Thus, $D_i(p)=\{x'\in\overline{C_i}\,|p\cdot x'\leqq 0\}(\ni 0)$ is non-empty even in the case of $0<m_i(p)$. Similarly when $0=m_i(p)$ occurs for $p\in S'$, then $D_i(p)=\{x'\in\overline{C_i}|p\cdot x'\leqq 0\}$ is also non-empty since $0\in\{x'\in\overline{C_i}\,|p\cdot x'\leqq 0\}\neq\emptyset$ holds as well. Thus in the case of

19) $\{x'\in|p\cdot x'\leqq 0\}\cap C_i=\emptyset$ may occur, and hence, the budget set in C_i may be empty as well.

$0 \leqq m_i(p)$, $D_i(p)$ is non-empty.

Note also that since $x' \in \overline{C_i}$ is expressed as a convex combination of some points in $\widetilde{C_i}$ and $\{0\}$, $0 \leqq m_i(p)$, i.e., $0 \leqq p \cdot z \; \forall z \in \widetilde{C_i}$, implies $0 \leqq p \cdot x' \; \forall x' \in \overline{C_i}$. In the case of $0 \geqq m_i(p)$, $\{x' \in \widetilde{C_i} \mid p \cdot x' \leqq 0\} \subset C_i$ is non-empty for $p \in S'$ from the definition of $m_i(p)$. Also when $0 > p \cdot x' \; \exists \; x' \in \overline{C_i}$ holds, there is $\alpha \in (0, 1]$ and $x'' \in \widetilde{C_i}$ with $\alpha x' = x''$ so that $0 > p \cdot x''$, and hence, $0 > m_i(p)$ holds. Thus $0 > m_i(p)$ is equivalent to $0 > p \cdot x'$ for some $x' \in \overline{C_i}$.

Lemma 3: *Under (3), (4), and (5), when* $x_i \in (D_i(p) \cap \widehat{C'_i})$ *satisfies* $0 > m_i(p)$ *for* $i \in I$, *then* $x'_1 \in P_i(x_i)$ *implies* $p \cdot x'_i > 0$. *Moreover, when non-satiation* $(P_i(x_i) \neq \emptyset)$ holds, $x'_i \in R_i(x_i)$ *implies* $p \cdot x'_i \geqq 0$

Proof. See Appendix ■

This lemma implies that for $i \in I$ satisfying $0 > m_i(p)$, $x' \in D_i(p)$ is indeed a demand point not only over $\widetilde{C_i}$ but also over C_i. That is, this lemma implies that a competitive equilibrium in the truncated economy is indeed a competitive equilibrium in the original untruncated economy. Thus, it is sufficient to show that the cheaper condition, $0 > m_i(p)$, and the demand condition over $\widetilde{C_i}$ hold $\forall i \in I$ in order to establish that the demand condition (ii) of competitive equilibrium holds $\forall i \in I$. Note that, as already mentioned, indirect resource relatedness (7) gives rise to non-satiation $(P_i(x'_i) \neq \emptyset)$ over $\widehat{C''_j}$ $\forall i \in I$.

Lemma 4: *Under (3), (4), and (5), when* $x_i \in (D_i(p) \cap \widehat{C'_i})$ *satisfies* $0 > m_i(p)$ *and non-satiation* $(P_i(x_i) \neq \emptyset)$, *then* $p \cdot x_i = 0$ *holds*.

Proof. See Appendix ■

This lemma implies that consumers with a cheaper point exhausts income at his demand point with price p. Note that $x_i \in (D_i(p) \cap \widehat{C'_i})$ satisfying $0 > m_i(p)$ implies $x_i \in \widetilde{C_i}$ so that in fact $x_i \in \widehat{C''_i} (= C_i \cap \widehat{C'_i})$ follows. The following is a simple consequence of this lemma.

Corollary: *Under (3), (4), (5), and (7), suppose that there is* $\{x_i\}_{i \in I}$ *satisfying* $x_i \in D_i(p) \; \forall i \in I$ *and* $x \equiv \sum_{i \in I} x_i \in ((\sum_{i \in I} D_i(p)) \cap Y)$, *and that there is some* $i' \in I$ *with* $0 > m_{i'}(p)$. *Then* $p \cdot x_{i'} = 0$ *holds.*

Proof. See Appendix ■

Although $i \in I \backslash I'$ in this corollary does not satisfy the cheaper point condition $0 > m_i(p)$, where I' is $\{i' \in I \mid m_{i'}(p) > 0\}$, the definition of $D_i(p)$, however, implies that $0 = p \cdot x_i$ holds for $x_i \in D_i(p)$. Thus, $x \equiv \sum_{i \in I} x_i \in ((\sum_{i \in I} D_i(p)) \cap Y)$ implies $p \cdot x_i = 0 \; \forall i \in I$ regardless of whether $0 \leqq m_i(p)$ or $0 > m_i(p)$ holds for $p \in S'$, and hence $p \cdot x = 0$ holds. Then since the definition of S' implies $p \cdot y \leqq 0 \; \forall y \in Y$, $p \cdot x = 0$ holds for $x \in Y$ implies that x maximizes the profit over Y. Thus, this price p is indeed a candidate for a competitive equilibrium price.

This suggests that as long as we can show the existence of such a aggregate consumption bundle $x' \equiv \sum_{i \in I} x'_i \in ((\sum_{i \in I} D_i(p)) \cap Y)$, with the aid of Debreu (1956)'s generalization of Gale-Nikaido's lemma, the profit maximization condition (iii) holds automatically. We do not need to worry about the profit maximization condition in the production side of the economy in this situation.

The basic fact on this modified demand correspondence is the following result on its continuity even without the individual survival assumption, $0 \in C_i$.

Lemma 5: *Under (1), (2), (3), (4), (5), and (7), the modified demand correspondence of the i-th consumer* $D_i : S' \to \widetilde{C}_i$ *is a non-empty, convex, and compact-valued upper hemi-continuous correspondence.*

Proof. See Appendix. ■

The next result is the most crucial implication of the generalized resource relatedness condition in the case without individual survival. Even without individual survival, a consumer who is resource related to a consumer with a cheaper point also has a cheaper point at a pseudo quasi-equilibrium. This is the generalization of the lemma in Arrow-Hahn (1971) to the case without individual survival.[20]

Lemma 6: *Under (1), (2), (3), (4), (5), and (7), when* $(x_i^*, y^*, p^*)_{i \in I}$ *with* $p^* \in R^N \backslash \{0\}$ *is a pseudo quasi-equilibrium and consumer i has a cheaper point, consumer i' who is resource related to consumer i also has a cheaper point. Similarly, when consumer i has a cheaper point, then consumer i' who is indirectly resource related to consumer i also has a cheaper point.*

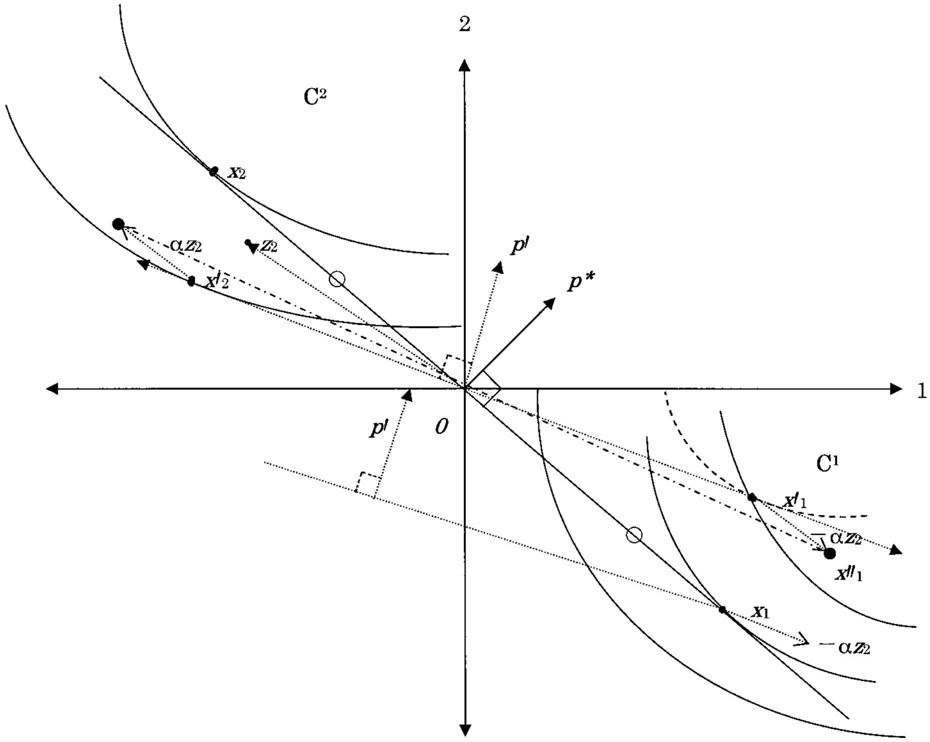
Proof. See Appendix ■

From this lemma, the sufficient income accruing from (6) spreads over the entire consumers under general resource relatedness (7) even in the case without individual survival, $0 \in C_i$. This is same as Lemma 8 , the one that irreducibility gives rise to in the case without individual survival.

The following figure illustrates this lemma in an exchange economy with two consumers without the individual survive assumption. From this figure, $p' \cdot x_1 < 0$ holds for p' and $p' \cdot x'_2 \geqq 0$ holds for $x'_2 \in C_2$. Thus, consumer 1 has a cheaper point but consumer 2 does not have any cheaper point at p'. Since (x'_1, x'_2) is feasible and consumer 2 is resource related to consumer 1 , $\exists z_2 \in C_2$, $x''_1 \in P(x'_1)$, $x''_2 \in C_2, x''_1 + x''_2 = -\alpha z_2$. In the figure, we use $x''_1 = x'_1 - \alpha z_2$ and $x''_2 = (x'_2 - \alpha z_2) - \alpha z_2 = x'_2$. Since (x'_1, x'_2, p') is supposed as a quasi-equilibrium and consumer 1 has a cheaper point, x'_1 is a demand point at p' although (x'_1, x'_2, p') is indeed not a quasi-equilibrium in the figure. Thus,

20) See, Allow-Hahn (1971, ch. 5, sec. 4, lemma 4).

Figure 3. An Illustration of Lemma 6



$p' \cdot x_1'' = -p' \cdot (x_2'' + \alpha z_2) > 0$ holds. Since $p' \cdot x_2'' = 0$ implies $p' \cdot z_2 < 0$, and consumer 2 turns out to have a cheaper point, a contradiction. This implies that when resource related holds the indifference curve must intersect the budget line from above at x_1' as in the figure since $p' \cdot x_1'' = -p' \cdot (x_2'' + \alpha z_2) > 0$ does not hold if the indifference curve through x_1', as shown as the dotted curve in the figure, is tangent to the budget line with p'.[21]

We define the modified aggregate demand correspondence of consumers $D: S' \to \overline{C}$ by $D(p) = \sum_{i \in I} D_i(p)$, where $\overline{C} = \sum_{i \in I} \overline{C_i}$. From the above lemma 5, this correspondence $D(\cdot)$ is non-empty, convex, and compact valued, and upper hemi-continuous. Since $p \cdot x = \sum_{i \in I} p \cdot x_i \leqq 0$, and hence $p \cdot x \leqq 0$ holds by the definition of $x_i \in D_i(p)$ for $i \in I$, $p \cdot D(p) \leqq 0$, i.e., $p \cdot z \leqq 0 \ \forall z \in D(p)$, holds $\forall p \in S'$. Note that since $D(\cdot)$ does not express aggregate excess demand, this inequality is not same as Walras' law, which Gale-Nikaido's lemma assumes. Here, on the other hand, uses this inequality and Debreu (1956)'s method in the proof of the generalization of Gale-Nikaido's lemma.

We also define the correspondence corresponding to a price adjustment

21) Since, we use $x_1'' = x_1' - \alpha z_2$, $x_2'' = x_2'$ in the figure, resource relatedness is indeed equivalent to irreducibility. The irreducibility condition is discussed in section 5.

rule in the Walrasian tatonnement process. Let $Q:\overline{C}\to S'$ be a correspondence defined by $Q(z)=\{p'\in S' \mid p'\cdot z \geqq p\cdot z \ \forall p \in S'\}$.

Lemma 7 : *Under (1), (2), (3), and (6), the price adjustment correspondence* $Q:\overline{C}\to S'$ *is a non-empty, convex, and compact-valued upper hemi-continuous correspondence.*

Proof. See Appendix. ■

Note that this price adjustment process is used to aggregate excess demand in the usual argument of Gale-Nikaido's lemma, on the other hand, here, it is used to aggregate demand instead of aggregate excess demand.

4. Existence of Competitive Equilibrium

With the aid of Debreu (1956)'s method employed in the proof of the generalization of Gale-Nikaido's lemma, we can establish the existence of competitive equilibrium in this setting without employing the inverse supply correspondence owing to McKenzie (1955, 1959, 1981), since the profit maximization condition (iii)′ holds automatically at its candidate as seen below.

Theorem 1 : *Under (1), (2), ⋯, and (7), there exists a competitive equilibrium in the economy.*

Proof. Define the correspondence $Q\times D: S'\times\overline{C}\to S'\times\overline{C}$ by $(Q\times D)(p, z)=Q(z)\times D(p)$ for $(p, z)\in S'\times\overline{C}$. It is non-empty, convex, and compact valued, and upper hemi-continuous. Then from Kakutani's fixed point theorem, there is $(\tilde{p}, \tilde{y})\in(Q\times D)(\tilde{p}, \tilde{y})=Q(\tilde{y})\times D(\tilde{p})(\subset(S'\times\overline{C}))$. Then $0\geqq\tilde{p}\cdot\tilde{y}\geqq p\cdot\tilde{y}$ holds $\forall p \in S'$ and hence $\tilde{y}\in((Y)^*)^*=Y$ follows. Since $\tilde{y}\in D(\tilde{p})$ implies $\tilde{y}=\tilde{x}=\sum_{i\in I}\tilde{x}_i$ for some $\tilde{x}_i\in D_i(\tilde{p})$, $i\in I$, $\tilde{x}\in[(\sum_{i\in I}D(\tilde{p}))\cap Y]$, thus, $(\tilde{x}_i, \tilde{y})_{i\in I}$ is a feasible pseudo-allocation in the economy. Note that $(\tilde{x}_i, \tilde{y})_{i\in I}$ is not necessarily an allocation at this moment. Then, as already shown after the corollary, $\tilde{p}\cdot\tilde{y}=\tilde{p}\cdot\tilde{x}=0$ holds and hence, $\tilde{y}$ satisfies the aggregate profit condition (iii) since $\tilde{p}\in S'$ implies $\tilde{p}\cdot y\leqq 0 \ \forall y\in Y$. Since (6) also implies that there are some $\overline{x}_i\in\widetilde{C}_i \ \forall i\in I$ satisfying $\overline{x}(\equiv\sum_{i\in I}\overline{x}_i)\in int(Y)$, $\tilde{p}\in S'$ implies $\tilde{p}\cdot\overline{x}=\tilde{p}\cdot(\sum_{i\in I}\overline{x}_i)<0$. Thus the set of consumers $I^1\equiv\{i\in I \mid \tilde{p}\cdot x_i<0$ for some $x_i\in\widetilde{C}_i(\subset\overline{C})\}$ is non-empty. Note first that since $(\tilde{x}_i, \tilde{y})_{i\in I}$ is a feasible pseudo-allocation, $\tilde{x}_i\in\widehat{C'_i}\cap\overline{C}$ and $\tilde{p}\cdot\tilde{x}_i=0$ hold for $i\in I$. Note also that, as already mentioned, $\tilde{x}_i\in D_i(\tilde{p})$ is indeed a demand point over C_i not only over $\widetilde{C}_i \ \forall i\in I^1$. That is, $\forall i\in I^1$, $x'_i\in C_i$ with $\tilde{p}\cdot x'_i\leqq 0$ implies $\tilde{x}_i\in R_i(x'_i)$. Moreover, since (5) and (7) imply the local non-satiation over $\widehat{C''_i}$, $x'_i\in C_i$ with $x'_i\in(R_i(\tilde{x}_i)\cap\widehat{C''_i})$ implies $\tilde{p}\cdot x'_i\geqq 0(=\tilde{p}\cdot\tilde{x}_i) \ \forall i\in I^1$.

Let $I^2\equiv I\backslash I^1$. Then, as already shown, $\tilde{x}_h\in\overline{C_h}$ and $0=\tilde{p}\cdot\tilde{x}_h$ holds $\forall h\in I^2$. Since (7) implies that $h\in I^2\equiv I\backslash I^1$ is indirectly connected to $i\in I^1$, lemma 6 implies that h also has a cheaper point over $\overline{C_h}$, and hence $\widetilde{x_h} \in (D_h(\widetilde{p}) \cap \widehat{C''_h})$ is indeed a demand point over $\widetilde{C_h}$ $\forall h \in I^2$. Of course, then $\widetilde{x_h} \in D_h(\widetilde{p})$ is a demand point over C_h not only over $\widetilde{C_h}$ $\forall h \in I^2$. Thus $\widetilde{x_i}$ is a demand point over C_i $\forall i \in I$ and the usual demand condition (ii) holds $\forall i \in I$. Moreover, since $\widetilde{x_i} \in C_i$ $\forall i \in I$ and $\widetilde{y} \in Y$ hold, $(\widetilde{x_i}, \widetilde{y})_{i \in I}$ is indeed a feasible allocation, and hence balanced condition (i) holds as well. Therefore the vector $(\widetilde{x_i}, \widetilde{y}, \widetilde{p})_{i \in I}$ is a competitive equilibrium in the economy. ■

The step to show $\widetilde{x} (= \widetilde{y}) \in Y$ in the above proof uses the proof of Debreu's (1956) generalization of Gale-Nikaido's lemma in the literature. It can be stated as follows.

Theorem (Debreu, 1956): *Let C be a closed, convex cone with vertex 0 in R^N, which is not a linear manifold; let Γ be its polar. If the correspondence ζ from $C \cap S^N$ to R^N is upper hemi-continuous and bounded, and if for every p in $C \cap S^N$ the set $\zeta(p)$ is non-empty, convex, and satisfies $p \cdot \zeta(p) \leqq 0$, then there is a $\tilde{p}$ in $C \cap S^N$ satisfying $\Gamma \cap \zeta(\tilde{p}) \neq \emptyset$.*[22]

See Debreu (1956) for this result. Debreu (1959, Gale-Nikaido's lemma p. 82) is a special case of this result where C is R^N_+ and Γ is $-(R^N_+)$. The production set here is a closed convex cone with its non-empty interior. Thus by regarding this production set as the dual cone of the price cone in the Debreu's result, we can dispense with McKenzie's (1955, 59, 81) inverse supply correspondence, and deal without free disposability as long as the local non-satiation and the aggregate adequacy assumption are met. In McKenzie (1955, 59) the inverse supply correspondence is originally used to get a composite mapping from the space of prices into itself. On the other hand, in Debreu (1956) the method of the proof uses a mapping from the Cartesian product of the space of prices and the space of feasible consumptions into itself. Then, since the production set is assumed to be closed convex cone, a corresponding equilibrium supply point is determined accordingly when the Debreu (1956)' approach is used. The approach with inverse supply correspondence itself, of course, is interesting and appeals a lot of economics sense.

5. Resource Relatedness and Irreducibility

This section discusses resource relatedness and irreducibility. One of the crucial aspects in the proof of the existence of competitive equilibrium in market economies is to assure that every consumer has cheaper points at a quasi-equilibrium. For this purpose, Arrow-Debreu (1954, Theorem 2) use, besides the individual survival assumption, the assumption that every consumer has a

22) Here S^N is the unit sphere in R^N.

productive goods which produces more of goods that are commonly desirable to every consumer. This corresponds to the fact that labors of consumers are useful in the world. For the same purpose, McKenzie (1959) use irreducibility, besides individual survival, that is based on the situation where each consumer has something, not necessarily labor, that is desirable to the rest of consumers. Debreu (1959) use the strong individual survival condition of Arrow-Debreu (1954, Theorem 1) where every consumer can provide some of all goods from his initial endowment. All of these suppose the individual survival assumption where the initial endowments of consumers are in their consumption sets, so that they can survive even without trading other consumers though markets.

The original definition of resource relatedness by Arrow-Hahn (1971) is a condition which generalizes the existence of productive goods used in Arrow-Dereu (1954, Theorem 2). We first consider the case with productive goods. Let D be the set of goods which are desirable commonly to all consumers. Then good l is defined as a productive goods when $\mathrm{y} \in \mathrm{Y}$ implies $\mathrm{y}_l \leqq 0$ and $\exists \mathrm{y}' \in \mathrm{Y}$ satisfying $\mathrm{y}_k \leqq \mathrm{y}'_k$ $\forall k \neq l$ and $\mathrm{y}_d < \mathrm{y}'_d$ $\exists\, d \neq D$. Let P be the set of the productive goods. The condition used in Arrow-Dereu (1954, Theorem 2) is that each consumer has a $\overline{\mathrm{x}_\mathrm{h}} \in (\mathrm{C}_\mathrm{h} \cap -(R^N_+) \setminus \{0\})$ satisfying $\overline{\mathrm{x}_{\mathrm{h}l}} < 0\ \exists\, l \in P$ and the economy satisfies $(\sum_{\mathrm{h}\in\mathrm{I}} \mathrm{C}_\mathrm{h} - \mathrm{Y}) \cap ((-(R^N_{++})) \neq \emptyset$, a version of the aggregate adequacy condition. Note first that $(\mathrm{C}_\mathrm{h} \cap (-(R^N_+) \setminus \{0\})) \neq \emptyset$ implies the individual survival assumption. Since the simplex is used for the price set in Arrow-Dereu (1954), this aggregate adequacy condition implies that some consumer satisfies the cheaper point condition at every price vector. Moreover, since the consumers with cheaper points are willing to pay for the desirable goods, the prices of these goods become positive and hence the imputation gives rise to that the prices of productive goods become positive as well. Then each consumer has at least one productive goods so that he turns out to have a cheaper point and a sufficient income level. Thus every consumer has a cheaper point at each price vector. Then the demand correspondence of each consumer is upper hemi-contious on the price set, and the proof goes along with the fixed point theorem argument. Thus the point in this argument is that each consumer has a goods whose price is positive all the time and he has a cheaper point all the time as well. This concept of productive goods and desirable goods is generalized to resource relatedness of Arrow-Hahn (1971) and irreducibility of McKenzie (1959), respectively.

The definition of resource relatedness in Arrow-Hahn (1971), where individual survival is assumed, is defined as following. The i-th consumer is called *resource related* to the j-th consumer when the following condition holds: There is $-\delta \mathrm{e}_\mathrm{i} \in \mathrm{C}_\mathrm{i}$ for $\delta > 0$ such that for any feasible allocation $(\mathrm{x}_\mathrm{h}, \mathrm{y})_{\mathrm{h}\in\mathrm{I}}$ there is $(x'_h)_{h\in I}$ satisfying $\sum_{h\in I}x'_h \in \sum_{h\in I}C_h \cap (Y+\delta e_i)$ with $x'_h \in R_h(x_h)$ holds for $h\in I$ and $x'_j \in P_j(x_j)$ holds. The basic idea of this definition is that when consumer i offers one goods that he can such as his labor to the production process in the entire economy, then the outcome is to make every consumer not worse off and the j-th consumer better off than before. This definition owes to the existence of productive goods and commonly desirable consumption goods used in Arrow-Debreu (1954) to weaken the assumption that every consumer can offer any goods to the one that every consumer can offer at least one goods. When the i-th consumer has a sufficient income at a quasi-equilibrium, he is willing to pay implicitly for the goods that j-th consumer can offer, and as a result, j-th consumer turns out to have a sufficient income as well. This is a main object to employ resource relatedness. Furthermore, the i-th consumer is called *indirectly resource related* to the j-th consumer when there is a series of consumers $\{h_i\}_{i=1}^m$ such that the i-th consumer is resource related to the h_1-th consumer, the h_1-th consumer is resource related to the h_2-th consumer, ……, the h_m-th consumer is resource related to the j-th consumer. That is, the i-th consumer is linked with the j-th consumer through resource related consumers between them.

In the case without individual survival, on the other hand, some of candidates for competitive equilibrium are only pseudo quasi-equilibria so that they may be only feasible pseudo-allocations. Thus, we need to apply the same argument even to feasible pseudo-allocations not only to feasible allocations. Moreover, since individual survival is assumed in the original definition and every consumer has at least some goods to offer, in the case without individual survival he may have only a bundle of goods to offer instead of some certain goods. Thus, the definition of resource relatedness in the case without individual surrival have to take account of these two aspects. It is defined as following. The i-th consumer is called *resource related* to the j-th consumer when the following condition holds: There is $z_i \in \overline{C_i}$ such that for any pseudo feasible allocation $(x_h, y)_{h\in I}$ with $x_j \in C_j$ there is $(x'_h)_{h\in I}$ satisfying $\sum_{h\in I}x'_h \in [(\sum_{h\in I^1}C_h) + (\sum_{h\in I^2}\overline{C_h})) \cap (Y-\delta z_i)](\exists\,\delta>0)$ such that $x'_h \in R_h(x_h)$ holds for $h\in I^1$, $x'_h \in \overline{C_h}$ holds for $h\in I^2$, and $x'_j \in P_j(x_j)$ holds, where $I^1 = \{h\in I : x_h \in C_h\}$ and $I^2 = \{h\in I : x_h \in \overline{C_h}\backslash C_h\}$. This is already used in section2.

The basic idea of this definition is analogous to the original and when the i-th consumer offers some bundle of goods that he can, instead of some certain goods in the original definition, to the production process in the entire economy, then an outcome is beneficial for the j-th consumer. Here x'_h as well as x_h is in $\overline{C_h}$ for $h\in I^2$ instead of $x'_h \in R_h(x_h)$ in the original definition. This means that consumers in I^2 gain small enough to stay within $\overline{C_h}$ from the i-th consumer's offer of some bundle of goods from $\overline{C_i}$ to the production process in the entire economy. When, however, $0 \in C_h$ and hence $\overline{C_h} = \widetilde{C}_h (\subset C_h)$ holds for $h \in I$, then feasible pseudo allocations are feasible allocations so that $I^2 = \{h \in I : x_h \in \overline{C_h} \backslash C_h\} = \emptyset$ or $I = I^1$ holds and $-\delta e_i$ is used as z_i. The original definition satisfies the general definition as a special case. Thus this definition of resource relatedness is considered as a generalization of the original resource relatedness to the case without the individual survival assumption.

The indirectly resource relatedness between the i-th consumer and the j-th consumer is defined analogously as above. The i-th consumer is called *indirectly resource related* to the j-th consumer when there are a series of consumers $\{h_i\}_{i=1}^{m}$ such that the i-th consumer is resource related to the h_1-th consumer, the h_1-th consumer is resource related to the h_2-th consumer, ……, the h_m-th consumer is resource related to the j-th consumer. That is, the i-th consumer is linked with the j-th consumer through resource related consumers between them.

In the proof of lemma 6, the crucial part with using the generalized resource relatedness is indeed that for a pseudo feasible allocation $(x_h, y)_{h \in I}$ with $x_j \in C_j$ there are $z_i \in \overline{C_i}$ and $(x'_h)_{h \in I}$ satisfying $\sum_{h \in I} x'_h \in [(\sum_{h \in I^1} C_h) + (\sum_{h \in I^2} \overline{C_h})) \cap (Y - \delta z_i)] (\exists \delta > 0)$ such that $x'_h \in R_h(x_h)$ holds for $h \in I^1$, $x'_h \in \overline{C_h}$ holds for $h \in I^2$, and $x'_j \in P_j(x_j)$ holds, where $I^1 = \{h \in I : x_h \in C_h\}$ and $I^2 = \{h \in I : x_h \in \overline{C_h} \backslash C_h\}$. We may use as the simplest generalized definition with using $I^2 = \{h \in I : x_h \in \overline{C_h} \backslash C_h\} = \emptyset$ or $I = I^1$ so that it is expressed as following : There is $z_i \in \overline{C_i}$ such that for any pseudo feasible allocation $(x_h, y)_{h \in I}$ with $x_j \in C_j$ there is $(x'_h)_{h \in I}$ satisfying $\sum_{h \in I} x'_h \in [(\sum_{h \in I} C_h) \cap (Y - \delta z_i)] (\exists \delta > 0)$ such that $x'_h \in R_h(x_h)$ holds for $h \in I$ and $x'_j \in P_j(x_j)$ holds. This definition supposes that the h-th consumer with $x_h \in \overline{C_h} \backslash C_h$ ends up with $x'_h \in C_h$ so that this consumer gains large enough to move into C_h from the i-th consumer's offer of some bundle of goods from $\overline{C_i}$ to the production process in the entire economy. There is, however, a problem with this definition to establish the result similar to lemma 6 since $R_h(x_n)$ is not defined for $x_h \in \overline{C_h} \backslash C_h$. We may use $x'_h \in R_h(x_h)$ only for h with $x_h \in \overline{C_h} \backslash C_h$. Note that the h-th consumer with $x_h \in \overline{C_h} \backslash C_h$ does not have cheaper points over $\overline{C_h}$ not over C_h at a price p, and hence $p^* \cdot x'_h > 0$ may occur for h with $x_h \in \overline{C_h} \backslash C_h$. He may, however, have a cheaper points over C_h. The proof of lemma 6 uses the fact that $x'_h \in R_h(x^*_h) \to p^* \cdot x'_h \geqq 0$, $p^* \cdot x'_h \geqq 0$, $x_{h'} \in \overline{C_{h'}}$, for h' with $x_{h'} \in \overline{C_{h'}} \backslash C_{h'}$, and $x'_j \in P_j(x^*_j) \to p^* \cdot x^*_j > 0$ to get $\sum_{h \in I} p^* \cdot x'_h > 0$. Thus when $x'_h \in \overline{C_h}$ for h' with $x_{h'} \in \overline{C_{h'}} \backslash C_{h'}$ is used, $\sum_{h \in I} p^* \cdot x^*{}_h > 0$ holds. This is a reason to use instead of $\sum_{h \in I} C_h$ in the definition of generalized resource relatedness in this paper. Thus, $x'_h \in \overline{C_h}$ is reguired for h with $x_h \in \overline{C_h} \backslash C_h$ in the definition of generalized resource relatedmess instead of $x'_h \in C_h$ in the simple generalization of the original definition.

We next consider the irreducibility of McKenzie (1959, 81). The original definition of irreducibility with individual survival is following :

Let (I^1, I^2) be a partition of the set of the consumers into two non-empty subsets. If $(x_i, y)_{i \in I}$ is a allocation such that, $x_i \in \widehat{C}_i$ for $i \in I^1$, $x_h \in \widehat{C}_h$ for $h \in I^2$, and $y \in Y$, then $\exists (\widetilde{x}_i)_{i \in I^1}$ such that $(x_i + \widetilde{x}_i)_{i \in I^1} \in \times_{i \in I^1} C_i$, $\exists\, \widetilde{y} \in Y$, $z \in \sum_{h \in I^2} C_h$, $\sum_{i \in I^1} \widetilde{x}_i = \widetilde{y} - y - \sum_{h \in I^2} \alpha_h z_h$ for some $\alpha_h > 0$, $x_i + \widetilde{x}_i \in R_i(x_i)$ $\forall\, i \in I^1$ and $\widetilde{x}_{i'} \in P_{i'}(x_{i'})$ at least some $i' \in I^1$.

Since the individual survival assumption is used, as in the case with original resource relatedness, $(x_i, y)_{i \in I}$ in this definition is a feasible allocation, i.e., $x_i \in \widehat{C}_i$ for $i \in I^1$, $x_h \in \widehat{C}_h$ for $h \in I^2$ and $\sum_{i \in I} x_i = y \in Y$. In the case of pure exchange economy with two consumers, it ends up with $x_1 + \widetilde{x}_1 = -(x_2 + \alpha_2 z_2) = x_1 - \alpha_2 z_2$. The definition of irreducibility in economies without individual survival is following :

(7′) Let (I^1, I^2) be a partition of the set of the consumers into two non-empty subsets. If $(x_i, y)_{i \in I}$ is a pseudo-allocation such that, $x_i \in \widetilde{C}_i$ for $i \in I^1$, $x_h \in \overline{C_h}$ for $h \in I^2$, and $y \in Y$, then $\exists (\widetilde{x}_i)_{i \in I^1}$ such that $(x_i + \widetilde{x}_i)_{i \in I^1} \in \times_{i \in I^1} C_i$, $\exists\, \widetilde{y} \in Y$, $z \in \sum_{h \in I^2} \overline{C_h}$, $\sum_{i \in I^1} \widetilde{x}_i = \widetilde{y} - y - (\sum_{h \in I^2} \alpha_h z_h)$ for some $\alpha_h > 0$, $(x_i + \widetilde{x}_i) \in R_i(x_i)$ $\forall\, i \in I^1$ and $\widetilde{x}_{i'} \in P_{i'}(x_{i'})$ at least some $i' \in I^1$.

Since the individual survival assumption is not used, as in the case with generalized resource relatedness, $(x_i, y)_{i \in I}$ in this definition is only a feasible pseudo-allocation, i.e., $x_i \in \widetilde{C}_i$ for $i \in I^1$, $x_h \in \overline{C_h}$ for $h \in I^2$. Thus, $\overline{C_h}$, the convex hull of 0 and $\widetilde{C}_i$, is used for $h \in I^2$ instead of C_h, the original consumption set, since the consumption bundle of some consumer at a candidate for a competitive equilibrium may be out of his consumption set due to the lack of the individual survival assumption, $0 \in C_i$. Moreover, $\widehat{C}_i$ in the definition of the original irreducibility condition is replaced with $\widetilde{C}_i$ for $i \in I^1$ in this generalized version of irreducibility. As in (7), generalized irreducibility implies that R_i is non-satiated over $\widetilde{C}_i$ when we choose $\{i\}$ as I^1 in the definition, and hence R_i is locally non-satiated over $\widehat{C}''_i$ as well, owing to (5) and $\widehat{C}''_i \subset \widetilde{C}_i$. Non-satiation over $\widetilde{C}_i$ is crucial to assure the non-emptiness of modified demand correspondences in lemma 3 . Also locally non-satiation over $\widehat{C}''_i$ is crucial to get that all income is spent at demand points. As in the generalized resource relatedness, this is one reason for this change in the definition of irreducibility due to the lack of the individual survival assumption, $0 \in C_i$.

McKenzie (1981, Theorem 1) proves in the case with ordered preferences the existence of competitive equilibrium with a version of irreducibility using only $\widehat{C}_h$ instead of $\overline{C_h}$ for $h \in I^2$. But it require a complicated argument that an allocation at a candidate for a competitive equilibrium is feasible.[23] For the same purpose we use $\widetilde{C}_i$ instead of $\widehat{C}_i$ for $i \in I^1$ besides $\overline{C_h}$ instead of $\widehat{C}_h$ for

$h \in I^2$. Then we can show below to dispense with this complicated argument although we use a stronger version of irreducibility. As mentioned above, McKenzie (1981, lemma 1) shows that a feasible pseudo-allocation $(\tilde{x}_i, \tilde{y})_{i \in I}$ is indeed a feasible allocation. The original irreducibility is shown to be enough to establish that I_2 is empty in the proof of McKenzie (1981, Theorem 1). Thus it is desirable to show that even in the approach of this paper a feasible pseudo-allocation $(\tilde{x}_i, \tilde{y})_{i \in I}$ is indeed a feasible allocation and hence that the original irreducibility is enough for emptiness of I_2 in the proof of Theorem 1. McKenzie (1981, lemma 1) uses utility representation owing to continuity and transitivity of preferences to get this result, so that of course the same approach works here as well. However, it is interesting whether the same result is obtained even without appealing to desirable to utility representation, and it is left here for a future research topic.

Since the resource relatedness condition is only employed in lemma 6 and the corollary to lemma 4, the other lemma still hold as well in the case with irreducibility. As to the corollary to lemma 4, resource relatedness is used to get non-satiation at $x_{i'} \in (D_{i'}(p) \cap \widehat{C'_{i'}})$, which is a condition of lemma 4. Since irreducibility also implies non-satiation over pseudo feasible allocations, the condition necessary to the corollary to lemma 4 follows as well, and the corollary to lemma 4 follows with irreducibility instead of resource relatedness. Thus, it is enough to show the result similar to lemma 6. In the case with irreducibility, the following lemma 8 is the lemma analogous to lemma 6 in the case with resource relatedness.

Lemma 8: *Under (1), (2), (3), (4), (5), (6), and (7′), when* $(x_i^*, y^*, p^*)_{i \in I}$ *with* $p^* \in R^N \setminus \{0\}$ *is a pseudo quasi-equilibrium, then each consumer has a cheaper point.*

Proof. See Appendix ■

Note that when irreducibility holds in the exchange economy with two consumers the similar situation as of Figure 3 occurs as well.[24)]

We can show the existence of competitive equilibrium in the economy with irreducibility.

Theorem 2: *Under (1), (2),⋯⋯ and (7'), there exists a competitive equilibrium in the economy.*

23) See McKenzie (1981, lemma 1 and its proof, p. 827-8). From this result, it is indeed possible to use the original irreducibility as in McKenzie (1981, Theorem 2) instead of stronger version of irreducibility here. This is an interesting isseu and left for a further development.

24) This irreducibility case corresponds to the case with $x''_2 = x'_2$, $x''_1 = x'_1 - \alpha z_2$ in Figure 3. This is indeed the situation where resource relatedness is explained in Figure 3. When $x'_2 + \alpha z_2 \in C_2$ holds, in fact strong irreducibility (McKenzie (2002, ch. 5)) holds as well since $(x'_2 + \alpha z_2) + (x'_1 - \alpha z_2) = 0$ holds.

Proof. Define the correspondence $Q \times D : S' \times \overline{C} \rightarrow S' \times \overline{C}$ by $(Q \times D)(p, z) = Q(z) \times D(p)$ for $(p, z) \in S' \times \overline{C}$, and is non-empty, convex, and compact valued, and upper hemi-continuous. Then from Kakutani's fixed point theorem, there is $(\tilde{p}, \tilde{y}) \in (Q \times D)(\tilde{p}, \tilde{y}) = Q(\tilde{y}) \times D(\tilde{p}) (\subset (S' \times \overline{C}))$. Then $0 \geqq \tilde{p} \cdot \tilde{y} \geqq p \cdot \tilde{y}$ holds $\forall p \in S'$ and hence $\tilde{y} \in ((Y)^*)^* = Y$ follows. Since $\tilde{y} \in D(\tilde{p})$ implies $\tilde{y} = \tilde{x} (\equiv \sum_{i \in I} \tilde{x}_i)$ for some $\tilde{x}_i \in D_i(\tilde{p}), i \in I$, $\tilde{x} \in [\sum_{i \in I} D_i(\tilde{p}) \cap Y]$, thus, $(\tilde{x}_i, \tilde{y}, \tilde{p})_{i \in I}$ is a pseudo quasi-equilibrium in the economy. Note that $(\tilde{x}_i, \tilde{y})_{i \in I}$ is not necessarily an allocation at this moment. Then, as already shown, $\tilde{p} \cdot \tilde{y} = \tilde{p} \cdot \tilde{x} = 0$ holds and hence, $\tilde{y}$ satisfies the aggregate profit condition (iii) since $\tilde{p} \in S'$ implies $\tilde{p} \cdot y \leqq 0 \, \forall y \in Y$. Lemma 8 then implies that $\tilde{x}_i \in \{D_i(\tilde{p}) \cap \tilde{C}_i\}$ holds and hence $\tilde{x}_i$ is a demand point over $\tilde{C}_i \ \forall i \in I$. Then $\tilde{x}_i$ is a demand point over $C_i \ \forall i \in I$. Thus the usual demand condition (ii) holds $\forall i \in I$. Moreover, since $\tilde{x}_i \in C_i \ \forall i \in I$ and $\tilde{y} \in Y$ hold, $(\tilde{x}_i, \tilde{y})_{i \in I}$ is indeed a feasible allocation, and hence balanced condition (1) holds as well. Therefore the vector $(\tilde{x}_i, \tilde{y}, \tilde{p})_{i \in I}$ is a competitive equilibrium in the economy. ■

6. Concluding Remarks

Since there are other issues to be discussed on the relation of resource relatedness as well as irreducibility with the existence of competitive equilibrium, these are briefly discussed below.

Without ordered preferences: From the works of Shafer-Sonnenshein (1975) and Gale-MasColell (1975), ordered preferences are eliminated in economies which have competitive equilibrium. McKenzie (1981, Theorem 3) then shows that economies without ordered preferences still have competitive equilibrium even the individual survival assumption is eliminated when the irreducibility assumption (McKenzie (1981, (6″))) used in the last section is employed. McKenzie (1981, Theorem 3), however, employs the excess demand approach as in this paper after deriving utility functions from non-ordered preferences, and does not employ the abstract economy approach used by Shafer-Sonnenshein (1975) and Gale-MasColell (1975). On the other hand, Kubota (1997a) establishes the existence of competitive equilibrium in economies without ordered preferences and the individual survival assumption but with the irreducibility assumption (McKenzie (1981, (6″))) used in the last section, with using the abstract economy approach of Shafer-Sonnenshein (1975) and Gale-MasColell (1975). Thus, it is an interesting exercise to establish the existence of competitive equilibrium in economies without ordered preferences and the individual survival assumption but with the generalized resource relatedness condition used in this paper instead of irreducibility, with using the abstract economy approach used in Kubota (1997a). On the other hand, Sonnenshein (1971) shows that convex preferences with open graph implies demand correspondences are upper hemi-continuos at prices where cheaper point condition holds. Thus, it seems an open question whether it is possible to use the simple excess demand approach with open graph property of P_h as in Sonnenshein (1971) not as in McKenzie (1981), instead of the abstract economy approach, with resource relatedness or irreducibility, to establish the existence of competitive equilibrium in economies without ordered preferences and the individual survival assumption.

Relative interiority: In this paper, the aggregate production set is assumed to have interior points in (6). As shown in McKenzie (1959, 1981), it is possible to dispense with this interiority assumption with the relative interior assumption such as $rel.int\ (\mathrm{Y}) \cap rel.int\ (\mathrm{C}) \neq \emptyset$. The essence of the argument is as following. First, introduce one extra goods so that the production set in the extended economy including the extra goods satisfies the interior assumption, then apply the existence result with the interiority assumption, and finally show that as the extended economy shrinks the limit of competitive equilibria in the extended economies is equivalent to a competitive equilibrium in the original economy without the extra goods. The same argument still holds here as well.

Infinitely many commodities: Bewley (1972) establishes the existence of competitive equilibrium in economies with infinitely many commodities. Bewley (1972) used L_∞ as the commodity space and shows that the limit of quasi-equilibrium in finite dimensional subeconomies is a competitive equilibrium in the original economy. It seems possible to establish the existence of quasi-equilibrium in economies without individual survival in L_∞ as the commodity space with using the same argument of Bewley (1972) showing the limit of quasi-equilibrium in finite dimensional subeconomies is a competitive equilibrium in the original economy since the existence of quasi-equilibrium in economies with finitely many commodities does not require the individual survival assumption. However, because of non-joint continuity of evaluation map in infinite dimensional spaces, it turns out that it is in fact impossible to show that the limit of pseudo-quasi-equilibrium in finite dimensional subeconomies is a quasi-equilibrium in the original infinite dimensional economy. Of course, if irreducibility is assumed to be satisfied in any finite dimensional subeconomies, then every pseudo-quasi-equilibrium in finite dimensional subeconomies is a competitive equilibrium, and hence so is the limit. However, this irreducibility in subeconomies is rather strong since irreducibility is assumed to hold only in the original infinite dimensional economy. It seems still an open question whether the individual survival assumption is necessary or not in economies with infinitely many commodities.

Appendix : Proofs of Lemmas

These proofs are all standard but given here for completeness.

Proof of lemma 1: Since (2) implies from the separation theorem of convex sets, there is some $p' \neq 0$ such that $p' \cdot y \leqq 0$ for $y \in Y$, $p' \in (Y)^* \backslash \{0\}$ holds. Then $\bar{y} \in int\,(Y) (\neq \emptyset)$ implies $p' \cdot \bar{y} < 0$, and hence $p'' \equiv -p'/p' \cdot \bar{y} \in S'$. Thus S' is non-empty. Since $(Y)^*$ and $\{p : p \cdot \bar{y} = -1\}$ are convex and closed, so is $S' \equiv (Y)^* \cap \{p : p \cdot \bar{y} = -1\}$. Thus it is enough to show that S' is bounded. Suppose that there is $\{p^\nu\}_{\nu=1}^{\infty} \subset S'$ with $\|p^\nu\| \to \infty$ as $\nu \to \infty$. Consider a sequence $\{p'^\nu\}_{\nu=1}^{\infty} \subset (Y)^*$, where p'^ν is defined as $p'^\nu/\|p^\nu\|$. Then the compactness of the unit ball and $\|p'^\nu\| = 1$ imply that there is $p^0 (\neq 0)$ such that $p'^\nu \to p^0$ as $\nu \to \infty$. Since $(Y)^*$ is closed, $p^0 \in (Y)^* \backslash \{0\}$ holds. Also consider $p'^\nu \cdot \bar{y} \equiv p^\nu \cdot \bar{y}/\|p^\nu\| = -/\|p^\nu\|$ for $\nu \in N = \{1, 2, \cdots\}$. Then $\|p^\nu\| \to \infty$ and $p'^\nu \to p^0$ imply that $p'^\nu \cdot \bar{y} \to p^0 \cdot \bar{y} = 0$ as $\nu \to \infty$. Then $p^0 \in (Y)^* \backslash \{0\}$ and $\bar{y} \in int\,(Y)$ imply that there is $\overline{y'} \in Y$ with $p^0 \cdot \overline{y'} > 0$. This is a contradiction to $p^0 \in (Y)^* \backslash \{0\}$. Thus S' is bounded, and hence compact. *Q. E. D.*

Proof of lemma 2: Since $\widehat{C_i'}$ is an intersection of convex sets, it is convex. Also (6) implies that there is some $\overline{x_i} \in C_i\ \forall i \in I$ satisfying $\bar{x} (\equiv \sum_{i \in I} \overline{x_i}) \in Y$. That is, $\bar{x} - \sum_{i \in I} \overline{x_i} = 0$ holds. Then $\overline{x_i} \in \widehat{C_i}$ and $\bar{x} \in \widehat{Y}$ hold, and $(\widehat{C_i} \subset) \widehat{C_i'}$ is non-empty. Let $x_i \in \widehat{C_i'}$. Then by the definition, $x_i = y - (a + \varepsilon e)$ for some $y \in Y$ and $\varepsilon > 0$, and hence $y \in \widehat{Y'}$. Since $C_i \subset a + R_+^N$ and $y \in \widehat{Y'}$ hold, $a \leqq x_i = y - (a + \varepsilon e) \leqq y$ holds. Thus when we show the boundedness of $\widehat{Y'}$, we can also show that of $\widehat{C'}$ as well. Suppose $\widehat{Y'}$ is not bounded, and hence there is a sequence $(y^\nu)_{\nu=1}^{\infty} \subset \widehat{Y'}$ such that $\|y^\nu\| \to \infty$ holds as $\nu \to \infty$. Note that the above inequality implies $y^\nu \geqq a\ \forall \nu \in N$. Let $\mu_\nu = \|y^\nu\|$. Since $\mu_\nu \to +\infty$ as $\nu \to +\infty$, $0 \leqq \frac{1}{\mu_\nu} \leqq 1$ holds for sufficiently large ν. Moreover the convexity of Y and $0 \in Y$ implies $\frac{1}{\mu_\nu} y^\nu = \frac{1}{\mu_\nu} y^\nu + (1 - \frac{1}{\mu_\nu}) 0 \in Y$ for sufficiently large ν. Note that $\left\| \frac{y^\nu}{\mu_\nu} \right\| = 1\ \forall \nu \in N$ by the definition implies $\frac{y^\nu}{\mu_\nu} \to y^0$ with $\|y^0\| = 1$ without loss of generality owing to the compactness of the unit sphere. Since Y is closed from (1), $y^0 \in Y$ follows. Also, $\frac{y^\nu}{\mu_\nu} \geqq \frac{a}{\mu_\nu}\ \forall \nu = 1, 2, \cdots$ implies $y^0 \geqq 0$ in the limit. Thus $y^0 = 0$ must hold from (2). This is, however, a contradiction to $\|y^0\| = 1$. Therefore $\widehat{Y'}$ is bounded, and hence so is each $\widehat{C_i'}$. *Q. E. D.*

Proof of lemma 3: Suppose that $x_i \in D_i(p) \cap \widehat{C_i'}$ satisfies $0 > m_i(p)$. Now suppose that there is $x_i' \in C_i$ with $x_i' \in P_i(x_i)$ and $p \cdot x_i' \leqq 0$.[25] Then, since $x_i \in \widehat{C_i'}$ $(\subset int\,(K))$ holds, from the definition of $\widetilde{C_i} (\equiv C_i \cap K)$, $\alpha x_i' + (1-\alpha) x_i \in (C_i \cap int\,(K)) \subset (C_i \cap K) = \widetilde{C_i}$ holds for some $\alpha \in (0, 1)$ sufficiently close to 0. Then $p \cdot (\alpha x_i' + (1-\alpha) x_i) \leqq 0$ holds, and (5), convexity of preferences, implies $\alpha x' + (1-\alpha) x_i \in P_i(x_i)$. This is, however, a contradiction to $x_i \in D_i(p)$. Thus, $x_i \in D_i(p)$ is indeed a demand point not only over $\widetilde{C_i}$ but also over C_i. Thus, $x_i' \in P_i(x_i)$ implies

25) Note that from the definition of $\widetilde{C_i}$, $x_i' \notin int\,(K)$ holds.

$p \cdot x'_i > 0$. Suppose next $x'_i \in R_i(x_i)$. Since $x'_i \in P_i(x_i)$ implies $p \cdot x'_i > 0$ from the above argument, it is enough to consider $x'_i \in I_i(x_i)$. Note that transitivity of preference with respect to R_i implies $I_i(x_i) = I_i(x'_i)$ and $P_i(x_i) = P_i(x'_i)$. Since non-satiation ($P_i(x_i) \neq \emptyset$) holds, there is $x''_i \in P_i(x_i)$ and hence $x''_i \in P_i(x'_i)$. Then convexity of preference (5) implies $(x''_i, x'_i) \subset P_i(x'_i)$ and hence $(x''_i, x'_i) \subset P_i(x_i)$, *i.e.*, $(\alpha x''_i + (1-\alpha) x'_i) \in P_i(x_i) \quad \forall \alpha \in (0, 1)$. Then the above argument implies $p \cdot (\alpha x''_i + (1-\alpha) x'_i) > 0$ for $\alpha \in (0, 1)$. Then letting $\alpha \to 0$ gives rise to $p \cdot x'_i \geqq 0$. Thus $x'_i \in R_i(x_i)$ implies $p \cdot x'_i \geqq 0$. ***Q. E. D.***

Proof of lemma 4: $x_i \in D_i(p)$ implies $p \cdot x_i \leqq 0$ from the budget constraint in the definition of $D_i(p)$. On the other hand, since non-satiation at x_i and $x_i \in R_i(x_i)$ hold, lemma 3 implies $p \cdot x_i \geqq 0$. Thus $p \cdot x_i = 0$ holds. ***Q. E. D.***

Proof of corollary: Suppose that there is $\{x_i\}_{i \in I}$ satisfying $x_i \in D_i(p) \ \forall i \in I$ and $x (\equiv \sum_{i \in I} x_i) \in [(\sum_{i \in I} D_i(p)) \cap Y]$. Then, $x_i \in D_i(p) \subset \overline{C_i} \ \forall i \in I$ from the definition of $D_i(p)$ implies $x \equiv \sum_{i \in I} x_i \subset [(\sum_{i \in I} \overline{C_i}) \cap Y]$, and hence, $(x_i, x)_{i \in I}$ is a feasible pseudo-allocation.[26] Suppose that some $i' \in I$ satisfies $0 > m_{i'}(p)$. Then $D_{i'}(p) \subset \widetilde{C}_{i'} \subset C_{i'}$ holds from the definition of $D_{i'}(p)$. Then since $x \in Y$ and $\sum_{i \in I \backslash \{i'\}} x_i \in \sum_{i \in I \backslash \{i'\}} C'_i$ implies $x_{i'} \in \widehat{C'_{i'}} (\equiv [C'_{i'} \cap (Y - (\sum_{i \in I \backslash \{i'\}} C'_i))]) \subset int(K)$, and hence $x_{i'} \in (D_{i'}(p) \cap \widehat{C'_{i'}})$ holds. Since non-satiation at $x_{i'}$ follows from (7), the conditions of lemma 4 hold and the result follows. ***Q. E. D.***

Proof of lemma 5: Note first that we already show the non-emptiness of $D_i(p)$ in the case of $0 \leqq m_i(p)$. Thus we need to show it in the case of $0 > m_i(p)$. Let $\boldsymbol{R}_i(x) = \{x' \in \widetilde{C}_i \mid p \cdot x' \leqq 0\} \cap \{x' \in \widetilde{C}_i \mid x' \in R_i(x)\}$ for $x \in \{z \in \widetilde{C}_i \mid p \cdot z \leqq 0\}$. Then $x \in R_i(x)$ imply the non-emptiness of $R_i(x)$ for $x \in \{z \in \widetilde{C}_i \mid p \cdot z \leqq 0\}$. Also the continuity of preferences and the compactness of $\widetilde{C}_i$ imply that $R_i(x)$ is a compact set. Since the preference $\boldsymbol{R}_i$ is transitive, $\cap_{j=1}^{n} R_i(x_j) \neq \emptyset \ \forall n \in N$, where $x_j \in \{z \in \widetilde{C}_i \mid p \cdot z \leqq 0\}$, $j = 1, \cdots, n$. Thus the finite intersection property of compact sets implies $\cap \{R_i(x) \mid x \in \widetilde{C}_i$ and $p \cdot x \leqq 0\} \neq \emptyset$. Since $D_i(p)$ is equal to $\cap \{R_i(x) \mid x \in \widetilde{C}_i$ and $p \cdot x \leqq 0\}$ from the transitivity of preferences, $D_i(p) \neq \emptyset$ follows in the case of $0 > m_i(p)$ as well.

To prove the upper hemi-continuity has two cases. Since $\overline{C_i}$ is compact, for this purpose, it is enough to show that $D_i(\cdot)$ has a closed graph. The first case is $0 > m_i(p^0)$ so that the cheaper point assumption is met at $p^0 \in S'$. Let $p^\nu \to p^0$ in S', $x^\nu \in D_i(p^\nu)$, and $x^\nu \to x^0$ in $\widetilde{C}_i$. Note $p^\nu \cdot x^\nu \leqq 0 \ \forall \nu = 1, 2, \cdots$ implies $p^0 \cdot x^0 \leqq 0$. Let x' be an arbitrary point in the budget set, $\{z \in \widetilde{C}_i \mid p^0 \cdot x \leqq 0\}$. Note that $0 > m_i(p^0)$ implies there is $\bar{x} \in \widetilde{C}_i$ such that $p^0 \cdot \bar{x} < 0$ and $p^\nu \cdot \bar{x} < 0$ for sufficiently large ν. Define $t^\nu = max\{t \in [0,1] : t x' + (1-t)\bar{x} \in \{z \in \widetilde{C}_i \mid p^\nu \cdot z \leqq 0\}\}$. Note that since $(x^\circ, x' \in) \widetilde{C}_i$ is convex, these are well defined and $x'^\nu \equiv t^\nu x' + (1-t^\nu)\bar{x} \in \{z \in \widetilde{C}_i \mid p^\nu \cdot z \leqq 0\}$ holds. Moreover, $t^\nu < 1$ implies $p^\nu \cdot x'^\nu = 0$ since $t^\nu < 1$

26) Note that $(x'_i, y')_{i \in I}$ is a feasible pseudo-allocation, although $x'_i \notin \widehat{C}_i$ occurs for some $i \in I$ due to the lack of the individual survival assumption. Thus $(x'_i, y')_{i \in I}$ may not be a feasible allocation.

implies $p^{\nu}\cdot x'>0$. Note that $t\in(t^{\nu},1)$ implies $tx'+(1-t)\bar{x}\in\widetilde{C}_i$. Suppose $t^{\nu}\to 1$ does not hold, and without loss of generality $t^{\nu}\to t<1$ holds. Then $x'^{\nu}\to tx'+(1-t)\bar{x}$ and $p^{\nu}\cdot x'^{\nu}=0$ for sufficiently large ν implies that $p^0\cdot(tx'+(1-t)\bar{x})=0$ in the limit. But $p^0\cdot\bar{x}<0$ and $t<1$ yield $p^0\cdot x'>0$ which is a contradiction to the definition of x'. Thus $t^{\nu}\to 1$ and hence $x'^{\nu}\to x'$ as $\nu\to\infty$ holds. Since $x^{\nu}\in D_i(p^{\nu})$ implies $x^{\nu}\in R_i(x'^{\nu})$, continuity of preferences gives rise to $x^0\in R_i(x')$, and so $x^0\in D_i(p^0)$. The second case is $0\leqq m_i(p^0)$. Then the closedness and hence upper hemi-continuity of $D_i(p^0)$ is trivial. Since $p^{\nu}\cdot x^{\nu}\leqq 0$ and $x^{\nu}\in\overline{C_i}$ from $x^{\nu}\in D_i(p^{\nu})$ $\forall\,\nu\in N$ implies $x^0\in\overline{C_i}$ and $p^0\cdot x^0\leqq 0$ by the compactness of $\overline{C_i}$ and the continuity of inner product, $x^0\in D_i(p^0)$ holds. Therefore $D_i(\cdot)$ is closed at each p^0 and hence it is upper hemi-continuous at each p^0 as well.

The convexity of $D_i(\cdot)$ in the first case follows from the convexity of $R_i(\cdot)$, a form of convexity of preferences derived from (5) as shown in footnote (15). The convexity of $D_i(\cdot)$ in the second case follows from its definition. *Q. E. D.*

Proof of lemma 6: Let $(x_h^*, y^*, p^*)_{h\in I}$ with $p^*\in R^N\backslash\{0\}$ be a pseudo quasi-equilibrium and consumer i have a cheaper point. Then since $(x_h^*, y^*)_{h\in I}$ is a pseudo feasible allocation, $x_h^*\in D_h(p^*)\subset\overline{C_h}$ and $x^*\equiv\sum_{h\in I}x_h^*\in Y$, and hence, $x_h^*\in\widehat{C'_h}$ hold. Moreover, since consumer i have a cheaper point, $x_i^*\in D_i(p^*)\subset C_i$ holds. Thus, $x_i^*\in\widehat{C''_i}$ $(=\widehat{C'_i}\cap C_i)$ holds and $P_i(x_i^*)\neq\emptyset$ from resource relatedness. Suppose consumer i' is resource related to consumer i. Then since $(x_h^*, y^*)_{h\in I}$ is a pseudo feasible allocation with $x_i^*\in C_i$, there are $z_{i'}\in\overline{C_{i'}}$ and $(x'_h)_{h\in I}$ satisfying $\sum_{h\in I}x'_h\in((\sum_{h\in I^1}C_h)+(\sum_{h\in I^2}\overline{C_h})\cap(Y-\delta z_{i'}))(\exists\,\delta>0)$ such that $x'_h\in R_h(x_h^*)$ hold for $h\in I^1$, $x'_h\in\overline{C_h}$ holds for $h\in I^2$, and $x'_i\in P_i(x_i^*)$ holds, where $I^1=\{h\in I: x_h^*\in C_h\}$ and $I^2=\{h\in I: x_h^*\in\overline{C_h}\backslash C_h\}$. Note first that at p^* each consumer has a cheaper point over $\overline{C_h}$ or not. Let $I'=\{h\in I: m(p^*)<0\}(\subset I^1)$ and $I''=\{h\in I: m(p^*)\geqq 0\}$ $=I\backslash I'(\supset I^2)$. Then $x_h^*\in D_h(p^*)$ implies that x_h^* is a demand point for $h\in I'$ with a cheaper point and $p^*\cdot x_h^*=0$ for $h\in I''$ without cheaper points. Of course, since i belongs to I', $x_i^*\in P_i(x_i^*)$ imply $p^*\cdot x_i^*>0$. Since $(x_i^*, y^*)_{i\in I}$ is a pseudo feasible allocation, non-satiation $P_h(x_h^*)\neq\emptyset$ holds for any $h\in I^1$, and hence for any $h\in I'$. Then $x'_h\in R_h(x_h^*)$ implies $p^*\cdot x'_h\geqq 0$ for $h\in I'$ from lemma 3. Also $x'_h\in\overline{C_h}$ implies $p^*\cdot x'_h\geqq 0$ for consumers $h\in I''$ without cheaper points over $\overline{C_h}$ at p^*, and hence $x'_h\in\overline{C_h}$ implies $p^*\cdot x'_h\geqq 0$ for consumers $h\in I^2$. Thus, $\sum_{h\in I}p^*\cdot x'_h>0$ holds. On the other hand, since $((\sum_{h\in I}x'_h)+\delta z_{i'})\in Y$ implies $p^*\cdot((\sum_{h\in I}x'_h)+\delta z_{i'})\leqq 0$ from the profit maximization condition (iii), $p^*\cdot\delta z_{i'}\leqq-(p^*\cdot\sum_{h\in I}x'_h)<0$ holds. Then $\delta>0$ implies $p^*\cdot z_{i'}<0$. Thus, consumer i' who is resource related to consumer i in fact has a cheaper point over $\overline{C_{i'}}$ and hence over $\widetilde{C}_{i'}$. When consumer i' is indirectly resource related to consumer i, then consumer i' is connected to consumer i through resource related consumers between them. Thus applying the above argument between two consumers who are resource related to the consumers between consumer i and i' turns out that consumer i' indeed has a cheaper point over $\overline{C_{i'}}$ and hence over $\widetilde{C}_{i'}$. ***Q. E. D.***

Proof of lemma 7: Since S' is convex and compact, and the inner product is linear and continuous, Q(z) is non-empty and convex. Suppose that $z^\nu \to z^0$ in $\overline{C}$, $p^\nu \to p^0$ in S' and $p^\nu \in Q(z^\nu)$. Then $p^\nu \cdot z^\nu \geqq p' \cdot z^\nu$ $\nu = 1, 2, \cdots$. hold $\forall p' \in S'$, and hence $p^0 \cdot z^0 \geqq p' \cdot z^0$ follows for any given p' in S'. This gives rise to $p^0 \in Q(z^0)$, and hence the upper hemi-continuity of $Q(\cdot)$ at z^0 since S' is compact. ***Q. E. D.***

Proof of lemma 8: Note first that (6) also implies that there is some $\overline{x}_i \in \widetilde{C}_i$ $\forall i \in I$ satisfying $\overline{x}(\equiv \sum_{i \in I} \overline{x}_i) \in int(Y)$. Then $\widetilde{p} \in S'$ implies $\widetilde{p} \cdot \overline{x} = \widetilde{p} \cdot (\sum_{i \in I} \overline{x}_i) < 0$. Thus the set of consumers $I^1 \equiv \{i \in I | \widetilde{p} \cdot x_i < 0 \text{ for some } x_i \in \widetilde{C}_i (\subset \overline{C_i})\}$ is non-empty. Note first that since $(\widetilde{x}_i, \widetilde{y})_{i \in I}$ is a feasible pseudo-allocation, corollary to lemma 4 implies $\widetilde{p} \cdot \widetilde{x}_i = 0$ for $i \in I^1$. Note also that, from lemma 3, $\widetilde{x}_i \in D_i(\widetilde{p})$ is indeed a demand point over C_i not only over $\widetilde{C}_i$ $\forall i \in I^1$. That is, $\forall i \in I^1$, $x_i' \in C_i$ with $\widetilde{p} \cdot x_i' \leqq 0$ implies $\widetilde{x}_i' \in R_i(x_i')$. Let $I_2 \equiv I \backslash I^1$. Then, from the definition of $D_i(\widetilde{p})$ for $h \in I_2$, $\widetilde{x}_h \in \overline{C_h}$ and $0 = \widetilde{p} \cdot \widetilde{x}_h$ holds $\forall h \in I^2$. Thus, $\widetilde{p} \cdot \widetilde{x}_i = 0$ holds $\forall i \in I$.

Suppose that $I^2 \equiv I \backslash I^1$ is non-empty. Since $(\widetilde{x}_i, \widetilde{y})_{i \in I}$ implies that $\widetilde{x}_i \in \widehat{C}_i''$ $= (\widehat{C}_i' \cap C_i)$ holds for $i \in I^1$ and $\widetilde{x}_h \in \overline{C_h}$ holds $\forall h \in I^2$, irreducibility assumption (7′) implies that there are some $(\widehat{x}_i)_{i \in I^1}$, $\widehat{y} \in Y$, and $\sum_{h \in I^2} z_h \in \sum_{h \in I^2} \overline{C_h}$ such that $(\widetilde{x}_i + \widehat{x}_i)_{i \in I^1} \in \times_{i \in I^1} C_i$, $\sum_{i \in I^1} \widehat{x}_i = \widehat{y} - \widetilde{y} - \sum_{h \in I^2} \alpha_h z_h$ for some $\alpha_h > 0$, and, $\widetilde{x}_i + \widehat{x}_i \in R_i(\widetilde{x}_i)$ $\forall i \in I^1$ and $\widetilde{x}_{i'} + \widehat{x}_{i'} \in P_{i'}(\widetilde{x}_{i'})$ for some $i' \in I^1$. Thus from lemma 3, $\widetilde{p} \cdot (\widetilde{x}_i + \widehat{x}_i) \geqq \widetilde{p} \cdot \widetilde{x}_i (=0)$ holds $\forall i \in I^1$, and $\widetilde{p} \cdot (\widetilde{x}_{i'} + \widehat{x}_{i'}) > \widetilde{p} \cdot \widetilde{x}_{i'} (=0)$ for some $i' \in I^1$. Then summation of these inequalities over I^1 gives rise to $\widetilde{p} \cdot \sum_{i \in I^1} \widehat{x}_i = \widetilde{p} \cdot (\widehat{y} - \widetilde{y} - \sum_{h \in I^2} \alpha_h z_h) > 0$. Since the profit maximization condition (iii) implies $\widetilde{p} \cdot (\widehat{y} - \widetilde{y}) \leqq 0$, $\sum_{h \in I^2} \alpha_h z_h < 0$ must hold. Then this implies that there is at least some $h' \in I^2$ such that $\widetilde{p} \cdot z_{h'} < 0$ for some $z_{h'} \in \overline{C_{h'}}$. Since $\overline{C_{h'}}$ is a convex combination of some points in $\widetilde{C}_{h'}$ and 0, $\widetilde{p} \cdot z_{h'} < 0$ implies that there is $z_{h'}' \in \widetilde{C}_{h'}$ satisfying $\widetilde{p} \cdot z_{h'}' < 0$. Thus $m_{h'}(\widetilde{p}) \equiv \mathit{min}\{\widetilde{p} \cdot z_{h'} | z_{h'} \in \widetilde{C}_{h'}\} < 0$ holds. This is, however, a contradiction to $h' \in I^2$ since it implies $m_{h'}(\widetilde{p}) \geqq 0$ by the definition of I^2. Thus I^2 must be empty, and hence every consumer in the economy has a cheaper point, i.e., his income strictly larger than his subsistence level which is equal to $m_i(\widetilde{p})$. ***Q. E. D.***

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