Semi-classical Asymptotics for the Partition Function of an Abstract Bose Field Model

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Semi-classical asymptotics for the partition function of an abstract Bose field model is considered.

Keywords: semi-classical asymptotics, Bose field, partition function, second quantization, Fock space.

I. Introduction

In quantum mechanics, in which a physical constant $\hbar := h/2\pi$ (h : the Planck constant) plays an important role, the limit $\hbar \to 0$ for various quantities (if it exists) is called the classical limit. Trace formulas in the abstract boson Fock space and the classical limit for the trace $Z(\beta\hbar)$ (the partition function) of the heat semigroup of a perturbed second quantization operator were derived by Arai [4], where $\beta > 0$ denotes the inverse temperature. Generally speaking, the classical limit is regarded as the zero-th order approximation in $\hbar$. From this point of view, it is interesting to derive higher order asymptotics of various quantities in $\hbar$. Such asymptotics are called semi-classical asymptotics. The purpose of this paper is to derive an asymptotic formula for $Z(\beta\hbar)$.

The outline of this paper is as follows. In Section II, we review some fundamental facts in the abstract boson Fock space over $\mathscr{H}_{\mathbb{C}}$, the complexification of a real separable Hilbert space $\mathscr{H}$. In particular, a differential structure over a class of locally convex spaces is introduced, which leads to the Q-space representation $L^2(E, d\mu)$ of the boson Fock space over $\mathscr{H}_{\mathbb{C}}$. The differentiation discussed in this section should be considered to be related to the infinite dimensional analysis in [2, 3]. In Section III, following [4], we review a classical limit in the abstract boson Fock space over a real separable Hilbert space $\mathscr{H}$. In Section IV, we introduce a class of locally convex spaces. This gives a general framework for the semi-classical analysis discussed in this paper. In the last section, we derive a semi-classical asymptotic formula for $Z(\beta\hbar)$ mentioned above. The present paper is based on [1].

I am very grateful to Professor Arai for his rigorous and hearty help. It is a real delight that the teachings of him, which has been shown to me responding to my consciousness on that occasion, let me discover some ideas in the present paper.

II. Preliminaries

Let $\mathscr{H}$ be a real separable Hilbert space. We denote by $\mathscr{H}_{\mathbb{C}}$ the complexification of $\mathscr{H}$. In general, we denote by $< \cdot, \cdot >$ and $\| \cdot \|$ the inner product and the norm of a Hilbert space.

We denote by $\mathfrak{S}_n$ the permutation group of n letters. For all $\sigma \in \mathfrak{S}_n$, there exists a unique unitary mapping U_σ on $\bigotimes^n \mathscr{H}_{\mathbb{C}}$ such that

$$U_\sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \ , \quad f_1, \cdots, f_n \in \mathscr{H}_{\mathbb{C}}.$$

We define S_n by

$$S_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} U_\sigma.$$

Then S_n is an orthogonal projection on $\bigotimes^n \mathscr{H}_{\mathbb{C}}$. We set

$$\bigotimes_{\mathrm{s}}^{n} \mathscr{H}_{\mathbb{C}} := S_n(\bigotimes^{n} \mathscr{H}_{\mathbb{C}}),$$

which is called the n-fold symmetric tensor product of $\mathscr{H}_{\mathbb{C}}$. Then $\bigotimes_{\mathrm{s}}^n \mathscr{H}_{\mathbb{C}}$ becomes a Hilbert space. We set

$$\bigotimes_{\mathrm{s}}^{0} \mathscr{H}_{\mathbb{C}} := \mathbb{C}.$$

We define $\mathscr{F}_{\mathrm{b}}(\mathscr{H}_{\mathbb{C}})$ by

$$\mathscr{F}_{\mathrm{b}}(\mathscr{H}_{\mathbb{C}}) := \bigoplus_{n=0}^{\infty} \bigotimes_{\mathrm{s}}^{n} \mathscr{H}_{\mathbb{C}}.$$

Then $\mathscr{F}_{\mathrm{b}}(\mathscr{H}_{\mathbb{C}})$ becomes a Hilbert space, which is called the boson Fock space over $\mathscr{H}_{\mathbb{C}}$.

For all $n \in \mathbb{Z}_+$ (the set of nonnegative integers), we define the mapping u_n from $\bigotimes_{\mathrm{s}}^n \mathscr{H}_{\mathbb{C}}$ to $\mathscr{F}_{\mathrm{b}}(\mathscr{H}_{\mathbb{C}})$ by

$$(u_n\psi)^{(m)} := \psi, \quad m = n,$$

$$(u_n\psi)^{(m)} := 0, \quad m \neq n, \ \psi \in \bigotimes_{\mathrm{s}}^{n} \mathscr{H}_{\mathbb{C}}.$$

Then u_n is a linear isometry from $\bigotimes_{\mathrm{s}}^n \mathscr{H}_{\mathbb{C}}$ to $\mathscr{F}_{\mathrm{b}}(\mathscr{H}_{\mathbb{C}})$. We define $\mathscr{F}_{\mathrm{b}}^{(n)}(\mathscr{H}_{\mathbb{C}})$ by

$$\mathscr{F}_{\mathrm{b}}^{(n)}(\mathscr{H}_{\mathbb{C}}) := u_n(\bigotimes_{\mathrm{s}}^{n} \mathscr{H}_{\mathbb{C}}).$$

Then $\mathscr{F}_{\mathrm{b}}^{(n)}(\mathscr{H}_{\mathbb{C}})$ can be identified with $\bigotimes_{\mathrm{s}}^n \mathscr{H}_{\mathbb{C}}$ by u_n. We define $\mathscr{F}_{\mathrm{b},0}(\mathscr{H}_{\mathbb{C}})$ by

$$\mathscr{F}_{\mathrm{b},0}(\mathscr{H}_{\mathbb{C}}) := \hat{\bigoplus}_{n=0}^{\infty} \mathscr{F}_{\mathrm{b}}^{(n)}(\mathscr{H}_{\mathbb{C}}),$$

where $\hat{\bigoplus}_{n=0}^{\infty}$ denotes algebraic infinite direct sum.

For all $f \in \mathscr{H}_{\mathbb{C}}$, we denote by $a(f)$ the boson annihilation operator in $\mathscr{F}_{\mathrm{b}}(\mathscr{H}_{\mathbb{C}})$ (cf. [6]), which is defined to be the closed linear operator in $\mathscr{F}_{\mathrm{b}}(\mathscr{H}_{\mathbb{C}})$ such that its adjoint $a(f)^*$ takes the following form (for a linear operator A, $D(A)$ denotes the domain of A):

$$D(a(f)^*) = \{\psi \in \mathscr{F}_b(\mathscr{H}_{\mathbb{C}}) | \sum_{n=1}^{\infty} n \|S_n(f \otimes \psi^{(n-1)})\|^2 < \infty\},$$

$$(a(f)^*\psi)^{(0)} = 0,$$
$$(a(f)^*\psi)^{(n)} = \sqrt{n} S_n(f \otimes \psi^{(n-1)}), \quad n \in \mathbb{N}.$$

We define $\Omega \in \mathscr{F}_{\mathrm{b}}^{(0)}(\mathscr{H}_{\mathbb{C}})$ by

$$\Omega := 1 \in \mathbb{C}.$$

We have the following proposition.

Proposition 2.1. (1) *For all* $f, g \in \mathscr{H}_{\mathbb{C}}$,

$$[a(f), a(g)^*]|_{\mathscr{F}_{\mathrm{b},0}(\mathscr{H}_{\mathbb{C}})} =< f, g >,$$

where for a linear operator A *and a subspace* $D \subset D(A)$, $A|_D$ *denotes the ristriction of* A *to* D. (2) *For all* $n \in \mathbb{N}$ *and* $f_1, \cdots, f_n \in \mathscr{H}_{\mathbb{C}}$, $a(f_1)^* \cdots a(f_n)^*\Omega \in \mathscr{F}_{\mathrm{b}}^{(n)}(\mathscr{H}_{\mathbb{C}})$, *and*

$$a(f_1)^* \cdots a(f_n)^*\Omega = \sqrt{n!} S_n(f_1 \otimes \cdots \otimes f_n).$$

Proof. See [6, Theorem 6.4]. □

Let $\{\mathscr{K}_n\}_{n\in\mathbb{Z}_+}$ be a family of Hilbert spaces, and $T^{(n)}$ be a densely defined linear operator in $\mathscr{K}_n$. We define $\bigoplus_{n=0}^{\infty} T^{(n)}$ by

$$D(\bigoplus_{n=0}^{\infty} T^{(n)}) := \{\psi \in \bigoplus_{n=0}^{\infty} \mathscr{K}_n | \psi^{(n)} \in D(T^{(n)}),\ \ n \in \mathbb{Z}_+\},$$

$$((\bigoplus_{n=0}^{\infty} T^{(n)})\psi)^{(n)} := T^{(n)}\psi^{(n)}, \quad\ n \in \mathbb{Z}_+.$$

Then $\bigoplus_{n=0}^{\infty} T^{(n)}$ is a linear operater in $\bigoplus_{n=0}^{\infty} \mathscr{K}_n$.

Let T be a densely defined closed linear operator in $\mathscr{H}_{\mathbb{C}}$. For all $n \in \mathbb{N}$, we set

$$T_0^{(n)} := \sum_{j=1}^{n} I \otimes \cdots \otimes \overbrace{T}^{j} \otimes \cdots \otimes I|_{\hat{\bigotimes}_{\mathrm{s}}^{n} D(T)},$$

where $\hat{\bigotimes}_{\mathrm{s}}^{n}$ denotes n-fold algebraic symmetric tensor product, and

$$T^{(n)} := \overline{T_0^{(n)}},$$

$$T^{(0)} := 0.$$

We define $d\Gamma(T)$ by

$$d\Gamma(T) := \bigoplus_{n=0}^{\infty} T^{(n)}.$$

Then $d\Gamma(T)$ is a linear operator in $\mathscr{F}_{\mathrm{b}}(\mathscr{H}_{\mathbb{C}})$, which is called the second quantization of T.

We have the following proposition.

PROPOSITION 2.2. *Let T be a self-adjoint operator in $\mathscr{H}_{\mathbb{C}}$.*

(1) *$d\Gamma(T)$ is self-adjoint.*

(2)

$$\sigma(d\Gamma(T)) = \{0\} \cup \overline{(\bigcup_{n=1}^{\infty}\{\sum_{j=1}^{n} \lambda_j | \lambda_j \in \sigma(T),\ \ j = 1, \cdots, n\})}.$$

$$\sigma_p(d\Gamma(T)) = \{0\} \cup (\bigcup_{n=1}^{\infty}\{\sum_{j=1}^{n} \lambda_j | \lambda_j \in \sigma_p(T),\ \ j = 1, \cdots, n\}).$$

Proof. See [5, Theorem 4.14]. □

Let $\mathscr{E}$ be a real locally convex space such that $\mathscr{E}$ is dense in $\mathscr{H}$ and the embedding mapping of $\mathscr{E}$ into $\mathscr{H}$ is continuous. Then we can see that

$$\mathscr{E} \subset \mathscr{H} \subset \mathscr{E}',$$

where $\mathscr{E}'$ denotes the topological dual of $\mathscr{E}$.

Following the fact that for all $\phi \in \mathscr{H}$ and $f \in \mathscr{E}$,

$$< \phi, f >= \phi(f),$$

for all $\phi \in \mathscr{E}'$ and $f \in \mathscr{E}$, we set

$$< \phi, f >:= \phi(f).$$

Let $\mathscr{B}$ be the Borel field generated by $\{< \cdot, f > | f \in \mathscr{E}\}$, and μ be a probability measure on $(\mathscr{E}', \mathscr{B})$ such that

$$\int_{\mathscr{E}'} e^{i\phi(f)} d\mu(\phi) = e^{-\|f\|^2_{\mathscr{H}}/2}, \quad f \in \mathscr{E}.$$

Then we have

$$\int_{\mathscr{E}'} \phi(f)^2 d\mu(\phi) = \|f\|^2_{\mathscr{H}}, \quad f \in \mathscr{E}.$$

Hence the mapping $f \longmapsto < \cdot, f >$ from $\mathscr{E}$ to $L^2(\mathscr{E}', d\mu)$ is continuous linear and it extends to the continuous linear mapping T from $\mathscr{H}$ to $L^2(\mathscr{E}', d\mu)$. For all $f \in \mathscr{H}$ and $\phi \in \mathscr{E}'$, we define $< \phi, f >$ by

$$< \phi, f >:= T(f)(\phi).$$

For all $f \in \mathscr{H}$ and $\phi \in \mathscr{E}'$, we define $\phi(f)$ by

$$\phi(f) :=< \phi, f > .$$

Then we have

$$\int_{\mathscr{E}'} e^{i\phi(f)} d\mu(\phi) = e^{-\|f\|^2_{\mathscr{H}}/2}, \quad f \in \mathscr{H}.$$

Let $\{E_n\}_{n\in\mathbb{N}}$ be a family of Banach spaces with the property that

$$E_{n+1} \subset E_n, \ \|\phi\|_n \le \|\phi\|_{n+1}, \quad \phi \in E_{n+1},$$

for all $n \in \mathbb{N}$, where $\|\cdot\|_n$ denotes the norm of E_n. Then, the topology defined by the norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ turns $\bigcap_{n\in\mathbb{N}} E_n$ into a Fréchet space. In particular, $\bigcap_{p\in\mathbb{N}} L^p(\mathscr{E}', d\mu)$ can be provided with the structure of Fréchet space.

We have the following proposition.

Proposition 2.3. *For all $f \in \mathscr{H}$, $< \cdot, f >\in \bigcap_{p\in\mathbb{N}} L^p(\mathscr{E}', d\mu)$, and the mapping $f \in \mathscr{H} \longmapsto < \cdot, f >$ from $\mathscr{H}$ to $\bigcap_{p\in\mathbb{N}} L^p(\mathscr{E}', d\mu)$ is continuous linear.*

Let F be a function on $\mathbb{R}^n$ and $G_1, \cdots, G_n$ be real valued functions on $\mathscr{E}'$. We define $F(G_1, \cdots, G_n)$ by

$$F(G_1, \cdots, G_n)(\phi) := F(G_1(\phi), \cdots, G_n(\phi)), \quad \phi \in \mathscr{E}'.$$

Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be a family of subsets of the linear space of the functions on $\mathbb{R}^n$. We define $\{\mathscr{F}_n\}_{n\in\mathbb{N}}(\mathscr{E}')$ by

$$\{\mathscr{F}_n\}_{n\in\mathbb{N}}(\mathscr{E}') := \mathscr{L}\{F(< \cdot, f_1 >, \cdots, < \cdot, f_n >), 1 | F \in \mathscr{F}_n, f_1, \cdots, f_n \in \mathscr{H}, n \in \mathbb{N}\}.$$

Let $\{\mathscr{P}_n\}_{n\in\mathbb{N}}$ be the family of the linear space of the polynomials of n real variables with complex coefficients. We define $\mathscr{P}(\mathscr{E}')$ by

$$\mathscr{P}(\mathscr{E}') := \{\mathscr{P}_n\}_{n\in\mathbb{N}}(\mathscr{E}').$$

Similarly, we define $\mathscr{S}(\mathscr{E}')$ by

$$\mathscr{S}(\mathscr{E}') := \{\mathscr{S}(\mathbb{R}^n)\}_{n\in\mathbb{N}}(\mathscr{E}'),$$

where $\mathscr{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. Then we have the following proposition.

Proposition 2.4. *$\mathscr{P}(\mathscr{E}')$ and $\mathscr{S}(\mathscr{E}')$ are dense in $L^2(\mathscr{E}', d\mu)$.*

Proof. See [6, Theorem 2.10].

□

Definition 2.5. *Let $\mathscr{D}(\mathscr{E}')$ be a linear subspace of the linear space of the functions on $\mathscr{E}'$, and $\{D_f\}_{f\in\mathscr{H}}$ be a family of linear mappings from $\mathscr{D}(\mathscr{E}')$ to itself. The pair $(\mathscr{D}(\mathscr{E}'), \{D_f\}_{f\in\mathscr{H}})$ is said to be a differential structure over $\mathscr{E}'$ if the following propoties are satisfied.*

(1) *For all $g \in \mathscr{H}$, $1, < \cdot, g > \in \mathscr{D}(\mathscr{E}')$,*

$$D_f 1 = 0, \; D_f(< \cdot, g >) = < f, g > .$$

(2) *For all $F, G \in \mathscr{D}(\mathscr{E}')$, $FG \in \mathscr{D}(\mathscr{E}')$,*

$$D_f(FG) = (D_f F)G \; + \; F(D_f G).$$

(3) *Let $n \in \mathbb{N}$ be arbitary. Then, for all differentiable functions F on $\mathbb{R}^n$ and all real valued functions $G_j \in \mathscr{D}(\mathscr{E}')$, $j = 1, \cdots, n$, $F(G_1, \cdots, G_n) \in \mathscr{D}(\mathscr{E}')$,*

$$D_f(F(G_1, \cdots, G_n)) = \sum_{j=1}^{n} (\partial_j F)(G_1, \cdots, G_n) D_f G_j, \quad f \in \mathscr{H}.$$

(4) *For all $F \in \mathscr{D}(\mathscr{E}')$, $F^* \in \mathscr{D}(\mathscr{E}')$ (F^* is the complex conjugate of F),*

$$D_f(F^*) = (D_f F)^*, \quad f \in \mathscr{H}.$$

We can see that $\mathscr{P}(\mathscr{E}') \cup \mathscr{S}(\mathscr{E}') \subset \mathscr{D}(\mathscr{E}')$.

DEFINITION 2.6. *Let F be a C^∞-function on $\mathbb{R}^n$. We say that F is in $\mathscr{T}(\mathbb{R}^n)$ if and only if*

(1) *for all affine mappings A_j on $\mathbb{R}$, $j = 1, \cdots, n$,*

$$\lim_{|t|\to\infty} (\partial^\alpha F)(A_1 t, \cdots, A_n t) e^{-at^2} = 0, \quad \alpha \in \mathbb{Z}_+^n, \ a > 0,$$

(2) *for all $f_1, \cdots, f_n \in \mathscr{H}$, and $\alpha \in \mathbb{Z}_+^n$, $(\partial^\alpha F)(< \cdot, f_1 >, \cdots, < \cdot, f_n >) \in \bigcap_{p\in\mathbb{N}} L^p(\mathscr{E}', d\mu)$, and the mapping $(f_1, \cdots, f_n) \longmapsto (\partial^\alpha F)(< \cdot, f_1 >, \cdots, < \cdot, f_n >)$ from $\mathscr{H}^n$ to $\bigcap_{p\in\mathbb{N}} L^p(\mathscr{E}', d\mu)$ is continuous.*

We define $\mathscr{T}(\mathscr{E}')$ by

$$\mathscr{T}(\mathscr{E}') := \{\mathscr{T}(\mathbb{R}^n)\}_{n\in\mathbb{N}}(\mathscr{E}').$$

Then we have the following proposition.

PROPOSITION 2.7. (1) *$\mathscr{T}(\mathscr{E}')$ is a linear subspace of $\mathscr{D}(\mathscr{E}') \cap L^2(\mathscr{E}', d\mu)$.*

(2) *$\mathscr{P}(\mathscr{E}') \cup \mathscr{S}(\mathscr{E}') \subset \mathscr{T}(\mathscr{E}')$.*

(3) *For all $F, G \in \mathscr{T}(\mathscr{E}')$ and all $f \in \mathscr{H}$,*

$$FG, \ D_f F, \ F^* \in \mathscr{T}(\mathscr{E}').$$

PROPOSITION 2.8. *Let $f \in \mathscr{H}$ and $F, G \in \mathscr{T}(\mathscr{E}')$. Then*

$$\int_{\mathscr{E}'} \phi(f) F G d\mu(\phi) = \int_{\mathscr{E}'} (D_f F) G d\mu + \int_{\mathscr{E}'} F (D_f G) d\mu.$$

For all $f \in \mathscr{H}$, we can regard D_f as a densely defined linear operator in $L^2(\mathscr{E}', d\mu)$ by

$$D(D_f) := \mathscr{T}(\mathscr{E}').$$

PROPOSITION 2.9. *For all $f \in \mathscr{H}$,*

$$\mathscr{T}(\mathscr{E}') \subset D({D_f}^*), \ (D_f)^*|_{\mathscr{T}(\mathscr{E}')} = \phi(f) - D_f.$$

Since $\mathscr{T}(\mathscr{E}')$ is dense in $L^2(\mathscr{E}', d\mu)$, D_f is closable.

PROPOSITION 2.10. *Let $f \in \mathscr{H}, F \in \mathscr{T}(\mathscr{E}')$ and $G \in D(\overline{D_f})$. Then*

$$\int_{\mathscr{E}'} \phi(f) F G d\mu(\phi) = \int_{\mathscr{E}'} (D_f F) G d\mu + \int_{\mathscr{E}'} F (\overline{D_f} G) d\mu.$$

For all $f_1, \cdots, f_n \in \mathscr{H}$, $D^*_{f_1} \cdots D^*_{f_n} 1 \in \mathscr{P}(\mathscr{E}')$, which is called the Wick product of the randum variables $< \cdot, f_1 >, \cdots, < \cdot, f_n >$. For all $\phi \in \mathscr{E}'$, we define $: \phi(f_1) \cdots \phi(f_n) :$ by

$$: \phi(f_1) \cdots \phi(f_n) : := (D^*_{f_1} \cdots D^*_{f_n} 1)(\phi).$$

Then we have the following proposition.

PROPOSITION 2.11. *Let $f_j, g_k \in \mathscr{H}$, $j = 1, \cdots, n$, $k = 1, \cdots, m$. Then*
(1) $[D_f, D^*_g]|_{\mathscr{T}(\mathscr{E}')} =< f, g >, \quad f, g \in \mathscr{H}$,
(2) $D_f : \phi(f_1) \cdots \phi(f_n) := \sum_{j=1}^n < f, f_j >: \phi(f_1) \cdots \hat{\phi(f_j)} \cdots \phi(f_n) :, \quad f \in \mathscr{H}$, *where $\hat{\phi(f_j)}$ indicates omission of $\phi(f_j)$.*
(3)

$$\begin{aligned}
&<: \phi(f_1) \cdots \phi(f_n) :, : \phi(g_1) \cdots \phi(g_m) :>_{L^2(\mathscr{E}', d\mu)} \\
&= \delta_{n,m} < a(f_1)^* \cdots a(f_n)^* \Omega, a(g_1)^* \cdots a(g_m)^* \Omega >_{\mathscr{F}_{\rm b}(\mathscr{H}_{\mathbb{C}})} .
\end{aligned}$$

We have the following theorem.

THEOREM 2.12. *There exists a unique unitary mapping U from $\mathscr{F}_{\rm b}(\mathscr{H}_{\mathbb{C}})$ to $L^2(\mathscr{E}', d\mu)$ such that*

$$U\Omega = 1,$$

$$U(a(f_1)^* \cdots a(f_n)^* \Omega) =: \phi(f_1) \cdots \phi(f_n) :, \quad f_1, \cdots, f_n \in \mathscr{H}$$

Proof. See [6, Theorem 6.34]. □

III. A Classical Limit in The Abstract Boson Fock Space

In this section we review a classical limit for the trace of a perturbed second quantization operator and some fundamental facts related to it, following the work of Arai [4].

Let $\mathscr{H}$ be a real separable Hilbert space, and A be a strictly positive self-adjoint operator acting in $\mathscr{H}$. We denote by $\{\mathscr{H}_s(A)\}_{s \in \mathbb{R}}$ the Hilbert scale associated with A [4]. For all $s \in \mathbb{R}$, the dual space of $\mathscr{H}_s(A)$ can be naturally identified with $\mathscr{H}_{-s}(A)$.

We denote by $\mathscr{I}_1(\mathscr{H})$ the ideal of the trace class operators on $\mathscr{H}$. Let $\gamma > 0$ be fixed. Throughout this paper, we assume the following.

Assumption I. $\quad A^{9-\gamma} \in \mathscr{I}_1(\mathscr{H})$.

Under Assumption I, the embedding mapping of $\mathscr{H}$ into

$$E := \mathscr{H}_{-\gamma}(A)$$

is Hilbert-Schmidt. Hence, by Minlos' theorem, there exists a unique probability measure μ on $(E, \mathscr{B})$ such that the Borel field $\mathscr{B}$ is generated by $\{\phi(f) | f \in \mathscr{H}_\gamma(A)\}$ and

$$\int_E e^{i\phi(f)} d\mu(\phi) = e^{-\|f\|^2_{\mathscr{H}}/2}, \quad f \in \mathscr{H},$$

where $\|\cdot\|_{\mathscr{H}}$ denotes the norm of $\mathscr{H}$.

The complex Hilbert space $L^2(E, d\mu)$ is canonically isomorphic (Theorem 2.12 with $\mathscr{E}' = E$) to the boson Fock space over $\mathscr{H}$, which is called the Q-space representation of it. We denote by $d\Gamma(A)$ the second quantization of A and set

$$H_0 = d\Gamma(A).$$

Then for all $\beta > 0$, $e^{-\beta H_0} \in \mathscr{I}_1(L^2(E, d\mu))$.

Definition 3.1. *A mapping V of a Banach space X into a Banach space Y is said to be polynomially continuous if there exists a polynomial P of two real variables with positive coefficients such that*

$$\|V(\phi) - V(\psi)\| \le P(\|\phi\|, \|\psi\|)\|\phi - \psi\|, \quad \phi, \psi \in X.$$

Let V be a real valued function on E. Throughout this paper, we assume the following.

Assumption II. *The function V is bounded from below, 3-times Fréchet differentiable, and V, V', V'', V''' are polynomially continuous.*

For $\hbar > 0$, we define $V_\hbar$ by

$$V_\hbar(\phi) := V(\sqrt{\hbar}\,\phi), \quad \phi \in E.$$

and set

$$H_\hbar := H_0 \dotplus \frac{1}{\hbar} V_\hbar,$$

where $\dotplus$ denotes the quadratic form sum.

Under Assumption I, II, for all $\beta > 0$, $e^{-\beta H_\hbar} \in \mathscr{I}_1(L^2(E, d\mu))$ [4]. The trace $\mathrm{Tr} e^{-\beta H_\hbar}$ is called the partition function of $H_\hbar$. For all $s \in \mathbb{R}$, $A^{s/2}$ is a continuous linear operator from $\mathscr{H}_s(A)$ to E and it extends to a continuous linear operator from $\mathscr{H}_{-\gamma+s}(A)$ to E.

Theorem 3.2. [4]. *Let* $\beta > 0$. *Then*

$$\lim_{\hbar \to 0} \frac{\mathrm{Tr}\ e^{-\beta\hbar H_\hbar}}{\mathrm{Tr}\ e^{-\beta\hbar H_0}} = \int_E \exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}} A^{-1/2}\phi\right)\right) d\mu(\phi).$$

We set

$$\Omega = E^3, \quad \nu = \mu \otimes \mu \otimes \mu.$$

Then ν is a probability measure on Ω.

Let $\{\lambda_n\}_{n=1}^\infty$ be the eigenvalues of A, and $\{e_n\}_{n=1}^\infty$ be the complete orthonormal system (CONS) of $\mathscr{H}$ with $Ae_n = \lambda_n e_n$, and

$$\sum_{n=1}^\infty \frac{1}{\lambda_n^{\gamma-9}} < \infty \tag{3.1}$$

Let φ be a bijection from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$. For all $n, m \in \mathbb{N}$, we set $f_{n,m} = e_{\varphi(n,m)}$. Then $\{f_{n,m}\}_{n,m=1}^\infty$ is a CONS of $\mathscr{H}$. For all $\phi \in E$, we define

$$\phi_n := \phi(e_n), \quad \phi_{n,m} := \phi(f_{n,m}).$$

Then $\{\phi_n\}_n$ and $\{\phi_{n,m}\}_{n,m}$ are families of independent Gaussian random variables such that for all $n, m, n', m' \in \mathbb{N}$,

$$\int_E \phi_n d\mu(\phi) = 0, \quad \int_E \phi_n\phi_m d\mu(\phi) = \delta_{nm} \tag{3.2}$$

$$\int_E \phi_{n,m}\phi_{n',m'} d\mu(\phi) = \delta_{nn'}\delta_{mm'}. \tag{3.3}$$

For all $m_1, \cdots, m_p \in \mathbb{N}$, we have

$$\sup_{n_1,\cdots,n_p\in\mathbb{N}} \int_E |\phi_{n_1}|^{m_1} \cdots |\phi_{n_p}|^{m_p} d\mu(\phi) < \infty. \tag{3.4}$$

For all $N, M \in \mathbb{N}$, we set

$$\begin{aligned} F_{N,M}(\varepsilon,\omega,s) &= \sqrt{\frac{2}{\beta}}\sum_{n=1}^N \frac{\phi_n}{\sqrt{\lambda_n}} e_n + \sum_{n=1}^N\sum_{m=1}^M \sqrt{\frac{4\varepsilon^2\lambda_n}{\beta(\varepsilon^2\lambda_n^2 + (2\pi m)^2)}}(\psi_{n,m}\cos(2\pi m s) \\ &+ \theta_{n,m}\sin(2\pi m s))e_n, \quad \varepsilon \geq 0,\ \omega = (\phi,\psi,\theta) \in \Omega,\ 0 \leq s \leq 1. \end{aligned} \tag{3.5}$$

Then we have

$$\frac{\mathrm{Tr} e^{-\beta\hbar H_\hbar}}{\mathrm{Tr} e^{-\beta\hbar H_0}} = \lim_{N,M\to\infty} \int_\Omega \exp\left(-\beta \int_0^1 V\left(F_{N,M}(\varepsilon,\omega,s)\right) ds\right) d\nu(\omega), \tag{3.6}$$

where $\varepsilon = \beta\hbar$ (See [4], Lemma 5.2, Lemma 5.3.).

IV. A Class of Locally Convex Spaces

We denote by $\mathbb{R}_+$ the set of the nonnegative real numbers.

Definition 4.1. *A mapping f from $\mathbb{R}_+$ to a locally convex space X is said to be locally bounded if for all $\delta > 0$ and every continuous seminorm p on X,*

$$p_\delta(f) := \sup_{0 \le \varepsilon \le \delta} p(f(\varepsilon)) < \infty.$$

We denote by $(X^{\mathbb{R}_+})_{\text{l.b.}}$ the linear space of the locally bounded mappings from $\mathbb{R}_+$ to X. The topology defined by the seminorms $\{p_\delta\}_{p,\delta}$ turns $(X^{\mathbb{R}_+})_{\text{l.b.}}$ into a locally convex space. If X is a Fréchet space, $(X^{\mathbb{R}_+})_{\text{l.b.}}$ is a Fréchet space.

Let $\{E_n\}_{n\in\mathbb{N}}$ be a family of Banach spaces with the property that

$$E_{n+1} \subset E_n, \ \|\phi\|_n \le \|\phi\|_{n+1}, \quad \phi \in E_{n+1},$$

for all $n \in \mathbb{N}$, where $\|\cdot\|_n$ denotes the norm of E_n. Then, the topology defined by the norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ turns $\bigcap_{n\in\mathbb{N}} E_n$ into a Fréchet space.

Let (X, P) be a probability space and Y be a Banach space. We denote by $L^p(X, dP; Y)$ the Banach space of the Y-valued L^p-functions on (X, P). Then $\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y)$ can be provided with the structure of Fréchet space.

Definition 4.2. *Let f be a mapping from $\mathbb{R}_+$ to $\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y)$. We say that f is in $(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}$ if and only if for each $\delta > 0$, there exists a nonnegative function $g \in \bigcap_{p\in\mathbb{N}} L^p(X, dP)$ such that*

$$\sup_{0 \le \varepsilon \le \delta} \|f(\varepsilon)(x)\|_Y \le g(x),$$

P-a.e.x.

The set $(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}$ is a linear subspace of $(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{l.b.}}$. In what follows, we omit x in $f(\varepsilon)(x)$.

Lemma 4.3. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{g_\lambda\}_{\lambda\in\Lambda}$ be nets in $(\bigcap_{p\in\mathbb{N}} L^p(X, dP))^{\mathbb{R}_+}_{\text{l.b.}}$. Suppose that*

$$\overline{\lim_{\lambda}} \sup_{0 \le \varepsilon \le \delta} \int_X |f_\lambda(\varepsilon)|^p dP < \infty \text{ and } g_\lambda \longrightarrow 0$$

in $(\bigcap_{p\in\mathbb{N}} L^p(X, dP))^{\mathbb{R}_+}_{\text{l.b.}}$, for all $p \in \mathbb{N}$ and $\delta > 0$. Then $f_\lambda g_\lambda \longrightarrow 0$ in $(\bigcap_{p\in\mathbb{N}} L^p(X, dP))^{\mathbb{R}_+}_{\text{l.b.}}$.

Proof. Let $p \in \mathbb{N}$ and $\delta > 0$. For each $\varepsilon \ge 0$, by the Schwarz inequality, we have

$$\int_X |f_\lambda(\varepsilon) g_\lambda(\varepsilon)|^p dP \ \le \ \left(\int_X |f_\lambda(\varepsilon)|^{2p} dP \right)^{1/2} \left(\int_X |g_\lambda(\varepsilon)|^{2p} dP \right)^{1/2}.$$

Hence we have

$$\sup_{0\le\varepsilon\le\delta}\int_X |f_\lambda(\varepsilon)g_\lambda(\varepsilon)|^p dP \le \left(\sup_{0\le\varepsilon\le\delta}\int_X |f_\lambda(\varepsilon)|^{2p} dP\right)^{1/2}\left(\sup_{0\le\varepsilon\le\delta}\int_X |g_\lambda(\varepsilon)|^{2p} dP\right)^{1/2}.$$

Then, by the assumption on f_λ and g_λ, we have $f_\lambda g_\lambda \longrightarrow 0$. □

Let $X_1, \cdots, X_n$ and Z be non-empty sets and G be a real-valued function on $X_1 \times \cdots \times X_n$ and F_j be a mapping from Z to X_j, $j = 1, \cdots, n$. We define $G(F_1, \cdots, F_n)$, the real-valued function on Z, by

$$G(F_1, \cdots, F_n)(z) = G(F_1(z), \cdots, F_n(z)), \quad z \in Z.$$

Lemma 4.4. *Let Q be a polynomial of n real variables and $F_j \in \left(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y)\right)^{\mathbb{R}_+}_{\text{l.b.}}$, $j = 1, \cdots, n$. Then, for all $\delta > 0$,*

$$\overline{\lim_{G_1\to F_1,\cdots,G_n\to F_n}} \sup_{0\le\varepsilon\le\delta}\int_X |Q(\|G_1(\varepsilon)\|, \cdots, \|G_n(\varepsilon)\|)| dP < \infty.$$

Proof. It is sufficient to consider the case where $Q(x_1, \cdots, x_n) = x_1^{p_1}\cdots x_n^{p_n}$, $x_1, \cdots, x_n \in \mathbb{R}$, $p_1, \cdots, p_n \in \mathbb{N}$. Let $G_j \in \left(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y)\right)^{\mathbb{R}_+}_{\text{l.b.}}$, $j = 1, \cdots, n$. By the Schwarz inequality, we have

$$\int_X \|G_1(\varepsilon)\|^{p_1}\cdots\|G_n(\varepsilon)\|^{p_n} dP \le \left(\int_X \|G_1(\varepsilon)\|^{2p_1} dP\right)^{1/2}\left(\int_X \|G_2(\varepsilon)\|^{2p_n}\cdots\|G_n(\varepsilon)\|^{2p_n} dP\right)^{1/2}.$$

Then, for all $\delta > 0$,

$$\sup_{0\le\varepsilon\le\delta}\int_X \|G_1(\varepsilon)\|^{p_1}\cdots\|G_n(\varepsilon)\|^{p_n} dP$$

$$\le \left(\sup_{0\le\varepsilon\le\delta}\int_X \|G_1(\varepsilon)\|^{2p_1} dP\right)^{1/2}\left(\sup_{0\le\varepsilon\le\delta}\int_X \|G_2(\varepsilon)\|^{2p_n}\cdots\|G_n(\varepsilon)\|^{2p_n} dP\right)^{1/2}.$$

By

$$\left(\sup_{0\le\varepsilon\le\delta}\int_X \|G_1(\varepsilon)\|^{2p_1} dP\right)^{1/2} \longrightarrow \left(\sup_{0\le\varepsilon\le\delta}\int_X \|F_1(\varepsilon)\|^{2p_1} dP\right)^{1/2},$$

as $G_1 \to F_1$, we inductively have

$$\overline{\lim_{G_1\to F_1,\cdots,G_n\to F_n}} \sup_{0\le\varepsilon\le\delta}\int_X \|G_1(\varepsilon)\|^{p_1}\cdots\|G_n(\varepsilon)\|^{p_n} dP < \infty.$$

□

Proposition 4.5. *Let Q be a polynomial of n real valuables. Then the mapping $(F_1, \cdots, F_n) \longmapsto Q(\|F_1\|, \cdots, \|F_n\|)$ from $\left((\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}\right)^n$ to $\left(\bigcap_{p\in\mathbb{N}} L^p(X, dP)\right)^{\mathbb{R}_+}_{\text{u.i.}}$ is continuous.*

Proof. We first show that the mapping in the Proposition 4.5 is well defined. Let $\delta > 0$, $p_1, \cdots, p_n \in \mathbb{N}$, and $F_j \in (\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}$, $j = 1, \cdots, n$. We assume that there exists a nonnegative function $g \in \bigcap_{p\in\mathbb{N}} L^p(X, dP)$ such that

$$\sup_{0\le\varepsilon\le\delta} (\|F_2(\varepsilon)\|^{p_2} \cdots \|F_n(\varepsilon)\|^{p_n}) \le g,$$

P-a.e.. By the assumption on F_1, there exists a nonnegative function $h \in \bigcap_{p\in\mathbb{N}} L^p(X, dP)$ such that

$$\sup_{0\le\varepsilon\le\delta} \|F_1(\varepsilon)\|^{p_1} \le h,$$

P-a.e.. Then, we have

$$\begin{aligned} \sup_{0\le\varepsilon\le\delta} \|F_1(\varepsilon)\|^{p_1} \cdots \|F_n(\varepsilon)\|^{p_n} &\le \sup_{0\le\varepsilon\le\delta} \|F_1(\varepsilon)\|^{p_1} \sup_{0\le\varepsilon\le\delta} \|F_2(\varepsilon)\|^{p_2} \cdots \|F_n(\varepsilon)\|^{p_n} \\ &\le hg, \end{aligned}$$

P-a.e..
By the Schwarz inequality, we have $hg \in \bigcap_{p\in\mathbb{N}} L^p(X, dP)$. Hence, we inductively have

$$\|F_1\|^{p_1} \cdots \|F_n\|^{p_n} \in \left(\bigcap_{p\in\mathbb{N}} L^p(X, dP)\right)^{\mathbb{R}_+}_{\text{u.i.}}.$$

Let $G_j \in (\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}$, $j = 1, \cdots, n$. Then

$$| \, \|F_1\|^{p_1} \cdots \|F_n\|^{p_n} - \|G_1\|^{p_1} \cdots \|G_n\|^{p_n} |$$

$$\le \sum_{j=1}^{n} \|G_1\|^{p_1} \cdots \|G_{j-1}\|^{p_{j-1}} \, | \, \|F_j\|^{p_j} - \|G_j\|^{p_j} \, | \, \|F_{j+1}\|^{p_{j+1}} \cdots \|F_n\|^{p_n}.$$

Then, there exist polynomials $\{Q_j\}_{j=1}^n$ of $2n$ variables with positive coefficients such that

$$| \, \|F_1\|^{p_1} \cdots \|F_n\|^{p_n} - \|G_1\|^{p_1} \cdots \|G_n\|^{p_n} |$$

$$\le \sum_{j=1}^{n} Q_j(\|F_1\|, \cdots, \|F_n\|, \|G_1\|, \cdots, \|G_n\|) \, \|F_j - G_j\|.$$

Applying Lemma 4.3 and Lemma 4.4, we have

$$\sum_{j=1}^{n} Q_j(\|F_1\|, \cdots, \|F_n\|, \|G_1\|, \cdots, \|G_n\|) \, \|F_j - G_j\| \longrightarrow 0,$$

as $F_1 \to G_1, \cdots, F_n \to G_n$. Hence the mapping in the Proposition 3.5 is continuous. □

Proposition 4.6. *Let Z_j be a Banach space ($j = 1, \cdots, n$), L be a continuous multilinear form on $Z_1 \times \cdots \times Z_n$, and V_j be a polynomially continuous mapping from Y to $Z_j (j = 1, \cdots, n)$. Then the mapping $(F_1, \cdots, F_n) \longmapsto L(V_1 \circ F_1, \cdots, V_n \circ F_n)$ from $\left((\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}\right)^n$ to $\left(\bigcap_{p\in\mathbb{N}} L^p(X, dP)\right)^{\mathbb{R}_+}_{\text{u.i.}}$ is continuous.*

Proof. We first show that the mapping in the Proposition 4.6 is well defined. Let $F_j \in (\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{u.i.}$, $j = 1, \cdots, n$. Then

$$|L(V_1 \circ F_1, \cdots, V_n \circ F_n)| \le \|L\| \|V_1 \circ F_1\| \cdots \|V_n \circ F_n\|.$$

Since V_j is polynomially bounded, there exists a polynomial Q of n real variables with positive coefficients such that

$$|L(V_1 \circ F_1, \cdots, V_n \circ F_n)| \le Q(\|F_1\|, \cdots, \|F_n\|).$$

By Proposition 4.5, $Q(\|F_1\|, \cdots, \|F_n\|) \in \left(\bigcap_{p\in\mathbb{N}} L^p(X, dP)\right)^{\mathbb{R}_+}_{\text{u.i.}}$. Hence we have $L(V_1 \circ F_1, \cdots, V_n \circ F_n) \in \left(\bigcap_{p\in\mathbb{N}} L^p(X, dP)\right)^{\mathbb{R}_+}_{\text{u.i.}}$.
Let $G_j \in (\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{u.i.}$, $j = 1, \cdots, n$. Then

$$|L(V_1 \circ F_1, \cdots, V_n \circ F_n) - L(V_1 \circ G_1, \cdots, V_n \circ G_n)|$$

$$\le \|L\| \sum_{j=1}^{n} \|V_1 \circ G_1\| \cdots \|V_{j-1} \circ G_{j-1}\| \|V_j \circ F_j - V_j \circ G_j\| \|V_{j+1} \circ F_{j+1}\| \cdots \|V_n \circ F_n\|.$$

Since V_j is polynomially continuous, there exist polynomials $\{Q_j\}_{j=1}^n$ of $2n$ real variables with positive coefficients such that

$$|L(V_1 \circ F_1, \cdots, V_n \circ F_n) - L(V_1 \circ G_1, \cdots, V_n \circ G_n)|$$

$$\le \sum_{j=1}^{n} Q_j(\|F_1\|, \cdots, \|F_n\|, \|G_1\|, \cdots, \|G_n\|) \, \|F_j - G_j\|.$$

Applying Lemma 4.3 and Proposition 4.5, we have

$$\sum_{j=1}^{n} Q_j(\|F_1\|, \cdots, \|F_n\|, \|G_1\|, \cdots, \|G_n\|) \, \|F_j - G_j\| \longrightarrow 0,$$

as $F_1 \to G_1, \cdots, F_n \to G_n$. Hence the mapping in the Proposition 3.6 is continuous. □

Let P_j be a probability measure on a set X_j, $j = 1, 2$. For $F \in (\bigcap_{p\in\mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2)))^{\mathbb{R}_+}_{\text{u.i.}}$, we define a mapping $\int_{X_2} F dP_2$ from $\mathbb{R}_+$ to the set of functions on X_1 by

$$\left(\int_{X_2} F dP_2\right)(\varepsilon) = \int_{X_2} F(\varepsilon) dP_2, \quad \varepsilon \geq 0.$$

By the property

$$\int_{X_1}\int_{X_2} |F(\varepsilon)|^p dP_2 dP_1 < \infty$$

for all $\varepsilon \geq 0$ and $p \in \mathbb{N}$, we have

$$\int_{X_2} |F(\varepsilon)|^p dP_2 < \infty,$$

P_1-a.e.. Hence $\int_{X_2} F dP_2$ is well defined.

Let $\delta > 0$. Then, by the assumption on F, there exists a nonnegative function $g \in \bigcap_{p\in\mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2))$ such that

$$\sup_{0\leq\varepsilon\leq\delta} |F(\varepsilon)| \leq g, \quad P_1 \otimes P_2 - \text{a.e.}$$

Then, we have

$$\begin{aligned} \sup_{0\leq\varepsilon\leq\delta} \left|\int_{X_2} F(\varepsilon) dP_2\right| &\leq \sup_{0\leq\varepsilon\leq\delta} \int_{X_2} |F(\varepsilon)| dP_2 \\ &\leq \int_{X_2} g\, dP_2, \end{aligned}$$

P_1-a.e.. For all $p \in \mathbb{N}$, by Jensen's inequality,

$$\begin{aligned} \int_{X_1} \left|\int_{X_2} g\, dP_2\right|^p dP_1 &\leq \int_{X_1}\int_{X_2} g^p\, dP_2 dP_1 \\ &< \infty. \end{aligned}$$

Hence we have $\int_{X_2} F dP_2 \in \left(\bigcap_{p\in\mathbb{N}} L^p(X_1, dP_1)\right)^{\mathbb{R}_+}_{u.i.}$.

Proposition 4.7. *The mapping* $F \longmapsto \int_{X_2} F dP_2$ *from* $(\bigcap_{p\in\mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2)))^{\mathbb{R}_+}_{\text{u.i.}}$ *to* $\left(\bigcap_{p\in\mathbb{N}} L^p(X_1, dP_1)\right)^{\mathbb{R}_+}_{\text{u.i.}}$ *is continuous linear.*

Proof. Let $F \in (\bigcap_{p\in\mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2)))^{\mathbb{R}_+}_{\text{u.i.}}$. Then, by Jensen's inequality,

$$\left|\int_{X_2} F(\varepsilon) dP_2\right|^p \leq \int_{X_2} |F(\varepsilon)|^p\, dP_2.$$

Hence, for all $\delta > 0$, we have

$$\begin{aligned} \sup_{0\leq\varepsilon\leq\delta} \int_{X_1} \left|\int_{X_2} F(\varepsilon) dP_2\right|^p dP_1 &\leq \sup_{0\leq\varepsilon\leq\delta} \int_{X_1}\int_{X_2} |F(\varepsilon)|^p dP_2 dP_1 \\ &\longrightarrow 0 \end{aligned}$$

as $F \to 0$ in $(\bigcap_{p\in\mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2)))^{\mathbb{R}_+}_{\text{l.b.}}$. Hence the mapping is continuous. □

V. An Asymptotic Formula

We set

$$Z(\varepsilon) = \lim_{N,M\to\infty} \int_\Omega \exp\left(-\beta \int_0^1 F_{N,M}(\varepsilon,\omega,s)ds\right) d\nu(\omega), \quad \varepsilon \geq 0, \tag{5.1}$$

(See (3.5) and (3.6)). In this section, we examine the differentiability of Z. For all $n, m \in \mathbb{N}$, we set

$$\alpha_{n,m}(\varepsilon) = \sqrt{\frac{4\varepsilon^2\lambda_n}{\beta(\varepsilon^2\lambda_n^2 \ + \ (2\pi m)^2)}}, \quad \varepsilon \geq 0.$$

Then, for all $\delta > 0$, there exists a constant $C > 0$ such that

$$|\alpha_{n,m}(\varepsilon)| \leq \frac{C\sqrt{\lambda_n}}{m}, \quad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{5.2}$$

$$|\alpha'_{n,m}(\varepsilon)| \leq \frac{C\sqrt{\lambda_n}}{m}, \quad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{5.3}$$

$$|\alpha''_{n,m}(\varepsilon)| \leq \frac{C\lambda_n^{5/2}}{m}, \quad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{5.4}$$

$$|\alpha'''_{n,m}(\varepsilon)| \leq \frac{C(\lambda_n^{5/2} \ + \ \lambda_n^{9/2})}{m}, \quad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{5.5}$$

For $n, m \in \mathbb{N}$, we set

$$\beta_{n,m}(\omega, s) = \psi_{n,m}\cos(2\pi ms) \ + \ \theta_{n,m}\sin(2\pi ms), \quad \omega \in \Omega,\ s \in \mathbb{R}.$$

We denote by $\mu^{(L)}_{[\,0,1\,]}$ the Lebesgue measure on $[\,0,1\,]$.

Lemma 5.1. *$\{F_{N,M}\}_{N,M\in\mathbb{N}}$, $\{F'_{N,M}\}_{N,M\in\mathbb{N}}$, $\{F''_{N,M}\}_{N,M\in\mathbb{N}}$, $\{F'''_{N,M}\}_{N,M\in\mathbb{N}}$ are Cauchy nets in* $(\bigcap_{p\in\mathbb{N}} L^p(\Omega \times [\,0,1\,], d(\nu \otimes \mu^{(L)}_{[\,0,1\,]}); E))^{\mathbb{R}_+}_{\text{u.i.}}$.

Proof. By (5.2), (5.3), (5.4), (5.5), $F_{N,M}, F'_{N,M}, F''_{N,M}, F'''_{N,M} \in (\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,], d(\nu\otimes \mu^{(L)}_{[\,0,1\,]}); E))^{\mathbb{R}_+}_{\text{u.i.}}$.

The set $\{\lambda_n^{\gamma/2} e_n\}_{n=1}^\infty$ is a CONS of E. Then, for $N, N' \in \mathbb{N}$ with $N > N'$,

$$\left\|\sum_{n=N'+1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n\right\|_E^2 = \sum_{n=N'+1}^{N} \frac{\phi_n^2}{\lambda_n^{\gamma+1}}.$$

Then, for all $p \in \mathbb{N}$,

$$\begin{aligned}\left\|\sum_{n=N'+1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n\right\|_E^{2p} &= \left(\sum_{n=N'+1}^{N} \frac{\phi_n^2}{\lambda_n^{\gamma+1}}\right)^p \\ &= \sum_{n_1,\cdots,n_p=N'+1}^{N} \frac{1}{\lambda_{n_1}^{\gamma+1}} \cdots \frac{1}{\lambda_{n_p}^{\gamma+1}} \phi_{n_1}^2 \cdots \phi_{n_p}^2.\end{aligned}$$

By (3.4) and the fact that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma+1}} < \infty,$$

we have

$$\int_{\Omega} \left\| \sum_{n=N'+1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n \right\|_E^{2p} d\nu(\phi) \longrightarrow 0,$$

as $N, N' \to 0$.

Let Λ_1, Λ_2 be finite subsets of $\mathbb{N}$. Then,

$$\left\| \sum_{n\in\Lambda_1} \left(\sum_{m\in\Lambda_2} \alpha_{n,m}(\varepsilon)\beta_{n,m} \right) e_n \right\|_E^2 = \sum_{n\in\Lambda_1} \frac{1}{\lambda_n^{\gamma}} \left(\sum_{m\in\Lambda_2} \alpha_{n,m}(\varepsilon)\beta_{n,m} \right)^2.$$

For all $p \in \mathbb{N}$,

$$\left\| \sum_{n\in\Lambda_1} \left(\sum_{m\in\Lambda_2} \alpha_{n,m}(\varepsilon)\beta_{n,m} \right) e_n \right\|_E^{2p}$$

$$= \sum_{n_1,\cdots,n_p\in\Lambda_1} \sum_{m_1,\cdots,m_p\in\Lambda_2} \sum_{l_1,\cdots,l_p\in\Lambda_2} \frac{1}{\lambda_{n_1}^{\gamma}} \cdots \frac{1}{\lambda_{n_p}^{\gamma}} \alpha_{n_1,m_1}(\varepsilon)\alpha_{n_1,l_1}(\varepsilon) \cdots \alpha_{n_p,m_p}(\varepsilon)\alpha_{n_p,l_p}(\varepsilon)$$
$$\times \beta_{n_1,m_1}\beta_{n_1,l_1} \cdots \beta_{n_p,m_p}\beta_{n_p,l_p}.$$

By (3.3), (5.2), and the fact that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} < \infty, \ \ \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-1}} < \infty,$$

we have

$$\|F_{N,M} - F_{N',M'}\|_{(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,],d(\nu\otimes\mu^{(L)}_{[\,0,1\,]});E))^{\mathbb{R}_+}_{\text{l.b.}}} \longrightarrow 0,$$

as $N, N', M, M' \to \infty$.

Similarly, by

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-1}} < \infty,$$

and (5.3), we have

$$\left\|F'_{N,M} - F'_{N',M'}\right\|_{(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,],d(\nu\otimes\mu^{(L)}_{[\,0,1\,]});E))^{\mathbb{R}_+}_{\text{l.b.}}} \longrightarrow 0,$$

as $N, N', M, M' \to \infty$. By

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-5}} < \infty,$$

and (5.4), we have

$$\left\|F''_{N,M} - F''_{N',M'}\right\|_{(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,],d(\nu\otimes\mu^{(L)}_{[\,0,1\,]});E))^{\mathbb{R}_+}_{\text{l.b.}}} \longrightarrow 0,$$

as $N, N', M, M' \to \infty$. By
$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-9}} < \infty,$$
and (5.5), we have
$$\left\| F'''_{N,M} - F'''_{N',M'} \right\|_{(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,],d(\nu\otimes\mu^{(L)}_{[\,0,1\,]});E))^{\mathbb{R}_+}_{\text{l.b.}}} \longrightarrow 0,$$
as $N, N', M, M' \to \infty$. $\square$

Lemma 5.2. *The mapping $F \longmapsto \exp\left(-\beta \int_0^1 V \circ F ds\right)$ from $(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,], d(\nu\otimes \mu^{(L)}_{[\,0,1\,]}); E))^{\mathbb{R}_+}_{\text{u.i.}}$ to $(\bigcap_{p\in\mathbb{N}} L^p(\Omega, d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$ is continuous.*

Proof. Let $F, G \in (\bigcap_{p\in\mathbb{N}} L^p(\Omega \times [\,0,1\,], d(\nu \otimes \mu^{(L)}_{[\,0,1\,]}; E))^{\mathbb{R}_+}_{\text{l.b.}}$. Since V is bounded from below, by the inequality
$$|e^x - e^y| \le (e^x + e^y)|x - y|, \quad x, y \in \mathbb{R},$$
there exists a constant $C \ge 0$ such that
$$\left|\exp\left(-\beta \int_0^1 V\circ F ds\right) - \exp\left(-\beta \int_0^1 V\circ G ds\right)\right| \le C \left|\int_0^1 V\circ F ds - \int_0^1 V \circ G ds\right|.$$
Hence, by Proposition 4.7,
$$\left|\exp\left(-\beta \int_0^1 V\circ F ds\right) - \exp\left(-\beta \int_0^1 V\circ G ds\right)\right| \longrightarrow 0,$$
as $F \to G$. $\square$

For all $N, M \in \mathbb{N}$, we set
$$G_{N,M}(\varepsilon, \omega) = \exp\left(-\beta \int_0^1 V\left(F_{N,M}(\varepsilon, \omega, s)\right) ds\right) \quad \varepsilon \ge 0,\ \omega \in \Omega.$$

Lemma 5.3. *$\{G_{N,M}\}_{N,M\in\mathbb{N}}, \{G'_{N,M}\}_{N,M\in\mathbb{N}}, \{G''_{N,M}\}_{N,M\in\mathbb{N}}, \{G'''_{N,M}\}_{N,M\in\mathbb{N}}$ are Cauchy nets in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega, d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$.*

Proof. By Lemma 5.1, Lemma 5.2 and the completeness of $((\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,], d(\nu\otimes \mu^{(L)}_{[\,0,1\,]}; E))^{\mathbb{R}_+}_{\text{l.b.}}$, $\{G_{N,M}\}_{N,M\in\mathbb{N}}$ is a Cauchy net in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega, d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$.
For all $\varepsilon \ge 0$,
$$G'_{N,M}(\varepsilon, \omega) = -\beta G_{N,M}(\varepsilon, \omega) \int_0^1 V'(F_{N,M}(\varepsilon, \omega, s))(F'_{N,M}(\varepsilon, \omega, s)) ds.$$
In general, for each n-times continuously Fréchet differentiable mapping F from a Banach space X to a Banach space Y, $\phi \in X$, and $n \in \mathbb{N}$, $F^{(n)}(\phi)$ is identified with an element in $\mathscr{L}^{(n)}(X^n, Y)$(the Banach space of continuous multilinear

mapping from X^n to Y). The mapping $(L, x_1, \cdots, x_n) \longmapsto L(x_1, \cdots, x_n)$ from $\mathscr{L}^{(n)}(X^n, Y) \times X^n$ to Y is continuous multilinear. Then, by Proposition 4.6, 4.7, and Lemma 5.1, $\{G'_{N,M}\}_{N,M\in\mathbb{N}}$ is a Cauchy net in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega, d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$.
Similarly and inductively, $\{G''_{N,M}\}_{N,M\in\mathbb{N}}$ and $\{G'''_{N,M}\}_{N,M\in\mathbb{N}}$ are Cauchy nets in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega, d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$. □

Now we have the following theorem.

THEOREM 5.4. *The function Z defined by* (5.1) is 3-times continuously differentiable with the following properties :

$$Z(0) = \int_E \exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}}A^{-1/2}\phi\right)\right) d\mu(\phi) \tag{5.6}$$

$$Z'(0) = 0 \tag{5.7}$$

$$Z''(0) = \sum_{m=1}^{\infty} \int_{E^2} d\mu(\phi)d\mu(\psi)\,(-\beta)\exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}}A^{-1/2}\phi\right)\right) \times V''\left(\sqrt{\frac{2}{\beta}}A^{-1/2}\phi\right)\left(A^{1/2}\left(\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{\infty}\psi_{n,m}e_n\right), A^{1/2}\left(\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{\infty}\psi_{n,m}e_n\right)\right) \tag{5.8}$$

Proof. By Lemma 5.1, Lemma 5.3, and the fact that $\alpha_{n,m}$ is infinitely differentiable for all $n, m \in \mathbb{N}$, $\int_\Omega H_{N,M}(\varepsilon, \omega)\, d\nu(\omega)$ with $H_{N,M} = G_{N,M}, G'_{N,M}, G''_{N,M}, G'''_{N,M}$ uniformly converges in ε. Hence one can interchange the limit $\lim_{N,M\to\infty}$ with differentiations in ε. Hence Z is 3-times continuously differentiable in $\mathbb{R}_+$.
By Theorem 3.2, we obtain (5.6).
For $\varepsilon \geq 0$

$$Z'(\varepsilon) = \lim_{N,M\to\infty} \int_\Omega G'_{N,M}(\varepsilon, \omega)d\nu(\omega)\,.$$

In particular

$$Z'(0) = \lim_{N,M\to\infty} \int_\Omega G'_{N,M}(0, \omega)d\nu(\omega),$$

and

$$\begin{aligned} G'_{N,M}(0,\omega) &= -\beta\, G_{N,M}(0,\omega)\int_0^1 V'\left(\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}\right)\left(\sum_{n=1}^{N}\alpha'_{n,m}(0)\beta_{n,m}(\omega,s)e_n\right) ds \\ &= -\beta\, G_{N,M}(0,\omega)V'\left(\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}\right)\left(\int_0^1\sum_{n=1}^{N}\alpha'_{n,m}(0)\beta_{n,m}(\omega,s)e_n ds\right). \end{aligned}$$

By the fact that

$$\int_0^1 \cos(2\pi ms)ds = \int_0^1 \sin(2\pi ms)ds = 0, \qquad m \in \mathbb{N},$$

we obtain (5.7). For $\varepsilon \geq 0$

$$\begin{aligned} G''_{N,M}(\varepsilon,\omega) &= -\beta\, G'_{N,M}(\varepsilon,\omega)\int_0^1 V'(F_{N,M}(\varepsilon,\omega,s))(F'_{N,M}(\varepsilon,\omega,s))ds \\ &\quad -\beta\, G_{N,M}(\varepsilon,\omega)\int_0^1 V''(F_{N,M}(\varepsilon,\omega,s))(F'_{N,M}(\varepsilon,\omega,s),F'_{N,M}(\varepsilon,\omega,s)) \\ &\quad +V'(F_{N,M}(\varepsilon,\omega,s))(F''_{N,M}(\varepsilon,\omega,s))ds. \end{aligned}$$

In particular,

$$\begin{aligned} & Z''(0) \\ = & \lim_{N,M\to\infty}\sum_{m=1}^{M}\int_{E^2} d\mu(\phi)d\mu(\psi)\,(-\beta)\exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\right)\right) \\ \times & \; V''\left(\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\right)\left(\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{N}\sqrt{\lambda_n}\psi_{n,m}e_n,\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{N}\sqrt{\lambda_n}\psi_{n,m}e_n\right), \end{aligned}$$

where we have used the fact that

$$\int_0^1 \cos(2\pi ms)\sin(2\pi ns)ds = 0,$$

$$\int_0^1 \cos(2\pi ms)\cos(2\pi ns)ds = \int_0^1 \sin(2\pi ms)\sin(2\pi ns)ds = \frac{\delta_{mn}}{2}, \quad n,m\in\mathbb{N}.$$

By (3.2), we have

$$\begin{aligned} \int_E \sum_{n=1}^{\infty}\frac{\phi_n^2}{\lambda_n^{\gamma+1}}d\mu(\phi) &= \sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\gamma+1}}\int_E \phi_n^2 d\mu(\phi) \\ &= \sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\gamma+1}} \\ &< \infty. \end{aligned}$$

Hence we have

$$A^{-1/2}\phi = \sum_{n=1}^{\infty}\frac{\phi_n}{\sqrt{\lambda_n}}e_n \in E, \quad \mu-a.e.\ \phi\in E.$$

Then, for all $p,N\in\mathbb{N}$,

$$\begin{aligned} \left\|\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\right\|_E^{2p} &= \left(\frac{2}{\beta}\sum_{n=1}^{N}\frac{\phi_n^2}{\lambda_n^{\gamma+1}}\right)^p \\ &\leq \left(\frac{2}{\beta}\right)^p\left(\sum_{n=1}^{\infty}\frac{\phi_n^2}{\lambda_n^{\gamma+1}}\right)^p \\ &= \left(\frac{2}{\beta}\right)^p\sum_{n_1,\cdots,n_p=1}^{\infty}\frac{1}{\lambda_{n_1}^{\gamma+1}}\cdots\frac{1}{\lambda_{n_p}^{\gamma+1}}\phi_{n_1}^2\cdots\phi_{n_p}^2, \end{aligned}$$

$\mu-$a.e. $\phi \in E$. Then, by (3.4), we have

$$\int_E \sup_{N\in\mathbb{N}} \left\| V''\left(\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\right)\right\|^2_{\mathscr{L}^{(2)}(E\times E,\mathbb{R})} d\mu(\phi) < \infty.$$

For all $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{\infty}\int_E \frac{\psi_{n,m}^2}{\lambda_n^{\gamma-1}}d\mu(\psi) &= \sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\gamma-1}}\int_E \psi_{n,m}^2 d\mu(\psi)\\ &= \sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\gamma-1}}\\ &< \infty.\end{aligned}$$

Then we have

$$A^{1/2}\left(\sum_{n=1}^{\infty}\psi_{n,m}e_n\right) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}\psi_{n,m}e_n \in E,$$

$\mu-$a.e. $\psi \in E$, $m \in \mathbb{N}$.
For all $N, m \in \mathbb{N}$,

$$\begin{aligned}\left\|\sum_{n=1}^{N}\sqrt{\lambda_n}\psi_{n,m}e_n\right\|^2 &= \sum_{n=1}^{N}\frac{\psi_{n,m}^2}{\lambda_n^{\gamma-1}}\\ &\leq \sum_{n=1}^{\infty}\frac{\psi_{n,m}^2}{\lambda_n^{\gamma-1}},\end{aligned}$$

$\mu-$a.e. $\psi \in E$, $m \in \mathbb{N}$.
Then, for all $N, m \in \mathbb{N}$,

$$\begin{aligned}\left\|\sum_{n=1}^{N}\sqrt{\lambda_n}\psi_{n,m}e_n\right\|^4 &\leq \left(\sum_{n=1}^{\infty}\frac{\psi_{n,m}^2}{\lambda_n^{\gamma-1}}\right)^2\\ &= \sum_{n_1,n_2=1}^{\infty}\frac{1}{\lambda_{n_1}^{\gamma-1}}\frac{1}{\lambda_{n_2}^{\gamma-1}}\psi_{n_1,m}^2\psi_{n_2,m}^2,\end{aligned}$$

$\mu-$a.e. $\psi \in E$, $m \in \mathbb{N}$.
Then, by (3.4), we have

$$\int_E \sup_{N\in\mathbb{N}}\left\|\sum_{n=1}^{N}\sqrt{\lambda_n}\psi_{n,m}e_n\right\|^4 d\mu(\psi) < \infty.$$

Since V is bounded from below, there exists a constant $C \geq 0$ such that

$$\begin{aligned} &\left| -\beta \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \right) \right. \\ &\left. \times V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \left(\sum_{n=1}^{N} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} \sqrt{\lambda_n} e_n, \sum_{n=1}^{N} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} \sqrt{\lambda_n} e_n \right) \right|^2 \\ &\leq C \sup_{N\in\mathbb{N}} \left\| V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \right\|^2 \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \sqrt{\lambda_n} \psi_{n,m} e_n \right\|^4 . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_{E^2} \sup_{N\in\mathbb{N}} \left\| V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \right\|^2 \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \sqrt{\lambda_n} \psi_{n,m} e_n \right\|^4 d\mu(\phi) d\mu(\psi) \\ = &\int_{E} \sup_{N\in\mathbb{N}} \left\| V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \right\|^2 d\mu(\phi) \int_{E} \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \sqrt{\lambda_n} \psi_{n,m} e_n \right\|^4 d\mu(\psi) < \infty. \end{aligned}$$

Hence, by the dominated convergence theorem, we have for all $M \in \mathbb{N}$,

$$\begin{aligned} &\sum_{m=1}^{M} \int_{E^2} d\mu(\phi) d\mu(\psi) \, (-\beta) \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n \right) \right) \\ \times \quad &V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n \right) \left(\frac{1}{\sqrt{\beta}\pi m} \sum_{n=1}^{N} \sqrt{\lambda_n} \psi_{n,m} e_n, \frac{1}{\sqrt{\beta}\pi m} \sum_{n=1}^{N} \sqrt{\lambda_n} \psi_{n,m} e_n \right) \\ \longrightarrow \quad &\sum_{m=1}^{M} \int_{E^2} d\mu(\phi) d\mu(\psi) \, (-\beta) \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) \\ \times \quad &V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \left(A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} e_n \right), A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} e_n \right) \right), \end{aligned}$$

as $N \to \infty$. There exists a constant $C \geq 0$ such that

$$\begin{aligned} &\left| \int_{E^2} d\mu(\phi) d\mu(\psi) \, (-\beta) \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) \right. \\ &\left. \times V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \left(A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} e_n \right), A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} e_n \right) \right) \right| \\ \leq \quad &C \int_{E^2} \left\| V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right\| \left\| A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} e_n \right) \right\|^2 d\mu(\phi) d\mu(\psi) \\ \leq \quad &\frac{C}{m^2} \left(\int_{E} \left\| V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right\| d\mu(\phi) \right) \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-1}} \int_{E} \psi_{n,m}^2 d\mu(\psi) \right) \\ = \quad &\frac{C}{m^2} \left(\int_{E} \left\| V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right\| d\mu(\phi) \right) \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-1}} \right), \end{aligned}$$

where we have used (3.2). By the fact that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} < \infty,$$

we obtain (5.8). □

Thus, we have an asymptotic formula for the partition function $\mathrm{Tr} e^{-\beta\hbar H_\hbar}$ as follows.

THEOREM 5.5. *For all $\beta > 0$,*

$$\begin{aligned}
&\frac{\mathrm{Tr} e^{-\beta\hbar H_\hbar}}{\mathrm{Tr} e^{-\beta\hbar H_0}} \\
= &\int_E \exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}} A^{-1/2}\phi\right)\right) d\mu(\phi) \\
&-\frac{\beta^3\hbar^2}{2} \sum_{m=1}^{\infty} \int_{E^2} d\mu(\phi) d\mu(\psi) \exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}} A^{-1/2}\phi\right)\right) \\
\times\ & V''\left(\sqrt{\frac{2}{\beta}} A^{-1/2}\phi\right)\left(A^{1/2}\left(\frac{1}{\sqrt{\beta}\pi m} \sum_{n=1}^{\infty} \psi_{n,m} e_n\right), A^{1/2}\left(\frac{1}{\sqrt{\beta}\pi m} \sum_{n=1}^{\infty} \psi_{n,m} e_n\right)\right) \\
&+o(\hbar^2)
\end{aligned}$$

as $\hbar \to 0$.

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