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The Lightcone Dualities for Submanifolds in the Sphere

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1 Introduction

In this thesis we consider submanifolds in the unit \( n \)-sphere from the view point of Legendrian dualities between pseudo-spheres in Minkowski space-time. In [5, 8] Izumiya introduced the mandala of Legendrian dualities between pseudo-spheres in Minkowski space-time. There are three kinds of pseudo-spheres in Minkowski space-time (i.e., the hyperbolic space, the de Sitter space and the lightcone). Especially, if we investigate spacelike submanifolds in the lightcone, this framework is essentially useful (see, also [13]). For the de Sitter space and the lightcone, there exist naturally embedded unit spheres. If we have a submanifold in the unit sphere, then we have the corresponding submanifolds in the embedded unit spheres in the lightcone or de Sitter space. Since these canoncial embeddings are isometries, the geometric structures of those two submanifolds are the same from the view point of the spherical geometry. We also have the dual hypersurfaces in the lightcone as an application of the duality theorem in [8]. We have two dual hypersurfaces depending on the embeddings of the sphere in the de Sitter sphere or the lightcone. On the lightcone, there is a projection onto the canonically embedded unit sphere (cf., §§2.1). We investigate the singular points of the dual hypersurfaces and the projection images of the singular value sets onto the unit sphere in the lightcone. Of course, the singular points of these two dual hypersurfaces are different in general. However, the situation depends on the codimension of the submanifold for the projections of the singular values. One of the consequences is that the projection of the singular values of these dual hypersurfaces are equal to the spherical focal set (or, the spherical evolute) for submanifolds of codimension one (cf., Theorem 3.1.7 and Theorem 5.2.5). However, these are different for submanifolds of higher codimensions (cf., Proposition 4.2.2).

In §3, we study the lightcone dualities for curves in the unit 2-sphere. The dual curve of a curve \( \gamma \) in the unit sphere is defined to be the front \( \gamma^\vee \) equidistant by \( \pi/2 \) from the original one. The dual curve \( \gamma^\vee \) can be considered as the Gauss map of the original curve \( \gamma \). Moreover the pair \((\gamma, \gamma^\vee)\) is a Legendrian immersion to the contact manifold \((\Delta, K)\) in the product of the spheres which gives the well-known spherical Legendrian duality. This means that the dual
curve $\gamma^\vee$ can be interpreted as the wavefront set of the Legendrian immersion

$$L_\gamma = (\gamma, \gamma^\vee) : I \rightarrow \Delta = \{(v, w) \in S^2 \times S^2 \mid v \cdot w = 0\}$$

to the contact manifold $(\Delta, K)$, where the contact structure $K$ is defined by the 1-form $\theta = (v_1 dw_1 + v_2 dw_2 + v_3 dw_3)|\Delta$. The duals in several different ambient spaces have been well-studied so far ([18, 4, 19, 21, 8, 16, 1], etc.).

On the other hand, the spherical evolute of $\gamma$ is naturally obtained as the envelope of the family of normal geodesics to $\gamma$. The evolute in Euclidean plane is naturally interpreted as a caustic [7]. The spherical evolute is also a caustic [17]. Moreover, it is the dual curve of the unit tangent vector of the original curve [18]. However, there might be no interpretation on the evolute from the view point of the dual of the original curve. In §3 we show that the spherical evolute of a given curve can be interpreted as the critical value sets of these dual surfaces (cf., Theorem 3.1.7). We also investigate the singularities of these curves and surfaces (cf., Theorem 3.3.4). As a consequence, we obtain an interesting correspondence among these singularities (cf., Remark 3.3.5).

In §4, we study the lightcone dualities for curves in the unit 3-sphere. In §3, the evolutes of curves in the unit 2-sphere has been investigated from the view point of the Legendrian duality [5, 8, 10]. It is known that the evolute of a curve in the unit 2-sphere is the dual of the tangent indicatrix of the original curve [18]. For a curve in the unit 3-sphere, however, the dual is a surface. Therefore, the dual of the tangent indicatrix of a curve is a surface which is called the focal surface (or, the focal set) of the original curve. The critical locus of the focal surface is the evolute of the original curve (cf., [18]). We remark that the focal set of a curve in the unit 2-sphere is a curve which is equal to the evolute.

For the de Sitter 4-space and the lightcone in Lorentz-Minkowski 5-space, there exist naturally embedded unit 3-spheres. The de Sitter 4-space corresponds to the cosmic model, and the lightcone also has its clear background in Physics [6]. In this section we investigate the curves in the unit 3-sphere in the framework of the theory of Legendrian dualities between pseudospheres in Lorentz-Minkowski 5-space([1, 4, 7, 8, 16, 18, 19, 21], etc.). If we have a regular
curve in the unit 3-sphere, then we have the regular curve in the embedded unit 3-sphere in
the lightcone or de Sitter space. Therefore, we naturally have the dual hypersurfaces in the
lightcone as an application of the duality theorem in [8]. There are two kinds of lightcone dual
hypersurfaces of a curve in the unit 3-sphere. We will give the classifications of the singularities
of these hypersurfaces. In physics, the singularities of the lightcone are also studied [20]. The
critical value sets of these two hypersurfaces are called the lightcone focal surfaces respectively.
The projections of these focal surfaces to unit 3-sphere are different surfaces. In [10] we have
shown that the projection images of the critical value sets of lightcone dual surfaces for a curve
in the unit 2-sphere coincide with the evolute of the original curve. Therefore, the situation of
curves in the unit 3-sphere is quite different from that of curves in the unit 2-sphere. However,
the projections of the critical sets of lightcone focal surfaces are equal to the evolute of the
curve. In order to clarify such situation, we introduce the notion of discriminant set of higher
order for unfoldings of functions of one-variable (see, §§4.4).

In §5, we study the lightcone dualities for hypersurfaces in the unit n-sphere. For de Sitter
space and the lightcone in Minkowski $(n + 2)$-space, there exist naturally embedded unit n-
spheres. Moreover, we have the canonical projection from the lightcone to the unit sphere
embedded in the lightcone. In this section, we investigate hypersurfaces in the unit n-sphere
in the framework of the theory of Legendrian dualities between pseudo-spheres in Minkowski
$(n + 2)$-space ([19, 21, 7, 8], etc.). If we have a hypersurface in the unit n-sphere, then we have
spacelike hypersurfaces in the embedded unit n-sphere in the lightcone and de Sitter space.
Therefore, we naturally have the dual hypersurfaces in the lightcone as an application of the
duality theorem in [8]. There are two kinds of lightcone dual hypersurfaces of a hypersurface
in the unit n-sphere. One is the dual of the hypersurface of the unit n-sphere embedded in de
Sitter space and another is the dual of the hypersurface of the unit n-sphere embedded in the
lightcone. By definition, these dual hypersurfaces are different.

On the other hand, we have studied the curves in the unit 2-sphere and the unit 3-sphere
from the view point of the Legendrian duality in §3 and §4 [10, 11]. In the unit 2-sphere, it
is known that the evolute of a curve in the unit 2-sphere is the dual of the tangent indicatrix
of the original curve [18]. We have shown that the projection images of the critical value sets of lightcone dual surfaces for a curve in the unit 2-sphere coincide with the evolute of the original curve in [10]. However, this fact doesn’t hold for a curve in unit 3-sphere (cf., [11]). For the curve case, these facts have been shown by the direct calculations in [10, 11]. We have not known the geometric reason why the situations are different. In order to clarify these situation, we investigate hypersurfaces in the unit $n$-sphere from the view point of the theory of Legendrian singularities. The curves in the unit 2-sphere can be considered as a special case of this paper. We can also show that the projection images of the critical value sets of two different lightcone dual hypersurfaces for a hypersurface in the unit $n$-sphere also coincide with the spherical evolute (cf., [17]) of the original hypersurface. We interpret geometric meanings of the singularities of those two lightcone dual hypersurfaces. Here, we remark that we do not have the notion of tangent indicatrices for higher dimensional submanifolds in the sphere. Therefore, the situation is completely different from the curve case.
2 Preliminary knowledge

2.1 The basic concepts

In this section we introduce the basic concepts in this thesis. Let $\mathbb{R}^{n+2}$ be an $(n+2)$-dimensional vector space. For any two vectors $x = (x_0, x_1, \ldots, x_{n+1})$, $y = (y_0, y_1, \ldots, y_{n+1})$ in $\mathbb{R}^{n+2}$, their pseudo scalar product is defined by $\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \ldots + x_{n+1} y_{n+1}$. Here, $(\mathbb{R}^{n+2}, \langle \cdot, \cdot \rangle)$ is called Lorentz-Minkowski $(n+2)$-space (simply, Minkowski $(n+2)$-space), which is denoted by $\mathbb{R}^{n+2}_1$. For any $(n+1)$ vectors $x_1, x_2, \ldots, x_{n+1} \in \mathbb{R}^{n+2}_1$, their pseudo vector product is defined by

$$x_1 \wedge x_2 \wedge \ldots \wedge x_{n+1} = \begin{vmatrix} -e_0 & e_1 & \cdots & e_{n+1} \\ x_1^0 & x_1^1 & \cdots & x_1^{n+1} \\ x_2^0 & x_2^1 & \cdots & x_2^{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n+1}^0 & x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{vmatrix},$$

where $\{e_0, e_1, \cdots, e_{n+1}\}$ is the canonical basis of $\mathbb{R}^{n+2}_1$ and $x_i = (x_i^0, x_i^1, \cdots, x_i^{n+1})$. A non-zero vector $x \in \mathbb{R}^{n+2}_1$ is called spacelike, lightlike or timelike if $\langle x, x \rangle > 0$, $\langle x, x \rangle = 0$ or $\langle x, x \rangle < 0$ respectively. The norm of $x \in \mathbb{R}^{n+2}_1$ is defined by $\| x \| = \sqrt{|\langle x, x \rangle|}$.

Let $\gamma : I \to \mathbb{R}^{n+2}_1$ be a regular curve in $\mathbb{R}^{n+2}_1$ (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), where $I$ is an open interval. For any $t \in I$, the curve $\gamma$ is called spacelike, lightlike or timelike if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ or $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$ respectively. We call $\gamma$ a nonlightlike curve if $\gamma$ is a spacelike or timelike curve. The arc-length of a nonlightlike curve $\gamma$ measured from $\gamma(t_0)(t_0 \in I)$ is $s(t) = \int_{t_0}^{t} \| \dot{\gamma}(t) \| \, dt$.

The parameter $s$ is determined such that $\| \gamma'(s) \| = 1$ for the nonlightlike curve, where $\gamma'(s) = \frac{d\gamma}{ds}$ is the unit tangent vector of $\gamma$ at $s$.

We define the de Sitter $(n+1)$-space by

$$S^{n+1}_1 = \{ x \in \mathbb{R}^{n+2}_1 | \langle x, x \rangle = 1 \}.$$
We define the **closed lightcone with the vertex** $\mathbf{a}$ by

$$LC_a = \{ \mathbf{x} \in \mathbb{R}_1^{n+2} | \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0 \}.$$ 

We define the **open lightcone at the origin** by

$$LC^* = \{ \mathbf{x} \in \mathbb{R}_1^{n+2} \setminus \{0\} | \langle \mathbf{x}, \mathbf{x} \rangle = 0 \}.$$ 

We consider a submanifold in the lightcone defined by

$$S_n^+ = \{ \mathbf{x} \in LC^* | x_0 = 1 \},$$

which is called the **lightcone unit sphere**. We have a projection $\pi : LC^* \rightarrow S_n^+$ defined by

$$\pi(\mathbf{x}) = \tilde{\mathbf{x}} = \left( 1, \frac{x_1}{x_0}, \ldots, \frac{x_{n+1}}{x_0} \right),$$

where $\mathbf{x} = (x_0, x_1, \ldots, x_{n+1})$. We also define the **$n$-dimensional Euclidean unit sphere** in $\mathbb{R}_0^{n+1}$ by

$$S_n^0 = \{ \mathbf{x} \in S_1^{n+1} | x_0 = 0 \},$$

where $\mathbb{R}_0^{n+1} = \{ \mathbf{x} \in \mathbb{R}_1^{n+2} | x_0 = 0 \}$.

### 2.2 The Legendrian duality theorem

We now review some properties of contact manifolds and Legendrian submanifolds. Let $N$ be a $(2n+1)$-dimensional smooth manifold and $K$ be a tangent hyperplane field on $N$. Locally, such a field is defined as the field of zeros of a 1-form $\alpha$. The tangent hyperplane field on $K$ is non-degenerate if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of $N$. We say that $(N, K)$ is a contact manifold if $K$ is a non-degenerate hyperplane field. In this case, $K$ is called a contact structure and $\alpha$ is a contact form. Let $\phi : N \rightarrow N'$ be a diffeomorphism between contact manifolds $(N, K)$ and $(N', K')$. We say that $\phi$ is a contact diffeomorphism if $d\phi(K) = K'$. Two contact manifolds $(N, K)$ and $(N', K')$ are contact diffeomorphic if there exists a contact diffeomorphism $\phi : N \rightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold $(N, K)$ is said to be Legendrian if $\dim L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \rightarrow M$ is
called a Legendrian fibration if its total space $E$ is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \to M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \to M$ is called a Legendrian map. The image of the Legendrian map $\pi \circ i$ is called a wavefront set of $i$ which is denoted by $W(i)$. For any $p \in E$, it is known that there is a local coordinate system $(x_1, \ldots, x_m, p_1, \ldots, p_m, z)$ around $p$ such that $\pi(x_1, \ldots, x_m, p_1, \ldots, p_m, z) = (x_1, \ldots, x_m, z)$ and the contact structure is given by the 1-form 

$$\alpha = dz - \sum_{i=1}^{m} p_i dx_i$$

(cf. [1], 20.3). One of the examples of Legendrian fibrations is given by the unit spherical tangent bundle of a Riemannian manifold. Let $M$ be a Riemannian manifold and $TM$ is its tangent bundle. Let $(v_1, \ldots, v_n)$ be coordinates on the fiber over $U$. Let $g_{ij}$ be the components of the metric $\langle \cdot , \cdot \rangle$ with respect to the above coordinates. Then the canonical one-form can be locally denoted by 

$$\theta = \sum_{i,j} g_{ij} v_j dq_i$$

where $q_i = x_i \circ \pi$ for the projection $\pi : TM \to M$. Let $\tilde{\pi} : S(TM) \to M$ be the unit spherical tangent bundle with respect to the metric $\langle \cdot , \cdot \rangle$. Then the restriction of $\theta$ onto $S(TM)$ gives a contact structure and $\tilde{\pi} : S(TM) \to M$ is a Legendrian fibration (cf., [2]).

We now show the basic theorem in this paper which is the fundamental tool for the study of spacelike submanifolds in pseudo-spheres in Minkowski space. We define one-forms $\langle dv, w \rangle = -w_0 dv_0 + \sum_{i=1}^{n} w_i dv_i$, $\langle v, dw \rangle = -v_0 dw_0 + \sum_{i=1}^{n} v_i dw_i$ on $\mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$ and consider the following four double fibrations with one-forms:

1. (a) $H^{n+1}(-1) \times S_1^{n+1} \supset \Delta_1 = \{(v, w) \mid \langle v, w \rangle = 0\}$,

(b) $\pi_{11} : \Delta_1 \to H^{n+1}(-1)$, $\pi_{12} : \Delta_1 \to S_1^{n+1}$,

(c) $\theta_{11} = (dv, w)|_{\Delta_3}$, $\theta_{12} = (v, dw)|_{\Delta_3}$.

2. (a) $H^{n+1}(-1) \times LC^* \supset \Delta_2 = \{(v, w) \mid \langle v, w \rangle = -1\}$,

(b) $\pi_{21} : \Delta_4 \to LC^*$, $\pi_{22} : \Delta_4 \to LC^*$,

(c) $\theta_{21} = (dv, w)|_{\Delta_2}$, $\theta_{22} = (v, dw)|_{\Delta_2}$.

3. (a) $LC^* \times S_1^3 \supset \Delta_3 = \{(v, w) \mid \langle v, w \rangle = 1\}$,

(b) $\pi_{31} : \Delta_3 \to LC^*$, $\pi_{32} : \Delta_3 \to S_1^3$.

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(c) \( \theta_{31} = \langle dv, w \rangle |_{\Delta_3}, \theta_{32} = \langle v, dw \rangle |_{\Delta_3} \).

(4)(a) \( LC^* \times LC^* \supset \Delta_4 = \{(v, w) \mid \langle v, w \rangle = -2\} \),

(b) \( \pi_{41} : \Delta_4 \rightarrow LC^*, \pi_{42} : \Delta_4 \rightarrow LC^* \),

(c) \( \theta_{41} = \langle dv, w \rangle |_{\Delta_4}, \theta_{42} = \langle v, dw \rangle |_{\Delta_4} \).

Here, \( \pi_{ii}(v, w) = v, \pi_{i2}(v, w) = w \) are the canonical projections. Moreover, \( \theta_{i1} = \langle dv, w \rangle |_{\Delta_i} \) and \( \theta_{i2} = \langle v, dw \rangle |_{\Delta_i} \) are the restrictions of the one-forms \( \langle dv, w \rangle \) and \( \langle v, dw \rangle \) on \( \Delta_i \). We remark that \( \theta_{i1}^{-1}(0) \) and \( \theta_{i2}^{-1}(0) \) define the same tangent hyperplane field over \( \Delta_i \) which is denoted by \( K_i \). The basic theorem in this thesis is the following theorem:

**Theorem 2.2.1.** Under the same notations as the previous paragraph, each \((\Delta_i; K_i) (i = 1, 2, 3, 4)\) is a contact manifold and both of \( \pi_{ij} (j = 1; 2) \) are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic each other.

The proof of this theorem can be found in [8]. In this thesis, we will only consider \((\Delta_3, K_3)\) and \((\Delta_4, K_4)\). If we have an isotropic mapping \( i : L \rightarrow \Delta_i \) (i.e., \( i^*\theta_{i1} = 0 \)), we say that \( \pi_{i1}(i(L)) \) and \( \pi_{i2}(i(L)) \) are \( \Delta_i \)-dual to each other \((i = 3, 4)\). For detailed properties of Legendrian fibrations, see [1].
3 Lightcone dualities for curves in the 2-sphere

3.1 Curves in the unit sphere and lightcone duals

Let \( \gamma : I \to S^2_+ \) be a regular curve. We have a map \( \Phi : S^2_+ \to S^2_0 \) defined by \( \Phi(v) = v - e_0 \), which is an isometry. Then we have a regular curve \( \overline{\gamma} : I \to S^2_0 \) defined by \( \overline{\gamma}(s) = \Phi(\gamma(s)) = \gamma(s) - e_0 \), so that \( \gamma \) and \( \overline{\gamma} \) have the same geometric properties as spherical curves. Since \( \overline{\gamma} \) is a spacelike curve, we can reparameterize it by the arc-length \( s \). So we have the unit tangent vector \( t(s) = \overline{\gamma}'(s) \) of \( \overline{\gamma}(s) \). We have another unit vector \( n(s) = \overline{\gamma}(s) \wedge e_0 \wedge t(s) \), then we have a pseudo-orthonormal frame \( \{ \overline{\gamma}, t, n \} \) of \( \mathbb{R}^3_0 \) along \( \overline{\gamma} \). By standard arguments, we have the following Frenet-Serret type formulae.

\[
\begin{align*}
\overline{\gamma}'(s) &= t(s) \\
t'(s) &= \kappa_g(s)n(s) - \overline{\gamma}(s) \\
n'(s) &= -\kappa_g(s)t(s)
\end{align*}
\]

where \( \kappa_g(s) = \langle t'(s), n(s) \rangle \). It is known that \( \{ \overline{\gamma}, t, n \} \) is called the Sabban frame of \( \overline{\gamma} \) [12].

Here, \( \kappa_g \) is the geodesic curvature of \( \overline{\gamma} \) in \( S^2_0 \). Under the above notation, the dual of \( \overline{\gamma} \) is given by \( \overline{\gamma}'(s) = n(s) \). Therefore, \( s_0 \) is a singular point of \( \overline{\gamma}' \) if and only if \( \kappa_g(s_0) = 0 \). Moreover, it has been known the following proposition [18, Proposition 1]:

**Proposition 3.1.1.** The dual curve \( \overline{\gamma}' \) has an ordinary cusp at \( s_0 \) if and only if \( \kappa_g(s_0) = 0 \) and \( \kappa'_g(s_0) \neq 0 \).

Here, we say that a curve has an ordinary cusp if it is locally diffeomorphic to the curve \( C = \{(x_1, x_2) \mid x_1^2 = x_2^3\} \) (cf., Fig.1).

The point \( \overline{\gamma}(s_0) \) with \( \kappa_g(s_0) = 0, \kappa'_g(s_0) \neq 0 \) is said to be an ordinary inflection of \( \overline{\gamma} \).

On the other hand, the evolute \( \varepsilon_{\overline{\gamma}} \) of the curve \( \overline{\gamma} \) is defined to be the envelope of the family of the normal geodesics to the curve \( \overline{\gamma} \). It is known that the evolute of \( \overline{\gamma} \) is the dual of the spherical curve given by the unit tangent vector \( t \) [18]. By the Frenet-Serret type formulae, we
The ordinary cusp

Fig. 1

have

\[
\mathcal{E}_\gamma^\pm(s) = \pm \mathcal{E}(s) = \frac{\pm (\kappa_g(s) \gamma(s) + \mathbf{n}(s))}{\sqrt{\kappa_g^2(s) + 1}}.
\]

For a general parametrized curve \( \gamma : I \rightarrow S^2 \), we can calculate that the geodesic curvature is given by \( \kappa_g(t) = \frac{\det(\gamma(t), \gamma(t), \gamma(t))}{\|\gamma(t)\|^3} \). Thus, we have the following proposition as a corollary of Proposition 3.2.1:

**Proposition 3.1.2.** The evolute \( \mathcal{E}_\gamma \) of the curve \( \gamma \) is singular at \( s_0 \) if and only if \( \kappa_g'(s_0) = 0 \). Moreover, \( \mathcal{E}_\gamma \) has an ordinary cusp at \( s_0 \) if and only if \( \kappa_g'(s_0) = 0 \) and \( \kappa_g''(s_0) \neq 0 \).

**Proof.** The geodesic curvature of \( t \) is given by

\[
\kappa_g[t](s) = \frac{\det(t(s), t'(s), t''(s))}{\|t'(s)\|^3} = \frac{\kappa_g'(s)}{\left(\sqrt{\kappa_g^2(s) + 1}\right)^3}.
\]

Hence, the first assertion holds. We also have

\[
\kappa_g[t]'(s) = \frac{\kappa_g''(s)(\kappa_g^2(s) + 1) - 3\kappa_g(s)(\kappa_g')^2(s)}{\left(\sqrt{\kappa_g^2(s) + 1}\right)^5}.
\]

It follows that \( \kappa_g[t](s_0) = 0 \) and \( \kappa_g[t]'(s_0) \neq 0 \) if and only if \( \kappa_g'(s_0) = 0 \) and \( \kappa_g''(s_0) \neq 0 \). This completes the proof.

The point \( \gamma(s_0) \) with \( \kappa_g'(s_0) = 0, \kappa_g''(s_0) \neq 0 \) is said to be an ordinary vertex of \( \gamma \).

On the other hand, \( \gamma \) is a curve in the Euclidean 3-space \( \mathbb{R}^3 \). Therefore, we have the Frenet frame \( \{T, N, B\} \) along \( \gamma \) as a space curve. Let \( \tau(s) \) be the torsion of \( \gamma \) at \( \gamma(s) \) with respect
to the Frenet frame \( \{ T, N, B \} \). By a straightforward calculation, we have
\[
\tau(s) = -\frac{\kappa'(s)}{\kappa_2'(s) + 1} \quad \text{and} \quad \tau'(s) = -\frac{\kappa''(s)(\kappa'_2(s) + 1) - 2\kappa(s)(\kappa'_2(s))}{(\kappa_2'(s) + 1)^2}.
\]
Therefore, we have the following corollary.

**Corollary 3.1.3.** The evolute \( \varepsilon_\gamma \) of the curve \( \gamma \) is singular at \( s_0 \) if and only if \( \tau(s_0) = 0 \).
Moreover, \( \varepsilon_\gamma \) has an ordinary cusp at \( s_0 \) if and only if \( \tau(s_0) = 0 \) and \( \tau'(s_0) \neq 0 \).

We call the point \( \gamma(s_0) \) with \( \tau(s_0) = 0 \), \( \tau'(s_0) \neq 0 \) an *ordinary flattening point* of \( \gamma \).

We now define surfaces in \( LC^* \) associated with the curves in \( S^2_+ \) or \( S^2_0 \). Let \( \gamma : I \rightarrow S^2_+ \) be a unit speed curve. We define \( LD_\gamma^\pm : I \times \mathbb{R} \rightarrow LC^* \) by
\[
LD_\gamma^\pm(s, u) = \gamma(s) + un(s) \pm \sqrt{u^2 + 1} e_0.
\]
We also define \( LD_\gamma : I \times \mathbb{R} \rightarrow LC^* \) by
\[
LD_\gamma(s, u) = \frac{u^2 - 4}{4} \gamma(s) + un(s) + \frac{u^2 + 4}{4} e_0.
\]
Then we have the following proposition.

**Proposition 3.1.4.** Under the above notations, we have the followings:

1. \( \gamma \) and \( LD_\gamma^\pm \) are \( \Delta_3 \)-dual to each other.
2. \( \gamma \) and \( LD_\gamma \) are \( \Delta_4 \)-dual to each other.

**Proof.** Consider the mapping \( \mathcal{L}_3(s, u) = (LD_\gamma^\pm(s, u), \gamma(s)) \). Then we have \( \langle LD_\gamma^\pm(s, u), \gamma(s) \rangle = \langle \gamma(s), \gamma(s) \rangle = 1 \) and \( \mathcal{L}_3^*\theta_{32} = \langle LD_\gamma^\pm(s, u), \gamma'(s) \rangle ds = \langle LD_\gamma^\pm(s, u), t(s) \rangle ds = 0 \). The assertion (1) holds.

We also consider the mapping \( \mathcal{L}_4(s, u) = (LD_\gamma(s, u), \gamma(s)) \). Since \( \langle \gamma(s), e_0 \rangle = 1 \) and \( \langle \gamma(s), \gamma(s) \rangle = 1 \), we have \( u^2 / 4 - 1 - (u^2 / 4 + 1) = -2 \). Moreover, we have \( \mathcal{L}_4^*\theta_{42} = \langle LD_\gamma(s, u), \gamma'(s) \rangle ds = \langle LD_\gamma(s, u), t(s) \rangle ds = 0 \). This completes the proof. \( \square \)

We call \( LD_\gamma^\pm \) the *Lightcone dual surface of the de Sitter spherical curve* \( \gamma \) and \( LD_\gamma \) the *Lightcone dual surface of the lightlike spherical curve* \( \gamma \). Then we have two mappings \( \pi \circ LD_\gamma^\pm : I \times \mathbb{R} \rightarrow LC^* \).
\( I \times \mathbb{R} \to S^2_+ \) and \( \pi \circ LD_{\gamma} : I \times \mathbb{R} \to S^2_+ \) defined by

\[
\pi \circ \overline{LD}_{\gamma}^\pm(s, u) = \pm \left( \frac{1}{\sqrt{u^2 + 1}} \vec{n}(s) + \frac{u}{\sqrt{u^2 + 1}} \vec{m}(s) \right) + e_0,
\]
\[
\pi \circ LD_{\gamma}(s, u) = \frac{(u^2 - 4)}{u^2 + 4} \vec{n}(s) + \frac{4u}{u^2 + 4} \vec{m}(s) + e_0.
\]

In this paper we consider the singularities of these dual surfaces and mappings. By the Frenet-Serret type formulae, we have

\[
\frac{\partial LD_{\gamma}^\pm}{\partial u}(s, u) = \vec{n}(s) \pm \frac{u}{\sqrt{u^2 + 1}} e_0,
\]
\[
\frac{\partial LD_{\gamma}^\pm}{\partial s}(s, u) = (1 - u\kappa_g(s)) \vec{t}(s),
\]
\[
\frac{\partial LD_{\gamma}}{\partial u}(s, u) = \frac{u}{2} \vec{n}(s) + \vec{m}(s) + \frac{u}{2} e_0,
\]
\[
\frac{\partial LD_{\gamma}}{\partial s}(s, u) = \left( \frac{u^2 - 4}{4} - u\kappa_g(s) \right) \vec{t}(s).
\]

Then we have the following proposition.

**Proposition 3.1.5.** Let \( \gamma : I \to S^2_+ \) be a unit speed curve. Then we have the followings:

1. \((s, u)\) is a singular point of \( \overline{LD}_{\gamma}^\pm \) if and only if \( \kappa_g(s) \neq 0 \) and \( u = 1/\kappa_g(s) \),
2. \((s, u)\) is a singular point of \( LD_{\gamma} \) if and only if \( u = 2(\kappa_g(s) \pm \sqrt{\kappa_g^2(s) + 1}) \).

**Proof.** By the above calculations, \( \partial \overline{LD}_{\gamma}^\pm/\partial u(s, u), \partial \overline{LD}_{\gamma}^\pm/\partial s(s, u) \) are linearly dependent if and only if \( 1 - \kappa_g(s) u = 0 \). The assertion (1) follows. By the similar reason, we have the assertion (2). \( \Box \)

For simplification, we denote \( \sigma_g^\pm(s) = \kappa_g(s) \pm \sqrt{\kappa_g^2(s) + 1} \). Therefore, the critical value sets of the above dual surfaces are given by

\[
C(\overline{LD}_{\gamma}^\pm) = \left\{ \vec{n}(s) + \frac{1}{\kappa_g(s)} \vec{m}(s) \pm \sqrt{\frac{\kappa_g^2(s) + 1}{\kappa_g^2(s)}} e_0 \mid s \in I, \kappa_g(s) \neq 0 \right\},
\]
\[
C(LD_{\gamma})^\pm = \{ ((\sigma_g^\pm(s))^2 - 1) \vec{n}(s) + 2\sigma_g^\pm(s) \vec{m}(s) + ((\sigma_g^\pm(s))^2 + 1) e_0 \mid s \in I \}.
\]
Then the projections of these critical value sets to $S^2_+$ are as follows:

$$
\pi(C(\overline{LD}_g^\pm)) = \left\{ \frac{\pm(\kappa_\gamma(s)\overline{\gamma}(s) + n(s))}{\sqrt{\kappa_\gamma^2(s) + 1}} + e_0 \mid s \in I, \kappa_\gamma(s) \neq 0 \right\},
$$

$$
\pi(C(LD_\gamma^\pm)) = \left\{ \frac{(\sigma_\gamma^\pm(s))^2 - 1}{(\sigma_\gamma^\pm(s))^2 + 1} \overline{\gamma}(s) + \frac{2\sigma_\gamma^\pm(s)}{(\sigma_\gamma^\pm(s))^2 + 1} n(s) + e_0 \mid s \in I \right\}.
$$

We remark that each of $\pm(\kappa_\gamma(s)\overline{\gamma}(s) + n(s))$ is the spherical evolute of $\overline{\gamma}$ (cf., [17]). This means that the spherical evolute is obtained from the critical value set of the lightcone dual surface of $\overline{\gamma}$. On the other hand, what is the curve $(\sigma_\gamma^\pm(s))^2 + 1 \overline{\gamma}(s) + \frac{2\sigma_\gamma^\pm(s)}{(\sigma_\gamma^\pm(s))^2 + 1} n(s)? Since 

$$
\sigma_\gamma^\pm(s) = \kappa_\gamma(s) \pm \sqrt{\kappa_\gamma^2(s) + 1},
$$

we have $(\sigma_\gamma^\pm(s))^2 = 2\kappa_\gamma(s)\sigma_\gamma^\pm(s) + 1$. Therefore, we have

$$
\left( (\sigma_\gamma^\pm(s))^2 - 1 \right)^2 = \frac{\kappa_\gamma^2(s)(\sigma_\gamma^\pm(s))^2}{\kappa_\gamma^2(s)(\sigma_\gamma^\pm(s))^2 + 2\kappa_\gamma(s)\sigma_\gamma^\pm(s) + 1} = \frac{\kappa_\gamma^2(s)}{\kappa_\gamma^2(s) + 1}
$$

and

$$
\left( \frac{2\sigma_\gamma^\pm(s)}{(\sigma_\gamma^\pm(s))^2 + 1} \right)^2 = \frac{2\kappa_\gamma(s)\sigma_\gamma^\pm(s) + 1}{(\kappa_\gamma(s)\sigma_\gamma^\pm(s) + 1)^2} = \frac{1}{\kappa_\gamma^2(s) + 1}.
$$

Thus we have the following proposition.

**Proposition 3.1.6.** Let $\gamma : I \rightarrow S^2_+$ be a unit speed curve. Then

$$
\frac{(\sigma_\gamma^\pm(s))^2 - 1}{(\sigma_\gamma^\pm(s))^2 + 1} \overline{\gamma}(s) + \frac{2\sigma_\gamma^\pm(s)}{(\sigma_\gamma^\pm(s))^2 + 1} n(s) = \pm(\kappa_\gamma(s)\overline{\gamma}(s) + n(s))
$$

The projection $\pi_\gamma = \Phi \circ \pi : S^2_+ \rightarrow S^2_0$. Then we have the following theorem as a corollary of Proposition 3.1.6

**Theorem 3.1.7.** Both of the projections of the critical value sets $C(\overline{LD}_g^\pm)$ and $C(LD_\gamma^\pm)$ in the unit sphere $S^2_0$ are the image of the evolute of $\overline{\gamma}$, that is

$$
\overline{\pi}(C(\overline{LD}_g^\pm)) = \overline{\pi}(C(LD_\gamma^\pm)) = \{ e_\gamma^\pm(s) \mid s \in I \}.
$$
3.2 Lightcone height functions

In order to study the singularities of Lightcone dual surfaces of spherical curves, we introduce two families of functions and apply the theory of unfoldings. Let $\gamma : I \rightarrow S^2_+$ be a unit speed curve, then we define two families of functions as follows:

$$H : I \times LC^* \rightarrow \mathbb{R}, \quad H(s, v) = \langle \gamma(s), v \rangle + 2.$$ 

We call $H$ a lightcone height function of the lightlike spherical curve $\gamma$. For any fixed $v \in LC^*$, we denote $h_v(s) = H(s, v)$. Then we have the following two propositions on $h_v$ and $\overline{h}_v$.

**Proposition 3.2.1.** Let $\gamma : I \rightarrow S^2_+$ be a unit speed curve, then we have the followings:

1. $\overline{h}_v(s) = 0$ if and only if there exist $\mu, \xi, \eta \in \mathbb{R}$ with $\eta^2 = 1 + \mu^2 + \xi^2$ such that $v = \gamma(s) + \mu t(s) + \xi n(s) + \eta e_0$.
2. $h_v(s) = \overline{h}_v(s) = 0$ if and only if there exist $\xi, \eta \in \mathbb{R}$ with $\eta^2 = 1 + \xi^2$ such that $v = \gamma(s) + \xi n(s) + \eta e_0$.
3. $\overline{h}_v(s) = \overline{h}_v'(s) = \overline{h}_v''(s) = 0$ if and only if $\kappa_g(s) \neq 0$ and
   
   $$v = \gamma(s) + \frac{1}{\kappa_g(s)} n(s) \pm \sqrt{\frac{\kappa_g^2(s) + 1}{\kappa_g^2(s)}} e_0.$$
4. $\overline{h}_v(s) = \overline{h}_v'(s) = \overline{h}_v''(s) = \overline{h}_v'''(s) = 0$ if and only if $\kappa_g(s) \neq 0$, $\kappa_g'(s) = 0$ and
   
   $$v = \gamma(s) + \frac{1}{\kappa_g(s)} n(s) \pm \sqrt{\frac{\kappa_g^2(s) + 1}{\kappa_g^2(s)}} e_0.$$
5. $\overline{h}_v(s) = \overline{h}_v'(s) = \overline{h}_v''(s) = \overline{h}_v'''(s) = \overline{h}_v^{(4)}(s) = 0$ if and only if $\kappa_g(s) \neq 0$, $\kappa_g'(s) = \kappa_g''(s) = 0$ and
   
   $$v = \gamma(s) + \frac{1}{\kappa_g(s)} n(s) \pm \sqrt{\frac{\kappa_g^2(s) + 1}{\kappa_g^2(s)}} e_0.$$
Proof. (1) Since \( v \in LC^* \), there exist \( \omega, \mu, \xi, \eta \in \mathbb{R} \) with \( \omega^2 + \mu^2 + \xi^2 - \eta^2 = 0 \) such that 
\[
v = \omega \gamma(s) + \mu t(s) + \xi n(s) + \eta e_0.\]
From \( \overline{h}_v(s) = (\gamma(s), v) - 1 = 0 \), we have \( \omega = 1 \). So 
\[
v = \gamma(s) + \mu t(s) + \xi n(s) + \eta e_0 \quad \text{and} \quad \eta^2 = 1 + \mu^2 + \xi^2.\]
The converse direction also holds.

(2) Since \( \overline{h}_v(s) = (t(s), v), \overline{h}_v(s) = \overline{h}_v(s) = 0 \) if and only if \( \overline{h}_v(s) = (t(s), v) = (t(s), \gamma(s) + \mu t + \xi n + \eta e_0) = 0 \). It follows from the fact \( \eta^2 = 1 + \xi^2 \) that \( \eta = \pm \sqrt{1 + \xi^2} \). Then we have 
\[
v = \gamma(s) + n(s) + \xi e_0 = \gamma(s) + n(s) \pm \sqrt{1 + \xi^2} e_0.\]

(3) Since \( \overline{h}_v(s) = (\kappa_\gamma(s)n(s) - \gamma(s), v), \overline{h}_v(s) = \overline{h}_v(s) = 0 \) if and only if \( \overline{h}_v(s) = (\kappa_\gamma(s)n(s) - \gamma(s), \gamma(s) + n(s) \pm \sqrt{1 + \xi^2} e_0) = \kappa_\gamma(s)/\kappa_\gamma(s) = 1 \). Then we have \( \kappa = 1/\kappa_\gamma(s), \)
\[
v = \gamma(s) + n(s)/\kappa_\gamma(s) \pm \sqrt{(1 + \kappa_\gamma^2(s))/\kappa_\gamma^2(s)} e_0, \kappa_\gamma(s) \neq 0.\]

(4) Since \( \overline{h}_v(s) = (\kappa_\gamma'(s)n(s) - (\kappa_\gamma^2(s) + 1)t(s), v), \overline{h}_v(s) = \overline{h}_v(s) = 0 \) if and only if \( \kappa_\gamma'(s)n(s) - (\kappa_\gamma^2(s) + 1)t(s), \gamma(s) + n(s)/\kappa_\gamma(s) \pm \sqrt{(1 + \kappa_\gamma^2(s))/\kappa_\gamma^2(s)} e_0 = \kappa_\gamma(s)/\kappa_\gamma(s) = 0 \). Then we have 
\[
v = \gamma(s) + n(s)/\kappa_\gamma(s) \pm \sqrt{(1 + \kappa_\gamma^2(s))/\kappa_\gamma^2(s)} e_0, \kappa_\gamma(s) \neq 0, \kappa_\gamma'(s) = 0, \kappa_\gamma''(s) = 0, \kappa_\gamma'''(s) = 0.\]

(5) Since \( \overline{h}_v(s) = (\kappa_\gamma''(s)n(s) - \gamma(s), v), \overline{h}_v(s) = \overline{h}_v(s) = 0 \) if and only if \( \kappa_\gamma''(s)n(s) - \gamma(s), \gamma(s) + n(s)/\kappa_\gamma(s) \pm \sqrt{(1 + \kappa_\gamma^2(s))/\kappa_\gamma^2(s)} e_0 = \kappa_\gamma(s)/\kappa_\gamma(s) = 0 \). Then we have 
\[
v = \gamma(s) + n(s)/\kappa_\gamma(s) \pm \sqrt{(1 + \kappa_\gamma^2(s))/\kappa_\gamma^2(s)} e_0, \kappa_\gamma(s) \neq 0, \kappa_\gamma'(s) = \kappa_\gamma''(s) = \kappa_\gamma'''(s) = 0.\]

\[\square\]

**Proposition 3.2.2.** Let \( \gamma : I \rightarrow S^2_+ \) be a unit speed curve, then we have the followings:

(1) \( h_v(s) = 0 \) if and only if \( v = \lambda \gamma(s) + \mu t + \xi n + (\lambda + 2)e_0 \), where \( \lambda, \mu, \xi \in \mathbb{R} \) and \( \mu^2 + \xi^2 - 4\lambda - 4 = 0 \).

(2) \( h_v(s) = h_v'(s) = 0 \) if and only if \( v = (\xi^2/4 - 1) \gamma(s) + \xi n + (\xi^2/4 + 1)e_0 \).

(3) \( h_v(s) = h_v'(s) = h_v''(s) = 0 \) if and only if 
\[
v = ((\sigma_\gamma^\pm(s))^2 - 1) \gamma(s) + 2\sigma_\gamma^\pm(s)n(s) + ((\sigma_\gamma^\pm(s))^2 + 1)e_0.\]

(4) \( h_v(s) = h_v'(s) = h_v''(s) = h_v'''(s) = 0 \) if and only if 
\[
v = ((\sigma_\gamma^\pm(s))^2 - 1) \gamma(s) + 2\sigma_\gamma^\pm(s)n(s) + ((\sigma_\gamma^\pm(s))^2 + 1)e_0\]
and \( \kappa_\gamma'(s) = 0 \).
(5) \(h_v(s) = h'_v(s) = h''_v(s) = h'''_v(s) = 0\) if and only if 
\[
\nu = ((\sigma^+_g(s))^2 - 1)\bar{\gamma}(s) + 2\sigma^+_g(s)n(s) + ((\sigma^+_g(s))^2 + 1)e_0.
\]
\(\kappa'_g(s) = 0\) and \(\kappa''_g(s) = 0\).

**Proof.** (1) Since \(\nu \in LC^\ast\), there exist \(\lambda, \mu, \xi, \eta \in \mathbb{R}\) with \(\lambda^2 + \mu^2 + \xi^2 - \eta^2 = 0\) such that
\[
\nu = \lambda\bar{\gamma}(s) + \mu t(s) + \xi n(s) + \eta e_0.
\] From \(h_v(s) = \langle \gamma(s), \nu \rangle + 2 = \langle \bar{\gamma}(s) + e_0, \lambda\bar{\gamma}(s) + \mu t(s) + \xi n(s) + \eta e_0 \rangle = \lambda - \eta + 2 = 0\), we have \(\eta = 2 + \lambda\). So \(\nu = \lambda\bar{\gamma}(s) + \mu t(s) + \xi n(s) + (2 + \lambda)e_0\) and \(\lambda^2 + \mu^2 + \xi^2 - (2 + \lambda)^2 = \mu^2 + \xi^2 - 4\lambda - 4 = 0\). The converse direction also holds.

(2) Since \(h'_v(s) = \langle t(s), \nu \rangle, h_v(s) = h'_v(s) = 0\) if and only if \(h'_v(s) = \langle t(s), \lambda\bar{\gamma}(s) + \mu t(s) + \xi n(s) + (2 + \lambda)e_0 \rangle = \mu = 0\). It follows from the fact \(\lambda^2 + \xi^2 - (2 + \lambda)^2 = \xi^2 - 4\lambda - 4 = 0\) that \(\lambda = \xi^2/4 - 1\). Then we have \(\nu = (\xi^2/4 - 1)\bar{\gamma}(s) + \xi n(s) + (\xi^2/4 + 1)e_0\).

(3) Since \(h''_v(s) = \langle \kappa_g(s)n(s) - \bar{\gamma}(s), \nu \rangle, h_v(s) = h'_v(s) = h''_v(s) = 0\) if and only if \(h''_v(s) = \langle \kappa_g(s)n(s) - \bar{\gamma}(s), (\xi^2/4 - 1)\bar{\gamma}(s) + \xi n(s) + (\xi^2/4 + 1)e_0 \rangle = \kappa_g(s)\xi - \xi^2/4 + 1 = 0\), then we have \(\xi = 2\kappa_g(s) + 2(\kappa^2_g(s) + 1)^{1/2} = 2\sigma^+_g(s)\), then we have \(\nu = ((\sigma^+_g(s))^2 - 1)\bar{\gamma}(s) + 2\sigma^+_g(s)\nu(s) + ((\sigma^+_g(s))^2 + 1)e_0\).

(4) Since \(h'''_v(s) = \langle \kappa'_g(s)n(s) - (\kappa^2_g(s) + 1)t(s), \nu \rangle, h_v(s) = h'_v(s) = h''_v(s) = h'''_v(s) = 0\) if and only if \(h'''_v(s) = \langle \kappa'_g(s)n(s) - (\kappa^2_g(s) + 1)t(s), ((\sigma^+_g(s))^2 - 1)\bar{\gamma}(s) + 2\sigma^+_g(s)n(s) + ((\sigma^+_g(s))^2 + 1)e_0 \rangle = 2\kappa'_g(s)\sigma^+_g(s) = 0\). For \(\sigma^+_g(s) \neq 0\), we have \(\kappa'_g(s) = 0\). Then we have \(\nu = ((\sigma^+_g(s))^2 - 1)\bar{\gamma}(s) + 2\sigma^+_g(s)n(s) + ((\sigma^+_g(s))^2 + 1)e_0\) and \(\kappa''_g(s) = 0\).

(5) Since \(h'''_v(s) = \langle (\kappa''_g(s) - \kappa^3_g(s) - \kappa_g(s))n(s) - 3\kappa_g(s)\kappa'_g(s)t(s) + (\kappa^2_g(s) + 1)\bar{\gamma}(s), \nu \rangle, h_v(s) = h'_v(s) = h''_v(s) = h'''_v(s) = 0\) if and only if \(h'''_v(s) = \langle (\kappa''_g(s) - \kappa^3_g(s) - \kappa_g(s))n(s) - 3\kappa_g(s)\kappa'_g(s)t(s) + (\kappa^2_g(s) + 1)\bar{\gamma}(s), ((\sigma^+_g(s))^2 - 1)\bar{\gamma}(s) + 2\sigma^+_g(s)n(s) + ((\sigma^+_g(s))^2 + 1)e_0 \rangle = 2\kappa''_g(s)\sigma^+_g(s) = 0\). For \(\sigma^+_g(s) \neq 0\), we have \(\kappa''_g(s) = 0\). Then we have \(\nu = ((\sigma^+_g(s))^2 - 1)\bar{\gamma}(s) + 2\sigma^+_g(s)n(s) + ((\sigma^+_g(s))^2 + 1)e_0, \kappa'_g(s) = 0\) and \(\kappa''_g(s) = 0\). \(\square\)

3.3 **Singularities of ligtcone duals of spherical curves**

In this section we classify the singularities of \(\overline{LD}^\pm_\gamma\) and \(LD_\gamma\) as an application of the unfolding theory of functions. Let \(F : (\mathbb{R} \times \mathbb{R}, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}\) be a function germ, we call \(F\)
an $r$-parameter unfolding of $f$, where $f(s) = F_{x_0}(s, x_0)$. If $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$, then we say that $f$ has $A_k$-singularity at $s_0$. Let $F$ be an $r$-parameter unfolding of $f$ and $f$ has $A_k$-singularity ($k \geq 1$) at $s_0$. We denote the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at $s_0$ as

$$j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s, x_0)\right)(s_0) = \sum_{j=1}^{k-1} \alpha_{ji}(s - s_0)^j, \ (i = 1, \cdots, r).$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is $k$ ($k \leq r$), then $F$ is called a versal unfolding of $f$, where $\alpha_{0i} = \partial F/\partial x_i(s_0, x_0)$. The discriminant set of $F$ is defined by

$$D_F = \left\{ x \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, \ F(s, x) = \frac{\partial F}{\partial s}(s, x) = 0 \right\}.$$

For the discriminant set of $F$, we have the following theorem [3].

**Theorem 3.3.1.** Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow \mathbb{R}$ be an $r$-parameter unfolding of $f$ which has $A_k$-singularity at $s_0$. Suppose $F$ is a versal unfolding of $f$, then we have the following assertions.

(a) If $k = 1$, then $D_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.

(b) If $k = 2$, then $D_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.

(c) If $k = 3$, then $D_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.

Here, $C$ is the ordinary cusp and $C \times \mathbb{R}$ is called a cuspidal edge (Fig.2). Moreover, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is called a swallow tail (Fig.3).

![The cuspidal edge](Figure 2) ![The swallow tail](Figure 3)

By Propositions 3.2.1 and 3.2.2, the discriminant set of $\Pi$ and $H$ are

$$D_{\Pi} = \{\gamma(s) + un(s) \pm \sqrt{u^2 + 1}e_0 \mid (s, u) \in I \times \mathbb{R}\},$$

$$D_{H} = \{(u^2/4 - 1)\gamma(s) + un(s) + (u^2/4 + 1)e_0 \mid s \in I, u \in \mathbb{R}\}.$$
These are the lightcone dual surface of $\gamma$ and the lightcone dual surface of $\gamma$ respectively. We have the following key propositions on $H$ and $\overline{H}$.

**Proposition 3.3.2.** If $h_{v_0}(s)$ has $A_k$-singularity ($k = 1, 2, 3$) at $s_0$, then $\overline{H}$ is a versal unfolding of $h_{v_0}$.

**Proof.** For $v \in LC^*$, we have $v = (\pm(v_1^2 + v_2^2 + v_3^2)^{1/2}, v_1, v_2, v_3)$, then

$$\overline{H}(s, v) = \langle \gamma(s), v \rangle - 1 = x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 - 1.$$  

Thus we have

$$\frac{\partial \overline{H}}{\partial v_1}(s, v) = x_1(s), \quad \frac{\partial \overline{H}}{\partial v_2}(s, v) = x_2(s), \quad \frac{\partial \overline{H}}{\partial v_3}(s, v) = x_3(s),$$

$$\frac{\partial^2 \overline{H}}{\partial s \partial v_1}(s, v) = x_1'(s), \quad \frac{\partial^2 \overline{H}}{\partial s \partial v_2}(s, v) = x_2'(s), \quad \frac{\partial^2 \overline{H}}{\partial s \partial v_3}(s, v) = x_3'(s),$$

$$\frac{\partial^3 \overline{H}}{\partial s^2 \partial v_1}(s, v) = x_1''(s), \quad \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_2}(s, v) = x_2''(s), \quad \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_3}(s, v) = x_3''(s).$$

For a fixed point $v_0 = (v_{00}, v_{01}, v_{02}, v_{03})$, the 2-jet of $\frac{\partial \overline{H}}{\partial v_i}(s, v_0)(i = 1, 2, 3)$ at $s_0$ is

$$j^2(\frac{\partial \overline{H}}{\partial v_i})(s, v_0)(s_0) = x_1'(s_0)(s - s_0) + x_2''(s_0)(s - s_0)^2/2, \quad (i = 1, 2, 3).$$

It is enough to show that the rank of the matrix $A$ is three, where

$$A = \begin{pmatrix}
    x_1(s_0) & x_2(s_0) & x_3(s_0) \\
    x_1'(s_0) & x_2'(s_0) & x_3'(s_0) \\
    x_1''(s_0) & x_2''(s_0) & x_3''(s_0)
\end{pmatrix}.$$  

Then we have

$$\det A = \langle e_0 \wedge \gamma(s_0) \wedge \gamma'(s_0), \gamma''(s_0) \rangle = \langle -n(s_0), \kappa_g(s_0)n(s_0) - \gamma(s_0) \rangle = -\kappa_g(s_0) \neq 0.$$  

So the rank of $A$ is three, this completes the proof. \qed

**Proposition 3.3.3.** If $h_{v_0}(s)$ has $A_k$-singularity ($k = 1, 2, 3$) at $s_0$, then $H$ is a versal unfolding of $h_{v_0}$.
Proof. For \( \mathbf{v} \in LC^* \), we have \( \mathbf{v} = (\pm (v_1^2 + v_2^2 + v_3^2)^{1/2}, v_1, v_2, v_3) \), then

\[
H(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle + 2 = \mp (v_1^2 + v_2^2 + v_3^2)^{1/2} + x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 + 2.
\]

Thus we have

\[
\frac{\partial H}{\partial v_1}(s, \mathbf{v}) = -v_1/v_0 + x_1(s), \quad \frac{\partial H}{\partial v_2}(s, \mathbf{v}) = -v_2/v_0 + x_2(s), \quad \frac{\partial H}{\partial v_3}(s, \mathbf{v}) = -v_3/v_0 + x_3(s),
\]

\[
\frac{\partial^2 H}{\partial s \partial v_1}(s, \mathbf{v}) = x'_1(s), \quad \frac{\partial^2 H}{\partial s \partial v_2}(s, \mathbf{v}) = x'_2(s), \quad \frac{\partial^2 H}{\partial s \partial v_3}(s, \mathbf{v}) = x'_3(s),
\]

\[
\frac{\partial^3 H}{\partial s^2 \partial v_1}(s, \mathbf{v}) = x''_1(s), \quad \frac{\partial^3 H}{\partial s^2 \partial v_2}(s, \mathbf{v}) = x''_2(s), \quad \frac{\partial^3 H}{\partial s^2 \partial v_3}(s, \mathbf{v}) = x''_3(s).
\]

For a fixed \( \mathbf{v}_0 = (v_{01}, v_{02}, v_{03}) \), the 2-jet of \( \frac{\partial H}{\partial \nu_i}(s, \mathbf{v}_0)(i = 1, 2, 3) \) at \( s_0 \) is

\[
j^{(2)} \frac{\partial H}{\partial v_i}(s, \mathbf{v}_0)(s_0) = x'_i(s_0)(s - s_0) + x''_i(s_0)(s - s_0)^2/2, \quad (i = 1, 2, 3).
\]

It is enough to show that the rank of the matrix \( A \) is three, where

\[
A = \begin{pmatrix}
-\frac{v_{01}}{v_0} + x_1(s_0) & -\frac{v_{02}}{v_0} + x_2(s_0) & -\frac{v_{03}}{v_0} + x_3(s_0) \\
x'_1(s_0) & x'_2(s_0) & x'_3(s_0) \\
x''_1(s_0) & x''_2(s_0) & x''_3(s_0)
\end{pmatrix}.
\]

By straight calculation, we have

\[
\det A = -\langle \mathbf{e}_0 \wedge \overline{n}'(s_0) \wedge \overline{n}''(s_0), \mathbf{v}_0 \rangle/v_0 + \langle \mathbf{e}_0 \wedge \overline{n}(s_0) \wedge \overline{n}(s_0), \overline{n}''(s_0) \rangle
\]

\[
= -\langle \mathbf{e}_0 \wedge \mathbf{t}(s_0) \wedge (\kappa_g(s_0) \mathbf{n}(s_0) - \overline{n}(s_0)), \mathbf{v}_0 \rangle/v_0 + \langle -\mathbf{n}(s_0), \kappa_g(s_0)\mathbf{n}(s_0) - \overline{n}(s_0) \rangle
\]

\[
= \langle \kappa_g(s_0)\overline{n}(s_0) + \mathbf{n}(s_0), \mathbf{v}_0 \rangle/v_0 - \kappa_g(s_0).
\]

Since \( \mathbf{v}_0 \in D_H \) is a singular point, we have

\[
\mathbf{v}_0 = ((\sigma^+_g(s_0))^2 - 1)\overline{n}(s_0) + 2\sigma^+_g(s_0)\mathbf{n}(s_0) + ((\sigma^+_g(s_0))^2 + 1)\mathbf{e}_0.
\]
Moreover we have
\[ v_{00} = (\sigma_g'(s_0))^2 + 1. \]

Therefore we have
\[ \det A = \pm 2(\kappa_g^2(s_0) + 1)^{1/2}/v_{00} \neq 0 \]

So the rank of \( A \) is three, this completes the proof.

We have the following theorem:

**Theorem 3.3.4.** Let \( \gamma : I \to S^2_+ \) be a unit speed curve.

(A) For the lightcone dual \( \overline{LD}_\gamma^\pm \) of \( \overline{\gamma} \), we have the following assertions:

1. The lightcone dual \( \overline{LD}_\gamma^\pm \) of \( \overline{\gamma} \) is locally diffeomorphic to the cuspidal edge \( C \times \mathbb{R} \) at \((s_0, u_0)\) if and only if \( k'_g(s_0) \neq 0 \) and \( u_0 = 1/\kappa_g(s_0) \).

2. The lightcone dual \( \overline{LD}_\gamma^\pm \) of \( \overline{\gamma} \) is locally diffeomorphic to the swallowtail at \((s_0, u_0)\) if and only if \( \kappa'_g(s_0) = 0, \kappa''_g(s_0) \neq 0 \) and \( u_0 = 1/\kappa_g(s_0) \).

(B) For the lightcone dual \( LD_\gamma \) of \( \gamma \), we have the following assertions:

1. The lightcone dual \( LD_\gamma \) of \( \gamma \) is locally diffeomorphic to the cuspidal edge \( C \times \mathbb{R} \) at \((s_0, u_0)\) if and only if \( k'_g(s_0) \neq 0 \) and \( u_0 = 2\sigma_g'(s_0) \).

2. The lightcone dual \( LD_\gamma \) of \( \gamma \) is locally diffeomorphic to the swallowtail at \((s_0, u_0)\) if and only if \( k'_g(s_0) = 0, \kappa''_g(s_0) \neq 0 \) and \( u_0 = 2\sigma_g'(s_0) \).

**Proof.** By Propositions 3.2.1 and 3.2.2, the discriminant sets of \( \overline{H} \) and \( H \) are the lightcone duals of \( \overline{\gamma} \) and \( \gamma \) respectively. By Propositions 3.2.1 and 3.2.2, both of \( \overline{h}_v \) and \( h_v \) have \( A_k \) singularities \((k = 1, 2, 3)\) respectively if and only if the above conditions on the geodesic curvatures hold. By Propositions 3.3.2 and 3.3.3, \( \overline{H} \) and \( H \) are versal unfoldings of \( \overline{h}_v \) and \( h_v \) at any point \( s_0 \in I \) respectively. We apply Theorem 3.3.1, so that we have the above assertions.

**Remark 3.3.5.** As a consequence of Proposition 3.1.2, Corollary 3.1.3 and Theorem 3.3.4, we can summarize the relationship between the singularities of \( \overline{e}_\gamma, \overline{LD}_\gamma^\pm \) and \( LD_\gamma \):
<table>
<thead>
<tr>
<th>( \gamma(s_0) ) in ( S^2_+ )</th>
<th>( \kappa'_g(s_0) \neq 0 )</th>
<th>( \kappa''_g(s_0) = 0, \kappa''_g(s_0) \neq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{\gamma}(s_0) ) in ( \mathbb{R}^3_0 )</td>
<td>( \tau(s_0) \neq 0 )</td>
<td>( \tau(s_0) = 0, \tau'(s_0) \neq 0 )</td>
</tr>
<tr>
<td>( \varepsilon_{\overline{\gamma}}(s_0) )</td>
<td>the regular point</td>
<td>the ordinary cusp</td>
</tr>
<tr>
<td>( LD_{\overline{\gamma}}(s_0, 1/\kappa_g(s_0)) )</td>
<td>the cuspidal edge</td>
<td>the swallowtail</td>
</tr>
<tr>
<td>( LD_{\gamma}(s_0, 2\sigma^\pm_g(s_0)) )</td>
<td>the cuspidal edge</td>
<td>the swallowtail</td>
</tr>
</tbody>
</table>

We can interpret the above correspondence between the cusps of the spherical evolute and the swallowtails of lightcone duals from the viewpoint of the theory of Legendrian/Lagrangian singularities. For spherical curves, however, we only need the curvature or the torsion. Such a framework is really needed for the higher dimensional case.
4 Lightcone dualities for curves in the 3-sphere

4.1 Curves in the unit 3-sphere and focal surfaces

Let $\gamma : I \to S_+^3$ be a regular curve. We have a map $\Phi : S_+^3 \to S_0^3$ defined by $\Phi(v) = v - e_0$, which is an isometry. Then we have a regular curve $\overline{\gamma} : I \to S_0^3$ defined by $\overline{\gamma}(s) = \Phi(\gamma(s)) = \gamma(s) - e_0$, so that $\gamma$ and $\overline{\gamma}$ have completely the same geometric properties as spherical curves. Since $\overline{\gamma}$ is a spacelike curve, we can reparameterize it by the arc-length $s$. So we have the unit tangent vector $t(s) = \gamma'(s)$ of $\gamma(s)$. Suppose that $\| \gamma'(s) \| \neq 1$. Then $\| \gamma'(s) + \overline{\gamma}(s) \| \neq 0$, so that we have another unit vector $n(s) = \gamma'(s) + \overline{\gamma}(s)$. We also define a unit vector by $b(s) = \gamma(s) \wedge e_0 \wedge t(s) \wedge n(s)$, then we have a pseudo-orthonormal frame field $\{\gamma(s), t(s), n(s), b(s)\}$ of $\mathbb{R}_0^4$ along $\overline{\gamma}(s)$. By standard arguments, we have the following Frenet-Serret type formulae.

\[
\begin{align*}
\overline{\gamma}'(s) &= t(s) \\
\gamma'(s) &= \kappa_g(s)n(s) - \overline{\gamma}(s) \\
n'(s) &= -\kappa_g(s)t(s) + \tau_g(s)b(s) \\
b'(s) &= -\tau_g(s)n(s)
\end{align*}
\]

where $\kappa_g(s) = \| \gamma'(s) + \overline{\gamma}(s) \|$ and $\tau_g(s) = -\det(\gamma(s), \overline{\gamma}'(s), \overline{\gamma}''(s), \overline{\gamma}'''(s)) / \kappa_g^2(s)$. We call $\{\gamma, t, n, b\}$ a Sabban frame of $\overline{\gamma}$ [12]. Here, $\kappa_g$ is called a geodesic curvature and $\tau_g$ a geodesic torsion of $\overline{\gamma}$ in $S_0^3$ respectively.

We now consider the focal surface of a curve $\overline{\gamma} : I \to S_0^3$ analogous to the case for curves in Euclidean space. We define $F^\pm : I \times J \to S_0^3$ by

\[F^\pm(s, u) = u\overline{\gamma}(s) + \frac{u}{\kappa_g(s)}n(s) \pm \frac{\sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)}}{\kappa_g(s)}b(s)\].

We call each image of $F^\pm$ the spherical focal surface of $\overline{\gamma}$. We remark that the focal surfaces of $\overline{\gamma}$ satisfies the equations $\langle \overline{\gamma}'(s), F^\pm(s, u) \rangle = \langle \overline{\gamma}'(s), F^\mp(s, u) \rangle = 0$. This means that each one of the focal surface $F^\pm(s, u)$ of $\gamma$ is the spherical dual of $t$ in the sense of [16]. By straightforward
calculations, we have

\[
\frac{\partial F^\pm}{\partial u}(s,u) = \frac{1}{\kappa_g(s)} n(s) \pm \frac{-u(\kappa_g^2(s) + 1)}{\kappa_g(s) \sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)}} b(s),
\]

\[
\frac{\partial F^\pm}{\partial s}(s,u) = -\frac{\kappa_g(s) u'\gamma(s) + \tau_g(s) \kappa_g'(s) \sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)} \pm u^2 \kappa_g'(s)}{\kappa_g^2(s) \sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)}} n(s)
\]

\[
+ \frac{u \kappa_g'(s) \sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)} \pm u^2 \kappa_g'(s)}{\kappa_g^2(s) \sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)}} b(s).
\]

It follows that \{\partial F^\pm/\partial u, \partial F^\pm/\partial s\} is linearly dependent if and only if

\[
\tau_g(s) \kappa_g(s) \sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s) + 1)} \pm u \kappa_g'(s) = 0,
\]

so that we have

\[
u = \frac{\pm \tau_g(s) \kappa_g^2(s)}{\sqrt{\kappa_g^2(s) + \kappa_g^4(s) \tau_g^2(s) + \kappa_g^2(s) \tau_g^2(s)}}.
\]

Therefore each critical value set of \(F^\pm\) is given by

\[
\varepsilon^\pm_\gamma(s) = \frac{\pm \tau_g(s) \kappa_g^2(s)}{\sqrt{\kappa_g^2(s) + \kappa_g^4(s) \tau_g^2(s) + \kappa_g^2(s) \tau_g^2(s)}} \left\{ \nabla(s) + \frac{1}{\kappa_g(s)} n(s) + \left( \frac{1}{\kappa_g(s)} \right)' \frac{1}{\tau_g(s)} b(s) \right\}.
\]

We remark that each curve of \(\varepsilon^\pm_\gamma\) satisfies the equations

\[
\langle \nabla(s), \varepsilon^\pm_\gamma(s) \rangle = \langle \nabla'(s), \varepsilon^\pm_\gamma(s) \rangle = \langle \nabla''(s), \varepsilon^\pm_\gamma(s) \rangle = 0.
\]

In [18] Porteous introduced the notion of the evolute of \(\gamma\) in the unit 3-sphere. He defined it as the curve satisfies the above equations, so that we call each image of \(\varepsilon^\pm_\gamma\) the spherical evolute of \(\gamma\) in the unit 3-sphere. We remark that \(\varepsilon^\pm_\gamma(s) = -\varepsilon^\pm_\gamma(s)\). For \(s = s_0\), we fix that \(v^+_0 = \varepsilon^\pm_\gamma(s_0)\) and \(\langle \gamma(s_0), \varepsilon^\pm_\gamma(s_0) \rangle = c^\pm\). Since \(v^-_0 = -v^+_0\) and \(c^- = -c^+\), we have a hyperplane

\[
HP(v^+_0, c^+) = \{ x \in \mathbb{R}^4_0 \mid \langle x, v^+_0 \rangle = c^+ \} = \{ x \in \mathbb{R}^4_0 \mid \langle x, v^-_0 \rangle = c^- \} = HP(v^-_0, c^-),
\]

so that we have a sphere

\[
S^2(v^+_0, c^+) = HP(v^+_0, c^+) \cap S^3_0.
\]
We call \( S^2(v_0^\pm, c^\pm) \) an osculating sphere of \( \gamma \) at \( s_0 \). Therefore the spherical evolutes \( \varepsilon^\pm_{\gamma}(s) \) are the loci of the centers of osculating spheres of \( \gamma \) respectively.

**Proposition 4.1.1.** There exists a sphere \( S^2(v, c) \subset S^3_0 \) such that \( \gamma(I) \subset S^2(v, c) \) if and only if both of the spherical evolutes \( \varepsilon^\pm_{\gamma} \) of \( \gamma \) are constant.

**Proof.** If one of the spherical evolutes \( \varepsilon^+_{\gamma} \) of \( \gamma \) is constant, we can set that \( \varepsilon^+_{\gamma}(s) = v^+ \). In this case another spherical evolute \( \varepsilon^-_{\gamma} \) is constant too. Then \( \langle \gamma(s), v^+ \rangle' = \langle \gamma(s), v^+ \rangle = \langle t(s), \varepsilon^+_{\gamma}(s) \rangle = 0 \), so we have \( \langle \gamma(s), v^+ \rangle = c^+ \) and \( \gamma(I) \subset S^2(v^+, c^+) \). On the contrary, if \( \gamma(I) \subset S^2(v, c) \), then at any point on \( \gamma \), the osculating spheres is \( S^2(v, c) \) itself. So the locus of the centers of osculating spheres of \( \gamma \) is \( v \) and \(-v \). Therefore, both of the spherical evolutes \( \varepsilon^\pm_{\gamma} \) of \( \gamma \) are constant. This completes the proof.

**4.2 Lightcone duals of curves in the unit 3-sphere**

We now define hypersurfaces in \( LC^* \) associated with the curves in \( S^3_+ \) or \( S^3_0 \). Let \( \gamma : I \rightarrow S^3_+ \) be a unit speed curve. We define \( \overline{LD}^\pm_{\gamma} : I \times \mathbb{R}^2 \rightarrow LC^* \) by

\[
\overline{LD}^\pm_{\gamma}(s, u, v) = \gamma(s) + un(s) + vb(s) \pm \sqrt{u^2 + v^2 + 1}e_0.
\]

We also define \( LD_{\gamma} : I \times \mathbb{R}^2 \rightarrow LC^* \) by

\[
LD_{\gamma}(s, u, v) = \frac{u^2 + v^2 - 4}{4} \gamma(s) + un(s) + vb(s) + \frac{u^2 + v^2 + 4}{4}e_0.
\]

Then we have the following proposition.

**Proposition 4.2.1.** Under the above notations, we have the followings:

1. \( \gamma \) and \( \overline{LD}^\pm_{\gamma} \) are \( \Delta_3 \)-dual to each other.
2. \( \gamma \) and \( LD_{\gamma} \) are \( \Delta_4 \)-dual to each other.

**Proof.** Consider the mapping \( \mathcal{L}_3(s, u, v) = (\overline{LD}^\pm_{\gamma}(s, u, v), \gamma(s)) \). Then we have

\[
\langle \overline{LD}^\pm_{\gamma}(s, u, v), \gamma(s) \rangle = \langle \gamma(s), \gamma(s) \rangle = 1
\]

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\[ L^*_3 \theta_{42} = \langle LD^+_\gamma(s, u, v), \gamma'(s) \rangle ds = \langle LD^+\gamma(s, u, v), t(s) \rangle ds = 0. \]

The assertion (1) holds.

We also consider the mapping \( L^*_4(s, u, v) = (LD_\gamma(s, u, v), \gamma(s)) \). Since \( \langle \gamma(s), e_0 \rangle = -1 \) and \( \langle \gamma(s), \gamma(s) \rangle = 1 \), we have
\[
\langle LD_\gamma(s, u, v), \gamma(s) \rangle = (u^2 + v^2)/4 - 1 - ((u^2 + v^2)/4 + 1) = -2.
\]
Moreover, we have
\[
L^*_4 \theta_{42} = \langle LD_\gamma(s, u, v), \gamma'(s) \rangle ds = \langle LD_\gamma(s, u, v), t(s) \rangle ds = 0.
\]

This completes the proof. \( \square \)

We call each one of \( LD^\pm_\gamma \) the Lightcone dual hypersurface of the de Sitter spherical curve \( \gamma \) and \( LD_\gamma \) the Lightcone dual hypersurface of the lightlike spherical curve \( \gamma \). Then we have two mappings \( \pi \circ LD^\pm_\gamma : I \times \mathbb{R}^2 \to S^3_+ \) and \( \pi \circ LD_\gamma : I \times \mathbb{R}^2 \to S^3_+ \) defined by
\[
\pi \circ LD^\pm_\gamma(s, u, v) = \pm \left( \frac{1}{\sqrt{u^2 + v^2 + 1}} \gamma(s) + \frac{u}{\sqrt{u^2 + v^2 + 1}} n(s) + \frac{v}{\sqrt{u^2 + v^2 + 1}} b(s) \right) + e_0,
\]
\[
\pi \circ LD_\gamma(s, u, v) = \frac{u^2 + v^2 - 4}{u^2 + v^2 + 4} \gamma(s) + \frac{4u}{u^2 + v^2 + 4} n(s) + \frac{4v}{u^2 + v^2 + 4} b(s) + e_0.
\]

In this paper we consider the singularities of these dual surfaces and mappings. By the Frenet-Serret type formulae, we have
\[
\frac{\partial LD^\pm_\gamma}{\partial u}(s, u, v) = n(s) \pm \frac{u}{\sqrt{1 + u^2 + v^2}} e_0,
\]
\[
\frac{\partial LD^\pm_\gamma}{\partial v}(s, u, v) = b(s) \pm \frac{v}{\sqrt{1 + u^2 + v^2}} e_0,
\]
\[
\frac{\partial LD_\gamma}{\partial s}(s, u, v) = (1 - u \kappa(s))t(s) - v \tau(s)n(s) + u \tau(s)b(s),
\]
\[
\frac{\partial LD_\gamma}{\partial u}(s, u, v) = \frac{u}{2} \gamma(s) + n(s) + \frac{u}{2} e_0,
\]
\[
\frac{\partial LD_\gamma}{\partial v}(s, u, v) = \frac{v}{2} \gamma(s) + b(s) + \frac{v}{2} e_0,
\]
\[
\frac{\partial LD_\gamma}{\partial s}(s, u, v) = \frac{u^2 + v^2 - 4u \kappa(s) - 4}{4} t(s) - v \tau(s)n(s) + u \tau(s)b(s).
\]

Then we have the following proposition.
Proposition 4.2.2. Let \( \gamma : I \rightarrow S^3_+ \) be a unit speed curve. Then we have the followings:

1. \((s, u, v)\) is a singular point of \( \overline{LD}_\gamma \) if and only if \( u = 1/\kappa_g(s) \).
2. \((s, u, v)\) is a singular point of \( LD_\gamma \) if and only if \( v = \pm \sqrt{4 + 4u\kappa_g(s) - u^2} \).

**Proof.** By the above calculations, \( \{\partial \overline{LD}_\gamma / \partial u(s, u, v), \partial \overline{LD}_\gamma / \partial v(s, u, v), \partial \overline{LD}_\gamma / \partial s(s, u, v)\} \) is linearly dependent if and only if \( u = 1/\kappa_g(s) \). The assertion (1) follows. By the similar reason, we have the assertion (2). This completes the proof. \( \square \)

Therefore, the critical value sets of the above dual surfaces are given by

\[
C(\overline{LD}^\pm_\gamma) = \left\{ \overline{\gamma}(s) + \frac{1}{\kappa_g(s)} n(s) + v b(s) \pm \sqrt{1 + \frac{\kappa_g^2(s) + v^2 \kappa_g^2(s)}{\kappa_g^2(s)}} e_0 \mid v \in \mathbb{R}, s \in I, \kappa_g(s) \neq 0 \right\},
\]

\[
C(LD_\gamma)^\pm = \{ \kappa_g(s) \overline{\gamma}(s) + un(s) \pm \sqrt{4 + 4u\kappa_g(s) - u^2} b(s) + (\kappa_g(s)u + 2)e_0 \mid u \in \mathbb{R}, s \in I \}.
\]

We respectively denote that

\[
LF^\pm_\overline{\gamma}(s, v) = \overline{\gamma}(s) + \frac{1}{\kappa_g(s)} n(s) + v b(s) \pm \sqrt{1 + \frac{\kappa_g^2(s) + v^2 \kappa_g^2(s)}{\kappa_g^2(s)}} e_0,
\]

\[
LF^\pm_\overline{\gamma}(s, u) = \kappa_g(s) \overline{\gamma}(s) + un(s) \pm \sqrt{4 + 4u\kappa_g(s) - u^2} b(s) + (\kappa_g(s)u + 2)e_0,
\]

where we have the relation \( v = \pm \sqrt{4 + 4u\kappa_g(s) - u^2} \). We respectively call each one of \( LF^\pm_\overline{\gamma} \) the lightcone focal surface of the de Sitter spherical curve \( \overline{\gamma} \) and each one of \( LF^\pm_\gamma \) the lightcone focal surface of the lightcone spherical curve \( \gamma \). Then the projections of these surfaces to \( S^3_+ \) are given as follows:

\[
\pi(C(\overline{LD}_\gamma^\pm)) = \left\{ \frac{\pm(\kappa_g(s) \overline{\gamma}(s) + n(s) + \kappa_g(s)b(s))}{\sqrt{1 + \kappa_g^2(s) + v^2 \kappa_g^2(s)}} + e_0 \mid v \in \mathbb{R}, s \in I, \kappa_g(s) \neq 0 \right\},
\]

\[
\pi(C(LD_\gamma)^\pm) = \left\{ \frac{u \kappa_g(s) \overline{\gamma}(s) + un(s) \pm \sqrt{4 + 4u\kappa_g(s) - u^2} b(s)}{\kappa_g(s)u + 2} + e_0 \mid u \in \mathbb{R}, s \in I \right\}.
\]

On the other hand, we define \( \overline{\pi} = \Phi \circ \pi :LC^* \rightarrow S^3_0 \). By the previous calculations, \( \overline{\pi}(C(\overline{LD}_\gamma^\pm)) \) is different from \( \overline{\pi}(C(LD_\gamma)^\pm) \). In [10], it was shown that the projections of the critical value sets of the lightcone dual surfaces of \( \gamma \) and \( \overline{\gamma} \) are the same for a curve \( \gamma : I \rightarrow S^2_+ \). Moreover, it is equal to the spherical evolute of \( \overline{\gamma} \). Therefore, the situation for curves in \( S^3_+ \) is quite different from that for curves in \( S^2_+ \).
4.3 Lightcone height functions

In order to study the singularities of Lightcone dual surfaces of spherical curves, we introduce two families of functions and apply the theory of unfoldings. Let \( \gamma : I \rightarrow S^3_+ \) be a unit speed curve, then we define two families of functions as follows:

\[
\overline{H} : I \times LC^* \rightarrow \mathbb{R}, \quad \overline{H}(s, v) = \langle \overline{\gamma}(s), v \rangle - 1,
\]
\[
H : I \times LC^* \rightarrow \mathbb{R}, \quad H(s, v) = \langle \gamma(s), v \rangle + 2.
\]

We call \( \overline{H} \) a lightcone height function of the de Sitter spherical curve \( \overline{\gamma} \). For any fixed \( v \in LC^* \), we denote \( \overline{h}_v(s) = \overline{H}(s, v) \). We call \( H \) a lightcone height function of the lightlike spherical curve \( \gamma \). For any fixed \( v \in LC^* \), we denote \( h_v(s) = H(s, v) \). Then we have the following two propositions on \( h_v \) and \( \overline{h}_v \).

For simplification, we denote \( \rho(s) = \sqrt{(\kappa_g^2(s) + \kappa^2(s) + \kappa_g^2(s)/\kappa^2(s))} \) and \( \sigma^\pm(s) = (\kappa_g^2(s)\tau_g(s) \pm \sqrt{\kappa^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa^2(s)\tau_g^2(s)}/(\kappa^2(s) + \kappa_g^2(s)\tau_g^2(s))) \).

**Proposition 4.3.1.** Let \( \gamma : I \rightarrow S^3_+ \) be a unit speed curve, then we have the followings:

1. \( \overline{h}_v(s) = 0 \) if and only if there exist \( \lambda, \mu, \xi, \eta \in \mathbb{R} \) with \( \eta^2 = 1 + \lambda^2 + \mu^2 + \xi^2 \) such that \( v = \overline{\gamma}(s) + \lambda t(s) + \mu m(s) + \xi b(s) + \eta e_0 \).
2. \( \overline{h}_v(s) = \overline{h}_v(s) = 0 \) if and only if there exist \( \mu, \xi, \eta \in \mathbb{R} \) with \( \eta^2 = 1 + \mu^2 + \xi^2 \) such that \( v = \overline{\gamma}(s) + \mu m(s) + \xi b(s) + \eta e_0 = \overline{\gamma}(s) + \mu m(s) + \xi b(s) \pm \sqrt{1 + \mu^2 + \xi^2} e_0 \).
3. \( \overline{h}_v(s) = \overline{h}_v(s) = \overline{h}_v(s) = 0 \) if and only if \( \kappa_g(s) \neq 0 \) and
   \[
   v = \overline{\gamma}(s) + \frac{1}{\kappa_g(s)} n(s) + \xi b(s) \pm \sqrt{\frac{1 + \kappa^2_g(s) + \kappa^2_g(s)}{\kappa^2_g(s)}} e_0.
   \]
4. \( \overline{h}_v(s) = \overline{h}_v(s) = \overline{h}_v(s) = 0 \) if and only if \( \kappa_g(s) \neq 0, \tau_g(s) \neq 0 \) and
   \[
   v = \overline{\gamma}(s) + \frac{1}{\kappa_g(s)} n(s) - \frac{\kappa^2_g(s)}{\kappa^2_g(s)\tau_g(s)} b(s) \pm \rho(s)e_0.
   \]
5. \( \overline{h}_v(s) = \overline{h}_v(s) = \overline{h}_v(s) = \overline{h}_v(s) = 0 \) if and only if \( \kappa_g(s) \neq 0, \tau_g(s) \neq 0, \)
   \[
   \left( \left( -1 \kappa_g(s) \right) \tau_g(s) \right) - \frac{\tau_g(s)}{\kappa_g(s)} = 0
   \]
and $$v = \gamma(s) + \frac{1}{\kappa_g(s)} n(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s) \tau_g(s)} b(s) \pm \rho(s)e_0.$$  

(6) \(\overline{h}_v(s) = \overline{h}'_v(s) = \overline{h}''_v(s) = \overline{h}'''(s) = \overline{h}^{(s)}_v(s) = 0\) if and only if \(\kappa_g(s) \neq 0, \tau_g(s) \neq 0\),

$$\left(\frac{-1}{\kappa_g(s)} \tau_g(s)\right) - \frac{\tau_g(s)}{\kappa_g(s)} = \left\{\left(\frac{-1}{\kappa_g(s)} \tau_g(s)\right) - \frac{\tau_g(s)}{\kappa_g(s)}\right\}' = 0$$

and

$$v = \gamma(s) + \frac{1}{\kappa_g(s)} n(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s) \tau_g(s)} b(s) \pm \rho(s)e_0.$$  

Proof. (1) Since \(v \in LC^*\), there exist \(\omega, \lambda, \mu, \xi, \eta \in \mathbb{R}\) with \(\omega^2 + \lambda^2 + \mu^2 + \xi^2 - \eta^2 = 0\) such that \(v = \omega \gamma(s) + \lambda t(s) + \mu n(s) + \xi b(s) + \eta e_0\). From \(\overline{h}_v(s) = \langle \gamma(s), v \rangle - 1 = 0\), we have \(\omega = 1\). So

\[
\overline{h}_v(s) = \langle \gamma(s) + \lambda t(s) + \mu n(s) + \xi b(s) + \eta e_0 \rangle = \lambda = 0.
\]

It follows from the fact \(\eta^2 = 1 + \mu^2 + \xi^2\) that \(\eta = \pm \sqrt{1 + \mu^2 + \xi^2}\). Then we have \(v = \gamma(s) + \mu n(s) + \xi b(s) + \eta e_0 = \gamma(s) + \mu n(s) + \xi b(s) \pm \sqrt{1 + \mu^2 + \xi^2} e_0\).

(3) Since \(\overline{h}_v(s) = \langle \kappa_g(s) n(s) - \gamma(s), v \rangle, \overline{h}_v(s) = \overline{h}'_v(s) = \overline{h}''_v(s) = 0\) if and only if

\[
\overline{h}'_v(s) = \langle \kappa_g(s) n(s) - \gamma(s), \gamma(s) + \mu n(s) + \xi b(s) \pm \sqrt{1 + \mu^2 + \xi^2} e_0 \rangle = \kappa_g(s) \mu - 1 = 0.
\]

Then we have \(\kappa_g(s) \neq 0, \mu = 1/\kappa_g(s)\) and

$$v = \gamma(s) + n(s)/\kappa_g(s) + \xi b(s) \pm \sqrt{(1 + \kappa_g^2(s) + \kappa_g^2(s) \xi^2)/\kappa_g^2(s)} e_0.$$  

(4) Since \(\overline{h}'''_v(s) = \langle \kappa_g'(s) n(s) - (\kappa_g^2(s) + 1) t(s) + \kappa_g(s) \tau_g(s) b(s), v \rangle, \overline{h}_v(s) = \overline{h}'_v(s) = \overline{h}''_v(s) = \overline{h}'''_v(s) = 0\) if and only if

$$\overline{h}'''_v(s) = \langle \kappa_g'(s) n(s) - (\kappa_g^2(s) + 1) t(s) + \kappa_g(s) \tau_g(s) b(s), \gamma(s) + n(s)/\kappa_g(s) + \xi b(s) \pm \sqrt{(1 + \kappa_g^2(s) + \kappa_g^2(s) \xi^2)/\kappa_g^2(s)} e_0 \rangle$$

$$= \kappa_g'(s)/\kappa_g(s) + \kappa_g(s) \tau_g(s) \xi = 0.$$  

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Then we have \( \kappa_g(s) \neq 0, \tau_g(s) \neq 0, \xi = -\kappa'_g(s)/\kappa_g^2(s)\tau_g(s) \) and \( \boldsymbol{v} = \boldsymbol{\tau}(s) + \boldsymbol{n}(s)/\kappa_g(s) - \kappa'_g(s)\boldsymbol{b}(s)/\kappa_g^2(s)\tau_g(s) \pm \rho(s)\boldsymbol{e}_0 \).

(5) Since \( \overline{h}_v^{(4)}(s) = (\kappa''_g(s) - \kappa'_g(s) - \kappa_g(s) - \kappa_g(s)\tau^2_g(s))\boldsymbol{n}(s) - 3\kappa_g(s)\kappa'_g(s)\boldsymbol{t}(s) + (2\kappa'_g(s)\tau_g(s) + \kappa_g(s)\tau^2_g(s))\boldsymbol{b}(s) + (1 + \kappa_g^2(s))\overline{\boldsymbol{\tau}}(s, \boldsymbol{v}) \), \( \overline{h}_v(s) = \overline{h}_v'(s) = \overline{h}_v''(s) = \overline{h}_v'''(s) = 0 \) if and only if

\[
\overline{h}_v^{(4)}(s) = \left( (\kappa''_g(s) - \kappa'_g(s) - \kappa_g(s) - \kappa_g(s)\tau^2_g(s))\boldsymbol{n}(s) - 3\kappa_g(s)\kappa'_g(s)\boldsymbol{t}(s) + (2\kappa'_g(s)\tau_g(s) + \kappa_g(s)\tau^2_g(s))\boldsymbol{b}(s) + (1 + \kappa_g^2(s))\overline{\boldsymbol{\tau}}(s) \right) = 0.
\]

This is equivalent to the condition \((\kappa''_g(s)/\kappa_g(s))' - \tau_g(s)/\kappa_g(s) = 0\). Then we have \( \kappa_g(s) \neq 0, \tau_g(s) \neq 0, \) \( ((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = 0 \) and \( \boldsymbol{v} = \boldsymbol{\tau}(s) + \boldsymbol{n}(s)/\kappa_g(s) - \kappa'_g(s)\boldsymbol{b}(s)/\kappa_g^2(s)\tau_g(s) \pm \rho(s)\boldsymbol{e}_0 \).

(6) Since \( \overline{h}_v^{(5)}(s) = ((\kappa'_g(s) + 2\kappa_g^2(s) + \kappa_g^2(s)\tau^2_g(s) + 1 - 3\kappa_g^2(s) - 4\kappa_g(s)\kappa''_g(s))\boldsymbol{t}(s) + (\kappa''_g(s) - \kappa'_g(s) - 6\kappa_g^2(s)\kappa'_g(s) - 3\kappa'_g(s)\tau^2_g(s) - 3\kappa_g(s)\tau_g(s))\boldsymbol{b}(s) + (3\kappa'_g(s)\tau_g(s) + 3\kappa'_g(s)\tau^2_g(s) + \kappa_g(s)\tau'_g(s) - \kappa_g(s)\tau^2_g(s) - \kappa_g(s)\tau^2_g(s))\boldsymbol{n}(s) + 5\kappa_g(s)\kappa'_g(s)\overline{\boldsymbol{\tau}}(s, \boldsymbol{v}) \), \( \overline{h}_v(s) = \overline{h}_v'(s) = \overline{h}_v''(s) = \overline{h}_v'''(s) = \overline{h}_v^{(4)}(s) = 0 \) if and only if

\[
\overline{h}_v^{(5)}(s) = \left( (\kappa'_g(s) + 2\kappa_g^2(s) + \kappa_g^2(s)\tau^2_g(s) + 1 - 3\kappa_g^2(s) - 4\kappa_g(s)\kappa''_g(s))\boldsymbol{t}(s) + (\kappa''_g(s) - \kappa'_g(s) - 6\kappa_g^2(s)\kappa'_g(s) - 3\kappa'_g(s)\tau^2_g(s) - 3\kappa_g(s)\tau_g(s))\boldsymbol{n}(s) + 5\kappa_g(s)\kappa'_g(s)\overline{\boldsymbol{\tau}}(s) \right) = 0.
\]
This is equivalent to the condition \(((-(1/\kappa_g(s))^\prime/\tau_g(s))\prime-\tau_g(s)/\kappa_g(s))\prime = 0\). Then we have \(\kappa_g(s) \neq 0, \tau_g(s) \neq 0\), \(((-(1/\kappa_g(s))^\prime/\tau_g(s))\prime-\tau_g(s)/\kappa_g(s))\prime = 0\) and \(v = \bar{\eta}(s) + n(s)/\kappa_g(s) - \kappa^\prime(s)b(s)/\kappa_g^2(s)\tau_g(s) \pm \rho(s)e_0\). This completes the proof. \(\square\)

**Proposition 4.3.2.** Let \(\gamma : I \rightarrow S^3_+\) be a unit speed curve, then we have the followings:

1. \(h_v(s) = 0\) if and only if \(v = \lambda \bar{\eta}(s) + \mu t(s) + \xi n(s) + \eta b(s) + (\lambda + 2)e_0\), where \(\lambda, \mu, \xi, \eta \in \mathbb{R}\) and \(\mu^2 + \xi^2 + \eta^2 - 4\lambda - 4 = 0\).

2. \(h_v(s) = h_v^\prime(s) = 0\) if and only if \(v = ((\xi^2 + \eta^2)/4 - 1) \bar{\eta}(s) + \xi n(s) + \eta b(s) + ((\xi^2 + \eta^2)/4 + 1)e_0\).

3. \(h_v(s) = h_v^\prime(s) = h_v^\prime\prime(s) = 0\) if and only if

\[
v = \kappa_g(s)\xi \bar{\eta}(s) + \xi n(s) \pm \sqrt{4 + 4\kappa_g(s)\xi - \xi^2 b(s)} + (\kappa_g(s)\xi + 2)e_0.
\]

4. \(h_v(s) = h_v^\prime(s) = h_v^\prime\prime(s) = h_v^\prime\prime\prime(s) = 0\) if and only if \(\kappa_g^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0\) and

\[
v = 2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s)\bar{\eta}(s) + 2\kappa_g(s)\tau_g(s)\sigma^\pm(s)n(s) - 2\kappa_g^\prime(s)\sigma^\pm(s)b(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s) + 2)e_0.
\]

5. \(h_v(s) = h_v^\prime(s) = h_v^\prime\prime(s) = h_v^\prime\prime\prime(s) = 0\) if and only if \(\kappa_g^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0\),

\[
\left(\left(\frac{-1}{\kappa_g(s)}\right)^\prime\frac{1}{\tau_g(s)}\right)^\prime - \tau_g(s)/\kappa_g(s) = 0
\]

and

\[
v = 2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s)\bar{\eta}(s) + 2\kappa_g(s)\tau_g(s)\sigma^\pm(s)n(s) - 2\kappa_g^\prime(s)\sigma^\pm(s)b(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s) + 2)e_0.
\]

6. \(h_v(s) = h_v^\prime(s) = h_v^\prime\prime(s) = h_v^\prime\prime\prime(s) = h_v^\prime\prime\prime\prime(s) = 0\) if and only if \(\kappa_g^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0\),

\[
\left(\left(\frac{-1}{\kappa_g(s)}\right)^\prime\frac{1}{\tau_g(s)}\right)^\prime - \tau_g(s)/\kappa_g(s) = \left\{\left(\left(\frac{-1}{\kappa_g(s)}\right)^\prime\frac{1}{\tau_g(s)}\right)^\prime - \tau_g(s)/\kappa_g(s)\right\}^\prime = 0
\]

and

\[
v = 2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s)\bar{\eta}(s) + 2\kappa_g(s)\tau_g(s)\sigma^\pm(s)n(s) - 2\kappa_g^\prime(s)\sigma^\pm(s)b(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^\pm(s) + 2)e_0.
\]

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Proof. (1) Since $\mathbf{v} \in LC^*$, there exist $\lambda, \mu, \xi, \eta, \omega \in \mathbb{R}$ with $\lambda^2 + \mu^2 + \xi^2 + \eta^2 - \omega^2 = 0$ such that $\mathbf{v} = \lambda \mathbf{g}(s) + \mu t(s) + \xi \mathbf{n}(s) + \eta \mathbf{b}(s) + \omega e_0$. From $h_\mathbf{v}(s) = \langle \mathbf{g}(s), \mathbf{v} \rangle + 2 = \langle \mathbf{g}(s) + e_0, \lambda \mathbf{g}(s) + \mu t(s) + \xi \mathbf{n}(s) + \eta \mathbf{b}(s) + \omega e_0 \rangle + 2 = \lambda - \omega + 2 = 0$, we have $\omega = 2 + \lambda$. So $\mathbf{v} = \lambda \mathbf{g}(s) + \mu t(s) + \xi \mathbf{n}(s) + \eta \mathbf{b}(s) + (2 + \lambda)e_0$ and $\lambda^2 + \mu^2 + \xi^2 + \eta^2 - (2 + \lambda)^2 = \mu^2 + \xi^2 + \eta^2 - 4\lambda - 4 = 0$. The converse direction also holds.

(2) Since $h'_\mathbf{v}(s) = \langle t(s), \mathbf{v} \rangle$, $h_\mathbf{v}(s) = h'_\mathbf{v}(s) = 0$ if and only if

$$h'_\mathbf{v}(s) = \langle t(s), \lambda \mathbf{g}(s) + \mu t(s) + \xi \mathbf{n}(s) + \eta \mathbf{b}(s) + (2 + \lambda)e_0 \rangle = \mu = 0.$$ 

By $\lambda^2 + \xi^2 + \eta^2 - (2 + \lambda)^2 = \xi^2 + \eta^2 - 4\lambda - 4 = 0$, we have $\lambda = (\xi^2 + \eta^2)/4 - 1$. So, $\mathbf{v} = ((\xi^2 + \eta^2)/4 - 1)\mathbf{g}(s) + \xi \mathbf{n}(s) + \eta \mathbf{b}(s) + ((\xi^2 + \eta^2)/4 + 1)e_0$.

(3) Since $h''_\mathbf{v}(s) = \langle \kappa_g(s)n(s) - \mathbf{g}(s), \mathbf{v} \rangle$, $h_\mathbf{v}(s) = h'_\mathbf{v}(s) = h''_\mathbf{v}(s) = 0$ if and only if

$$h''_\mathbf{v}(s) = \left(\kappa_g(s)n(s) - \mathbf{g}(s), \left(\frac{\xi^2 + \eta^2}{4} - 1\right)\mathbf{g}(s) + \xi \mathbf{n}(s) + \eta \mathbf{b}(s) + \left(\frac{\xi^2 + \eta^2}{4} + 1\right)e_0\right)$$

so that we have $\eta = \pm \sqrt{4 + 4\kappa_g(s)\xi - \xi^2}$ and $\mathbf{v} = \kappa_g(s)\mathbf{g}(s) + \xi \mathbf{n}(s) \pm \sqrt{4 + 4\kappa_g(s)\xi - \xi^2} \mathbf{b}(s) + (\kappa_g(s)\xi + 2)e_0$.

(4) Since $h'''_\mathbf{v}(s) = \langle \kappa'_g(s)n(s) - (\kappa^2_g(s) + 1)t(s) + \kappa_g(s)\tau_g(s)b(s), \mathbf{v} \rangle$, $h_\mathbf{v}(s) = h'_\mathbf{v}(s) = h''_\mathbf{v}(s) = h'''_\mathbf{v}(s) = 0$ if and only if

$$h'''_\mathbf{v}(s) = \left(\kappa'_g(s)n(s) - (\kappa^2_g(s) + 1)t(s) + \kappa_g(s)\tau_g(s)b(s), \kappa_g(s)\xi \mathbf{g}(s) + \xi \mathbf{n}(s) \pm \sqrt{4 + 4\kappa_g(s)\xi - \xi^2 \mathbf{b}(s) + (\kappa_g(s)\xi + 2)e_0}\right)$$

so that we have $\kappa_g^2(s) + \kappa^4_g(s)\tau^2_g(s) \neq 0$, $\xi = 2\kappa_g(s)\tau_g(s)\sigma^\pm(s)$ and $\mathbf{v} = 2\kappa^2_g(s)\tau_g(s)\sigma^\pm(s)\mathbf{g}(s) + 2\kappa_g(s)\tau_g(s)\sigma^\pm(s)n(s) - 2\kappa_g(s)\sigma^\pm(s)b(s) + (2\kappa^2_g(s)\tau_g(s)\sigma^\pm(s) + 2)e_0$.

(5) Since $h''_4(s) = \langle (\kappa''_g(s) - \kappa^3_g(s) - \kappa_g(s)\tau^2_g(s))n(s) - 3\kappa_g(s)\kappa'_g(s)t(s) + (2\kappa'_g(s)\tau_g(s) + \kappa_g(s)\tau^2_g(s))b(s) + (1 + \kappa^2_g(s))\mathbf{g}(s), \mathbf{v} \rangle$, $h_\mathbf{v}(s) = h'_\mathbf{v}(s) = h''_\mathbf{v}(s) = h'''_\mathbf{v}(s) = h''''_\mathbf{v}(s) = 0$ if and only if

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\[ h^{(4)}_u(s) = \left( (\kappa''_g(s) - \kappa^2_g(s) - \kappa_g(s)\tau^2_g(s))n(s) - 3\kappa_g(s)\kappa'_g(s)t(s) \right. \\
+ (2\kappa'_g(s)\tau_g(s) + \kappa_g(s)\tau'_g(s))b(s) + (1 + \kappa'_g(s))\gamma(s), \\
\kappa_g(s)\xi\gamma(s) + \xi n(s) + \eta b(s) + (\kappa_g(s)\xi + 2)e_0 \bigg) = \left( (\kappa''_g(s) - \kappa_g(s)\tau^2_g(s)) + (2\kappa'_g(s)\tau_g(s) + \kappa_g(s)\tau'_g(s))\eta = 0. \right. \]

By the above condition, we have the equation \((\kappa''_g(s) - \kappa_g(s)\tau^2_g(s))/(2\kappa'_g(s)\tau_g(s) + \kappa_g(s)\tau'_g(s)) = \kappa'_g(s)/\kappa_g(s)\tau_g(s). \) It is equivalent to \(((1/\kappa_g(s))'/\tau_g(s)' - \tau_g(s)/\kappa_g(s) = 0. \) Then we have \( \kappa''_g(s) + \kappa'_g(s)\tau^2_g(s) \neq 0, \) \((1/\kappa_g(s))'/\tau_g(s)' - \tau_g(s)/\kappa_g(s) = 0 \) and \( \nu = 2\kappa'_g(s)\tau_g(s)\sigma^\pm(s)\gamma(s) + 2\kappa_g(s)\sigma^\pm(s)\nu = h^{(4)}_u(s) = h^{(5)}_u(s) = h^{(6)}_u(s) = h^{(7)}_u(s) = 0 \) if and only if

\[ h^{(5)}_u(s) = \left( (\kappa'_g(s) + 2\kappa'_g(s) + \kappa_g(s)\tau^2_g(s) + 1 - 3\kappa'_g(s) - 4\kappa_g(s)\kappa''_g(s))t(s) \right. \\
+ (6\kappa'_g(s) - \kappa'_g(s) - 6\kappa'_g(s)\kappa'_g(s) - 3\kappa'_g(s)\tau^2_g(s) - 3\kappa'_g(s)\tau_g(s)\tau'_g(s))n(s) + (3\kappa'_g(s)\tau_g(s) + 3\kappa'_g(s)\tau'_g(s) + \kappa_g(s)\tau''_g(s) - \kappa_g(s)\tau_g(s) - \kappa'_g(s)\tau_g(s) - \kappa'_g(s)\tau^2_g(s) + \kappa_g(s)\kappa'_g(s)\tau_g(s) + 5\kappa_g(s)\kappa'_g(s)\gamma(s), \right. \\
\nu \right) = h^{(5)}_u(s) = h^{(6)}_u(s) = h^{(7)}_u(s) = 0. \]

By the above condition, we have the equation \((\kappa''_g(s) - \kappa'_g(s) - \kappa^2_g(s)\kappa'_g(s) - 3\kappa'_g(s)\tau^2_g(s) - 3\kappa'_g(s)\tau_g(s)\tau'_g(s))/3\kappa'_g(s)\tau_g(s) + 3\kappa'_g(s)\tau'_g(s) + \kappa_g(s)\tau''_g(s) - \kappa_g(s)\tau_g(s) - \kappa'_g(s)\tau_g(s) - \kappa'_g(s)\tau^2_g(s) - \kappa_g(s)\tau_g(s)\tau'_g(s)\) = \( \kappa'_g(s)/\kappa_g(s)\tau_g(s). \) It is equivalent to \(((1/\kappa_g(s))'/\tau_g(s)' - \tau_g(s)/\kappa_g(s) = 0 \) and \( \nu = 2\kappa'_g(s)\tau_g(s)\sigma^\pm(s)\gamma(s) + 2\kappa_g(s)\tau_g(s)\sigma^\pm(s)n(s) - 2\kappa'_g(s)\sigma^\pm(s)b(s) + (2\kappa'_g(s)\tau_g(s)\sigma^\pm(s) + 2)e_0. \) This completes the proof. \( \square \)
According to the assertions of Propositions 4.3.1 and 4.3.2, we define an invariant
\[ \kappa_S(s) = \left( \left( \frac{-1}{\kappa_g(s)} \right) \frac{1}{\tau_g(s)} \right)' - \frac{\tau_g(s)}{\kappa_g(s)}, \]
which we call a **spherical curvature** of \( \overline{\gamma} \). We have the following proposition.

**Proposition 4.3.3.** For a unit speed curve \( \gamma : I \to S^2_+ \), both of the spherical evolutes \( \varepsilon^\pm_\gamma(s) \) are constant if and only if \( \kappa_S \equiv 0 \).

**Proof.** \( \varepsilon^\pm_\gamma(s) = \pm (\kappa_g(s)\kappa'_g(s)(2\kappa^2_g(s)\tau_g(s)+\kappa_g(s)\kappa'_g(s)\tau'_g(s)+\kappa^2_g(s)\tau^3_g(s)-\kappa_g(s)\kappa''_g(s)\tau_g(s))\{\overline{\tau}(s)+\frac{\mathbf{n}(s)}{\kappa_g(s)}+(\frac{1}{\kappa_g(s)})'\mathbf{b}(s)/\tau_g(s)\} / (\kappa^2_g(s)+\kappa'_g(s)\tau^2_g(s)+\kappa^2_g(s)\tau^2_g(s))^{3/2}+(2\kappa^2_g(s)\tau_g(s)+\kappa_g(s)\kappa'_g(s)\tau'_g(s)+\kappa^2_g(s)\tau^3_g(s)-\kappa_g(s)\kappa''_g(s)\tau_g(s))\mathbf{b}(s)/\tau_g(s)\tau_g(s)(\kappa^2_g(s)+\kappa'_g(s)\tau^2_g(s)+\kappa^2_g(s)\tau^2_g(s))^{1/2} \).

On the other hand \( \kappa_S(s) = (-1/\kappa_g(s))'\tau_g(s)-\tau_g(s)/\kappa_g(s) = (\kappa_g(s)\kappa'_g(s)\tau_g(s)-2\kappa^2_g(s)\tau_g(s)-\kappa_g(s)\kappa'_g(s)\tau'_g(s)-\kappa^2_g(s)\tau^3_g(s))/\kappa^2_g(s)\tau^2_g(s) = 0 \). So \( \varepsilon^\pm_\gamma(s) \equiv 0 \) if and only if \( \kappa_S \equiv 0 \). This completes the proof.

### 4.4 Singularity of lightcone duals of spherical curves

In this section we classify the singularities of \( LD^\pm_\gamma \) and \( LD_\gamma \) as an application of the unfolding theory of functions. Let \( F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R} \) be a function germ, we call \( F \) an **\( r \)-parameter unfolding** of \( f \), where \( f(s) = F_{x_0}(s, x_0) \). The **discriminant set** of \( F \) is defined by
\[ D_F = \left\{ x \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, \ F(s, x) = \frac{\partial F}{\partial s}(s, x) = 0 \right\}. \]

By Propositions 4.3.1, (2) and 4.3.2, (2), the discriminant sets of \( \overline{H} \) and \( H \) are
\[ D_{\overline{H}} = \{ \overline{\tau}(s) + u\mathbf{n}(s) + v\mathbf{b}(s) \pm \sqrt{u^2 + v^2 + 1}e_0 \mid s \in I, u, v \in \mathbb{R} \}, \]
\[ D_H = \{ (u^2 + v^2 - 4)\overline{\tau}(s)/4 + u\mathbf{n}(s) + v\mathbf{b}(s) + (u^2 + v^2 + 4)e_0/4 \mid s \in I, u, v \in \mathbb{R} \}. \]
These are the lightcone dual surfaces of \( \overline{\gamma} \) and the lightcone dual surface of \( \gamma \) respectively. Moreover, the both assertions (4) of Propositions 4.3.1 and 4.3.2 describe the singularities of the lightcone focal surfaces of \( \gamma \) and \( \overline{\gamma} \) respectively.
Proposition 4.4.1. The critical value sets of \( LF_+_7 \) and \( LF^-_7 \) are given as follows:

\[
C(LF_+_7) = \left\{ \varpi(s) + \frac{1}{\kappa_g(s)} \mathbf{n}(s) - \frac{\kappa'_g(s)}{\kappa^2_g(s)\tau_g(s)} \mathbf{b}(s) \pm \rho(s) e_0 \mid s \in I \right\},
\]

\[
C(LF^-_7) = \left\{ 2\kappa_g^2(s)\tau_g(s)\sigma^+(s)\varpi(s) + 2\kappa_g(s)\tau_g(s)\sigma^+(s)\mathbf{n}(s) - 2\kappa'_g(s)\sigma^+(s)\mathbf{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^+(s) + 2)\mathbf{e}_0 \mid s \in I \right\}.
\]

Then we have the following theorem as a corollary.

Theorem 4.4.2. Both of the projections of the critical value sets \( C(LF_+_7) \) and \( C(LF^-_7) \) in the unit 3-sphere \( S^3 \) are the images of the spherical evolutes of \( \varpi \), that is

\[
\pi(C(LF_+_7)) = \pi(C(LF^-_7)) = \{ \varepsilon^+_7(s) \mid s \in I \}.
\]

Proof. We know that

\[
\pi(C(LF_+_7)) = \left\{ \pm \left( \frac{\varpi(s)}{\rho(s)} + \frac{\mathbf{n}(s)}{\rho(s)\kappa_g(s)} - \frac{\kappa'_g(s)\mathbf{b}(s)}{\rho(s)\kappa^2_g(s)\tau_g(s)} \right) \mid s \in I \right\}
\]

and

\[
\pi(C(LF^-_7)) = \left\{ \frac{\kappa^2_g(s)\tau_g(s)\sigma^+(s)\varpi(s) + \kappa_g(s)\tau_g(s)\sigma^+(s)\mathbf{n}(s) - \kappa'_g(s)\sigma^+(s)\mathbf{b}(s)}{\kappa^2_g(s)\tau_g(s)\sigma^+(s) + 1} \mid s \in I \right\}.
\]

By straightforward calculations, we have

\[
\frac{\kappa^2_g(s)\tau_g(s)\sigma^+(s)}{\kappa^2_g(s)\tau_g(s)\sigma^+(s) + 1} = \frac{\kappa^2_g(s)\tau_g(s)(\kappa^2_g(s)\tau_g(s) \pm \sqrt{\kappa^2_g(s) + \kappa^2_g(s)\tau^2_g(s) + \kappa^4_g(s)\tau^2_g(s)})}{\kappa^2_g(s) + \kappa^2_g(s)\tau^2_g(s) + \kappa^4_g(s)\tau^2_g(s) + \kappa^2_g(s)\tau^2_g(s) + \kappa^4_g(s)\tau^2_g(s)} = \pm 1/\rho(s).
\]

Similarly, we can calculate that

\[
\frac{\kappa_g(s)\tau_g(s)\sigma^+(s)}{\kappa^2_g(s)\tau_g(s)\sigma^+(s) + 1} = \pm 1/\rho(s)\kappa_g(s),
\]

\[
\frac{\kappa'_g(s)\sigma^+(s)}{\kappa^2_g(s)\tau_g(s)\sigma^+(s) + 1} = \pm \kappa'_g(s)/\rho(s)\kappa^2_g(s)\tau_g(s).
\]
So we have
\[ \pi(C(LF^±_γ)) = \pi(C(LF^±_γ)) = \{ e^±_γ(s) \mid s \in I \}. \]

In order to understand the geometric properties of the discriminant set of order \( \ell \), we introduce an equivalence relation among the unfoldings of functions. This completes the proof. 

Inspired by Propositions 4.3.1, 4.3.2 and Theorem 4.4.2, we define the following set:
\[ D^\ell_F = \left\{ x \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, F(s, x) = \frac{\partial F}{\partial s}(s, x) = \cdots = \frac{\partial^\ell F}{\partial s^\ell}(s, x) = 0 \right\}, \]
which is called a discriminant set of order \( \ell \). Of course, \( D^1_F = D_F \).

Let \( F \) and \( G \) be \( r \)-parameter unfoldings of \( f(s) \) and \( g(s) \), respectively. We say that \( F \) and \( G \) are \( P-R \)-equivalent if there exists a diffeomorphism germ \( \Phi : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow (\mathbb{R} \times \mathbb{R}^r, (s'_0, x'_0)) \) of the form \( \Phi(s, x) = (\Phi_1(s, x), \phi(x)) \) such that \( G \circ \Phi = F \). By straightforward calculations, we have the following proposition.

Proposition 4.4.3. Let \( F \) and \( G \) be \( r \)-parameter unfoldings of \( f(s) \) and \( g(s) \), respectively. If \( F \) and \( G \) are \( P-R \)-equivalent by a diffeomorphism germ \( \Phi : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow (\mathbb{R} \times \mathbb{R}^r, (s'_0, x'_0)) \) of the form \( \Phi(s, x) = (\Phi_1(s, x), \phi(x)) \), then \( \phi(D^\ell_F) = D^\ell_G \) as set germs.

By Propositions 4.3.1 and 4.3.2, we have the following proposition.

Proposition 4.4.4. Under the same notations as in the previous paragraphs, we have
\[
D^\ell_F = D^1_F = \text{Images } LF^±_γ, \quad D^2_F = \text{Images } LF^±_γ, \quad \pi(D^2_F) = \text{Images } e^±_γ, \\
D^\ell_H = D^1_H = \text{Image } LD^±_γ, \quad D^2_H = \text{Images } LF^±_γ, \quad \pi(D^2_H) = \text{Images } e^±_γ.
\]

For a function \( f(s) \), we say that \( f \) has \( A_k \)-singularity at \( s_0 \) if \( f^{(p)}(s_0) = 0 \) for all \( 1 \leq p \leq k \) and \( f^{(k+1)}(s_0) \neq 0 \). Let \( F \) be an \( r \)-parameter unfolding of \( f \) and \( f \) has \( A_k \)-singularity \( (k \geq 1) \) at \( s_0 \). We denote the \((k-1)\)-jet of the partial derivative \( \partial F/\partial x_i \) at \( s_0 \) as
\[
\left( \frac{\partial F}{\partial x_i}(s, x_0) \right)(s_0) = \sum_{j=1}^{k-1} \alpha_{ji}(s - s_0)^j, \quad (i = 1, \cdots, r).
\]
If the rank of \( k \times r \) matrix \((\alpha_{0i}, \alpha_{ji})\) is \( k \) \((k \leq r)\), then \( F \) is called a versal unfolding of \( f \), where \( \alpha_{0i} = \partial F/\partial x_i(s_0, x_0) \). We have the following classification theorem of versal unfoldings [3, Page 149, 6.6].
**Theorem 4.4.5.** Let \( F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R} \) be an \( r \)-parameter unfolding of \( f \) which has \( A_k \)-singularity at \( s_0 \). Suppose \( F \) is a versal unfolding of \( f \), then \( F \) is \( P,R \)-equivalent to one of the following unfoldings:

(a) \( k = 1 \), \( s^2 + x_1 \),
(b) \( k = 2 \), \( s^3 + x_1 + sx_2 \),
(c) \( k = 3 \), \( s^4 + x_1 + sx_2 + s^2x_3 \),
(d) \( k = 4 \), \( s^5 + x_1 + sx_2 + s^2x_3 + s^3x_4 \).

We have the following classification result as a corollary of the above theorem.

**Corollary 4.4.6.** Let \( F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R} \) be an \( r \)-parameter unfolding of \( f \) which has \( A_k \)-singularity at \( s_0 \). Suppose \( F \) is a versal unfolding of \( f \), then we have the following assertions:

(a) If \( k = 1 \), then \( D_F \) is diffeomorphic to \( \{0\} \times \mathbb{R}^{r-1} \) and \( D_F^2 = \emptyset \).
(b) If \( k = 2 \), then \( D_F \) is diffeomorphic to \( C(2,3) \times \mathbb{R}^{r-2} \), \( D_F^2 \) is diffeomorphic to \( \{0\} \times \mathbb{R}^{r-2} \) and \( D_F^3 = \emptyset \).
(c) If \( k = 3 \), then \( D_F \) is diffeomorphic to \( SW \times \mathbb{R}^{r-3} \), \( D_F^2 \) is diffeomorphic to \( C(2,3,4) \times \mathbb{R}^{r-3} \), \( D_F^3 \) is diffeomorphic to \( \{0\} \times \mathbb{R}^{r-3} \) and \( D_F^4 = \emptyset \).
(d) If \( k = 4 \), then \( D_F \) is locally diffeomorphic to \( BF \times \mathbb{R}^{r-4} \), \( D_F^2 \) is diffeomorphic to \( C(BF) \times \mathbb{R}^{r-4} \), \( D_F^3 \) is diffeomorphic to \( C(2,3,4,5) \times \mathbb{R}^{r-4} \), \( D_F^4 \) is diffeomorphic to \( \{0\} \times \mathbb{R}^{r-4} \) and \( D_F^5 = \emptyset \).

We remark that all of diffeomorphisms in the above assertions are diffeomorphism germs.

Here, we respectively call \( C(2,3) = \{(x_1, x_2) \mid x_1 = u^2, x_2 = u^3\} \) a \( (2,3)-\text{cusp} \), \( C(2,3,4) = \{(x_1, x_2, x_3) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4\} \) a \( (2,3,4)-\text{cusp} \), \( C(2,3,4,5) = \{(x_1, x_2, x_3, x_4) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4, x_4 = u^5\} \) a \( (2,3,4,5)-\text{cusp} \), \( SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\} \) a swallow tail, \( BF = \{(x_1, x_2, x_3, x_4) \mid x_1 = 5u^4 + 3vu^2 + 2wu_2, x_2 = 4u^5 + 2vu^3 + wu^2, x_3 = u, x_4 = v\} \) a butterfly and \( C(BF) = \{(x_1, x_2, x_3, x_4) \mid x_1 = 6u^5 + u^3v, x_2 = 25u^4 + 9u^2v, x_3 = 10u^3 + 3uv, x_4 = v\} \) a c-butterfly (i.e., the critical value set of the butterfly).

We have the following key propositions on \( H \) and \( \overline{H} \).

**Proposition 4.4.7.** If \( \overline{h_{s_0}} \) has \( A_k \)-singularity \((k = 1, 2, 3, 4)\) at \( s_0 \), then \( \overline{H} \) is a versal unfolding of \( \overline{h_{s_0}} \).

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Proposition 4.4.8. For $v \in LC^*$, we have $v = (\pm(v_1^2 + v_2^2 + v_3^2 + v_4^2)^{1/2}, v_1, v_2, v_3, v_4)$. We denote that $\tilde{\gamma}(s) = (0, x_1(s), x_2(s), x_3(s)x_4(s))$. Then

$$\overline{H}(s, v) = \langle \tilde{\gamma}(s), v \rangle - 1 = x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 + x_4(s)v_4 - 1.$$  

Thus we have

$$\frac{\partial \overline{H}}{\partial v_1}(s, v) = x_1(s), \frac{\partial \overline{H}}{\partial v_2}(s, v) = x_2(s), \frac{\partial \overline{H}}{\partial v_3}(s, v) = x_3(s), \frac{\partial \overline{H}}{\partial v_4}(s, v) = x_4(s),$$

$$\frac{\partial^2 \overline{H}}{\partial s \partial v_1}(s, v) = x'_1(s), \frac{\partial^2 \overline{H}}{\partial s \partial v_2}(s, v) = x'_2(s), \frac{\partial^2 \overline{H}}{\partial s \partial v_3}(s, v) = x'_3(s), \frac{\partial^2 \overline{H}}{\partial s \partial v_4}(s, v) = x'_4(s),$$

$$\frac{\partial^3 \overline{H}}{\partial s^2 \partial v_1}(s, v) = x''_1(s), \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_2}(s, v) = x''_2(s), \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_3}(s, v) = x''_3(s), \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_4}(s, v) = x''_4(s),$$

$$\frac{\partial^4 \overline{H}}{\partial s^3 \partial v_1}(s, v) = x'''_1(s), \frac{\partial^4 \overline{H}}{\partial s^3 \partial v_2}(s, v) = x'''_2(s), \frac{\partial^4 \overline{H}}{\partial s^3 \partial v_3}(s, v) = x'''_3(s), \frac{\partial^4 \overline{H}}{\partial s^3 \partial v_4}(s, v) = x'''_4(s).$$

For a fixed point $v_0 = (v_{00}, v_{01}, v_{02}, v_{03}, v_{04})$, the 3-jet of $\partial \overline{H}/\partial v_i(s, v_0)(i = 1, 2, 3, 4)$ at $s_0$ is

$$j^3(\partial \overline{H}/\partial v_i)(s, v_0)(s_0) = x'_i(s_0)(s - s_0) + x''_i(s_0)(s - s_0)^2/2 + x'''_i(s_0)(s - s_0)^3/6, \ (i = 1, 2, 3, 4).$$

It is enough to show that the rank of the matrix $A$ is 4, where

$$A = \begin{pmatrix} x_1(s_0) & x_2(s_0) & x_3(s_0) & x_4(s_0) \\ x'_1(s_0) & x'_2(s_0) & x'_3(s_0) & x'_4(s_0) \\ x''_1(s_0) & x''_2(s_0) & x''_3(s_0) & x''_4(s_0) \\ x'''_1(s_0) & x'''_2(s_0) & x'''_3(s_0) & x'''_4(s_0) \end{pmatrix}. $$

Then we have

$$\det A = \langle e_0 \wedge \tilde{\gamma}(s_0) \wedge \tilde{\gamma}'(s_0) \wedge \tilde{\gamma}''(s_0), \tilde{\gamma}'''(s_0) \rangle = -\kappa^2(s_0)\tau(s_0) \neq 0.$$ 

So the rank of $A$ is 4. This completes the proof. 

Proposition 4.4.8. If $h_{v_0}$ has $A_k$-singularity ($k = 1, 2, 3, 4$) at $s_0$, then $H$ is a versal unfolding of $h_{v_0}$.
Proof. For \( v \in LC^* \), we have \( v = (v_0, v_1, v_2, v_3, v_4) = (\pm (v_1^2 + v_2^2 + v_3^2 + v_4^2)^{1/2}, v_1, v_2, v_3, v_4) \). We denote that \( \Psi(s) = (1, x_1(s), x_2(s), x_3(s)x_4(s)) \). Then we have
\[
H(s, v) = (\gamma(s), v) + 2 = \mp(v_1^2 + v_2^2 + v_3^2 + v_4^2)^{1/2} + x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 + x_4(s)v_4 + 2.
\]
Thus we have
\[
\frac{\partial H}{\partial v_1}(s, v) = -v_1/v_0 + x_1(s), \quad \frac{\partial H}{\partial v_2}(s, v) = -v_2/v_0 + x_2(s),
\]
\[
\frac{\partial H}{\partial v_3}(s, v) = -v_3/v_0 + x_3(s), \quad \frac{\partial H}{\partial v_4}(s, v) = -v_4/v_0 + x_4(s),
\]
\[
\frac{\partial^2 H}{\partial s\partial v_1}(s, v) = x_1'(s), \quad \frac{\partial^2 H}{\partial s\partial v_2}(s, v) = x_2'(s), \quad \frac{\partial^2 H}{\partial s\partial v_3}(s, v) = x_3'(s), \quad \frac{\partial^2 H}{\partial s\partial v_4}(s, v) = x_4'(s),
\]
\[
\frac{\partial^3 H}{\partial s^2\partial v_1}(s, v) = x_1''(s), \quad \frac{\partial^3 H}{\partial s^2\partial v_2}(s, v) = x_2''(s), \quad \frac{\partial^3 H}{\partial s^2\partial v_3}(s, v) = x_3''(s), \quad \frac{\partial^3 H}{\partial s^2\partial v_4}(s, v) = x_4''(s),
\]
\[
\frac{\partial^4 H}{\partial s^3\partial v_1}(s, v) = x_1'''(s), \quad \frac{\partial^4 H}{\partial s^3\partial v_2}(s, v) = x_2'''(s), \quad \frac{\partial^4 H}{\partial s^3\partial v_3}(s, v) = x_3'''(s), \quad \frac{\partial^4 H}{\partial s^3\partial v_4}(s, v) = x_4'''(s).
\]
For a fixed \( v_0 = (v_{00}, v_{01}, v_{02}, v_{03}, v_{04}) \), the 3-jet of \( \partial H/\partial v_i(s, v_0) \)(\( i = 1, 2, 3, 4 \)) at \( s_0 \) is
\[
j^{(3)}\frac{\partial H}{\partial v_i}(s, v_0)(s_0) = x_i'(s_0)(s - s_0) + x_i''(s_0)(s - s_0)^2/2 + x_i'''(s_0)(s - s_0)^3/6, \quad (i = 1, 2, 3, 4).
\]
It is enough to show that the rank of the matrix \( B \) is 4, where
\[
B = \begin{pmatrix}
-v_01/v_{00} + x_1(s_0) & -v_02/v_{00} + x_2(s_0) & -v_03/v_{00} + x_3(s_0) & -v_04/v_{00} + x_4(s_0) \\
x_1'(s_0) & x_2'(s_0) & x_3'(s_0) & x_4'(s_0) \\
x_1''(s_0) & x_2''(s_0) & x_3''(s_0) & x_4''(s_0) \\
x_1'''(s_0) & x_2'''(s_0) & x_3'''(s_0) & x_4'''(s_0)
\end{pmatrix}.
\]
By straightforward calculations, we have
\[
\det B = \langle e_0 \wedge \Psi'(s_0) \wedge \Psi''(s_0) \wedge \Psi'''(s_0), v_0 \rangle/v_{00} + \langle e_0 \wedge \Psi'(s_0) \wedge \Psi''(s_0) \wedge \Psi'''(s_0), \Psi''(s_0) \rangle
\]
\[
= \langle \kappa_g(s_0)^2 \tau_g(s_0) \Psi(s_0), v_0 \rangle/v_{00} - \langle \kappa_g(s_0)^2 b(s_0), v_0 \rangle/v_{00} + \langle \kappa_g(s_0) \tau_g(s_0) n(s_0), v_0 \rangle/v_{00} - \kappa_g(s_0)^2 \tau_g(s_0).
\]
In this case, \( h_{v_0}(s) \) has \( A_4 \)-singularity, then we have
\[
v_0 = 2\kappa_g(s_0)^2 \tau_g(s_0) \sigma^+(s_0) \Psi(s_0) + 2\kappa_g(s_0)^2 \tau_g(s_0) \sigma^+(s_0) n(s_0) - 2\kappa_g(s_0)^2 \sigma^+(s_0) b(s_0)
\]
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Moreover we have
\[ v_{00} = 2\kappa_g^2(s_0)\tau_g(s_0)\sigma^\pm(s_0) + 2e_0. \]

Therefore by calculation, we have
\[
\det B = \pm \frac{\kappa_g^2(s_0)\tau_g^2(s_0) + \kappa_g^4(s_0)}{\sqrt{\kappa_g^2(s_0) + \kappa_g^2(s_0)\tau_g^2(s_0) + \kappa_g^4(s_0)\tau_g^2(s_0) \pm \kappa_g^2(s_0)\tau_g(s_0)}} \neq 0
\]
So the rank of \( B \) is 4. This completes the proof.

We have the following theorem:

**Theorem 4.4.9.** Let \( \gamma: I \rightarrow S^3_+ \) be a unit speed curve.

(A) For each one of the lightcone duals \( \overline{LD}_\gamma^\pm \) of \( \gamma \), we have the following assertions:

1. Each one of the lightcone duals \( \overline{LD}_\gamma^\pm \) of \( \gamma \) is locally diffeomorphic to \( C(2,3) \times \mathbb{R}^2 \) at \((s_0, u_0, v_0)\) if and only if
   \[ \kappa_g(s_0) \neq 0, \ u_0 = \frac{1}{\kappa_g(s_0)} \text{ and } v_0 \neq \left( \frac{1}{\kappa_g(s_0)} \right)' \frac{1}{\tau_g(s_0)}. \]
   In this case, each one of \( LF_\gamma^\pm \) is non-singular and each one of Images \( \varepsilon_\gamma^\pm \) is empty.

2. Each one of the lightcone duals \( \overline{LD}_\gamma^\pm \) of \( \gamma \) is locally diffeomorphic to \( SW \times \mathbb{R} \) at \((s_0, u_0, v_0)\) if and only if
   \[ \kappa_g(s_0) \neq 0, \ \tau_g(s_0) \neq 0, \ u_0 = \frac{1}{\kappa_g(s_0)}, \ v_0 = \left( \frac{1}{\kappa_g(s_0)} \right)' \frac{1}{\tau_g(s_0)} \text{ and } \kappa_S(s_0) \neq 0. \]
   In this case, each one of \( LF_\gamma^\pm \) is locally diffeomorphic to \( C(2,3,4) \times \mathbb{R} \) and each one of Images \( \varepsilon_\gamma^\pm \) is a regular curve.

3. Each one of the lightcone duals \( \overline{LD}_\gamma^\pm \) of \( \gamma \) is locally diffeomorphic to \( BF \) at \((s_0, u_0, v_0)\) if and only if
   \[ \kappa_g(s_0) \neq 0, \ \tau_g(s_0) \neq 0, \ u_0 = \frac{1}{\kappa_g(s_0)}, \ v_0 = \left( \frac{1}{\kappa_g(s_0)} \right)' \frac{1}{\tau_g(s_0)}, \ \kappa_S(s_0) = 0 \text{ and } \kappa'_S(s_0) \neq 0. \]
   In this case, each one of \( LF_\gamma^\pm \) is locally diffeomorphic to \( C(BF) \times \mathbb{R} \) and each one of Images \( \varepsilon_\gamma^\pm \) is locally diffeomorphic to the projection of the \( C(2,3,4,5) \)-cusp.
(B) For the lightcone dual $LD_\gamma$ of $\gamma$, we have the following assertions:

1. The lightcone dual $LD_\gamma$ of $\gamma$ is locally diffeomorphic to $C(2,3) \times \mathbb{R}^2$ at $(s_0, u_0, v_0)$ if and only if

$$u_0 \neq 2 \kappa_\gamma(s_0) \tau_\gamma(s_0) \sigma^\pm(s_0) \text{ and } v_0 = \pm \sqrt{4 + 4 \kappa_\gamma(s_0) u_0 - u_0^2}.$$ 

In this case, each one of $LF^\pm_\gamma$ is non-singular and each one of Images $\varepsilon^\pm_\gamma$ is empty.

2. The lightcone dual $LD_\gamma$ of $\gamma$ is locally diffeomorphic to $SW \times \mathbb{R}$ at $(s_0, u_0, v_0)$ if and only if

$$\kappa'^2_\gamma(s_0) + \kappa^2_\gamma(s_0) \tau^2_\gamma(s_0) \neq 0, \quad u_0 = 2 \kappa_\gamma(s_0) \tau_\gamma(s_0) \sigma^\pm(s_0), \quad v_0 = -2 \kappa'_\gamma(s_0) \sigma^\pm(s_0) \text{ and } \kappa_S(s_0) \neq 0.$$ 

In this case, each one of $LF^\pm_\gamma$ is locally diffeomorphic to $C(2,3,4) \times \mathbb{R}$ and each one of Images $\varepsilon^\pm_\gamma$ is a regular curve.

3. The lightcone dual $LD_\gamma$ of $\gamma$ is locally diffeomorphic to $BF$ at $(s_0, u_0, v_0)$ if and only if

$$\kappa'^2_\gamma(s_0) + \kappa^2_\gamma(s_0) \tau^2_\gamma(s_0) \neq 0, \quad u_0 = 2 \kappa_\gamma(s_0) \tau_\gamma(s_0) \sigma^\pm(s_0), \quad v_0 = -2 \kappa'_\gamma(s_0) \sigma^\pm(s_0),$$

$$\kappa_S(s_0) = 0 \text{ and } \kappa'_S(s_0) \neq 0.$$ 

In this case, each one of $LF^\pm_\gamma$ is locally diffeomorphic to $C(BF) \times \mathbb{R}$ and each one of Images $\varepsilon^\pm_\gamma$ is locally diffeomorphic to the projection of the $C(2,3,4,5)$-cusp.

**Proof.** By Propositions 4.3.1 and 4.3.2, the discriminant sets of $\overline{H}$ and $H$ are the lightcone duals of $\overline{\gamma}$ and $\gamma$ respectively. By Propositions 4.3.1 and 4.3.2, both of $\overline{h}_{v_0}$ and $h_{v_0}$ have $A_k$ singularities ($k = 1, 2, 3, 4$) respectively if and only if the above conditions on the geodesic curvatures and geodesic torsion hold. By Propositions 4.4.7 and 4.4.8, $\overline{H}$ and $H$ are versal unfoldings of $\overline{h}_{v_0}$ and $h_{v_0}$ at any point $s_0 \in I$ respectively. We apply Corollary 4.4.6, so that we have the above assertions. This completes the proof. \qed
5 Lightcone dualities for hypersurfaces in the sphere

5.1 The hypersurfaces in the unit n-sphere

Let \( x : U \longrightarrow S^n_+ \) be an embedding from an open set \( U \subset \mathbb{R}^{n-1} \). We identify \( M = x(U) \) with \( U \) through the embedding \( x \). Obviously, the tangent space \( T_pM \) are all spacelike (i.e., consists only spacelike vectors), so \( M \) is a spacelike hypersurface in \( S^n_+ \subset \mathbb{R}^{n+2}_1 \). We have a map \( \Phi : S^n_+ \rightarrow S^n_0 \) defined by \( \Phi(v) = v - e_0 \), which is an isometry. Then we have a hypersurface \( \overline{x} : U \rightarrow S^n_0 \) defined by \( \overline{x}(u) = \Phi(x(u)) = x(u) - e_0 \), so that \( x \) and \( \overline{x} \) have the same geometric properties as spherical hypersurfaces. For any \( p = x(u) \), we can construct a unit normal vector \( n(u) \) as

\[
    n(u) = \frac{\overline{x}(u) \wedge e_0 \wedge x_{u_1}(u) \wedge \ldots \wedge x_{u_{n-1}}(u)}{||\overline{x}(u) \wedge e_0 \wedge x_{u_1}(u) \wedge \ldots \wedge x_{u_{n-1}}(u)||}
\]

We have \( \langle n(u), n(u) \rangle = 1 \), \( \langle e_0, e_0 \rangle = -1 \) and \( \langle e_0, n \rangle = \langle n, x_{u_i} \rangle = \langle n, x \rangle = 0 \). The system \( \{e_0, n(u), \overline{x}(u), x_{u_1}(u), \ldots, x_{u_{n-1}}(u)\} \) is a basis of \( T_p\mathbb{R}^{n+2}_1 \). We define a map \( G : U \longrightarrow S^n_0 \) by \( G(u) = n(u) \). We call it the Gauss map of the hypersurface \( M = x(U) \). We have a linear mapping provided by the derivation of the Gauss map at \( p \in M \), \( dG(u) : T_pM \longrightarrow T_pM \).

We call the linear transformation \( S_p = -dG(u) \) the shape operator of \( M \) at \( p = x(u) \). The eigenvalues of \( S_p \) denoted by \( \{\kappa_i(p)\}_{i=1}^{n-1} \) are called the principal curvatures of \( M \) at \( p \). The Gauss-Kronecker curvature of \( M \) at \( p \) is defined to be \( K(p) = \det S_p \). A point \( p \) is called an umbilic point if all the principal curvatures coincide at \( p \) and thus we have \( S_p = \kappa(p)\text{id}_{T_pM} \) for some \( \kappa(p) \in \mathbb{R} \). We say that \( M \) is totally umbilic if all the points on \( M \) are umbilic. Since \( x \) is a spacelike embedding, we have a Riemannian metric (or the first fundamental form) on \( M \) given by \( ds^2 = \sum_{i,j=1}^{n-1} g_{ij} du_i du_j \), where \( g_{ij}(u) = \langle x_{u_i}(u), x_{u_j}(u) \rangle \) for any \( u \in U \). The second fundamental form on \( M \) is given by \( h_{ij}(u) = -\langle n_{u_i}(u), x_{u_j}(u) \rangle \) at any \( u \in U \). Under the above notations, we have the following Weingarten formula [16]:

\[
    G_{u_i} = -\sum_{j=1}^{n-1} h^j_i x_{u_j}(i = 1, \ldots, n - 1),
\]

where \( (h^j_i) = (h_{ik}) (g^{kj}) \) and \( (g^{kj}) = (g_{kj})^{-1} \). This formula induces an explicit expression of the Gauss-Kronecker curvature in terms of the Riemannian metric and the second fundamental
invariant given by $K = \det(h_{ij}/\det(g_{ab})$. A point $p$ is a parabolic point if $K(p) = 0$. A point $p$ is a flat point if it is an umbilic point and $K(p) = 0$.

In [17] the spherical evolute of a hypersurface has been introduced and investigated the singularities. Then each spherical evolute of $M = \overline{x}(U)$ is defined to be

$$\varepsilon_{\pm}^M = \bigcup_{i=1}^{n-1} \left\{ \pm \left( \sqrt{\frac{\kappa_i^2(p)}{1 + \kappa_i^2(p)}} \overline{x}(u) + \sqrt{\frac{1}{1 + \kappa_i^2(p)}} \overline{n}(u) \right) \mid p = x(u) \in M = \overline{x}(U) \right\}.$$ 

5.2 The lightcone dual hypersurfaces and the lightcone height functions

We now define hypersurfaces in $\text{LC}^*$ associated with the hypersurfaces in $S^+_n$ or $S^0_n$. Let $x : U \to S^+_n$ be a hypersurface. We define $LD_{\pm}^M : U \times \mathbb{R} \to \text{LC}^*$ by

$$LD_{\pm}^M(u, \mu) = \overline{x}(u) + \mu \overline{n}(u) \pm \sqrt{\mu^2 + 1} e_0.$$ 

We also define $LD_M : U \times \mathbb{R} \to \text{LC}^*$ by

$$LD_M(u, \mu) = (\mu^2/4 - 1)\overline{x}(u) + \mu \overline{n}(u) + (\mu^2/4 + 1)e_0.$$ 

Then we have the following proposition.

**Proposition 5.2.1.** Under the above notations, we have the followings:

1. $x$ and $LD_{\pm}^M$ are $\Delta_3$-dual to each other.
2. $x$ and $LD_M$ are $\Delta_4$-dual to each other.

**Proof.** Consider the mapping $\mathcal{L}_3 : U \times \mathbb{R} \to \Delta_3$ defined by $\mathcal{L}_3(u, \mu) = (LD_{\pm}^M(u, \mu), \overline{x}(u))$. Then we have

$$\langle LD_{\pm}^M(u, \mu), \overline{x}(u) \rangle = \langle \overline{x}(u), \overline{x}(u) \rangle = 1$$

and

$$\mathcal{L}_3^*\theta_{32} = \langle LD_{\pm}^M(u, \mu), d\overline{x}(u) \rangle = \sum_{i=1}^{n-1} \langle LD_{\pm}^M(u, \mu), x_{u_i} \rangle du_i = 0.$$
The assertion (1) holds. We also consider the mapping \( \mathcal{L}_4 : U \times \mathbb{R} \rightarrow \Delta_4 \) defined by \( \mathcal{L}_4(u, \mu) = (LD_M(u, \mu), x(u)) \). Since \( \langle x(u), e_0 \rangle = -1 \) and \( \langle x(u), \overline{x}(u) \rangle = 1 \), we have \( \langle LD_M(u, \mu), x(u) \rangle = \mu^2/4 - 1 - (\mu^2/4 + 1) = -2 \). Moreover, we have

\[
\mathcal{L}_4^* \theta_{42} = \langle LD_M(u, \mu), dx(u) \rangle = \sum_{i=1}^{n-1} \langle LD_M(u, \mu), x_u \rangle du_i = 0.
\]

This completes the proof. \( \square \)

We call each one of \( \overline{LD}_\pm^\pi \) the \textit{lightcone dual hypersurface along} \( \overline{M} \subset S^0_n \) and \( LD_M \) the \textit{lightcone dual hypersurface along} \( M \subset S^+_n \). Then we have two mappings \( \pi \circ \overline{LD}_\pm^\pi : U \times \mathbb{R} \rightarrow S^+_n \) and \( \pi \circ LD_M : U \times \mathbb{R} \rightarrow S^+_n \) defined by

\[
\pi \circ \overline{LD}_\pm^\pi(u, \mu) = \pm \left( \frac{1}{\sqrt{\mu^2 + 1}} x(u) + \frac{\mu}{\sqrt{\mu^2 + 1}} n(u) \right) + e_0,
\]

\[
\pi \circ LD_M(u, \mu) = \frac{\mu^2 - 4}{\mu^2 + 4} x(u) + \frac{4\mu}{\mu^2 + 4} n(u) + e_0.
\]

Let \( x : U \rightarrow S^+_n \) be a hypersurface in the lightcone unit sphere. Then we define two families of functions as follows:

\[
\overline{H} : U \times LC^* \rightarrow \mathbb{R}, \quad \overline{H}(u, v) = \langle x(u), v \rangle - 1,
\]

\[
H : U \times LC^* \rightarrow \mathbb{R}, \quad H(u, v) = \langle x(u), v \rangle + 2.
\]

We call \( \overline{H} \) a \textit{lightcone height function} of the de Sitter spherical hypersurface \( \overline{M} \). For any fixed \( v_0 \in LC^* \), we denote \( \overline{H}_{v_0}(u) = \overline{H}(u, v_0) \). We also call \( H \) a \textit{lightcone height function} of the lightlike spherical hypersurface \( M \). For any fixed \( v_0 \in LC^* \), we denote \( h_{v_0}(u) = H(u, v_0) \).

**Proposition 5.2.2.** Let \( \overline{M} \) be a hypersurface in \( S^0_n \) and \( \overline{H} \) the lightcone height function on \( \overline{M} \). For \( p = x(u) \) and \( \overline{p} = \overline{x}(u) \neq \overline{v}^\pm \), we have the followings:

1. \( \overline{H}_{\overline{v}}^\pm(u) = \partial \overline{H}_{\overline{v}}^\pm / \partial u_i(u) = 0(i = 1, \ldots, n - 1) \) if and only if \( \overline{v}^\pm = \overline{LD}_\pm^\pi(u, \mu) \) for some \( \mu \in \mathbb{R} \backslash \{0\} \).

2. \( \overline{H}_{\overline{v}}^\pm(u) = \partial \overline{H}_{\overline{v}}^\pm / \partial u_i(u) = 0(i = 1, \ldots, n - 1) \) and \det Hess \((\overline{H}_{\overline{v}}^\pm)(u) = 0 \) if and only if \( \overline{v}^\pm = \overline{LD}_\pm^\pi(u, \mu), 1/\mu \) is one of the non-zero principle curvatures \( \kappa_i(p) \) of \( M \).
Proof. (1) Since $\overline{v} \in LC^*$, there exist $\lambda, \mu, \xi_i, (i = 1, \ldots, n - 1), \eta \in \mathbb{R}$ such that $\overline{v} = \lambda x(u) + \mu n(u) + \sum_{i=1}^{n-1} \xi_i x(u) + \eta e_0$ with $\lambda^2 + \mu^2 + \sum_{i,j=1}^{n-1} \xi_i \xi_j g_{ij}(u) - \eta^2 = 0$. The condition $\overline{h}_v(u) = \langle \overline{x}(u), \overline{v} \rangle - 1 = \langle \overline{x}(u), \lambda \overline{x}(u) + \mu n(u) + \sum_{i=1}^{n-1} \xi_i x(u) + \eta e_0 \rangle - 1 = \lambda - 1 = 0$ implies $\lambda = 1$, so that $v = \overline{x}(u) + \mu n(u) + \sum_{i=1}^{n-1} \xi_i x(u) + \eta e_0$ and $1 + \mu^2 + \sum_{i,j=1}^{n-1} \xi_i \xi_j g_{ij}(u) - \eta^2 = 0$. Therefore, $\overline{h}_v(u) = \partial h_v/\partial u_i(u) = 0$ if and only if $\partial h_v/\partial u_i(u) = \langle x(u), v \rangle = \langle x(u), \overline{x}(u) + \mu n(u) + \sum_{i=1}^{n-1} \xi_i x(u) + \eta e_0 \rangle = \sum_{j=1}^{n} g_{ij} \xi_j = 0$. Since $g_{ij}$ is positive definite, we have $\xi_j = 0(j = 1, \ldots, n - 1)$. Then we have $1 + \mu^2 - \eta^2 = 0$, so that $\eta = \pm \sqrt{1 + \mu^2}$. Thus, we have $\overline{v} = \overline{x}(u) + \mu n(u) \pm \sqrt{1 + \mu^2} e_0$. The converse direction also holds.

(2) Suppose that $\overline{h}_{v \pm}(u) = \partial \overline{h}_{v \pm}/\partial u_i(u) = 0$. Then we have $\text{Hess} (\overline{h}_{v \pm})(u) = (\langle x_{u_i u_j}(u), v \rangle) = \langle (x_{u_i u_j}(u), \overline{x}(u) + \mu n(u) \pm \sqrt{1 + \mu^2} e_0) \rangle = \langle (x_{u_i u_j}(u), \overline{x}(u)) + \mu (x_{u_i u_j}(u), n(u)) = -(g_{ij}(u)) + \mu (h_{ij}(u)) \rangle$. It follows that $\det \text{Hess} (\overline{h}_{v \pm})(u) = 0$ if and only if $\det \text{Hess} (\overline{h}_{v \pm})(u)(g_{ij}(u))^{-1} / \mu = \det(\overline{h}_{v \pm}^2(u)) - I / \mu = 0$. Thus, $\det (\overline{h}_{v \pm})(u) = 0$ if and only if $1/\mu$ is one of the non-zero principle curvatures of $M$ at $p$. □

Proposition 5.2.3. Let $M$ be a hypersurface in $S_+^n$ and $H$ be the lightcone height function on $M$. For $p = x(u) \neq v$, we have the followings.

(1) $h_v(u) = \partial h_v/\partial u_i(u) = 0, (i = 1, \ldots, n - 1)$ if and only if $v = LD_M(u, \mu)$ for some $\mu \in \mathbb{R} \setminus \{0\}$.

(2) $h_v(u) = \partial h_v/\partial u_i(u) = 0, (i = 1, \ldots, n - 1)$ and $\det \text{Hess} (h_v)(u) = 0$ if and only if $v = LD_M(u, \mu, (\mu/4 - 1/\mu))$ is one the non-zero principle curvatures $\kappa_i(p)$ of $M$.

Proof. (1) Since $v \in LC^*$, there exist $\lambda, \mu, \xi_i, (i = 1, \ldots, n - 1), \eta \in \mathbb{R}$ such that $v = \lambda x(u) + \mu n(u) + \sum_{i=1}^{n-1} \xi_i x(u) + \eta e_0$ with $\lambda^2 + \mu^2 + \sum_{i,j=1}^{n-1} \xi_i \xi_j g_{ij}(u) - \eta^2 = 0$. The condition $h_v(u) = \langle x(u), v \rangle + 2 \langle x(u), e_0 \rangle, \lambda x(u) + \mu n(u) + \sum_{i=1}^{n-1} \xi_i x(u) + \eta e_0 \rangle + 2 = \lambda - \eta + 2 = 0$ means that $\eta = 2 + \lambda$, so that $v = \lambda x(u) + \mu n(u) + \sum_{i=1}^{n-1} \xi_i x(u) + (2 + \lambda) e_0$ and $\lambda^2 + \mu^2 + \sum_{i,j=1}^{n-1} \xi_i \xi_j g_{ij}(u) - (2 + \lambda)^2 = \mu^2 + \sum_{i,j=1}^{n-1} \xi_i \xi_j g_{ij}(u) - 4 \lambda - 4 = 0$. Therefore, $h_v(u) = \partial h_v/\partial u_i(u) = 0$ if and only if $\partial h_v/\partial u_i(u) = \langle x(u), v \rangle = \langle x(u), \lambda x(u) + \mu n(u) + \sum_{i=1}^{n-1} \xi_i x(u) + (2 + \lambda) e_0 \rangle =$
\[ \sum_{j=1}^{n-1} g_{ij} \xi_j = 0. \] Since \( g_{ij} \) is positive definite, we have \( \xi_j = 0 \) \((j = 1, \ldots, n - 1)\). Then we have \[ \mu^2 - 4\lambda - 4 = 0, \] so that \( \lambda = \mu^2/4 - 1 \). Thus, we have \( \nu = (\mu^2/4 - 1)\varphi(\mu) + \mu u(\mu) + (\mu^2/4 + 1)e_0 \).

The converse direction also holds.

(2) Suppose that \( h_v(u) = \partial h_v/\partial u_i(u) = 0 \). Then we have

\[
\text{Hess}(h_v)(u) = \langle \varphi_u, u \rangle \frac{1}{\kappa_i(p)} n(u) \pm \sqrt{1/\kappa_i^2(p) + 1} e_0 | u \in U \}
\]

Therefore, \( \det \text{Hess}(h_v)(u) = 0 \) if and if \( \det \text{Hess}(h_v)(d_j(u)^{-1}/\mu = \det((h_v^2(u)) - (\mu/4 - 1/\mu)I) = 0 \), so that \( \det \text{Hess}(h_v)(u) = 0 \) if and only if \( (\mu/4 - 1/\mu) \) is one of the non-zero principle curvatures of \( M \) at \( p \).

Let \((u, \mu)\) be a singular point of each one of \( L^D_\mu \). By Proposition 5.2.2, we have \( 1/\mu = \kappa_i(p)(1 \leq i \leq n - 1) \), where \( \kappa_i(p) \) is one of the non-zero principle curvatures of \( M \) at \( p = \varphi(u) \). It follows that \( \mu = 1/\kappa_i(p) \). Therefore the critical value sets of \( L^D_\mu \) are given by

\[
C(L^D_\mu) = \bigcup_{i=1}^{n-1} \left\{ \varphi(u) + \frac{1}{\kappa_i(p)} n(u) \pm \sqrt{1/\kappa_i^2(p) + 1} e_0 \bigg| u \in U \right\}
\]

Let \((u, \mu)\) be a singular point of \( L_M(u, \mu) \). By Proposition 5.2.3, we have \( \mu/4 - 1/\mu = \kappa_i(p)(1 \leq i \leq n - 1) \). It follows that we have \( \mu = 2(\kappa_i(p) \pm \sqrt{1 + \kappa_i^2(p)}) \). For simplification, we write that \( \sigma^\pm(\kappa_i(p)) = \kappa_i(p) \pm \sqrt{1 + \kappa_i^2(p)} \). Then the critical value sets of \( L^M_\nu \) are given by

\[
C(L^M_\mu) = \bigcup_{i=1}^{n-1} \left\{ ((\sigma^\pm(\kappa_i(p))^2 - 1)\varphi(u) + 2\sigma^\pm(\kappa_i(p)) n(u) + ((\sigma^\pm(\kappa_i(p))^2 + 1)e_0 \bigg| u \in U \right\}
\]

We respectively denote that

\[
LF^\pm_M = \bigcup_{i=1}^{n-1} \left\{ \varphi(u) + \frac{1}{\kappa_i(p)} n(u) \pm \sqrt{1/\kappa_i^2(p) + 1} e_0 \bigg| u \in U \right\}
\]

\[
LF^\pm_M = \bigcup_{i=1}^{n-1} \left\{ ((\sigma^\pm(\kappa_i(p))^2 - 1)\varphi(u) + 2\sigma^\pm(\kappa_i(p)) n(u) + ((\sigma^\pm(\kappa_i(p))^2 + 1)e_0 \bigg| u \in U \right\}
\]
We respectively call each one of $LF_M^\pm$ the lightcone focal hypersurface of the de Sitter spherical hypersurface $\mathbf{x}$ and each one of $LF_M^{\pm}$ the lightcone focal hypersurface of the lightcone spherical hypersurface $\mathbf{x}$. Then the projections of these surfaces to $S^n_+$ are given as follows:

$$
\pi(C(\overline{LD}_M^{\pm})) = \bigcup_{i=1}^{n-1} \left\{ \pm \left( \sqrt{\frac{\kappa_i^2(p)}{1+\kappa_i^2(p)}} \mathbf{x}(u) + \sqrt{\frac{1}{1+\kappa_i^2(p)}} \mathbf{n}(u) \right) + e_0 \mid u \in U \right\},
$$

$$
\pi(C(\overline{LD}_M)^{\pm}) = \bigcup_{i=1}^{n-1} \left\{ \frac{(\sigma^{\pm}(\kappa_i(p)))^2 - 1}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1} \mathbf{x}(u) + \frac{2\sigma^{\pm}(\kappa_i(p))}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1} \mathbf{n}(u) + e_0 \mid u \in U \right\}.
$$

By definition, we have $\varepsilon_M^\pm = \Phi \circ \pi(C(\overline{LD}_M^{\pm}))$, where $\varepsilon_M^\pm$ is the spherical evolute of $M = \mathbf{x}(U)$.

This means that the spherical evolutes are obtained from the critical value sets of the lightcone dual hypersurfaces of $M = \mathbf{x}(U)$. Since $\sigma^{\pm}(\kappa_i(p)) = \kappa_i(p) \pm \sqrt{1 + \kappa_i^2(p)}$, we have $\sigma^{\pm}(\kappa_i(p))^2 = 2\kappa_i(p)(\sigma^{\pm}(\kappa_i(p)) + 1)$. By straightforward calculations, we have

$$
\left( \frac{(\sigma^{\pm}(\kappa_i(p)))^2 - 1}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1} \right)^2 = \frac{\kappa_i^2(p)(\sigma^{\pm}(\kappa_i(p)))^2}{\kappa_i^2(p)(\sigma^{\pm}(\kappa_i(p)))^2 + (\sigma^{\pm}(\kappa_i(p)))^2} = \frac{\kappa_i^2(p)}{1 + \kappa_i^2(p)}
$$

and

$$
\left( \frac{2\sigma^{\pm}(\kappa_i(p))}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1} \right)^2 = \frac{(\sigma^{\pm}(\kappa_i(p)))^2}{\kappa_i^2(p)(\sigma^{\pm}(\kappa_i(p)))^2 + (\sigma^{\pm}(\kappa_i(p)))^2} = \frac{1}{1 + \kappa_i^2(p)}.
$$

Thus we have the following proposition.

**Proposition 5.2.4.** Let $\mathbf{x} : U \rightarrow S^n_+$ be a hypersurface in $S^n_+$. Then

$$
\frac{(\sigma^{\pm}(\kappa_i(p)))^2 - 1}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1} \mathbf{x}(u) + \frac{2\sigma^{\pm}(\kappa_i(p))}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1} \mathbf{n}(u) = \pm \left( \sqrt{\frac{\kappa_i^2(p)}{1 + \kappa_i^2(p)}} \mathbf{x}(u) + \sqrt{\frac{1}{1 + \kappa_i^2(p)}} \mathbf{n}(u) \right).
$$

We define $\tilde{\pi} = \Phi \circ \pi : LC^* \rightarrow S^n_0$. Then we have the following theorem as a corollary of Proposition 5.2.4.

**Theorem 5.2.5.** Both of the projections of the critical value sets $C(\overline{LD}_M^{\pm})$ and $C(\overline{LD}_M)^{\pm}$ in the unit sphere $S^n_0$ are the images of the spherical evolutes of $M$.

$$
\tilde{\pi}(C(\overline{LD}_M^{\pm})) = \tilde{\pi}(C(\overline{LD}_M)^{\pm}) = \varepsilon_M^{\pm},
$$

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5.3 The lightcone dual hypersurfaces as wave fronts

We now naturally interpret the lightcone dual hypersurfaces of the submanifolds in $S^n_+$ as wave front sets in the theory of Legendrian singularities. Let $\pi : PT^*(LC^*) \rightarrow LC^*$ be the projective cotangent bundles with canonical contact structures. Consider the tangent bundle $\tau : TPT^*(LC^*) \rightarrow PT^*(LC^*)$ and the differential map $d\pi : TPT^*(LC^*) \rightarrow T(LC^*)$ of $\pi$. For any $X \in TPT^*(LC^*)$, there exists an element $\alpha \in T^*(LC^*)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_v(LC^*)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we have the canonical contact structure on $PT^*(LC^*)$ by

$$K = \{ X \in TPT^*(LC^*) \mid \tau(X)(d\pi(X)) \} = 0.$$  

On the other hand, we consider a point $v = (v_0, v_1, \ldots, v_{n+1}) \in LC^*$, then we have $v_0 = \pm \sqrt{v_1^2 + \ldots + v_{n+1}^2}$. So we adopt the coordinate system $(v_1, \ldots, v_{n+1})$ of $LC^*$. For the local coordinate neighborhood $(U, (\pm \sqrt{v_1^2 + \ldots + v_{n+1}^2}, v_1, \ldots, v_{n+1}))$ in $LC^*$, we have a trivialization $PT^*(LC^*) \equiv LC^* \times P(\mathbb{R}^n)^*$ and we call $((\pm \sqrt{v_1^2 + \ldots + v_{n+1}^2}, v_1, \ldots, v_{n+1}), [\xi_1 : \cdots : \xi_{n+1}])$ homogeneous coordinates of $PT^*(LC^*)$, where $[\xi_1 : \cdots : \xi_{n+1}]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^n)^*$. It is easy to show that $X \in K_{(v, [\xi])}$ if and only if $\sum_1^{n+1} \mu_i \xi_i = 0$, where $d\pi(X) = \sum_1^{n+1} \mu_i \partial / \partial v_i \in T_v LC^*$. An immersion $i : L \rightarrow PT^*(LC^*)$ is said to be a Legendrian immersion if $\dim L = n$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called the Legendrian map and we call the set $W(i)=$image$\pi \circ i$ the wave front of $i$. Moreover, $i$(or the image of $i$) is called the Legendrian lift of $W(i)$. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a function germ. We say that $F$ is a Morse family of hypersurfaces if the map germ $\Delta^* F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}^{k+1}, 0)$ defined by $\Delta^* F = (F, \partial F / \partial u_1, \cdots, \partial F / \partial u_k)$ is nonsingular. In this case, we have the following smooth $(n-1)$-dimensional smooth submanifold.

$$\Sigma_s(F) = \left\{ (u, v) \in (\mathbb{R}^k \times \mathbb{R}^n, 0) \mid F(u, v) = \frac{\partial F}{\partial u_1}(u, v) = \cdots = \frac{\partial F}{\partial u_k}(u, v) = 0 \right\} = (\Delta^* F)^{-1}(0).$$
The map germ \( \mathcal{L}_F : (\Sigma_*(F), 0) \to PT^*\mathbb{R}^n \) defined by

\[
\mathcal{L}_F(u, v) = \left( v, \left[ \frac{\partial F}{\partial v_1}(u, v) : \cdots : \frac{\partial F}{\partial v_n}(u, v) \right] \right).
\]

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd and Zakalyukin [1, 23].

**Proposition 5.3.1.** All Legendrian submanifold germs in \( PT^*\mathbb{R}^n \) are constructed by the above method.

We call \( F \) a generating family of \( \mathcal{L}_F(\Sigma_*(F)) \). Therefore the wave front of \( \mathcal{L}_F \) is

\[
W(\mathcal{L}_F) = \left\{ v \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^k \text{ such that } F(u, v) = \frac{\partial F}{\partial u_1}(u, v) = \cdots = \frac{\partial F}{\partial u_k}(u, v) = 0 \right\}.
\]

We claim here that we have a trivialization as follows:

\[
\Phi : PT^*(LC^*) \equiv LC^* \times P(\mathbb{R}^n)^*, \Phi(\sum_{i=1}^{n+1} \xi_idv_i) = (v_0, v_1, \cdots, v_{n+1}, [\xi_1 : \cdots : \xi_{n+1}])
\]

by using the above coordinate system.

**Proposition 5.3.2.** The lightcone height function \( H : U \times LC^* \to \mathbb{R} \) is a Morse family of the hypersurfaces around \((u, v) \in \Sigma_*(H)\).

**Proof.** Without the loss of the generality, we consider the future component \( LC^+_\mathbb{R} \). For any \( v = (v_0, v_1, \cdots, v_{n+1}) \in LC^*, \) we have \( v_0 = \sqrt{v_1^2 + \cdots + v_{n+1}^2} \). For \( x(u) = (1, x_1(u), \cdots, x_{n+1}(u)) \in S^n_+ \), we have

\[
H(u, v) = -\sqrt{v_1^2 + \cdots + v_{n+1}^2} + x_1(u)v_1 + \cdots + x_{n+1}(u)v_{n+1} + 2.
\]

We have to prove the mapping

\[
\Delta^*H = \left( H, \frac{\partial H}{\partial u_1}, \cdots, \frac{\partial H}{\partial u_{n-1}} \right)
\]

is non-singular at any point on \((\Delta^*H)^{-1}(0)\). If \((u, v) \in (\Delta^*H)^{-1}(0)\), then \( v = LD_M(u, \mu) \) by Proposition 5.2.3. The Jacobian matrix of \( \Delta^*H \) is given as follows:
A = \begin{pmatrix}
(x_{u_1}, v) & \cdots & (x_{u_{n-1}}, v) & -v_1/v_0 + x_1 & \cdots & -v_{n+1}/v_0 + x_{n+1} \\
(x_{u_1}, v) & \cdots & (x_{u_{n-1}}, v) & x_{1u_1} & \cdots & x_{n+1u_1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(x_{u_{n-1}}, v) & \cdots & (x_{u_{n-1}u_{n-1}}, v) & x_{1u_{n-1}} & \cdots & x_{n+1u_{n-1}}
\end{pmatrix},

We now prove that \( \text{rank} \ A = n \). For \((0, x_1, \cdots, x_{n+1}) = x \) and \((0, v_1/v_0, \cdots, v_{n+1}/v_0) = v/v_0 - e_0 = (\mu^2 - 4)x/(\mu^2 + 4) + 4\mu n/(\mu^2 + 4) \), we have

\[(0, -v_1/v_0 + x_1, \cdots, -v_{n+1}/v_0 + x_{n+1}) = x - v/v_0 + e_0 = 8x/(\mu^2 + 4) - 4\mu n/(\mu^2 + 4).\]

Since \(\{8x/(\mu^2 + 4) - 4\mu n/(\mu^2 + 4), x_{u_1}, \cdots, x_{u_{n-1}}\} \) are linearly independent, \( \text{rank} \ A = n \). This completes the proof.

By the similar arguments to the above proof, we have the following proposition.

**Proposition 5.3.3.** The lightcone height function \( \overline{H} : U \times LC^* \longrightarrow \mathbb{R} \) is a Morse family of the hypersurfaces around \((u, v) \in \Sigma_s(\overline{H})\).

Here, we consider the Legendrian immersion

\[\mathcal{L}_4 : (u, \mu) \longrightarrow \Delta_4, \quad \mathcal{L}_4(u, \mu) = (LD_M(u, \mu), x(u)).\]

We define the following mapping:

\[\Psi : \Delta_4 \longrightarrow LC^* \times P(\mathbb{R}^n)^*, \quad \Psi(v, w) = (v, [v_0w_1 - v_1w_0 : \cdots : v_0w_{n+1} - v_{n+1}w_0]).\]

For the canonical contact form \( \theta = \sum_{i=1}^{n+1} \xi_i dv_i \) on \( PT^*(LC^*) \), we have \( \Psi^* \theta = (v_0w_1 - v_1w_0)dv_1 + \cdots + (v_0w_{n+1} - v_{n+1}w_0)dv_{n+1}|_{\Delta_4} = v_0(-w_0dv_0 + w_1dv_1 + \cdots + w_{n+1}dv_{n+1}) - w_0(-v_0dv_0 + v_1dv_1 + \cdots + v_{n+1}dv_{n+1})|_{\Delta_4} = v_0\langle w, dv \rangle|_{\Delta_4} = v_0\theta_{42}|_{\Delta_4}. \) Thus \( \Psi \) is a contact morphism.

**Theorem 5.3.4.** For any hypersurface \( x : U \longrightarrow S^n_+ \), the lightcone height function \( H : U \times LC^* \longrightarrow \mathbb{R} \) is a generating family of the Legendrian immersion \( \mathcal{L}_4 \).
Proof. Since \( H \) is a Morse family of hypersurfaces, we have a Legendrian immersion \( \mathcal{L}_H : \Sigma_*(H) \rightarrow PT^*(LC^*) \) defined by \( \mathcal{L}_H(u,v) = (v, |\partial H/\partial v_1(u,v) : \cdots : \partial H/\partial v_{n+1}(u,v)|) \), where \( v = (v_0, \ldots, v_{n+1}) \) and \( \Sigma_*(H) = \{ (u,v) \in U \times LC^* \mid u \in U, v = LD_M(u,\mu), \mu \in \mathbb{R} \} \). We observe that \( H \) is a generating family of the Legendrian submanifold \( \mathcal{L}_H(\Sigma_*(H)) \) whose wave front is the image of \( LD_M \). We have

\[
\frac{\partial H}{\partial v_i}(u,v) = -\frac{l_i(u,\mu)}{l_0(u,\mu)} + x_i(u)(i = 1, \ldots, n+1),
\]

where \( x(u) = (1, x_1(u), \ldots, x_{n+1}(u)) \) and \( v = LD_M(u,\mu) = (l_0(u,\mu), \ldots, l_{n+1}(u,\mu)) \). It follows that

\[
\mathcal{L}_H(u,LD_M(u,\mu)) = (LD_M(u,\mu), [x_1(u)l_0(u,\mu) - l_1(u,\mu) : \cdots : x_{n+1}(u)l_0(u,\mu) - l_{n+1}(u,\mu)]).
\]

Therefore we have \( \Psi \circ \mathcal{L}_4(u,\mu) = \mathcal{L}_H(u,\mu) \). This completes the proof. \[\square\]

Similarly, we consider the Legendrian immersions \( \mathcal{L}_3^+ : (u,\mu) \rightarrow \Delta_3 \) defined by \( \mathcal{L}_3^+(u,\mu) = (LD_{\Delta_3}^+(u,\mu), \vec{x}(u)) \). Then we have the following theorem.

**Theorem 5.3.5.** For any hypersurface \( \vec{x} : U \rightarrow S_0^n \), the lightcone height function \( \overline{H} : U \times LC^* \rightarrow \mathbb{R} \) is a generating family of the Legendrian immersions \( \mathcal{L}_3^+ \).

### 5.4 Contact with parabolic \((n-1)\)-spheres and parabolic \(n\)-hyperquadrics

Before we start to consider the contact between hypersurfaces in the sphere with parabolic \((n-1)\)-sphere and parabolic \(n\)-hyperquadrics, we briefly review the theory of contact due to Montaldi[15]. Let \( X_i, Y_i(i = 1, 2) \) be submanifolds of \( \mathbb{R}^n \) with \( \dim X_1 = \dim X_2 \) and \( \dim Y_1 = \dim Y_2 \). We say that the contact of \( X_1 \) and \( Y_1 \) at \( y_1 \) is *the same type as the contact of \( X_2 \) and \( Y_2 \) at \( y_2 \) if there is a diffeomorphism \( \Phi : (\mathbb{R}^n, y_1) \rightarrow (\mathbb{R}^n, y_2) \) such that \( \Phi(X_1) = X_2 \) and \( \Phi(Y_1) = Y_2 \). In this case, we write \( K(X_1, Y_1, y_1) = K(X_2, Y_2, y_2) \). Of course, in the definition, \( \mathbb{R}^n \) can be replaced by any manifold. Two function germs \( f_i : (\mathbb{R}^n, a_i) \rightarrow \mathbb{R} \) are called \( K \)-equivalent if there is a diffeomorphism germ \( \Phi : (\mathbb{R}^n, a_1) \rightarrow (\mathbb{R}^n, a_2) \), and a function germ \( \lambda : (\mathbb{R}^n, a_1) \rightarrow \mathbb{R} \) with \( \lambda(a_1) \neq 0 \) such that \( f_1 = \lambda \cdot (f_2 \circ \Phi) \).

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Theorem 5.4.1 (Montaldi [15]). Let $X_i$, $Y_i$ (for $i=1,2$) be submanifolds of $\mathbb{R}^n$ with $\text{dim}X_1=\text{dim}X_2$ and $\text{dim}Y_1=\text{dim}Y_2$. Let $g_i : (X_i, x_i) \rightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \rightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1, y_1) = K(X_2, Y_2, y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $K$-equivalent.

Returning to the lightcone dual hypersurface $LD_M$, we now consider the function $h : S^n_+ \times LC^* \rightarrow \mathbb{R}$ defined by $h(u, v) = \langle u, v \rangle + 2$ and the function $g : LC^* \times LC^* \rightarrow \mathbb{R}$ defined by $g(u, v) = \langle u, v \rangle + 2$. For a given $v_0 \in LC^*$, we denote $h_{v_0}(u) = h(u, v_0)$ and $g_{v_0}(u) = g(u, v_0)$, then we have $h_{v_0}^{-1}(0) = S^n_+ \cap HP(v_0, -2)$ and $g_{v_0}^{-1}(0) = LC^* \cap HP(v_0, -2)$. For any $u_0 \in U$, $\mu_0 \in \mathbb{R}$, we take the point $v_0 = LD_M(u_0, \mu_0)$. Then we have

$$g_{v_0} \circ x(u_0) = g \circ (x \times id_{LC^*})(u_0, v_0) = h_{v_0} \circ x(u_0) = h \circ (x \times id_{LC^*})(u_0, v_0) = H(u_0, v_0) = 0.$$ 

We also have

$$\frac{\partial(g_{v_0} \circ x)}{\partial u_i}(u_0) = \frac{\partial(h_{v_0} \circ x)}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, v_0) = 0$$

for $i = 1, \cdots, n-1$. This means that the $(n-1)$-sphere $h_{v_0}^{-1}(0) = S^n_+ \cap HP(v_0, -2)$ is tangent to $M = x(U)$ at $p_0 = x(u_0)$. In this case, we call it the **lightcone tangent parabolic $(n-1)$-sphere of $M$ at $p_0$**, which is denoted by $TPS^n_{+1}(x, u_0)$. The $n$-hyperquadric $g_{v_0}^{-1}(0) = LC^* \cap HP(v_0, -2)$ is also tangent to $M$ at $p_0$. In this case, we call it the **lightcone tangent parabolic $n$-hyperquadric of $M$ at $p_0$**, which is denoted by $TPH^n(x, u_0)$. For the lightcone dual surfaces $LD^+_M$, we consider a function $\overline{h} : S^n_0 \times LC^* \rightarrow \mathbb{R}$ defined by $\overline{h}(u, v) = \langle u, v \rangle - 1$ and a function $\overline{g} : S^{n+1}_1 \times LC^* \rightarrow \mathbb{R}$ defined by $\overline{g}(u, v) = \langle u, v \rangle - 1$. For a given $v_0 \in LC^*$, we denote that $\overline{h}_{v_0}(u) = \overline{h}(u, v_0)$ and $\overline{g}_{v_0}(u) = \overline{g}(u, v_0)$. Then we have $\overline{h}_{v_0}^{-1}(0) = S^n_0 \cap HP(v_0, 1)$ and $\overline{g}_{v_0}^{-1}(0) = S^{n+1}_1 \cap HP(v_0, 1)$. For any $u_0 \in U$ and the points $\overline{v}_0^\pm = LD^+_M(u_0, \mu_0)$, we have

$$\overline{g}_{v_0}^\pm \circ x(u_0) = \overline{g} \circ (x \times id_{LC^*})(u_0, v_0^\pm) = \overline{h}_{v_0}^\pm \circ x(u_0) = \overline{h} \circ (x \times id_{LC^*})(u_0, v_0^\pm) = H(u_0, v_0^\pm) = 0.$$ 

We also have

$$\frac{\partial(\overline{g}_{v_0}^\pm \circ x)}{\partial u_i}(u_0) = \frac{\partial(\overline{h}_{v_0}^\pm \circ x)}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, v_0^\pm) = 0$$

for $i = 1, \cdots, n-1$. It follows that each one of the $(n-1)$-sphere $\overline{h}_{v_0}^{-1}(0) = S^n_0 \cap HP(\overline{v}_0^\pm, 1)$ is tangent to $\overline{M}$ at $\overline{v}_0 = \overline{x}(u_0)$. In this case, we call each one the **de-Sitter tangent parabolic**
denote the \((n-1)\)-sphere of \(\overline{M}\) at \(\overline{p}_0\), which are denoted by \(TPS_0^{n-1\pm}(x, u_0)\). Also we have each of the \(n\)-hyperquadric \(\overline{a}^{-1}_0(0) = S_1^{n-1} \cap HP(\overline{w}_0^\pm, 1)\) is tangent to \(\overline{M}\) at \(\overline{p}_0\). In this case, we call each one the de-Sitter tangent parabolic \(n\)-hyperquadric of \(\overline{M}\) at \(\overline{p}_0\), which are denoted by \(TPS_1^{n\pm}(\overline{x}, u_0)\).

Let \(x_i : (U, u_i) \rightarrow (S_+^n, p_i)(i = 1, 2)\) be hypersurface germs. For \(v_i = LD_M(u_i, \mu_i)\), we denote \(h_{i,v_i} : (U, u_i) \rightarrow (\mathbb{R}, 0)\) by \(h_{i,v_i} = H(u_i, v_i)\). Then we have \(h_{i,v_i}(u) = (\beta_{i,v_i} \circ x_i)(u) = (g_{i,v_i} \circ x_i)(u)\). For \(\overline{v}_i = \overline{LD}_M(u_i, \mu_i)\), we denote \(\overline{h}_{i,v_i} : (U, u_i) \rightarrow (\mathbb{R}, 0)\) by \(\overline{h}_{i,v_i}(u) = \overline{H}(u_i, \overline{v}_i)\). Then we have \(\overline{h}_{i,v_i}(u) = (\overline{f}_{i,v_i} \circ \overline{x}_i)(u) = (\overline{g}_{i,v_i} \circ \overline{x}_i)(u)\). By Theorem 5.4.1, we have the following proposition.

**Proposition 5.4.2.** Let \(x_i : (U, u_i) \rightarrow (S_+^n, p_i)(i = 1, 2)\) be hypersurface germs. For \(v_i = LD_M(u_i, \mu_i)\), the following conditions are equivalent:

1. \(K(x_1(U), TPS_0^{n-1\pm}(x_1, u_1), v_1) = K(x_2(U), TPS_0^{n-1\pm}(x_2, u_2), v_2)\).
2. \(K(x_1(U), TPH^n(x_1, u_1), v_1) = K(x_2(U), TPH^n(x_2, u_2), v_2)\).
3. \(h_{1,v_1}\) and \(h_{2,v_2}\) are \(K\)-equivalent.

Moreover, for \(\overline{v}_i = \overline{LD}_M(u_i, \mu_i)\), the following conditions are equivalent:

4. \(K(x_1(U), TPS_0^{n-1\pm}(x_1, u_1), \overline{v}_i) = K(x_2(U), TPS_0^{n-1\pm}(x_2, u_2), \overline{v}_i)\).
5. \(K(x_1(U), TPS_0^{n\pm}(x_1, u_1), \overline{v}_i) = K(x_2(U), TPS_0^{n\pm}(x_2, u_2), \overline{v}_i)\).
6. \(\overline{h}_{1,v_i}\) and \(\overline{h}_{2,v_i}\) are \(K\)-equivalent.

On the other hand, we return to the review on the theory of Legendrian singularities. We introduce a natural equivalence relation among Legendrian submanifold germs. Let \(F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)\) be Morse families of hypersurfaces. Then we say that \(\mathcal{L}_F(\Sigma_*(F))\) and \(\mathcal{L}_G(\Sigma_*(G))\) are **Legendrian equivalent** if there exists a contact diffeomorphism germ \(H : (PT^*\mathbb{R}^n, z) \rightarrow (PT^*\mathbb{R}^n, z')\) such that \(H\) preserves fibers of \(\pi\) and that \(H(\mathcal{L}_F(\Sigma_*(F))) = \mathcal{L}_G(\Sigma_*(G))\), where \(z = \mathcal{L}_F(0), z' = \mathcal{L}_G(0)\). By using the Legendrian equivalence, we can define the notion of Legendrian stability for Legendrian submanifold germs by the ordinary way (see [1][Part III]). We can interpret the Legendrian equivalence by using the notion of generating families. We denote by \(\mathcal{E}_n\) the local ring of function germs \((\mathbb{R}^n, 0) \rightarrow \mathbb{R}\) with the unique maximal ideal \(\mathfrak{m}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}\). Let \(F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)\) be function
germs. We say that $F$ and $G$ are $P$-$K$-equivalent if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Psi(q, x) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \rightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$. We say that $F$ is an infinitesimally $K$-versal deformation of $f = F|_{\mathbb{R}^k \times \{0\}}$ if

$$\mathcal{E}_k = T_\varepsilon(K)(f) + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k \times \{0\}}, \ldots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k \times \{0\}} \right\rangle_{\mathbb{R}},$$

where

$$T_\varepsilon(K)(f) = \left\langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k} \right\rangle_{\mathbb{R}}.$$

The main result in the theory of Legendrian singularities ([1], §20.8 and [23], THEOREM 2) is the following:

**Proposition 5.4.3** (Arnol’d, Zakalyukin). Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be Morse families and we denote the corresponding Legendrian immersion germs by $L_F, L_G$. Then

(1) $L_F$ and $L_G$ are Legendrian equivalent if and only if $F$ and $G$ are $P$-$K$-equivalent.

(2) $L_F$ is Legendrian stable if and only if $F$ is $K$-versal deformation of $f$.

Since $F$ and $G$ are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, 0)$, we do not need the notion of stably $P$-$K$-equivalences under this situation [23, page 27]. For any map germ $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$, we define the local ring of $f$ by $Q_r(f) = \mathcal{E}_n/(f^*(\mathcal{M}_p)\mathcal{E}_n + \mathcal{M}_{r+1}^n)$. We have the following classification result of Legendrian stable germs (cf. [9, Proposition A.4]) which is the key for the purpose in this section.

**Proposition 5.4.4.** Let $F, G : (\mathbb{R}^n \times \mathbb{R}^k, 0) \rightarrow (\mathbb{R}, 0)$ be Morse families. Suppose that Legendrian immersion germs $L_F$ and $L_G$ are Legendrian stable, then the following conditions are equivalent.

(1) $W(L_F)$ and $W(L_G)$ are diffeomorphic as set germs.

(2) $L_F$ and $L_G$ are Legendrian equivalent.

(3) $Q_{n+1}(f)$ and $Q_{n+1}(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|_{\mathbb{R}^k \times \{0\}}$ and $g = G|_{\mathbb{R}^k \times \{0\}}$. 

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Let \( Q_{n+1}(x, u_0) \) be the local ring of the function germ \( h_{v_0} : (U, u_0) \to \mathbb{R} \) defined by

\[
Q_{n+1}(x, u_0) = C^\infty_{u_0}(U)/(h_{v_0})C^\infty_{u_0}(U) + \mathfrak{m}^{n+2}_{n-1},
\]

and \( Q^\pm_{n+1}(x, u_0) \) be the local rings of the function germs \( h_{v^\pm 0} : (U, u_0) \to \mathbb{R} \) defined by

\[
Q^\pm_{n+1}(x, u_0) = C^\infty_{u_0}(U)/(h_{v^\pm 0})C^\infty_{u_0}(U) + \mathfrak{m}^{n+2}_{n-1},
\]

where \( v_0 = LD_M(u_0, \mu_0) \), \( v^\pm 0 = LD^\pm_M(u_0, \mu_0) \) and \( C^\infty_{u_0}(U) \) is the local ring of function germs at \( u_0 \) with the unique maximal ideal \( \mathfrak{m}_{n-1} \).

**Theorem 5.4.5.** Let \( x_i : (U, u_i) \to (S^n, p_i)(i = 1, 2) \) be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent.

1. The lightcone hypersurface germs \( LD_M(U \times \mathbb{R}) \) and \( LD_M(U \times \mathbb{R}) \) are diffeomorphic.
2. Legendrian immersion germs \( L^1_i \) and \( L^0_i \) are Legendrian equivalent.
3. The lightcone height functions germs \( H_1 \) and \( H_2 \) are \( P,K \)-equivalent.
4. \( h_{1, v_1} \) and \( h_{2, v_2} \) are \( K \)-equivalent.
5. \( K(x_1(U), TPS^1\mathfrak{m}^{-1}(x_1, u_1), v_1) = K(x_2(U), TPS^1\mathfrak{m}^{-1}(x_2, u_2), v_2). \)
6. \( K(x_1(U), TPH^n(x_1, u_1), v_1) = K(x_2(U), TPH^n(x_2, u_2), v_2). \)
7. Local rings \( Q_{n+1}(x_1, u_1) \) and \( Q_{n+1}(x_2, u_2) \) are isomorphic as \( \mathbb{R} \)-algebras.

**Proof.** By Proposition 5.4.3 and Proposition 5.4.4, the conditions (1)~(3) and (7) are equivalent. By definition, the condition (3) implies the condition (4). By Proposition 5.4.3, \( H_i \) is a \( K \)-versal deformation of \( h_{i, v_i} \). We can apply the uniqueness result of \( K \)-versal deformations (cf., [14]), so that the condition (4) implies the condition (3). By Theorem 5.4.1, the conditions (4)~(6) are equivalent. This completes the proof.

**Theorem 5.4.6.** Let \( x_i : (U, u_i) \to (S^n, p_i)(i = 1, 2) \) be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent.

1. The lightcone hypersurface germs \( LD_M(U \times \mathbb{R}) \) and \( LD_M(U \times \mathbb{R}) \) are diffeomorphic.
(2) Legendrian immersion germs $\mathcal{L}_3^{1\pm}$ and $\mathcal{L}_3^{2\pm}$ are Legendrian equivalent.
(3) The lightcone height functions germs $\overline{H}_1$ and $\overline{H}_2$ are $\mathcal{P}$-$\mathcal{K}$-equivalent.
(4) $\overline{t}_{1,\sigma_1^{\pm}}$ and $\overline{t}_{2,\sigma_2^{\pm}}$ are $\mathcal{K}$-equivalent.
(5) $K(\overline{x}_1(U), TPS_0^{n-1}\pm(\overline{x}_1, u_1), \overline{v}_1) = K(\overline{x}_2(U), TPS_0^{n-1}\pm(\overline{x}_2, u_2), \overline{v}_2)$.
(6) $K(\overline{x}_1(U), TPS_1^{n}\pm(\overline{x}_1, u_1), \overline{v}_1^\pm) = K(\overline{x}_2(U), TPS_1^{n}\pm(\overline{x}_2, u_2), \overline{v}_2^\pm)$.
(7) Local rings $Q_{n+1}^\pm(\overline{x}_1, u_1)$ and $Q_{n+1}^\pm(\overline{x}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.

The proof is similar to the proof of the above theorem, so that we omit it.

**Lemma 5.4.7.** Let $x : U \rightarrow S^n_+$ be a hypersurface germ such that the corresponding Legendrian immersion germs $\mathcal{L}_4$ and $\mathcal{L}_3^{\pm}$ are Legendrian stable. Then at the singular point $v_0 = LD_M(u_0, 2\sigma^\pm(\kappa_i(p_0)))(1 \leq i \leq n-1)$ of $LD_M$ and the singular points $v_0^\pm = LD_M^\pm(u_0, 1/\kappa_i(p_0))$ of $LD_M^\pm$, we have the following equivalent assertions:

1. The lightcone hypersurface germs $LD_M(U \times \mathbb{R})$ and $LD_M^\pm(U \times \mathbb{R})$ are diffeomorphic.
2. Legendrian immersion germs $\mathcal{L}_3^{\pm}$ and $\mathcal{L}_4$ are Legendrian equivalent.
3. The lightcone height functions germs $H$ and $\overline{H}$ are $\mathcal{P}$-$\mathcal{K}$-equivalent.
4. $h_{v_0}$ and $\overline{t}_{v_0}^\pm$ are $\mathcal{K}$-equivalent.
5. $K(x(U), TPS^n_1(x, u_0), v_0) = K(\overline{x}(U), TPS^n_1(\overline{x}, u_0), \overline{v}_1^\pm)$.
6. $K(x(U), \text{TPH}^n(x, u_0), v_0) = K(\overline{x}(U), TPS^n_1(\overline{x}, u_0), \overline{v}_1^\pm)$.
7. Local rings $Q_{n+1}^\pm(\overline{x}, u_0)$ and $Q_{n+1}(x, u_0)$ are isomorphic as $\mathbb{R}$-algebras.

**Proof.** By definition, we have $h_{v_0}(u) = (x(u), ((\sigma^\pm(\kappa_i(p_0))^2 - 1))x(u) + 2\sigma^\pm(\kappa_i(p_0))n(u_0) + (\sigma^\pm(\kappa_i(p_0))^2 + 1)e_0) + 2$, so that

$$\frac{h_{v_0}(u)}{(\sigma^\pm(\kappa_i(p_0)))^2 + 1} = (\overline{x}(u) + e_0) \pm \left( \sqrt{\frac{\kappa_i^2(p_0)}{1 + \kappa_i^2(p_0)}}x(u) + \sqrt{\frac{1}{1 + \kappa_i^2(p_0)}}n(u_0) \right) + e_0$$

and

$$= (\overline{x}(u) \pm \left( \sqrt{\frac{\kappa_i^2(p_0)}{1 + \kappa_i^2(p_0)}}x(u) + \sqrt{\frac{1}{1 + \kappa_i^2(p_0)}}n(u_0) \right) + e_0) \pm \frac{\kappa_i(p_0)}{\sqrt{1 + \kappa_i^2(p_0)}}.$$
We also have
\[
\overline{h}^\pm_{v_0}(u) = \langle \overline{x}(u), x(u_0) \rangle + \frac{1}{\kappa_i(p_0)} n(u_0) \pm \frac{1}{\kappa_i^2(p_0) + 1} e_0 - 1
\]
and
\[
\pm \frac{\kappa_i(p_0) \overline{h}^\pm_{v_0}(u)}{\sqrt{\kappa_i^2(p_0) + 1}} = \langle \overline{x}(u), \pm \left( \sqrt{\frac{\kappa_i^2(p_0)}{1 + \kappa_i^2(p_0)}} \overline{x}(u) + \sqrt{\frac{1}{1 + \kappa_i^2(p_0)}} n(u_0) \right) + e_0 \rangle \pm \frac{\kappa_i(p_0)}{\sqrt{1 + \kappa_i^2(p_0)}}.
\]

Therefore, we have
\[
\overline{h}^\pm_{v_0} = \frac{\pm \sqrt{\kappa_i^2(p_0) + 1}}{\kappa_i(p_0)((\sigma^\pm(\kappa_i(p_0)))^2 + 1)} h_{v_0}.
\]

This means that the assertion (4) holds. By the uniqueness of the $K$-versal deformation, we have the assertion (3). By Proposition 5.4.3, we have the assertion (2). By Proposition 5.4.4, we have the assertions (1) and (7). On the other hand, for $g_{v_0} \circ x = \overline{h}_{v_0} \circ x = h_{v_0}$ and $\overline{g}_{v_0} \circ \overline{x} = \overline{h}_{v_0} \circ \overline{x} = \overline{h}_{v_0}^\pm$, by Theorem 5.4.1, we have the assertions (5) and (6). This completes the proof.

By Lemma 5.4.7, we have our main result as the following theorem.

**Theorem 5.4.8.** Let $x_i : (U, u_i) \rightarrow (S^3_{+}, p_i)(i = 1, 2)$ be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. At the singular points $\overline{v}^\pm_i = \overline{LD}_M^\pm(u_0, 1/\kappa_j(p))(1 \leq j \leq n - 1)$ of $\overline{LD}_M^\pm$, and the singular points $v_i = LD_M(u_0, 2\sigma^\pm(\kappa_i(p)))$ of $LD_M$, the conditions (1) $\sim$ (7) in Theorem 5.4.5 and the conditions (1) $\sim$ (7) in Theorem 5.4.6 are all equivalent.

5.5 Surfaces in the 3-sphere

In this section, we stick to the case $n = 3$. We consider the surfaces in the 3-sphere as a special case of the previous sections. First we consider the generic properties of spacelike submanifolds in the unit sphere $S^3_0$. We consider the space of embeddings $\text{Emb}(U, S^3_0)$ with Whitney $C^\infty$-topology. We also consider the function $\overline{H} : S^3_0 \times LC^* \rightarrow \mathbb{R}$ which is given by $\overline{H}(u, v) = \langle u, v \rangle - 1$. We claim that $\overline{h}_u$ is a submersion for any $v \in LC^*$, where $\overline{h}_u(u) = \overline{H}(u, v)$.
For any \( \overline{x} \in \text{Emb}(U, S^3_0) \), we have \( \overline{H} = \overline{H} \circ (\overline{x} \times id_{LC^*}) \). We have the \( k \)-jet extension

\[
j^k \overline{H} : U \times LC^* \to J^k(U, \mathbb{R})
\]
defined by \( j^k \overline{H}(u, v) = j^k \overline{T}_v(u) \). We consider the trivialization \( J^k(U, \mathbb{R}) = U \times \mathbb{R} \times J^k(2, 1) \).

For any submanifold \( Q \subset J^k(2, 1) \), we denote \( \tilde{Q} = U \times 0 \times Q \). Then we have the following proposition as a corollary of [22, Lemma 6].

**Proposition 5.5.1.** Let \( Q \) be a submanifold of \( J^k(2, 1) \). Then the set

\[
T_Q = \{ \overline{x} \in \text{Emb}(U, S^3_0) \mid j^k \overline{H} \text{ is transversal to } \tilde{Q} \}
\]
is a residual subset of \( \text{Emb}(U, S^3_0) \). If \( Q \) is a closed set, then \( T_Q \) is open.

By the previous arguments and the Appendix of [9], we have the following theorem.

**Theorem 5.5.2.** There exists an open dense subset \( \mathcal{O} \subset \text{Emb}(U, S^3_0) \) such that for any \( \overline{x} \in \mathcal{O} \), the corresponding Legendrian immersion germs \( L^\pm_3 \) at any point are Legendrian stable.

If we consider \( \mathcal{H} : S^3_+ \times LC^* \to \mathbb{R} \) defined by \( \mathcal{H}(u, v) = \langle u, v \rangle + 2 \) instead of \( \overline{H} : S^3_0 \times LC^* \to \mathbb{R} \), we can show that the corresponding Legendrian immersion germ \( L_4 \) at any point is Legendrian stable for a generic hypersurface \( \overline{x} : U \to S^3_+ \).

We now classify the singularities of the lightcone dual hypersurfaces. Here we only consider the case for \( \overline{M} = \overline{x}(U) \) in \( S^3_0 \). By Proposition 5.4.5, a \( \mathcal{K} \)-invariant for the height function \( \overline{T}_v \) is an invariant for the diffeomorphism class of the singularities of the lightcone duals of a hypersurface in \( S^3_0 \). Let \( \overline{x} : U \to S^3_0 \) be an embedding from an open set \( U \subset \mathbb{R}^2 \), we define the \( \mathcal{K} \)-codimension (or Tyurina number) of the function germ \( \overline{T}_{\tau^+_{u_0}} \) by

\[
\overline{H}\text{-ord}^\pm(\overline{x}, u_0) = \dim C_{u_0}^\infty / \langle \overline{T}_{\tau^+_{u_0}}, \partial \overline{T}_{\tau^+_{u_0}} / \partial u_i \rangle_{C_{u_0}^\infty}.
\]

We call it the *order of contact* of \( M \) with parabolic \((n-1)\)-spheres and parabolic \( n \)-hyperquadrics. We also define the corank of the function germ \( \overline{T}_{\tau^\pm_{u_0}} \) by

\[
\overline{H}\text{-corank}^\pm(\overline{x}, u_0) = 2 - \text{rank Hess}(\overline{T}_{\tau^\pm_{u_0}})(u_0).
\]
By Theorem 5.3.5, Theorem 5.5.2 and Proposition 5.4.3, the lightcone height function $\overline{H}$ is a $\mathcal{K}$-versal deformation of $\overline{h}_{\tau_0^+}$ at each point $(u_0, \overline{v}_0^+) \in U \times LC^*$. Therefore we can apply the classification of $\mathcal{K}$-versal deformations of function germs up to 4-parameters[1]. Suppose that the lightcone height function $\overline{H}$ is a $\mathcal{K}$-versal deformation of $h_{\tau_0} \pm 0$ at each point $(u_0, v_0 \pm 0) \in U \times LC^*$. Then it is $\mathcal{P}$-$\mathcal{K}$-equivalent to one of the following germs:

\[(A_k) \quad F(u_1, u_2, \lambda) = u_1^{k+1} \pm u_2^2 + \lambda_1 + \lambda_2 u_1 + \cdots + \lambda_k u_1^{k-1}, (1 \leq k \leq 4),\]

\[(D_4^+) \quad F(u_1, u_2, \lambda) = u_1^3 + u_2^3 + \lambda_1 + \lambda_2 u_1 + \lambda_3 u_2 + \lambda_4 u_1 u_2,\]

\[(D_4^-) \quad F(u_1, u_2, \lambda) = u_1^3 - u_1 u_2^2 + \lambda_1 + \lambda_2 u_1 + \lambda_3 u_2 + \lambda_4 (u_1^2 + u_2^2).\]

For any $F(u_1, u_2, \lambda)$, we have

\[W(L_F) = \left\{ \lambda \in \mathbb{R}^4 \mid \exists u \in \mathbb{R}^2 \text{ such that } F(u, \lambda) = \frac{\partial F}{\partial u_1}(u, \lambda) = \frac{\partial F}{\partial u_2}(u, \lambda) = 0 \right\}.\]

Let $f_i : (N_i, x_i) \rightarrow (P_i, y_i) (i = 1, 2)$ be $C^\infty$ map germs. We say that $f_1$ and $f_2$ are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi : (N_1, x_1) \rightarrow (N_2, x_2)$ and $\psi : (P_1, y_1) \rightarrow (P_2, y_2)$ such that $\psi \circ f_1 = f_2 \circ \phi$. Then we have the following theorem.

**Theorem 5.5.3.** There exists an open dense subset $\mathcal{O} \subset \text{Emb}_{ap}(U, S^3_0)$ such that for any $\overline{x} \in \mathcal{O}$, we have the following classification:

(a) If $\overline{H}$-corank$^\pm(\overline{x}, u_0) = 1$, then there are two distinct principle curvatures $\kappa_1$ and $\kappa_2$. In this case $\overline{H}$-ord$^\pm(\overline{x}, u_0) \leq 4$ and we have the following:

\[(A_1) \quad \text{If } \overline{H}$-ord$^\pm(\overline{x}, u_0) = 1, \text{ then each one of } \overline{LD}_{\overline{x}}^\pm \text{ is } \mathcal{A}$-equivalent to\]

\[f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0).\]

\[(A_2) \quad \text{If } \overline{H}$-ord$^\pm(\overline{x}, u_0) = 2, \text{ then each one of } \overline{LD}_{\overline{x}}^\pm \text{ is } \mathcal{A}$-equivalent to\]

\[f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3).\]

The image of $f$ is diffeomorphic to $C \times \mathbb{R}^2$. 59
(A3) If $H$-ord$^\pm(x, u_0) = 3$, then each one of $\mathcal{LD}^\pm_{\mathbb{M}}$ is $\mathcal{A}$-equivalent to
\[
f(u_1, u_2, u_3) = (4u_1^3 + 2u_1 u_2, 3u_1^4 + u_2 u_1^2, u_2, u_3).
\]
The image of $f$ is diffeomorphic to $SW \times \mathbb{R}$.

(A4) If $H$-ord$^\pm(x, u_0) = 4$, then each one of $\mathcal{LD}^\pm_{\mathbb{M}}$ is $\mathcal{A}$-equivalent to
\[
f(u_1, u_2, u_3) = (5u_1^4 + 3u_2 u_1^2 + 2u_1 u_3, 4u_1^5 + 2u_2 u_1^3 + u_3 u_1^2, u_2, u_3).
\]
The image of $f$ is diffeomorphic to $BF$.

(b) If $\overline{H}$-corank$^\pm(\mathbf{x}, u_0) = 2$ and the principle curvature $\kappa \neq 0$, then $u_0$ is a non-flat umbilic point. In this case we have $H$-ord$^\pm(\mathbf{x}, u_0) = 2$ and the following two cases:

\[
(D_4^+) \text{ Each one of } \mathcal{LD}^\pm_{\mathbb{M}} \text{ is } \mathcal{A} \text{-equivalent to } \\
f(u_1, u_2, u_3) = (2(u_1^3 + u_2^3) + u_1 u_2 u_3, 3u_1^2 + u_2 u_3, 3u_2^2 + u_1 u_3, u_3).
\]

\[
(D_4^-) \text{ Each one of } \mathcal{LD}^\pm_{\mathbb{M}} \text{ is } \mathcal{A} \text{-equivalent to } \\
f(u_1, u_2, u_3) = (2(u_1^3 - u_1 u_2^2) + (u_1^2 + u_2^2) u_3, u_2^2 - 3u_1^2 - 2u_1 u_3, u_1 u_2 - u_2 u_3, u_3).
\]

Here, $C = \{(x_1, x_2) \mid x_1 = u^2, x_2 = u^3\}$ is the ordinary cusp, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is called a swallowtail and $BF = \{(x_1, x_2, x_3, x_4) \mid x_1 = 5u^4 + 3uv^2 + 2uw, x_2 = 4u^5 + 2uv^3 + wu^2, x_3 = u, x_4 = v\}$ is called a butterfly.

**Proof.** By Theorem 5.5.2, there exists an open dense subset $\mathcal{O} \subset Emb_{op}(U, \mathcal{S}_0^3)$ such that for any $\mathbf{x} \in \mathcal{O}$, the corresponding Legendrian immersion germs $\mathcal{L}_x^\pm$ at any point are Legendrian stable. Therefore, the height function $\overline{H}$ is $\mathcal{P}$-$\mathcal{K}$-equivalent to one of the germs of $(A_k)$ ($k = 1, 2, 3, 4$) and $D_4^\pm$. If we consider the germ $F(u_1, u_2, \lambda) = u_1^3 \pm u_2^3 + \lambda_3 + \lambda_4 u_1$, then we have
\[
W(\mathcal{L}_F) = \{(2u_1^3, -3u_1^2, \lambda_3, \lambda_4) \mid (u_1, \lambda_3, \lambda_4) \in \mathbb{R}^3\},
\]
so that the corresponding Legendrian map germ is $(A_2) f(u_1, u_2, u_3) = (3u_1^3, 2u_1^2, u_2, u_3)$. Suppose that $\overline{H}$ is $\mathcal{P}$-$\mathcal{K}$-equivalent to $F$ of type $(A_2)$. By Propositions 5.4.3 and 5.4.4, $\mathcal{LD}^\pm_{\mathbb{M}}$ is $\mathcal{A}$-equivalent to $(A_2)$. Of course, the image of $f$ is $C \times \mathbb{R}^2$. Moreover, the $\mathcal{K}$-codimension of $f(u_1, u_2) = u_1^3 \pm u_2^3$ is 2, so that we have $H$-ord$^\pm(x, u_0) = 2$. The proof of the other assertions are similar to this case. Therefore, we omit it. \qed
By Lemma 5.4.7, the lightcone dual surface $LD_M$ of $x : U \rightarrow S^3_+$ is locally diffeomorphic to the lightcone dual surfaces $\overline{LD}^+_M$. Therefore, we obtain exactly the same assertions as the above theorem for the lightcone dual surface $LD_M$.

References


