The Lightcone Dualities for Submanifolds in the Sphere

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Contents

1 Introduction

In this thesis we consider submanifolds in the unit n-sphere from the view point of Legendrian dualities between pseudo-spheres in Minkowski space-time. In [5, 8] Izumiya introduced the mandala of Legendrian dualities between psudo-spheres in Minkowski space-time. There are three kinds of pseudo-spheres in Minkowski space-time (i.e., the hyperbolic space, the de Sitter space and the lightcone). Especially, if we investigate spacelike submanifolds in the lightcone, this framework is essentially useful (see, also [13]). For the de Sitter space and the lightcone, there exist naturally embedded unit spheres. If we have a submanifold in the unit sphere, then we have the corresponding submanifolds in the embedded unit spheres in the lightcone or de Sitter space. Since these canoncial embeddings are isometries, the geometric structures of those two submanifolds are the same from the view point of the spherical geometry. We also have the dual hypersurfaces in the lightcone as an application of the duality theorem in [8]. We have two dual hypersurfaces depending on the embeddings of the sphere in the de Sitter sphere or the lightcone. On the lightcone, there is a projection onto the canonically embedded unit sphere (cf., §§2.1). We investigate the singular points of the dual hypersurfaces and the projection images of the singular value sets onto the unit sphere in the lightcone. Of coruse, the singular points of these two dual hypersurfaces are different in general. However, the situation depends on the codimension of the submanifold for the projections of the singular values. One of the consequences is that the projetion of the singular values of these dual hypersurfaces are equal to the spherical focal set (or , the spherical evolute) for submanifolds of codimension one (cf., Theorem 3.1.7 and Theorem 5.2.5). However, these are different for submanifolds of higher codimensions (cf., Proposition 4.2.2).

In §3, we study the lightcone dualities for curves in the unit 2-sphere. The *dual curve* of a curve $\boldsymbol{\gamma}$ in the unit sphere is defined to be the front $\boldsymbol{\gamma}^\vee$ equidistant by $\pi/2$ from the original one. The dual curve $\boldsymbol{\gamma}^\vee$ can be considered as the Gauss map of the original curve $\boldsymbol{\gamma}$. Moreover the pair $(\boldsymbol{\gamma}, \boldsymbol{\gamma}^\vee)$ is a Legendrian immersion to the contact manifold (Δ, K) in the product of the spheres which gives the well-known spherical Legendrian duality. This means that the dual curve $\boldsymbol{\gamma}^\vee$ can be interpreted as the wavefront set of the Legendrian immersion

$$\mathcal{L}_\gamma = (\boldsymbol{\gamma}, \boldsymbol{\gamma}^\vee) : I \longrightarrow \Delta = \{(\boldsymbol{v}, \boldsymbol{w}) \in S^2 \times S^2 \mid \boldsymbol{v} \cdot \boldsymbol{w} = 0\}$$

to the contact manifold (Δ, K), where the contact structure K is defined by the 1-form $\theta = (v_1dw_1+v_2dw_2+v_3dw_3)|\Delta$. The duals in several different ambient spaces have been well-studied so far ([18, 4, 19, 21, 8, 16, 1], etc.).

On the other hand, the spherical evolute of $\boldsymbol{\gamma}$ is naturally obtained as the envelope of the family of normal geodesics to $\boldsymbol{\gamma}$. The evolute in Euclidean plane is naturally interpreted as a caustic [7]. The spherical evolute is also a caustic [17]. Moreover, it is the dual curve of the unit tangent vector of the original curve [18]. However, there might be no interpretation on the evolute from the view point of the dual of the original curve. In §3 we show that the spherical evolute of a given curve can be interpreted as the critical value sets of these dual surfaces (cf., Theorem 3.1.7). We also investigate the singularities of these curves and surfaces (cf., Theorem 3.3.4). As a consequence, we obtain an interesting correspondence among these singularities (cf., Remark 3.3.5).

In §4, we study the lightcone dualities for curves in the unit 3-sphere. In §3, the evolutes of curves in the unit 2-sphere has been investigated from the view point of the Legendrian duality [5, 8, 10]. It is known that the evolute of a curve in the unit 2-sphere is the dual of the tangent indicatrix of the original curve [18]. For a curve in the unit 3-sphere, however, the dual is a surface. Therefore, the dual of the tangent indicatrix of a curve is a surface which is called the *focal surface* (or, the *focal set*) of the original curve. The critical locus of the focal surface is the evolute of the original curve (cf., [18]). We remark that the focal set of a curve in the unit 2-sphere is a curve which is equal to the evolute.

For the de Sitter 4-space and the lightcone in Lorentz-Minkowski 5-space, there exist naturally embedded unit 3-spheres. The de Sitter 4-space corresponds to the cosmic model, and the lightcone also has its clear background in Physics [6]. In this section we investigate the curves in the unit 3-sphere in the framework of the theory of Legendrian dualities between pseudo-spheres in Lorentz-Minkowski 5-space([1, 4, 7, 8, 16, 18, 19, 21], etc.). If we have a regular curve in the unit 3-sphere, then we have the regular curve in the embedded unit 3-sphere in the lightcone or de Sitter space. Therefore, we naturally have the dual hypersurfaces in the lightcone as an application of the duality theorem in [8]. There are two kinds of lightcone dual hypersurfaces of a curve in the unit 3-sphere. We will give the classifications of the singularities of these hypersurfaces. In physics, the singularities of the lightcone are also studied [20]. The critical value sets of these two hypersurfaces are called the lightcone focal surfaces respectively. The projections of these focal surfaces to unit 3-sphere are different surfaces. In [10] we have shown that the projection images of the critical value sets of lightcone dual surfaces for a curve in the unit 2-sphere coincide with the evolute of the original curve. Therefore, the situation of curves in the unit 3-sphere is quite different from that of curves in the unit 2-sphere. However, the projections of the critical sets of lightcone focal surfaces are equal to the evolute of the curve. In order to clarify such situation, we introduce the notion of discriminant set of higher order for unfoldings of functions of one-variable (see, §§4.4).

In §5, we study the lightcone dualities for hypersurfaces in the unit n-sphere. For de Sitter space and the lightcone in Minkowski $(n+2)$-space, there exist naturally embedded unit n-spheres. Moreover, we have the canonical projection from the lightcone to the unit sphere embedded in the lightcone. In this section, we investigate hypersurfaces in the unit n-sphere in the framework of the theory of Legendrian dualities between pseudo-spheres in Minkowski $(n+2)$-space ([19, 21, 7, 8], etc.). If we have a hypersurface in the unit n-sphere, then we have spacelike hypersurfaces in the embedded unit n-sphere in the lightcone and de Sitter space. Therefore, we naturally have the dual hypersurfaces in the lightcone as an application of the duality theorem in [8]. There are two kinds of lightcone dual hypersurfaces of a hypersurface in the unit n-sphere. One is the dual of the hypersurface of the unit n-sphere embedded in de Sitter space and another is the dual of the hypersurface of the unit n-sphere embedded in the lightcone. By definition, these dual hypersurfaces are different.

On the other hand, we have studied the curves in the unit 2-sphere and the unit 3-sphere from the view point of the Legendrian duality in §3 and §4 [10, 11]. In the unit 2-sphere, it is known that the evolute of a curve in the unit 2-sphere is the dual of the tangent indicatrix of the original curve [18]. We have shown that the projection images of the critical value sets of lightcone dual surfaces for a curve in the unit 2-sphere coincide with the evolute of the original curve in [10]. However, this fact doesn't hold for a curve in unit 3-sphere (cf., [11]). For the curve case, these facts have been shown by the direct calculations in [10, 11]. We have not known the geometric reason why the situations are different. In order to clarify these situation, we investigate hypersurfaces in the unit n-sphere from the view point of the theory of Legendrian singularities. The curves in the unit 2-sphere can be considered as a special case of this paper. We can also show that the projection images of the critical value sets of two different lightcone dual hypersurfaces for a hypersurface in the unit n-sphere also coincide with the spherical evolute (cf., [17]) of the original hypersurface. We interpret geometric meanings of the singularities of those two lightcone dual hypersurfaces. Here, we remark that we do not have the notion of tangent indicatrices for higher dimensional submanifolds in the sphere. Therefore, the situation is completely different from the curve case.

2 Preliminary knowledge

2.1 The basic concepts

In this section we introduce the basic concepts in this thesis. Let $\mathbb{R}^{n+2}$ be an $(n+2)$-dimensional vector space. For any two vectors $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n+1}), \boldsymbol{y} = (y_0, y_1, \ldots, y_{n+1})$ in $\mathbb{R}^{n+2}$, their *pseudo scalar product* is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + x_1y_1 + \ldots + x_{n+1}y_{n+1}$. Here, $(\mathbb{R}^{n+2}, \langle , \rangle)$ is called *Lorentz-Minkowski* $(n+2)$*-space* (simply, *Minkowski* $(n+2)$*-space*), which is denoted by $\mathbb{R}^{n+2}_1$. For any $(n+1)$ vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{n+1} \in \mathbb{R}^{n+2}_1$, their *pseudo vector product* is defined by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \ldots \wedge \boldsymbol{x}_{n+1} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_{n+1} \\ x_1^0 & x_1^1 & \cdots & x_1^{n+1} \\ x_2^0 & x_2^1 & \cdots & x_2^{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n+1}^0 & x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}\}$ is the canonical basis of $\mathbb{R}^{n+2}_1$ and $\boldsymbol{x}_i = (x_i^0, x_i^1, \cdots, x_i^{n+1})$. A non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+2}_1$ is called spacelike, lightlike or timelike if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of $\boldsymbol{x} \in \mathbb{R}^{n+2}_1$ is defined by $\| \boldsymbol{x} \| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}^{n+2}_1$ be a regular curve in $\mathbb{R}^{n+2}_1$ (i.e., $\dot{\boldsymbol{\gamma}}(t) \neq \boldsymbol{0}$ for any $t \in I$), where I is an open interval. For any $t \in I$, the curve $\boldsymbol{\gamma}$ is called *spacelike*, *lightlike* or *timelike* if $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle > 0, \langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle = 0$ or $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle < 0$ respectively. We call $\boldsymbol{\gamma}$ a *nonlightlike curve* if $\boldsymbol{\gamma}$ is a spacelike or timelike curve. The arc-length of a nonlightlike curve $\boldsymbol{\gamma}$ measured from $\boldsymbol{\gamma}(t_0)(t_0 \in I)$ is $s(t) = \int_{t_0}^t \| \dot{\boldsymbol{\gamma}}(t) \| \, \mathrm{d}t$.

The parameter s is determined such that $\| \boldsymbol{\gamma}'(s) \| = 1$ for the nonlightlike curve, where $\boldsymbol{\gamma}'(s) = \mathrm{d}\boldsymbol{\gamma}/\mathrm{d}s(s)$ is the unit tangent vector of $\boldsymbol{\gamma}$ at s.

We define the *de Sitter* $(n+1)$*-space* by

$$S^{n+1}_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+2}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}.$$

We define the *closed lightcone with the vertex* $\boldsymbol{a}$ by

$$LC_a = \{\boldsymbol{x} \in \mathbb{R}^{n+2}_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\}.$$

We define the *open lightcone at the origin* by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+2}_1 \backslash \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We consider a submanifold in the lightcone defined by

$$S^n_+ = \{\boldsymbol{x} \in LC^* \mid x_0 = 1\},$$

which is called the *lightcone unit sphere*. We have a projection $\pi : LC^* \longrightarrow S^n_+$ defined by

$$\pi(\boldsymbol{x}) = \widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_{n+1}}{x_0}\right),$$

where $\boldsymbol{x} = (x_0, x_1, \dots x_{n+1})$. We also define the *n-dimensional Euclidean unit sphere* in $\mathbb{R}^{n+1}_0$ by

$$S^n_0 = \{\boldsymbol{x} \in S^{n+1}_1 \mid x_0 = 0\},$$

where $\mathbb{R}^{n+1}_0 = \{\boldsymbol{x} \in \mathbb{R}^{n+2}_1 \mid x_0 = 0\}$.

2.2 The Legendrian duality theorem

We now review some properties of contact manifolds and Legendrian submanifolds. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally, such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field on K is non-degenerate if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a contact manifold if K is a non-degenerate hyperplane field. In this case, K is called a contact structure and α is a contact form. Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a contact diffeomorphism if $d\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are contact diffeomorphic if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be Legendrian if $\dim L = n$ and $di_x(T_x L) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is called a Legendrian fibration if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \longrightarrow M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E, \pi \circ i : L \longrightarrow M$ is called a Legendrian map. The image of the Legendrian map $\pi \circ i$ is called a wavefront set of i which is denoted by $W(i)$. For any $p \in E$, it is known that there is a local coordinate system $(x_1, \dots, x_m, p_1, \dots, p_m, z)$ around p such that $\pi(x_1, \dots, x_m, p_1, \dots, p_m, z) = (x_1, \dots, x_m, z)$ and the contact structure is given by the 1-form

$$\alpha = dz - \sum_1^m p_i dx_i$$

(cf. [1], 20.3). One of the examples of Legendrian fibrations is given by the unit spherical tangent bundle of a Riemannian manifold. Let M be a Riemannian manifold and TM is its tangent bundle. Let $(x_1, \dots, x_n)$ be local coordinates on a neighbourhood U of M and $(v_1, \dots, v_n)$ be coordinates on the fiber over U. Let g_{ij} be the components of the metric $\langle , \rangle$ with respect to the above coordinates. Then the canonical one-form can be locally denoted by $\theta = \sum_{i,j} g_{ij} v_j dq_i$ where $q_i = x_i \circ \pi$ for the projection $\pi : TM \longrightarrow M$. Let $\tilde{\pi} : S(TM) \longrightarrow M$ be the unit spherical tangent bundle with respect to the metric $\langle , \rangle$. Then the restriction of θ onto $S(TM)$ gives a contact structure and $\tilde{\pi} : S(TM) \longrightarrow M$ is a Legendrian fibration (cf., [2]).

We now show the basic theorem in this paper which is the fundamental tool for the study of spacelike submanifolds in pseudo-spheres in Minkowski space. We define one-forms $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$, $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$ on $\mathbb{R}_1^{n+2} \times \mathbb{R}_1^{n+2}$ and consider the following four double fibrations with one-forms:

(1)(a) $H^{n+1}(-1) \times S_1^{n+1} \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,

(b) $\pi_{11} : \Delta_1 \longrightarrow H^{n+1}(-1), \pi_{12} : \Delta_1 \longrightarrow S_1^{n+1}$,

(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_3, \theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_3$.

(2)(a) $H^{n+1}(-1) \times LC^* \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1\}$,

(b) $\pi_{21} : \Delta_4 \longrightarrow LC^*, \pi_{22} : \Delta_4 \longrightarrow LC^*$,

(c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_2, \theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_2$.

(3)(a) $LC^* \times S_1^3 \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 1\}$,

(b) $\pi_{31} : \Delta_3 \longrightarrow LC^*, \pi_{32} : \Delta_3 \longrightarrow S_1^3$,

(c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_3, \theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_3$.

(4)(a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = -2\}$,

(b) $\pi_{41} : \Delta_4 \longrightarrow LC^*, \pi_{42} : \Delta_4 \longrightarrow LC^*$,

(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_4, \theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_4$.

Here, $\pi_{i1}(v, w) = v$, $\pi_{i2}(v, w) = w$ are the canonical projections. Moreover, $\theta_{i1} = \langle dv, w\rangle \mid_{\triangle_i}$ and $\theta_{i2} = \langle v, dw\rangle \mid_{\triangle_i}$ are the restrictions of the one-forms $\langle dv, w\rangle$ and $\langle v, dw\rangle$ on $\triangle_i$. We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over $\triangle_i$ which is denoted by K_i. The basic theorem in this thesis is the following theorem:

Theorem 2.2.1. Under the same notations as the previous paragraph, each $(\triangle_i; K_i)(i = 1, 2, 3, 4)$ is a contact manifold and both of $\pi_{ij}(j = 1; 2)$ are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic each other.

The proof of this theorem can be found in [8]. In this thesis, we will only consider (Δ_3, K_3) and (Δ_4, K_4). If we have an isotropic mapping $i : L \to \Delta_i$ (i.e., $i^*\theta_{i1} = 0$), we say that $\pi_{i1}(i(L))$ and $\pi_{i2}(i(L))$ are Δ_i-*dual* to each other $(i = 3, 4)$. For detailed properties of Legendrian fibrations, see [1].

3 Lightcone dualities for curves in the 2-sphere

3.1 Curves in the unit sphere and lightcone duals

Let $\boldsymbol{\gamma} : I \longrightarrow S^2_+$ be a regular curve. We have a map $\Phi : S^2_+ \to S^2_0$ defined by $\Phi((v)) = \boldsymbol{v} - \boldsymbol{e}_0$, which is an isometry. Then we have a regular curve $\overline{\boldsymbol{\gamma}} : I \to S^2_0$ defined by $\overline{\boldsymbol{\gamma}}(s) = \Phi(\boldsymbol{\gamma}(s)) = \boldsymbol{\gamma}(s) - \boldsymbol{e}_0$, so that $\boldsymbol{\gamma}$ and $\overline{\boldsymbol{\gamma}}$ have the same geometric properties as spherical curves. Since $\overline{\boldsymbol{\gamma}}$ is a spacelike curve, we can reparameterize it by the arc-length s. So we have the unit tangent vector $\boldsymbol{t}(s) = \overline{\boldsymbol{\gamma}}'(s)$ of $\overline{\boldsymbol{\gamma}}(s)$. We have another unit vector $\boldsymbol{n}(s) = \overline{\boldsymbol{\gamma}}(s) \wedge \boldsymbol{e}_0 \wedge \boldsymbol{t}(s)$, then we have a pseudo-orthonormal frame $\{\overline{\boldsymbol{\gamma}}, \boldsymbol{t}, \boldsymbol{n}\}$ of $\mathbb{R}^3_0$ along $\overline{\boldsymbol{\gamma}}$. By standard arguments, we have the following *Frenet-Serret type formulae.*

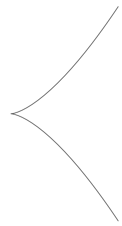
$$\begin{cases} \overline{\boldsymbol{\gamma}}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = \kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s) \\ \boldsymbol{n}'(s) = -\kappa_g(s)\boldsymbol{t}(s) \end{cases},$$

where $\kappa_g(s) = \langle \boldsymbol{t}'(s), \boldsymbol{n}(s)\rangle$. It is known that $\{\overline{\boldsymbol{\gamma}}, \boldsymbol{t}, \boldsymbol{n}\}$ is called the *Sabban frame* of $\overline{\boldsymbol{\gamma}}$ [12]. Here, κ_g is the *geodesic curvature* of $\overline{\boldsymbol{\gamma}}$ in S^2_0. Under the above notation, the dual of $\overline{\boldsymbol{\gamma}}$ is given by $\overline{\boldsymbol{\gamma}}^\vee(s) = \boldsymbol{n}(s)$. Therefore, s_0 is a singular point of $\overline{\boldsymbol{\gamma}}^\vee$ if and only if $\kappa_g(s_0) = 0$. Moreover, it has been known the following proposition [18, Proposition 1]:

Proposition 3.1.1. The dual curve $\overline{\boldsymbol{\gamma}}^\vee$ has an ordinary cusp at s_0 if and only if $\kappa_g(s_0) = 0$ and $\kappa_g'(s_0) \neq 0$.

Here, we say that a curve has an *ordinary cusp* if it is locally diffeomorphic to the curve $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$ (cf., Fig.1).
The point $\overline{\boldsymbol{\gamma}}(s_0)$ with $\kappa_g(s_0) = 0, \kappa_g'(s_0) \neq 0$ is said to be an *ordinary inflection* of $\overline{\boldsymbol{\gamma}}$.

On the other hand, the evolute $\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}$ of the curve $\overline{\boldsymbol{\gamma}}$ is defined to be the envelope of the family of the normal geodesics to the curve $\overline{\boldsymbol{\gamma}}$. It is known that the evolute of $\overline{\boldsymbol{\gamma}}$ is the dual of the spherical curve given by the unit tangent vector $\boldsymbol{t}$ [18]. By the Frenet-Serret type formulae, we



The ordinary cusp

Fig. 1

have

$$\boldsymbol{\varepsilon}^{\pm}_{\overline{\boldsymbol{\gamma}}}(s) = \pm \boldsymbol{t}^{\vee}(s) = \frac{\pm(\kappa_g(s)\overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s))}{\sqrt{\kappa_g^2(s)+1}}.$$

For a general parametrized curve $\boldsymbol{\gamma} : I \longrightarrow S^2_+$, we can calculate that the geodesic curvature is given by $\kappa_g(t) = \dfrac{\det(\boldsymbol{\gamma}(t), \dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t))}{\|\dot{\boldsymbol{\gamma}}(t)\|^3}$. Thus, we have the following proposition as a corollary of Proposition 3.2.1:

Proposition 3.1.2. The evolute $\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}$ of the curve $\overline{\boldsymbol{\gamma}}$ is singular at s_0 if and only if $\kappa_g'(s_0) = 0$. Moreover, $\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}$ has an ordinary cusp at s_0 if and only if $\kappa_g'(s_0) = 0$ and $\kappa_g''(s_0) \neq 0$.

Proof. The geodesic curvature of $\boldsymbol{t}$ is given by

$$\kappa_g[\boldsymbol{t}](s) = \frac{\det(\boldsymbol{t}(s), \boldsymbol{t}'(s), \boldsymbol{t}''(s))}{\|\boldsymbol{t}'(s)\|^3} = \frac{\kappa_g'(s)}{\left(\sqrt{\kappa_g^2(s)+1}\right)^3}.$$

Hence, the first assertion holds. We also have

$$\kappa_g[\boldsymbol{t}]'(s) = \frac{\kappa_g''(s)(\kappa_g^2(s)+1) - 3\kappa_g(s)(\kappa_g')^2(s)}{\left(\sqrt{\kappa_g^2(s)+1}\right)^5}.$$

It follows that $\kappa_g[\boldsymbol{t}](s_0) = 0$ and $\kappa_g[\boldsymbol{t}]'(s_0) \neq 0$ if and only if $\kappa_g'(s_0) = 0$ and $\kappa_g''(s_0) \neq 0$. This completes the proof. □

The point $\boldsymbol{\gamma}(s_0)$ with $\kappa_g'(s_0) = 0, \kappa_g''(s_0) \neq 0$ is said to be an *ordinary vertex* of $\boldsymbol{\gamma}$.

On the other hand, $\overline{\boldsymbol{\gamma}}$ is a curve in the Euclidean 3-space $\mathbb{R}^3_0$. Therefore, we have the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ along $\overline{\boldsymbol{\gamma}}$ as a space curve. Let $\tau(s)$ be the *torsion* of $\overline{\boldsymbol{\gamma}}$ at $\overline{\boldsymbol{\gamma}}(s)$ with respect to the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$. By a straightforward calculation, we have

$$\tau(s) = -\frac{\kappa_g'(s)}{\kappa_g^2(s)+1} \text{ and } \tau'(s) = -\frac{\kappa_g''(s)(\kappa_g^2(s)+1) - 2\kappa_g(s)(\kappa_g')^2(s)}{(\kappa_g^2(s)+1)^2}.$$

Therefore, we have the following corollary.

Corollary 3.1.3. The evolute $\boldsymbol{\varepsilon}_{\overline{\gamma}}$ of the curve $\overline{\boldsymbol{\gamma}}$ is singular at s_0 if and only if $\tau(s_0) = 0$. Moreover, $\boldsymbol{\varepsilon}_{\overline{\gamma}}$ has an ordinary cusp at s_0 if and only if $\tau(s_0) = 0$ and $\tau'(s_0) \neq 0$.

We call the point $\overline{\boldsymbol{\gamma}}(s_0)$ with $\tau(s_0) = 0, \tau'(s_0) \neq 0$ an *ordinary flattening point* of $\overline{\boldsymbol{\gamma}}$.

We now define surfaces in LC^* associated with the curves in S^2_+ or S^2_0. Let $\boldsymbol{\gamma} : I \longrightarrow S^2_+$ be a unit speed curve. We define $\overline{LD}^{\pm}_{\overline{\gamma}} : I \times \mathbb{R} \longrightarrow LC^*$ by

$$\overline{LD}^{\pm}_{\overline{\gamma}}(s,u) = \overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) \pm \sqrt{u^2+1}\boldsymbol{e}_0.$$

We also define $LD_{\boldsymbol{\gamma}} : I \times \mathbb{R} \longrightarrow LC^*$ by

$$LD_{\boldsymbol{\gamma}}(s,u) = \frac{u^2-4}{4}\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) + \frac{u^2+4}{4}\boldsymbol{e}_0.$$

Then we have the following proposition.

Proposition 3.1.4. Under the above notations, we have the followings:
(1) $\overline{\boldsymbol{\gamma}}$ and $\overline{LD}^{\pm}_{\overline{\gamma}}$ are Δ_3-dual to each other.
(2) $\boldsymbol{\gamma}$ and $LD_{\boldsymbol{\gamma}}$ are Δ_4-dual to each other.

Proof. Consider the mapping $\mathscr{L}_3(s,u) = (\overline{LD}^{\pm}_{\overline{\gamma}}(s,u), \overline{\boldsymbol{\gamma}}(s))$. Then we have $\langle \overline{LD}^{\pm}_{\overline{\gamma}}(s,u), \overline{\boldsymbol{\gamma}}(s)\rangle = \langle \overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}(s)\rangle = 1$ and $\mathscr{L}_3^*\theta_{32} = \langle \overline{LD}^{\pm}_{\overline{\gamma}}(s,u), \overline{\boldsymbol{\gamma}}'(s)\rangle ds = \langle \overline{LD}^{\pm}_{\overline{\gamma}}(s,u), \boldsymbol{t}(s)\rangle ds = 0$. The assertion (1) holds.

We also consider the mapping $\mathscr{L}_4(s,u) = (LD_{\boldsymbol{\gamma}}(s,u), \boldsymbol{\gamma}(s))$. Since $\langle \boldsymbol{\gamma}(s), \boldsymbol{e}_0\rangle = 1$ and $\langle \boldsymbol{\gamma}(s), \overline{\boldsymbol{\gamma}}(s)\rangle = 1$, we have $u^2/4 - 1 - (u^2/4+1) = -2$. Moreover, we have $\mathscr{L}_4^*\theta_{42} = \langle LD_{\boldsymbol{\gamma}}(s,u)), \boldsymbol{\gamma}'(s)\rangle ds = \langle LD_{\boldsymbol{\gamma}}(s,u), \boldsymbol{t}(s)\rangle ds = 0$. This completes the proof. □

We call $\overline{LD}^{\pm}_{\overline{\gamma}}$ the *Lightcone dual surface of the de Sitter spherical curve* $\overline{\boldsymbol{\gamma}}$ and $LD_{\boldsymbol{\gamma}}$ the *Lightcone dual surface of the lightlike spherical curve* $\boldsymbol{\gamma}$. Then we have two mappings $\pi \circ \overline{LD}^{\pm}_{\overline{\gamma}}$:

$I \times \mathbb{R} \to S^2_+$ and $\pi \circ LD_{\boldsymbol{\gamma}} : I \times \mathbb{R} \to S^2_+$ defined by

$$\begin{aligned} \pi \circ \overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}(s,u) &= \pm\left(\frac{1}{\sqrt{u^2+1}}\overline{\boldsymbol{\gamma}}(s) + \frac{u}{\sqrt{u^2+1}}\boldsymbol{n}(s)\right) + \boldsymbol{e}_0, \\ \pi \circ LD_{\boldsymbol{\gamma}}(s,u) &= \frac{(u^2-4)}{u^2+4}\overline{\boldsymbol{\gamma}}(s) + \frac{4u}{u^2+4}\boldsymbol{n}(s) + \boldsymbol{e}_0. \end{aligned}$$

In this paper we consider the singularities of these dual surfaces and mappings. By the Frenet-Serret type formulae, we have

$$\begin{aligned} \frac{\partial \overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}}{\partial u}(s,u) &= \boldsymbol{n}(s) \pm \frac{u}{\sqrt{u^2+1}}\boldsymbol{e}_0, \\ \frac{\partial \overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}}{\partial s}(s,u) &= (1 - u\kappa_g(s))\boldsymbol{t}(s), \\ \frac{\partial LD_{\boldsymbol{\gamma}}}{\partial u}(s,u) &= \frac{u}{2}\overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s) + \frac{u}{2}\boldsymbol{e}_0, \\ \frac{\partial LD_{\boldsymbol{\gamma}}}{\partial s}(s,u) &= \left(\frac{u^2-4}{4} - u\kappa_g(s)\right)\boldsymbol{t}(s). \end{aligned}$$

Then we have the following proposition.

Proposition 3.1.5. Let $\boldsymbol{\gamma} : I \longrightarrow S^2_+$ be a unit speed curve. Then we have the followings:
(1) (s,u) is a singular point of $\overline{LD}^{\pm}_{\overline{\gamma}}$ if and only if $\kappa_g(s) \neq 0$ and $u = 1/\kappa_g(s)$,
(2) (s,u) is a singular point of LD_γ if and only if $u = 2(\kappa_g(s) \pm \sqrt{\kappa_g^2(s)+1})$.

Proof. By the above calculations, $\partial\overline{LD}^{\pm}_{\overline{\gamma}}/\partial u(s,u), \partial\overline{LD}^{\pm}_{\overline{\gamma}}/\partial s(s,u)$ are linearly dependent if and only if $1 - \kappa_g(s)u = 0$. The assertion (1) follows. By the similar reason, we have the assertion (2). □

For simplification, we denote $\sigma_g^{\pm}(s) = \kappa_g(s) \pm \sqrt{\kappa_g^2(s)+1}$. Therefore, the critical value sets of the above dual surfaces are given by

$$\begin{aligned} C(\overline{LD}^{\pm}_{\overline{\gamma}}) &= \left\{ \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) \pm \sqrt{\frac{\kappa_g^2(s)+1}{\kappa_g^2(s)}}\boldsymbol{e}_0 \;\middle|\; s \in I, \kappa_g(s) \neq 0 \right\}, \\ C(LD_\gamma)^{\pm} &= \{((\sigma_g^{\pm}(s))^2 - 1)\overline{\boldsymbol{\gamma}}(s) + 2\sigma_g^{\pm}(s)\boldsymbol{n}(s) + ((\sigma_g^{\pm}(s))^2+1)\boldsymbol{e}_0 \;| s \in I \;\}. \end{aligned}$$

Then the projections of these critical value sets to S^2_+ are as follows:

$$\begin{aligned}\pi(C(\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}})) &= \left\{ \frac{\pm(\kappa_g(s)\overline{\boldsymbol{\gamma}}(s)+\boldsymbol{n}(s))}{\sqrt{\kappa_g^2(s)+1}} + \boldsymbol{e}_0 \;\middle|\; s\in I, \kappa_g(s)\neq 0 \right\}, \\ \pi(C(LD_{\boldsymbol{\gamma}})^{\pm}) &= \{ \frac{(\sigma_g^{\pm}(s))^2-1}{(\sigma_g^{\pm}(s))^2+1}\overline{\boldsymbol{\gamma}}(s) + \frac{2\sigma_g^{\pm}(s)}{(\sigma_g^{\pm}(s))^2+1}\boldsymbol{n}(s)+\boldsymbol{e}_0 \; | s\in I \; \}.\end{aligned}$$

We remark that each of $\dfrac{\pm(\kappa_g(s)\overline{\boldsymbol{\gamma}}(s)+\boldsymbol{n}(s))}{\sqrt{\kappa_g^2(s)+1}}$ is the spherical evolute of $\overline{\boldsymbol{\gamma}}$(cf., [17]). This means that the spherical evolute is obtained from the critical value set of the lightcone dual surface of $\overline{\boldsymbol{\gamma}}$. On the other hand, what is the curve $\dfrac{(\sigma_g^{\pm}(s))^2-1}{(\sigma_g^{\pm}(s))^2+1}\overline{\boldsymbol{\gamma}}(s) + \dfrac{2\sigma_g^{\pm}(s)}{(\sigma_g^{\pm}(s))^2+1}\boldsymbol{n}(s)$? Since $\sigma_g^{\pm}(s) = \kappa_g(s) \pm \sqrt{\kappa_g^2(s)+1}$, we have $(\sigma_g^{\pm}(s))^2 = 2\kappa_g(s)\sigma_g^{\pm}(s)+1$. Therefore, we have

$$\left(\frac{(\sigma_g^{\pm}(s))^2-1}{(\sigma_g^{\pm}(s))^2+1}\right)^2 = \frac{\kappa_g^2(s)(\sigma_g^{\pm}(s))^2}{\kappa_g^2(s)(\sigma_g^{\pm}(s))^2+2\kappa_g(s)\sigma_g^{\pm}(s)+1} = \frac{\kappa_g^2(s)}{\kappa_g^2(s)+1}$$

and

$$\left(\frac{2\sigma_g^{\pm}(s)}{(\sigma_g^{\pm}(s))^2+1}\right)^2 = \frac{2\kappa_g(s)\sigma_g^{\pm}(s)+1}{(\kappa_g(s)\sigma_g^{\pm}(s)+1)^2} = \frac{1}{\kappa_g^2(s)+1}.$$

Thus we have the following proposition.

Proposition 3.1.6. Let $\boldsymbol{\gamma} : I \longrightarrow S^2_+$ be a unit speed curve. Then

$$\frac{(\sigma_g^{\pm}(s))^2-1}{(\sigma_g^{\pm}(s))^2+1}\overline{\boldsymbol{\gamma}}(s) + \frac{2\sigma_g^{\pm}(s)}{(\sigma_g^{\pm}(s))^2+1}\boldsymbol{n}(s) = \frac{\pm(\kappa_g(s)\overline{\boldsymbol{\gamma}}(s)+\boldsymbol{n}(s))}{\sqrt{\kappa_g^2(s)+1}}.$$

We define $\widetilde{\pi} = \Phi\circ\pi : S^2_+ \to S^2_0$. Then we have the following theorem as a corollary of Proposition 3.1.6

Theorem 3.1.7. Both of the projections of the critical value sets $C(\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}})$ and $C(LD_{\boldsymbol{\gamma}})^{\pm}$ in the unit sphere S^2_0 are the image of the evolute of $\overline{\boldsymbol{\gamma}}$, that is

$$\widetilde{\pi}(C(\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}})) = \widetilde{\pi}(C(LD_{\boldsymbol{\gamma}})^{\pm}) = \{\boldsymbol{\varepsilon}^{\pm}_{\overline{\boldsymbol{\gamma}}}(s) \mid s\in I\ \}.$$

3.2 Lightcone height functions

In order to study the singularities of Lightcone dual surfaces of spherical curves, we introduce two families of functions and apply the theory of unfoldings. Let $\boldsymbol{\gamma} : I \longrightarrow S^2_+$ be a unit speed curve, then we define two families of functions as follows:

$$\overline{H} : I \times LC^* \longrightarrow \mathbb{R}, \qquad \overline{H}(s, \boldsymbol{v}) = \langle \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v} \rangle - 1,$$

$$H : I \times LC^* \longrightarrow \mathbb{R}, \qquad H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle + 2.$$

We call $\overline{H}$ a *lightcone height function* of the de Sitter spherical curve $\overline{\boldsymbol{\gamma}}$. For any fixed $\boldsymbol{v} \in LC^*$, we denote $\overline{h}_{\boldsymbol{v}}(s) = \overline{H}(s, \boldsymbol{v})$. We call H a *lightcone height function* of the lightlike spherical curve $\boldsymbol{\gamma}$. For any fixed $\boldsymbol{v} \in LC^*$, we denote $h_{\boldsymbol{v}}(s) = H(s, \boldsymbol{v})$. Then we have the following two propositions on $h_{\boldsymbol{v}}$ and $\overline{h}_{\boldsymbol{v}}$.

Proposition 3.2.1. Let $\boldsymbol{\gamma} : I \longrightarrow S^2_+$ be a unit speed curve, then we have the followings:

(1) $\overline{h}_{\boldsymbol{v}}(s) = 0$ if and only if there exist $\mu, \xi, \eta \in \mathbb{R}$ with $\eta^2 = 1 + \mu^2 + \xi^2$ such that $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \mu \boldsymbol{t}(s) + \xi \boldsymbol{n}(s) + \eta \boldsymbol{e}_0$.

(2) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = 0$ if and only if there exist $\xi, \eta \in \mathbb{R}$ with $\eta^2 = 1 + \xi^2$ such that $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \xi \boldsymbol{n}(s) + \eta \boldsymbol{e}_0$.

(3) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g(s) \neq 0$ and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)} \boldsymbol{n}(s) \pm \sqrt{\frac{\kappa_g^2(s) + 1}{\kappa_g^2(s)}} \boldsymbol{e}_0.$$

(4) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = \overline{h}'''_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g(s) \neq 0$, $\kappa'_g(s) = 0$ and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)} \boldsymbol{n}(s) \pm \sqrt{\frac{\kappa_g^2(s) + 1}{\kappa_g^2(s)}} \boldsymbol{e}_0.$$

(5) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = \overline{h}'''_{\boldsymbol{v}}(s) = \overline{h}^{(4)}_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g(s) \neq 0$, $\kappa'_g(s) = \kappa''_g(s) = 0$ and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)} \boldsymbol{n}(s) \pm \sqrt{\frac{\kappa_g^2(s) + 1}{\kappa_g^2(s)}} \boldsymbol{e}_0.$$

Proof. (1) Since $\boldsymbol{v} \in LC^*$, there exist $\omega, \mu, \xi, \eta \in \mathbb{R}$ with $\omega^2 + \mu^2 + \xi^2 - \eta^2 = 0$ such that $\boldsymbol{v} = \omega\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{e}_0$. From $\overline{h}_{\boldsymbol{v}}(s) = \langle\overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle - 1 = 0$, we have $\omega = 1$. So $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{e}_0$ and $\eta^2 = 1 + \mu^2 + \xi^2$. The converse direction also holds.

(2) Since $\overline{h}'_{\boldsymbol{v}}(s) = \langle\boldsymbol{t}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = 0$ if and only if $\overline{h}'_{\boldsymbol{v}}(s) = \langle\boldsymbol{t}(s), \boldsymbol{v}\rangle = \langle\boldsymbol{t}(s), \overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t} + \xi\boldsymbol{n} + \eta\boldsymbol{e}_0\rangle = \mu = 0$. It follows from the fact $\eta^2 = 1 + \xi^2$ that $\eta = \pm\sqrt{1+\xi^2}$. Then we have $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{e}_0 = \overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) \pm \sqrt{1+\xi^2}\boldsymbol{e}_0$.

(3) Since $\overline{h}''_{\boldsymbol{v}}(s) = \langle\kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = 0$ if and only if $\overline{h}''_{\boldsymbol{v}}(s) = \langle\kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) \pm \sqrt{1+\xi^2}\boldsymbol{e}_0\rangle = \kappa_g(s)\xi - 1 = 0$. Then we have $\xi = 1/\kappa_g(s)$, $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) \pm \sqrt{(1+\kappa_g^2(s))/\kappa_g^2(s)}\boldsymbol{e}_0$ and $\kappa_g(s) \neq 0$.

(4) Since $\overline{h}'''_{\boldsymbol{v}}(s) = \langle\kappa'_g(s)\boldsymbol{n}(s) - (\kappa_g^2(s)+1)\boldsymbol{t}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = \overline{h}'''_{\boldsymbol{v}}(s) = 0$ if and only if $\langle\kappa'_g(s)\boldsymbol{n}(s) - (\kappa_g^2(s)+1)\boldsymbol{t}(s), \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) \pm \sqrt{(1+\kappa_g^2(s))/\kappa_g^2(s)}\boldsymbol{e}_0\rangle = \kappa'_g(s)/\kappa_g(s) = 0$. Then we have $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) \pm \sqrt{(1+\kappa_g^2(s))/\kappa_g^2(s)}\boldsymbol{e}_0$, $\kappa_g(s) \neq 0$ and $\kappa'_g(s) = 0$.

(5) Since $\overline{h}^{(4)}_{\boldsymbol{v}}(s) = \langle(\kappa''_g(s) - \kappa_g^3(s) - \kappa_g(s))\boldsymbol{n}(s) - 3\kappa_g(s)\kappa'_g(s)\boldsymbol{t}(s) + (1+\kappa_g^2(s))\overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = \overline{h}'''_{\boldsymbol{v}}(s) = \overline{h}^{(4)}_{\boldsymbol{v}}(s) = 0$ if and only if $\overline{h}^{(4)}_{\boldsymbol{v}}(s) = \langle(\kappa''_g(s) - \kappa_g^3(s) - \kappa_g(s))\boldsymbol{n}(s) - 3\kappa_g(s)\kappa'_g(s)\boldsymbol{t}(s) + (1+\kappa_g^2(s))\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) \pm \sqrt{(1+\kappa_g^2(s))/\kappa_g^2(s)}\boldsymbol{e}_0\rangle = \kappa''_g(s)/\kappa_g(s) = 0$. Then we have $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) \pm \sqrt{(1+\kappa_g^2(s))/\kappa_g^2(s)}\boldsymbol{e}_0$, $\kappa_g(s) \neq 0, \kappa'_g(s) = 0$ and $\kappa''_g(s) = 0$. □

Proposition 3.2.2. Let $\boldsymbol{\gamma} : I \longrightarrow S^2_+$ be a unit speed curve, then we have the followings:

(1) $h_{\boldsymbol{v}}(s) = 0$ if and only if $\boldsymbol{v} = \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t} + \xi\boldsymbol{n} + (\lambda+2)\boldsymbol{e}_0$, where $\lambda, \mu, \xi \in \mathbb{R}$ and $\mu^2 + \xi^2 - 4\lambda - 4 = 0$.

(2) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = 0$ if and only if $\boldsymbol{v} = (\xi^2/4 - 1)\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n} + (\xi^2/4 + 1)\boldsymbol{e}_0$.

(3) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = 0$ if and only if

$$\boldsymbol{v} = ((\sigma_g^{\pm}(s))^2 - 1)\overline{\boldsymbol{\gamma}}(s) + 2\sigma_g^{\pm}(s)\boldsymbol{n}(s) + ((\sigma_g^{\pm}(s))^2 + 1)\boldsymbol{e}_0.$$

(4) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = 0$ if and only if

$$\boldsymbol{v} = ((\sigma_g^{\pm}(s))^2 - 1)\overline{\boldsymbol{\gamma}}(s) + 2\sigma_g^{\pm}(s)\boldsymbol{n}(s) + ((\sigma_g^{\pm}(s))^2 + 1)\boldsymbol{e}_0$$

and $\kappa'_g(s) = 0$.

(5) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = h^{(4)}_{\boldsymbol{v}}(s) = 0$ if and only if

$$\boldsymbol{v} = ((\sigma_g^{\pm}(s))^2 - 1)\overline{\boldsymbol{\gamma}}(s) + 2\sigma_g^{\pm}(s)\boldsymbol{n}(s) + ((\sigma_g^{\pm}(s))^2 + 1)\boldsymbol{e}_0,$$

$\kappa'_g(s) = 0$ and $\kappa''_g(s) = 0$.

Proof. (1) Since $\boldsymbol{v} \in LC^*$, there exist $\lambda, \mu, \xi, \eta \in \mathbb{R}$ with $\lambda^2 + \mu^2 + \xi^2 - \eta^2 = 0$ such that $\boldsymbol{v} = \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{e}_0$. From $h_{\boldsymbol{v}}(s) = \langle\boldsymbol{\gamma}(s), \boldsymbol{v}\rangle + 2 = \langle\overline{\boldsymbol{\gamma}}(s) + \boldsymbol{e}_0, \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{e}_0\rangle = \lambda - \eta + 2 = 0$, we have $\eta = 2 + \lambda$. So $\boldsymbol{v} = \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + (2 + \lambda)\boldsymbol{e}_0$ and $\lambda^2 + \mu^2 + \xi^2 - (2 + \lambda)^2 = \mu^2 + \xi^2 - 4\lambda - 4 = 0$. The converse direction also holds.

(2) Since $h'_{\boldsymbol{v}}(s) = \langle\boldsymbol{t}(s), \boldsymbol{v}\rangle$, $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = 0$ if and only if $h'_{\boldsymbol{v}}(s) = \langle\boldsymbol{t}(s), \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + (2 + \lambda)\boldsymbol{e}_0\rangle = \mu = 0$. It follows from the fact $\lambda^2 + \xi^2 - (2 + \lambda)^2 = \xi^2 - 4\lambda - 4 = 0$ that $\lambda = \xi^2/4 - 1$. Then we have $\boldsymbol{v} = (\xi^2/4 - 1)\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) + (\xi^2/4 + 1)\boldsymbol{e}_0$.

(3) Since $h''_{\boldsymbol{v}}(s) = \langle\kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = 0$ if and only if $h''_{\boldsymbol{v}}(s) = \langle\kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), (\xi^2/4 - 1)\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) + (\xi^2/4 + 1)\boldsymbol{e}_0\rangle = \kappa_g(s)\xi - \xi^2/4 + 1 = 0$, then we have $\xi = 2\kappa_g(s) \pm 2(\kappa_g^2(s) + 1)^{1/2} = 2\sigma_g^{\pm}(s)$, then we have $\boldsymbol{v} = ((\sigma_g^{\pm}(s))^2 - 1)\overline{\boldsymbol{\gamma}}(s) + 2\sigma_g^{\pm}(s)\boldsymbol{n}(s) + ((\sigma_g^{\pm}(s))^2 + 1)\boldsymbol{e}_0$.

(4) Since $h'''_{\boldsymbol{v}}(s) = \langle\kappa'_g(s)\boldsymbol{n}(s) - (\kappa_g^2(s) + 1)\boldsymbol{t}(s), \boldsymbol{v}\rangle$, $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = 0$ if and only if $h'''_{\boldsymbol{v}}(s) = \langle\kappa'_g(s)\boldsymbol{n}(s) - (\kappa_g^2(s) + 1)\boldsymbol{t}(s), ((\sigma_g^{\pm}(s))^2 - 1)\overline{\boldsymbol{\gamma}}(s) + 2\sigma_g^{\pm}(s)\boldsymbol{n}(s) + ((\sigma_g^{\pm}(s))^2 + 1)\boldsymbol{e}_0\rangle = 2\kappa'_g(s)\sigma_g^{\pm}(s) = 0$. For $\sigma_g^{\pm}(s) \neq 0$, we have $\kappa'_g(s) = 0$. Then we have $\boldsymbol{v} = ((\sigma_g^{\pm}(s))^2 - 1)\overline{\boldsymbol{\gamma}}(s) + 2\sigma_g^{\pm}(s)\boldsymbol{n}(s) + ((\sigma_g^{\pm}(s))^2 + 1)\boldsymbol{e}_0$ and $\kappa'_g(s) = 0$.

(5) Since $h^{(4)}_{\boldsymbol{v}}(s) = \langle(\kappa''_g(s) - \kappa_g^3(s) - \kappa_g(s))\boldsymbol{n}(s) - 3\kappa_g(s)\kappa'_g(s)\boldsymbol{t}(s) + (\kappa_g^2(s) + 1)\overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = h^{(4)}_{\boldsymbol{v}}(s) = 0$ if and only if $h^{(4)}_{\boldsymbol{v}}(s) = \langle(\kappa''_g(s) - \kappa_g^3(s) - \kappa_g(s))\boldsymbol{n}(s) - 3\kappa_g(s)\kappa'_g(s)\boldsymbol{t}(s) + (\kappa_g^2(s) + 1)\overline{\boldsymbol{\gamma}}(s), ((\sigma_g^{\pm}(s))^2 - 1)\overline{\boldsymbol{\gamma}}(s) + 2\sigma_g^{\pm}(s)\boldsymbol{n}(s) + ((\sigma_g^{\pm}(s))^2 + 1)\boldsymbol{e}_0\rangle = 2\kappa''_g(s)\sigma_g^{\pm}(s) = 0$. For $\sigma_g^{\pm}(s) \neq 0$, we have $\kappa''_g(s) = 0$. Then we have $\boldsymbol{v} = ((\sigma_g^{\pm}(s))^2 - 1)\overline{\boldsymbol{\gamma}}(s) + 2\sigma_g^{\pm}(s)\boldsymbol{n}(s) + ((\sigma_g^{\pm}(s))^2 + 1)\boldsymbol{e}_0$, $\kappa'_g(s) = 0$ and $\kappa''_g(s) = 0$. □

3.3 Singularities of ligtcone duals of spherical curves

In this section we classify the singularities of $\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ and $LD_{\boldsymbol{\gamma}}$ as an application of the unfolding theory of functions. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be a function germ, we call F

an r-*parameter unfolding* of f, where $f(s) = F_{\boldsymbol{x}_0}(s, \boldsymbol{x}_0)$. If $f^{(p)}(s_0) = 0$ for all $1 \le p \le k$ and $f^{(k+1)}(s_0) \neq 0$, then we say that f has A_k-*singularity* at s_0. Let F be an r-parameter unfolding of f and f has A_k-singularity ($k \ge 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 as

$$j^{(k-1)} \left(\frac{\partial F}{\partial x_i}(s, \boldsymbol{x}_0) \right)(s_0) = \sum_{j=1}^{k-1} \alpha_{ji}(s - s_0)^j, \quad (i = 1, \cdots, r).$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k $(k \le r)$, then F is called a *versal unfolding* of f, where $\alpha_{0i} = \partial F/\partial x_i(s_0, \boldsymbol{x}_0)$. The *discriminant set* of F is defined by

$$D_F = \left\{ \boldsymbol{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R},\ F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s}(s, \boldsymbol{x}) = 0 \right\}.$$

For the discriminant set of F, we have the following theorem [3].

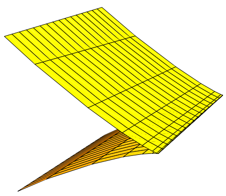
Theorem 3.3.1. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of f which has A_k-singularity at s_0. Suppose F is a versal unfolding of f, then we have the following assertions.

(a) If $k = 1$, then D_F is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.

(b) If $k = 2$, then D_F is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.

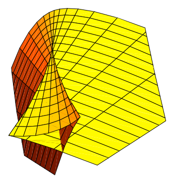
(c) If $k = 3$, then D_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.

Here, C is the ordinary cusp and $C \times \mathbb{R}$ is called a *cuspidal edge* (Fig.2). Moreover, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is called a *swallow tail* (Fig.3).



The cuspidal edge

Figure 2



The swallow tail

Figure 3

By Propositions 3.2.1 and 3.2.2, the discriminant set of $\overline{H}$ and H are

$$D_{\overline{H}} = \{\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) \pm \sqrt{u^2+1}\boldsymbol{e}_0 \mid (s, u) \in I \times \mathbb{R}\},$$

$$D_H = \{(u^2/4 - 1)\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) + (u^2/4 + 1)\boldsymbol{e}_0 \mid s \in I, u \in \mathbb{R}\}.$$

These are the lightcone dual surface of $\overline{\boldsymbol{\gamma}}$ and the lightcone dual surface of $\boldsymbol{\gamma}$ respectively. We have the following key propositions on H and $\overline{H}$.

Proposition 3.3.2. If $\overline{h}_{\boldsymbol{v_0}}(s)$ has A_k-singularity $(k = 1, 2, 3)$ at s_0, then $\overline{H}$ is a versal unfolding of $\overline{h}_{\boldsymbol{v_0}}$.

Proof. For $\boldsymbol{v} \in LC^*$, we have $\boldsymbol{v} = (\pm(v_1^2 + v_2^2 + v_3^2)^{1/2}, v_1, v_2, v_3)$, then

$$\overline{H}(s, \boldsymbol{v}) = \langle \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v} \rangle - 1 = x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 - 1.$$

Thus we have

$$\frac{\partial \overline{H}}{\partial v_1}(s, \boldsymbol{v}) = x_1(s), \frac{\partial \overline{H}}{\partial v_2}(s, \boldsymbol{v}) = x_2(s), \frac{\partial \overline{H}}{\partial v_3}(s, \boldsymbol{v}) = x_3(s),$$
$$\frac{\partial^2 \overline{H}}{\partial s \partial v_1}(s, \boldsymbol{v}) = x_1'(s), \ \frac{\partial^2 \overline{H}}{\partial s \partial v_2}(s, \boldsymbol{v}) = x_2'(s), \ \frac{\partial^2 \overline{H}}{\partial s \partial v_3}(s, \boldsymbol{v}) = x_3'(s),$$
$$\frac{\partial^3 \overline{H}}{\partial s^2 \partial v_1}(s, \boldsymbol{v}) = x_1''(s), \ \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_2}(s, \boldsymbol{v}) = x_2''(s), \ \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_3}(s, \boldsymbol{v}) = x_3''(s).$$

For a fixed point $\boldsymbol{v}_0 = (v_{00}, v_{01}, v_{02}, v_{03})$, the 2-jet of $\partial \overline{H}/\partial v_i(s, \boldsymbol{v}_0)(i = 1, 2, 3)$ at s_0 is

$$j^{(2)} \frac{\partial \overline{H}}{\partial v_i}(s, \boldsymbol{v}_0)(s_0) = x_i'(s_0)(s - s_0) + x_i''(s_0)(s - s_0)^2/2, \ (i = 1, 2, 3).$$

It is enough to show that the rank of the matrix A is three, where

$$A = \begin{pmatrix} x_1(s_0) & x_2(s_0) & x_3(s_0) \\ x_1'(s_0) & x_2'(s_0) & x_3'(s_0) \\ x_1''(s_0) & x_2''(s_0) & x_3''(s_0) \end{pmatrix}.$$

Then we have

$$\det A = \langle \boldsymbol{e}_0 \wedge \overline{\boldsymbol{\gamma}}(s_0) \wedge \overline{\boldsymbol{\gamma}}'(s_0), \overline{\boldsymbol{\gamma}}''(s_0) \rangle = \langle -\boldsymbol{n}(s_0), \kappa_g(s_0)\boldsymbol{n}(s_0) - \overline{\boldsymbol{\gamma}}(s_0) \rangle = -\kappa_g(s_0) \neq 0.$$

So the rank of A is three, this completes the proof. □

Proposition 3.3.3. If $h_{\boldsymbol{v_0}}(s)$ has A_k-singularity $(k = 1, 2, 3)$ at s_0, then H is a versal unfolding of $h_{\boldsymbol{v_0}}$.

Proof. For $\boldsymbol{v} \in LC^*$, we have $\boldsymbol{v} = (\pm(v_1^2 + v_2^2 + v_3^2)^{1/2}, v_1, v_2, v_3)$, then

$$H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle + 2 = \mp(v_1^2 + v_2^2 + v_3^2)^{1/2} + x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 + 2.$$

Thus we have

$$\frac{\partial H}{\partial v_1}(s, \boldsymbol{v}) = -v_1/v_0 + x_1(s), \frac{\partial H}{\partial v_2}(s, \boldsymbol{v}) = -v_2/v_0 + x_2(s), \frac{\partial H}{\partial v_3}(s, \boldsymbol{v}) = -v_3/v_0 + x_3(s),$$

$$\frac{\partial^2 H}{\partial s \partial v_1}(s, \boldsymbol{v}) = x_1'(s),\ \frac{\partial^2 H}{\partial s \partial v_2}(s, \boldsymbol{v}) = x_2'(s),\ \frac{\partial^2 H}{\partial s \partial v_3}(s, \boldsymbol{v}) = x_3'(s),$$
$$\frac{\partial^3 H}{\partial s^2 \partial v_1}(s, \boldsymbol{v}) = x_1''(s),\ \frac{\partial^3 H}{\partial s^2 \partial v_2}(s, \boldsymbol{v}) = x_2''(s),\ \frac{\partial^3 H}{\partial s^2 \partial v_3}(s, \boldsymbol{v}) = x_3''(s).$$

For a fixed $\boldsymbol{v}_0 = (v_{00}, v_{01}, v_{02}, v_{03})$, the 2-jet of $\frac{\partial H}{\partial v_i}(s, \boldsymbol{v}_0)(i = 1, 2, 3)$ at s_0 is

$$j^{(2)}\frac{\partial H}{\partial v_i}(s, \boldsymbol{v}_0)(s_0) = x_i'(s_0)(s - s_0) + x_i''(s_0)(s - s_0)^2/2,\ (i = 1, 2, 3).$$

It is enough to show that the rank of the matrix A is three, where

$$A = \begin{pmatrix} -v_{01}/v_{00} + x_1(s_0) & -v_{02}/v_{00} + x_2(s_0) & -v_{03}/v_{00} + x_3(s_0) \\ x_1'(s_0) & x_2'(s_0) & x_3'(s_0) \\ x_1''(s_0) & x_2''(s_0) & x_3''(s_0) \end{pmatrix}.$$

By straight calculation, we have

$$\det A = -\langle \boldsymbol{e}_0 \wedge \overline{\boldsymbol{\gamma}}'(s_0) \wedge \overline{\boldsymbol{\gamma}}''(s_0), \boldsymbol{v}_0 \rangle / v_{00} + \langle \boldsymbol{e}_0 \wedge \overline{\boldsymbol{\gamma}}(s_0) \wedge \overline{\boldsymbol{\gamma}}'(s_0), \overline{\boldsymbol{\gamma}}''(s_0) \rangle$$

$$= -\langle \boldsymbol{e}_0 \wedge \boldsymbol{t}(s_0) \wedge (\kappa_g(s_0)\boldsymbol{n}(s_0) - \overline{\boldsymbol{\gamma}}(s_0)), \boldsymbol{v}_0 \rangle / v_{00} + \langle -\boldsymbol{n}(s_0), \kappa_g(s_0)\boldsymbol{n}(s_0) - \overline{\boldsymbol{\gamma}}(s_0) \rangle$$

$$= \langle \kappa_g(s_0)\overline{\boldsymbol{\gamma}}(s_0) + \boldsymbol{n}(s_0), \boldsymbol{v}_0 \rangle / v_{00} - \kappa_g(s_0).$$

Since $\boldsymbol{v}_0 \in D_H$ is a singular point, we have

$$\boldsymbol{v}_0 = ((\sigma_g^{\pm}(s_0))^2 - 1)\overline{\boldsymbol{\gamma}}(s_0) + 2\sigma_g^{\pm}(s_0)\boldsymbol{n}(s_0) + ((\sigma_g^{\pm}(s_0))^2 + 1)\boldsymbol{e}_0.$$

Moreover we have

$$v_{00} = (\sigma_g^{\pm}(s_0))^2 + 1.$$

Therefore we have

$$\det A = \pm 2(\kappa_g^2(s_0) + 1)^{1/2}/v_{00} \neq 0$$

So the rank of A is three, this completes the proof. □

We have the following theorem:

Theorem 3.3.4. Let $\boldsymbol{\gamma} : I \longrightarrow S_+^2$ be a unit speed curve.

(A) For the lightcone dual $\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ of $\overline{\boldsymbol{\gamma}}$, we have the following assertions:

(1) The lightcone dual $\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ of $\overline{\boldsymbol{\gamma}}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, u_0) if and only if $k_g'(s_0) \neq 0$ and $u_0 = 1/\kappa_g(s_0)$.

(2) The lightcone dual $\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ of $\overline{\boldsymbol{\gamma}}$ is locally diffeomorphic to the swallowtail at (s_0, u_0) if and only if $\kappa_g'(s_0) = 0$, $\kappa_g''(s_0) \neq 0$ and $u_0 = 1/\kappa_g(s_0)$.

(B) For the lightcone dual $LD_{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$, we have the following assertions:

(1) The lightcone dual $LD_{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, u_0) if and only if $k_g'(s_0) \neq 0$ and $u_0 = 2\sigma_g^{\pm}(s_0)$.

(2) The lightcone dual $LD_{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallowtail at (s_0, u_0) if and only if $\kappa_g'(s_0) = 0$, $\kappa_g''(s_0) \neq 0$ and $u_0 = 2\sigma_g^{\pm}(s_0)$.

Proof. By Propositions 3.2.1 and 3.2.2, the discriminant sets of $\overline{H}$ and H are the lightcone duals of $\overline{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}$ respectively. By Propositions 3.2.1 and 3.2.2, both of $\overline{h}_{\boldsymbol{v}_0}$ and $h_{\boldsymbol{v}_0}$ have A_k singularities $(k = 1, 2, 3)$ respectively if and only if the above conditions on the geodesic curvatures hold. By Propositions 3.3.2 and 3.3.3, $\overline{H}$ and H are versal unfoldings of $\overline{h}_{\boldsymbol{v}_0}$ and $h_{\boldsymbol{v}_0}$ at any point $s_0 \in I$ respectively. We apply Theorem 3.3.1, so that we have the above assertions. □

Remark 3.3.5. As a consequence of Proposition 3.1.2, Corollary 3.1.3 and Theorem 3.3.4, we can summarize the relationship between the singularities of $\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}, \overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ and $LD_{\boldsymbol{\gamma}}$:

$\boldsymbol{\gamma}(s_0)$ in S^2_+	$\kappa'_g(s_0) \neq 0$	$\kappa'_g(s_0) = 0, \kappa''_g(s_0) \neq 0$
$\overline{\boldsymbol{\gamma}}(s_0)$ in $\mathbb{R}^3_0$	$\tau(s_0) \neq 0$	$\tau(s_0) = 0, \tau'(s_0) \neq 0$
$\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}(s_0)$	the regular point	the ordinary cusp
$\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}(s_0, 1/\kappa_g(s_0))$	the cuspidal edge	the swallowtail
$LD_{\boldsymbol{\gamma}}(s_0, 2\sigma^{\pm}_g(s_0))$	the cuspidal edge	the swallowtail

We can interpret the above correspondence between the cusps of the spherical evolute and the swallowtails of lightcone duals from the view point of the theory of Legendrian/Lagrangian singularities. For spherical curves, however, we only need the curvature or the torsion. Such a framework is really needed for the higher dimensional case.

4 Lightcone dualities for curves in the 3-sphere

4.1 Curves in the unit 3-sphere and focal surfaces

Let $\boldsymbol{\gamma} : I \longrightarrow S^3_+$ be a regular curve. We have a map $\Phi : S^3_+ \to S^3_0$ defined by $\Phi(\boldsymbol{v}) = \boldsymbol{v} - \boldsymbol{e}_0$, which is an isometry. Then we have a regular curve $\overline{\boldsymbol{\gamma}} : I \to S^3_0$ defined by $\overline{\boldsymbol{\gamma}}(s) = \Phi(\boldsymbol{\gamma}(s)) = \boldsymbol{\gamma}(s) - \boldsymbol{e}_0$, so that $\boldsymbol{\gamma}$ and $\overline{\boldsymbol{\gamma}}$ have completely the same geometric properties as spherical curves. Since $\overline{\boldsymbol{\gamma}}$ is a spacelike curve, we can reparameterize it by the arc-length s. So we have the unit tangent vector $\boldsymbol{t}(s) = \overline{\boldsymbol{\gamma}}'(s)$ of $\overline{\boldsymbol{\gamma}}(s)$. Suppose that $\|\boldsymbol{t}'(s)\| \neq 1$. Then $\|\boldsymbol{t}'(s) + \overline{\boldsymbol{\gamma}}(s)\| \neq 0$, so that we have another unit vector $\boldsymbol{n}(s) = \frac{\boldsymbol{t}'(s)+\overline{\boldsymbol{\gamma}}(s)}{\|\boldsymbol{t}'(s)+\overline{\boldsymbol{\gamma}}(s)\|}$. We also define a unit vector by $\boldsymbol{b}(s) = \overline{\boldsymbol{\gamma}}(s) \wedge \boldsymbol{e}_0 \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$, then we have a pseudo-orthonormal frame field $\{\overline{\boldsymbol{\gamma}}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ of $\mathbb{R}^4_0$ along $\overline{\boldsymbol{\gamma}}(s)$. By standard arguments, we have the following *Frenet-Serret type formulae.*

$$\begin{cases} \overline{\boldsymbol{\gamma}}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = \kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s) \\ \boldsymbol{n}'(s) = -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{b}(s) \\ \boldsymbol{b}'(s) = -\tau_g(s)\boldsymbol{n}(s) \end{cases},$$

where $\kappa_g(s) = \|\boldsymbol{t}'(s)+\overline{\boldsymbol{\gamma}}(s)\|$ and $\tau_g(s) = -\det(\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}'(s), \overline{\boldsymbol{\gamma}}''(s), \overline{\boldsymbol{\gamma}}'''(s))/\kappa_g^2(s)$. We call $\{\overline{\boldsymbol{\gamma}}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ a *Sabban frame* of $\overline{\boldsymbol{\gamma}}$ [12]. Here, κ_g is called a *geodesic curvature* and τ_g a *geodesic torsion* of $\overline{\boldsymbol{\gamma}}$ in S^3_0 respectively.

We now consider the focal surface of a curve $\overline{\boldsymbol{\gamma}} : I \to S^3_0$ analogous to the case for curves in Euclidean space. We define $F^{\pm} : I \times J \to S^3_0$ by

$$F^{\pm}(s,u) = u\overline{\boldsymbol{\gamma}}(s) + \frac{u}{\kappa_g(s)}\boldsymbol{n}(s) \pm \frac{\sqrt{\kappa_g^2(s) - u^2(\kappa_g^2(s)+1)}}{\kappa_g(s)}\boldsymbol{b}(s).$$

We call each image of $F^{\pm}$ the *spherical focal surface of* $\overline{\boldsymbol{\gamma}}$. We remark that the focal surfaces of $\overline{\boldsymbol{\gamma}}$ satisfies the equations $\langle \overline{\boldsymbol{\gamma}}'(s), F^{\pm}(s,u)\rangle = \langle \overline{\boldsymbol{\gamma}}''(s), F^{\pm}(s,u)\rangle = 0$. This means that each one of the focal surface $F^{\pm}(s,u)$ of $\boldsymbol{\gamma}$ is the spherical dual of $\boldsymbol{t}$ in the sense of [16]. By straightforward calculations, we have

$$
\begin{aligned}
\frac{\partial F^{\pm}}{\partial u}(s,u) &= \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) \pm \frac{-u(\kappa_g^2(s)+1)}{\kappa_g(s)\sqrt{\kappa_g^2(s)-u^2(\kappa_g^2(s)+1)}}\boldsymbol{b}(s),\\
\frac{\partial F^{\pm}}{\partial s}(s,u) &= -\frac{u\kappa_g'(s) \pm \tau_g(s)\kappa_g(s)\sqrt{\kappa_g^2(s)-u^2(\kappa_g^2(s)+1)}}{\kappa_g^2(s)}\boldsymbol{n}(s)\\
&\quad + \frac{u\tau_g(s)\kappa_g(s)\sqrt{\kappa_g^2(s)-u^2(\kappa_g^2(s)+1)} \pm u^2\kappa_g'(s)}{\kappa_g^2(s)\sqrt{\kappa_g^2(s)-u^2(\kappa_g^2(s)+1)}}\boldsymbol{b}(s).
\end{aligned}
$$

It follows that $\{\partial F^{\pm}/\partial u, \partial F^{\pm}/\partial s\}$ is linearly dependent if and only if

$$
\tau_g(s)\kappa_g(s)\sqrt{\kappa_g^2(s)-u^2(\kappa_g^2(s)+1)} \pm u\kappa_g'(s) = 0,
$$

so that we have

$$
u = \frac{\pm\tau_g(s)\kappa_g^2(s)}{\sqrt{\kappa_g'^2(s)+\kappa_g^4(s)\tau_g^2(s)+\kappa_g^2(s)\tau_g^2(s)}}.
$$

Therefore each critical value set of $F^{\pm}$ is given by

$$
\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s) = \frac{\pm\tau_g(s)\kappa_g^2(s)}{\sqrt{\kappa_g'^2(s)+\kappa_g^4(s)\tau_g^2(s)+\kappa_g^2(s)\tau_g^2(s)}}\left\{\overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) + \left(\frac{1}{\kappa_g(s)}\right)'\frac{1}{\tau_g(s)}\boldsymbol{b}(s)\right\}.
$$

We remark that each curve of $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}$ satisfies the equations

$$
\langle \overline{\boldsymbol{\gamma}}'(s), \boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s)\rangle = \langle \overline{\boldsymbol{\gamma}}''(s), \boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s)\rangle = \langle \overline{\boldsymbol{\gamma}}'''(s), \boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s)\rangle = 0.
$$

In [18] Porteous introduced the notion of the evolute of $\overline{\boldsymbol{\gamma}}$ in the unit 3-sphere. He defined it as the curve satisfies the above equations, so that we call each image of $\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}^{\pm}$ the *spherical evolute* of $\overline{\boldsymbol{\gamma}}$ in the unit 3-sphere. We remark that $\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}^{-}(s) = -\boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}^{+}(s)$. For $s = s_0$, we fix that $\boldsymbol{v}_0^{\pm} = \boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s_0)$ and $\langle \overline{\boldsymbol{\gamma}}(s_0), \boldsymbol{\varepsilon}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s_0)\rangle = c^{\pm}$. Since $\boldsymbol{v}_0^- = -\boldsymbol{v}_0^+$ and $c^- = -c^+$, we have a hyperplane

$$
HP(\boldsymbol{v}_0^+, c^+) = \{\boldsymbol{x} \in \mathbb{R}_0^4 \mid \langle \boldsymbol{x}, \boldsymbol{v}_0^+\rangle = c^+\ \} = \{\boldsymbol{x} \in \mathbb{R}_0^4 \mid \langle \boldsymbol{x}, \boldsymbol{v}_0^-\rangle = c^-\ \} = HP(\boldsymbol{v}_0^-, c^-),
$$

so that we have a sphere

$$
S^2(\boldsymbol{v}_0^{\pm}, c^{\pm}) = HP(\boldsymbol{v}_0^{\pm}, c^{\pm}) \cap S_0^3.
$$

We call $S^2(\boldsymbol{v}_0^{\pm}, c^{\pm})$ an *osculating sphere* of $\overline{\boldsymbol{\gamma}}$ at s_0. Therefore the spherical evolutes $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}(s)$ are the loci of the centers of osculating spheres of $\overline{\boldsymbol{\gamma}}$ respectively.

Proposition 4.1.1. There exists a sphere $S^2(\boldsymbol{v}, c) \subset S_0^3$ such that $\overline{\boldsymbol{\gamma}}(I) \subset S^2(\boldsymbol{v}, c)$ if and only if both of the spherical evolutes $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}$ of $\overline{\boldsymbol{\gamma}}$ are constant.

Proof. If one of the spherical evolutes $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{+}$ of $\overline{\boldsymbol{\gamma}}$ is constant, we can set that $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{+}(s) = \boldsymbol{v}^+$. In this case another spherical evolute $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{-}$ is constant too. Then $\langle \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}^+ \rangle' = \langle \overline{\boldsymbol{\gamma}}'(s), \boldsymbol{v}^+ \rangle = \langle \boldsymbol{t}(s), \boldsymbol{\varepsilon}_{\overline{\gamma}}^{+}(s) \rangle = 0$, so we have $\langle \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}^+ \rangle = c^+$ and $\overline{\boldsymbol{\gamma}}(I) \subset S^2(\boldsymbol{v}^+, c^+)$. On the contrary, if $\overline{\boldsymbol{\gamma}}(I) \subset S^2(\boldsymbol{v}, c)$, then at any point on $\overline{\boldsymbol{\gamma}}$, the osculating spheres is $S^2(\boldsymbol{v}, c)$ itself. So the locus of the centers of osculating spheres of $\overline{\boldsymbol{\gamma}}$ is $\boldsymbol{v}$ and $-\boldsymbol{v}$. Therefore, both of the spherical evolutes $\boldsymbol{\varepsilon}_{\overline{\gamma}}^{\pm}$ of $\overline{\boldsymbol{\gamma}}$ are constant. This completes the proof. □

4.2 Lightcone duals of curves in the unit 3-sphere

We now define hypersurfaces in LC^* associated with the curves in S_+^3 or S_0^3. Let $\boldsymbol{\gamma} : I \longrightarrow S_+^3$ be a unit speed curve. We define $\overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm} : I \times \mathbb{R}^2 \longrightarrow LC^*$ by

$$\overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s, u, v) = \overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) + v\boldsymbol{b}(s) \pm \sqrt{u^2 + v^2 + 1}\boldsymbol{e}_0.$$

We also define $LD_{\boldsymbol{\gamma}} : I \times \mathbb{R}^2 \longrightarrow LC^*$ by

$$LD_{\boldsymbol{\gamma}}(s, u, v) = \frac{u^2 + v^2 - 4}{4}\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) + v\boldsymbol{b}(s) + \frac{u^2 + v^2 + 4}{4}\boldsymbol{e}_0.$$

Then we have the following proposition.

Proposition 4.2.1. Under the above notations, we have the followings:
(1) $\overline{\boldsymbol{\gamma}}$ and $\overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}$ are Δ_3-dual to each other.
(2) $\boldsymbol{\gamma}$ and $LD_{\boldsymbol{\gamma}}$ are Δ_4-dual to each other.

Proof. Consider the mapping $\mathscr{L}_3(s, u, v) = (\overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s, u, v), \overline{\boldsymbol{\gamma}}(s))$. Then we have

$$\langle \overline{LD}_{\overline{\boldsymbol{\gamma}}}^{\pm}(s, u, v), \overline{\boldsymbol{\gamma}}(s) \rangle = \langle \overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}(s) \rangle = 1$$

and

$$\mathscr{L}_3^*\theta_{32} = \langle \overline{LD}^{\pm}_{\overline{\gamma}}(s,u,v), \overline{\gamma}'(s)\rangle ds = \langle \overline{LD}^{\pm}_{\overline{\gamma}}(s,u,v), \boldsymbol{t}(s)\rangle ds = 0.$$

The assertion (1) holds.

We also consider the mapping $\mathscr{L}_4(s,u,v) = (LD_{\boldsymbol{\gamma}}(s,u,v), \boldsymbol{\gamma}(s))$. Since $\langle \boldsymbol{\gamma}(s), \boldsymbol{e}_0\rangle = -1$ and $\langle \boldsymbol{\gamma}(s), \overline{\boldsymbol{\gamma}}(s)\rangle = 1$, we have $\langle LD_{\boldsymbol{\gamma}}(s,u,v), \boldsymbol{\gamma}(s)\rangle = (u^2+v^2)/4 - 1 - ((u^2+v^2)/4+1) = -2$. Moreover, we have

$$\mathscr{L}_4^*\theta_{42} = \langle LD_{\boldsymbol{\gamma}}(s,u,v)), \boldsymbol{\gamma}'(s)\rangle ds = \langle LD_{\boldsymbol{\gamma}}(s,u,v), \boldsymbol{t}(s)\rangle ds = 0.$$

This completes the proof. $\square$

We call each one of $\overline{LD}^{\pm}_{\overline{\gamma}}$ the *Lightcone dual haypersurface of the de Sitter spherical curve* $\overline{\boldsymbol{\gamma}}$ and $LD_{\boldsymbol{\gamma}}$ the *Lightcone dual hypersurface of the lightlike spherical curve* $\boldsymbol{\gamma}$. Then we have two mappings $\pi \circ \overline{LD}^{\pm}_{\overline{\gamma}} : I \times \mathbb{R}^2 \to S^3_+$ and $\pi \circ LD_{\boldsymbol{\gamma}} : I \times \mathbb{R}^2 \to S^3_+$ defined by

$$\begin{aligned}
\pi \circ \overline{LD}^{\pm}_{\overline{\gamma}}(s,u,v) &= \pm\left(\frac{1}{\sqrt{u^2+v^2+1}}\overline{\boldsymbol{\gamma}}(s) + \frac{u}{\sqrt{u^2+v^2+1}}\boldsymbol{n}(s) + \frac{v}{\sqrt{u^2+v^2+1}}\boldsymbol{b}(s)\right) + \boldsymbol{e}_0,\\
\pi \circ LD_{\boldsymbol{\gamma}}(s,u,v) &= \frac{u^2+v^2-4}{u^2+v^2+4}\overline{\boldsymbol{\gamma}}(s) + \frac{4u}{u^2+v^2+4}\boldsymbol{n}(s) + \frac{4v}{u^2+v^2+4}\boldsymbol{b}(s) + \boldsymbol{e}_0.
\end{aligned}$$

In this paper we consider the singularities of these dual surfaces and mappings. By the Frenet-Serret type formulae, we have

$$\begin{aligned}
\frac{\partial \overline{LD}^{\pm}_{\overline{\gamma}}}{\partial u}(s,u,v) &= \boldsymbol{n}(s) \pm \frac{u}{\sqrt{1+u^2+v^2}}\boldsymbol{e}_0,\\
\frac{\partial \overline{LD}^{\pm}_{\overline{\gamma}}}{\partial v}(s,u,v) &= \boldsymbol{b}(s) \pm \frac{v}{\sqrt{1+u^2+v^2}}\boldsymbol{e}_0,\\
\frac{\partial \overline{LD}^{\pm}_{\overline{\gamma}}}{\partial s}(s,u,v) &= (1-u\kappa_g(s))\boldsymbol{t}(s) - v\tau_g(s)\boldsymbol{n}(s) + u\tau_g(s)\boldsymbol{b}(s),\\
\frac{\partial LD_{\boldsymbol{\gamma}}}{\partial u}(s,u,v) &= \frac{u}{2}\overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s) + \frac{u}{2}\boldsymbol{e}_0,\\
\frac{\partial LD_{\boldsymbol{\gamma}}}{\partial v}(s,u,v) &= \frac{v}{2}\overline{\boldsymbol{\gamma}}(s) + \boldsymbol{b}(s) + \frac{v}{2}\boldsymbol{e}_0,\\
\frac{\partial LD_{\boldsymbol{\gamma}}}{\partial s}(s,u,v) &= \frac{u^2+v^2-4u\kappa_g(s)-4}{4}\boldsymbol{t}(s) - v\tau_g(s)\boldsymbol{n}(s) + u\tau_g(s)\boldsymbol{b}(s).
\end{aligned}$$

Then we have the following proposition.

Proposition 4.2.2. Let $\boldsymbol{\gamma} : I \longrightarrow S^3_+$ be a unit speed curve. Then we have the followings:
(1) (s,u,v) is a singular point of $\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ if and only if $u = 1/\kappa_g(s)$.
(2) (s,u,v) is a singular point of $LD_{\boldsymbol{\gamma}}$ if and only if $v = \pm\sqrt{4 + 4u\kappa_g(s) - u^2}$.

Proof. By the above calculations, $\{\partial\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}/\partial u(s,u,v), \partial\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}/\partial v(s,u,v), \partial\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}/\partial s(s,u,v)\ \}$ is linearly dependent if and only if $u = 1/\kappa_g(s)$.The assertion (1) follows. By the similar reason, we have the assertion (2). This completes the proof. □

Therefore, the critical value sets of the above dual surfaces are given by

$$\begin{aligned} C(\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}) &= \left\{ \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) + v\boldsymbol{b}(s) \pm \sqrt{\frac{1+\kappa_g^2(s) + v^2\kappa_g^2(s)}{\kappa_g^2(s)}}\boldsymbol{e}_0 \ \Big|\ v \in \mathbb{R}, s \in I, \kappa_g(s) \neq 0 \right\}, \\ C(LD_{\boldsymbol{\gamma}})^{\pm} &= \{\kappa_g(s)u\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) \pm \sqrt{4 + 4u\kappa_g(s) - u^2}\boldsymbol{b}(s) + (\kappa_g(s)u + 2)\boldsymbol{e}_0 | u \in \mathbb{R}, s \in I\ \}. \end{aligned}$$

We respectively denote that

$$\begin{aligned} LF^{\pm}_{\overline{\boldsymbol{\gamma}}}(s,v) &= \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) + v\boldsymbol{b}(s) \pm \sqrt{\frac{1+\kappa_g^2(s) + v^2\kappa_g^2(s)}{\kappa_g^2(s)}}\boldsymbol{e}_0, \\ LF^{\pm}_{\boldsymbol{\gamma}}(s,u) &= \kappa_g(s)u\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) \pm \sqrt{4 + 4u\kappa_g(s) - u^2}\boldsymbol{b}(s) + (\kappa_g(s)u + 2)\boldsymbol{e}_0, \end{aligned}$$

where we have the relation $v = \pm\sqrt{4 + 4u\kappa_g(s) - u^2}$. We respectively call each one of $LF^{\pm}_{\overline{\boldsymbol{\gamma}}}$ the *lightcone focal surface* of the de Sitter spherical curve $\overline{\boldsymbol{\gamma}}$ and each one of $LF^{\pm}_{\boldsymbol{\gamma}}$ the *ligtcone focal surface* of the lightcone spherical curve $\boldsymbol{\gamma}$. Then the projections of these surfaces to S^3_+ are given as follows:

$$\begin{aligned} \pi(C(\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}})) &= \left\{ \frac{\pm(\kappa_g(s)\overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s) + v\kappa_g(s)\boldsymbol{b}(s))}{\sqrt{1+\kappa_g^2(s) + v^2\kappa_g^2(s)}} + \boldsymbol{e}_0 \ \Big|\ v \in \mathbb{R}, s \in I, \kappa_g(s) \neq 0 \right\}, \\ \pi(C(LD_{\boldsymbol{\gamma}})^{\pm}) &= \left\{ \frac{u\kappa_g(s)\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) \pm \sqrt{4 + 4u\kappa_g(s) - u^2}\boldsymbol{b}(s)}{\kappa_g(s)u + 2} + \boldsymbol{e}_0 \ |u \in \mathbb{R}, s \in I \right\}. \end{aligned}$$

On the other hand, we define $\widetilde{\pi} = \Phi \circ \pi : LC^* \to S^3_0$. By the previous calculations, $\widetilde{\pi}(C(\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}))$ is different from $\widetilde{\pi}(C(LD_{\boldsymbol{\gamma}})^{\pm})$. In [10], it was shown that the projections of the critical value sets of the lightcone dual surfaces of $\boldsymbol{\gamma}$ and $\overline{\boldsymbol{\gamma}}$ are the same for a curve $\boldsymbol{\gamma} : I \to S^2_+$. Moreover, it is equal to the spherical evolute of $\overline{\boldsymbol{\gamma}}$. Therefore, the situation for curves in S^3_+ is quite different from that for curves in S^2_+.

4.3 Lightcone height functions

In order to study the singularities of Lightcone dual surfaces of spherical curves, we introduce two families of functions and apply the theory of unfoldings. Let $\boldsymbol{\gamma} : I \longrightarrow S^3_+$ be a unit speed curve, then we define two families of functions as follows:

$$\overline{H} : I \times LC^* \longrightarrow \mathbb{R}, \qquad \overline{H}(s, \boldsymbol{v}) = \langle \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v} \rangle - 1,$$

$$H : I \times LC^* \longrightarrow \mathbb{R}, \qquad H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle + 2.$$

We call $\overline{H}$ a *lightcone height function* of the de Sitter spherical curve $\overline{\boldsymbol{\gamma}}$. For any fixed $\boldsymbol{v} \in LC^*$, we denote $\overline{h}_{\boldsymbol{v}}(s) = \overline{H}(s, \boldsymbol{v})$. We call H a *lightcone height function* of the lightlike spherical curve $\boldsymbol{\gamma}$. For any fixed $\boldsymbol{v} \in LC^*$, we denote $h_{\boldsymbol{v}}(s) = H(s, \boldsymbol{v})$. Then we have the following two propositions on $h_{\boldsymbol{v}}$ and $\overline{h}_{\boldsymbol{v}}$.

For simplification, we denote $\rho(s) = \sqrt{(\kappa_g^4(s)\tau_g^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g'^2(s))/\kappa_g^4(s)\tau_g^2(s)}$ and $\sigma^{\pm}(s) = (\kappa_g^2(s)\tau_g(s) \pm \sqrt{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s)})/(\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s))$.

Proposition 4.3.1. Let $\boldsymbol{\gamma} : I \longrightarrow S^3_+$ be a unit speed curve, then we have the followings:

(1) $\overline{h}_{\boldsymbol{v}}(s) = 0$ if and only if there exist $\lambda, \mu, \xi, \eta \in \mathbb{R}$ with $\eta^2 = 1 + \lambda^2 + \mu^2 + \xi^2$ such that $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \lambda \boldsymbol{t}(s) + \mu \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) + \eta \boldsymbol{e}_0$.

(2) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = 0$ if and only if there exist $\mu, \xi, \eta \in \mathbb{R}$ with $\eta^2 = 1 + \mu^2 + \xi^2$ such that $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \mu \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) + \eta \boldsymbol{e}_0 = \overline{\boldsymbol{\gamma}}(s) + \mu \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) \pm \sqrt{1 + \mu^2 + \xi^2}\boldsymbol{e}_0$.

(3) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g(s) \neq 0$ and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)} \boldsymbol{n}(s) + \xi \boldsymbol{b}(s) \pm \sqrt{\frac{1 + \kappa_g^2(s) + \kappa_g^2(s)\xi^2}{\kappa_g^2(s)}} \boldsymbol{e}_0.$$

(4) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = \overline{h}'''_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g(s) \neq 0$, $\tau_g(s) \neq 0$ and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)} \boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)} \boldsymbol{b}(s) \pm \rho(s) \boldsymbol{e}_0.$$

(5) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = \overline{h}'''_{\boldsymbol{v}}(s) = \overline{h}^{(4)}_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g(s) \neq 0$, $\tau_g(s) \neq 0$,

$$\left(\left(\frac{-1}{\kappa_g(s)} \right)' \frac{1}{\tau_g(s)} \right)' - \frac{\tau_g(s)}{\kappa_g(s)} = 0$$

and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)}\boldsymbol{b}(s) \pm \rho(s)\boldsymbol{e}_0.$$

(6) $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = \overline{h}'''_{\boldsymbol{v}}(s) = \overline{h}^{(4)}_{\boldsymbol{v}}(s) = \overline{h}^{(5)}_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g(s) \neq 0$, $\tau_g(s) \neq 0$,

$$\left(\left(\frac{-1}{\kappa_g(s)}\right)'\frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)} = \left\{\left(\left(\frac{-1}{\kappa_g(s)}\right)'\frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)}\right\}' = 0$$

and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)}\boldsymbol{b}(s) \pm \rho(s)\boldsymbol{e}_0.$$

Proof. (1) Since $\boldsymbol{v} \in LC^*$, there exist $\omega, \lambda, \mu, \xi, \eta \in \mathbb{R}$ with $\omega^2 + \lambda^2 + \mu^2 + \xi^2 - \eta^2 = 0$ such that $\boldsymbol{v} = \omega\overline{\boldsymbol{\gamma}}(s) + \lambda\boldsymbol{t}(s) + \mu\boldsymbol{n}(s) + \xi\boldsymbol{b}(s) + \eta\boldsymbol{e}_0$. From $\overline{h}_{\boldsymbol{v}}(s) = \langle\overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle - 1 = 0$, we have $\omega = 1$. So $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \lambda\boldsymbol{t}(s) + \mu\boldsymbol{n}(s) + \xi\boldsymbol{b}(s) + \eta\boldsymbol{e}_0$ and $\eta^2 = 1 + \lambda^2 + \mu^2 + \xi^2$. The converse direction also holds.

(2) Since $\overline{h}'_{\boldsymbol{v}}(s) = \langle\boldsymbol{t}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = 0$ if and only if

$$\overline{h}'_{\boldsymbol{v}}(s) = \langle\boldsymbol{t}(s), \boldsymbol{v}\rangle = \langle\boldsymbol{t}(s), \overline{\boldsymbol{\gamma}}(s) + \lambda\boldsymbol{t}(s) + \mu\boldsymbol{n}(s) + \xi\boldsymbol{b}(s) + \eta\boldsymbol{e}_0\rangle = \lambda = 0.$$

It follows from the fact $\eta^2 = 1 + \mu^2 + \xi^2$ that $\eta = \pm\sqrt{1 + \mu^2 + \xi^2}$. Then we have $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{n}(s) + \xi\boldsymbol{b}(s) + \eta\boldsymbol{e}_0 = \overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{n}(s) + \xi\boldsymbol{b}(s) \pm \sqrt{1 + \mu^2 + \xi^2}\boldsymbol{e}_0$.

(3) Since $\overline{h}''_{\boldsymbol{v}}(s) = \langle\kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = 0$ if and only if

$$\overline{h}''_{\boldsymbol{v}}(s) = \langle\kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{n}(s) + \xi\boldsymbol{b}(s) \pm \sqrt{1 + \mu^2 + \xi^2}\boldsymbol{e}_0\rangle = \kappa_g(s)\mu - 1 = 0.$$

Then we have $\kappa_g(s) \neq 0$, $\mu = 1/\kappa_g(s)$ and

$$\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) + \xi\boldsymbol{b}(s) \pm \sqrt{(1 + \kappa_g^2(s) + \kappa_g^2(s)\xi^2)/\kappa_g^2(s)}\boldsymbol{e}_0.$$

(4) Since $\overline{h}'''_{\boldsymbol{v}}(s) = \langle\kappa_g'(s)\boldsymbol{n}(s) - (\kappa_g^2(s) + 1)\boldsymbol{t}(s) + \kappa_g(s)\tau_g(s)\boldsymbol{b}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}'_{\boldsymbol{v}}(s) = \overline{h}''_{\boldsymbol{v}}(s) = \overline{h}'''_{\boldsymbol{v}}(s) = 0$ if and only if

$$\begin{aligned}\overline{h}'''_{\boldsymbol{v}}(s) = \langle&\kappa_g'(s)\boldsymbol{n}(s) - (\kappa_g^2(s) + 1)\boldsymbol{t}(s) + \kappa_g(s)\tau_g(s)\boldsymbol{b}(s),\\ &\overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) + \xi\boldsymbol{b}(s) \pm \sqrt{(1 + \kappa_g^2(s) + \kappa_g^2(s)\xi^2)/\kappa_g^2(s)}\boldsymbol{e}_0\rangle\\ &= \kappa_g'(s)/\kappa_g(s) + \kappa_g(s)\tau_g(s)\xi = 0.\end{aligned}$$

Then we have $\kappa_g(s) \neq 0$, $\tau_g(s) \neq 0$, $\xi = -\kappa_g'(s)/\kappa_g^2(s)\tau_g(s)$ and $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) - \kappa_g'(s)\boldsymbol{b}(s)/\kappa_g^2(s)\tau_g(s) \pm \rho(s)\boldsymbol{e}_0$.

(5) Since $\overline{h}_{\boldsymbol{v}}^{(4)}(s) = \langle(\kappa_g''(s)-\kappa_g^3(s)-\kappa_g(s)-\kappa_g(s)\tau_g^2(s))\boldsymbol{n}(s)-3\kappa_g(s)\kappa_g'(s)\boldsymbol{t}(s)+(2\kappa_g'(s)\tau_g(s)+\kappa_g(s)\tau_g'(s))\boldsymbol{b}(s)+(1+\kappa_g^2(s))\overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = \overline{h}_{\boldsymbol{v}}''(s) = \overline{h}_{\boldsymbol{v}}'''(s) = \overline{h}_{\boldsymbol{v}}^{(4)}(s) = 0$ if and only if

$$\begin{aligned}\overline{h}_{\boldsymbol{v}}^{(4)}(s) = \Big\langle &(\kappa_g''(s) - \kappa_g^3(s) - \kappa_g(s) - \kappa_g(s)\tau_g^2(s))\boldsymbol{n}(s) - 3\kappa_g(s)\kappa_g'(s)\boldsymbol{t}(s) \\ &+(2\kappa_g'(s)\tau_g(s) + \kappa_g(s)\tau_g'(s))\boldsymbol{b}(s) + (1+\kappa_g^2(s))\overline{\boldsymbol{\gamma}}(s), \\ &\qquad \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)}\boldsymbol{b}(s) \pm \rho(s)\boldsymbol{e}_0\Big\rangle \\ &= \frac{\kappa_g(s)\kappa_g''(s)\tau_g(s) - 2\kappa_g'^2(s)\tau_g(s) - \kappa_g(s)\kappa_g'(s)\tau_g'(s) - \kappa_g^2(s)\tau_g^3(s)}{\kappa_g^2(s)\tau_g(s)} = 0.\end{aligned}$$

This is equivalent to the condition $((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = 0$. Then we have $\kappa_g(s) \neq 0, \tau_g(s) \neq 0$, $((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = 0$ and $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) - \kappa_g'(s)\boldsymbol{b}(s)/\kappa_g^2(s)\tau_g(s) \pm \rho(s)\boldsymbol{e}_0$.

(6)Since $\overline{h}_{\boldsymbol{v}}^{(5)}(s) = \langle(\kappa_g^4(s) + 2\kappa_g^2(s) + \kappa_g^2(s)\tau_g^2(s) + 1 - 3\kappa_g'^2(s) - 4\kappa_g(s)\kappa_g''(s))\boldsymbol{t}(s) + (\kappa_g'''(s) - \kappa_g'(s) - 6\kappa_g^2(s)\kappa_g'(s) - 3\kappa_g'(s)\tau_g^2(s) - 3\kappa_g(s)\tau_g(s)\tau_g'(s))\boldsymbol{n}(s) + (3\kappa_g''(s)\tau_g(s) + 3\kappa_g'(s)\tau_g'(s) + \kappa_g(s)\tau_g''(s) - \kappa_g(s)\tau_g(s) - \kappa_g^3(s)\tau_g(s) - \kappa_g(s)\tau_g^3(s))\boldsymbol{b}(s) + 5\kappa_g(s)\kappa_g'(s)\overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $\overline{h}_{\boldsymbol{v}}(s) = \overline{h}_{\boldsymbol{v}}'(s) = \overline{h}_{\boldsymbol{v}}''(s) = \overline{h}_{\boldsymbol{v}}'''(s) = \overline{h}_{\boldsymbol{v}}^{(4)}(s) = \overline{h}_{\boldsymbol{v}}^{(5)}(s) = 0$ if and only if

$$\begin{aligned}\overline{h}_{\boldsymbol{v}}^{(5)}(s) = \Big\langle &(\kappa_g^4(s) + 2\kappa_g^2(s) + \kappa_g^2(s)\tau_g^2(s) + 1 - 3\kappa_g'^2(s) - 4\kappa_g(s)\kappa_g''(s))\boldsymbol{t}(s) \\ &+(\kappa_g'''(s) - \kappa_g'(s) - 6\kappa_g^2(s)\kappa_g'(s) - 3\kappa_g'(s)\tau_g^2(s) - 3\kappa_g(s)\tau_g(s)\tau_g'(s))\boldsymbol{n}(s) \\ &+(3\kappa_g''(s)\tau_g(s) + 3\kappa_g'(s)\tau_g'(s) + \kappa_g(s)\tau_g''(s) - \kappa_g(s)\tau_g(s) - \kappa_g^3(s)\tau_g(s) - \kappa_g(s)\tau_g^3(s))\boldsymbol{b}(s) \\ &+5\kappa_g(s)\kappa_g'(s)\overline{\boldsymbol{\gamma}}(s), \\ &\qquad \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)}\boldsymbol{b}(s) \pm \rho(s)\boldsymbol{e}_0\Big\rangle \\ &= \frac{1}{\kappa_g^2(s)\tau_g(s)}(\kappa_g'''(s)\kappa_g(s)\tau_g(s) - 2\kappa_g(s)\kappa_g'(s)\tau_g^3(s) - 3\kappa_g^2(s)\tau_g^2(s)\tau_g'(s) - 3\kappa_g'(s)\kappa_g''(s)\tau_g(s) \\ &-3\kappa_g'^2(s)\tau_g'(s) - \kappa_g(s)\kappa_g'(s)\tau_g''(s)) \\ &= \frac{(\kappa_g(s)\kappa_g''(s)\tau_g(s) - 2\kappa_g'^2(s)\tau_g(s) - \kappa_g(s)\kappa_g'(s)\tau_g'(s) - \kappa_g^2(s)\tau_g^3(s))'}{\kappa_g^2(s)\tau_g(s)} = 0.\end{aligned}$$

This is equivalent to the condition $((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = (((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s))' = 0$. Then we have $\kappa_g(s) \neq 0, \tau_g(s) \neq 0$, $((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = (((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s))' = 0$ and $\boldsymbol{v} = \overline{\boldsymbol{\gamma}}(s) + \boldsymbol{n}(s)/\kappa_g(s) - \kappa_g'(s)\boldsymbol{b}(s)/\kappa_g^2(s)\tau_g(s) \pm \rho(s)\boldsymbol{e}_0$. This completes the proof. □

Proposition 4.3.2. Let $\boldsymbol{\gamma} : I \longrightarrow S^3_+$ be a unit speed curve, then we have the followings:

(1) $h_{\boldsymbol{v}}(s) = 0$ if and only if $\boldsymbol{v} = \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + (\lambda + 2)\boldsymbol{e}_0$, where $\lambda, \mu, \xi, \eta \in \mathbb{R}$ and $\mu^2 + \xi^2 + \eta^2 - 4\lambda - 4 = 0$.

(2) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = 0$ if and only if $\boldsymbol{v} = ((\xi^2 + \eta^2)/4 - 1)\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n} + \eta\boldsymbol{b}(s) + ((\xi^2 + \eta^2)/4 + 1)\boldsymbol{e}_0$.

(3) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = 0$ if and only if

$$\boldsymbol{v} = \kappa_g(s)\xi\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) \pm \sqrt{4 + 4\kappa_g(s)\xi - \xi^2}\boldsymbol{b}(s) + (\kappa_g(s)\xi + 2)\boldsymbol{e}_0.$$

(4) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0$ and

$$\boldsymbol{v} = 2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s) - 2\kappa_g'(s)\sigma^{\pm}(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 2)\boldsymbol{e}_0.$$

(5) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = h^{(4)}_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0$,

$$\left(\left(\frac{-1}{\kappa_g(s)}\right)' \frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)} = 0$$

and

$$\boldsymbol{v} = 2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s) - 2\kappa_g'(s)\sigma^{\pm}(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 2)\boldsymbol{e}_0.$$

(6) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = h^{(4)}_{\boldsymbol{v}}(s) = h^{(5)}_{\boldsymbol{v}}(s) = 0$ if and only if $\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0$,

$$\left(\left(\frac{-1}{\kappa_g(s)}\right)' \frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)} = \left\{\left(\left(\frac{-1}{\kappa_g(s)}\right)' \frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)}\right\}' = 0$$

and

$$\boldsymbol{v} = 2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s) - 2\kappa_g'(s)\sigma^{\pm}(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 2)\boldsymbol{e}_0.$$

Proof. (1) Since $\boldsymbol{v} \in LC^*$, there exist $\lambda, \mu, \xi, \eta, \omega \in \mathbb{R}$ with $\lambda^2 + \mu^2 + \xi^2 + \eta^2 - \omega^2 = 0$ such that $\boldsymbol{v} = \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + \omega\boldsymbol{e}_0$. From $h_{\boldsymbol{v}}(s) = \langle\boldsymbol{\gamma}(s), \boldsymbol{v}\rangle + 2 = \langle\overline{\boldsymbol{\gamma}}(s) + \boldsymbol{e}_0, \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + \omega\boldsymbol{e}_0\rangle + 2 = \lambda - \omega + 2 = 0$, we have $\omega = 2 + \lambda$. So $\boldsymbol{v} = \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + (2+\lambda)\boldsymbol{e}_0$ and $\lambda^2 + \mu^2 + \xi^2 + \eta^2 - (2+\lambda)^2 = \mu^2 + \xi^2 + \eta^2 - 4\lambda - 4 = 0$. The converse direction also holds.

(2) Since $h'_{\boldsymbol{v}}(s) = \langle\boldsymbol{t}(s), \boldsymbol{v}\rangle$, $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = 0$ if and only if

$$h'_{\boldsymbol{v}}(s) = \langle\boldsymbol{t}(s), \lambda\overline{\boldsymbol{\gamma}}(s) + \mu\boldsymbol{t}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + (2+\lambda)\boldsymbol{e}_0\rangle = \mu = 0.$$

By $\lambda^2 + \xi^2 + \eta^2 - (2+\lambda)^2 = \xi^2 + \eta^2 - 4\lambda - 4 = 0$, we have $\lambda = (\xi^2 + \eta^2)/4 - 1$. So, $\boldsymbol{v} = ((\xi^2 + \eta^2)/4 - 1)\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + ((\xi^2 + \eta^2)/4 + 1)\boldsymbol{e}_0$.

(3) Since $h''_{\boldsymbol{v}}(s) = \langle\kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = 0$ if and only if

$$\begin{aligned} h''_{\boldsymbol{v}}(s) &= \left\langle \kappa_g(s)\boldsymbol{n}(s) - \overline{\boldsymbol{\gamma}}(s), \left(\frac{\xi^2+\eta^2}{4} - 1\right)\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + \left(\frac{\xi^2+\eta^2}{4} + 1\right)\boldsymbol{e}_0 \right\rangle \\ &= (4\kappa_g(s)\xi - \xi^2 - \eta^2 + 4)/4 = 0, \end{aligned}$$

so that we have $\eta = \pm\sqrt{4 + 4\kappa_g(s)\xi - \xi^2}$ and $\boldsymbol{v} = \kappa_g(s)\xi\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) \pm \sqrt{4 + 4\kappa_g(s)\xi - \xi^2}\boldsymbol{b}(s) + (\kappa_g(s)\xi + 2)\boldsymbol{e}_0$.

(4) Since $h'''_{\boldsymbol{v}}(s) = \langle\kappa'_g(s)\boldsymbol{n}(s) - (\kappa_g^2(s)+1)\boldsymbol{t}(s) + \kappa_g(s)\tau_g(s)\boldsymbol{b}(s), \boldsymbol{v}\rangle$, $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = 0$ if and only if

$$\begin{aligned} h'''_{\boldsymbol{v}}(s) &= \Big\langle \kappa'_g(s)\boldsymbol{n}(s) - (\kappa_g^2(s)+1)\boldsymbol{t}(s) + \kappa_g(s)\tau_g(s)\boldsymbol{b}(s), \\ &\qquad \kappa_g(s)\xi\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) \pm \sqrt{4 + 4\kappa_g(s)\xi - \xi^2}\boldsymbol{b}(s) + (\kappa_g(s)\xi + 2)\boldsymbol{e}_0 \Big\rangle \\ &= \kappa'_g(s)\xi + \kappa_g(s)\tau_g(s)\eta = \kappa'_g(s)\xi \pm \kappa_g(s)\tau_g(s)\sqrt{4 + 4\kappa_g(s)\xi - \xi^2} = 0, \end{aligned}$$

so that we have $\kappa'^2_g(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0$, $\xi = 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)$ and $\boldsymbol{v} = 2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s) - 2\kappa'_g(s)\sigma^{\pm}(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 2)\boldsymbol{e}_0$.

(5) Since $h^{(4)}_{\boldsymbol{v}}(s) = \langle(\kappa''_g(s) - \kappa_g^3(s) - \kappa_g(s) - \kappa_g(s)\tau_g^2(s))\boldsymbol{n}(s) - 3\kappa_g(s)\kappa'_g(s)\boldsymbol{t}(s) + (2\kappa'_g(s)\tau_g(s) + \kappa_g(s)\tau'_g(s))\boldsymbol{b}(s) + (1 + \kappa_g^2(s))\overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = h^{(4)}_{\boldsymbol{v}}(s) = 0$ if and only if

$$\begin{aligned} h_{\boldsymbol{v}}^{(4)}(s) &= \langle (\kappa_g''(s) - \kappa_g^3(s) - \kappa_g(s) - \kappa_g(s)\tau_g^2(s))\boldsymbol{n}(s) - 3\kappa_g(s)\kappa_g'(s)\boldsymbol{t}(s) \\ &\quad + (2\kappa_g'(s)\tau_g(s) + \kappa_g(s)\tau_g'(s))\boldsymbol{b}(s) + (1+\kappa_g^2(s))\overline{\boldsymbol{\gamma}}(s), \\ &\qquad\qquad \kappa_g(s)\xi\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + (\kappa_g(s)\xi + 2)\boldsymbol{e}_0\rangle \\ &= (\kappa_g''(s) - \kappa_g(s)\tau_g^2(s))\xi + (2\kappa_g'(s)\tau_g(s) + \kappa_g(s)\tau_g'(s))\eta = 0. \end{aligned}$$

By the above condition, we have the equation $(\kappa_g''(s) - \kappa_g(s)\tau_g^2(s))/(2\kappa_g'(s)\tau_g(s) + \kappa_g(s)\tau_g'(s)) = \kappa_g'(s)/\kappa_g(s)\tau_g(s)$. It is equivalent to $((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = 0$. Then we have $\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0$, $((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = 0$ and $\boldsymbol{v} = 2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s) - 2\kappa_g'(s)\sigma^{\pm}(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 2)\boldsymbol{e}_0$.

(6) Since $h_{\boldsymbol{v}}^{(5)}(s) = \langle (\kappa_g^4(s) + 2\kappa_g^2(s) + \kappa_g^2(s)\tau_g^2(s) + 1 - 3\kappa_g'^2(s) - 4\kappa_g(s)\kappa_g''(s))\boldsymbol{t}(s) + (\kappa_g'''(s) - \kappa_g'(s) - 6\kappa_g^2(s)\kappa_g'(s) - 3\kappa_g'(s)\tau_g^2(s) - 3\kappa_g(s)\tau_g(s)\tau_g'(s))\boldsymbol{n}(s) + (3\kappa_g''(s)\tau_g(s) + 3\kappa_g'(s)\tau_g'(s) + \kappa_g(s)\tau_g''(s) - \kappa_g(s)\tau_g(s) - \kappa_g^3(s)\tau_g(s) - \kappa_g(s)\tau_g^3(s))\boldsymbol{b}(s) + 5\kappa_g(s)\kappa_g'(s)\overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle$, $h_{\boldsymbol{v}}(s) = h_{\boldsymbol{v}}'(s) = h_{\boldsymbol{v}}''(s) = h_{\boldsymbol{v}}'''(s) = h_{\boldsymbol{v}}^{(4)}(s) = h_{\boldsymbol{v}}^{(5)}(s) = 0$ if and only if

$$\begin{aligned} h_{\boldsymbol{v}}^{(5)}(s) &= \langle (\kappa_g^4(s) + 2\kappa_g^2(s) + \kappa_g^2(s)\tau_g^2(s) + 1 - 3\kappa_g'^2(s) - 4\kappa_g(s)\kappa_g''(s))\boldsymbol{t}(s) \\ &\quad + (\kappa_g'''(s) - \kappa_g'(s) - 6\kappa_g^2(s)\kappa_g'(s) - 3\kappa_g'(s)\tau_g^2(s) - 3\kappa_g(s)\tau_g(s)\tau_g'(s))\boldsymbol{n}(s) \\ &\quad + (3\kappa_g''(s)\tau_g(s) + 3\kappa_g'(s)\tau_g'(s) + \kappa_g(s)\tau_g''(s) - \kappa_g(s)\tau_g(s) - \kappa_g^3(s)\tau_g(s) - \kappa_g(s)\tau_g^3(s))\boldsymbol{b}(s) \\ &\quad + 5\kappa_g(s)\kappa_g'(s)\overline{\boldsymbol{\gamma}}(s), \kappa_g(s)\xi\overline{\boldsymbol{\gamma}}(s) + \xi\boldsymbol{n}(s) + \eta\boldsymbol{b}(s) + (\kappa_g(s)\xi + 2)\boldsymbol{e}_0\rangle \\ &= (\kappa_g'''(s) - \kappa_g'(s) - \kappa_g^2(s)\kappa_g'(s) - 3\kappa_g'(s)\tau_g^2(s) - 3\kappa_g(s)\tau_g(s)\tau_g'(s))\xi \\ &\quad + (3\kappa_g''(s)\tau_g(s) + 3\kappa_g'(s)\tau_g'(s) + \kappa_g(s)\tau_g''(s) - \kappa_g(s)\tau_g(s) - \kappa_g^3(s)\tau_g(s) - \kappa_g(s)\tau_g^3(s))\eta = 0. \end{aligned}$$

By the above condition, we have the equation $(\kappa_g'''(s) - \kappa_g'(s) - \kappa_g^2(s)\kappa_g'(s) - 3\kappa_g'(s)\tau_g^2(s) - 3\kappa_g(s)\tau_g(s)\tau_g'(s))/(3\kappa_g''(s)\tau_g(s) + 3\kappa_g'(s)\tau_g'(s) + \kappa_g(s)\tau_g''(s) - \kappa_g(s)\tau_g(s) - \kappa_g^3(s)\tau_g(s) - \kappa_g(s)\tau_g^3(s)) = \kappa_g'(s)/\kappa_g(s)\tau_g(s)$. It is equivalent to $((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = (((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s))' = 0$. Then we have $\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) \neq 0$, $((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = (((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s))' = 0$ and $\boldsymbol{v} = 2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s) - 2\kappa_g'(s)\sigma^{\pm}(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 2)\boldsymbol{e}_0$. This completes the proof. $\square$

According to the assertions of Propositions 4.3.1 and 4.3.2, we define an invariant

$$\kappa_S(s) = \left(\left(\frac{-1}{\kappa_g(s)}\right)' \frac{1}{\tau_g(s)}\right)' - \frac{\tau_g(s)}{\kappa_g(s)},$$

which we call a *spherical curvature* of $\overline{\boldsymbol{\gamma}}$. We have the following proposition.

Proposition 4.3.3. For a unit speed curve $\boldsymbol{\gamma} : I \to S^3_+$, both of the spherical evolutes $\boldsymbol{\varepsilon}^{\pm}_{\overline{\gamma}}(s)$ are constant if and only if $\kappa_S \equiv 0$.

Proof. $\boldsymbol{\varepsilon}'^{\pm}_{\overline{\gamma}}(s) = \pm(\kappa_g(s)\kappa_g'(s)(2\kappa_g'^2(s)\tau_g(s)+\kappa_g(s)\kappa_g'(s)\tau_g'(s)+\kappa_g^2(s)\tau_g^3(s)-\kappa_g(s)\kappa_g''(s)\tau_g(s))\{\overline{\boldsymbol{\gamma}}(s)+\boldsymbol{n}(s)/\kappa_g(s)+(1/\kappa_g(s))'\boldsymbol{b}(s)/\tau_g(s)\}/(\kappa_g'^2(s)+\kappa_g^4(s)\tau_g^2(s)+\kappa_g^2(s)\tau_g^2(s))^{3/2}+(2\kappa_g'^2(s)\tau_g(s)+\kappa_g(s)\kappa_g'(s)\tau_g'(s)+\kappa_g^2(s)\tau_g^3(s)-\kappa_g(s)\kappa_g''(s)\tau_g(s))\boldsymbol{b}(s)/\kappa_g(s)\tau_g(s)(\kappa_g'^2(s)+\kappa_g^4(s)\tau_g^2(s)+\kappa_g^2(s)\tau_g^2(s))^{1/2})$.

On the other hand $\kappa_S(s) = ((-1/\kappa_g(s))'/\tau_g(s))' - \tau_g(s)/\kappa_g(s) = (\kappa_g(s)\kappa_g''(s)\tau_g(s) - 2\kappa_g'^2(s)\tau_g(s) - \kappa_g(s)\kappa_g'(s)\tau_g'(s) - \kappa_g^2(s)\tau_g^3(s))/\kappa_g^3(s)\tau_g^2(s) = 0$. So $\boldsymbol{\varepsilon}'^{\pm}_{\overline{\gamma}} \equiv 0$ if and only if $\kappa_S \equiv 0$. This completes the proof. □

4.4 Singularities of lightcone duals of spherical curves

In this section we classify the singularities of $\overline{LD}^{\pm}_{\overline{\gamma}}$ and $LD_{\boldsymbol{\gamma}}$ as an application of the unfolding theory of functions. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be a function germ, we call F an *r-parameter unfolding* of f, where $f(s) = F_{\boldsymbol{x}_0}(s, \boldsymbol{x}_0)$. The *discriminant set* of F is defined by

$$D_F = \left\{ \boldsymbol{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R},\ F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s}(s, \boldsymbol{x}) = 0 \right\}.$$

By Propositions 4.3.1, (2) and 4.3.2, (2), the discriminant sets of $\overline{H}$ and H are

$$\begin{aligned} D_{\overline{H}} &= \{\overline{\boldsymbol{\gamma}}(s) + u\boldsymbol{n}(s) + v\boldsymbol{b}(s) \pm \sqrt{u^2+v^2+1}\,\boldsymbol{e}_0 \mid s \in I, u, v \in \mathbb{R}\}, \\ D_H &= \{(u^2+v^2-4)\overline{\boldsymbol{\gamma}}(s)/4 + u\boldsymbol{n}(s) + v\boldsymbol{b}(s) + (u^2+v^2+4)\boldsymbol{e}_0/4 \mid s \in I, u, v \in \mathbb{R}\}. \end{aligned}$$

These are the lightcone dual surfaces of $\overline{\boldsymbol{\gamma}}$ and the lightcone dual surface of $\boldsymbol{\gamma}$ respectively. Moreover, the both assertions (4) of Propositions 4.3.1 and 4.3.2 describe the singularities of the lightcone focal surfaces of $\boldsymbol{\gamma}$ and $\overline{\boldsymbol{\gamma}}$ respectively.

Proposition 4.4.1. The critical value sets of $LF^{\pm}_{\overline{\gamma}}$ and $LF^{\pm}_{\gamma}$ are give as follows:

$$\begin{aligned} C(LF^{\pm}_{\overline{\gamma}}) &= \left\{ \overline{\boldsymbol{\gamma}}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)}\boldsymbol{b}(s) \pm \rho(s)\boldsymbol{e}_0 \;\Big|\; s \in I \right\}, \\ C(LF^{\pm}_{\gamma}) &= \{ 2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\boldsymbol{\gamma}}(s) + 2\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s) \\ &\qquad -2\kappa_g'(s)\sigma^{\pm}(s)\boldsymbol{b}(s) + (2\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 2)\boldsymbol{e}_0 \mid s \in I \}. \end{aligned}$$

Then we have the following theorem as a corollary.

Theorem 4.4.2. Both of the projections of the critical value sets $C(LF^{\pm}_{\overline{\gamma}})$ and $C(LF^{\pm}_{\gamma})$ in the unit 3-sphere S^3_0 are the images of the spherical evolutes of $\overline{\boldsymbol{\gamma}}$, that is

$$\widetilde{\pi}(C(LF^{\pm}_{\overline{\gamma}})) = \widetilde{\pi}(C(LF^{\pm}_{\gamma})) = \{ \boldsymbol{\varepsilon}^{\pm}_{\overline{\gamma}}(s) \mid s \in I \}.$$

Proof. We know that

$$\widetilde{\pi}(C(LF^{\pm}_{\overline{\gamma}})) = \left\{ \pm\left(\frac{\overline{\boldsymbol{\gamma}}(s)}{\rho(s)} + \frac{\boldsymbol{n}(s)}{\rho(s)\kappa_g(s)} - \frac{\kappa_g'(s)\boldsymbol{b}(s)}{\rho(s)\kappa_g^2(s)\tau_g(s)} \right) \;\Big|\; s \in I \right\}$$

and

$$\widetilde{\pi}(C(LF^{\pm}_{\gamma})) = \left\{ \frac{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)\overline{\boldsymbol{\gamma}}(s) + \kappa_g(s)\tau_g(s)\sigma^{\pm}(s)\boldsymbol{n}(s) - \kappa_g'(s)\sigma^{\pm}(s)\boldsymbol{b}(s)}{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s) + 1} \;\Big|\; s \in I \right\}.$$

By straightforward calculations, we have

$$\begin{aligned} &\frac{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)}{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)+1} \\ &= \frac{\kappa_g^2(s)\tau_g(s)(\kappa_g^2(s)\tau_g(s) \pm \sqrt{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s)})}{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s) \pm \kappa_g^2(s)\tau_g(s)\sqrt{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s)}} \\ &= \frac{\pm\kappa_g^2(s)\tau_g(s)}{\sqrt{\kappa_g'^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)\tau_g^2(s)}} = \frac{\pm 1}{\rho(s)}. \end{aligned}$$

Similarly, we can calculate that

$$\frac{\kappa_g(s)\tau_g(s)\sigma^{\pm}(s)}{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)+1} = \frac{\pm 1}{\rho(s)\kappa_g(s)},$$

$$\frac{\kappa_g'(s)\sigma^{\pm}(s)}{\kappa_g^2(s)\tau_g(s)\sigma^{\pm}(s)+1} = \frac{\pm\kappa_g'(s)}{\rho(s)\kappa_g^2(s)\tau_g(s)}.$$

So we have

$$\widetilde{\pi}(C(LF^{\pm}_{\overline{\gamma}})) = \widetilde{\pi}(C(LF^{\pm}_{\boldsymbol{\gamma}})) = \{\boldsymbol{\varepsilon}^{\pm}_{\overline{\gamma}}(s) \mid s \in I\ \}.$$

In order to understand the geometric properties of the discriminant set of order ℓ, we introduce an equivalence relation among the unfoldings of functions. This completes the proof. □

Inspired by Propositions 4.3.1, 4.3.2 and Theorem 4.4.2, we define the following set:

$$D^{\ell}_{F} = \left\{ \boldsymbol{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R},\ F(s,\boldsymbol{x}) = \frac{\partial F}{\partial s}(s,\boldsymbol{x}) = \cdots = \frac{\partial^{\ell} F}{\partial s^{\ell}}(s,\boldsymbol{x}) = 0 \right\},$$

which is called a *discriminant set of order ℓ*. Of course, $D^1_F = D_F$. Let F and G be r-parameter unfoldings of $f(s)$ and $g(s)$, respectively. We say that F and G are *P-$\mathcal{R}$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}\times\mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow (\mathbb{R}\times\mathbb{R}^r, (s'_0, \boldsymbol{x}'_0))$ of the form $\Phi(s,\boldsymbol{x}) = (\Phi_1(s,\boldsymbol{x}), \phi(\boldsymbol{x}))$ such that $G \circ \Phi = F$. By straightforward calculations, we have the following proposition.

Proposition 4.4.3. Let F and G be r-parameter unfoldings of $f(s)$ and $g(s)$, respectively. If F and G are P-$\mathcal{R}$-equivalent by a diffeomorphism germ $\Phi : (\mathbb{R}\times\mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow (\mathbb{R}\times\mathbb{R}^r, (s'_0, \boldsymbol{x}'_0))$ of the form $\Phi(s,\boldsymbol{x}) = (\Phi_1(s,\boldsymbol{x}), \phi(\boldsymbol{x}))$, then $\phi(D^{\ell}_F) = D^{\ell}_G$ as set germs.

By Propositions 4.3.1 and 4.3.2, we have the following proposition.

Proposition 4.4.4. Under the same notations as in the previous paragraphs, we have

$$D_{\overline{H}} = D^1_{\overline{H}} = \text{Images}\,\overline{LD}^{\pm}_{\overline{\gamma}},\ D^2_{\overline{H}} = \text{Images}\,LF^{\pm}_{\overline{\gamma}},\ \widetilde{\pi}(D^3_{\overline{H}}) = \text{Images}\,\boldsymbol{\varepsilon}^{\pm}_{\overline{\gamma}},$$
$$D_H = D^1_H = \text{Image}\,LD_{\boldsymbol{\gamma}},\ D^2_H = \text{Images}\,LF^{\pm}_{\boldsymbol{\gamma}},\ \widetilde{\pi}(D^3_H) = \text{Images}\,\boldsymbol{\varepsilon}^{\pm}_{\overline{\gamma}}.$$

For a function $f(s)$, we say that f has *A_k-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \le p \le k$ and $f^{(k+1)}(s_0) \neq 0$. Let F be an r-parameter unfolding of f and f has A_k-singularity $(k \ge 1)$ at s_0. We denote the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 as

$$j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s,\boldsymbol{x}_0)\right)(s_0) = \sum_{j=1}^{k-1} \alpha_{ji}(s-s_0)^j, \quad (i = 1,\cdots,r).$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k $(k \le r)$, then F is called a *versal unfolding* of f, where $\alpha_{0i} = \partial F/\partial x_i(s_0, \boldsymbol{x}_0)$. We have the following classification theorem of versal unfoldings [3, Page 149, 6.6].

Theorem 4.4.5. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of f which has A_k-singularity at s_0. Suppose F is a versal unfolding of f, then F is P-$\mathcal{R}$-equivalent to one of the following unfoldings:

(a) $k = 1$, $\pm s^2 + x_1$,

(b) $k = 2$, $s^3 + x_1 + sx_2$,

(c) $k = 3$, $\pm s^4 + x_1 + sx_2 + s^2 x_3$,

(d) $k = 4$, $s^5 + x_1 + sx_2 + s^2 x_3 + s^3 x_4$.

We have the following classification result as a corollary of the above theorem.

Corollary 4.4.6. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of f which has A_k-singularity at s_0. Suppose F is a versal unfolding of f, then we have the following assertions:
(a) If $k = 1$, then D_F is diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$ and $D_F^2 = \emptyset$.
(b) If $k = 2$, then D_F is diffeomorphic to $C(2, 3) \times \mathbb{R}^{r-2}$, D_F^2 is diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-2}$ and $D_F^3 = \emptyset$.
(c) If $k = 3$, then D_F is diffeomorphic to $SW \times \mathbb{R}^{r-3}$, D_F^2 is diffeomorphic to $C(2, 3, 4) \times \mathbb{R}^{r-3}$, D_F^3 is diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-3}$ and $D_F^4 = \emptyset$.
(d) If $k = 4$, then D_F is locally diffeomorphic to $BF \times \mathbb{R}^{r-4}$, D_F^2 is diffeomorphic to $C(BF) \times \mathbb{R}^{r-4}$, D_F^3 is diffeomorphic to $C(2, 3, 4, 5) \times \mathbb{R}^{r-4}$, D_F^4 is diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-4}$ and $D_F^5 = \emptyset$.

We remark that all of diffeomorphisms in the above assertions are diffeomorphism germs.

Here, we respectively call $C(2, 3) = \{(x_1, x_2) \mid x_1 = u^2, x_2 = u^3\}$ a $(2, 3)$-*cusp*, $C(2, 3, 4) = \{(x_1, x_2, x_3) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4\}$ a $(2, 3, 4)$-*cusp*, $C(2, 3, 4, 5) = \{(x_1, x_2, x_3, x_4) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4, x_4 = u^5\}$ a $(2, 3, 4, 5)$-*cusp*, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ a *swallow tail*, $BF = \{(x_1, x_2, x_3.x_4) \mid x_1 = 5u^4 + 3vu^2 + 2wu, x_2 = 4u^5 + 2vu^3 + wu^2, x_3 = u, x_4 = v\}$ a *butterfly* and $C(BF) = \{(x_1, x_2, x_3, x_4) \mid x_1 = 6u^5 + u^3 v, x_2 = 25u^4 + 9u^2 v, x_3 = 10u^3 + 3uv, x_4 = v\}$ a *c-butterfly* (i.e., the critical value set of the butterfly).

We have the following key propositions on H and $\overline{H}$.

Proposition 4.4.7. If $\overline{h}_{\boldsymbol{v_0}}$ has A_k-singularity $(k = 1, 2, 3, 4)$ at s_0, then $\overline{H}$ is a versal unfolding of $\overline{h}_{\boldsymbol{v_0}}$.

Proof. For $\boldsymbol{v} \in LC^*$, we have $\boldsymbol{v} = (\pm(v_1^2+v_2^2+v_3^2+v_4^2)^{1/2}, v_1, v_2, v_3, v_4)$. We denote that $\overline{\boldsymbol{\gamma}}(s) = (0, x_1(s), x_2(s), x_3, (s)x_4(s))$. Then

$$\overline{H}(s,\boldsymbol{v}) = \langle \overline{\boldsymbol{\gamma}}(s), \boldsymbol{v}\rangle - 1 = x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 + x_4(s)v_4 - 1.$$

Thus we have

$$\frac{\partial \overline{H}}{\partial v_1}(s,\boldsymbol{v}) = x_1(s), \frac{\partial \overline{H}}{\partial v_2}(s,\boldsymbol{v}) = x_2(s), \frac{\partial \overline{H}}{\partial v_3}(s,\boldsymbol{v}) = x_3(s), \frac{\partial \overline{H}}{\partial v_4}(s,\boldsymbol{v}) = x_4(s),$$

$$\frac{\partial^2 \overline{H}}{\partial s \partial v_1}(s,\boldsymbol{v}) = x_1'(s), \frac{\partial^2 \overline{H}}{\partial s \partial v_2}(s,\boldsymbol{v}) = x_2'(s),\ \frac{\partial^2 \overline{H}}{\partial s \partial v_3}(s,\boldsymbol{v}) = x_3'(s),\ \frac{\partial^2 \overline{H}}{\partial s \partial v_4}(s,\boldsymbol{v}) = x_4'(s),$$

$$\frac{\partial^3 \overline{H}}{\partial s^2 \partial v_1}(s,\boldsymbol{v}) = x_1''(s), \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_2}(s,\boldsymbol{v}) = x_2''(s),\ \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_3}(s,\boldsymbol{v}) = x_3''(s),\ \frac{\partial^3 \overline{H}}{\partial s^2 \partial v_4}(s,\boldsymbol{v}) = x_4''(s),$$

$$\frac{\partial^4 \overline{H}}{\partial s^3 \partial v_1}(s,\boldsymbol{v}) = x_1'''(s), \frac{\partial^4 \overline{H}}{\partial s^3 \partial v_2}(s,\boldsymbol{v}) = x_2'''(s),\ \frac{\partial^4 \overline{H}}{\partial s^3 \partial v_3}(s,\boldsymbol{v}) = x_3'''(s),\ \frac{\partial^4 \overline{H}}{\partial s^3 \partial v_4}(s,\boldsymbol{v}) = x_4'''(s).$$

For a fixed point $\boldsymbol{v}_0 = (v_{00}, v_{01}, v_{02}, v_{03}, v_{04})$, the 3-jet of $\partial\overline{H}/\partial v_i(s,\boldsymbol{v}_0)(i = 1,2,3,4)$ at s_0 is

$$j^{(3)}\frac{\partial \overline{H}}{\partial v_i}(s,\boldsymbol{v}_0)(s_0) = x_i'(s_0)(s-s_0) + x_i''(s_0)(s-s_0)^2/2 + x_i'''(s_0)(s-s_0)^3/6,\ (i=1,2,3,4).$$

It is enough to show that the rank of the matrix A is 4, where

$$A = \begin{pmatrix} x_1(s_0) & x_2(s_0) & x_3(s_0) & x_4(s_0) \\ x_1'(s_0) & x_2'(s_0) & x_3'(s_0) & x_4'(s_0) \\ x_1''(s_0) & x_2''(s_0) & x_3''(s_0) & x_4''(s_0) \\ x_1'''(s_0) & x_2'''(s_0) & x_3'''(s_0) & x_4'''(s_0) \end{pmatrix}.$$

Then we have

$$\det A = \langle \boldsymbol{e}_0 \wedge \overline{\boldsymbol{\gamma}}(s_0) \wedge \overline{\boldsymbol{\gamma}}'(s_0) \wedge \overline{\boldsymbol{\gamma}}''(s_0), \overline{\boldsymbol{\gamma}}'''(s_0)\rangle = -\kappa_g^2(s_0)\tau_g(s_0) \neq 0.$$

So the rank of A is 4. This completes the proof. □

Proposition 4.4.8. If $h_{\boldsymbol{v}_0}$ has A_k-singularity $(k = 1,2,3,4)$ at s_0, then H is a versal unfolding of $h_{\boldsymbol{v}_0}$.

Proof. For $\boldsymbol{v} \in LC^*$, we have $\boldsymbol{v} = (v_0, v_1, v_2, v_3, v_4) = (\pm(v_1^2 + v_2^2 + v_3^2 + v_4^2)^{1/2}, v_1, v_2, v_3, v_4)$. We denote that $\overline{\boldsymbol{\gamma}}(s) = (1, x_1(s), x_2(s), x_3, (s)x_4(s))$. Then we have

$$H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle + 2 = \mp(v_1^2 + v_2^2 + v_3^2 + v_4^2)^{1/2} + x_1(s)v_1 + x_2(s)v_2 + x_3(s)v_3 + x_4(s)v_4 + 2.$$

Thus we have

$$\begin{gathered}
\frac{\partial H}{\partial v_1}(s, \boldsymbol{v}) = -v_1/v_0 + x_1(s), \frac{\partial H}{\partial v_2}(s, \boldsymbol{v}) = -v_2/v_0 + x_2(s), \\
\frac{\partial H}{\partial v_3}(s, \boldsymbol{v}) = -v_3/v_0 + x_3(s), \frac{\partial H}{\partial v_4}(s, \boldsymbol{v}) = -v_4/v_0 + x_4(s), \\
\frac{\partial^2 H}{\partial s \partial v_1}(s, \boldsymbol{v}) = x_1'(s), \frac{\partial^2 H}{\partial s \partial v_2}(s, \boldsymbol{v}) = x_2'(s), \frac{\partial^2 H}{\partial s \partial v_3}(s, \boldsymbol{v}) = x_3'(s), \ \frac{\partial^2 H}{\partial s \partial v_4}(s, \boldsymbol{v}) = x_4'(s), \\
\frac{\partial^3 H}{\partial s^2 \partial v_1}(s, \boldsymbol{v}) = x_1''(s), \frac{\partial^3 H}{\partial s^2 \partial v_2}(s, \boldsymbol{v}) = x_2''(s), \frac{\partial^3 H}{\partial s^2 \partial v_3}(s, \boldsymbol{v}) = x_3''(s), \ \frac{\partial^3 H}{\partial s^2 \partial v_4}(s, \boldsymbol{v}) = x_4''(s), \\
\frac{\partial^4 H}{\partial s^3 \partial v_1}(s, \boldsymbol{v}) = x_1'''(s), \frac{\partial^4 H}{\partial s^3 \partial v_2}(s, \boldsymbol{v}) = x_2'''(s), \frac{\partial^4 H}{\partial s^3 \partial v_3}(s, \boldsymbol{v}) = x_3'''(s), \ \frac{\partial^4 H}{\partial s^3 \partial v_4}(s, \boldsymbol{v}) = x_4'''(s).
\end{gathered}$$

For a fixed $\boldsymbol{v}_0 = (v_{00}, v_{01}, v_{02}, v_{03}, v_{04})$, the 3-jet of $\partial H/\partial v_i(s, \boldsymbol{v}_0)(i = 1, 2, 3, 4)$ at s_0 is

$$j^{(3)}\frac{\partial H}{\partial v_i}(s, \boldsymbol{v}_0)(s_0) = x_i'(s_0)(s - s_0) + x_i''(s_0)(s - s_0)^2/2 + x_i'''(s_0)(s - s_0)^3/6, \ (i = 1, 2, 3, 4).$$

It is enough to show that the rank of the matrix B is 4, where

$$B = \begin{pmatrix} -v_{01}/v_{00} + x_1(s_0) & -v_{02}/v_{00} + x_2(s_0) & -v_{03}/v_{00} + x_3(s_0) & -v_{04}/v_{00} + x_4(s_0) \\ x_1'(s_0) & x_2'(s_0) & x_3'(s_0) & x_4'(s_0) \\ x_1''(s_0) & x_2''(s_0) & x_3''(s_0) & x_4''(s_0) \\ x_1'''(s_0) & x_2'''(s_0) & x_3'''(s_0) & x_4'''(s_0) \end{pmatrix}.$$

By straightforward calculations, we have

$$\det B = \langle \boldsymbol{e}_0 \wedge \overline{\boldsymbol{\gamma}}'(s_0) \wedge \overline{\boldsymbol{\gamma}}''(s_0) \wedge \overline{\boldsymbol{\gamma}}'''(s_0), \boldsymbol{v}_0 \rangle / v_{00} + \langle \boldsymbol{e}_0 \wedge \overline{\boldsymbol{\gamma}}(s_0) \wedge \overline{\boldsymbol{\gamma}}'(s_0) \wedge \overline{\boldsymbol{\gamma}}''(s_0), \overline{\boldsymbol{\gamma}}'''(s_0) \rangle$$

$$= \langle \kappa_g^2(s_0)\tau_g(s_0)\overline{\boldsymbol{\gamma}}(s_0), \boldsymbol{v}_0 \rangle / v_{00} - \langle \kappa_g'(s_0)\boldsymbol{b}(s_0), \boldsymbol{v}_0 \rangle / v_{00} + \langle \kappa_g(s_0)\tau_g(s_0)\boldsymbol{n}(s_0), \boldsymbol{v}_0 \rangle / v_{00} - \kappa_g^2(s_0)\tau_g(s_0).$$

In this case, $h_{\boldsymbol{v_0}}(s)$ has A_4-singularity, then we have

$$\boldsymbol{v}_0 = 2\kappa_g^2(s_0)\tau_g(s_0)\sigma^{\pm}(s_0)\overline{\boldsymbol{\gamma}}(s_0) + 2\kappa_g(s_0)\tau_g(s_0)\sigma^{\pm}(s_0)\boldsymbol{n}(s_0) - 2\kappa_g'(s_0)\sigma^{\pm}(s_0)\boldsymbol{b}(s_0)$$

$$+(2\kappa_g^2(s_0)\tau_g(s_0)\sigma^{\pm}(s_0)+2)\boldsymbol{e}_0.$$

Moreover we have

$$v_{00} = 2\kappa_g^2(s_0)\tau_g(s_0)\sigma^{\pm}(s_0)+2.$$

Therefore by calculation, we have

$$\det B = \pm\frac{\kappa_g^2(s_0)\tau_g^2(s_0)+\kappa_g'^2(s_0)}{\sqrt{\kappa_g'^2(s_0)+\kappa_g^2(s_0)\tau_g^2(s_0)+\kappa_g^4(s_0)\tau_g^2(s_0)}\pm\kappa_g^2(s_0)\tau_g(s_0)} \neq 0$$

So the rank of B is 4. This completes the proof. $\square$

We have the following theorem:

Theorem 4.4.9. Let $\boldsymbol{\gamma}: I \longrightarrow S^3_+$ be a unit speed curve.

(A) For each one of the lightcone duals $\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ of $\overline{\boldsymbol{\gamma}}$, we have the following assertions:

(1) Each one of the lightcone duals $\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ of $\overline{\boldsymbol{\gamma}}$ is locally diffeomorphic to $C(2,3)\times\mathbb{R}^2$ at (s_0,u_0,v_0) if and only if

$$\kappa_g(s_0)\neq 0,\ u_0=\frac{1}{\kappa_g(s_0)} \text{ and } v_0\neq\left(\frac{1}{\kappa_g(s_0)}\right)'\frac{1}{\tau_g(s_0)}.$$

In this case, each one of $LF^{\pm}_{\overline{\boldsymbol{\gamma}}}$ is non-singular and each one of Images $\boldsymbol{\varepsilon}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ is empty.

(2) Each one of the lightcone duals $\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ of $\overline{\boldsymbol{\gamma}}$ is locally diffeomorphic to $SW\times\mathbb{R}$ at (s_0,u_0,v_0) if and only if

$$\kappa_g(s_0)\neq 0,\ \tau_g(s_0)\neq 0,\ u_0=\frac{1}{\kappa_g(s_0)},\ v_0=\left(\frac{1}{\kappa_g(s_0)}\right)'\frac{1}{\tau_g(s_0)} \text{ and } \kappa_S(s_0)\neq 0.$$

In this case, each one of $LF^{\pm}_{\overline{\boldsymbol{\gamma}}}$ is locally diffeomorphic to $C(2,3,4)\times\mathbb{R}$ and each one of Images $\boldsymbol{\varepsilon}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ is a regular curve.

(3) Each one of the lightcone duals $\overline{LD}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ of $\overline{\boldsymbol{\gamma}}$ is locally diffeomorphic to BF at (s_0,u_0,v_0) if and only if

$$\kappa_g(s_0)\neq 0,\ \tau_g(s_0)\neq 0,\ u_0=\frac{1}{\kappa_g(s_0)},\ v_0=\left(\frac{1}{\kappa_g(s_0)}\right)'\frac{1}{\tau_g(s_0)},\ \kappa_S(s_0)=0 \text{ and } \kappa_S'(s_0)\neq 0.$$

In this case, each one of $LF^{\pm}_{\overline{\boldsymbol{\gamma}}}$ is locally diffeomorphic to $C(BF)\times\mathbb{R}$ and each one of Images $\boldsymbol{\varepsilon}^{\pm}_{\overline{\boldsymbol{\gamma}}}$ is locally diffeomorphic to the projection of the $C(2,3,4,5)$-cusp.

(B) For the lightcone dual LD_{γ} of $\boldsymbol{\gamma}$, we have the following assertions:

(1) The lightcone dual LD_{γ} of $\boldsymbol{\gamma}$ is locally diffeomorphic to $C(2,3)\times\mathbb{R}^2$ at (s_0,u_0,v_0) if and only if

$$u_0 \neq 2\kappa_g(s_0)\tau_g(s_0)\sigma^{\pm}(s_0) \text{ and } v_0 = \pm\sqrt{4+4\kappa_g(s_0)u_0-u_0^2}.$$

In this case, each one of $LF^{\pm}_{\gamma}$ is non-singular and each one of Images $\boldsymbol{\varepsilon}^{\pm}_{\overline{\gamma}}$ is empty.

(2) The lightcone dual LD_{γ} of $\boldsymbol{\gamma}$ is locally diffeomorphic to $SW\times\mathbb{R}$ at (s_0,u_0,v_0) if and only if

$$\kappa_g'^2(s_0)+\kappa_g^2(s_0)\tau_g^2(s_0)\neq 0,\ u_0 = 2\kappa_g(s_0)\tau_g(s_0)\sigma^{\pm}(s_0),\ v_0 = -2\kappa_g'(s_0)\sigma^{\pm}(s_0) \text{ and } \kappa_S(s_0)\neq 0.$$

In this case, each one of $LF^{\pm}_{\gamma}$ is locally diffeomorphic to $C(2,3,4)\times\mathbb{R}$ and each one of Images $\boldsymbol{\varepsilon}^{\pm}_{\overline{\gamma}}$ is a regular curve.

(3) The lightcone dual LD_{γ} of $\boldsymbol{\gamma}$ is locally diffeomorphic to BF at (s_0,u_0,v_0) if and only if

$$\kappa_g'^2(s_0)+\kappa_g^2(s_0)\tau_g^2(s_0)\neq 0,\ u_0 = 2\kappa_g(s_0)\tau_g(s_0)\sigma^{\pm}(s_0),\ v_0 = -2\kappa_g'(s_0)\sigma^{\pm}(s_0),$$
$$\kappa_S(s_0)=0 \text{ and } \kappa_S'(s_0)\neq 0.$$

In this case, each one of $LF^{\pm}_{\gamma}$ is locally diffeomorphic to $C(BF)\times\mathbb{R}$ and each one of Images $\boldsymbol{\varepsilon}^{\pm}_{\overline{\gamma}}$ is locally diffeomorphic to the projection of the $C(2,3,4,5)$-cusp.

Proof. By Propositions 4.3.1 and 4.3.2, the discriminant sets of $\overline{H}$ and H are the lightcone duals of $\overline{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}$ respectively. By Propositions 4.3.1 and 4.3.2, both of $\overline{h}_{\boldsymbol{v}_0}$ and $h_{\boldsymbol{v}_0}$ have A_k singularities $(k=1,2,3,4)$ respectively if and only if the above conditions on the geodesic curvatures and geodesic torsion hold. By Propositions 4.4.7 and 4.4.8, $\overline{H}$ and H are versal unfoldings of $\overline{h}_{\boldsymbol{v}_0}$ and $h_{\boldsymbol{v}_0}$ at any point $s_0\in I$ respectively. We apply Corollary 4.4.6, so that we have the above assertions. This completes the proof. □

5 Lightcone dualities for hypersurfaces in the sphere

5.1 The hypersurfaces in the unit n-sphere

Let $\boldsymbol{x} : U \longrightarrow S^n_+$ be an embedding from an open set $U \subset \mathbb{R}^{n-1}$. We identify $M = \boldsymbol{x}(U)$ with U through the embedding $\boldsymbol{x}$. Obviously, the tangent space T_pM are all spacelike (i.e., consists only spacelike vectors), so M is a spacelike hypersuface in $S^n_+ \subset \mathbb{R}^{n+2}_1$. We have a map $\Phi : S^n_+ \to S^n_0$ defined by $\Phi(\boldsymbol{v}) = \boldsymbol{v} - \boldsymbol{e}_0$, which is an isometry. Then we have a hypersurface $\overline{\boldsymbol{x}} : U \to S^n_0$ defined by $\overline{\boldsymbol{x}}(\boldsymbol{u}) = \Phi(\boldsymbol{x}(\boldsymbol{u})) = \boldsymbol{x}(\boldsymbol{u}) - \boldsymbol{e}_0$, so that $\boldsymbol{x}$ and $\overline{\boldsymbol{x}}$ have the same geometric properties as spherical hypersurfaces. For any $p = \boldsymbol{x}(\boldsymbol{u})$, we can construct a unit normal vector $\boldsymbol{n}(\boldsymbol{u})$ as

$$\boldsymbol{n}(\boldsymbol{u}) = \frac{\overline{\boldsymbol{x}}(\boldsymbol{u}) \wedge \boldsymbol{e}_0 \wedge \boldsymbol{x}_{u_1}(\boldsymbol{u}) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(\boldsymbol{u})}{\|\overline{\boldsymbol{x}}(\boldsymbol{u}) \wedge \boldsymbol{e}_0 \wedge \boldsymbol{x}_{u_1}(\boldsymbol{u}) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(\boldsymbol{u})\|}.$$

We have $\langle \boldsymbol{n}(\boldsymbol{u}), \boldsymbol{n}(\boldsymbol{u})\rangle = 1$, $\langle \boldsymbol{e}_0, \boldsymbol{e}_0\rangle = -1$ and $\langle \boldsymbol{e}_0, \boldsymbol{n}\rangle = \langle \boldsymbol{n}, \boldsymbol{x}_{u_i}\rangle = \langle \boldsymbol{n}, \boldsymbol{x}\rangle = 0$. The system $\{\boldsymbol{e}_0, \boldsymbol{n}(\boldsymbol{u}), \overline{\boldsymbol{x}}(\boldsymbol{u}), \boldsymbol{x}_{u_1}(\boldsymbol{u}), \ldots, \boldsymbol{x}_{u_{n-1}}(\boldsymbol{u})\}$ is a basis of $T_p\mathbb{R}^{n+2}_1$. We define a map $G : U \longrightarrow S^n_0$ by $G(\boldsymbol{u}) = \boldsymbol{n}(\boldsymbol{u})$. We call it the *Gauss map* of the hypersurface $M = \boldsymbol{x}(U)$. We have a linear mapping provided by the derivation of the Gauss map at $p \in M$, $dG(\boldsymbol{u}) : T_pM \longrightarrow T_pM$. We call the linear transformation $S_p = -dG(\boldsymbol{u})$ the *shape operator* of M at $p = \boldsymbol{x}(\boldsymbol{u})$. The eigenvalues of S_p denoted by $\{\kappa_i(p)\}_{i=1}^{n-1}$ are called the *principal curvatures* of M at p. The *Gauss-Kronecker curvature* of M at p is defined to be $K(p) = \det S_p$. A point p is called an *umbilic point* if all the principal curvatures coincide at p and thus we have $S_p = \kappa(p)\mathrm{id}_{T_pM}$ for some $\kappa(p) \in \mathbb{R}$. We say that M is *totally umbilic* if all the points on M are umbilic. Since $\boldsymbol{x}$ is a spacelike embedding, we have a *Riemannian metric* (or the *first fundamental form*) on M given by $ds^2 = \sum_{i,j=1}^{n-1} g_{ij} du_i du_j$, where $g_{ij}(\boldsymbol{u}) = \langle \boldsymbol{x}_{u_i}(\boldsymbol{u}), \boldsymbol{x}_{u_j}(\boldsymbol{u})\rangle$ for any $\boldsymbol{u} \in U$. The second fundamental form on M is given by $h_{ij}(\boldsymbol{u}) = -\langle \boldsymbol{n}_{u_i}(\boldsymbol{u}), \boldsymbol{x}_{u_j}(\boldsymbol{u})\rangle$ at any $\boldsymbol{u} \in U$. Under the above notations, we have the following Weingarten formula [16]:

$$G_{u_i} = -\sum_{j=1}^{n-1} h_i^j \boldsymbol{x}_{u_j} (i = 1, \ldots, n-1),$$

where $(h_i^j) = (h_{ik})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$. This formula induces an explicit expression of the Gauss-Kronecker curvature in terms of the Riemannian metric and the second fundamental invariant given by $K = det(h_{ij}/det(g_{\alpha\beta})$. A point p is a parabolic point if $K(p) = 0$. A point p is a *flat point* if it is an umbilic point and $K(p) = 0$.

In [17] the spherical evolute of a hypersurface has been introduced and investigated the singularities. Then each *spherical evolute* of $\overline{M} = \overline{\boldsymbol{x}}(U)$ is defined to be

$$\boldsymbol{\varepsilon}^{\pm}_{\overline{M}} = \bigcup_{i=1}^{n-1} \left\{ \pm \left(\sqrt{\frac{\kappa_i^2(p)}{1+\kappa_i^2(p)}} \overline{\boldsymbol{x}}(\boldsymbol{u}) + \sqrt{\frac{1}{1+\kappa_i^2(p)}} \boldsymbol{n}(\boldsymbol{u}) \right) \;\middle|\; p = \boldsymbol{x}(\boldsymbol{u}) \in M = \boldsymbol{x}(U) \right\}.$$

5.2 The lightcone dual hypersurfaces and the lightcone height functions

We now define hypersurfaces in LC^* associated with the hypersurfaces in S^n_+ or S^n_0. Let $\boldsymbol{x} : U \longrightarrow S^n_+$ be a hypersurface. We define $\overline{LD}^{\pm}_{M} : U \times \mathbb{R} \longrightarrow LC^*$ by

$$\overline{LD}^{\pm}_{M}(\boldsymbol{u}, \mu) = \overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu \boldsymbol{n}(\boldsymbol{u}) \pm \sqrt{\mu^2 + 1}\boldsymbol{e}_0.$$

We also define $LD_M : U \times \mathbb{R} \longrightarrow LC^*$ by

$$LD_M(\boldsymbol{u}, \mu) = (\mu^2/4 - 1)\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu \boldsymbol{n}(\boldsymbol{u}) + (\mu^2/4 + 1)\boldsymbol{e}_0.$$

Then we have the following proposition.

Proposition 5.2.1. Under the above notations, we have the followings:
(1) $\overline{\boldsymbol{x}}$ and $\overline{LD}^{\pm}_{M}$ are Δ_3-dual to each other.
(2) $\boldsymbol{x}$ and LD_M are Δ_4-dual to each other.

Proof. Consider the mapping $\mathcal{L}_3 : U \times \mathbb{R} \longrightarrow \Delta_3$ defined by $\mathcal{L}_3(\boldsymbol{u}, \mu) = (\overline{LD}^{\pm}_{M}(\boldsymbol{u}, \mu), \overline{\boldsymbol{x}}(\boldsymbol{u}))$. Then we have

$$\langle \overline{LD}^{\pm}_{M}(\boldsymbol{u}, \mu), \overline{\boldsymbol{x}}(\boldsymbol{u}) \rangle = \langle \overline{\boldsymbol{x}}(\boldsymbol{u}), \overline{\boldsymbol{x}}(\boldsymbol{u}) \rangle = 1$$

and

$$\mathcal{L}_3^* \theta_{32} = \langle \overline{LD}^{\pm}_{M}(\boldsymbol{u}, \mu), d\overline{\boldsymbol{x}}(\boldsymbol{u}) \rangle = \sum_{i=1}^{n-1} \langle \overline{LD}^{\pm}_{M}(\boldsymbol{u}, \mu), \boldsymbol{x}_{u_i} \rangle du_i = 0.$$

The assertion (1) holds. We also consider the mapping $\mathcal{L}_4 : U \times \mathbb{R} \longrightarrow \Delta_4$ defined by $\mathcal{L}_4(\boldsymbol{u}, \mu) = (LD_M(\boldsymbol{u}, \mu), \boldsymbol{x}(\boldsymbol{u}))$. Since $\langle \boldsymbol{x}(\boldsymbol{u}), \boldsymbol{e}_0 \rangle = -1$ and $\langle \boldsymbol{x}(\boldsymbol{u}), \overline{\boldsymbol{x}}(\boldsymbol{u}) \rangle = 1$, we have $\langle LD_M(\boldsymbol{u}, \mu), \boldsymbol{x}(\boldsymbol{u}) \rangle = \mu^2/4 - 1 - (\mu^2/4 + 1) = -2$. Moreover, we have

$$\mathcal{L}_4^* \theta_{42} = \langle LD_M(\boldsymbol{u}, \mu), d\boldsymbol{x}(\boldsymbol{u}) \rangle = \sum_{i=1}^{n-1} \langle LD_M(\boldsymbol{u}, \mu), \boldsymbol{x}_{u_i} \rangle du_i = 0.$$

This completes the proof. □

We call each one of $\overline{LD}^{\pm}_{\overline{M}}$ the *lightcone dual hypersurface along* $\overline{M} \subset S^n_0$ and LD_M the *lightcone dual hypersurface along* $M \subset S^n_+$. Then we have two mappings $\pi \circ \overline{LD}^{\pm}_{\overline{M}} : U \times \mathbb{R} \longrightarrow S^n_+$ and $\pi \circ LD_M : U \times \mathbb{R} \longrightarrow S^n_+$ defined by

$$\begin{aligned} \pi \circ \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}, \mu) &= \pm \left(\frac{1}{\sqrt{\mu^2+1}} \overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{\mu}{\sqrt{\mu^2+1}} \boldsymbol{n}(\boldsymbol{u}) \right) + \boldsymbol{e}_0, \\ \pi \circ LD_M(\boldsymbol{u}, \mu) &= \frac{\mu^2-4}{\mu^2+4} \overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{4\mu}{\mu^2+4} \boldsymbol{n}(\boldsymbol{u}) + \boldsymbol{e}_0. \end{aligned}$$

Let $\boldsymbol{x} : U \longrightarrow S^n_+$ be a hypersurface in the lightcone unit sphere. Then we define two families of functions as follows:

$$\overline{H} : U \times LC^* \longrightarrow \mathbb{R},\ \overline{H}(\boldsymbol{u}, \overline{\boldsymbol{v}}) = \langle \overline{\boldsymbol{x}}(\boldsymbol{u}), \overline{\boldsymbol{v}} \rangle - 1,$$

$$H : U \times LC^* \longrightarrow \mathbb{R},\ H(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{x}(\boldsymbol{u}), \boldsymbol{v} \rangle + 2.$$

We call $\overline{H}$ a *lightcone height function* of the de Sitter spherical hypersurface $\overline{M}$. For any fixed $\overline{\boldsymbol{v}}_0 \in LC^*$, we denote $\overline{h}_{\overline{v}_0}(\boldsymbol{u}) = \overline{H}(\boldsymbol{u}, \overline{\boldsymbol{v}}_0)$. We also call H a *lightcone height function* of the lightlike spherical hypersurface M. For any fixed $\boldsymbol{v}_0 \in LC^*$, we denote $h_{v_0}(\boldsymbol{u}) = H(\boldsymbol{u}, \boldsymbol{v}_0)$.

Proposition 5.2.2. Let $\overline{M}$ be a hypersurface in S^n_0 and $\overline{H}$ the lightcone height function on $\overline{M}$. For $p = \boldsymbol{x}(\boldsymbol{u})$ and $\overline{p} = \overline{\boldsymbol{x}}(\boldsymbol{u}) \neq \overline{\boldsymbol{v}}^{\pm}$, we have the followings:

(1) $\overline{h}_{\overline{v}^{\pm}}(\boldsymbol{u}) = \partial \overline{h}_{\overline{v}^{\pm}} / \partial u_i(\boldsymbol{u}) = 0 (i = 1, \dots, n-1)$ if and only if

$$\overline{\boldsymbol{v}}^{\pm} = \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}, \mu) \text{ for some } \mu \in \mathbb{R} \backslash \{\boldsymbol{0}\}.$$

(2) $\overline{h}_{\overline{v}^{\pm}}(\boldsymbol{u}) = \partial \overline{h}_{\overline{\boldsymbol{v}}^{\pm}} / \partial u_i(\boldsymbol{u}) = 0 (i = 1, \dots, n-1)$ and det Hess $(\overline{h}_{\overline{v}^{\pm}})(\boldsymbol{u}) = 0$ if and only if

$$\overline{\boldsymbol{v}}^{\pm} = \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}, \mu),\ 1/\mu \text{ is one of the non-zero principle curvatures } \kappa_i(p) \text{ of } M.$$

Proof. (1) Since $\overline{\boldsymbol{v}} \in LC^*$, there exist $\lambda, \mu, \xi_i, (i = 1, \ldots, n-1), \eta \in \mathbb{R}$ such that $\overline{\boldsymbol{v}} = \lambda\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i \boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_0$ with $\lambda^2 + \mu^2 + \sum_{i,j=1}^{n-1} \xi_i\xi_j g_{ij}(\boldsymbol{u}) - \eta^2 = 0$. The condition $\overline{h}_v(\boldsymbol{u}) = \langle\overline{\boldsymbol{x}}(\boldsymbol{u}), \overline{\boldsymbol{v}}\rangle - 1 = \langle\overline{\boldsymbol{x}}(\boldsymbol{u}), \lambda\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_0\rangle - 1 = \lambda - 1 = 0$ implies $\lambda = 1$, so that $\boldsymbol{v} = \overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_0$ and $1 + \mu^2 + \sum_{i,j=1}^{n-1} \xi_i\xi_j g_{ij}(\boldsymbol{u}) - \eta^2 = 0$. Therefore, $\overline{h}_{\overline{v}}(\boldsymbol{u}) = \partial\overline{h}_{\overline{v}}/\partial u_i(\boldsymbol{u}) = 0$ if and only if $\partial\overline{h}_{\overline{v}}/\partial u_i(\boldsymbol{u}) = \langle\boldsymbol{x}_{u_i}(\boldsymbol{u}), \overline{\boldsymbol{v}}\rangle = \langle\boldsymbol{x}_{u_i}(\boldsymbol{u}), \overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_0\rangle = \sum_{j=1}^{n-1} g_{ij}\xi_j = 0$. Since g_{ij} is positive definite, we have $\xi_j = 0 (j = 1, \ldots, n-1)$. Then we have $1 + \mu^2 - \eta^2 = 0$, so that $\eta = \pm\sqrt{1+\mu^2}$. Thus, we have $\overline{\boldsymbol{v}}^{\pm} = \overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) \pm \sqrt{1+\mu^2}\boldsymbol{e}_0$. The converse direction also holds.

(2) Suppose that $\overline{h}_{\overline{v}^{\pm}}(\boldsymbol{u}) = \partial\overline{h}_{\overline{v}^{\pm}}/\partial u_i(\boldsymbol{u}) = 0$. Then we have $\text{Hess}\,(\overline{h}_{\overline{v}^{\pm}})(\boldsymbol{u}) = (\langle\boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \overline{\boldsymbol{v}}\rangle) = (\langle\boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) \pm \sqrt{1+\mu^2}\boldsymbol{e}_0\rangle) = (\langle\boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \overline{\boldsymbol{x}}(\boldsymbol{u})\rangle) + \mu(\langle\boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \boldsymbol{n}(\boldsymbol{u})\rangle) = -(g_{ij}(\boldsymbol{u})) + \mu(h_{ij}(\boldsymbol{u}))$. It follows that $\det\text{Hess}\,(\overline{h}_{\overline{v}^{\pm}})(\boldsymbol{u}) = 0$ if and if $\det\text{Hess}\,(\overline{h}_{\overline{v}^{\pm}})(\boldsymbol{u})(g_{ij}(\boldsymbol{u}))^{-1}/\mu = \det((h_i^j(\boldsymbol{u})) - I/\mu) = 0$. Thus, $\det\text{Hess}\,(\overline{h}_{\overline{v}^{\pm}})(\boldsymbol{u})$=0 if and only if $1/\mu$ is one of the non-zero principle curvatures of M at p. □

Proposition 5.2.3. Let M be a hypersurface in S_+^n and H be the lightcone height function on M. For $p = \boldsymbol{x}(\boldsymbol{u}) \neq \boldsymbol{v}$, we have the followings.
(1) $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0,\ (i = 1, \ldots, n-1)$ if and only if

$$\boldsymbol{v} = LD_M(\boldsymbol{u}, \mu) \text{ for some } \mu \in \mathbb{R}\backslash\{\boldsymbol{0}\}.$$

(2) $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0,\ (i = 1, \ldots, n-1)$ and $\det\text{ Hess }(h_v)(\boldsymbol{u}) = 0$ if and only if

$$\boldsymbol{v} = LD_M(\boldsymbol{u}, \mu),\ (\mu/4 - 1/\mu) \text{ is one the non-zero principle curvatures } \kappa_i(p) \text{ of } M.$$

Proof. (1) Since $\boldsymbol{v} \in LC^*$, there exist $\lambda, \mu, \xi_i, (i = 1, \ldots, n-1), \eta \in \mathbb{R}$ such that $\boldsymbol{v} = \lambda\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_0$ with $\lambda^2 + \mu^2 + \sum_{i,j=1}^{n-1} \xi_i\xi_j g_{ij}(\boldsymbol{u}) - \eta^2 = 0$. The condition $h_v(\boldsymbol{u}) = \langle\boldsymbol{x}(\boldsymbol{u}), \boldsymbol{v}\rangle + 2 = \langle\overline{\boldsymbol{x}}(\boldsymbol{u}) + \boldsymbol{e}_0, \lambda\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_0\rangle + 2 = \lambda - \eta + 2 = 0$ means that $\eta = 2 + \lambda$, so that $\boldsymbol{v} = \lambda\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + (2+\lambda)\boldsymbol{e}_0$ and $\lambda^2 + \mu^2 + \sum_{i,j=1}^{n-1} \xi_i\xi_j g_{ij}(\boldsymbol{u}) - (2+\lambda)^2 = \mu^2 + \sum_{i,j=1}^{n-1} \xi_i\xi_j g_{ij}(\boldsymbol{u}) - 4\lambda - 4 = 0$. Therefore, $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0$ if and only if $\partial h_v/\partial u_i(\boldsymbol{u}) = \langle\boldsymbol{x}_{u_i}(\boldsymbol{u}), \boldsymbol{v}\rangle = \langle\boldsymbol{x}_{u_i}(\boldsymbol{u}), \lambda\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + (2+\lambda)\boldsymbol{e}_0\rangle =$

$\sum_{j=1}^{n-1} g_{ij}\xi_j = 0$. Since g_{ij} is positive definite, we have $\xi_j = 0$ $(j = 1, \ldots, n-1)$. Then we have $\mu^2 - 4\lambda - 4 = 0$, so that $\lambda = \mu^2/4 - 1$. Thus, we have $\boldsymbol{v} = (\mu^2/4-1)\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + (\mu^2/4+1)\boldsymbol{e}_0$. The converse direction also holds.

(2) Suppose that $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0$. Then we have

$$\begin{aligned}
\operatorname{Hess}(h_v)(\boldsymbol{u}) &= (\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \boldsymbol{v}\rangle) \\
&= (\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), (\mu^2/4-1)\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + (\mu^2/4+1)\boldsymbol{e}_0\rangle) \\
&= (\mu^2/4-1)(\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \overline{\boldsymbol{x}}(\boldsymbol{u})\rangle) + \mu(\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \boldsymbol{n}(\boldsymbol{u})\rangle) \\
&= (1-\mu^2/4)(g_{ij}(\boldsymbol{u})) + \mu(h_{ij}(\boldsymbol{u})).
\end{aligned}$$

Therefore, $\det \operatorname{Hess}(h_v)(\boldsymbol{u}) = 0$ if and if $\det \operatorname{Hess}(h_v)(\boldsymbol{u})(g_{ij}(\boldsymbol{u}))^{-1}/\mu = \det((h_i^j)(\boldsymbol{u}) - (\mu/4 - 1/\mu)I) = 0$, so that $\det \operatorname{Hess}(h_v)(\boldsymbol{u}){=}0$ if and only if $(\mu/4 - 1/\mu)$ is one of the non-zero principle curvatures of M at p. □

Let $(\boldsymbol{u}, \mu)$ be a singular point of each one of $\overline{LD}^{\pm}_{\overline{M}}$. By Proposition 5.2.2, we have $1/\mu = \kappa_i(p)(1 \le i \le n-1)$, where $\kappa_i(p)$ is one of the non-zero principle curvatures of M at $p = \boldsymbol{x}(\boldsymbol{u})$. It follows that $\mu = 1/\kappa_i(p)$. Therefore the critical value sets of $\overline{LD}^{\pm}_{\overline{M}}$ are given by

$$C(\overline{LD}^{\pm}_{\overline{M}}) = \bigcup_{i=1}^{n-1} \left\{ \overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{1}{\kappa_i(p)}\boldsymbol{n}(\boldsymbol{u}) \pm \sqrt{\frac{1}{\kappa_i^2(p)} + 1}\boldsymbol{e}_0 \,\middle|\, \boldsymbol{u} \in U \right\}.$$

Let $(\boldsymbol{u}, \mu)$ be a singular point of $LD_M(\boldsymbol{u}, \mu)$. By Proposition 5.2.3, we have $\mu/4 - 1/\mu = \kappa_i(p)(1 \le i \le n-1)$. It follows that we have $\mu = 2(\kappa_i(p) \pm \sqrt{1 + \kappa_i^2(p)})$. For simplification, we write that $\sigma^{\pm}(\kappa_i(p)) = \kappa_i(p) \pm \sqrt{1+\kappa_i^2(p)}$. Then the critical value sets of LD_M are given by

$$C(LD_M)^{\pm} = \bigcup_{i=1}^{n-1} \{((\sigma^{\pm}(\kappa_i(p)))^2 - 1)\overline{\boldsymbol{x}}(\boldsymbol{u}) + 2\sigma^{\pm}(\kappa_i(p))\boldsymbol{n}(\boldsymbol{u}) + ((\sigma^{\pm}(\kappa_i(p)))^2 + 1)\boldsymbol{e}_0 \mid \boldsymbol{u} \in U\ \}.$$

We respectively denote that

$$LF^{\pm}_{\overline{M}} = \bigcup_{i=1}^{n-1} \left\{ \overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{1}{\kappa_i(p)}\boldsymbol{n}(\boldsymbol{u}) \pm \sqrt{\frac{1}{\kappa_i^2(p)} + 1}\boldsymbol{e}_0 \,\middle|\, \boldsymbol{u} \in U \right\},$$

$$LF^{\pm}_{M} = \bigcup_{i=1}^{n-1} \{((\sigma^{\pm}(\kappa_i(p)))^2 - 1)\overline{\boldsymbol{x}}(\boldsymbol{u}) + 2\sigma^{\pm}(\kappa_i(p))\boldsymbol{n}(\boldsymbol{u}) + ((\sigma^{\pm}(\kappa_i(p)))^2 + 1)\boldsymbol{e}_0 \mid \boldsymbol{u} \in U\ \}.$$

We respectively call each one of $LF^{\pm}_{\overline{M}}$ the *lightcone focal hypersurface* of the de Sitter spherical hypersurface $\overline{\boldsymbol{x}}$ and each one of $LF^{\pm}_{M}$ the *ligtcone focal hypersurface* of the lightcone spherical hypersurface $\boldsymbol{x}$. Then the projections of these surfaces to S^n_+ are given as follows:

$$\pi(C(\overline{LD}^{\pm}_{\overline{M}})) = \bigcup_{i=1}^{n-1} \left\{ \pm \left(\sqrt{\frac{\kappa_i^2(p)}{1+\kappa_i^2(p)}}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \sqrt{\frac{1}{1+\kappa_i^2(p)}}\boldsymbol{n}(\boldsymbol{u}) \right) + \boldsymbol{e}_0 \middle| \ \boldsymbol{u} \in U \right\},$$

$$\pi(C(LD_M)^{\pm}) = \bigcup_{i=1}^{n-1} \left\{ \frac{(\sigma^{\pm}(\kappa_i(p)))^2 - 1}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{2\sigma^{\pm}(\kappa_i(p))}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\boldsymbol{n}(\boldsymbol{u}) + \boldsymbol{e}_0 \mid \boldsymbol{u} \in U \right\}.$$

By definition, we have $\boldsymbol{\varepsilon}^{\pm}_{\overline{M}} = \Phi \circ \pi(C(\overline{LD}^{\pm}_{\overline{M}}))$, where $\boldsymbol{\varepsilon}^{\pm}_{\overline{M}}$ is the spherical evolute of $\overline{M} = \overline{\boldsymbol{x}}(U)$. This means that the spherical evolutes are obtained from the critical value sets of the lightcone dual hypersurfaces of $\overline{M} = \overline{\boldsymbol{x}}(U)$. Since $\sigma^{\pm}(\kappa_i(p)) = \kappa_i(p) \pm \sqrt{1+\kappa_i^2(p)}$, we have$(\sigma^{\pm}(\kappa_i(p)))^2 = 2\kappa_i(p)\sigma^{\pm}(\kappa_i(p)) + 1$. By straightforward calculations, we have

$$\left(\frac{(\sigma^{\pm}(\kappa_i(p)))^2 - 1}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\right)^2 = \frac{\kappa_i^2(p)(\sigma^{\pm}(\kappa_i(p)))^2}{\kappa_i^2(p)(\sigma^{\pm}(\kappa_i(p)))^2 + (\sigma^{\pm}(\kappa_i(p)))^2} = \frac{\kappa_i^2(p)}{1+\kappa_i^2(p)}$$

and

$$\left(\frac{2\sigma^{\pm}(\kappa_i(p))}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\right)^2 = \frac{(\sigma^{\pm}(\kappa_i(p)))^2}{\kappa_i^2(p)(\sigma^{\pm}(\kappa_i(p)))^2 + (\sigma^{\pm}(\kappa_i(p)))^2} = \frac{1}{1+\kappa_i^2(p)}.$$

Thus we have the following proposition.

Proposition 5.2.4. Let $\boldsymbol{x} : U \longrightarrow S^n_+$ be a hypersurface in S^n_+. Then

$$\frac{(\sigma^{\pm}(\kappa_i(p)))^2 - 1}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{2\sigma^{\pm}(\kappa_i(p))}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\boldsymbol{n}(\boldsymbol{u}) = \pm \left(\sqrt{\frac{\kappa_i^2(p)}{1+\kappa_i^2(p)}}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \sqrt{\frac{1}{1+\kappa_i^2(p)}}\boldsymbol{n}(\boldsymbol{u}) \right).$$

We define $\tilde{\pi} = \Phi \circ \pi : LC^* \longrightarrow S^n_0$. Then we have the following theorem as a corollary of Proposition 5.2.4.

Theorem 5.2.5. Both of the projections of the critical value sets $C(\overline{LD}^{\pm}_{\overline{M}})$ and $C(LD_M)^{\pm}$ in the unit sphere S^n_0 are the images of the spherical evolutes of $\overline{M}$.

$$\tilde{\pi}(C(\overline{LD}^{\pm}_{\overline{M}})) = \tilde{\pi}(C(LD_M)^{\pm}) = \boldsymbol{\varepsilon}^{\pm}_{\overline{M}}.$$

5.3 The lightcone dual hypersurfaces as wave fronts

We now naturally interpret the lightcone dual hypersurfaces of the submanifolds in S^n_+ as wave front sets in the theory of Legendrian singularities. Let $\overline{\pi} : PT^*(LC^*) \longrightarrow LC^*$ be the projective cotangent bundles with canonical contact structures. Consider the tangent bundle $\tau : TPT^*(LC^*) \longrightarrow PT^*(LC^*)$ and the differential map $d\overline{\pi} : TPT^*(LC^*) \longrightarrow T(LC^*)$ of $\overline{\pi}$. For any $X \in TPT^*(LC^*)$, there exists an element $\alpha \in T^*(LC^*)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_{\boldsymbol{v}}(LC^*)$, the property $\alpha(V) = 0$ dose not depend on the choice of representative of the class $[\alpha]$. Thus we have the canonical contact structure on $PT^*(LC^*)$ by

$$K = \{X \in TPT^*(LC^*) \mid \tau(X)(d\overline{\pi}(X))\} = 0.$$

On the other hand, we consider a point $\boldsymbol{v} = (v_0, v_1, \ldots, v_{n+1}) \in LC^*$, then we have $v_0 = \pm\sqrt{v_1^2 + \ldots + v_{n+1}^2}$. So we adopt the coordinate system $(v_1, \ldots, v_{n+1})$ of LC^*. For the local coordinate neighborhood $(U, (\pm\sqrt{v_1^2 + \ldots + v_{n+1}^2}, v_1, \ldots, v_{n+1}))$ in LC^*, we have a trivialization $PT^*(LC^*) \equiv LC^* \times P(\mathbb{R}^n)^*$ and we call $((\pm\sqrt{v_1^2 + \ldots + v_{n+1}^2}, v_1, \ldots, v_{n+1}), [\xi_1 : \cdots : \xi_{n+1}])$ homogeneous coordinates of $PT^*(LC^*)$, where $[\xi_1 : \cdots : \xi_{n+1}]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^n)^*$. It is easy to show that $X \in K_{(\boldsymbol{v},[\xi])}$ if and only if $\sum_1^{n+1} \mu_i \xi_i = 0$, where $d\overline{\pi}(X) = \sum_1^{n+1} \mu_i \partial/\partial v_i \in T_{\boldsymbol{v}} LC^*$. An immersion $i : L \longrightarrow PT^*(LC^*)$ is said to be a *Legendrian immersion* if dim $L = n$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. The map $\overline{\pi} \circ i$ is also called the *Legendrian map* and we call the set $W(i)$=image$\overline{\pi} \circ i$ the *wave front* of i. Moreover, i(or the image of i) is called the *Legendrian lift* of $W(i)$. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* of hypersurfaces if the map germ $\Delta^* F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^{k+1}, \mathbf{0})$ defined by $\Delta^* F = (F, \partial F/\partial u_1, \cdots, \partial F/\partial u_k)$. is nonsingular. In this case, we have the following smooth $(n-1)$-dimensional smooth submanifold.

$$\Sigma_*(F) = \left\{ (\boldsymbol{u}, \boldsymbol{v}) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \mid F(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial F}{\partial u_1}(\boldsymbol{u}, \boldsymbol{v}) = \cdots = \frac{\partial F}{\partial u_k}(\boldsymbol{u}, \boldsymbol{v}) = 0 \right\} = (\Delta^* F)^{-1}(\mathbf{0}).$$

The map germ $\mathcal{L}_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\mathcal{L}_F(\boldsymbol{u}, \boldsymbol{v}) = \left(\boldsymbol{v}, \left[\frac{\partial F}{\partial v_1}(\boldsymbol{u}, \boldsymbol{v}) : \ldots : \frac{\partial F}{\partial v_n}(\boldsymbol{u}, \boldsymbol{v}) \right] \right).$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd and Zakalyukin [1, 23].

Proposition 5.3.1. All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.

We call F a generating family of $\mathcal{L}_F(\Sigma_*(F))$. Therefore the wave front of $\mathcal{L}_F$ is

$$W(\mathcal{L}_F) = \left\{ \boldsymbol{v} \in \mathbb{R}^n \mid \exists \boldsymbol{u} \in \mathbb{R}^k \text{ such that } F(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial F}{\partial u_1}(\boldsymbol{u}, \boldsymbol{v}) = \ldots = \frac{\partial F}{\partial u_k}(\boldsymbol{u}, \boldsymbol{v}) = 0 \right\}.$$

We claim here that we have a trivialization as follows:

$$\Phi : PT^*(LC^*) \equiv LC^* \times P(\mathbb{R}^n)^*, \Phi([\sum_{i=1}^{n+1} \xi_i dv_i]) = (v_0, v_1, \cdots, v_{n+1}), [\xi_1 : \cdots \xi_{n+1}])$$

by using the above coordinate system.

Proposition 5.3.2. The lightcone height function $H : U \times LC^* \longrightarrow \mathbb{R}$ is a Morse family of the hypersurfaces around $(\boldsymbol{u}, \boldsymbol{v}) \in \Sigma_*(H)$.

Proof. Without the loss of the generality, we consider the future component LC^*_+. For any $\boldsymbol{v} = (v_0, v_1, \cdots, v_{n+1}) \in LC^*$, we have $v_0 = \sqrt{v_1^2 + \cdots + v_{n+1}^2}$. For $\boldsymbol{x}(\boldsymbol{u}) = (1, x_1(\boldsymbol{u}), \cdots, x_{n+1}(\boldsymbol{u})) \in S^n_+$, we have

$$H(\boldsymbol{u}, \boldsymbol{v}) = -\sqrt{v_1^2 + \cdots + v_{n+1}^2} + x_1(\boldsymbol{u})v_1 + \cdots + x_{n+1}(\boldsymbol{u})v_{n+1} + 2.$$

We have to prove the mapping

$$\triangle^* H = \left(H, \frac{\partial H}{\partial u_1}, \cdots, \frac{\partial H}{\partial u_{n-1}} \right)$$

is non-singular at any point on $(\Delta^* H)^{-1}(\mathbf{0})$. If $(\boldsymbol{u}, \boldsymbol{v}) \in (\Delta^* H)^{-1}(\mathbf{0})$, then $\boldsymbol{v} = LD_M(\boldsymbol{u}, \mu)$ by Proposition 5.2.3. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$A = \begin{pmatrix} \langle \boldsymbol{x}_{u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_{n-1}}, \boldsymbol{v} \rangle & -v_1/v_0 + x_1 & \cdots & -v_{n+1}/v_0 + x_{n+1} \\ \langle \boldsymbol{x}_{u_1 u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_1 u_{n-1}}, \boldsymbol{v} \rangle & x_{1u_1} & \cdots & x_{n+1u_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \boldsymbol{x}_{u_{n-1} u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_{n-1} u_{n-1}}, \boldsymbol{v} \rangle & x_{1u_{n-1}} & \cdots & x_{n+1u_{n-1}} \end{pmatrix},$$

We now prove that rank $A = n$. For $(0, x_1, \cdots, x_{n+1}) = \overline{\boldsymbol{x}}$ and $(0, v_1/v_0, \cdots, v_{n+1}/v_0) = \boldsymbol{v}/v_0 - \boldsymbol{e}_0 = (\mu^2 - 4)\overline{\boldsymbol{x}}/(\mu^2 + 4) + 4\mu\boldsymbol{n}/(\mu^2 + 4)$, we have

$$(0, -v_1/v_0 + x_1, \cdots, -v_{n+1}/v_0 + x_{n+1}) = \overline{\boldsymbol{x}} - \boldsymbol{v}/v_0 + \boldsymbol{e}_0 = 8\overline{\boldsymbol{x}}/(\mu^2 + 4) - 4\mu\boldsymbol{n}/(\mu^2 + 4).$$

Since $\{8\overline{\boldsymbol{x}}/(\mu^2 + 4) - 4\mu\boldsymbol{n}/(\mu^2 + 4), \boldsymbol{x}_{u_1}, \cdots, \boldsymbol{x}_{u_{n-1}}\}$ are linearly independent, rank $A = n$. This completes the proof. $\square$

By the similar arguments to the above proof, we have the following proposition.

Proposition 5.3.3. The lightcone height function $\overline{H} : U \times LC^* \longrightarrow \mathbb{R}$ is a Morse family of the hypersurfaces around $(\boldsymbol{u}, \boldsymbol{v}) \in \Sigma_*(\overline{H})$.

Here, we consider the Legendrian immersion

$$\mathcal{L}_4 : (\boldsymbol{u}, \mu) \longrightarrow \Delta_4, \ \mathcal{L}_4(\boldsymbol{u}, \mu) = (LD_M(\boldsymbol{u}, \mu), \boldsymbol{x}(\boldsymbol{u})).$$

We define the following mapping:

$$\Psi : \Delta_4 \longrightarrow LC^* \times P(\mathbb{R}^n)^*, \Psi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, [v_0 w_1 - v_1 w_0 : \cdots : v_0 w_{n+1} - v_{n+1} w_0]).$$

For the canonical contact form $\theta = \sum_{i=1}^{n+1} \xi_i dv_i$ on $PT^*(LC^*)$, we have $\Psi^*\theta = (v_0 w_1 - v_1 w_0)dv_1 + \cdots + (v_0 w_{n+1} - v_{n+1} w_0)dv_{n+1}|_{\Delta_4} = v_0(-w_0 dv_0 + w_1 dv_1 + \cdots + w_{n+1} dv_{n+1}) - w_0(-v_0 dv_0 + v_1 dv_1 + \cdots + v_{n+1} dv_{n+1})|_{\Delta_4} = v_0\langle \boldsymbol{w}, d\boldsymbol{v} \rangle|_{\Delta_4} = v_0 \theta_{42}|_{\Delta_4}$. Thus Ψ is a contact morphism.

Theorem 5.3.4. For any hypersurface $\boldsymbol{x} : U \longrightarrow S^n_+$, the lightcone height function $H : U \times LC^* \longrightarrow \mathbb{R}$ is a generating family of the Legendrian immersion $\mathcal{L}_4$.

Proof. Since H is a Morse family of hypersurfaces, we have a Legendrian immersion $\mathcal{L}_H : \Sigma_*(H) \longrightarrow PT^*(LC^*)$ defined by $\mathcal{L}_H(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{v}, [\partial H/\partial v_1(\boldsymbol{u}, \boldsymbol{v}) : \cdots : \partial H/\partial v_{n+1}(\boldsymbol{u}, \boldsymbol{v})])$, where $\boldsymbol{v} = (v_0, \ldots, v_{n+1})$ and $\Sigma_*(H) = \{(\boldsymbol{u}, \boldsymbol{v}) \in U \times LC^* \mid \boldsymbol{u} \in U, \boldsymbol{v} = LD_M(\boldsymbol{u}, \mu), \mu \in \mathbb{R}\}$. We observe that H is a generating family of the Legendrian submanifold $\mathcal{L}_H(\Sigma_*(H))$ whose wave front is the image of LD_M. We have

$$\frac{\partial H}{\partial v_i}(\boldsymbol{u}, \boldsymbol{v}) = -\frac{l_i(\boldsymbol{u}, \mu)}{l_0(\boldsymbol{u}, \mu)} + x_i(\boldsymbol{u})(i = 1, \cdots, n+1),$$

where $\boldsymbol{x}(\boldsymbol{u}) = (1, x_1(\boldsymbol{u}), \cdots, x_{n+1}(\boldsymbol{u}))$ and $\boldsymbol{v} = LD_M(\boldsymbol{u}, \mu) = (l_0(\boldsymbol{u}, \mu), \cdots, l_{n+1}(\boldsymbol{u}, \mu))$. It follows that

$$\mathcal{L}_H(\boldsymbol{u}, LD_M(\boldsymbol{u}, \mu)) = (LD_M(\boldsymbol{u}, \mu), [x_1(\boldsymbol{u})l_0(\boldsymbol{u}, \mu) - l_1(\boldsymbol{u}, \mu) : \cdots : x_{n+1}(\boldsymbol{u})l_0(\boldsymbol{u}, \mu) - l_{n+1}(\boldsymbol{u}, \mu)]).$$

Therefore we have $\Psi \circ \mathcal{L}_4(\boldsymbol{u}, \mu) = \mathcal{L}_H(\boldsymbol{u}, \mu)$. This completes the proof. □

Similarly, we consider the Legendrian immersions $\mathcal{L}_3^{\pm} : (\boldsymbol{u}, \mu) \longrightarrow \Delta_3$ defined by $\mathcal{L}_3^{\pm}(\boldsymbol{u}, \mu) = (\overline{LD}_M^{\pm}(\boldsymbol{u}, \mu), \overline{\boldsymbol{x}}(\boldsymbol{u}))$. Then we have the following theorem.

Theorem 5.3.5. For any hypersurface $\overline{\boldsymbol{x}} : U \longrightarrow S_0^n$, the lightcone height function $\overline{H} : U \times LC^* \longrightarrow \mathbb{R}$ is a generating family of the Legendrian immersions $\mathcal{L}_3^{\pm}$.

5.4 Contact with parabolic $(n-1)$-spheres and parabolic n-hyperquadrics

Before we start to consider the contact between hypersurfaces in the sphere with parabolic $(n-1)$-sphere and parabolic n-hyperquadrics, we briefly review the theory of contact due to Montaldi[15]. Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with dim X_1=dim X_2 and dimY_1=dim Y_2. We say that the contact of X_1 and Y_1 at y_1 is *the same type as the contact of* X_2 *and* Y_2 *at* y_2 if there is a diffeomorphism $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case, we write $K(X_1, Y_1, y_1) = K(X_2, Y_2, y_2)$. Of course, in the definition, $\mathbb{R}^n$ can be replaced by any manifold. Two function germs $f_i : (\mathbb{R}^n, a_i) \longrightarrow \mathbb{R}(i = 1, 2)$ are called $\mathcal{K}$-equivalent if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, a_1) \longrightarrow (\mathbb{R}^n, a_2)$, and a function germ $\lambda : (\mathbb{R}^n, a_1) \longrightarrow \mathbb{R}$ with $\lambda(a_1) \neq 0$ such that $f_1 = \lambda \cdot (f_2 \circ \Phi)$.

Theorem 5.4.1 (Montaldi [15])**.** Let X_i, Y_i(for i=1,2) be submanifolds of $\mathbb{R}^n$ with dimX_1=dimX_2 and dimY_1=dimY_2. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, \mathbf{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1, y_1) = K(X_2, Y_2, y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

Returning to the lightcone dual hypersurface LD_M, we now consider the function $\mathfrak{h} : S^n_+ \times LC^* \longrightarrow \mathbb{R}$ defined by $\mathfrak{h}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 2$ and the function $\mathfrak{g} : LC^* \times LC^* \longrightarrow \mathbb{R}$ defined by $\mathfrak{g}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 2$. For a given $\boldsymbol{v}_0 \in LC^*$, we denote $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathfrak{h}(\boldsymbol{u}, \boldsymbol{v_0})$ and $\mathfrak{g}_{v_0}(\boldsymbol{u}) = \mathfrak{g}(\boldsymbol{u}, \boldsymbol{v_0})$, then we have $\mathfrak{h}_{v_0}^{-1}(0) = S^n_+ \cap HP(\boldsymbol{v_0}, -2)$ and $\mathfrak{g}_{v_0}^{-1}(0) = LC^* \cap HP(\boldsymbol{v_0}, -2)$. For any $\boldsymbol{u}_0 \in U$, $\mu_0 \in \mathbb{R}$, we take the point $\boldsymbol{v}_0 = LD_M(\boldsymbol{u}_0, \mu_0)$. Then we have

$$\mathfrak{g}_{v_0} \circ \boldsymbol{x}(\boldsymbol{u}_0) = \mathfrak{g} \circ (\boldsymbol{x} \times id_{LC^*})(\boldsymbol{u}_0, \boldsymbol{v}_0) = \mathfrak{h}_{v_0} \circ \boldsymbol{x}(\boldsymbol{u}_0) = \mathfrak{h} \circ (\boldsymbol{x} \times id_{LC^*})(\boldsymbol{u}_0, \boldsymbol{v}_0) = H(\boldsymbol{u}_0, \boldsymbol{v}_0) = 0.$$

We also have

$$\frac{\partial(\mathfrak{g}_{v_0} \circ \boldsymbol{x})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial(\mathfrak{h}_{v_0} \circ \boldsymbol{x})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial H}{\partial u_i}(\boldsymbol{u}_0, \boldsymbol{v}_0) = 0$$

for $i = 1, \cdots, n-1$. This means that the $(n-1)$-sphere $\mathfrak{h}_{v_0}^{-1}(0) = S^n_+ \cap HP(\boldsymbol{v_0}, -2)$ is tangent to $M = \boldsymbol{x}(U)$ at $p_0 = \boldsymbol{x}(\boldsymbol{u}_0)$. In this case, we call it the *lightcone tangent parabolic* $(n-1)$*-sphere* of M at p_0, which is denoted by $TPS^{n-1}_+(\boldsymbol{x}, \boldsymbol{u}_0)$. The n-hyperquadric $\mathfrak{g}_{v_0}^{-1}(0) = LC^* \cap HP(\boldsymbol{v_0}, -2)$ is also tangent to M at p_0. In this case, we call it the *lightcone tangent parabolic n-hyperquadric* of M at p_0, which is denoted by $TPH^n(\boldsymbol{x}, \boldsymbol{u}_0)$. For the lightcone dual surfaces $\overline{LD}^{\pm}_{\overline{M}}$, we consider a function $\overline{\mathfrak{h}} : S^n_0 \times LC^* \longrightarrow \mathbb{R}$ defined by $\overline{\mathfrak{h}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - 1$ and a function $\overline{\mathfrak{g}} : S^{n+1}_1 \times LC^* \longrightarrow \mathbb{R}$ defined by $\overline{\mathfrak{g}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - 1$. For a given $\boldsymbol{v}_0 \in LC^*$, we denote that $\overline{\mathfrak{h}}_{v_0}(\boldsymbol{u}) = \overline{\mathfrak{h}}(\boldsymbol{u}, \boldsymbol{v_0})$ and $\overline{\mathfrak{g}}_{v_0}(\boldsymbol{u}) = \overline{\mathfrak{g}}(\boldsymbol{u}, \boldsymbol{v_0})$. Then we have $\overline{\mathfrak{h}}_{v_0}^{-1}(0) = S^n_0 \cap HP(\boldsymbol{v_0}, 1)$ and $\overline{\mathfrak{g}}_{v_0}^{-1}(0) = S^{n+1}_1 \cap HP(\boldsymbol{v_0}, 1)$. For any $\boldsymbol{u}_0 \in U$ and the points $\overline{\boldsymbol{v}}^{\pm}_0 = \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}_0, \mu_0)$, we have

$$\overline{\mathfrak{g}}_{\overline{v}^{\pm}_0} \circ \overline{\boldsymbol{x}}(\boldsymbol{u}_0) = \overline{\mathfrak{g}} \circ (\overline{\boldsymbol{x}} \times id_{LC^*})(\boldsymbol{u}_0, \overline{\boldsymbol{v}}^{\pm}_0) = \overline{\mathfrak{h}}_{\overline{v}^{\pm}_0} \circ \overline{\boldsymbol{x}}(\boldsymbol{u}_0) = \overline{\mathfrak{h}} \circ (\overline{\boldsymbol{x}} \times id_{LC^*})(\boldsymbol{u}_0, \overline{\boldsymbol{v}}^{\pm}_0) = \overline{H}(\boldsymbol{u}_0, \overline{\boldsymbol{v}}^{\pm}_0) = 0.$$

We also have

$$\frac{\partial(\overline{\mathfrak{g}}_{\overline{v}^{\pm}_0} \circ \overline{\boldsymbol{x}})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial(\overline{\mathfrak{h}}_{\overline{v}^{\pm}_0} \circ \overline{\boldsymbol{x}})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial \overline{H}}{\partial u_i}(\boldsymbol{u}_0, \overline{\boldsymbol{v}}^{\pm}_0) = 0$$

for $i = 1, \cdots, n-1$. It follows that each one of the $(n-1)$-sphere $\overline{\mathfrak{h}}_{\overline{v}^{\pm}_0}^{-1}(0) = S^n_0 \cap HP(\overline{\boldsymbol{v}}^{\pm}_0, 1)$ is tangent to $\overline{M}$ at $\overline{p}_0 = \overline{\boldsymbol{x}}(\boldsymbol{u}_0)$. In this case, we call each one the *de-Sitter tangent parabolic*

$(n-1)$-*sphere* of $\overline{M}$ at $\overline{p}_0$, which are denoted by $TPS_0^{n-1\pm}(\boldsymbol{x}, \boldsymbol{u}_0)$. Also we have each of the n-hyperquadric $\overline{\mathfrak{g}}^{-1}_{\overline{\boldsymbol{v}}_0^\pm}(0) = S_1^{n+1} \cap HP(\overline{\boldsymbol{v}}_{\boldsymbol{0}}^{\pm}, 1)$ is tangent to $\overline{M}$ at $\overline{p}_0$. In this case, we call each one the *de-Sitter tangent parabolic n-hyperquadric* of $\overline{M}$ at $\overline{p}_0$, which are denoted by $TPS_1^{n\pm}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0)$.

Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (S_+^n, p_i)(i = 1, 2)$ be hypersurface germs. For $\boldsymbol{v}_i = LD_{M_i}(\boldsymbol{u}_i, \mu_i)$, we denote $h_{i,v_i} : (U, \boldsymbol{u}_i) \longrightarrow (\mathbb{R}, 0)$ by $h_{i,v_i}(\boldsymbol{u}_i) = H(\boldsymbol{u}_i, \boldsymbol{v}_i)$. Then we have $h_{i,v_i}(\boldsymbol{u}) = (\mathfrak{h}_{i,v_i} \circ \boldsymbol{x}_i)(\boldsymbol{u}) = (\mathfrak{g}_{i,v_i} \circ \boldsymbol{x}_i)(\boldsymbol{u})$. For $\overline{\boldsymbol{v}}_i^{\pm} = \overline{LD}^{\pm}_{M_i(}(\boldsymbol{u}_i, \mu_i)$, We denote $\overline{h}_{i,\overline{v}_i^\pm} : (U, \boldsymbol{u}_i) \longrightarrow (\mathbb{R}, 0)$ by $\overline{h}_{i,\overline{v}_i^\pm}(\boldsymbol{u}_i) = \overline{H}(\boldsymbol{u}_i, \overline{\boldsymbol{v}}_i^{\pm})$. Then we have $\overline{h}_{i,\overline{v}_i^\pm}(\boldsymbol{u}) = (\overline{\mathfrak{h}}_{i,\overline{v}_i^\pm} \circ \overline{\boldsymbol{x}}_i)(\boldsymbol{u}) = (\overline{\mathfrak{g}}_{i,\overline{v}_i^\pm} \circ \overline{\boldsymbol{x}}_i)(\boldsymbol{u})$. By Theorem 5.4.1, we have the following proposition.

Proposition 5.4.2. Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (S_+^n, p_i)(i = 1, 2)$ be hypersurface germs. For $\boldsymbol{v}_i = LD_{M_i}(\boldsymbol{u}_i, \mu_i)$, the following conditions are equivalent:

(1) $K(\boldsymbol{x}_1(U), TPS_+^{n-1}(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPS_+^{n-1}(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.

(2) $K(\boldsymbol{x}_1(U), TPH^n(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPH^n(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.

(3) h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent.

Moreover, for $\overline{\boldsymbol{v}}_i^{\pm} = \overline{LD}^{\pm}_{M_i}(\boldsymbol{u}_i, \mu_i)$, the following conditions are equivalent:

(4) $K(\boldsymbol{x}_1(U), TPS_0^{n-1\pm}(\boldsymbol{x}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1^{\pm}) = K(\boldsymbol{x}_2(U), TPS_0^{n-1\pm}(\boldsymbol{x}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2^{\pm})$.

(5) $K(\boldsymbol{x}_1(U), TPS_1^{n\pm}(\boldsymbol{x}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1^{\pm}) = K(\boldsymbol{x}_2(U), TPS_1^{n\pm}(\boldsymbol{x}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2^{\pm})$.

(6) $\overline{h}_{1,\overline{v}_1^\pm}$ and $\overline{h}_{2,\overline{v}_2^\pm}$ are $\mathcal{K}$-equivalent.

On the other hand, we return to the review on the theory of Legendrian singularities. We introduce a natural equivalence relation among Legendrian submanifold germs. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then we say that $\mathcal{L}_F(\Sigma_*(F))$ and $\mathcal{L}_G(\Sigma_*(G))$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, z) \longrightarrow (PT^*\mathbb{R}^n, z')$ such that H preserves fibers of π and that $H(\mathcal{L}_F(\Sigma_*(F))) = \mathcal{L}_G(\Sigma_*(G))$, where $z = \mathcal{L}_F(0), z' = \mathcal{L}_G(0)$. By using the Legendrian equivalence, we can define the notion of Legendrian stability for Legendrian submanifold germs by the ordinary way (see, [1][Part III]). We can interpret the Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \boldsymbol{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}, \boldsymbol{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(\boldsymbol{q}, \boldsymbol{x}) = (\psi_1(\boldsymbol{q}, \boldsymbol{x}), \psi_2(\boldsymbol{x}))$ for $(\boldsymbol{q}, \boldsymbol{x}) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$. We say that F is an *infinitesimally $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \ldots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

The main result in the theory of Legendrian singularities ([1], §20.8 and [23], THEOREM 2) is the following:

Proposition 5.4.3 (Arnol'd, Zakalyukin)**.** Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be Morse families and we denote the corresponding Legendrian immersion germs by $\mathcal{L}_F, \mathcal{L}_G$. Then

(1) $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent if and only if F and G are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(2) $\mathcal{L}_F$ is Legendrian stable if and only if F is $\mathcal{K}$-versal deformation of f.

Since F and G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do not need the notion of stably P-$\mathcal{K}$-equivalences under this situation [23, page 27]. For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define *the local ring of f* by $Q_r(f) = \mathcal{E}_n / (f^*(\mathfrak{M}_p)\mathcal{E}_n + \mathfrak{M}_n^{r+1})$. We have the following classification result of Legendrian stable germs (cf. [9, Proposition A.4]) which is the key for the purpose in this section.

Proposition 5.4.4. Let $F, G : (\mathbb{R}^n \times \mathbb{R}^k, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be Morse families. Suppose that Legendrian immersion germs $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian stable, then the following conditions are equivalent.

(1) $W(\mathcal{L}_F)$ and $W(\mathcal{L}_G)$ are diffeomorphic as set germs.

(2) $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent.

(3) $Q_{n+1}(f)$ and $Q_{n+1}(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ and $g = G|_{\mathbb{R}^k \times \{\mathbf{0}\}}$.

Let $Q_{n+1}(\boldsymbol{x}, u_0)$ be the local ring of the function germ $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ defined by

$$Q_{n+1}(\boldsymbol{x}, \boldsymbol{u}_0) = C^{\infty}_{u_0}(U)/(\langle h_{v_0}\rangle_{C^{\infty}_{u_0}(U)} + \mathfrak{M}^{n+2}_{n-1}),$$

and $Q^{\pm}_{n+1}(\overline{\boldsymbol{x}}, u_0)$ be the local rings of the function germs $\overline{h}_{\overline{v}^{\pm}_0} : (U, u_0) \longrightarrow \mathbb{R}$ defined by

$$Q^{\pm}_{n+1}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0) = C^{\infty}_{u_0}(U)/(\langle \overline{h}_{\overline{v}^{\pm}_0}\rangle_{C^{\infty}_{u_0}(U)} + \mathfrak{M}^{n+2}_{n-1}),$$

where $\boldsymbol{v}_0 = LD_M(\boldsymbol{u}_0, \mu_0)$, $\overline{\boldsymbol{v}}^{\pm}_0 = \overline{LD}^{\pm}_{M}(\boldsymbol{u}_0, \mu_0)$ and $C^{\infty}_{u_0}(U)$ is the local ring of function germs at $\boldsymbol{u}_0$ with the unique maximal ideal $\mathfrak{M}_{n-1}$.

Theorem 5.4.5. Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (S^n_+, p_i)(i = 1, 2)$ be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent.

(1) The lightcone hypersurface germs $LD_{M_1}(U \times \mathbb{R})$ and $LD_{M_2}(U \times \mathbb{R})$ are diffeomorphic.

(2) Legendrian immersion germs $\mathcal{L}^1_4$ and $\mathcal{L}^2_4$ are Legendrian equivalent.

(3) The lightcone height functions germs H_1 and H_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(4) h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent.

(5) $K(\boldsymbol{x}_1(U), TPS^{n-1}_+(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPS^{n-1}_+(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.

(6) $K(\boldsymbol{x}_1(U), TPH^n(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPH^n(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.

(7) Local rings $Q_{n+1}(\boldsymbol{x}_1, \boldsymbol{u}_1)$ and $Q_{n+1}(\boldsymbol{x}_2, \boldsymbol{u}_2)$ are isomorphic as $\mathbb{R}$-algebras.

Proof. By Proposition 5.4.3 and Proposition 5.4.4, the conditions (1)∼(3)and (7) are equivalent. By definition, the condition (3) implies the condition (4). By Proposition 5.4.3, H_i is a $\mathcal{K}$-versal deformation of h_{i,v_i}. We can apply the uniqueness result of $\mathcal{K}$-versal deformations (cf., [14]), so that the condition (4) implies the condition (3). By Theorem 5.4.1, the conditions (4)∼(6) are equivalent. This completes the proof. □

Theorem 5.4.6. Let $\overline{\boldsymbol{x}}_i : (U, \boldsymbol{u}_i) \longrightarrow (S^n_0, p_i)(i = 1, 2)$ be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent.

(1) The lightcone hypersurface germs $\overline{LD}^{\pm}_{M_1}(U \times \mathbb{R})$ and $\overline{LD}^{\pm}_{M_2}(U \times \mathbb{R})$ are diffeomorphic.

(2) Legendrian immersion germs $\mathcal{L}_3^{1\pm}$ and $\mathcal{L}_3^{2\pm}$ are Legendrian equivalent.

(3) The lightcone height functions germs $\overline{H}_1$ and $\overline{H}_2$ are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(4) $\overline{h}_{1,\overline{v}_1^\pm}$ and $\overline{h}_{2,\overline{v}_2^\pm}$ are $\mathcal{K}$-equivalent.

(5) $K(\overline{\boldsymbol{x}}_1(U), TPS_0^{n-1\pm}(\overline{\boldsymbol{x}}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1^\pm) = K(\overline{\boldsymbol{x}}_2(U), TPS_0^{n-1\pm}(\overline{\boldsymbol{x}}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2^\pm)$.

(6) $K(\overline{\boldsymbol{x}}_1(U), TPS_1^{n\pm}(\overline{\boldsymbol{x}}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1^\pm) = K(\overline{\boldsymbol{x}}_2(U), TPS_1^{n\pm}(\overline{\boldsymbol{x}}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2^\pm)$.

(7) Local rings $Q_{n+1}^\pm(\overline{\boldsymbol{x}}_1, \boldsymbol{u}_1)$ and $Q_{n+1}^\pm(\overline{\boldsymbol{x}}_2, \boldsymbol{u}_2)$ are isomorphic as $\mathbb{R}$-algebras.

The proof is similar to the proof of the above theorem, so that we omit it.

Lemma 5.4.7. Let $\boldsymbol{x} : U \longrightarrow S_+^n$ be a hypersurface germ such that the corresponding Legendrian immersion germs $\mathcal{L}_4$ and $\mathcal{L}_3^\pm$ are Legendrian stable. Then at the singular point $\boldsymbol{v}_0 = LD_M(\boldsymbol{u}_0, 2\sigma^\pm(\kappa_i(p_0)))(1 \le i \le n-1)$ of LD_M and the singular points $\overline{\boldsymbol{v}}_0^\pm = \overline{LD}_M^\pm(\boldsymbol{u}_0, 1/\kappa_i(p_0))$ of $\overline{LD}_M^\pm$, we have the following equivalent assertions:

(1) The lightcone hypersurface germs $LD_M(U \times \mathbb{R})$ and $\overline{LD}_M^\pm(U \times \mathbb{R})$ are diffeomorphic.

(2) Legendrian immersion germs $\mathcal{L}_3^\pm$ and $\mathcal{L}_4$ are Legendrian equivalent.

(3) The lightcone height functions germs H and $\overline{H}$ are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(4) h_{v_0} and $\overline{h}_{\overline{v}_0^\pm}$ are $\mathcal{K}$-equivalent.

(5) $K(\boldsymbol{x}(U), TPS_+^{n-1}(\boldsymbol{x}, \boldsymbol{u}_0), \boldsymbol{v}_0) = K(\overline{\boldsymbol{x}}(U), TPS_0^{n-1\pm}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0), \overline{\boldsymbol{v}}_0^\pm)$.

(6) $K(\boldsymbol{x}(U), TPH^n(\boldsymbol{x}, \boldsymbol{u}_0), \boldsymbol{v}_0) = K(\overline{\boldsymbol{x}}(U), TPS_1^{n\pm}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0), \overline{\boldsymbol{v}}_0^\pm)$.

(7) Local rings $Q_{n+1}^\pm(\overline{\boldsymbol{x}}, \boldsymbol{u}_0)$ and $Q_{n+1}(\boldsymbol{x}, \boldsymbol{u}_0)$ are isomorphic as $\mathbb{R}$-algebras.

Proof. By definition, we have $h_{v_0}(\boldsymbol{u}) = \langle \boldsymbol{x}(\boldsymbol{u}), ((\sigma^\pm(\kappa_i(p_0)))^2 - 1)\overline{\boldsymbol{x}}(\boldsymbol{u}_0) + 2\sigma^\pm(\kappa_i(p_0))\boldsymbol{n}(\boldsymbol{u}_0) + (\sigma^\pm(\kappa_i(p_0)))^2 + 1)\boldsymbol{e}_0\rangle + 2$, so that

$$\frac{h_{v_0}(\boldsymbol{u})}{(\sigma^\pm(\kappa_i(p_0)))^2 + 1} = \langle \overline{\boldsymbol{x}}(\boldsymbol{u}) + \boldsymbol{e}_0, \pm\left(\sqrt{\frac{\kappa_i^2(p_0)}{1+\kappa_i^2(p_0)}}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \sqrt{\frac{1}{1+\kappa_i^2(p_0)}}\boldsymbol{n}(\boldsymbol{u}_0)\right) + \boldsymbol{e}_0\rangle$$

and

$$+\frac{1}{\sqrt{1+\kappa_i^2(p_0)}(\sqrt{1+\kappa_i^2(p_0)} \pm \kappa_i(p_0))}$$

$$= \langle \overline{\boldsymbol{x}}(\boldsymbol{u}), \pm\left(\sqrt{\frac{\kappa_i^2(p_0)}{1+\kappa_i^2(p_0)}}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \sqrt{\frac{1}{1+\kappa_i^2(p_0)}}\boldsymbol{n}(\boldsymbol{u}_0)\right) + \boldsymbol{e}_0\rangle \mp \frac{\kappa_i(p_0)}{\sqrt{1+\kappa_i^2(p_0)}}.$$

We also have

$$\overline{h}_{\overline{v}_0^\pm}(\boldsymbol{u}) = \langle \overline{\boldsymbol{x}}(\boldsymbol{u}), \overline{\boldsymbol{x}}(\boldsymbol{u}_0) + \frac{1}{\kappa_i(p_0)}\boldsymbol{n}(\boldsymbol{u}_0) \pm \sqrt{\frac{1}{\kappa_i^2(p_0)} + 1}\boldsymbol{e}_0\rangle - 1$$

and

$$\pm\frac{\kappa_i(p_0)\overline{h}_{\overline{v}_0^\pm}(\boldsymbol{u})}{\sqrt{\kappa_i^2(p_0)+1}} = \langle \overline{\boldsymbol{x}}(\boldsymbol{u}), \pm\left(\sqrt{\frac{\kappa_i^2(p_0)}{1+\kappa_i^2(p_0)}}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \sqrt{\frac{1}{1+\kappa_i^2(p_0)}}\boldsymbol{n}(\boldsymbol{u}_0)\right) + \boldsymbol{e}_0\rangle \mp \frac{\kappa_i(p_0)}{\sqrt{1+\kappa_i^2(p_0)}}.$$

Therefore, we have

$$\overline{h}_{\overline{v}_0^\pm} = \frac{\pm\sqrt{\kappa_i^2(p_0)+1}}{\kappa_i(p_0)((\sigma^\pm(\kappa_i(p_0)))^2+1)}h_{v_0}.$$

This means that the assertion (4) holds. By the uniqueness of the $\mathcal{K}$-versal deformation, we have the assertion (3). By Proposition 5.4.3, we have the assertion (2). By Proposition 5.4.4, we have the assertions (1) and (7). On the other hand, for $\mathfrak{g}_{v_0} \circ \boldsymbol{x} = \mathfrak{h}_{v_0} \circ \boldsymbol{x} = h_{v_0}$ and $\overline{\mathfrak{g}}_{\overline{v}_0^\pm} \circ \overline{\boldsymbol{x}} = \overline{\mathfrak{h}}_{\overline{v}_0^\pm} \circ \overline{\boldsymbol{x}} = \overline{h}_{\overline{v}_0^\pm}$, by Theorem 5.4.1, we have the assertions (5) and (6). This completes the proof. □

By Lemma 5.4.7, we have our main result as the following theorem.

Theorem 5.4.8. Let $\boldsymbol{x}_i : (U, \boldsymbol{u}_i) \longrightarrow (S^n_+, p_i)(i = 1, 2)$ be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. At the singular points $\overline{\boldsymbol{v}}_i^\pm = \overline{LD}^\pm_{\overline{M}}(\boldsymbol{u}_0, 1/\kappa_j(p))(1 \le j \le n-1)$ of $\overline{LD}^\pm_{\overline{M}}$, and the singular points $\boldsymbol{v}_i = LD_M(\boldsymbol{u}_0, 2\sigma^\pm(\kappa_j(p)))$ of LD_M, the conditions (1) $\sim$ (7) in Theorem 5.4.5 and the conditions (1) $\sim$ (7) in Theorem 5.4.6 are all equivalent.

5.5 Surfaces in the 3-sphere

In this section, we stick to the case $n = 3$. We consider the surfaces in the 3-sphere as a special case of the previous sections. First we consider the generic properties of spacelike submanifolds in the unit sphere S^3_0. We consider the space of embeddings $\mathrm{Emb}(U, S^3_0)$ with Whitney C^∞-topology. We also consider the function $\overline{\mathcal{H}} : S^3_0 \times LC^* \longrightarrow \mathbb{R}$ which is given by $\overline{\mathcal{H}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v}\rangle - 1$. We claim that $\overline{\mathfrak{h}}_v$ is a submersion for any $\boldsymbol{v} \in LC^*$, where $\overline{\mathfrak{h}}_v(\boldsymbol{u}) = \overline{\mathcal{H}}(\boldsymbol{u}, \boldsymbol{v})$.

For any $\overline{\boldsymbol{x}} \in \mathrm{Emb}(U, S^3_0)$, we have $\overline{H} = \overline{\mathcal{H}} \circ (\overline{\boldsymbol{x}} \times id_{LC^*})$. We have the k-jet extension

$$j^k_1\overline{H} : U \times LC^* \longrightarrow J^k(U, \mathbb{R})$$

defined by $j^k_1\overline{H}(\boldsymbol{u}, \boldsymbol{v}) = j^k\overline{h}_v(\boldsymbol{u})$. We consider the trivialization $J^k(U, \mathbb{R}) = U \times \mathbb{R} \times J^k(2, 1)$. For any submanifold $Q \subset J^k(2, 1)$, we denote $\tilde{Q} = U \times 0 \times Q$. Then we have the following proposition as a corollary of [22, Lemma 6].

Proposition 5.5.1. Let Q be a submanifold of $J^k(2, 1)$. Then the set

$$T_Q = \{\overline{\boldsymbol{x}} \in \mathrm{Emb}(U, S^3_0) \mid j^k_1\overline{H} \text{ is transversal to } \tilde{Q}\}$$

is a residual subset of $\mathrm{Emb}(U, S^3_0)$. If Q is a closed set, then T_Q is open.

By the previous arguments and the Appendix of [9], we have the following theorem.

Theorem 5.5.2. There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}(U, S^3_0)$ such that for any $\overline{\boldsymbol{x}} \in \mathcal{O}$, the corresponding Legendrian immersion germs $\mathcal{L}^{\pm}_3$ at any point are Legendrian stable.

If we consider $\mathcal{H} : S^3_+ \times LC^* \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 2$ instead of $\overline{\mathcal{H}} : S^3_0 \times LC^* \longrightarrow \mathbb{R}$, we can show that the corresponding Legendrian immersion germ $\mathcal{L}_4$ at any point is Legendrian stable for a generic hypersurface $\boldsymbol{x} : U \longrightarrow S^3_+$.

We now classify the singularities of the lightcone dual hypersurfaces. Here we only consider the case for $\overline{M} = \overline{\boldsymbol{x}}(U)$ in S^3_0. By Proposition 5.4.5, a $\mathcal{K}$-invarint for the height function $\overline{h}_v$ is an invariant for the diffeomorphism class of the singularities of the lightcone duals of a hypersurface in S^3_0. Let $\overline{\boldsymbol{x}} : U \longrightarrow S^3_0$ be an embedding from an open set $U \subset \mathbb{R}^2$, we define the $\mathcal{K}$-codimension (or Tyurina number) of the function germ $\overline{h}_{\overline{v}^{\pm}_0}$ by

$$\overline{H}\text{-ord}^{\pm}(\overline{\boldsymbol{x}}, u_0) = \dim C^{\infty}_{u_0} / \langle \overline{h}_{\overline{v}^{\pm}_0}, \partial \overline{h}_{\overline{v}^{\pm}_0} / \partial u_i \rangle_{C^{\infty}_{u_0}}.$$

We call it the *order of contact* of M with parabolic $(n-1)$-spheres and parabolic n-hyperquadrics. We also define the corank of the function germ $\overline{h}_{\overline{v}^{\pm}_0}$ by

$$\overline{H}\text{-corank}^{\pm}(\overline{\boldsymbol{x}}, u_0) = 2 - \text{rank Hess}(\overline{h}_{\overline{v}^{\pm}_0})(u_0).$$

By Theorem 5.3.5, Theorem 5.5.2 and Proposition 5.4.3, the lightcone height function $\overline{H}$ is a $\mathcal{K}$-versal deformation of $\overline{h}_{\overline{v}_0^{\pm}}$ at each point $(\boldsymbol{u}_0, \overline{\boldsymbol{v}}_0^{\pm}) \in U \times LC^*$. Therefore we can apply the classification of $\mathcal{K}$-versal deformations of function germs up to 4-parameters[1]. Suppose that the lightcone height function $\overline{H}$ is a $\mathcal{K}$-versal deformation of $\overline{h}_{\overline{v}_0^{\pm}}$ at each point $(\boldsymbol{u}_0, \overline{\boldsymbol{v}}_0^{\pm}) \in U \times LC^*$. Then it is P-$\mathcal{K}$-equivalent to one of the following germs:

$$(A_k) \quad F(u_1, u_2, \boldsymbol{\lambda}) = u_1^{k+1} \pm u_2^2 + \lambda_1 + \lambda_2 u_1 + \cdots + \lambda_k u_1^{k-1}, (1 \leq k \leq 4),$$

$$(D_4^+) \quad F(u_1, u_2, \boldsymbol{\lambda}) = u_1^3 + u_2^3 + \lambda_1 + \lambda_2 u_1 + \lambda_3 u_2 + \lambda_4 u_1 u_2,$$

$$(D_4^-) \quad F(u_1, u_2, \boldsymbol{\lambda}) = u_1^3 - u_1 u_2^2 + \lambda_1 + \lambda_2 u_1 + \lambda_3 u_2 + \lambda_4 (u_1^2 + u_2^2).$$

For any $F(u_1, u_2, \boldsymbol{\lambda})$, we have

$$W(\mathcal{L}_F) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^4 \mid \exists \boldsymbol{u} \in \mathbb{R}^2 \text{ such that } F(\boldsymbol{u}, \boldsymbol{\lambda}) = \frac{\partial F}{\partial u_1}(\boldsymbol{u}, \boldsymbol{\lambda}) = \frac{\partial F}{\partial u_2}(\boldsymbol{u}, \boldsymbol{\lambda}) = 0 \right\}.$$

Let $f_i : (N_i, x_i) \longrightarrow (P_i, y_i) (i = 1, 2)$ be C^∞ map germs. We say that f_1 and f_2 are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi : (N_1, x_1) \longrightarrow (N_2, x_2)$ and $\psi : (P_1, y_1) \longrightarrow (P_2, y_2)$ such that $\psi \circ f_1 = f_2 \circ \phi$. Then we have the following theorem.

Theorem 5.5.3. There exists an open dense subset $\mathcal{O} \subset Emb_{sp}(U, S_0^3)$ such that for any $\overline{\boldsymbol{x}} \in \mathcal{O}$, we have the following classification:

(a) If $\overline{H}$-corank$^{\pm}(\overline{\boldsymbol{x}}, u_0) = 1$, then there are two distinct principle curvatures κ_1 and κ_2. In this case H-ord$^{\pm}(\boldsymbol{x}, u_0) \leq 4$ and we have the following:

(A_1) If H-ord$^{\pm}(\boldsymbol{x}, u_0) = 1$, then each one of $\overline{LD_M^{\pm}}$ is $\mathcal{A}$-equivalent to

$$f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0).$$

(A_2) If H-ord$^{\pm}(\boldsymbol{x}, u_0) = 2$, then each one of $\overline{LD_M^{\pm}}$ is $\mathcal{A}$-equivalent to

$$f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3).$$

The image of f is diffeomorphic to $C \times \mathbb{R}^2$.

(A_3) If H-ord$^{\pm}(\boldsymbol{x}, u_0) = 3$, then each one of $\overline{LD}^{\pm}_{M}$ is $\mathcal{A}$-equivalent to

$$f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3).$$

The image of f is diffeomorphic to $SW \times \mathbb{R}$.

(A_4) If H-ord$^{\pm}(\boldsymbol{x}, u_0) = 4$, then each one of $\overline{LD}^{\pm}_{M}$ is $\mathcal{A}$-equivalent to

$$f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_2, u_3).$$

The image of f is diffeomorphic to BF.

(b) If $\overline{H}$-corank$^{\pm}(\overline{\boldsymbol{x}}, u_0) = 2$ and the principle curvature $\kappa \neq 0$, then $\boldsymbol{u}_0$ is a non-flat umbilic point. In this case we have H-ord$^{\pm}(\boldsymbol{x}, u_0) = 4$ and the following two cases:

(D_4^+) Each one of $\overline{LD}^{\pm}_{M}$ is $\mathcal{A}$-equivalent to

$$f(u_1, u_2, u_3) = (2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3).$$

(D_4^-) Each one of $\overline{LD}^{\pm}_{M}$ is $\mathcal{A}$-equivalent to

$$f(u_1, u_2, u_3) = (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3).$$

Here, $C = \{(x_1, x_2) \mid x_1 = u^2, x_2 = u^3\}$ is the *ordinary cusp*, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is called a *swallowtail* and $BF = \{(x_1, x_2, x_3, x_4) \mid x_1 = 5u^4 + 3vu^2 + 2wu, x_2 = 4u^5 + 2vu^3 + wu^2, x_3 = u, x_4 = v\}$ is called a *butterfly*.

Proof. By Theorem 5.5.2, there exists an open dense subset $\mathcal{O} \subset Emb_{sp}(U, S^3_0)$ such that for any $\overline{\boldsymbol{x}} \in \mathcal{O}$, the corresponding Legendrian immersion germs $\mathcal{L}^{\pm}_3$ at any point are Legendrian stable. Therefore, the height function $\overline{H}$ is P-$\mathcal{K}$-equivalent to one of the germs of (A_k) ($k = 1, 2, 3, 4$) and $D_4^{\pm}$. If we consider the germ $F(u_1, u_2, \boldsymbol{\lambda}) = u_1^3 \pm u_2^2 + \lambda_1 + \lambda_2u_1$, then we have

$$W(\mathcal{L}_F) = \{(2u_1^3, -3u_1^2, \lambda_3, \lambda_4) \mid (u_1, \lambda_3, \lambda_4) \in \mathbb{R}^3\},$$

so that the corresponding Legendrian map germ is (A_2) $f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3)$. Suppose that $\overline{H}$ is $\mathcal{P}$-$\mathcal{K}$-equivalent to F of type (A_2). By Propositions 5.4.3 and 5.4.4, $\overline{LD}^{\pm}_{M}$ is $\mathcal{A}$-equivalent to (A_2). Of course, the image of f is $C \times \mathbb{R}^2$. Moreover, the $\mathcal{K}$-codimension of $f(u_1, u_2) = u_1^3 \pm u_2^2$ is 2, so that we have H-ord$^{\pm}(\boldsymbol{x}, u_0) = 2$. The proof of the other assertions are similar to this case. Therefore, we omit it. □

By Lemma 5.4.7, the lightcone dual surface LD_M of $\boldsymbol{x} : U \longrightarrow S^3_+$ is locally diffeomorphic to the lightcone dual surfaces $\overline{LD}^{\pm}_{\overline{M}}$. Therefore, we obtain exactly the same assertions as the above theorem for the lightcone dual surface LD_M.

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