# On global solutions for the Constantin-Lax-Majda equation with a generalized viscosity term

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**Abstract**

We consider a one-dimensional model for the three-dimensional vorticity equation of incompressible and viscous fluids. This model is obtained by adding a generalized viscous diffusion term to the Constantin-Lax-Majda equation, which was introduced as a model for the 3-D Euler equation[2]. It is shown in [6] that the solution of the model equation blows up in finite time for sufficiently small viscosity, however large diffusion term it may has. In the present article, we discuss the existence of a unique global solution for large viscosity.

## 1 Introduction

The vorticity equation of three-dimensional inviscid and incompressible fluids is given by

$$\begin{cases} \dfrac{Dw}{Dt} &= w\nabla v, \qquad x \in \mathbf{R^3},\ t > 0, \\ w(x,0) &= w_0(x), \end{cases} \tag{1}$$

where $\frac{D}{Dt}$ is the convective derivative, $w$ is the vorticity and $v$ is the velocity. Since $w = \nabla \times v$ and div $v = 0$, the velocity is recovered from the vorticity by the Biot-Savart integral;

$$v(x,t) = -\frac{1}{4\pi}\int \frac{(x-y)}{|x-y|^3} \times w(y,t)dy. \tag{2}$$

The primary mathematical problem with regard to the equations (1) is to verify whether smooth solution exists for all time or it acquires singularity in finite time. In two dimensional space, the quadratic term $w\nabla v \equiv 0$ and the vorticity is conserved, which guarantees existence of global solution. On the other hand, well-posedness of the 3-D Euler equations is an open problem, since the quadratic term $w\nabla v \neq 0$ in three dimensional space and it allows the vorticity to grow. According to Beale et al.[1], if a smooth solution acquires a singularity in finite time $T^*$, the maximum vorticity necessarily diverges as $t \nearrow T^*$. This means the growth of the vorticity, and so the quadratic term $w\nabla v$, plays an important role in the mathematical problem.

Constantin, Lax and Majda[2] rewrote the quadratic term $w\nabla v$ in an operator form $\mathrm{D}(w)w$ by using (2), where $\mathrm{D}(w)$ is a strongly singular integral operator acting on $w$. By considering some properties of the operator, they proposed a

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one-dimensional model for the vorticity equation (1), which we call Constantin-Lax-Majda (CLM) equation:

$$\frac{\partial \omega}{\partial t} = \omega H(\omega), \qquad x \in \mathbf{R},\ t > 0.$$

The operator $H$ is the Hilbert transform, which is a scalar one-dimensional analogue of the operator $\mathrm{D}(w)$. The model successfully described the growth of the magnitude of vorticity for the 3-D Euler equations. See [2] for the details.

Now, we consider the CLM equation with a viscosity term. This equation can be a one-dimensional model for the 3-D Navier-Stokes equations. Since well-posedness of the 3-D Navier-Stokes equations is also an open problem in the field of the mathematical fluid dynamics, the one-dimensional model is supposed to give a key to the problem.

One candidate for the viscosity term is a simple diffusion:

$$\frac{\partial \omega}{\partial t} = \omega H(\omega) + \nu \omega_{xx}, \qquad x \in \mathbf{R},\ t > 0.$$

Schochet gave an explicit solution of this equation by using pole expansion method[7]. He showed that the solution blew up in finite time and the velocity recovered from the solution also blew up at the same time. Besides, the solution blew up before the solution of the original CLM equation. He concluded that the model with the simple diffusion term was less successful than the CLM equation, since the vorticity and the velocity never blows up simultaneously in the 3-D Navier-Stokes equations and the solution of 3-D Navier-Stokes equation is more regular than that of 3-D Euler equation.

Murthy[3] and Wegert & Murthy[8] proposed a non-local viscosity term, $\nu H(\omega_x)$ instead of the diffusion. They obtained an explicit solution that blew up after the solution of the CLM equation blew up. Thus, they improved one of the drawbacks of Schochet's model.

However, it is still unknown what kind of the viscosity term is appropriate for the model of the 3-D Navier-Stoke equation. In the present article, we consider the CLM equation with a generalized viscosity term:

$$\begin{cases} \dfrac{\partial \omega}{\partial t} &= \omega \mathrm{H}(\omega) - \nu(-\Delta)^{\frac{\alpha}{2}}\omega, \qquad x \in \mathbf{R},\ t > 0, \\ \omega(x,0) &= \omega_0(x), \end{cases} \tag{3}$$

where a positive real $\nu$ represents the viscosity coefficient and $\alpha \in \mathbf{R}$ is order of derivative. Note that this model includes the Schochet's ($\alpha = 2$) and the Murthys' ($\alpha = 1$). In the paper[6], S- proved that for an arbitrary $\alpha$, there exists a small viscosity coefficient $\nu_0$ such that solutions for almost all initial data blow up in finite time for $0 < \nu < \nu_0$. This means the quadratic term $\omega\mathrm{H}(\omega)$ has strong nonlinearity so that arbitrary diffusion can't control its growth if the viscosity coefficient is small.

In the present article, we consider a counter-question: do solutions exist for all time if the viscosity coefficient is sufficiently large? The paper consists of five sections. We give a spectral representation of the CLM equation (3) and give a formal solution in §2, by the spectral method used in the previous paper[6]. In §3, we prove existence of global solutions for $\alpha \geq 1$. In §4, we show that the solution always blows up for arbitrary large $\nu$, i.e. no existence of global solution, when $\alpha < 0$. Final section is conclusion.

## 2 Spectral Method

In this article, we consider the periodic boundary condition in $x$, i.e. $\omega(x+2\pi,t)=\omega(x,t)$. Thus we can represent the solution by the following Fourier series;

$$\omega(x,t)=\sum_{n=-\infty}^{\infty}\omega_n(t)\mathrm{e}^{\mathrm{i}nx},\quad \omega_n(t)\in\mathbf{C}. \tag{4}$$

Then, the Hilbert transform of (4) is expressed as follows[4].

$$\mathrm{H}(\omega)=\sum_{n=-\infty}^{\infty}\mathrm{i}\ \mathrm{sgn}(n)\omega_n(t)\mathrm{e}^{\mathrm{i}nx},$$

where the function $\mathrm{sgn}(n)$ is

$$\mathrm{sgn}(n)=\begin{cases}1 & \text{if } n>0,\\ 0 & \text{if } n=0,\\ -1 & \text{if } n<0.\end{cases}$$

The paper[6] showed that the quadratic term $\omega\mathrm{H}(\omega)$ was simply represented by

$$\omega\mathrm{H}(\omega) = \sum_{n=1}^{\infty}\left\{\left(\mathrm{i}\sum_{k=0}^{n}\omega_k\omega_{n-k}\right)\mathrm{e}^{\mathrm{i}nx}-\left(\mathrm{i}\sum_{k=0}^{n}\omega_{-k}\omega_{-n+k}\right)\mathrm{e}^{-\mathrm{i}nx}\right\}. \tag{5}$$

Substituting (4) and (5) into (3) and equating coefficients of $\mathrm{e}^{\mathrm{i}nx}$, we obtain ordinary differential equations for $\{\omega_n(t)\}_{n\geq 1}$, $\{\omega_{-n}(t)\}_{n\geq 1}$ and $\omega_0(t)$:

$$\begin{cases}\dfrac{\mathrm{d}\omega_n}{\mathrm{d}t} = -\nu n^{\alpha}\omega_n+\mathrm{i}\displaystyle\sum_{k=0}^{n}\omega_k\omega_{n-k},\\ \dfrac{\mathrm{d}\omega_0}{\mathrm{d}t} = -\nu n^{\alpha}\omega_0,\\ \dfrac{\mathrm{d}\omega_{-n}}{\mathrm{d}t} = -\nu n^{\alpha}\omega_{-n}-\mathrm{i}\displaystyle\sum_{k=0}^{n}\omega_{-k}\omega_{-n+k}.\end{cases} \tag{6}$$

Now, we assume that initial condition of $\omega$ is symmetric with respect to the origin.

$$\omega(x,0)=\sum_{n=0}^{\infty}A_n\sin nx. \tag{7}$$

Then, it follows from the equations (6) that the solution is symmetric with respect to the origin for all time. That is to say,

$$\omega_n(t)+\omega_{-n}(t)=0,\qquad \text{for}\qquad t>0.$$

Because of the symmetry, the function $\mathrm{i}\omega_n(t)$ becomes real-valued and thus the equations (6) and (7) are reduced to

$$\frac{\mathrm{d}p_n^{(\alpha)}}{\mathrm{d}t} = -\nu n^{\alpha}p_n^{(\alpha)}+\sum_{k=1}^{n-1}p_k^{(\alpha)}p_{n-k}^{(\alpha)}, \tag{8}$$

$$p_n^{(\alpha)}(0) = \frac{A_n}{2},\qquad n=1,2,\cdots, \tag{9}$$

where $p_n^{(\alpha)}(t)\equiv\mathrm{i}\omega_n(t)$.

Let $p_n^{(\alpha)}(t; A_n)$ denotes the solutions of (8) with the initial conditions (9), which are given recursively by

$$p_1^{(\alpha)}(t; A_1) = \frac{A_1}{2}\mathrm{e}^{-\nu t}, \tag{10}$$

$$p_n^{(\alpha)}(t; A_n) = \frac{A_n}{2}\mathrm{e}^{-n^\alpha \nu t} + \mathrm{e}^{-n^\alpha \nu t}\int_0^t \mathrm{e}^{n^\alpha \nu s}\sum_{k=1}^{n-1} p_k^{(\alpha)} p_{n-k}^{(\alpha)} ds. \tag{11}$$

Note that the solution of the equation (3) for the initial condition (7) is represented formally by

$$\omega(x,t) = \sum_{n=1}^{\infty} 2p_n^{(\alpha)}(t; A_n)\sin nx.$$

# 3 Global existence of solution for $\alpha \geq 1$

First of all, we consider local existence of solution of (3) for $\alpha \geq 1$ in the framework of the abstract evolution equations. The linear operator $A_0 = \nu(-\Delta)^{\frac{\alpha}{2}}$ is the infinitesimal generator of a $C^0$ group on $L^2$. The initial value problem with the periodic boundary condition on $\Omega = [-\pi, \pi)$ can be rewritten by

$$\left\{ \begin{array}{rcl} \dfrac{\partial \omega}{\partial t} + A_0\omega &=& F(\omega), \qquad \text{in } L^2(\Omega),\ t > 0, \\ \omega(0) &=& \omega_0, \end{array} \right. \tag{12}$$

where the nonlinear mapping $F(\omega) = \omega H(\omega)$. In order to show existence of a local solution, we check the required properties of the nonlinear operator $F$.

**Lemma 1** *The nonlinear operator $F(u)$ maps $H^1(\Omega)$ into $H^1(\Omega)$ and satisfies the following properties; for $u, v \in H^1(\Omega)$ there exist constants $C$ and $C'$ such that*

$$||F(u)||_{H^1} \leq C||u||_{H^1} \tag{13}$$

$$||F(u) - F(v)||_{H^1} \leq C'||u - v||_{H^1}. \tag{14}$$

*Proof* Note first that the Hilbert transform is a unitary operator on $L^2$. Hence, if $u \in H^1$ then $Hu \in H^1$ and satisfies

$$||Hu||_{H^1} = ||u||_{H^1}, \qquad \text{for} \qquad u \in H^1(\Omega). \tag{15}$$

Moreover, it follows from the Sobolev theorem that $H^1(\Omega) \subset C^0(\Omega)$ and there is a constant $C$ such that

$$||u||_{L^\infty} \leq C||u||_{H^1}, \qquad \text{for} \qquad u \in H^1(\Omega). \tag{16}$$

Let $D$ denotes the first order differential operator. Then we have for each $u \in H^1$,

$$\begin{array}{rcl} \displaystyle\int_\Omega |D(F(u))|^2 dx &\leq& \displaystyle\int_\Omega |Du|^2|H(u)|^2 dx + \int_\Omega |u|^2|D(Hu)|^2 dx \\ &\leq& ||Hu||_{L^\infty}^2 \displaystyle\int_\Omega |Du|^2 dx + ||u||_{L^\infty}^2 \int_\Omega |D(Hu)|^2 dx \\ &\leq& ||Hu||_{H^1}^2 ||u||_{H^1}^2 + ||u||_{H^1}^2 ||Hu||_{H^1}^2 \\ &=& (||Hu||_{H^1}^2 + ||u||_{H^1}^2)||u||_{H^1}^2, \end{array}$$

and therefore

$$||F(u)||_{H^1} \leq C||u||_{H^1}.$$

As for the second inequality (14), it follows from

$$F(u) - F(v) = uHu - vHv = uH(u-v) + (u-v)Hv,$$

(15) and (16) that

$$\begin{aligned}
\int |D(uHu - vHv)|^2 dx &\leq \int |Du|^2 |H(u-v)|^2 dx + \int |u|^2 |D(H(u-v))|^2 dx \\
&\quad + \int |D(u-v)|^2 |Hv|^2 dx + \int |u-v|^2 |D(Hv)|^2 dx \\
&\leq ||H(u-v)||^2_{L^\infty} ||u||^2_{H^1} + ||u||^2_{L^\infty} ||H(u-v)||^2_{H^1} \\
&\quad + ||Hv||^2_{L^\infty} ||u-v||^2_{H^1} + ||u-v||^2_{L^\infty} ||Hv)||^2_{H^1} \\
&\leq C||H(u-v)||^2_{H^1} ||u||^2_{H^1} + C||u||^2_{H^1} ||H(u-v)||^2_{H^1} \\
&\quad + C||Hv||^2_{H^1} ||u-v||^2_{H^1} + C||u-v||^2_{H^1} ||Hv||^2_{H^1} \\
&\leq C(||u||^2_{H^1} + ||v||^2_{H^1}) ||u-v||^2_{H^1}.
\end{aligned}$$

and thus there exists a constant $C'$ such that

$$||F(u) - F(v)||_{H^1} \leq C' ||u-v||_{H^1}. \qquad \square$$

It follows from Lemma 1 that the initial value problem (12) has a unique local solution for $\alpha \geq 1$.

**Proposition 2** *For every $\omega_0 \in H^\alpha(\Omega) = D(A_0)$, there exist a time $T_m$ and a unique solution of the initial value problem (12) such that*

$$\omega \in C^1\left([0, T_m) : L^2(\Omega)\right) \cap C\left([0, T_m) : H^\alpha(\Omega)\right).$$

In order to show that this local solution is global, it suffices to prove that for every $T > 0$ if $\omega$ is a solution of the initial value problem on $[0, T)$ then there exists a constant $C$ such that $||\omega(t)||_{H^\alpha} < C$.

Now we prove two comparisons in terms of $\alpha$ and the initial conditions. In what follows, we assume that the initial conditions $A_n$ are non-negative for $n \geq 1$. The first comparison is obtained in terms of $\alpha$.

**Proposition 3** *If $\alpha_2 \geq \alpha_1$, then $p_n^{(\alpha_1)}(t; A_n) \geq p_n^{(\alpha_2)}(t; A_n) \geq 0$ for $n \geq 1$ and $t \geq 0$.*

*Proof.* For $n = 1$, we have $p_1^{(\alpha_1)}(t; A_1) = p_1^{(\alpha_2)}(t; A_1) > 0$. We assume that $p_k^{(\alpha_1)}(t; A_k) \geq p_k^{(\alpha_2)}(t; A_k) \geq 0$ for $k = 1, \cdots, n-1$. It is easy to show that $p_n^{(\alpha_1)}(t; A_n)$ and $p_n^{(\alpha_2)}(t; A_n)$ are non-negative for all time because of (11). Then, for a function $f_n(t) = p_n^{(\alpha_1)}(t; A_n) - p_n^{(\alpha_2)}(t; A_n)$, we have

$$\begin{aligned}
\frac{d}{dt}\left(\mathrm{e}^{\nu n^{\alpha_1} t} f_n\right) &= \mathrm{e}^{\nu n^{\alpha_1} t}\left(\nu n^{\alpha_1} f_n + \frac{df_n}{dt}\right) \\
&= \mathrm{e}^{\nu n^{\alpha_1} t}\left(\nu n^{\alpha_1} p_n^{(\alpha_1)} + \frac{d}{dt} p_n^{(\alpha_1)} - \nu n^{\alpha_1} p_n^{(\alpha_2)} - \frac{d}{dt} p_n^{(\alpha_2)}\right) \\
&= \mathrm{e}^{\nu n^{\alpha_1} t}\left\{\sum_{k=1}^{n-1}\left(p_k^{(\alpha_1)} p_{n-k}^{(\alpha_1)} - p_k^{(\alpha_2)} p_{n-k}^{(\alpha_2)}\right) + \nu(n^{\alpha_2} - n^{\alpha_1}) p_n^{(\alpha_2)}\right\} > 0,
\end{aligned}$$

since $\alpha_2 > \alpha_1$. Consequently, we obtain $f_n(t) \geq 0$ for $t \geq 0$, since $f_n(0) = 0$. $\square$

The second comparison is about the initial conditions.

**Proposition 4** *If $A_n \geq \tilde{A}_n \geq 0$, then $p_n^{(\alpha)}(t; A_n) \geq p_n^{(\alpha)}(t; \tilde{A}_n) \geq 0$ for $n \geq 1$ and $t \geq 0$.*

*Proof.* This is easily proved by the mathematical induction. For $n = 1$, we have $p_1^{(\alpha)}(t; A_1) \geq p_1^{(\alpha)}(t; \tilde{A}_1) \geq 0$. If $p_k^{(\alpha)}(t; A_k) \geq p_k^{(\alpha)}(t; \tilde{A}_k) \geq 0$ holds for $k = 1, \cdots, n-1$, then we have

$$\begin{aligned} p_n^{(\alpha)}(t; A_n) &= \frac{A_n}{2}\mathrm{e}^{-n^\alpha \nu t} + \mathrm{e}^{-n^\alpha \nu t}\int_0^t \mathrm{e}^{n^\alpha \nu s}\sum_{k=1}^{n-1} p_k^{(\alpha)}(s; A_k)p_{n-k}^{(\alpha)}(s; A_k)ds \\ &\geq \frac{\tilde{A}_n}{2}\mathrm{e}^{-n^\alpha \nu t} + \mathrm{e}^{-n^\alpha \nu t}\int_0^t \mathrm{e}^{n^\alpha \nu s}\sum_{k=1}^{n-1} p_k^{(\alpha)}(s; \tilde{A}_k)p_{n-k}^{(\alpha)}(s; \tilde{A}_{n-k})ds \\ &= p_n^{(\alpha)}(t; \tilde{A}_n) \geq 0. \qquad \square \end{aligned}$$

Now suppose that $\omega(x, 0) \in H^\alpha$ and $M = ||\omega(x, 0)||_{H^\alpha}$, Then we have an explicit representation of (10) and (11) with $A_n = M$.

**Lemma 5** *When $\alpha = 1$, the functions (10) and (11) for $A_n = M$ $(n \geq 1)$ are represented explicitly by*

$$p_n^{(1)}(t; M) = \left(\frac{M}{2}\right)\left(1 + \frac{M}{2}t\right)^{n-1}\mathrm{e}^{-n\nu t}. \tag{17}$$

*Proof.* For $n = 1$, (17) is nothing but (10). If the functions $p_k^{(1)}(t; M)$ for $k = 1, \cdots, n-1$ are expressed by (17), then it follows from (11) that $p_n^{(\alpha)}(t; M)$ becomes

$$\begin{aligned} p_n^{(1)}(t; M) &= \frac{M}{2}\mathrm{e}^{-n\nu t} + \mathrm{e}^{-n\nu t}\int_0^t \mathrm{e}^{n\nu s}\sum_{k=1}^{n-1}\left(\frac{M}{2}\right)^2\left(1 + \frac{M}{2}t\right)^{n-2}\mathrm{e}^{-n\nu s}ds \\ &= \frac{M}{2}\mathrm{e}^{-n\nu t} + \left(\frac{M}{2}\right)^2\mathrm{e}^{-n\nu t}\int_0^t (n-1)\left(1 + \frac{M}{2}t\right)^{n-2}ds \\ &= \frac{M}{2}\mathrm{e}^{-n\nu t} + \left(\frac{M}{2}\right)\mathrm{e}^{-n\nu t}\left[\left(1 + \frac{M}{2}s\right)^{n-1}\right]_0^t \\ &= \left(\frac{M}{2}\right)\left(1 + \frac{M}{2}t\right)^{n-1}\mathrm{e}^{-n\nu t}. \qquad \square \end{aligned}$$

The function

$$r(t) = \left(1 + \frac{M}{2}t\right)\mathrm{e}^{\nu t}$$

has the maximum at $t^* = \frac{1}{\nu} - \frac{2}{M}$. If $t^* < 0$, i.e. $\nu > \frac{M}{2}$, then $r(t)$ is monotonically decreasing for $t \geq 0$. Since $r(0) = 1$, we have $r(t) < 1$ for $t > 0$. Therefore, since $0 \leq A_n \leq M$ for $n \geq 1$, it follows from the Parseval's equality, Proposition 4 and Proposition 5 that

$$\begin{aligned} ||\omega(x, t)||_{H^\alpha}^2 &= \pi\sum_{n=1}^\infty \left(1 + n^{2\alpha}\right)\left\{2p_n^{(\alpha)}(t; A_n)\right\}^2 \\ &\leq \pi\sum_{n=1}^\infty \left(1 + n^{2\alpha}\right)\left\{2p_n^{(1)}(t; A_n)\right\}^2 \\ &\leq \pi\sum_{n=1}^\infty \left(1 + n^{2\alpha}\right)\left\{2p_n^{(1)}(t; M)\right\}^2 \\ &\leq \pi M^2\mathrm{e}^{-2\nu t}\left\{\sum_{n=1}^\infty \left(1 + n^{2\alpha}\right)r^{2(n-1)}(t)\right\} < \infty, \end{aligned}$$

for $\nu > \frac{M}{2}$, $\alpha \geq 1$ and $t > 0$. Consequently, we have the following lemma.

**Lemma 6** *When $\alpha \geq 1$, if $\omega(x,0) \in H^\alpha(\Omega)$, then there exists a constant $C(\nu)$ such that $||\omega(x,t)||_{H^\alpha} < C$ for $\nu > \frac{M}{2}$.*

Then, Lemma 1 and Lemma 6 yield the existence of a unique global solution.

**Theorem 7** *Suppose that $\omega_0 \in H^\alpha(\Omega)$ and all of its initial Fourier coefficients are non-negative. Then the initial value problem (12) has a unique global solution,*

$$u \in C\left([0,\infty) : H^\alpha(\Omega)\right) \cap C^1\left([0,\infty) : L^2(\Omega)\right),$$

*for $\alpha \geq 1$ and $\nu > \frac{1}{2}||\omega_0||_{H^\alpha}$.*

Wegert and Murthy[8] showed that global solutions of the modified CLM equation, which corresponds to out equation with $\alpha = 1$, existed for given non-constant Hölder continuous $2\pi$-periodic initial functions when the viscosity coefficient is sufficiently large. Their result is part of Theorem 7. On the other hand, Schochet[7] and Wegert and Murthy[8] gave solutions that blew up in finite time for small viscosity coefficient for $\alpha = 2$ and $\alpha = 1$, respectively. These solutions are also included in the blow-up results for the general case, $\alpha \in \mathbf{R}$, in [6].

# 4 No global solution for $\alpha \leq 0$

When $\alpha < 0$, for large viscosity coefficient $\nu$, there exists an integer $N_0$ such that the dissipative viscosity term in (8), $-\nu n^\alpha p_n^{(\alpha)}$, becomes negligible for all $n \geq N_0$. Therefore, we expect that the solutions of (8) behave like those of the CLM equation with no viscosity term, $\nu = 0$ asymptotically as $n \to \infty$. This suggests the solution of (3) blows up in finite time no matter how large the viscosity coefficient is. This is not trivial fact, since we are unable to construct a lower comparison function of $p_n^{(\alpha)}$ uniformly in $n$ like in the previous paper[6] because such integer $N_0$ depends on $\nu$. The purpose of the section is to verify this expectation.

First of all, we introduce the following CLM equation with a damping term.

$$\frac{\partial \omega}{\partial t} = \omega \mathrm{H}(\omega) - \beta\omega, \quad x \in \mathbf{R},\ t > 0, \tag{18}$$

where real $\beta > 0$ is the damping coefficient. In the same way as in §2, we consider the periodic boundary condition. Suppose that the solution is symmetric with respect to the origin and is represented by

$$\omega(x,t) = \sum_{n=1}^{\infty} 2\tilde{p}_n(t) \sin nx.$$

then the damping equation is reduced to

$$\frac{\mathrm{d}\tilde{p}_n}{\mathrm{d}t} = -\beta\tilde{p}_n + \sum_{k=1}^{n-1} \tilde{p}_k \tilde{p}_{n-k}, \qquad n = 1, 2, \cdots, \tag{19}$$

We give solutions of the equations (19) for an initial data $\tilde{p}_n(0) = \epsilon\delta_{nl}$, in which $\epsilon$ is a positive real, $l$ ($\geq 1$) is a fixed integer and $\delta$ is the Kronecker's delta.

**Lemma 8** *The solutions of (19) with $\tilde{p}_n(0) = \epsilon\delta_{nl}$ are represented by*

$$\begin{cases} \tilde{p}_n(t;\epsilon\delta_{nl}) &= 0, & \text{for } n \bmod l \neq 0, \\ \tilde{p}_n(t;\epsilon\delta_{nl}) &= \epsilon\left(\dfrac{\epsilon}{\beta}\right)^{m-1} e^{-\beta t}\left(1 - e^{-\beta t}\right)^{m-1}, & \text{for } n = ml, m \in \mathbf{N}. \end{cases} \tag{20}$$

*Proof.* This is proved by the mathematical induction. Firstly, for $1 \le n \le l-1$, since $\tilde{p}_n(0) = 0$, we have $\tilde{p}_n(t) = 0$. When $n = l$, it follows from $\tilde{p}_l(0) = \epsilon$ and $\sum_{k=1}^{l-1} \tilde{p}_k \tilde{p}_{l-k} \equiv 0$ that $\tilde{p}_l(t) = \epsilon e^{-\beta t}$.

Secondly, suppose that (20) holds for $1 \le n \le ml$, then for $n = ml + q$, $(1 \le q \le l-1)$,

$$\sum_{k=1}^{n-1} \tilde{p}_k \tilde{p}_{n-k} = \sum_{k=1}^{m} \tilde{p}_{kl} \tilde{p}_{ml+q-kl} = 0,$$

since $\tilde{p}_k = 0$ for $k \bmod l \neq 0$ and $\tilde{p}_{(m-k)l+1} = 0$. On the other hand, when $n = (m+1)l$ we have

$$\begin{aligned}
\sum_{k=1}^{n-1} \tilde{p}_k \tilde{p}_{n-k} &= \sum_{k=1}^{m} \tilde{p}_{kl} \tilde{p}_{n-kl} = \sum_{k=1}^{m} \tilde{p}_{kl} \tilde{p}_{(m+1-k)l} \\
&= \sum_{k=1}^{m} \epsilon^2 \left(\frac{\epsilon}{\beta}\right)^{m-1} \mathrm{e}^{-2\beta t} \left(1 - \mathrm{e}^{-\beta t}\right)^{m-1} \\
&= m\epsilon^2 \left(\frac{\epsilon}{\beta}\right)^{m-1} \mathrm{e}^{-2\beta t} \left(1 - \mathrm{e}^{-\beta t}\right)^{m-1}.
\end{aligned}$$

Consequently, since $\tilde{p}_n(0) = 0$, we have

$$\begin{aligned}
\tilde{p}_n(t) &= \mathrm{e}^{-\beta t} \int_0^t \mathrm{e}^{\beta s} \sum_{k-1}^{n-1} \tilde{p}_k \tilde{p}_{n-k} ds \\
&= \epsilon^2 \left(\frac{\epsilon}{\beta}\right)^{m-1} \mathrm{e}^{-\beta t} \int_0^t m\mathrm{e}^{-\beta s} \left(1 - \mathrm{e}^{-\beta s}\right)^{m-1} ds \\
&= \epsilon^2 \left(\frac{\epsilon}{\beta}\right)^{m-1} \mathrm{e}^{-\beta t} \left[\frac{\left(1 - \mathrm{e}^{-\beta s}\right)^m}{\beta}\right]_0^t \\
&= \epsilon \left(\frac{\epsilon}{\beta}\right)^{m} \mathrm{e}^{-\beta t} \left(1 - \mathrm{e}^{-\beta t}\right)^{m}. \qquad \square
\end{aligned}$$

Using this lemma, we prove the following.

**Lemma 9** *Let $\alpha < 0$. For arbitrary $\nu$ and $\epsilon > 0$, there exists a positive integer $N_0$ such that the solution of (3) for the initial condition $\omega(x,0) = 2\epsilon \sin N_0 x$ blows up in finite time in $L^2$ norm.*

*Proof.* Since $\alpha < 0$, for given $\nu$ and $\epsilon$, there exists a positive integer $N_0$ and a real $\beta$ such that for all $n \ge N_0$,

$$\nu n^\alpha < \beta < \epsilon. \tag{21}$$

It follows from (21) and the comparison of the dissipative terms in (8) and (19) that $p_n^{(\alpha)}(t; \epsilon\delta_{N_0 n}) \ge \tilde{p}_n(t; \epsilon\delta_{N_0 n})$. (This is easily proved in the same way as Proposition 3.) Consequently, we obtain

$$\begin{aligned}
||\omega(x,t)||_{L^2}^2 &= \pi \sum_{n=1}^{\infty} |p_n^{(\alpha)}(t; \epsilon\delta_{N_0 n})|^2 \\
&\ge \pi \sum_{n=1}^{\infty} |\tilde{p}_n(t)|^2 \\
&= \pi \sum_{m=1}^{\infty} \left\{ \epsilon \left(\frac{\epsilon}{\beta}\right)^{m-1} \mathrm{e}^{-\beta t} \left(1 - \mathrm{e}^{-\beta t}\right)^{m-1} \right\}^2.
\end{aligned}$$

The function

$$g(t) = \frac{\epsilon}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right)$$

exceeds one at $T_0 = \frac{1}{\beta}\log\frac{\epsilon}{\epsilon-\beta}$, since $\beta < \epsilon$. Therefore, $||\omega(x,t)||_{L^2}^2 \nearrow \infty$ as $t \nearrow T_0$. □

The comparison in terms of the initial conditions (Proposition 4) and this lemma show that for given non-negative initial conditions $A_n \geq 0$, the viscosity coefficient $\nu$ and an arbitrary $\epsilon$, there exists a positive integer $N_0$ such that the solution of (3) for the slightly perturbed initial condition $A_n + \epsilon\delta_{N_0 n}$ blows up in finite time. Since $\epsilon$ is arbitrary, we obtain the following theorem.

**Theorem 10** *When $\alpha < 0$, the solution of (3) for initial condition with non-negative Fourier coefficients $A_n \geq 0$ always blows up in finite time in $L^2$ norm.*

We remark that the condition $A_n \geq 0$ in this theorem is technically required because of analytic proof based on the comparison of the spectra, which was similarly used by Palais[5] to show blow-up of solutions for a certain class of equations. This condition seems to be inessential, because, as we have shown numerically in the paper[6], some solutions with negative initial Fourier coefficients can blow up in finite time.

# 5 Conclusion

We consider the global existence of solutions for the Constantin-Lax-Majda equation with the generalized viscosity term, which is proposed as a one-dimensional model for the 3-D Navier-Stokes equations. When the order of diffusion term $\alpha > 1$, the solution, whose initial Fourier coefficients are non-negative, exists globally in time for sufficiently large viscosity coefficient. On the other hand, when $\alpha < 0$ there exists no global solution no matter how large the viscosity coefficient is. It means that the quadratic term $H(\omega)\omega$ has such strong nonlinearity that weak diffusion is impossible to control its growth.

In 3-D Navier-Stokes equations, the solution exists globally in time regardless of the viscosity coefficient when $\alpha > \frac{5}{4}$. Hence, the present global result and the blow-up result in [6], which showed that the solution of the CLM equation blew up in finite time in $L^2$ for all order of diffusion term if the viscosity coefficient was sufficiently small, indicate that the CLM equation with the generalized viscosity term fails to catch this analytic property of the 3-D Navier-Stokes equations.


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