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Equivalence Problems of the Geometric Structures admitting Differential Filtrations

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Equivalence Problems of the Geometric Structures admitting Differential Filtrations

By Tohru MORIMOTO

0. Introduction. Succeeding to Lie and Cartan, many authors have treated general equivalence problems of geometric structures. It is mainly due to Singer-Sternberg [S-S] that some important ideas of Cartan are settled in the framework of G-structures, in particular, the finite dimensionality of the automorphism group of any G-structure of finite type and the analytic theory for involutive G-structures. In his series of papers ([T1], [T2], [T3]), Tanaka has investigated geometric structures admitting differential systems and developed prolongation method based on differential systems, to obtain the finite dimensionality of the automorphism group of any G-structure of finite type in his generalized sense and the existence of Cartan connections for many goemetric structures, especially for those associated with simple graded Lie algebras.

In our previous paper [M1], we have developed a general method to treat the equivalence problem of the G-structures including intransitive cases, in which played an important rôle the introduction of the higher order non-commutative frame bundles (in other words semi-holonimic frame bundles) and their structure functions. According to this method one can solve in principle equivalence problem of any G-structure in a neighbourhood of any generic point in the analytic category.

However it was not so clear how our method was related with the other prolongation methods, especially with those of Tanaka.

Recently, in order to apply our method to geometric structures admitting differential systems (or a little more generally differential filtration), we have refined it by introducing the higher order non-commutative frame bundles associated with differential filtrations, and we have obtained a prolongation scheme which generalizes in the unified manner all the prolongation methods mentioned above. In particular, first of all we have a generalization of the analytic theory of Singer-Sternberg, based on the vanishing of the generalized Spencer cohomology group in higher degree ([M2]). Secondly we obtain a general criterion for the existence of Cartan connections associated with geometric structures, which not only covers all existence theorems of Cartan connections that we know, but also may be applied to other geometric structures.

In this note we shall outline very briefly the above prolongation scheme, of which details will be found in our forthcoming paper, and some algebraic aspect may be referred to our preprint [M2].

1. <u>Differential filtration</u>. Let M be a differentiable manifold. A vector subbundle D of the tangent bundle TM is usually called a differential system on M. Somewhat more

generally we define a <u>differential</u> (resp. <u>tangential</u>) <u>filtration on</u> M <u>of depth</u> μ to be a sequence $D = \{D^p\}_{p\in Z}$ of subbundles of TM satisfying the following conditions i), ii), iii) (resp. i), ii)):

i) $D^p \supset D^{p+1}$ for $p \in Z$,

ii) $D^{-\mu} = TM$, $D^0 = 0$,

iii) $[\underline{D}^p, \underline{D}^q] \subset \underline{D}^{p+q}$ for $p, q \in Z$,

where $\underline{D}^p$ denotes the sheaf of the germs of sections of D^p.

If $\mu = 1$, D is called trivial. We say that D is generated by D^{-1} if $\underline{D}^p = [\underline{D}^{p+1}, \underline{D}^{-1}] + \underline{D}^{p+1}$ for $p \leq -2$.

Let $V = (V, \{F^pV\}_{p\in Z})$ be a filtered vector space. A tangential filtration D is called of type V if rank D^p = dim F^pV for all $p \in Z$. If D is a differential filtration on M, we can associate with each $x \in M$ a graded Lie algebra $gr\, D(x) = \oplus\, gr_p D(x)$, by putting $gr_p D(x) = D^p(x)/D^{p+1}(x)$. If $gr\, D(x)$ are all isomorphic to a graded Lie algebra m, D is called <u>regular of type</u> m.

Our main objects of this note are the geometric structures admitting differential filtrations as underlying structures. Many examples of such structures may be found not only in geometries but also through geometrization of differential equations.

2. <u>Extensions of a filtered vector space</u>. Before going further into geometric investigation we need some algebraic preparations. In what follows we shall use freely the following notation and convention: Filtrations are always assumed to be

descending. If $\mathbf{V} = (V, \{F^pV\})$, $\mathbf{W} = (W, \{F^pW\})$ are filtered vector spaces, naturally induced filtrations on various associated spaces are also denoted by F. For instance, we set

$$F^p(V\oplus W) = F^pV \oplus F^pW, \quad F^p(V\otimes W) = \bigoplus_{r+s=p} F^rV \otimes F^sW,$$

$$F^p\mathrm{Hom}(V,W) = \{\alpha \in \mathrm{Hom}(V,W) \,|\, \alpha(F^iV) \subset F^{i+p}W, \ \forall i\}$$

$$F^pGL(V) = \{\alpha \in GL(V) \,|\, \alpha - 1_V \in F^p\mathrm{Hom}(V,V)\}$$

We also write:

$$GL(\mathbf{V}) = F^0GL(V), \quad g\ell(\mathbf{V}) = F^0\,\mathrm{Hom}(V,V)$$

$$GL(\mathbf{V})^{(k)} = GL(\mathbf{V})/F^{k+1}GL(V)$$

$$g\ell(\mathbf{V})^{(k)} = g\ell(\mathbf{V})/F^{k+1}g\ell(V)$$

If $\nu = \oplus\nu_p$, $\omega = \oplus\omega_p$ are graded vector spaces, we set

$$\mathrm{Hom}(\nu,\omega)_p = \{\alpha \in \mathrm{Hom}(\nu,\omega) \,|\, \alpha(\nu_i) \subset \omega_{i+p}, \ \forall i\}$$

Now let $\mathbf{V} = (V, \{F^pV\})$ be a finite dimensional filtered vector space over $\mathbf{R}$ such that $F^0V = 0$ and $F^{-\mu}V = V$ for some $\mu > 0$. Let G be a Lie group with the Lie algebra denoted by g. Suppose that we are given a representation of G on

$$E = V \oplus g.$$

We call (E, G) an <u>extension</u> of $\mathbf{V}$ if the following holds:

i) $a\cdot A = \mathrm{Ad}(a)A$ for $a \in G$, $A \in g$,

ii) $a(F^pV) \subset F^pV \oplus g$ for $a \in G$, $p \in \mathbf{Z}$,

iii) $G \cong \mathrm{proj\ lim}\ G/F^kG$,

where $\{F^kG\}$ is defined by the following conditions:

$$\begin{cases} F^pG = G & (p \leq 0) \\ 1 \to F^{k+1}G \to G \to GL(E^{(k-1)})^{(k)} & \text{(exact)} \\ E^{(k-1)} = V \oplus g/F^k g, \quad F^k g = \text{Lie alg}(F^kG) \end{cases}$$

Here we emphasize that we do not assume G finite dimensional but belonging to the category of groups obtained by projective limits of finite dimensional Lie groups, so that the above definition has a sense.

Example 1. $E = gr\ V \oplus g\ell(V)^{(0)}$, $G = GL(V)^{(0)}$, where $GL(V)^{(0)}$ is identified with $GL(gr\ V)$, the group of automorphisms of the graded vector space $gr\ V$.

Example 2. Let $m = \bigoplus_{p<0} m_p$ be a graded Lie algebra and $\mathcal{M}$ be a Lie group with the Lie algebra m. Let $D^p(\mathcal{M})$ be the differential system defined by the left invariant vector fields belonging to $\bigoplus_{q \geq p} m_q$, and put $D(\mathcal{M}) = \{D^p(\mathcal{M})\}$. We define $\overline{E}(m)$ to be the Lie algebra of formal vector fields leaving $D(\mathcal{M})$ invariant and $\overline{G}(m)$ to be the group of formal isomorphism of $D(\mathcal{M})$ fixing e. Then $(\overline{E}(m), \overline{G}(m))$ is an extension. If m is trivial (i.e. $m = m_{-1} = V$) the structure of $\overline{E}(v)$ is easy to see, but in general the structure of $\overline{E}(m)$ is not a priori clear. (See Theorem 1).

Example 3. Let $(L, \{L^p\})$ be a transitive filtered Lie algebra (TFLA) of depth $\mu \geq 1$ (See [M2]), that is, a filtered Lie algebra such that i) $L = L^{-\mu}$, ii) $\dim L^p/L^{p+1} < \infty$, iii) $\bigcap L^p = 0$, iv) for any $p \geq 0$,

$$L^{p+1} = \{x \in L^p \mid [x, L^i] \subset L^{p+i+1},\ \forall i < 0\}.$$

If G is a Lie group having L^0 as its Lie algebra. Then (L, G) is an extension.

Let (E, G) be an extension of V. We define $[\ ,\ ] \in \mathrm{Hom}(\Lambda^2 E, E)$ by $[A, B]$ = the bracket of $\mathfrak{g}$ for $A, B \in \mathfrak{g}$, $[A, v] = A\cdot v$, $A \in \mathfrak{g}$, $v \in V$ and $[v, w] = 0$ for $v, w \in V$. Note that the bracket does not in general satisfy the Jacobi identity. If there exists $c \in F^0\mathrm{Hom}(\Lambda^2 V, E) (\subset F^0\mathrm{Hom}(\Lambda^2 E, E))$ such that $\gamma = c + [\ ]$ satisfies the Jacobi identity and that G is an automorphism group of the Lie algebra (E, γ), we say that (E, G) is holonomic. Note that in this case (E, γ) is a TFLA.

The above examples are all holonomic extensions. But we need to consider non-holonomic ones in order to realize TALA's in a universal space.

3. Universal extensions. Let V be a filtered vector space as above. Suppose $\alpha_0 \in \mathrm{Hom}(\Lambda^2 \mathrm{gr}\, V, \mathrm{gr}\, V)_0$ defines a Lie algebra structure on $\mathrm{gr}\, V$. We denote by $\mathfrak{m}$ this graded Lie algebra. Now let us introduce the following universal extensions of V:

$$(E(V), G(V)) \supset (E(\mathfrak{m}), G(\mathfrak{m})) \supset (\overline{E}(\mathfrak{m}), \overline{G}(\mathfrak{m})).$$

Proposition 3.1. There exists, uniquely up to isomorphisms, a universal extension $(E(V), G(V))$ of V, namely if (E, G) is an extension of V then (E, G) is canonically embedded into $(E(V), G(V))$.

The construction of $(E(V), G(V))$ is already suggested in our definion of the extension. We put $G^{(k)}(V) = G(V)/F^{k+1}G(V)$. We see immediately that $G^{(0)}(V) = GL(V)^{(0)}$.

Given a Lie subgroup $G_0 \subset G^{(0)}(V)$, similarly as Proposition 3.1, we see that there exists a universal extension $(E(V,\mathfrak{g}_0), G(V,G_0))$ such that $G^{(0)}(V, G_0) = G_0$. In particular, denoting by $G_0(m) \subset G^{(0)}(V)$ the group of the automorphisms of m, we set

$$(E(m), G(m)) = (E(V,\mathfrak{g}_0(m)), G(V,G_0(m))).$$

Identifying V with m, we define the bracket $[\ ,\ ]'$ of $E(m) = m \oplus \mathfrak{g}(m)$ by $[\ ,\ ]' = [\ ,\ ] + \alpha_0$.

<u>Proposition</u> 3.2. For a Lie subgroup $G_0 \subset G_0(m)$, there exists a universal extension $(\overline{E}(m,\mathfrak{g}_0), \overline{G}(m,G_0))$ such that $\overline{G}^{(0)}(m,G_0) = G_0$ and that it is holonomic with respect to the bracket $[\ ,\ ]'$ of $E(m)$.

We set: $(\overline{E}(m), \overline{G}(m)) = (\overline{E}(m,\mathfrak{g}_0(m)), \overline{G}(m,G_0(m)))$.

<u>Proposition</u> 3.3. Let L be a TFLA such that $gr\_L = \bigoplus_{p<0} gr_p L$ is isomorphic to m, then L can be realized as a holonomic extension embedded in $E(m)$ and gr L can be embedded in $\overline{E}(m)$.

<u>Remark</u>. The extension of Example 2 is, in fact, isomorphic to $(\overline{E}(m), \overline{G}(m))$ defined above.

<u>Definition</u> 3.1. $G_0 \subset G_0(m)$ is said to be finite type of order k if $F^k\overline{G}(m, G_0) = 1$.

4. <u>Extensions and the non-commutative frame bundle of</u> (M, D)

Let D be a tangential filtration on M of type V and let (E, G) be an extension of V. A principal fibre bundle P with the base space M, the structure group G, and the projection $\pi : P \to M$, equipped with an E-valued 1-form θ on P is called an <u>extension</u> of (M, D) of type (E, V, G) (or (E,G)) if the following holds:

i) θ is an absolute parallelism of P,

ii) $\theta(\tilde{A}) = A$, for $A \in \mathfrak{g}$,

iii) $R_a^*\theta = a^{-1}\theta$, $a \in G$

iv) For each $z \in P$, $\theta_z^{(-1)}$ is an isomorphism

$(T_xM, \{D^p(x)\}) \to V$ of filtered vector spaces, where $x = \pi(z)$ and $\theta_z^{(-1)}$ is defined by the following commutative diagram:

$$\begin{array}{ccc} T_zP & \xrightarrow{\theta_z} & E \\ \downarrow & \pi_* & \downarrow \\ T_xM & \xrightarrow{\theta_z^{(-1)}} & V \end{array}$$

Given an extension $(P,M,G;\ \theta)$, we can write

$$(4.1) \qquad d\theta + \frac{1}{2}\gamma(\theta, \theta) = 0,$$

with a $\mathrm{Hom}(\Lambda^2E,E)$-valued function γ on P, which satisfies, by iii),

$$R_a^*\gamma = \rho(a)^{-1}\gamma, \quad a \in G,$$

where ρ denotes the natural representation of G on $\mathrm{Hom}(\Lambda^2E, E)$. From ii) and iii) it follows that for any $z \in P$,

$$\begin{cases} \gamma(z)(A,\ v) = A\cdot v = [A,\ v], & A \in \mathfrak{g},\ v \in V \\ \gamma(z)(A,\ B) = [A,\ B], & A,\ B \in \mathfrak{g}. \end{cases}$$

Thus we can rewrite (4.1) as

$$d\theta + \frac{1}{2}[\theta,\ \theta] + \frac{1}{2}c(\theta,\ \theta) = 0,$$

with a $\mathrm{Hom}(\Lambda^2 V, E)$-valued function c on P. We call c (or γ) the <u>structure function</u> of $(P,\ \theta)$.

<u>Proposition</u> 4.1. For a tangential filtration D on M of type V, there exists uniquely up to isomorphism a universal extention $(\mathcal{R}(M,D;\ V);\ \theta)$ of $(M,\ D)$ of type $(E(V),\ G(V))$. If $(P,\ \theta)$ is an extension of $(M,\ D)$ then there exists a unique embedding $f : P \to \mathcal{R}(M,D;\ V)$ such that $f^*\theta_{\mathcal{R}} = \theta_P$.

We set: $\mathcal{R}^{(k)}(M,D;\ V) = \mathcal{R}(M,D;\ V)/F^{k+1}G(V)$, which is a principal fibre bundle over M with the structure group $G^{(k)}(V)$, and is called the <u>non-commutative frame bundle</u> of order $k+1$ of $(M,\ D)$.

Let $\hat{\mathcal{R}}^{(0)}$ be the set of all linear frames $z : T_xM \to V$ such that $z(D^p(x)) = V^p$, then $\hat{\mathcal{R}}^0$ is a principal fibre bundle over M with the structure group $GL(V)$. We have:

$$\mathcal{R}^{(0)}(M,D;\ V) = \hat{\mathcal{R}}^{(0)}/F^1GL(V).$$

In particular if D is trivial, $\mathcal{R}^{(0)}(M,D;\ V)$ is nothing but the usual linear frame bundle of M.

<u>Definition</u> 4.1. A subbundle $(B_0,\ M,\ G_0)$ of $\mathcal{R}^{(0)}(M,D;\ V)$ is called a 1-st order <u>geometric structure</u> admitting D.

Proposition 4.2. Let (P, θ) be an extension of (M, D) of type (E, G) with the structure function c. Then D is a differential filtration if and only if $c(z) \in F^0\mathrm{Hom}(\Lambda^2 V, E(V))$ for all $z \in P$. If it is the case $c(z)$ induces $c_0(z) \in \mathrm{Hom}(\Lambda^2 \mathrm{gr}\, V, \mathrm{gr}\, V)_0$, by which $\mathrm{gr}\, V$ is made into a graded Lie algebra isomorphic to $\mathrm{gr}\, D(x)$, where $x = \pi(z)$.

Proposition 4.3. Let D be a differential filtration regular of type $m = (\mathrm{gr}\, V, \alpha_0)$, then there exists a universal extension $(\mathcal{R}(M,D;\, m), M, G(m);\, \theta)$ such that

$$c_0(z) = \alpha_0 \qquad \text{for all } z \in \mathcal{R}(M,D;\, m).$$

Moreover to every subbundle B_0 of $\mathcal{R}^{(0)}(M,D;\, V)$ corresponds in a canonical manner a subbundle B_0^* of $\mathcal{R}^{(0)}(M,D;\, m)$.

Thus, as long as we are interested in regular differential filtration, it suffices to work in $\mathcal{R}(M,D;\, m)$ called the reduced frame bundle. Note that the structure equation of an extension $(P;\, \theta) \subset \mathcal{R}(M,D;\, m)$ may be written as:

$$d\theta + \frac{1}{2}c'(\theta, \theta) + \frac{1}{2}[\theta, \theta]' = 0,$$

with an $F^1\mathrm{Hom}(\Lambda^2 m, E)$-valued function c' on P called the reduced structure function of P.

5. Equivalence problem. To investigate the equivalence problem of 1-st order geometric structures admitting regular differential filtration of type m, we wish to associate to each subbundle (B, M, G_0) of $\mathcal{R}^{(0)}(M,D;\, m)$, in a natural manner, an

extension $(P,M,G;\ \theta)$ of $(M,\ D)$ having as nice structure as possible.

In the case when G_0 is not necessarily of finite tyep, we have the following:

<u>Theorem</u> 1. Assume $(B,\ M,\ G_0)$ transitive, then we can associate with it, up to conjugates, an extension $(P,M,G;\ \theta)$ of $(M,\ D)$ contained in $\mathcal{R}(M,D;\ m)$ with constant reduced structure function c'.

Since c is constant, $L = (E = m\oplus g,\ \gamma = c'+[\ ,\ ]')$ is a TFLA of depth μ, and P may be viewed as a local Lie group with the L-valued Maurer-Cartan form θ. If $\dim L < \infty$, such $(P,\ \theta)$ is quite familiar. In the case $\dim L = \infty$, according to our algebraic study [M2], L is completely determined by its truncated Lie algebra $\mathrm{Trun}^k L$ for k large enough by virtue of the vanishing of the generalized Spencer cohomology group. Geometrically this means that $(P,M,G;\ \theta)$ is completely determined by the involutive subbundle $(P^{(k)},\ M,\ G^{(k)})$ of $\mathcal{R}^{(k)}(M,D;\ m)$. In the analytic category this solves our local equivalence problem of $(B,\ M,\ G_0)$. Note also the the Lie algebra of formal infinitesimal automorphisms of B is isomorphic to L.

On the other hand, in the C^∞-category the equivalence problem is difficult when $\dim L = \infty$, but we have the following conjecture which is valid for many examples:

<u>Conjecture</u>. Let L be a TFLA of depth μ, and $(P,\ \theta)$, $(P',\ \theta')$ be local Lie groups of type L. If L is graded, then $(P,\ \theta)$ and $(P',\ \theta')$ are locally isomorphic.

If $\mu = 1$, this is know to be true. (See [G], [MP])

Remark. So far we have assumed the transitivity of B for the sake of simplicity. To treat the intransitive case we need the similar works as done in [M1].

In the case G_0 is of finite type, we have:

Theorem 2 (Singer-Sternberg and Tanaka). If $G_0 \subset G^{(0)}(m)$ is of finite type of order k. Then to each subbundle $(B, M, G_0) \subset \mathcal{R}^{(0)}(M,D;\ m)$, there corresponds an absolute parallelism $(B^{(k+\mu-1)}, \theta)$. In particular the automorphism group of B is finite dimensional.

In should be remarked that $B^{(k+\mu-1)}$ is constructed by the series: $M \leftarrow B \leftarrow B^{(1)} \leftarrow \cdots \leftarrow B^{(k+\mu-1)}$, each $B^{(i+1)} \rightarrow B^{(i)}$ is a principal bundle, but not so for $B^{(k+\mu-1)} \rightarrow M$. But under the condition (CI) which is going to be described, we can construct $B^{(k+\mu-1)}$ so as to be a principal bundle over M with the structure group $\overline{G}(m, G_0)$, so that $(B^{(k+\mu-1)}, \theta)$ becomes a Cartan connection, namely an extension $(P,M,G;\ \theta)$ of type (E, G) such that (E, G) is holonomic.

Since $\overline{E}(m, \mathfrak{g}_0)$ is a TFLA (as a matter of fact, graded) containing m, we can define a generalized Spencer complex (cf. [M2]):

$$\mathrm{Hom}(\Lambda^p m,\ \overline{E}(m,\mathfrak{g}_0)) \xrightarrow{\ \partial\ } \mathrm{Hom}(\Lambda^{p+1} m,\ \overline{E}(m,\mathfrak{g}_0)).$$

Recall also that $\overline{G}(m, G_0)$ acts naturally on $\mathrm{Hom}(\Lambda m, \overline{E})$.

We say that G_0 satisfies the condition (CI) if there exists a $\overline{G}(m,G_0)$-invariant subspace W such that

$$F^1\mathrm{Hom}(\Lambda^2 m,\ \overline{E}(m,\mathfrak{g}_0)) = \partial F^1\mathrm{Hom}(m,\ \overline{E}(m,\mathfrak{g}_0)) \oplus W.$$

Then we have:

Theorem 3. If G_0 satisfies (CI), then with each subbundle $(B, M, G_0) \subset \mathcal{R}^{(0)}(M,D; m)$, we can associate canonically an extension $(P,M,\overline{G}(m,G_0); \theta)$ of type $(\overline{E}(m,\mathfrak{g}_0), \overline{G}(m,G_0))$ such that the reduced structure function c' of P takes values in W.

Though we have limited ourselves to 1-st order geometric structures, it is not difficult to extend the above theorems to higher order geometric structures.

As far as we know, Theorem 3 not only covers all the existence theorems of Cartan connections known until now, riemannian, conformal, projective connections, and the Cartan connections associated with simple graded Lie algebras [T3], but also give a new example of Cartan connection.

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