PATCH LOADING – STABILITY ANALYSIS WITH EXACT IN-PLANE STRESS FUNCTIONS

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ABSTRACT

Past studies on the stability of rectangular plates under the influence of variable loads were based on assumptions of simplified stress distributions, which put the question of the accuracy of the results thus obtained.

The procedure of applying the exact stress functions on the problem of elastic stability of the plate with different boundary conditions under effects of patch loading is presented in this paper.

Mathieu (1890) obtained the exact solution for the plane-strain state for a rectangular element for certain types of variable stresses on the boundaries. Baker and Pavlovic (1993), following Mathieu's results, analyzed the general problem of a rectangular plate loaded by completely arbitrary distributions of (normal and/or shear) stresses along the edges of the plate. Their method was based on splitting the solution into eight fundamental problems. Superposition of these basic cases, enables the definition of internal stress distributions for any type of external load.

The problem of the elastic stability of rectangular plates with different boundary conditions under patch loading is investigated using the Ritz energy technique. The strain energy due to bending of the plate is defined in the traditional way. On the other hand, the exact stress distribution of Mathieu's theory of elasticity is introduced through the potential energy of the plate associated with the work done by external loads. By adopting the exact stresses within a plate under any type of external loads and using the double Fourier series to represent any possible buckled profile, the buckling loads can be obtained in a very accurate way.

Results for the critical load obtained by presented analytical approach are reaffirmed by numerical finite-element (FE) runs.

Keywords: elastic stability of plates, exact stress function, mixed boundary conditions, patch loading

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1. INTRODUCTION

In steel structures, buckling problem of the high steel girders under variable external loads is still very interesting topic. Presently available literature abounds with data regarding this problem, but mostly obtained by numerical or experimental methods. Analytical approach has been avoided mostly because of unknown stress distribution.

In the series of papers based on Mathieu's method from 1890, Pavlovic, Baker and Tahan (1993) and later Liu (2006) and Mijuskovic (2008) developed very precise approach for exact stress function determination for main case of rectangular plate under arbitrary external load. Existence of such solutions created the basis for the analysis of very complex stability problems in real steel structures.

Analytical approach to critical load determination based on exact stress functions implementation, is verified for relatively simple case of plate under (DEA) compression (Liu 2006, Mijuskovic 2008, Mijuskovic et al. 2012). In this paper the next step is introduced through a significantly complicated problem of the plate under locally distributed stress (patch loading) applied on the upper flange of the steel girder. That way, the applicability and accuracy of introduced analytical approach can be proven on a more demanding and near to real life engineering problems.

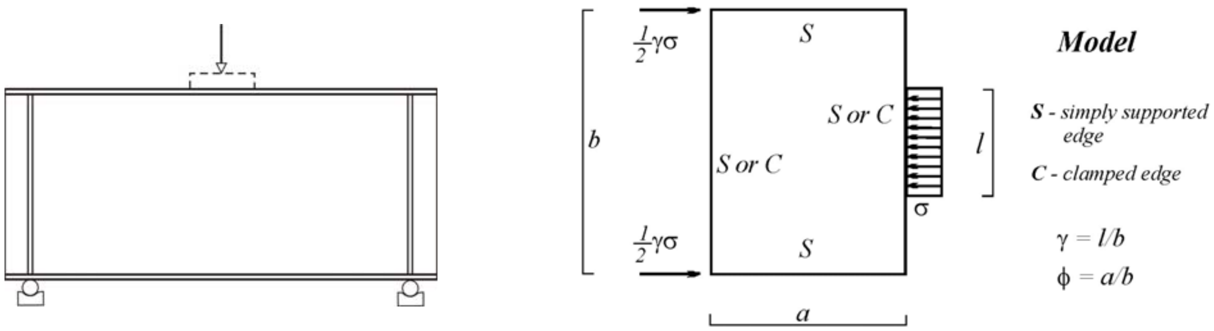


Figure 1.1: Basic model for patch-loading analysis

The case of patch loading can be analyzed by using different mathematical models which are describing the mentioned problem with different levels of accuracy. Considering models with different levels of complexity, it is possible to compare the results and analyze contribution of individual parameters to the value of the critical load.

In this paper, the first, basic mathematical model is chosen to represent buckling problems of plates under locally distributed compression (Figure 1.1). As shown in Figures 2.1 and 2.2, superposition of two fundamental load types (DEA and DEB) is used to describe initial model for the case of patch loading.

The next step would be raising model to a more complex level through introduction of the shear stresses along vertical stiffeners with task to equilibrate external loads (the third fundamental load SEB).The final goal would be defining and analyzing model with effects of shear stress at the flange-web junction (load SOA) whose distribution depends on the rigidity of the flange. Until now, such effect has never been discussed.

Comparative analysis of the three models defining stability problem of rectangular plates with different boundary conditions under patch loading, can point to interesting conclusions about the relevance of various parameters and their influence on the value of the critical load.

2. BASIC OUTLINE

2.1. Introduction

Analytical approach to stability problems of the plates due to the patch-loading begins with determination of exact stress functions for selected model.

In the previous papers (Baker et al. 1993, Liu 2006) it has been already explained that any arbitrary load (normal and/or shear) which acts along the edges of the plate, can be described by the chosen functions (even and/or odd in relation to the coordinate axes), so the total solution is obtained by the adequate combination of eight basic cases (Figure 2.1).

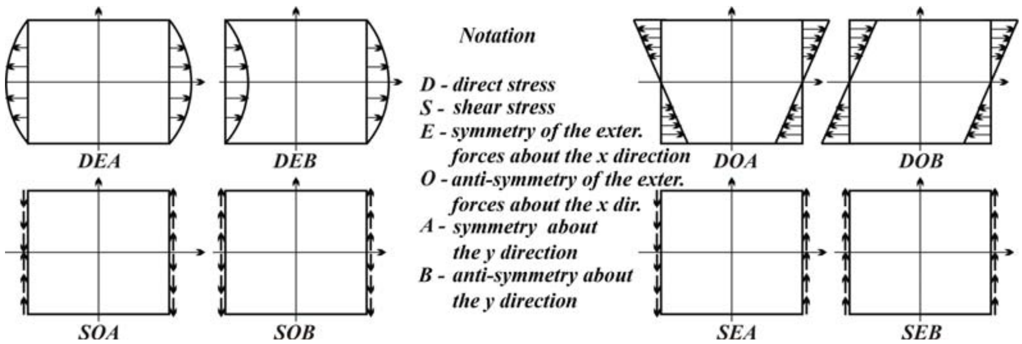


Figure 2.1: Eight basic load cases

For the presented initial model, external load is obtained by combining symmetrical (DEA) and anty-symmetrical (DEB) basic types (Figure 2.2). Since the results for stress functions for the DEA and DEB cases can be found in literature (Liu 2006, Mijuskovic 2012, 2013) only Mathieu's basic approach will be presented in this paper.

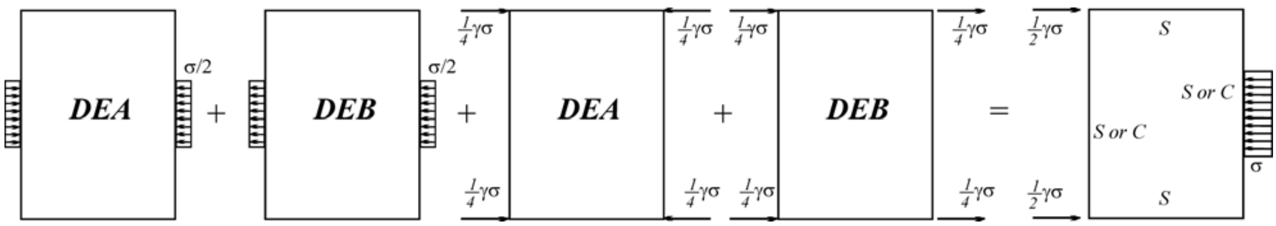


Figure 2.2: Creation of model by combination of two basic load cases

In the Figure 2.2 the procedure for obtaining the exact stress distribution for the adopted model is explained by superposition of the adequate DEA and DEB solutions. The possibility to achieve exact stress functions for complex cases of plates under patch loading guarantees accurate analytical approach to critical load determination. So far, in the literature, only in the researches of Pavlovic and Liu (2006), it is possible to find analytical results for buckling loads, but exclusively for simply supported plates. Up to now, for this load case and the plates with different boundary conditions, there are no precise analytical solutions.

All the results in this paper are reaffirmed by numerical finite-element (ANSYS) runs.

2.2. Mathieu's solution

Although basic equations can be found in literature, before proceeding with solution it is necessary to summarize the main governing expressions of two-dimensional elasticity, since Mathieu's notation and approach (XIX century work) depart from current conventions.

In his paper (1890), Mathieu expressed the known equilibrium equations, without the presence of body forces, in terms of displacements:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \quad and \quad \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \quad Mathieu \quad \Rightarrow \quad \Delta u=-\frac{1}{\varepsilon}\frac{d\nu}{dx} \quad and \quad \Delta v=-\frac{1}{\varepsilon}\frac{d\nu}{dy} \tag{1}$$

where:

Δ - Laplase's operator,

u, v - displacements along the *x* and *y* directions respectively,

$\nu=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$ - volumetric dilatation, (2)

$\varepsilon=\mu/(\lambda+\mu)$ - parameter related to Lamé's constants. (3)

With the quite simple mathematical operations system (1) can be transformed into the following form:

$$\Delta\nu=0 \tag{4}$$

Mathieu's approach to the 2D elasticity problem starts with the careful selection of two ordinary Fourier series for ν (4) with infinite unknown coefficients, taking into account the symmetry or anti-symmetry of the stresses with respect to the *x* and *y* directions.

$$\nu=\nu_1+\nu_2 \tag{5}$$

The following step presents the introduction of the function *F* (F_1+F_2), from the conditions that the equation is fulfilled:

$$\Delta F=-\frac{1}{\varepsilon}\nu \quad \Rightarrow \quad \Delta F_1=-\frac{1}{\varepsilon}\nu_1 \quad and \quad \Delta F_2=-\frac{1}{\varepsilon}\nu_2 \tag{6}$$

Finally, when displacements *u* and *v* are determined

$$u=\frac{dF}{dx}+\alpha\int \nu_1 dx \quad and \quad v=\frac{dF}{dy}+\alpha\int \nu_2 dy \tag{7}$$

where $\alpha = \dfrac{(\lambda + 2\mu)}{\mu}$ is constant expressed in terms of Lamé's parameters,

normal stresses N_1 and N_2 are defined along the axes x and y, as well as the in-plane shear stress T_3.

$$N_1 = \lambda v + 2\mu\alpha v_1 + 2\mu \frac{d^2F}{dx^2}, \quad N_2 = \lambda v + 2\mu\alpha v_2 + 2\mu \frac{d^2F}{dy^2} \tag{8a}$$

$$T_3 = \mu\left[2\frac{d^2F}{dxdy} + \alpha\int \frac{dv_1}{dy}dx + \alpha\int \frac{dv_2}{dx}dy\right] \tag{8b}$$

As it is pointed above, solutions for the basic cases DEA and DEB has already been presented (Liu 2006, Mijuskovic et al. 2012 and 2013), and in this paper, special attention is paid to buckling analysis of the plates under patch loading.

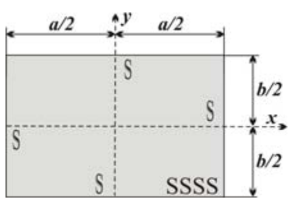
2.3. Analytical approach to plate buckling

The problem of the elastic stability of rectangular plates with different boundary conditions is investigated using the Ritz energy technique. The strain energy due to bending of the plate is defined in the traditional way. On the other hand, the exact stress distribution of Mathieu's theory of elasticity is introduced through the potential energy of the plate associated with the work done by external loads. By adopting the exact stresses within a plate under patch loading and using the double Fourier series to represent any possible buckled profile, the buckling loads can be obtained in a very accurate way. Analytical approach to plate buckling under patch loading is presented in the examples of the rectangular simply supported plates (SSSS) as well as in plates with two edges simply supported and other two clamped (CSCS). In order to verify the results from analytical method, the finite-element method (ANSYS) is used to produce buckling coefficients for the considered problem. Presently available literature has no records on analytical solutions dealing with the subject.

2.3.1. The adopted deflection series

In order to guarantee the accuracy, the double Fourier series are used to represent buckled profiles of the two chosen types of plates (9 - 10). These series satisfy all boundary conditions, term by term, and, as it has been previously shown, are capable of representing any possible buckled profiles for very wide range of aspect ratios and load cases.

Case 1

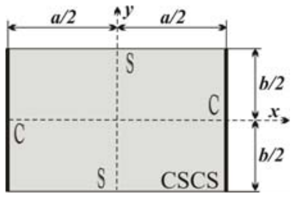


edges $x = \pm a/2$ simply supported (S)
edges $y = \pm b/2$ simply supported (S)

Figure 2.4. Simply supported plate SSSS

$$w = \sum_{m=1}^{s}\sum_{n=1}^{s} W_{mn} \sin\frac{m\pi}{a}\left(x + \frac{a}{2}\right)\sin\frac{n\pi}{b}\left(y + \frac{b}{2}\right) \tag{9}$$

Case 2



edges x = ± a/2 clamped (C)

edges y = ± b/2 simply supported (S)

Figure 2.5. Plate with mixed boundary conditions CSCS

$$w = \sum_{m=1}^{s}\sum_{n=1}^{s} W_{mn}\left(\cos\frac{(m-1)\pi}{a}\left(x+\frac{a}{2}\right) - \cos\frac{(m+1)\pi}{a}\left(x+\frac{a}{2}\right)\right)\sin\frac{n\pi}{b}\left(y+\frac{b}{2}\right) \tag{10}$$

2.3.2. Strain energy due to bending

During the evaluation of the total potential energy of the plate, the first step is defining the strain energy due to plate bending in the traditional way:

$$U = \frac{1}{2}D\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right]dxdy \tag{11}$$

where D is flexural rigidity of the plate.

The part of the potential energy of the plate associated with the work done by external loads is presented by the expression (12). In this expression, the stresses within the plate N_1, N_2 and T_3 are given by equations (Mijuskovic et al. 2012 and 2013) that represent solutions of the Mathieu's exact approach:

$$V = -\frac{t}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left[N_1\left(\frac{\partial w}{\partial x}\right)^2 + N_2\left(\frac{\partial w}{\partial y}\right)^2 + 2T_3\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dxdy \tag{12}$$

Introducing the exact stress functions makes the expression for the work done by external forces more complex. It presents the basic difference in relation to the all previous analyses of the stability of plates which are not simply supported along all edges

2.3.3. Formulation of eigenvalue problem

Finally, after the definition of the strain energy of the plate bending *U*, and of the value which responds to the work done by external forces *V*, the total potential energy of the system can be written in form:

$$\Pi = U + V \tag{13}$$

From the minimum potential energy principle, the condition (13) is given by

$$\frac{\partial \Pi}{\partial W_{mn}} = \frac{\partial U}{\partial W_{mn}} + \frac{\partial V}{\partial W_{mn}} \tag{14}$$

which basically represents linear system of $m \cdot n$ homogenous equations per unknown coefficients W_{mn}. The existence of nontrivial solution, expressed through condition that the determinant of the system is equal to zero, leads to the solution of the classical eigenvalue problem. In its scope, the lowest value has the only practical importance, which presents the requested critical load. Surely, the usage of the corresponding software (MATHEMATICA) was necessary in the solving process because of the complexity of the analytical procedure. The complexity directly depends on the adopted number of terms of the stress functions, as well as of number of terms of the deflection functions.

3. RESULTS AND CONCLUSIONS

With the detailed analysis of the presented results for buckling coefficients (Tables 1 and 2), it is very easy to notice very good behavior of analytical solution for both types of boundary conditions in the complete considered ranges of the plate (ϕ = 0.3 –1) and load (γ = 0.1 – 1) aspect ratios.
Tables 1 and 2, which present values of buckling coefficients of two types of plates (SSSS, CSCS) under patch loading refer to the maximal discrepancy of 0.95% (CSCS ϕ = 0.3 and γ = 1) in relation to the results evaluated by the application of the finite element method.
It is important to point out that for the problems regarding stability of the plates, buckling coefficients obtained by finite element method are below exact values, as a result of limited number of terms in interpolation functions. Knowing that, small existing discrepancy between presented results confirms accuracy of the analytical approach.

Table 1 – Buckling coefficients for plate SSSS (ϕ = 0.3 - 1, γ = 0.1 – 1)

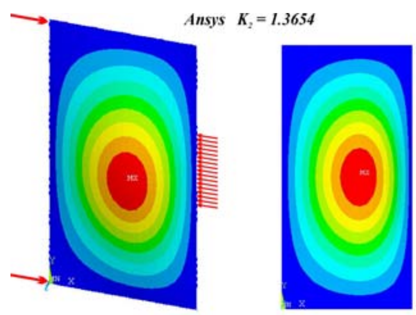
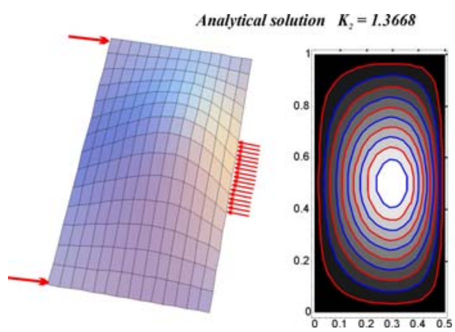
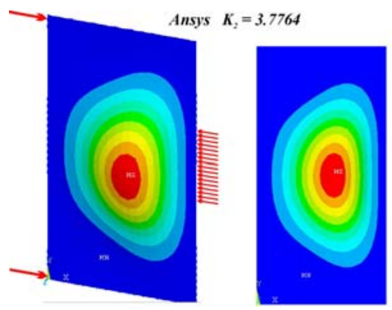
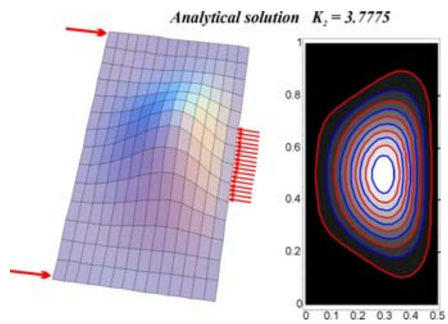
Plate SSSS – patch loading							Example ϕ = 0.5 i γ = 0.3
$K_2 = K\phi^2\gamma$	ϕ = 0.3	ϕ =0.5	ϕ =0.7	ϕ =0.9	ϕ =1.0	**Results**	
γ = 0.1	0.6838	1.2217	1.7687	2.4838	2.9493	A. Solution	
	0.6819	1.2202	1.7659	2.4780	2.9415	MKE (Ansys)	
	(-0.274)	(-0.127)	(-0.162)	(-0.231)	(-0.263)	Disc. (%)	
γ = 0.3	0.8751	1.3668	1.8982	2.6271	3.1069	A. Solution	
	0.8732	1.3654	1.8951	2.6210	3.0987	MKE (Ansys)	
	(-0.215)	(-0.101)	(-0.160)	(-0.233)	(-0.264)	Disc. (%)	
γ = 0.4	1.0082	1.4733	2.0022	2.7453	3.2367	A. Solution	
	1.0064	1.4720	1.9990	2.7387	3.2278	MKE (Ansys)	
	(-0.179)	(-0.088)	(-0.161)	(-0.238)	(-0.273)	Disc. (%)	
γ = 0.5	1.1572	1.5970	2.1307	2.8919	3.3966	A. Solution	
	1.1556	1.5956	2.1272	2.8849	3.3872	MKE (Ansys)	
	(-0.139)	(-0.085)	(-0.168)	(-0.242)	(-0.278)	Disc. (%)	
γ = 0.7	1.4804	1.8952	2.4626	3.2621	3.7912	A. Solution	
	1.4797	1.8935	2.4583	3.2538	3.7802	MKE (Ansys)	
	(-0.050)	(-0.089)	(-0.174)	(-0.254)	(-0.290)	Disc. (%)	
γ = 0.9	1.8073	2.2774	2.8881	3.6964	4.2275	A. Solution	
	1.8062	2.2750	2.8830	3.6868	4.2150	MKE (Ansys)	
	(-0.060)	(-0.105)	(-0.177)	(-0.258)	(-0.296)	Disc. (%)	
γ = 1.0	1.9862	2.4990	3.1227	3.9122	4.4299	A. Solution	
	1.9840	2.4962	3.1172	3.9022	4.4169	MKE (Ansys)	
	(-0.113)	(-0.111)	(-0.176)	(-0.257)	(-0.293)	Disc. (%)	

Table 2 – Buckling coefficients for plate CSCS (ϕ = 0.3 - 1, γ = 0.1 – 1)

Plate CSCS – patch loading							Example ϕ = 0.5 i γ = 0.3
$K_2 = K\phi^2\gamma$	ϕ = 0.3	ϕ =0.5	ϕ =0.7	ϕ =0.9	ϕ =1.0	**Results**	
γ = 0.1	1.7457	3.2504	4.3787	5.2158	5.7355	A. Solution	
	1.7416	3.2476	4.3782	5.2111	5.7277	MKE (Ansys)	
	(-0.238)	(-0.081)	(-0.012)	(-0.092)	(-0.136)	Disc. (%)	
γ = 0.3	2.3899	3.7775	4.7011	5.4687	5.9707	A. Solution	
	2.3837	3.7764	4.7004	5.4643	5.9634	MKE (Ansys)	
	(-0.262)	(-0.031)	(-0.016)	(-0.079)	(-0.122)	Disc. (%)	
γ = 0.4	2.7009	4.1655	4.9395	5.6664	6.1541	A. Solution	
	2.6881	4.1648	4.9386	5.6622	6.1388	MKE (Ansys)	
	(-0.475)	(-0.017)	(-0.018)	(-0.074)	(-0.248)	Disc. (%)	
γ = 0.5	2.9147	4.5962	5.2230	5.9072	6.3744	A. Solution	
	2.8993	4.5921	5.2209	5.9029	6.3665	MKE (Ansys)	
	(-0.528)	(-0.090)	(-0.041)	(-0.073)	(-0.123)	Disc. (%)	
γ = 0.7	3.3238	5.2103	5.9119	6.4977	6.8948	A. Solution	
	3.3013	5.2003	5.9082	6.4927	6.8865	MKE (Ansys)	
	(-0.676)	(-0.193)	(-0.063)	(-0.077)	(-0.121)	Disc. (%)	
γ = 0.9	3.5465	5.6027	6.7277	7.1673	7.4418	A. Solution	
	3.5160	5.5796	6.7230	7.1617	7.4328	MKE (Ansys)	
	(-0.861)	(-0.413)	(-0.069)	(-0.078)	(-0.121)	Disc. (%)	
γ = 1.0	3.6470	5.7828	7.1434	7.5021	7.6974	A. Solution	
	3.6124	5.7543	7.1368	7.4965	7.6884	MKE (Ansys)	
	(-0.948)	(-0.493)	(-0.092)	(-0.076)	(-0.117)	Disc. (%)	

At the end, the main conclusion can be that obtained exact stress functions, as well as adopted deflection functions, for initial mathematical model, are capable to describe the behavior of the plates under patch loading and produce very accurate solutions. Now it is possible to go a step further and build new, more advanced models, by introducing shear stresses along the shorter plate edges and/or shear effects on the flange-web junction. Until now, such effect has never been discussed analytically.

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