AN ASSESSMENT OF LIMIT LOADS OF CRACKED STRUCTURES USING EXTENDED ISOGEOMETRIC ANALYSIS

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ABSTRACT

A novel numerical procedure using eXtended IsoGeometric Analysis (XIGA) in combination with second-order cone programming (SOCP) is applied to assess the limit loads of the cracked structures. Isogeometric analysis (IGA) utilizing the non-uniform rational B-spline functions from Computer Aided Design (CAD) tools is incorporated with enrichment functions through the partition of unity method (PUM) in order to model the discontinuous phenomenon in the cracked structures. The obtained optimization formulation is then transformed into the form of a SOCP problem so that it can be solved by highly efficient interior-point solvers. Two numerical examples are provided to show excellent performance of the present method compared with other published solutions in the literature.

Keywords: Isogeometric Analysis (IGA), NURBS, cracked structure, limit analysis, SOCP.

1. INTRODUCTION

Cracks are found in the structures during fabrication, construction and service life. Beside the Linear Elastic Fracture Mechanics (LEFM), limit analysis plays a significant role in the estimation of elastic-plastic fracture toughness and safety assessment of fracture failure (Yan 1999). Therefore, limit analysis is extensively studied with the first research of Hill (1952), Ewing (1974), Miller (1988), etc. However, the analytical method is not applicable for complicated problems. Hence, development of numerical methods has been performed with various methods including the ones proposed, for example, by Yan and Nguyen-Dang (1999) and Vu (2001)… In their methods, the geometries need to be discretized with special elements (Barsoum 1977) near the crack tips in order to capture the singular stresses. It makes the computational cost very expensive, especially for complex cracked structures. In another approach, Moes and Ted (1999) introduced the extended finite element method (XFEM) which enriches the displacement field by special functions to describe the singular and discontinuous phenomena on uniform meshes that do not conform to the cracks.

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In traditional FE methods, the discretized geometry through mesh generation often leads to geometrical errors. Furthermore, the communication of geometry model and mesh generation during analysis consumes much time (Hughes 2005). To overcome this disadvantage, Hughes et al. (2005) proposed a new computational method - the so-called isogeometric analysis (IGA) to closely link the gap between Computer Aided Design (CAD) and Finite Element Analysis (FEA). IGA utilizes the same non-uniform rational B-spline (NURBS) basic functions in describing the geometry and constructing finite approximation for analysis. The exact geometry is therefore maintained at the coarsest level of discretization and the re-meshing is performed on this coarsest level without any further communication with CAD geometry.

In this paper, we present a novel numerical procedure to assess the limit loads of the cracked structures based on a combination XIGA with second-order cone programming (SOCP). Enrichment functions through the partition of unity method are used to model the discontinuous phenomena. After that the optimization formulation can be solved by highly efficient interior-point solvers from SOCP. Two numerical examples under plane stress condition are provided to show the excellent performance of the present method compared with other published solutions in the literature.

2. BRIEF ON LIMIT ANALYSIS

Consider a rigid-perfectly plastic body defined in a domain $\mathbf{\Omega}$ with a boundary Γ such that $\Gamma = \Gamma_{\mathbf{u}} \cup \Gamma_{\mathbf{t}} \cup \Gamma_c$, $\Gamma_{\mathbf{u}} \cap \Gamma_{\mathbf{t}} \cap \Gamma_c = \varnothing$ where $\Gamma_{\mathbf{u}}$, Γ_t, Γ_c are the Dirichlet and Neumann boundary and crack surface, respectively. The body subjected to body forces $\mathbf{f}$ and to surface tractions $\mathbf{g}$ on the free portion Γ_t.

Let $\dot{\mathbf{u}} = [\dot{u} \quad \dot{v}]^T$ be plastic velocity that belongs to a space kinematically admissible velocity fields, where $\dot{u}, \dot{v}$ are the velocities in *x*- and *y*-direction, respectively. The external work rate associated with a virtual plastic flow $\dot{\mathbf{u}}$ is expressed in the linear form as

$$F(\dot{\mathbf{u}}) = \int_{\Omega} \mathbf{f}^T \dot{\mathbf{u}} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{g}^T \dot{\mathbf{u}} \, \mathrm{d}\Gamma \tag{1}$$

Based on the plastic upper boundary limit analysis problem, the collapse load multiplier $\lambda^+ \in \mathbb{R}^+$ can be determined by solving the following mathematical programming

$$\lambda^+ = \min_{\dot{\mathbf{u}} \in C} \int_{\Omega} D\left(\dot{\boldsymbol{\varepsilon}}\right) \mathrm{d}\Omega \quad \text{, where } C = \left\{ \dot{\mathbf{u}} \in V \middle| F(\dot{\mathbf{u}}) = 1 \right\} \tag{2}$$

The plastic dissipation $D(\dot{\boldsymbol{\varepsilon}})$ is defined by

$$D\left(\dot{\boldsymbol{\varepsilon}}\right) = \max_{\psi(\sigma) \le 0} \ \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \equiv \boldsymbol{\sigma}_{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \tag{3}$$

where $\boldsymbol{\sigma}$ is the admissible stresses contained within the convex yield surface and $\boldsymbol{\sigma}_{\varepsilon}$ describes the stresses on the yield surface associated with any strain rate $\dot{\boldsymbol{\varepsilon}}$ through the plasticity condition.

In this work, we use the von Mises failure criterion for plane stress problems. The plastic dissipation can be rewritten as a function of strain rate as (Capsoni and Corradi 1997)

$$D(\dot{\boldsymbol{\varepsilon}}) = \sigma_0 \sqrt{\dot{\boldsymbol{\varepsilon}}^T \Theta \dot{\boldsymbol{\varepsilon}}} \quad \text{with } \Theta = \frac{1}{3}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

where σ_0 is the yield stress.

3. XIGA FOMULATION FOR LIMIT ANALYSIS

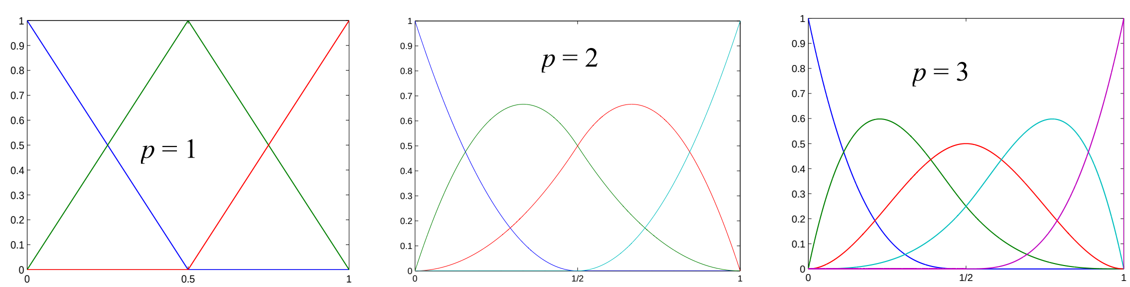
3.1. NURBS basic functions

To build a B-spline in 1D, we firstly define two positive integers: a polynomial degree p and number of control point n and a knot vector $\Xi = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$, where $\xi_i \in \mathbb{R}, i = 1,...n + p + 1$ called i^{th} knot lies in the parametric space.

The B-spline basis functions $N_{i,p}(\xi)$ are defined by the following recursion formula

$$\begin{gathered} N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \\ \text{as } p = 0 \quad N_{i,0}(\xi) = \begin{cases} 1 & \text{if} \quad \xi_i \le \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \end{gathered} \tag{5}$$

Figure 1 illustrates a set of 1D linear, quadratic and cubic B-spline basis functions according to open uniform knot vectors $\Xi = \{0, 0, 0.5, 1, 1\}$, $\Xi = \{0, 0, 0, 0.5, 1, 1, 1\}$, $\Xi = \{0, 0, 0, 0, 0.5, 1, 1, 1, 1\}$, respectively. It is clear that the B-spline basis functions are C^{p-1} continuous.



(a) (b) (c)

Figure 1. Some B-spline basis functions: (a) linear, (b) quadratic, and (c) cubic functions.

In 2D problems, the B-spline functions are defined by the tensor product of two 1D basis functions $N_{i,p}(\xi)$ and $M_{j,q}(\eta)$ according to parametric dimensions ξ and η with two knot vectors

$\boldsymbol{\Xi} = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$ and $\mathbf{H} = \{\eta_1, \eta_2, ..., \eta_{m+q+1}\}$ are expressed as follows:

$$N_I^b(\xi,\eta) = N_{i,p}(\xi) M_{j,q}(\eta) \tag{6}$$

To present exactly curved geometries (e.g., circles, cylinders, spheres, etc.) NURBS are used. Be different from B-spline, each control point of NURBS has additional value called an individual weight w_I. Then the NURBS function can be expressed as

$$N_I(\xi,\eta) = N_I^b w_I \Big/ \sum_{I=1}^{m\times n} N_I^b(\xi,\eta) w_I \tag{7}$$

It notes that B-spline function is only the special case of the NURBS function when the individual weight of control points is constant (Thai et al. 2011; Tran et al. 2013).

3.2. Extended Isogeometric finite element method

Being different from XFEM which using the Lagrange polynomial, XIGA (Luycker 2011; Nguyen 2012) utilizes NURBS basis functions that are common in CAD geometry to approximate the displacement field for a solid with traction-free crack as follows:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I\in S} N_I(\mathbf{x})\mathbf{u}_I + \sum_{J\in S^c} N_J(\mathbf{x}) H(\mathbf{x})\mathbf{a}_J + \sum_{K\in S^f} N_K(\mathbf{x}) \sum_{\alpha=1}^{4} B_\alpha(\mathbf{x})\mathbf{b}_K^\alpha \tag{8}$$

where $N_{(I,J,K)}$ are the NURBS functions. Besides, the standard DOFs $\mathbf{u}_I$ in the set of the finite element nodes S like standard FEM, the extra DOFs $\mathbf{a}_J$, $\mathbf{b}_K^\alpha$ are added. The set S^c includes the nodes whose support is cut by the crack and S^f is the set of nodes whose support contains the crack tip (as shown in Figure 2).

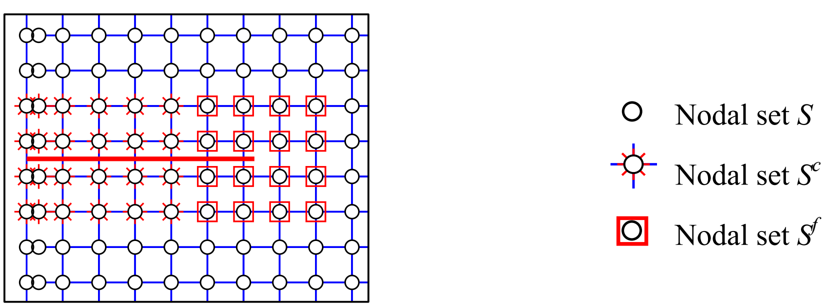


Figure 2. Illustration of the nodal sets S, S^c, S^f for a cubic NURBS mesh.

To present the discontinuous displacement along the crack path, the Heaviside function $H(\mathbf{x})$ is used

$$H(\mathbf{x}) = \begin{cases} +1 & \text{above crack} \\ -1 & \text{below crack} \end{cases} \tag{9}$$

And for linear elastic crack, four branch functions used to describe the singular stress at crack tip are given below

$$[B_1, B_2, B_3, B_4] = \left[\sqrt{r}\sin\frac{\theta}{2},\quad \sqrt{r}\cos\frac{\theta}{2},\quad \sqrt{r}\sin\frac{\theta}{2}\sin\theta,\quad \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right] \tag{10}$$

where r and θ are polar coordinates in the local crack tip coordinate system.

3.3. XIGA – based approximated formulation

Using the NURBS basis functions, the velocity of the displacement field is approximated as:

$$\dot{\mathbf{u}}^h = \sum_{I=1}^{m\times n} N_I(\xi,\eta)\dot{\mathbf{q}}_I \tag{11}$$

where $\dot{\mathrm{q}}_I = [\dot{u}_I \quad \dot{v}_I]^T$ are the velocities of nodal displacement associated with the control point I.

The strain rate is written as:

$$\dot{\boldsymbol{\varepsilon}} = \sum_{I=1}^{m\times n} \mathbf{B}_I \dot{\mathbf{q}}_I \tag{12}$$

where the strain matrix $\mathbf{B}$ is given by:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}^{sta} & \mathbf{B}^{enr} \end{bmatrix} \tag{13}$$

in which $\mathbf{B}^{sta}$ and $\mathbf{B}^{enr}$ are the standard and enriched part of matrix $\mathbf{B}$

$$\mathrm{B}_I^{sta} = \begin{bmatrix} N_{I,x} & 0 \\ 0 & N_{I,y} \\ N_{I,y} & N_{I,x} \end{bmatrix}, \quad \mathrm{B}_I^{enr} = \begin{bmatrix} N_{I,x}\psi_I + N_I\psi_{I,x} & 0 \\ 0 & N_{I,y}\psi_I + N_I\psi_{I,y} \\ N_{I,y}\psi_I + N_I\psi_{I,y} & N_{I,x}\psi_I + N_I\psi_{I,x} \end{bmatrix} \tag{14}$$

in which ψ_I may represent either the Heaviside function H or the branch functions B_α. Substituting Eq.(12) into Eq. (2) the optimization problem can be reformulated as

$$\lambda^+ = \min \sum_{e}^{nel} \sum_{i=1}^{n_{GP}} \sigma_0 w_i \sqrt{\dot{\boldsymbol{\varepsilon}}_i^{eT} \Theta \dot{\boldsymbol{\varepsilon}}_i^e} \qquad \text{s.t} \begin{cases} \dot{\mathbf{u}} = 0 & \text{on } \Gamma_u \\ F(\dot{\mathbf{u}}) = 1 \end{cases} \tag{15}$$

The above limit analysis problem is a non-linear optimization problem with equality constraints. It can be solved using a general non-linear optimization solver such as a sequential quadratic programming algorithm or a direct iterative algorithm (Capsoni and Corradi 1997). In particular, the optimization problem can be reduced to the problem of minimizing a sum of norms (Andersen 1998). In fact a problem of this sort can be reformed as a SOCP problem.

4. SOLUTION PROCEDURE USING SECOND – ORDER CONE PROGRAMING

Since Θ is a positive definite matrix, the plastic dissipation function in Eq.(4) can be formulated in a sum of norms, as described below

$$D = \sigma_0 \|\boldsymbol{\rho}\| \quad \text{where } \boldsymbol{\rho} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\boldsymbol{\varepsilon}} \tag{16}$$

By introducing auxiliary variables $t_1,\ t_2,\ \ldots\ ,t_{nel \times n_{GP}}$, optimization problem in Eq. (15) can be cast as a SOCP problem:

$$\lambda^+ = \min \sum_{i=1}^{NG} \sigma_0 \mathrm{w}_i t_i \qquad \text{s.t} \begin{cases} \dot{\mathbf{u}} = 0 & \text{on } \Gamma_u \\ F(\dot{\mathbf{u}}) = 1 & \\ \|\boldsymbol{\rho}_i\| \le t_i & i = 1, 2, \ldots, \ nel \times n_{GP} \end{cases} \tag{17}$$

5. NUMERICAL RESULTS

5.1. Centre – cracked plate subjected to tension

Let consider a centre – cracked plate under tension as depicted in ***Figure 3***. The exact limit load factor $\lambda = \sigma_{\lim it} / \sigma$ was calculated as (Miller 1988)

$$\lambda = 1 - a / b \tag{18}$$

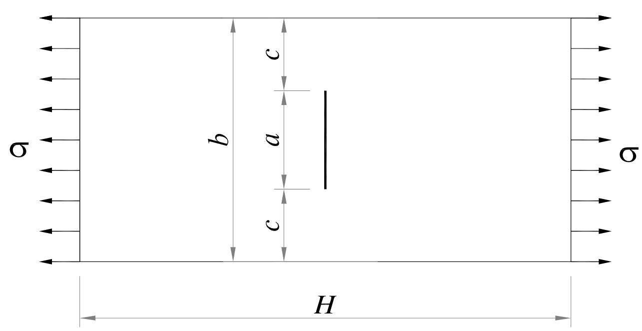


Figure 3: The model of centre-cracked plate.

The problem has been solved numerically by XIGA with a uniform mesh of 14x31 cubic NURBS elements shown in Figure 4a. Table 1 shows the comparison of present limit load factor to the analytical solution (Miler 1988) and that from XEFM (Tran et al. 2012). It can be seen that the present method gives the upper convergence to the exact solution as we increase the order of basis function or the number of elements. Although using less number of elements, XIGA also gains the better results than XFEM. At the collapse state, the collapse mechanism and plastic dissipation distribution in the plate are shown in Figure 4b according to ratio of length crack to width *a/b*=0.5.

Table 1: The collapse limit factors of centre cracked plate via ratios *a/b*

Method	Ratios *a/b*							
	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Analytical solution	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
XFEM (18x36)	0.8503	-	0.6352	-	0.4381	-	0.2312	-
XFEM (24x48)	0.8324	-	0.6346	-	0.4327	-	0.2239	-
XFEM (30x60)	0.8324	-	0.6311	-	0.4285	-	0.2218	-
XIGA(*p*=1)	**0.8706**	**0.7633**	**0.6737**	**0.5681**	**0.4606**	**0.3702**	**0.2613**	**0.1497**
XIGA(*p*=2)	**0.8203**	**0.7272**	**0.6323**	**0.5286**	**0.424**	**0.3393**	**0.2242**	**0.1187**
XIGA(*p*=3)	**0.8197**	**0.7215**	**0.6209**	**0.5255**	**0.427**	**0.3207**	**0.2204**	**0.1191**

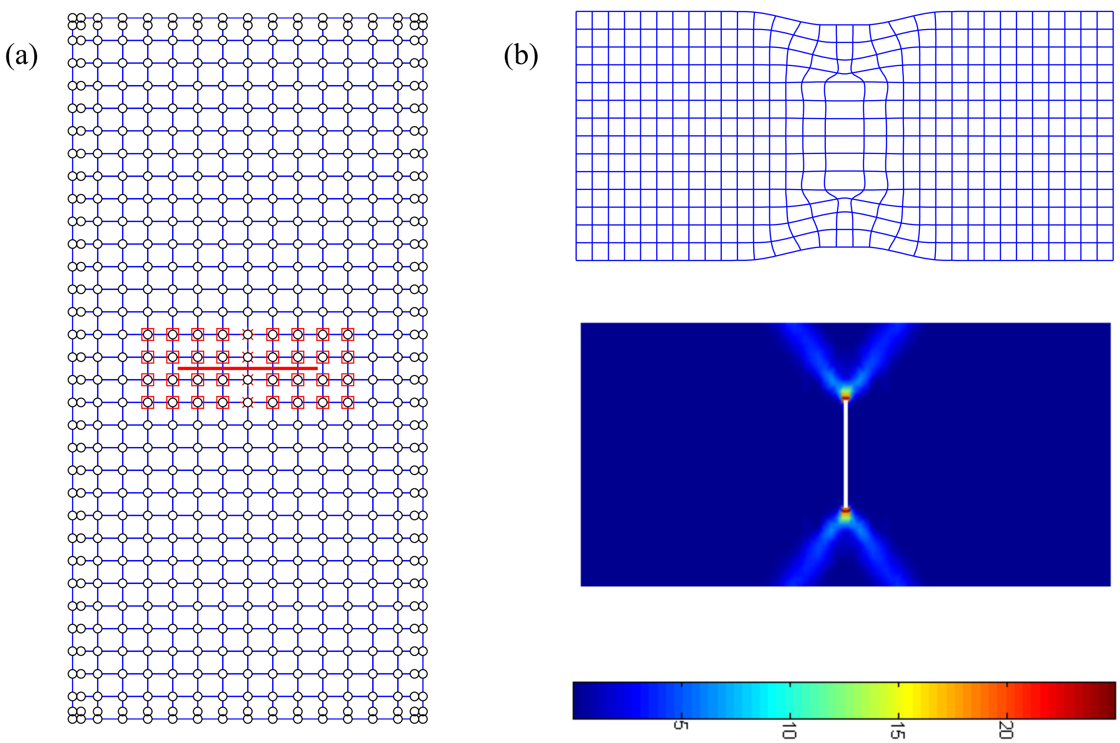


Figure 4: The center-cracked plate with *a/b*=0.5, *H/b*=2: (a) The meshing; (b) The collapse mechanism and plastic dissipation distribution.

5.2. Edge – crack subjected to tension

Next, a plate revealed in Figure 5 with length H, width b and a single – edge cracked length a subjected to a tension stress is studied. This benchmark problem is solved by analytical method (Ewing and Richards 1974) with the limit load factor defined as:

$$\lambda = 1 - x - x^2 \qquad \text{as } x = a/b \le 0.146 \text{ (short – crack)}$$

$$\lambda = \sqrt{\left(-\gamma x + 0.5(\gamma - 1)\right)^2 + \gamma\left(1 - x\right)^2} - \left(\gamma x - 0.5(\gamma - 1)\right) \text{ otherwise with } \gamma = 2/\sqrt{3} \qquad (19)$$

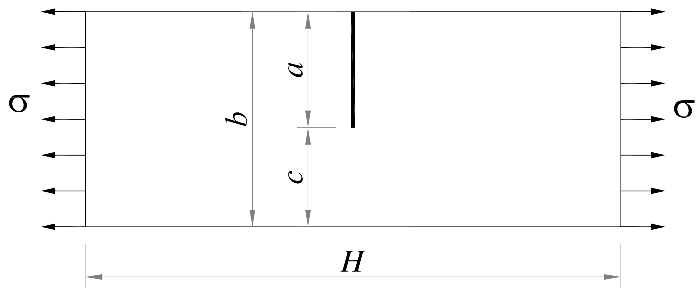


Figure 5: The model of edge – cracked plate.

Herein, this problem is investigated with various aspect ratios such as: *H/b* = 1, 2, and 3, respectively. The limit load factors via the length crack to width ratios *a/b* are plotted in Figure 6a. It is revealed that in case of short – crack, the square plate gains the lower results compared to analytical solution (Ewing and Richards 1974). However, as the ratio *a/b* is enough large, the limit loads are independent on aspect ratios. It is seen again that the obtained results are upper convergence and get excellent agreement with that by Ewing and Richards (1974). To close this section, Figure 6b is produced to display the collapse mechanism and plastic dissipation distribution at rectangular plate with *H/b*=2.

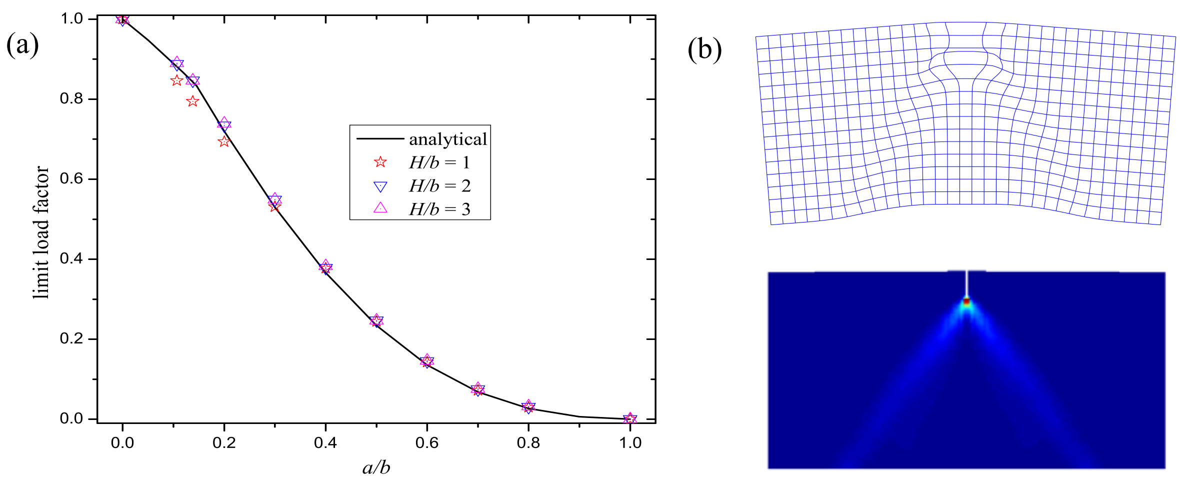


Figure 6: (a) The limit load factor via *a/b* ratio; (b) The collapse mechanism and plastic dissipation distribution of edge – cracked plate with *a/b*=0.138 and *H/b* = 2.

6. CONCLUSIONS

An efficient approach that uses eXtended IsoGeometric Analysis (XIGA) in combination with second-order cone programming (SOCP) has been proposed to access the limit load of cracked structures. Using extra enriched functions in displacement filed, present method can capture the discontinuous phenomenon along crack path and the singularity at crack tip. In addition, by utilizing NURBS functions instead of Lagrange functions, the obtain results are dramatically improved and more accurate compared to XFEM whether the coarser mesh is used.

The obtained results are excellent agreement with analytical solutions for the plane stress problems. It is believed that proposed method can be very promising to provide the high accuracy for plane strain problems, 3D problems or shakedown analysis.

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