Boundary values of quasiconformal mappings

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BOUNDARY VALUES OF QUASICONFORMAL MAPPINGS

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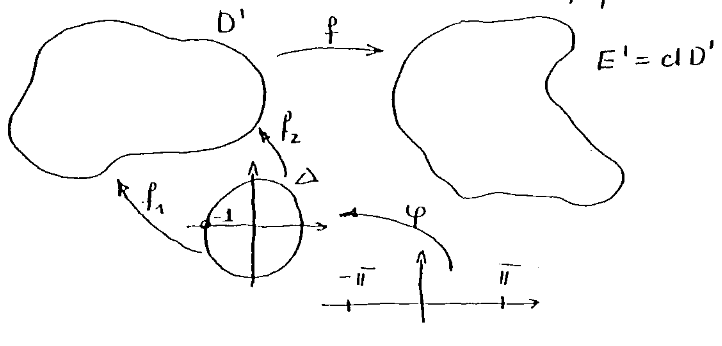
Notations.

$\mathbb{E}$ - closed plane,
$\mathbb{C}$ - open, complex plane,
$\bar{\mathbb{C}}$ - complex plane compactified with one point at infinity,
$\mathbb{R}$ - real line,
E, E' - sets,
$\operatorname{fr} E$ - topological boundary of E, ∂E - oriented boundary of E,
$\operatorname{cl} E$ - the closure of E,
$\operatorname{int} D$ - the interior of D,
D - Jordan domain,
Jordan curves - a homeomorphic image of a circle,
Jordan arc - a homeomorphic image of a segment, i.e. a connected subset of $\mathbb{R}$, which does not reduce to a point,
(a,b) - open segment,
$\langle a,b\rangle$ - closed segment
$\Delta = \{z : |z| \le 1\}$, $\Delta_r = \{z : r \le |z| \le 1\}$,
$\{a_n\}$ - a sequence of numbers, f, g - functions, f^{-1} - inverse funct.
$\{f_n\}$ - a sequence of functions,
$f_n \to f$ - pointwise convergence, $f_n \rightrightarrows f$ - almost uniform convergence.

<u>A sense-preserving homeomorphism.</u>

Suppose that D is a Jordan domain and f is a homeomorphism such that $f: \operatorname{cl} D \to E$. Then there exists a Jordan domain D' such, that $E = \operatorname{cl} D'$, $f: D \to D'$, $f_{|\operatorname{fr} D}: \operatorname{fr} D \to \operatorname{fr} D'$



By the definition of a Jordan domain, there exists a homeomorphis f_1, f_2 such that $f_i : \mathrm{fr}\,\Delta \to \mathrm{fr}\,D$, $i = 1,2$, and $\arg f_2^{-1} \circ f_1|_{\mathrm{fr}\Delta \setminus \{-1\}}$ is an increasing function of $\arg z$, $z \in \mathrm{fr}\,\Delta \setminus \{-1\}$.

It is well known that also $g_1 = f \circ f_1$ and $g_2 = f \circ f_2$ are homeomorphisms such that $\arg g_2^{-1} \circ g_1|_{\mathrm{fr}\Delta \setminus \{-1\}}$ is an increasin function of $\arg z$, $z \in \mathrm{fr}\,\Delta \setminus \{-1\}$.

Hence f induces a mapping between the orientation of $\mathrm{fr}\,D$ and the orientation of $\mathrm{fr}\,D'$. If both the orientations are positive or negative with respect to the corresponding domain then $f : D \to D'$ is said to be <u>sense-preserving</u>.

More general if $f : E \to E'$ is a homeomorphism of two sets then f is said to be sense-preserving if $f|_D$ is sense-preservin for every Jordan domain D, such that $\mathrm{cl}\,D \subset E$.

Proposition 1 (Newmann [ ])

If $f : E \to E'$ is a homeomorphism and E is either a domain or the closure of a Jordan domain and there exist a Jordan domain D such that $\mathrm{cl}\,D \subset E$ and $f|_D$ is sense-preserving, then f is sense-preserving on E.

It is worth-while to note that if f is a sense-preserving homeomorphism, then also is f^{-1}. If f_1 and f_2 are sense-preserving homeomorphisms then also $f_2 \circ f_1$ is the sense-preservin homeomorphism provided the composition makes sens.

<u>Differentiability</u>

Suppose now that E is either an open set or the closure of a Jordan domain. A mapping $f : E \to E$ is said to be differentiable at $z_0 \in \mathrm{int}\,E$ ($z_0, f(z_0) \neq \infty$) if

(1) $$f(z) = f(z_0) + f_z(z_0)(z_0 - z) + f_{\bar z}(z_0)(\bar z_0 - \bar z) + o(z - z_0)$$

where: $f_z = \frac{1}{2}(f_x - i f_y)$, $f_{\bar z} = \frac{1}{2}(f_x + i f_y)$, $z = x + iy$.

Also $\overline{(f_z)} = (\bar f)_{\bar z}$, $\overline{(f_{\bar z})} = (\bar f)_z$,

$$(g \circ f)_z = g_z \circ f \cdot f_z + g_{\bar z} \circ f \cdot \bar f_z ,$$
$$(g \circ f)_{\bar z} = g_z \circ f \cdot f_{\bar z} + g_{\bar z} \circ f \cdot \bar f_{\bar z} .$$

A mapping f is said to be differentiable at $z = \infty$ if f^* defined by $f^*(z) = f(1/z)$ is differentiable at $z = 0$. A mapping f is said to be differentiable at z_0 such that $f(z_0) = \infty$ if f^{**} defined by $f^{**}(z) = 1/f(z)$ is differentiable at z_0.

A mapping f is differentiable on a set E if it is differentiable at every point of E.

The directional derivatives $f_{|\alpha}$ are defined by $f_{|\alpha} = e^{-i\alpha}[f_t(z + te^{i\alpha})]_i$ where α is real and $t \geq 0$.

The Jacobian of f will be denoted by J or J_f. If $z = \infty$, or $f(z_0) = \infty$ for some $z_0 \in E$, we will not define J_f but only the $\operatorname{sgn} J_f$ $= \operatorname{sgn} J_{f^{**}}(z_0)$.

It is easily verified that if $f : E \to \mathbb{C}$, where $E \subset \mathbb{C}$, is differentiabl then

(2) $$f_{|\alpha} = e^{-i\alpha}(f_x \cos\alpha + f_y \sin\alpha) = f_z + e^{-2i\alpha} f_{\bar z} ,$$

(3) $$\max_\alpha |f_{|\alpha}| = |f_z| + |f_{\bar z}| , \quad \min_\alpha |f_{|\alpha}| = ||f_z| - |f_{\bar z}|| ,$$

(4) $$J_f = |f_z|^2 - |f_{\bar z}|^2 = \operatorname{sgn} J_f \max_\alpha |f_{|\alpha}| \cdot \min_\alpha |f_{|\alpha}| = \frac{i}{2}(f_x \cdot \bar f_y - f_y \cdot$$

If f is differentiable at $z_0 \in \operatorname{int} E$ and $\operatorname{sgn} J_f = 0$, then f is called regular at z_0 and z_0 is called a regular point of f. A mapping f is called regular in E if it is regular at every point $z \in \operatorname{int} E$.

A C^1 class regular homeomorphism is called a diffeomorph. In the case when $z_0 = \infty$ or $f(z_0) = \infty$ the definition of C^1 fund may be extended similiarly to that of differentiability

Suppose now that $f: E \to E'$ is a homeomorphism and E is either a domain or the closure of a Jordan domain Then by Newman's result quoted above, if $\operatorname{sgn} J(z) = 1$ at some regular point z, then f is sense-preserving and, conversely, if f is a sense-preserving then $\operatorname{sgn} J_f(z) = 1$ at any regular $z \in \operatorname{int} E$.

Suppose that f is a diffeomorphism and D is a domain The ratio

(5) $$p(z) = p_f(z) = \frac{\max_\alpha |f_\alpha(z)|}{\min_\alpha |f_\alpha(z)|} = \frac{|f_{\bar z}(z)| + |f_z|}{\big||f_z| - |f_{\bar z}|\big|}, \quad z \in D$$

is called the dilatation of f at z. *Clearly it is bounded on every* compact subset of D and invariant under conformal mappings. i.e. if g_1 and g_2 are conformal such that $f \circ g_1$ and $g_2 \circ f$ makes sens, then $p_{f \circ g_1} = p_f = p_{g_2 \circ f}$.

The last conclusion anables us to extend the definition of p to the cases $z_0 = \infty$ and $f(z_0) = \infty$ analogously to the definition of differentiability was extended.

Definition (*Grötzsch*)

A sense-preserving diffeomorphism $f: D \to D'$, where D is a domain and there exist constant number Q, $1 \le Q < \infty$ such there

(6) $$\sup p_f(z) \le Q$$

is called a regular Q-quasiconformal *mapping (Q-qc)*.

This very natural definition has the disadvantage that the class of regular Q-qc mappings is not close with respect to almost uniform convergence. Thus

Definition (*Gehring, Lehto*)

A homeomorphism $f: D \to D'$, D-being *a domain, is said* to be a Q-qc mapping, if *there exist a sequence of regular* of regular quasiconformal mappings $f_n : D \to D'$ such that $f_n \rightrightarrows f$, and for a.e. z for which there exists finite partial derivatives $f_z, f_{\bar z}$ we have $f_{nz}/f_{n\bar z} \to f_z/f_z$.

Properties :

1° 1-qc mapping $f : D \to D'$ is identical with a class of conformal mappings,

2° if f is a Q-qc mapping then also f^{-1} is Q-qc mapping

3° if f_1 is a Q_1-qc mapping and f_2 is a Q_2-qc mapping then also $f_1 \circ f_2$ is a $Q_1 \cdot Q_2$-qc mapping

4° it is enough to consider classes of qc mapping on special domains like; unit disc Δ, an annulus Δ_N, etc. (canonical domain which are conformally equivalent to a class o domains).

Complex dilatation and Beltrami equation

If f is a sense-preserving homeomorphism differentiable almost everywhere on a set E, where E is either an open set or a closure of a Jordan domain, then the mapping given by it differential taken at a differentiable point $z_0 \in \mathrm{Int}\,E$ has the form

$$dw = f_z\,dz + f_{\bar z}\,d\bar z \tag{7}$$

and map a unit (infinitezimal) circle $|dz| = d\varrho$ onto the elipse with half-axis $a = |f_z| + |f_{\bar z}|$, $b = |f_z| - |f_{\bar z}|$ and the center at $f(z_0)$.

Writting down the dilatation p_f in the form

$$p_f = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} = \frac{1 + \left|\frac{f_{\bar z}}{f_z}\right|}{1 - \left|\frac{f_{\bar z}}{f_z}\right|} = \frac{1 + |\mu|}{1 - |\mu|},$$

where $\mu = f_{\bar z}/f_{\bar z}$, we attain at the complex function μ

which is called <u>complex dilatation</u> of f. If f is regular Q-qc mapping then $|\mu| < \frac{Q-1}{Q+1}$. Starting from a given complex dilatation we may consider the differential equation $f_{\bar z} = \mu f_z$ which is an elliptic type equation known as the Beltrami differential equation. This is when μ satisfies certain conditions. Confining ourselves to the case of the unit disc we presente the most general result of Bojarski which in our context is as

<u>Proposition 2</u>

If μ is a measurable function defined in Δ and such that $\|\mu\|_\infty = \inf_E \sup_{z\in\Delta\setminus E} |\mu(z)|$, where the infimum is taken over all set $E\subset$ with Lebesgue measure $|E|=0$, is bounded from 1 i.e. $\|\mu\|_\infty < 1$ Then there exist a unique solution $w=f(z)$ of

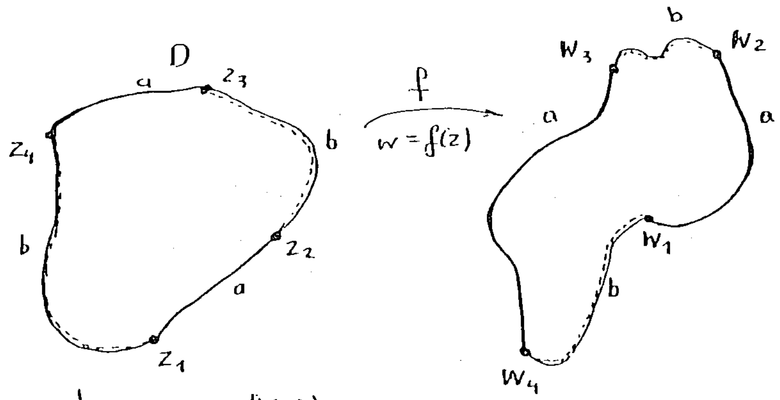
$$\text{(B)} \qquad f_{\bar z} = \mu f_z$$

which is a Q-qc selfmapping of Δ with $Q = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}$ and normalization condition $f(0)=0$, $f(1)=1$ or, equivalently $f(1)=1$, $f(i)=i$, $f(-1)=-1$.

<u>*Geometrical characterization of quasiconformal mappings.*</u>

A <u>quadrilateral</u> consist of a Jordan domain D and a sequence z_1, z_2, z_3, z_4 of boundary points of D. The points z_i, $i=1,2,3,4$ are called <u>vertices</u> of the quadrilateral. We confine ourselves to the only quadrilaterals $D(z_1, z_2, z_3, z_4)$ whose sequence of vertices agrees with positive orientation with respect to D. The vertices divide the boundary of D into four Jordan arcs, the sides of quadrilateral, the arcs $\widehat{z_1 z_2}$ and $\widehat{z_3 z_4}$ are called the <u>*a-sides*</u> and the other two arcs, the <u>*b-sides*</u> of D.

It is well known that image of D under sense-preserving homeomorphism is a quadrilateral.



$f(\partial D) = \partial D'$, $w_i = f(z_i)$

It is not, in general, possible to map given quadrilateral onto another, since the image of three boundary points determine the mapping uniqety. All quadrilaterals are therefore devoted into several classes which are conformally equivalent.

It follows from Riemann mapping theorem that every quadrilateral $D(z_1, z_2, z_3, z_4)$ can be mapped conformally onto a quadrilateral $H(-1/k, -1, 1, 1/k)$, where $0<k<1$ and H is the upper half-plane. From the classical theory of the elliptic integrals we furter obtain, that the function

$$(8) \qquad \zeta(z) = \int_0^z \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

maps $H(-1/k, -1, 1, 1/k)$ conformally onto a quadrilateral which consist of rectangle and its corners. Composing the above mappings, we can map an arbitrary quadrilateral conformally onto a rectangle. Such a mapping will be called *the canonical mapping*. All canonical rectangles of a given quadrilateral $D(z_1, z_2, z_3, z_4)$ have therefore the same ratio of sides $M(D) = \frac{a}{b}$ - conformal module of it.

Two quadrilaterals are conformally equivalent *if and only if* they possess the same ratio of a and b. If one carries out a reorregement of the vertices of D, the the module behaves as follows

$$M(D(z_1, z_2, z_3, z_4)) = M(D(z_3, z_4, z_1, z_2)) = 1/M(D(z_2, z_3, z_4, z_1)).$$

Definition.

A sense-preserving homeomorphism of E where E is either a domain or a closure of a Jordan domain, is called Q-qc mapping if there exist a finite number $Q \geq 1$ such that

$$(9) \qquad (1/Q)\, M(D) \leq M(f(D)) \leq Q \cdot M(D)$$

holds for all quadrilaterals D, $\mathrm{cl}D \subset E$.

Boundary values problem.

This is a problem of existence and continuity of the boundary values under quasiconformal mappings. But the solvability of boundar value problem means the possibility of constructing a quasiconforma mapping with given boundary values.

The simple example of the mapping $f(z) = z - \varrho(1-|z|^2)$, ϱ-suff. small shows that the identity mapping $f(e^{i\theta}) = e^{i\theta}$, $-\pi < \theta \leq \pi$ has many solution in the case of the unit disc, when the solution must be qc. It has, of coure, exactly one, when we confine ourselfs to the conformal automorphism.

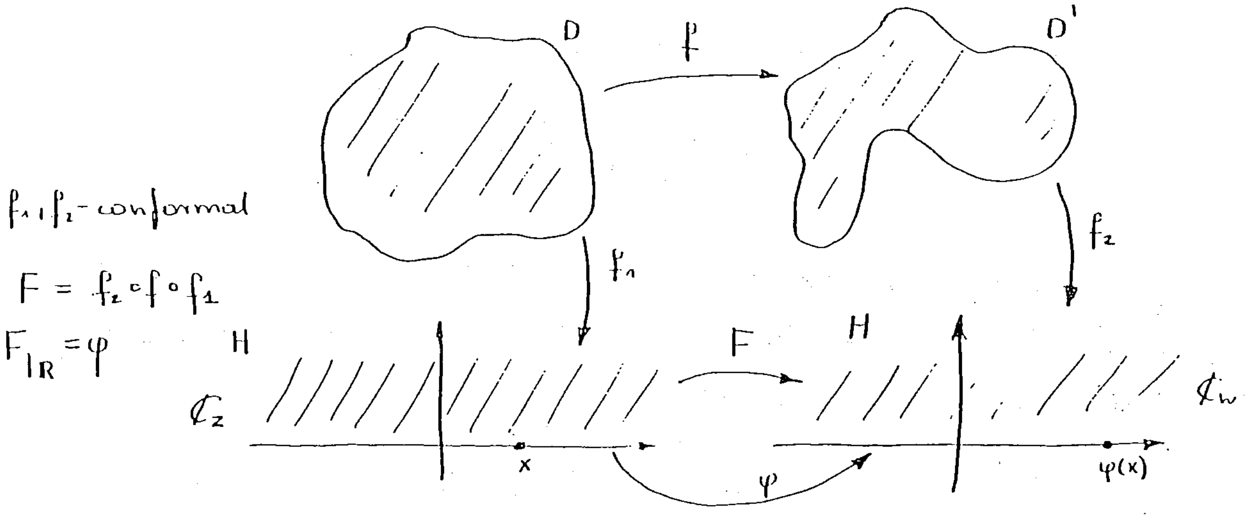
Let D and D' be n-tuply connected domains which both have n Jordan curves as boundary components. Then every conformal mapping $f: D \to D'$ can be extended to a homeomorphism of its closures. If these domains are Jordan domains and the orientation of z_1, z_2, z_3 and w_1, w_2, w_3 are both positive with respect to the corresponding domains, then there exist a unique conformal mapping $f: D \to D'$: $f_{|\mathrm{fr}\, D}$: $\mathrm{fr}\, D \to \mathrm{fr}\, D'$ - is the homeomorphism and $f(z_i) = w_i$, $i = 1,2,3$. For qc mapping we state the main result as

Proposition 3 If D and D' are Jordan domains then every Q-qc mapping $f: D \to D'$ can be extended to a homeomorphisn of $\mathrm{cl}D \to \mathrm{cl}D'$.

We restrict ourselves to Q-qc mapping between Jordan domains. It follows from Proposition 3 that every quasiconforma mapping can be extended to a homeomorphism between closure of these domains. By the orientation-preserving theoren (which is in our context as a proposition) the extension is sense-preserving, and the boundary value problem therefore reads as follows.

Let D and D' be two Jordan domains with boundries C and C' respectively, and φ be a homeomorphism of C onto C' which preserves the positive orientation. It is required to find necessary and sufficient condition on φ such that there exists a qc mapping $f: D \to D'$ with the boundary values $f(z) = \varphi(z)$.

In view of the invariance of qc mapping under compositio with conformal mappings the problem of characterizing the induced homeomorphism φ can be reduced to the case, when $D = D' = H = \{z : \operatorname{Im} z > 0\}$. Then the boundary correspondence is determined by a monotone continuous function φ in this sens that the point $(x, 0)$ corresponds to $(\varphi(x), 0)$. It is sufficiently to consider the case when φ is strictly increasin



By the definition of Q-quasiconformality based on the moduli of a quadrilateral we see that if F i a Q-qc self-mapping of the upper half-plane H then

$$\frac{1}{Q} M(G) \le M(f(G)) \le Q\, M(G)$$

holds for all quadrilaterals G, $\mathrm{cl}\, G \subset H$. Since H is the Jordan domain than we may consider this definition not only for a domain but also for closure of a Jordan domain. Then (9) must also be true when $G = H$. The condition (9) assumes a weaker form, better suited to our applications, if we fix the vertex z_4 of a quadrilateral and choose the other three vertices such that G is conformally equivalent to a square. Then $M(G) = 1$ so G' must satisfy the condition

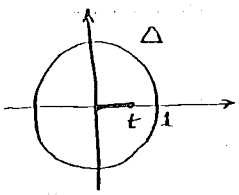
$$(9') \qquad \frac{1}{Q} \le M(G') \le Q \,.$$

On the other hand, this necessary condition is also sufficient in the following sense: it guarantees the existence of qc, though not necessarily Q-qc mapping. The order of quasiconformality may increase.

To transform condition (9') to a simple form we set $z_4 = \infty = x_4$. and normalize all our mapping so that they preserve the point at infinity. The quadrilateral $H(x_1, x_2, x_3, \infty)$ is conformally equivalent to a square if $x_3 - x_2 = x_2 - x_1$. Then we write $x_2 = x$, $x_1 = x + t$, $x_3 = x - t$, where $t > 0$ since the orientation is positive with respect to H. For the moduli of quadrilateral $H' = H(x_1', x_2', x_3', \infty)$ we have

$$(10) \qquad M(H') = \frac{2}{\pi}\mu\left(\sqrt{\frac{x_3' - x_2'}{x_3' - x_1'}}\right)$$

where $\mu(t)$ denotes the conformal moduli of the unit dis sliced along real line from 0 to t, $0 < t < 1$ and is strictly decreasing with limits ∞ at 0 and 0 at 1



$\mu(t)$ – module of double connected domain. which is $\Delta \setminus \{\langle 0, t\rangle\}$

The function μ can be also expressed in therms of elliptic integral $K(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$ and $\mu(t) = \frac{\pi}{2} \frac{K(\sqrt{1-t^2})}{K(t)}$.

If we set this value into (9') we obtain for the boundary values of Q-qc mapping $F: H \to H$, $F(\infty) = \infty$, the double inequality

$$\mu^{-1}\left(\frac{\pi Q}{2}\right) \leq \sqrt{\frac{\varphi(x+t) - \varphi(x)}{\varphi(x+t) - \varphi(x-t)}} \leq \mu^{-1}\left(\frac{\pi}{2Q}\right)$$

To obtain this condition in the symmetric form we write

$$(11) \qquad \lambda(Q) = \frac{1}{\left(\mu^{-1}\left(\frac{\pi Q}{2}\right)\right)^2} - 1$$

Then since $\mu\left(1/\sqrt{1+\lambda(Q)}\right) = \frac{\pi Q}{2}$ and $\mu\left(1/\sqrt{1+\lambda(1/Q)}\right) = \frac{\pi}{2Q}$, then $\mu\left(1/\sqrt{1+\lambda(Q)}\right) \cdot \mu\left(1/\sqrt{1+\lambda(1/Q)}\right) = \frac{\pi^2}{4}$ do not depends on Q and we may replace $\lambda(Q)$ by $1/\lambda(1/Q)$ by which $\lambda(1/Q) = 1/\lambda(Q)$.

Thus we obtain

Theorem

The boundary value of a Q-qc self-mapping F of the upper half-plane preserving the point at ∞ satisfy the double inequality

$$(12) \qquad 1/\lambda(Q) \leq \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \leq \lambda(Q)$$

for all real x and $t > 0$.

Solution of the boundary values problem

In 1956 Beurling and Ahlfors constructed a quasiconformal self-mapping of the upper half-plane with given boundary values But the construction is only possible when the boundary values satisfy a condition of (12) form. Then they have show that this condition is also sufficient, although the maximal dilatation of the constructed mapping turns out to be larger then the value arrising from (9). Moreover if $Q(\tilde{f})$ is the qc dilatation of f, then $Q(\tilde{f}) \geq 1 + 0.2284 \log \varrho$ for each
qc extension $\tilde{f}$ of a function satisfying (12) when ϱ is instead of λ(...

Theorem (Beurling - Ahlfors)

Let φ be a strictly increasing, continuous function of the real which satisfies the condition

$$(13) \qquad \frac{1}{\varrho} \le \frac{\varphi(x+t)-\varphi(x)}{\varphi(x)-\varphi(x-t)} \le \varrho$$

for every real $x, t>0$. Then there exist a qc selfmapping F^* of the upper half-plane, which has the boundary values $F^*_{|\mathrm{Fr}H} = F^*_{|\mathbb{R}} = \varphi$ and a maximal dilatation less then a bound dependin on ϱ.

Proof.

Let us consider a function of the form $F(z)=u+iv$, where $u=\frac{1}{2}(\alpha+\beta)$, $v=\frac{1}{2}(\alpha-\beta)$, where

$$(14) \qquad \alpha(x,y) = \int_0^1 \varphi(x+ty)\,dt \quad , \quad \beta(x,y) = \int_0^1 \varphi(x-ty)\,dt \,.$$

This function F^* is defined and continuous in the whole finite $\mathbb{C}_z$ plane. Since φ is strictly increasing, F^* corries every point of the upper half-plane $\operatorname{Im} z>0$ into the upper half-plane $\operatorname{Im} w>0$, where $w=u+iv$. On the real line $F^*(x)=\varphi(x)$ and at conjugate points $z=x+iy$, $\bar z = x-iy$ we have $F^*(\bar z) = \overline{F^*(z)}$. We have to prove that F^* is a quasiconformal mapping of the whole plane, then the image domain will also consist of the whole plane, and it follows from the above that F^* maps the upper-half plane H onto itself and has the right boundary values $\varphi(x)$.

We show first that F^* is a homeomorphism of the finite plane. It is enough to prove that F^* assumes every value at no more than one point. It is easy to see that the continuity of the inverse $(F^*)^{-1}$ of F^* will follow from the continuity of one-to-one character and continuity of F^*. Let z_1 and z_2 be two points such that $F^*(z_1)=F^*(z_2)$. It follows from the above that z_1 and z_2 lie in the same half-plane bounded by $y=0$. Since $F^*(\bar z) = \overline{F^*(z)}$ we may suppose that y_1 and y_2 are positive

Writing down (14) in the form

(15) $$\alpha(x,y) = \frac{1}{y}\int_x^{x+y} \varphi(\xi)\,d\xi\ , \quad \beta(x,y) = \frac{1}{y}\int_{x-y}^{x} \varphi(\xi)\,d\xi$$

we see that the mean values of φ are equal on the intervals (x_1, x_1+y_1) and (x_2, x_2+y_2) as well as on (x_1-y_1, x_1) and (x_2-y_2, x_2). Since φ is strictly increasing, one of the first mentioned intervals must be contained in the other, and the same holds for the last two. If, for example $x_2 \geq x_1$ it follows that $x_2+y_2 \leq x_1+y_1$ and $x_2-y_2 \leq x_1-y_1$. This implies that $z_1 = z_2$, so the mapping F^* is a homeomorphism, becouse local homeomorphism of the closed $\mathbb{C}$ onto itself is the global one.

Now we show that F^* is quasiconformal in the upper half-plan From (15) it follows that α and β are continuously differentiable and have the partial derivetives

$$\alpha_x(x,y) = \frac{1}{y}\left(\varphi(x+y) - \varphi(x)\right), \quad \beta_x(x,y) = \frac{1}{y}\left(\varphi(x) - \varphi(x-y)\right),$$

$$\alpha_y(x,y) = \frac{1}{y}\left(\varphi(x+y) - \alpha(x,y)\right), \quad \beta_y(x,y) = \frac{1}{y}\left(\varphi(x-y) - \beta(x,y)\right)$$

Since φ is strictly increasing then

$$\alpha_x > 0\ , \quad \beta_x > 0\ , \quad \alpha_y > 0\ , \quad \beta_y < 0\ .$$

The Jacobian $J = \frac{1}{2}(\alpha_y\beta_x - \alpha_x\beta_y)$ of F^* is therefore positive. Hence all points of the upper half-plane are regular for F^*.

To obtain the uper bound for the dilatation quotient $p_{F^*} = (|F_z| + |F_{\bar z}|)/(|F_z| - |F_{\bar z}|)$ we write

$$p_{F^*} \leq 2\,\frac{|F_z|^2 + |F_{\bar z}|^2}{|F_z|^2 - |F_{\bar z}|^2} = \frac{\alpha_x^2+\beta_x^2+\alpha_y^2+\beta_y^2}{\alpha_y\beta_x - \alpha_x\beta_y} = \frac{\alpha_x\beta_x}{-\alpha_y\beta_y}\cdot\frac{\dfrac{\alpha_x}{\beta_x} + \dfrac{\alpha_y^2}{\alpha_x\beta_y} + \dfrac{\beta_x}{\alpha_x} + \dfrac{\beta_y^2}{\alpha_x\beta_x}}{-\dfrac{\beta_x}{\beta_y} + \dfrac{\alpha_x}{\alpha_y}}\ .$$

To estimate the partial derivatives of α and β we make use of the assumption (13).

From

$$\varphi(x+\tfrac{1}{2}y)-\varphi(x) \le \rho\left(\varphi(x+y)-\varphi(x+\tfrac{1}{2}y)\right)$$

it follows that

$$\varphi(x+y)-\varphi(x) \le (1+\rho)\left(\varphi(x+y)-\varphi(x+\tfrac{1}{2}y)\right).$$

The the double inequality

$$\alpha_x(x,y) > \alpha_y(x,y) = \frac{1}{y}\int_0^1 \left(\varphi(x+y)-\varphi(x+ty)\right)dt \geq$$

$$\geq \frac{1}{2y}\left(\varphi(x+y)-\varphi(x+\tfrac{1}{2}y)\right) \geq \frac{\alpha_x(x,y)}{2(1+\rho)} .$$

In the same manner we obtain

$$\beta_x(x,y) > -\beta_y(x,y) \geq \frac{1}{2y}\left(\varphi(x-\tfrac{1}{2}y)-\varphi(x-y)\right) \geq \frac{\beta_x(x,y)}{2(1+\rho)} .$$

Finally, it follows from (13) that

$$\frac{\beta_x}{\rho} \le \alpha_x \le \rho\beta_x$$

and by combining these inequalities we arrive at

$$P_{F^*} \le 8\rho(1+\rho)^2. \tag{16}$$

Because of the symmetry property of $F^*(\bar z)=\overline{F^*(z)}$ it follow that (16) holds in the lower half-plane also. By this F^* is therefore a quasiconformal mapping of the whole plane with maximal dilatation not exceeding $8\rho(1+\rho)^2$.

The bound $8\rho(1+\rho)$ was several times improved but the last know as the good (still not the best one) has form $\min\{\rho^{\frac{3}{2}}, 2\rho-1\}$ (Lehtinen).

Quasisymmetric functions.

The condition (13) indicates that φ possesses a degree of approximate symmetry, and for this reasone we refer to φ as a quasisymmetric function (mapping), when $\inf \rho =$ is called the dilatation of φ.

Qualitatively, the properties of ρ-qs functions parallel rather closely to those of quasiconformal mappings. However, they have some shortcomings not shared by qc mappings. We list a number of very basic properties:

1° 1-qs function is linear mapping,

2° the composition of a ρ-qs function with a linear function yields a ρ-qs function

3° an unfortunate feature of qs functions is that the dilatation of a function is not necessarily the same as for its inverse, nor is dilatation of a composed function bounded by product of the dilatations whereas both conditions hold for quasiconformal mappi For example, if $\varphi(x) = x^{\alpha}$ $(x>0)$, then $\rho(f) = 2^{\alpha}-1$, if $\alpha>1$ and $\rho(f) = 1/2^{\alpha}-1$ if $0<\alpha<1$. Hence, if $\alpha = 2$, then $\rho(f)=3$ and $\rho(f^{-1}) = \sqrt{2}+1$. Similiarly $\rho(f(f)) = 15$ but $\rho(f)\cdot\rho(f) = 9$ But the exact bounds for $\rho(f^{-1})$ and $\rho(f\circ g)$ are alrea known

4. Bojarski has pointed out that there exist $p>1$ such that the Jacobian of each Q-qk mapping is locally p-integrabl The result is false for quasisymmetric function and says that for each $\rho>1$, each $p>1$, and each compact set E of positive measure, there exists a function φ, ρ-qs on the entire line, such that

$$\int_E [\varphi'(x)]^p dx = \infty$$

Sapporo, 17 June, 1989