A MODEL FOR PREDICTION OF ANISOTROPIC BEHAVIOR OF SOFT TISSUES UNDER LOADING

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ABSTRACT

Many materials which are widely used in industries such as building, aircraft, aerospace, marine and biological technology can undergo large nonlinear elastic deformations and have inherent anisotropic characteristics. To be able to utilize these materials efficiently and economically, better analytical methods and designs are needed. Hyperelastic constitutive modeling has appeared as an effective tool in continuum mechanics for characterizing large deformation materials. In this paper a hyperelastic model proposed for anisotropic materials is applied for example to soft tissues at large strains. The model is an extension of a model successfully used for the analysis of isotropic hyperelastic materials. The formulation is based on the invariant theory and polyconvexity. The model is decomposed into an isotropic part and an anisotropic part. The numerical results show a very good agreement with the experimental results.

Keywords: soft tissues, anisotropy, hyperelasticity, finite strain.

1. INTRODUCTION

It is well known that soft biological tissues can undergo large nonlinear elastic deformations and have inherent anisotropic characteristics (Fung 1993). Modeling of these materials has received increasing interest from research community due to an increase in the use of virtual human modeling.Hyperelastic constitutive modeling has appeared as an effective tool for characterizing complex behavior of soft tissue (Spencer 1980). Behavior of hyperelastic materials are generally described by a strain energy function. The constitutive stress-strain relations and elastic moduli can be derived directly from a strain energy function. A strain energy function can be derived by using phenomenological or microstructural approaches. Although the parameters in microstructural models have a physical meaning, identification of these parameters might be a problem. In general, the macroscopic behavior of soft tissues can be satisfactorily described by phenomenological models (Ehret and Itskov 2007). Phenomenological models have thus widely been adopted.

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In this paper a hyperelastic model proposed for anisotropic materials is presented. The model is an extension of a model proposed by Attard (2003) which has been successfully used for the analysis of isotropic hyperelastic materials (Attard and Hunt 2004). The model is formulated within the framework of the invariant theory by representing the anisotropy using an isotropic tensor function through the so-called structural tensors and is based on polyconvexity and coercivity conditions so as to guarantee the existence of solutions. In order to include the isotropic model, the model is decomposed into an isotropic and anisotropic component. For example, the model is fitted to uniaxial tension tests of soft tissues. The model can be classified as a phenomenological model.

2. HYPERELASTICITY

A hyperelastic material is a material with elastic behavior that can be described by a strain energy function with respect to the initial volume in terms of the deformation gradient $W = W(\mathbf{F})$. According to the objectivity condition the strain energy function has to be independent of superposed rigid body motions. This condition can automatically be met by representing the strain energy function in terms of the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ so that (see e.g. Truesdell and Noll 1965)

$$W = W\left(\mathbf{F}\right) = W(\mathbf{C}). \tag{1}$$

The condition of material symmetry requires that the strain energy function be invariant under transformations with elements of the material symmetry group that describe the anisotropic class of the material. According to Rychlewski's theorem (see e.g. Zhang and Rychlewski 1990), this condition is satisfied if and only if the strain energy can be represented as an isotropic tensor-valued function of arguments containing the so-called structural tensors which reflect the symmetry group of the material. The strain energy function has thus to satisfy

$$W = W(\mathbf{C}, \mathbf{G}_i) = W(\mathbf{QCQ}^T, \mathbf{QG}_i\mathbf{Q}^T), \tag{2}$$

where $\mathbf{G}_i, i = 1, 2, ..., n$ are the structural tensors and $\mathbf{Q}$ is an element of the material symmetry group and is a proper orthogonal tensor. The anisotropy can be characterized by certain directions, lines or planes associated with some unit vectors defined in the undeformed state (Zheng and Boehler 1994). This leads to a definition for $\mathbf{G}_i$ of the form

$$\mathbf{G}_i = \mathbf{m}_i \otimes \mathbf{m}_i, \tag{3}$$

where each $\mathbf{m}_i$ is a unit vector. The $\mathbf{G}_i$ have the property

$$\mathrm{tr}\left[\mathbf{G}_i\right] = 1. \tag{4}$$

According to the invariant theory, the invariants of $\mathbf{C}$ and $\mathbf{G}_i$ are required. For the formulation of the strain energy function, only a set of invariants is determined (Schröder and Neff 2003). Due to the property of the structural tensors in equation (4), the strain energy function can be expressed in terms of the invariants of the argument tensors $(\mathbf{C}, \mathbf{G}_i)$ as

$$W = W(I_1, I_2, I_3, J_{4i}, J_{5i}), \tag{5}$$

where I_1, I_2 and I_3 are the principal invariants of $\mathbf{C}$, J_{4i} and J_{5i} are the mixed invariants for $\mathbf{C}$ and $\mathbf{G}_i$. They have the explicit expressions as, where λ_{pi} are the principal stretches,

$$I_1 = \mathrm{tr}[\mathbf{C}] = (\lambda_{p1})^2 + (\lambda_{p2})^2 + (\lambda_{p3})^2, \tag{6}$$

$$I_2 = \mathrm{tr}\left[\mathrm{Cof}[\mathbf{C}]\right] = (\lambda_{p1}\lambda_{p2})^2 + (\lambda_{p2}\lambda_{p3})^2 + (\lambda_{p3}\lambda_{p1})^2, \tag{7}$$

$$I_3 = \det[\mathbf{C}] = (\lambda_{p1}\lambda_{p2}\lambda_{p3})^2 \tag{8}$$

$$J_{4i} = \mathrm{tr}[\mathbf{C}\mathbf{G}_i], \quad J_{5i} = \mathrm{tr}\left[\mathbf{C}^2\mathbf{G}_i\right]. \tag{9}$$

3. POLYCONVEX STRAIN ENERGY FUNCTIONS

The polyconvexity condition in the sense of Ball (1997) has been proved to be able to serve for both sequentially weakly lower semicontinuous and coercivity conditions which guarantee the existence of solutions. Furthermore, ellipticity condition is also guaranteed (Schröder and Neff 2003). Hence, the proposed strain energy function is formulated on the basis of the polyconvexity.

3.1. Polyconvexity

A strain energy function is said to be polyconvex if and only if there exists a convex function with the arguments of $\mathbf{F}$, $\mathrm{Cof}[\mathbf{F}]$ and $\det[\mathbf{F}]$ in such a way that the strain energy function $W = W(\mathbf{F})$ can satisfy

$$W = W(\mathbf{F}) = W\left(\mathbf{F}, \mathrm{Cof}[\mathbf{F}], \det[\mathbf{F}]\right). \tag{10}$$

A subclass of the polyconvexity in equation (10) is the additive polyconvex functions of the form

$$W\left(\mathbf{F}, \mathrm{Cof}[\mathbf{F}], \det[\mathbf{F}]\right) = W_1(\mathbf{F}) + W_2\left(\mathrm{Cof}[\mathbf{F}]\right) + W_2\left(\det[\mathbf{F}]\right). \tag{11}$$

If each $W_i, i = 1, 2, 3$ is convex then the strain energy function is polyconvex (Schröder and Neff 2003). That means the definition of the polyconvexity requires the convexity properties of the arguments of the strain energy function equation (5). Indeed it can be shown that the invariants

I_1, I_2, I_3 and J_{4i} are convex with respect to $\mathbf{F}$, $\mathrm{Cof}[\mathbf{F}]$, $\det[\mathbf{F}]$ and $\mathbf{F}$, respectively, but the invariant J_{5i} is not convex with respect to $\mathbf{F}$ (Schröder and Neff 2003) and a convex mixed invariant derived by use of the Cayley-Hamilton theorem has to be used instead. The convex mixed invariant can be expressed in terms of J_{5i} and other invariants as

$$K_{5i} = \mathrm{tr}\left[\mathrm{Cof}\left[\mathbf{C}\right]\mathbf{G}_i\right] = J_{5i} - I_1 J_{4i} + I_2 \mathrm{tr}\left[\mathbf{G}_i\right]. \tag{12}$$

Therefore, to satisfy the polyconvexity condition, the expression for the strain energy function in equation (5) is replaced by

$$W = W(I_1, I_2, I_3, J_{4i}, K_{5i}) . \tag{13}$$

The additive representation of the polyconvexity, equation (11), is then utilized to additively decompose the strain energy function, equation (13), into an isotropic part W_{iso} and an anisotropic part W_{aniso}. This decomposition of the strain energy function allows a variety of combinations of the two parts. In addition, the two parts are associated with scalar weight factors w_i representing a dispersion of components as experimentally observed (Gasser et al. 2006), i.e.

$$W = w_0 \cdot W_{\mathrm{iso}}\left(I_1, I_2, I_3\right) + \sum_{r=1}^{m}\sum_{i=1}^{n} w_i^r \cdot W_{\mathrm{aniso}}^r\left(I_3, J_{4i}, K_{5i}\right), \qquad \sum_{r=1}^{m}\sum_{i=1}^{n} w_i^r = 1 - w_0 \tag{14}$$

$$W_{\mathrm{iso}}\left(I_1, I_2, I_3\right) = W_{\mathrm{iso}}^{\mathrm{inc}}\left(I_1, I_2, I_3\right) + W_{\mathrm{iso}}^{\mathrm{com}}\left(I_3\right), \tag{15}$$

$$W_{\mathrm{aniso}}\left(I_3, J_{4i}, K_{5i}\right) = \sum_{r=1}^{m}\sum_{i=1}^{n} D^r\left[W_{\mathrm{aniso}}^{Ir}\left(I_3\right) + W_{\mathrm{aniso}}^{Jr}\left(J_{4i}\right) + W_{\mathrm{aniso}}^{Kr}\left(K_{5i}\right)\right], \tag{16}$$

where $W_{\mathrm{iso}}^{\mathrm{inc}}$ is an incompressible component associated with constrained volume change or volume constant distortion, $W_{\mathrm{iso}}^{\mathrm{com}}$ is a compressible component associated with specific volume change and D^r are material constants.

3.2. Natural state conditions

In this section, the isotropic and anisotropic stain energy functions are analyzed with respect to the natural state conditions, i.e. the stress and energy have to be zero in the undeformed configuration.

3.2.1. Isotropic strain energy function

For W_{iso} of equation (15), the second Piola-Kirchhoff stress tensor are given by

$$\boldsymbol{\pi}_{\text{iso}} = 2\frac{\partial W_{\text{iso}}}{\partial \mathbf{C}} = 2\left[\left(\frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_1} + \frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_2} I_1\right)\mathbf{I} - \frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_2}\mathbf{C} + \frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_3} I_3 \mathbf{C}^{-1} + \frac{\partial W_{\text{iso}}^{\text{com}}}{\partial I_3} I_3 \mathbf{C}^{-1}\right], \tag{17}$$

with $\partial I_1 / \partial \mathbf{C} = \mathbf{I}$, $\partial I_2 / \partial \mathbf{C} = I_1 \mathbf{I} - \mathbf{C}$ and $\partial I_3 / \partial \mathbf{C} = I_3 \mathbf{C}^{-1}$. In order to consider the stress condition for the natural state, $\mathbf{C} = \mathbf{I}$ is set and $\boldsymbol{\pi}_{\text{iso}}\big|_{\mathbf{C}=\mathbf{I}} = \mathbf{0}$ is required. Thus, the stress-free conditions are

$$\frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_1} + 2\frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_2} + \frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_3} = -\frac{\partial W_{\text{iso}}^{\text{com}}}{\partial I_3}. \tag{18}$$

For the energy condition at undeformed configuration, $\mathbf{C} = \mathbf{I}$ is set and $W_{\text{iso}}\big|_{\mathbf{C}=\mathbf{I}} = \mathbf{0}$ is required. Thus, the energy-free conditions are

$$W_{\text{iso}}^{\text{inc}}\left(3,3,1\right) = W_{\text{iso}}^{\text{com}}(1) = 0 \tag{19}$$

3.2.2. Anisotropic strain energy function

For W_{aniso} of equation (16), the second Piola-Kirchhoff stress tensor are given by

$$\boldsymbol{\pi}_{\text{aniso}} = 2\frac{\partial W_{\text{aniso}}}{\partial \mathbf{C}} = 2\sum_{r=1}^{m}\sum_{i=1}^{n} D^r \left[\left(\frac{\partial W_{\text{aniso}}^{Ir}}{\partial I_3} I_3 + \frac{\partial W_{\text{aniso}}^{Kr}}{\partial K_{5i}} K_{5i}\right)\mathbf{C}^{-1} + \frac{\partial W_{\text{aniso}}^{Jr}}{\partial J_{4i}}\mathbf{G}_i - \frac{\partial W_{\text{aniso}}^{Kr}}{\partial K_{5i}} I_3 \mathbf{C}^{-1}\mathbf{G}_i\mathbf{C}^{-1}\right] \tag{20}$$

with $\partial I_3 / \partial \mathbf{C} = I_3 \mathbf{C}^{-1}$, $\partial J_{4i} / \partial \mathbf{C} = \mathbf{G}_i$ and $\partial K_{5i} / \partial \mathbf{C} = K_{5i}\mathbf{C}^{-1} - I_3 \mathbf{C}^{-1}\mathbf{G}_i\mathbf{C}^{-1}$. For the stress condition at the natural state $\mathbf{C} = \mathbf{I}$ is set and $\boldsymbol{\pi}_{\text{aniso}}\big|_{\mathbf{C}=\mathbf{I}} = \mathbf{0}$ is required. Thus, the stress-free conditions are

$$\frac{\partial W_{\text{aniso}}^{Kr}\left(1\right)}{\partial K_{5i}} = \frac{\partial W_{\text{aniso}}^{Jr}\left(1\right)}{\partial J_{4i}} = -\frac{\partial W_{\text{aniso}}^{Ir}\left(1\right)}{\partial I_3} = 1, \quad r = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, n \tag{21}$$

For the energy-free reference configuration condition, $\mathbf{C} = \mathbf{I}$ is set and $W_{\text{aniso}}\big|_{\mathbf{C}=\mathbf{I}} = \mathbf{0}$ is required. Thus, the energy-free conditions are

$$W_{\text{aniso}}^{Ir}\left(1\right) = W_{\text{aniso}}^{Jr}\left(1\right) = W_{\text{aniso}}^{Kr}\left(1\right) = 0 \tag{22}$$

4. NUMERICAL EXAMPLES

The proposed model equation (14) is specified and applied to a set of experimental data on human arterial tissue performed by Holzapfel et al. (2005). Arteries were split into the adventitial, medial and intimal layer and tested in cyclic uniaxial quasi-static tension. The load was applied such that the principal axes of deformation coincided with the circumferential, axial and radial direction of the vessel. The stress response for loading in circumferential and axial direction was recorded.

In order to describe the mechanical behavior of the tissue samples, an incompressible fiber-reinforced composite is often assumed for each layer with two mechanically equivalent families of collagen fibers that form symmetrical helices tilted by an angle ±φ against the circumferential direction (Ehret and Itskov 2007, Holzapfel et al. 2005). This assumption is also adopted in this study. The orientations of the two fiber families can be given by the vectors

$$\mathbf{m}_1 = \cos\varphi\mathbf{e}_\theta + \sin\varphi\mathbf{e}_z, \quad \mathbf{m}_2 = \cos\varphi\mathbf{e}_\theta - \sin\varphi\mathbf{e}_z, \tag{23}$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are unit vectors in the circumferential and axial direction of the artery, respectively. For the isotropic term, the function proposed by Attard (2003) is adopted herein. The function and their derivatives which satisfies equations (18) and (19) are in the form

$$W_{\text{iso}}^{\text{inc}}(I_1, I_2, I_3) = \frac{A_1}{2}(I_1 - 3) + \frac{B_1}{2}\left(\frac{I_2}{I_3} - 3\right), \quad \frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_1} = \frac{A_1}{2}, \quad \frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_2} = \frac{B_1}{2I_3}, \quad \frac{\partial W_{\text{iso}}^{\text{inc}}}{\partial I_3} = -\frac{B_1 I_2}{2I_3^2}, \tag{24}$$

where A_1, B_1 and C_1 are material constants. For the anisotropic term, with the incompressibility constraint $I_3 = 1$, the W_{aniso} in equation (16) becomes

$$W_{\text{aniso}}^{\text{inc}}\left(J_{4i}, K_{5i}^{\text{inc}}\right) = \sum_{r=1}^{m}\sum_{i=1}^{n} D^r \left[W_{\text{aniso}}^{Jr}\left(J_{4i}\right) + W_{\text{aniso}}^{Kr}\left(K_{5i}^{\text{inc}}\right)\right], \tag{25}$$

$$K_{5i}^{\text{inc}} = \text{tr}\left[\mathbf{C}^{-1}\mathbf{G}_i\right] \tag{26}$$

where the invariants K_{5i}^{inc} result from K_{5i} under the constraint $I_3 = 1$. And the exponential functions are utilized herein. The function and their derivatives which satisfies equations (21) and (22) are in the form

$$\begin{aligned} W_{\text{aniso}}^{Jr}(J_{4i}) &= \frac{1}{A_r}\left[e^{A_r(J_{4i}-1)} - 1\right], \quad \frac{\partial W_{\text{aniso}}^{Jr}}{\partial J_{4i}} = e^{A_r(J_{4i}-1)}, \\ W_{\text{aniso}}^{Kr}(K_{5i}^{\text{inc}}) &= \frac{1}{B_r}\left[e^{B_r\left(K_{5i}^{\text{inc}}-1\right)} - 1\right], \quad \frac{\partial W_{\text{aniso}}^{Kr}}{\partial K_{5i}^{\text{inc}}} = e^{B_r\left(K_{5i}^{\text{inc}}-1\right)}. \end{aligned} \tag{27}$$

Applying equations (3), (9) and (26), the two generalized invariants can be expressed in terms of the principal stretches under the incompressibility constraint $I_3 = \lambda_\theta\lambda_z\lambda_r = 1$ as

$$J_{4i} = \lambda_\theta^2 \cos^2\varphi + \lambda_z^2 \sin^2\varphi,$$
$$K_{5i}^{\text{inc}} = \lambda_\theta^{-2} \cos^2\varphi + \lambda_z^{-2} \sin^2\varphi, \tag{28}$$

where λ_θ, λ_z and λ_r are the principal stretches in circumferential, axial and radial direction, respectively. Considering one single term *m* = 1 and *n* = 2 with taking into account the mechanical equivalence of the fiber families, so that $w_2^1 = w_1^1$, $J_{42} = J_{41}$ and $K_{52}^{\text{inc}} = K_{51}^{\text{inc}}$, the strain energy function is then given by

$$W = \left(1 - 2w_1^1\right)\left[\frac{A_1}{2}\left(I_1 - 3\right) + \frac{B_1}{2}\left(\frac{I_2}{I_3} - 3\right)\right] + 2w_1^1 D^1\left[\frac{1}{A_1}\left(e^{A_1\left(J_{41}-1\right)} - 1\right) + \frac{1}{B_1}\left(e^{B_1\left(K_{51}^{\text{inc}}-1\right)} - 1\right)\right]. \tag{29}$$

For uniaxial tension test, the lateral directions are stress free and the Cauchy stresses in circumferential and axial direction can be calculated, respectively, by

$$\sigma_\theta = \lambda_\theta \frac{\partial W}{\partial \lambda_\theta}, \qquad \sigma_z = \lambda_z \frac{\partial W}{\partial \lambda_z}. \tag{30}$$

The model was fitted and compared with the experimental results. The five model parameters (A_1, B_1, D_1, w_1^1 and φ) for each layer were determined by using a least squares regression analysis of the experimental data implemented using the commercial package MATLAB. The comparisons of the results as Cauchy stress versus stretch diagrams are shown in Figure 1. The values of the model parameters as well as the sum squared error (SSE) and R-square (R^2) obtained from the fittings are also presented in Figure 1. For all three layers, i.e. intimal, medial and adventitial layer, the comparisons are in good agreement with the experiemental results with R^2 = 0.9978, 0.9995 and 0.9981, respectively.

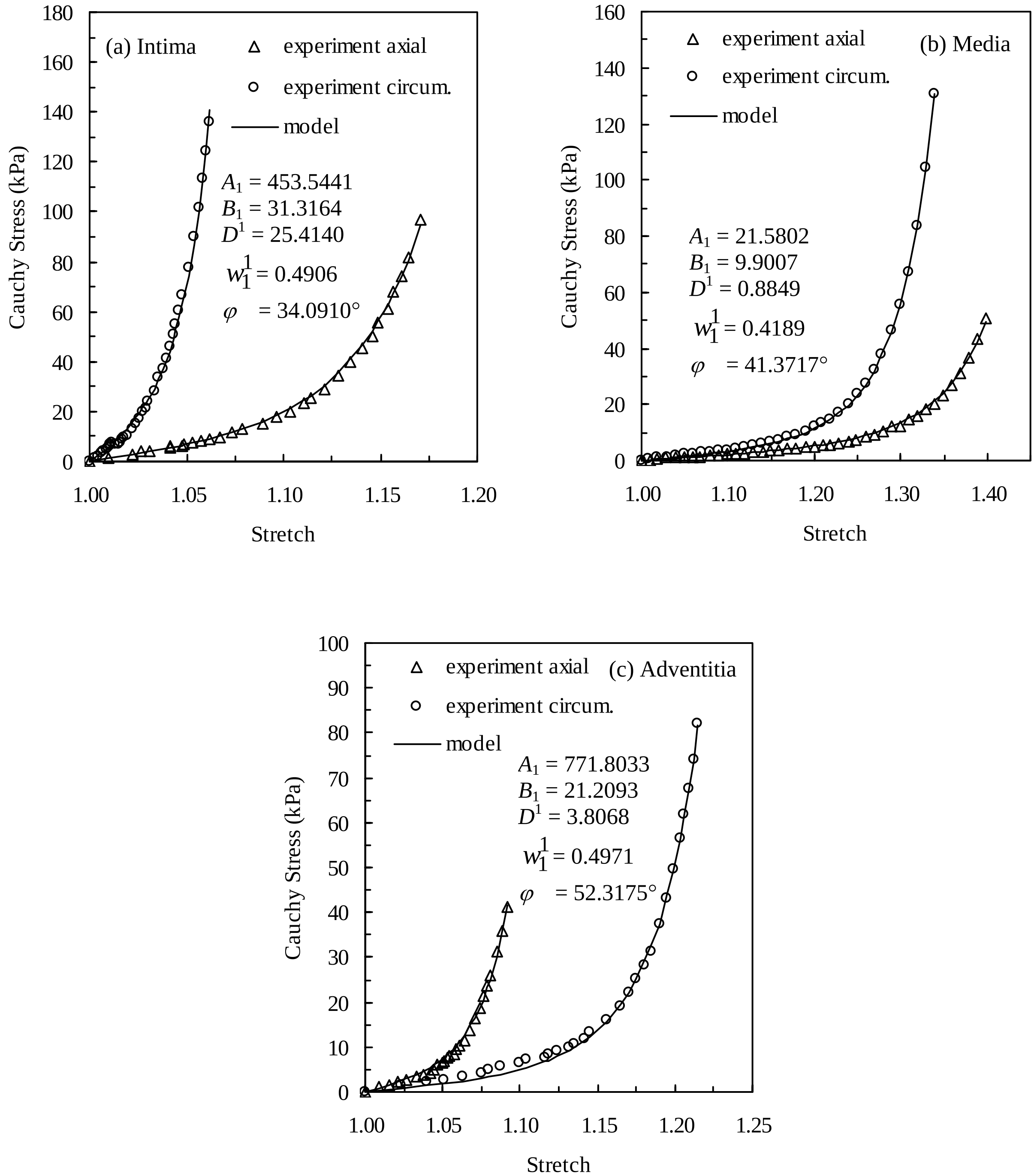

Figure 1: Comparison of the propose model with the experimental data.

5. CONCLUSIONS

In this paper a hyperelastic model for anisotropic materials has been formulated and proposed. The proposed strain energy function is decomposed into an isotropic part and an anisotropic part which allows for a variety of combinations of isotropic and anisotropic functions. The model is based on the framework of the invariant theory and polyconvexity which guarantee the existence of solutions. For example, the model was able to describe the hyperelastic behavior in soft tissues under uniaxial tension tests. With a few model parameters, the numerical results showed a good agreement with the experimental results.

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