第１５回偏微分方程式論
札幌シンポジウム

（代表者 上見　練太郎）

予稿集

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第１５回偏微分方程式論
札幌シンポジウム

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代表者　上見　練太郎

記

１．　日　時　　１９９１年2月１４日（木）～ 2月１６日（土）

２．　場　所　　北海道大学理学部数学教室　４－５０８室

３．　講　演

２月１４日（木）　　ページ

９：３０～１０：３０　Solonnikov, V. A.（ﾚｰﾆﾝｸﾞﾗｰﾄﾞ.ｽﾁｪｸﾛﾌ研）

Free boundary problem of Navier-Stokes equation　・・・

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Uniqueness and Widder's theorems for the heat equation on Riemannian manifolds

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1 Introduction

Let (M, g) be a complete Riemannian manifold, and Δ be the Laplace-Beltrami operator on M. We consider the uniqueness of solutions for the heat equation

$$u_t = \Delta u \tag{1}$$

on $M \times (0, T)$. Equivalently we discuss whether $u \equiv 0$ is the unique solution of (1) satisfying

$$u(x, 0) \equiv 0. \tag{2}$$

As well known, in case $M = \mathbf{R}^n$ with the standard metric, the answer is negative unless we impose some additional assumption on u. For instance Tychonoff [8] showed the following result.

Theorem 1. *Let M be $\mathbf{R}^n$, and $u \in C(M \times [0, T))$ be a solution of (1) – (2). If u satisfies*

$$|u(x, t)| \leq \exp\{C(1 + |x|^2)\}, \tag{3}$$

then $u \equiv 0$.

In respect of the square power this result is the best possible.
Subsequently Widder [9] studied the uniqueness of non-negative solution.

Theorem 2. *Let M be $\mathbf{R}^n$, and $u \in C(M \times [0, T))$ be a non-negative solution of (1) (2). Then $u \equiv 0$.*

Strictly speaking, in [8] and [9] they discussed in case $n = 1$, however, we can show results in the multi-dimensional case.

For general manifolds we know the uniqueness of bounded solutions [2, 3, 4]. For example,

Theorem 3. *Let (M, g) be an n-dimensional complete Riemannian manifold, where $n \geq 2$. We denote by $V_p(R)$ the volume of the geodesic ball of radius R centered at $p \in M$. We assume that there exist $p \in M$ and $C > 0$ such that*

$$V_p(R) \leq \exp\{C(1 + R^2)\}$$

holds for any $R > 0$. If $u \in C(M \times [0, T))$ is a bounded (weak) solution of (1) – (2), then $u \equiv 0$.

Theorems 1 and 3 are apparently independent, however, they have a similarity, that is, the right-hand sides of growth conditions are in the form of exp{quadratic expression of distance (or norm)}.

Our aim is twofold. One is to establish the uniqueness result from which Theorems 1 and 3 follow as the special cases, and the other is to study the condition on M such that Widder's Theorem 2 is valid.

2 Results

We can show the following. For details see [7].

Theorem 4. *We assume that $u \in C(M \times [0, T))$ satisfies (2) and either*

$$u_t = \Delta u \qquad or \qquad u_t \leq \Delta u, \quad u \geq 0$$

in the weak sense. If there exist $p \in M$, $k \geq 2$, and $C > 0$ such that

$$\int_{B_p(R+1) \setminus B_p(R)} |u(x,t)|^k dv_g \leq \exp\{C(1 + R^2)\}$$

holds for any $R > 0$, then $u \equiv 0$. Here dv_g is the volume element of (M, g).

This is our main theorem. We sketch its proof. Without loss of generality u may be assumed real-valued. Let $q \in M$ and $t \in (0, T_0)$ ($T_0 = \min\{T, 1/8C\}$). Our purpose is to show $u(q,t) = 0$. We compute the right-hand side of

$$0 \leq \int_0^T \int_M \varphi_R^2(x) \exp\{g(x,s)\} u(x,s) \{\Delta u(x,s) - u_s(x,s)\} dv_g ds$$

by the integration by parts. Here

$$g(x,s) = -\frac{d^2(q,x)}{4(2t-s)} \qquad (x \in M, \quad 0 \leq s \leq t, \quad d = \text{the distance function}),$$

and $\varphi_R(x)$ is a cut-off function having properties

$$\varphi_R(x) = \begin{cases} 0 & \text{on} \quad M \setminus B_p(R+1), \\ 1 & \text{on} \quad B_p(R), \end{cases}$$

$$0 \leq \varphi_R(x) \leq 1,$$

$$|\nabla \varphi_R(x)| \leq 3.$$

We choose $R > \max\left\{\sqrt{t/4}, d(p,q)\right\}$. and then we get

$$\int_{B_q(\sqrt{t/4})} u^2(x,\tau) dv_g \leq 36 \exp\left\{\frac{1}{16} - \frac{R^2}{8t}\right\} \int_0^t \int_{B_q(R+1)\setminus B_q(R)} u^2(x,s) dv_g ds.$$

On the other hand, Moser's [6, Theorem 3] iteration scheme asserts that

$$u^2(q,t) \leq C(q,t) \int_0^t \int_{B_q(\sqrt{t/4})} u^2(x,\tau) dv_g d\tau.$$

Combining these estimates with our assumption of Theorem, we have

$$\begin{aligned} u^2(q,t) \quad &\leq \quad 36C(p,q,t) \exp\left\{\frac{1}{16} - \frac{R^2}{8t} + C\left\{1 + (R + d(p,q) + 1)^2\right\}\right\} \\ &\to \quad 0 \qquad \text{as} \qquad R \to \infty. \end{aligned}$$

Here we use $t < 1/8C$. The step by step argument yields our assertion. □

It is obvious that Theorem 4 implies Theorems 1 and 3. We can obtain the maximum principle from this result. Assume that $u \in C(M \times [0,T))$ is a weak subsolution of the heat

equation, and that its initial value $u(x,0)$ is bounded from above. Let define a function v by

$$v(x,t) = \max\left\{u(x,t) - \sup_{x\in M} u(x,0), 0\right\}.$$

It is easy to see that $v^{k/2}$ $(k \geq 2)$ is a non-negative weak subsolution with zero initial value. Therefore we have the following fact.

Theorem 5. *We assume that $u \in C(M \times [0,T))$ satisfies*

$$u_t \leq \Delta u$$

in the weak sense, and that $u(x,0)$ is bounded from above. Let v be as above. If there exist $p \in M$, $k \geq 2$, and $C > 0$ such that

$$\int_{B_p(R+1)\setminus B_p(R)} v(x,t)^k dv_g \leq \exp\{C(1+R^2)\}$$

holds for any $R > 0$, then

$$u(x,t) \leq \sup_{x\in M} u(x,0).$$

As an application of the maximum principle to non-linear problems, we can establish the uniqueness of solutions for the Eells-Sampson equation (gradient flow for total energy of maps between two manifolds).

Li and Yau [5] established the parabolic Harnack inequality for the heat equation on Riemannian manifolds. By making use of Theorem 4 and this inequality we can show Widder's theorem on Riemannian manifold provided its Ricci curvature decays to $-\infty$ subquadratically.

Theorem 6. *Let Ric_M be the Ricci curvature of M, and $K_p(R) = -\inf_{B_p(R)} \mathrm{Ric}_M$. If there exist $p \in M$ and $C > 0$ such that*

$$K_p(R) \leq C(1+R^2)$$

holds for any $R > 0$, then $u \equiv 0$ is the unique non-negative solution of (1) – (2).

In respect of the square power of R, it follows from Azencott's example [1, §§7.7 – 7.9] that these results are the best possible.

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Nonlinear Heat Equations Describing Deformations of a Metric

Nobumitsu Nakauchi

[1] Basic Notations

M : a n-dimensional compact smooth manifold;

$g = (g_{ij})$: a metric on M ; $(g^{ij}) = (g_{ij})^{-1}$: the inverse matrix ;

$R_g = (R_{ijkl})$: the curvature tensor of g ;

$Ric_g = (R_{jk})$: the Ricci curvature of g ;

$Scal_g = (R)$: the scalar curvature of g ;

dv_g : the volume element w.r.t. g ;

$Vol_g(M) = \int_M dv_g$: the volume of M w.r.t. g .

$$R_{ijkl} = \frac{1}{2}\left\{ \frac{\partial^2 g_{ik}}{\partial x_j \partial x_l} + \frac{\partial^2 g_{jl}}{\partial x_i \partial x_k} - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_l} - \frac{\partial^2 g_{il}}{\partial x_j \partial x_k} \right\} + g_{ab}\left\{ \Gamma^b_{ik}\Gamma^a_{jl} - \Gamma^b_{il}\Gamma^a_{jk} \right\} .$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{ia}\left\{ \frac{\partial g_{aj}}{\partial x_k} + \frac{\partial g_{ak}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_a} \right\} , \ R_{jk} = g^{il} R_{ijkl} , \ R = g^{jk} R_{jk} .$$

[2] A functional

Put

$$\mathcal{M} \stackrel{def}{=} \{\, g \,:\, \text{a metric on } M \text{ s.t. } Vol_g(M) \;=\; 1 \,\}.$$

This is a family of all metrics on M with a normalized volume. The functional $\mathcal{F}$ over $\mathcal{M}$, called *total scalar curvature*, is defined by

$$\mathcal{F}(g) \stackrel{def}{=} \int_M Scal_g \, dv_g .$$

Fact 1.

$$\{\, \text{critical points of } \mathcal{F} \,\} \;=\; \{\, \text{Einstein metrics} \,\}.$$

Fact 2.

$$\begin{aligned} \text{The gradient flow of } \mathcal{F} \;&=\; \textbf{Ricci flow} \;\oplus\; \textbf{conformal deformation} \\ &=\; \textbf{R.Hamilton} \;\oplus\; \textbf{Yamabe}. \end{aligned}$$

[3] Ricci flow (Hamilton's equation)

Gradient flow equation

$$\frac{\partial g}{\partial t} \;=\; -2\, Ric_g \;+\; Scal_g \; g .$$

Ricci flow equation (unnormalized)

$$\frac{\partial g}{\partial t} = -2\, Ric_g .$$

Ricci flow equation (normalized)

$$\frac{\partial g}{\partial t} = -2\, Ric_g + \frac{2}{n}\, s_g\, g ,$$

where s_g denotes the average of the scalar curvature, i.e.

$$s_g = \frac{\int_M Scal_g\, dv_g}{\int_M dv_g} .$$

Transformation: unnormalized $\Longleftrightarrow$ normalized

$$(*) \qquad \begin{cases} g^n = \psi(t^u)\, g^u \\ t^n = \int^{t^u} \psi(t^u)\, d\, t^u , \end{cases}$$

where the superscripts "n" and "u" correspond to "normalized" and "unnormalized" respectively.

Short time existence

For any initial metric, there exists a solution in a short time.

Long time existence

A "long-time solution" means here the the solution such that the orbit of this flow reaches an Einstein metric or an Einstein-like metric. For long-time existence, there are three main results as follows:

(1) (Hamilton [3]):

In case dim $M = 3$, for any initial metric g_0 with $Ric_{g_0} > 0$, there exists a long-time solution, and the solution g_t converges to a metric of constant curvature as t tends to the maximum existence time T.

(2) (Hamilton [4]):

In case dim $M = 4$, for any initial metric g_0 satisfying that the curvature operator of g_0, there exists a long-time solution, and the solution g_t converges to a metric of constant curvature as t tends to the maximum existence time T.

(3) (Huisken [5], Margerin [9], Nishikawa [12]):

In general dimension, if the initial metric g_0 is close in some sense to one with a constant curvature, there exists a long-time solution, and the solution g_t converges to a metric of constant curvature as t tends to the maximum existence time T.

There does not exist, in general, a long-time solution: It is impossible to go beyond the maximum existence time T before a "long time". So we want to ask:

Question : *What occurs in solutions as* $t \to T$ *?*

We give a partial answer for this question:

Theorem (Nakauchi [10]). *Let M be an n-dim. compact smooth manifold ($n \geq 5$). Let g_t be a solution of unnormalized Ricci flow equation such that the curvature operator of the initial metric g_0 is positive. Let $T(< \infty)$ denote the maximum existence time. We assume the following two conditions:*

$$\text{(A)} \qquad \int_{T-\delta}^{T} \int_M \| R_{g_t} \|^{\frac{n}{2}+1} dv_{g_t} \, dt < \infty \qquad (\exists \delta > 0),$$

where $\| R_g \|^2 = g^{ip} g^{jq} g^{kr} g^{ls} R_{ijkl} R_{pqrs}$.

$$\text{(B)} \qquad \lim_{t \to T} Vol_{g_t}(M) > 0.$$

Then there exist

(i) *a set $\mathcal{S}$ of points $x_1, \ldots, x_k$ of M*

and

(ii) *positive real numbers $a_1, \ldots, a_k$*

satisfying the following two conditions:

(1) *The metric g_t converges smoothly to a metric g^* on $M - \mathcal{S}$ as $t \to T$, where g^* has positive curvature operator.*

(2) *The measure $\| R_{g_t} \|^{\frac{n}{2}} dv_{g_t}$ converges weakly to $\| R_{g^*} \|^{\frac{n}{2}} dv_{g^*} + \Sigma_{i=1}^{k} a_i \delta_{x_i}$ as $t \to T$, where δ_{x_i} denotes the Dirac mass supported at x_i.*

Remarks.

(1) The integral in the condition (A) is invariant under the scale-chang (*).

(2) In condition (A), $\frac{n}{2} + 1 = \frac{n+2}{2}$ is regarded as the "critical exponent" in the *space-time* integral, while $\frac{n}{2}$ is critical in the *space* integral.

(3) For any solution g_t of the unnormalized equation, the volume $Vol_{g_t}(M)$ is decreasing as t increases. So the limit $\lim_{t \to T} Vol_{g_t}(M)$ always exists.

[4] Yamabe flow

In his study on conformal deformations ([15]), Yamabe attempted to minimize Yamabe functional on an n-dimensional compact Riemannian manifold M $(n \geq 3)$:

$$\mathcal{Y}(u) \stackrel{def}{=} \frac{\int_M \left(4\frac{n-1}{n-2}\|\nabla u\|^2 + Ru^2\right)}{\left\{\int_M |u|^{2^*}\right\}^{\frac{2}{2^*}}} \qquad (1)$$

$$\left(2^* = \frac{2n}{n-2}\right)$$

for $u(\not\equiv 0) \in W^{1,2}(M)$, where $W^{1,2}(M)$ denotes the Sobolev space whose elements and their derivatives belong to $L^2(M)$, and R is any given smooth function on M.

Yamabe claimed that the infimum (called the *Yamabe invariant*)

$$\mu(M) \stackrel{def}{=} \inf\{\mathcal{Y}(u)\ ;\ u \in C^\infty(M),\ u \not\equiv 0\}\ (> -\infty)$$

is always attained. Trudinger [14] found a gap in his proof, and improved it when the Yamabe invariant is bounded from above by some (small) constant. Aubin [1] showed that if $n \geq 6$ and M is not locally conformally flat, then there exists a minimizer of the Yamabe functional. Finally, Schoen [13] proved the remaining case, and Yamabe problem was completely solved.

The above approach is based on the direct method, i.e. method of convergence of appropriate minimizing sequences. From the viewpoint of calculus of variations, it is important to consider a gradient flow instead of such a

sequence; the gradient of the functional (1) is

$$grad\,\mathcal{Y}(u) = \frac{2}{\|u\|^2_{L^{2^*}}} \left\{ 4\frac{n-1}{n-2}\,\Delta u - Ru + \frac{\mathcal{Y}(u)}{\|u\|^{2^*-2}_{L^{2^*}}}\,|u|^{2^*-2}\,u \right\}.$$

Since the Yamabe functional is invariant by the multiplication of constants to u, we may normalize the norm $\|u\|_{L^{2^*}}$. Set

$$\mathcal{L} \stackrel{def}{=} \{\, u \in W^{1,2}(M)\ ;\ \|u\|_{L^{2^*}} = 1,\ u \not\equiv 0 \,\}.$$

The gradient of the functional restricted to this normalized subspace $\mathcal{L}$ is given by

$$grad\left(\mathcal{Y}|_{\mathcal{L}}\right)(u) = 2\left\{ 4\frac{n-1}{n-2}\,\Delta u - Ru + \mathcal{Y}(u)\,|u|^{2^*-2}\,u \right\}.$$

Thus we have the *normalized Yamabe flow*:

$$\frac{\partial u}{\partial t} = 4\frac{n-1}{n-2}\,\Delta u - Ru + \mathcal{Y}(u)\,|u|^{2^*-2}\,u. \tag{2}$$

We consider the initial value problem for (2) with the initial data:

$$u(\ ,0) = u_0 \in C^\infty(M). \tag{3}$$

Inoue [6] constructed a weak solution in a Sobolev space for the initial value problem (2) and (3). (See Theorem 2.2 in [6].) We show the existence of a smooth solution:

Theorem (Nakauchi [11]). *There exists a positive constant $C(n)$ depending only on n with the following property: Let M be an n-dimensional compact Riemannian manifold such that $\mu(M) < C(n)$. Then for any initial data $u_0 \in C^\infty(M)$ such that $\mu(M) < \mathcal{Y}(u_0) < C(n)$, there exists a smooth solution u of the initial value problem* (2) and (3) *for $t \in [0, \infty)$. Furthermore $u(\ , t)$ converges to a solution $u(\ , \infty)$ of the Yamabe problem as $t \to \infty$.*

Remark. we can show that

$$C(n) \geq \frac{(n+2)^2 - 16}{(n+2)^2 + 16}\, \mu(S^n),$$

where S^n denotes the unit sphere. Note that, in general, $\mu(M) \leq \mu(S^n) = n(n-1)\,Vol(S^n)^{\frac{2}{n}}$, where $Vol(S^n)$ denotes the volume of S^n (See Lee-Parker [8].)

We show the following result in general:

Theorem (Nakauchi [11]). *Let u be a smooth solution of the initial value problem* (2) and (3). *Let T be a maximum existence time. Then there exist a finite set $\mathcal{S}$ of points $x_1, \ldots, x_k$ of M such that $u(\ , t)$ converges smoothly to a smooth function w on $M - \mathcal{S}$ as $t \to T$.*

Remark. In Theorem 2, we do not assume that $\mathcal{Y}(u_0) \leq \mu(S^n)$. If $\mathcal{Y}(u_0) > \mu(S^n)$, Yamabe solutions may bubble out.

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Blow-up Sets for Semilinear Parabolic Equation and Asymptotic Behaviors of Interfaces

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We consider the Cauchy problem

(1) $\partial_t \beta(u) = \Delta u + f(u)$ in $(x,t) \in \mathrm{R}^N \times (0,T)$,

(2) $u(x,0) = u_0(x)$ in $x \in \mathrm{R}^N$,

where $\partial_t = \partial/\partial t$, Δ is the N-dimensional Laplacian and $\beta(v)$, $f(v)$ with $v \geq 0$ and $u_0(x)$ are nonnegative functions.

Equation (1) describes the combustion process in a stationary medium, in which the thermal conductivity $\beta'(u)^{-1}$ and the volume heat source $f(u)$ are depending in a nonlinear way on the temperature $\beta(u) = \beta(u(x,t))$ of the medium. We assume

(A1) $\beta(v), f(v) \in C^\infty(\mathrm{R}_+) \cap C(\overline{\mathrm{R}}_+)$; $\beta(v) > 0$, $\beta'(v) > 0$, $\beta''(v) \leq 0$ and $f(v) > 0$ for $v > 0$; $\lim_{v\to\infty} \beta(v) = \infty$; $f \circ \beta^{-1}$ is locally Lipschitz continuous in $[0,\infty)$.

(A2) $u_0(x) \geq 0$, $\not\equiv 0$ and $\in B(\mathrm{R}^N)$ (bounded continuous in R^N).

With these conditions the above Cauchy problem has a unique local solution $u(x,t) \geq 0$ (in time) which satisfies (1) in $\mathrm{R}^N \times (0,T)$ in a weak sense. If $u(x,t)$ does not exist globally in time, its existence time T is defined by

(3) $T = \sup\{\tau > 0;\ u(x,t) \text{ is bounded in } \mathrm{R}^N \times [0,\tau]\}$.

In this case we say that u is a *blow-up solution* and T is the *blow-up time*.

Our main purpose is the study of blow-up solutions near the blow-up time. Especially, we are interested in the shape of the *blow-up set* which lokates the "hot-spots" at the blow-up time. In addition,

since equation (1) has a property of finte propagation, there are several interesting subjects such as the regularity of the *interface* and its asymptotic behavior near the blow-up time.

To deal with the finite propagation of solutions and the regularity of interfaces, we require

(A3) $\beta(0) = f(0) = 0$; $\int_0^1 \frac{dv}{\beta(v)} < \infty$; $\frac{f(v)}{\beta(v)\beta'(v)}$ is bounded near $v = 0$.

(A4) $u_0(x) > 0$ in $x \in D$ and $= 0$ in $x \notin D$, where $D \subset \mathrm{R}^N$ is a bounded convex set with smooth boundary ∂D.

We put

(4) $$\Omega(t) = \{x \in \mathrm{R}^N;\ u(x,t) > 0\}, \quad \Gamma(t) = \partial\Omega(t)$$

for $t \in (0,T)$. Then the interface Γ is given by

(5) $$\Gamma = \bigcup_{0 \le t < T} \Gamma(t) \times \{t\}.$$

Theorem 1. *Assume* (A1)~(A4). *Let u be any weak solution of problem* (1),(2). (I) *Then $\Omega(t)$ forms a bounded set in R^N which is nondecreasing in t:*

(6) $$\Omega(t_1) \subset \Omega(t_2) \quad \text{if } t_1 < t_2.$$

(II) *There exists a continuous function $\mathcal{F}$: $\partial D \times [0,T) \to \mathrm{R}^N$ such that*

(7) $$\Gamma(t) = \{x = \mathcal{F}(y,t);\ y \in \partial D\} \text{ for each } t \in [0,T).$$

(III) *For each $t \in (0,T)$, $\mathcal{F}(\cdot,t)$: $\partial D \to \Gamma(t)$ is bicontinuous.*

(IV) *If $\mathcal{F}(\bar{y},\bar{t}) \notin \bar{D}$ for some $(\bar{y},\bar{t}) \in \partial D \times (0,T)$, then $\mathcal{F}(y,\bar{t})$ is Lipschitz continuous in $y \in \partial D$ in a neighborhood of $\bar{y}$.*

Note that in the case of the porous medium equation

(8) $$\partial_t (u^{1/m}) = \Delta u \quad (m > 1) \text{ in } (x,t) \in \mathrm{R}^N \times (0,\infty),$$

there are many works studying the interface. Among them Caffarelli et al [1] proved that $\mathcal{F}(y,t)$ is Lipschitz continuous in $(y,t) \in \partial D \times (0,\infty)$ in a neighborhood of $(\bar{y},\bar{t})$. To obtain a more regularity in t

in our case, it seems necessary to know suitable exact solutions of (1) whose space-time structure reflects the most important properties of general solutions.

Next, we restrict our concern to blow-up solutions of (1),(2) requiring the following additional conditions.

(A5) $\int_1^\infty \frac{\beta'(v)}{f(v)}\,dv < \infty$.

(A4)' There exists a convex domain $D \subset \mathrm{R}^N$ with smooth boundary $\partial\Omega$ such that $u_0(x) > 0$ in $x \in D$ and for any $y \in \partial\Omega$, $u_0(y+sn(y))$ is nonincreasing in $s > 0$, where $n(y)$ denotes the outer unit normal to the boundary.

(A5) is known as a blow-up condition (cf.,e.g. [5]). We shall classify the blow-up solutions by the following three conditions.

(A6) (sublinear case) $f(v) = o(v)$ as $v \to \infty$.

(A7) (asymptotic linear case) There exist γ, $C > 0$ such that $f(v) \le \gamma v + C$ for each $v > 0$.

(A8) (superlinear case) There exists a function $\Phi(v)$ such that

(i) $\Phi(v) > 0$, $\Phi'(v) > 0$ and $\Phi''(v) \ge 0$ for $v > 0$;

(ii) $\int_1^\infty \frac{dv}{\Phi(v)} < \infty$;

(iii) there is constants $c > 0$ and $v_0 > 0$ such that $f'(v)\Phi(v) - f(v)\Phi'(v) \ge c\Phi(v)\Phi'(v)$ for $v > v_0$.

Remark. Equation (1) with power nonlinearities

(9) $$\partial_t(u^{1/m}) = \Delta u + u^{p/m} \quad \text{in } (x,t) \in \mathrm{R}^N\times(0,T)$$

satisfies (A1),(A3) and (A5) if $m > 1$ and $p > 1$, and satisfies (A6) (or (A7)) if $1 < p < m$ (or $1 < p \le m$). (A8) is originally introduced in [3] to semilinear parabolic equations. (9) satisfies (A8) if $p > m$.

The blow-up set of u is defined as

(10) $S = \{x \in \mathrm{R}^N;$ there is a sequence $(x_n, t_n) \in \mathrm{R}^N\times(0,T)$ such that $x_n \to x$, $t_n \uparrow T$ and $u(x_n, t_n) \to \infty$ as $n \to \infty\}$.

Our results are summarized in the following three theorems.

Theorem 2. *Assume* (A1),(A2),(A4)',(A5) and (A6). *Let u be a blow-up solution of* (1),(2). (I) *Then*

(11) $$S = \mathrm{R}^N,$$

and the way of blow-up is uniform in each compact set K of R^N:

(12) $$\liminf_{t \uparrow T} \inf_{x \in K} u(x,t) = \infty.$$

(II) *Assume further* (A3) and (A4). *Then the support $\overline{\Omega}(t)$ of $u(x,t)$ grows to* R^N *as $t \uparrow T$, in other words,*

(13) $$\liminf_{t \uparrow T} \inf_{y \in \partial D} |\mathcal{I}(y,t)| = \infty.$$

Theorem 3. *Assume* (A1),(A2),(A4)',(A5) *and* (A7). *Let u be a blow-up solution of* (1),(2). *We choose $R_\gamma > 0$ so that γ is the principal eigenvalue of $-\Delta$ in $B(3R_\gamma) = \{x \in \mathrm{R}^N;\ |x| < 3R_\gamma\}$ with zero Dirichlet condition. Suppose that D in* (A4) *is included in $B(R_\gamma)$. Then we have*

(14) $$S \supset B(R_\gamma),$$

and u blows up uniformly in each compact set of $B(R_\gamma)$.

Theorem 4. *Assume* (A1),(A2),(A4)',(A5),(A8) *and the following*

(A9) $\Delta u_0(x) + f(u_0(x)) \geq 0$ *in the distribution sense in* R^N.

Let u be a blow-up solution of (1),(2). (I) *Then*

(15) $$S \subset \overline{D}.$$

(II) *Assume further* (A3) *and* (A4). *Then the support $\overline{\Omega}(t)$ of $u(x,t)$ remains bounded as $t \uparrow T$, in other words,*

(16) $$\limsup_{t \uparrow T} \sup_{y \in \partial D} |\mathcal{I}(y,t)| < \infty.$$

Corollary 5. *Assume* (A1),(A2),(A5),(A8),(A9) *and the following*

(A4)" $u_0(x) = u_0(r)$, *where* $r = |x|$; $u_0(r) > 0$ *in* $0 \leq r < R$, *and* $= 0$ *in* $r \geq R$; $u_0'(r) < 0$ *in* $0 < r < R$.

Let $u = u(r,t)$ be a blow-up solution of (1),(2). *Then*

(17) $S = \{0\}$.

We are based on comparison and reflection principles. The main proof is done by reduction to absurdity. To do so, for Theorems 2~3, a nonblow-up result to the Diriclet ploblem in a bounded domain plays a key role. For Theorem 4 and Corollary 5, we can follow the argument of [3] (see also [2]). In case $N = 1$, more precise results have been done in [7] (cf., also [4]).

As for the details of the above results, see [6].

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On Stabilities of Difference Solutions for a Degenerate Parabolic Equation

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In this article, we consider the stability of a difference analogue to the initial value problem of a nonlinear degenerate parabolic equation

$$(1) \qquad u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{(|\nabla u|^\sigma + \delta)^{\frac{1}{\sigma}}}\right), \qquad (t,x) \in Q = (0,\infty) \times \mathbb{R}^N$$

with the initial data

$$(2) \qquad u(0,x) = u_0, \qquad x \in \mathbb{R}^N.$$

Here, δ is a constant with $\delta \leqslant 0$, and $\nabla u = (u_{x_1}, \cdots, u_{x_N})$ is the gradient of $u(t,x)$.

The equation (1) can be changed into

$$(1') \qquad u_t = \sum_{p,q=1}^{N} a_{pq}(\nabla u) u_{x_p x_q}$$

where

(3)

$$a_{pq} = a_{pq}(\nabla u) = \frac{|\nabla u|}{(|\nabla u|^\sigma + \delta)^{\frac{1}{\sigma}}}\left(\delta_{pq} - \frac{|\nabla u|^{\sigma-2}}{|\nabla u|^\sigma + \delta} u_{x_p} u_{x_q}\right), \qquad p,q = 1, \cdots, N$$

and δ_{pq} is the Kronecker delta.

We now state our difference scheme for (1) (or, its another form (1′)) as below, with $N = 2$ for simplicity. Denoting by x and y the spatial variables in $\mathbb{R}^2$, we shall use notations x_j and y_k to denote the spatial coordinates of the net points here and hereafter. The difference analogue to (1) and (2) is given by

$$
\begin{aligned}
&\frac{u_{jk}^{n+1} - u_{jk}^{n}}{\tau} = \sum_{p,q=1}^{2} a_{pq}(\mathrm{D}\, u_{jk}^{n})\, \mathrm{D}_{pq}^{2}\, u_{jk}^{n+\theta}, \\
&\qquad\qquad n = 0, 1, 2, \cdots; \quad j, k = 0, \pm 1, \pm 2, \cdots; \\
&u_{jk}^{0} = u_0(x_j, y_k), \quad j, k = 0, \pm 1, \pm 2, \cdots .
\end{aligned}
\tag{4}
$$

In (4), we have introduced the following notation for convenience.

Notation:

$\tau > 0$: increment of the time variable t;

$t_n = n\tau$: nth time step;

h_1, h_2: mesh sizes of x and y directions, respectively;

$(x_j, y_k) = (jh_1, kh_2)$: net point in $\mathbb{R}^2$, $j, k = 0, \pm 1, \pm 2, \cdots$;

u_{jk}^{n}: approximate value of the difference solution to $u(t_n, x_j, y_k)$;

$\mathrm{D}\, u_{jk}^{n} = (\mathrm{D}_x\, u_{jk}^{n}, \mathrm{D}_y\, u_{jk}^{n}) \in \mathbb{R}^2$, where $\mathrm{D}_x\, u_{jk}^{n} = (u_{j+1,k}^{n} - u_{j-1,k}^{n})/(2h_1)$ and $\mathrm{D}_y\, u_{jk}^{n} = (u_{j,k+1}^{n} - u_{j,k-1}^{n})/(2h_2)$ are the approximations to $u_x(t_n, x_j, y_k)$ and $u_y(t_n, x_j, y_k)$, respectively (both are central differences);

$\mathrm{D}_{11}^{2}\, u_{jk}^{n} = (u_{j+1,k}^{n} - 2u_{jk}^{n} + u_{j-1,k}^{n})/h_1^2$,

$\mathrm{D}_{22}^{2}\, u_{jk}^{n} = (u_{j,k+1}^{n} - 2u_{jk}^{n} + u_{j,k-1}^{n})/h_2^2$ and

$\mathrm{D}_{12}^{2}\, u_{jk}^{n} = (u_{j+1,k+1}^{n} - u_{j-1,k+1}^{n} - u_{j+1,k-1}^{n} + u_{j-1,k-1}^{n})/(4h_1h_2)$

are the approximations to u_{xx}, u_{yy} and u_{xy} at (t_n, x_j, y_k), respectively;

$u_{jk}^{n+\theta} = \theta u_{jk}^{n+1} + (1-\theta)u_{jk}^{n}$, linear combination of u_{jk}^{n+1} with u_{jk}^{n} for $\theta \in [0,1]$. The difference equation (4) is explicit for u^{n+1} if $\theta = 0$, while implicit if $0 < \theta \leqslant 1$.

Put $b_{jk}^{n} = |\,\mathrm{D}\,u_{jk}^{n}|/(|\,\mathrm{D}\,u_{jk}^{n}|^{\sigma} + \delta)^{1/\sigma}$. Then for the coefficients in (4) it is easy to get

$$a_{12} = a_{21}, \quad a_{11}a_{22} - a_{12}^2 \geqslant 0;$$

$$0 \leqslant a_{pp} \leqslant b_{jk}^{n} \leqslant 1, \quad p = 1,2;$$

$$0 \leqslant b_{jk}^{n} \leqslant a_{11} + a_{22} \leqslant 2b_{jk}^{n} \leqslant 2$$

since $0 \leqslant b_{jk}^{n} \leqslant 1$.

As a sufficient condition for the stability of the nonlinear difference equation (4), we get the following

THEOREM. *The difference equation (4) is stable if either*

$$\frac{1}{2} \leqslant \theta \leqslant 1 \tag{5}$$

or

$$\lambda \leqslant \frac{1}{4-8\theta} \quad \textit{when} \quad 0 \leqslant \theta < \frac{1}{2} \tag{6}$$

holds true, where

$$\lambda = \frac{\tau}{h_1^2} + \frac{\tau}{h_2^2} > 0$$

is a constant.

REMARK: It is worthwhile to note that the stability condition here is independent of δ.

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On the local energy decay property for the elastic wave equation with the Neumann boundary condition

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Abstract

Let Ω be an exterior domain in $\mathbf{R}^n$ $(n \geq 3)$ with smooth and compact boundary Γ. We set

$$A(\partial_x)u = \sum_{i,j=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j}u, \qquad u = {}^t(u_1, u_2, \cdots, u_n)$$

where a_{ij} are $n \times n$ matrices whose (p,q)–components a_{ipjq} are of the form

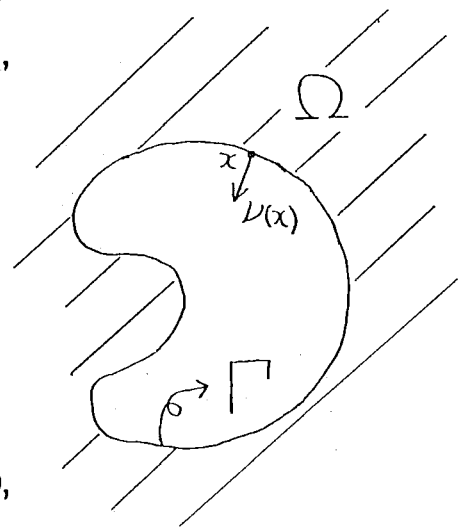
$$a_{ipjq} = \lambda\delta_{ip}\delta_{jq} + \mu(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp}).$$

We consider the isotropic elastic wave equation with the Neumann boundary condition

$$\text{(N.P)} \qquad \begin{cases} (A(\partial_x) - \partial_t^2)u(t,x) = 0 & \text{in} \quad \mathbf{R} \times \Omega, \\ N(\partial_x)u(t,x) = 0 & \text{on} \quad \mathbf{R} \times \Gamma, \\ u(0,x) = f_0(x), \quad \partial_t u(0,x) = f_1(x) & \text{on} \quad \Omega. \end{cases}$$

Here $N(\partial_x)$ is the conormal derivative of $A(\partial_x)$, that is,

$$N(\partial_x)u(x) = \sum_{i,j=1}^{n} \nu_i(x)a_{ij}\partial_{x_j}u(x)|_\Gamma,$$



where $\nu(x) = {}^t(\nu_1(x), \nu_2(x), \cdots, \nu_n(x))$ is the unit outer normal vector to Ω at $x \in \Gamma$. We assume that the Lamé constants λ and μ are independent of the variables t and x and satisfy

$$\lambda + \frac{2}{n}\mu > 0 \quad \text{and} \quad \mu > 0,$$

which imply that the energy form associated with the operator $A(\partial_x)$ is positive definite.

Under that assumption, Iwashita and Shibata [6] examine the analytic continuation of the resorvent. Shibata and Soga [13] develop the scattering theory obtained by Lax and Phillips [8]. In [13], it is shown that for any initial data f_0, f_1 and any bounded domain D the local energy for the solution of (N.P) in $D \cap \Omega$

$$\mathbb{E}(u, D, t) = \frac{1}{2} \int_{D \cap \Omega} \{ \sum_{i,p,j,q=1}^{n} a_{ipjq} \partial_{x_j} u_q(t,x) \overline{\partial_{x_i} u_p(t,x)} + |\partial_t u(t,x)|^2 \} \, dx.$$

decays as t tends to infinity (i.e. $\lim_{t \to \infty} \mathbb{E}(u, D, t) = 0$ holds). We would like to examine more precise properties of the local energy decay.

PROBLEM. *Whether the problem (N.P) has uniform rate of the local energy decay.*

DEFINITION. *We say that the problem (N.P) has the uniform local energy decay property of strong type when for any bounded domains D and D_0, there exists a bounded, continuous and non-negative valued function $p(t)$ defined on $[0, \infty)$ satisfying*

$$\int_0^\infty p(t)^{1/2} \, dt < \infty \quad \text{and} \quad \int_0^\infty \int_s^\infty p(t)^{1/2} \, dtds < \infty$$

such that

$$\mathbb{E}(u, D, t) \leq p(t) \mathbb{E}(u, \Omega, 0) \qquad \text{for any } t \geq 0$$

holds for any solution of (N.P) with an initial data $f_0, f_1 \in C_0^\infty(D_0 \cap \overline{\Omega})$.

Remark 1. Usually, we say that the problem (N.P) has the uniform local energy decay property when for any bounded domains D and D_0, there exists a continuous and non-negative valued function $p(t)$ defined on $[0, \infty)$ satisfying $\lim_{t \to \infty} p(t) = 0$ such that $\mathbb{E}(u, D, t) \leq p(t) \mathbb{E}(u, \Omega, 0)$ holds for any $t \geq 0$ and any solution of (N.P) with an initial data $f_0, f_1 \in C_0^\infty(D_0 \cap \overline{\Omega})$.

Remark 2. In the case of the scalar-valued wave equation with the Dirichlet or the Neumann boundary condition or the elastic wave equation with the Dirichlet boundary condition, if the obstacle $\mathbb{R}^n \backslash \Omega$ satisfies a non-trapping condition in some sence (e.g. the obstacle is convex), then the uniform local energy decay property in the sence of Remark 1 holds. Furthermore, we can be taken $p(t)$ as $p(t) = C \exp(-\alpha t)$ $(\alpha > 0)$

for n is odd, $p(t) = C(1+t)^{-2(n-1)}$ for n is even (cf. e.g. Vainberg [14], Morawetz [10, 11], Ralston [12], Kapitanov [7], and Iwashita and Shibata [6]). In particular, these initial boundary value problems have the uniform local energy decay property of strong type in Definition if the obstacle is non-trapping. Hence, the uniform local energy decay property of strong type is not a meaningless condition.

For the problem (N.P), there is the interesting phenomenon, or the existence of the Rayleigh surface wave which seems to propagate along the boundary, and it does not occur for the cases of the problems stated in Remark 2. In particular, in the case of the half space in $\mathbf{R}^3_+$, the Rayleigh surface wave is represented explicitly and it is shown that its energy concentrates near the boundary $\partial\mathbf{R}^3_+$ as t tends to infinity (cf. Achenbach [1] and Guillot [2]). Hence, we can expect that the local energy does not decay uniformly. Indeed, Ikehata and Nakamura [5] show that the problem (N.P) does not have the uniform local energy decay property if Γ is the unit sphere in $\mathbf{R}^3$. They also prove more precise results, however, they essentially use the fact that the boundary is the sphere because they represent the solution of (N.P) by using special functions. Hence, it seems that we do not use their methods in the case of the general smooth and compact boundary. Thus, our result is a generalization of Ikehata and Nakamura's one.

THEOREM.
The problem (N.P) does not have the uniform local energy decay property of strong type.

It is well known that the total energy $E(u,\Omega,t)$ of the solution $u(t,x)$ of the problem (N.P) is conserved and if the space dimension n is odd then the Cauchy problem for the operator $A(\partial_x) - \partial_t^2$ satisfies Heygens' Principle. Hence, the Morawetz argument due to Morawetz [11] is available for the problem (N.P). Thus, we can show that if (N.P) has the uniform local energy decay property, then we can take $p(t) = C\exp(-\alpha t)$ $(\alpha > 0)$, which implies that

COROLLARY. *If n is odd, then the problem (N.P) does not have the uniform local energy decay property.*

We shall prove Theorem by contradiction. The procedure of the proof is as follows.

Step 1. We denote the outgoing (resp. incoming) Neumann operator denoted by T^+ (resp. T^-). First, we can prove that the Neumann operators have the following estimates, which are key results for the proof of Theorem.

PROPOSITION 1. *If the problem (N.P) has the uniform local energy decay property of strong type, then we obtain*

$$\|f^{\pm}\|_{L^2(\mathbb{R}\times\Gamma)} \leq C\|T^{\pm}f^{\pm}\|_{L^2(\mathbb{R}\times\Gamma)}$$
$$\text{for any} \quad f^{\pm} \in C^{\infty}_{\pm}(\mathbb{R}\times\Gamma) \quad \text{with} \quad T^{\pm}f^{\pm} \in C^{\infty}_{0}(\mathbb{R}\times\Gamma),$$

where $C^{\infty}_{\pm}(\mathbb{R}\times\Gamma) = \{ f \in C^{\infty}(\mathbb{R}\times\Gamma) \mid$ *there exists* $t_1 \in \mathbb{R}$ *such that* $f(t,x) = 0$ *for* $\pm t < t_1 \}$.

Step 2. On the other hand, however, in the elliptic region the Neumann operators are the first order classical pseudo-differential operator on $\mathbb{R}\times\Gamma$ of real principal type. Hence, we can construct the asymptotic null solution of the Neumann operator T^{+} (, that is, the function g satisfying that $T^{+}g = O(k^{-1})$, where k is the wave number), and its principal part does not vanish.

Step 3. But, using the estimate obtained in Step 1, we can show that the principal part of the asymptotic null solution must be zero, which is contradiction.

In the above procedure, if we can construct an asymptotic null solution described in Step 2 in the time interval $(-\infty,\infty)$, then it is not difficult to perform Step 3. But, the construction of the time global asymtotic null solution does not seem easy, and this causes the main difficulty for the proof of Theorem.

In our case, however, we can construct an asymptotic null solution in the time interval $[-T_0, T_0]$ for any fixed $T_0 > 0$ by using the Maslov method originally due to V.P.Maslov (for the Maslov method see Maslov and Fedoriuk [9] or Ichinose [3, 4]). Furthermore, that asymptotic null solution is sufficient to prove Theorem, because we can carry out the time global construction of the principal part of that solution. Noting the methods of the construction of the asymptotic solution, we can parameterize the principal part by $(s,x) \in \mathbb{R}\times\Gamma$. We denote the principal part by $\Psi(s,x) \in C^{\infty}(\mathbb{R}\times\Gamma)$. Using the estimate in Proposition 1, we can get the following estimate of $\Psi(s,x)$, and it is available to accomplish Step 3.

PROPOSITION 2. *If the problem (N.P) has the uniform local energy decay property of strong type, then we have*

$$\int_{[t_0+2,T_0-2]\times\Gamma} |\Psi(\pm s,x)|^2\, dV_{\mathbb{R}\times\Gamma}$$
$$\leq C_0 \int_{[t_0+\frac{3}{2},t_0+2]\times\Gamma} |\Psi(\pm s,x)|^2\, dV_{\mathbb{R}\times\Gamma}$$
$$\text{for any } t_0, T_0 \in \mathbb{R} \text{ with } t_0 < -2, |t_0 - 1| < T_0,$$

where $dV_{\mathbf{R}\times\Gamma}$ is the volume element of $\mathbf{R}\times\Gamma$ and a constant C_0 depends only upon Γ, λ and μ.

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A minimizing problem for a functional with a characteristic function

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1. Introduction

There are some results obtained by H.W.Alt, L.A.Caffarelli and A.Friedman about functionals with a variable boundary(See [1] and [2].).

Their problems are as follows: for $u : \Omega \to R$, $\Omega \subset R^n$, consider the functional:

$$I(u) = \int_\Omega \Big(F(|\nabla u|^2) + Q^2(x)\chi_{u>0} \Big) dL^n,$$

where L^n is n dimensional Lebesgue measure and $Q(x)$ is a given measurable function with $0 < Q_{min} \leq Q(x) \leq Q_{max}$ and χ denotes a characteristic function and $\Omega(\subset R^n)$ is an open and connected domain (may be unbounded) with Lipschitz boundary. Here and in the sequel we denote $\{x \in \Omega;\ u(x) > 0\} = \Omega(u > 0)$, and $\chi_{u>0}$ is the function of the set $\Omega(u > 0)$. In [1], the case $F(t) = t$, and in [2], it was treated the case $F(t)$ belonging to $C^{2,1}[0,\infty)$, with $F(0) = 0$ and $0 < c \leq \frac{\partial F}{\partial t} \leq C$ and $0 \leq \frac{1}{1+t}\frac{\partial^2 F}{\partial t^2} \leq C$. They proved that if $Q(x)$ is Hölder continuous, roughly speaking, the free boundary $\Omega \cap \partial\Omega(u > 0)$ is a $C^{1,\beta}$ -curve in any compact subset of Ω, provided that u is a minimizer of I. These results are applied to solve the Jet problem and the Cavitational flow problem(See [3-7] and [11]).

We extend their result ([1] and [2]) to the following nonlinear problem. Consider the minimizing problem:

$$J(u) = \int_\Omega \Big(a^{ij}(u) D_i u D_j u + Q^2 \chi_{u>0} \Big) dL^n,$$

under the same assumption for χ and Ω as in [1] and here Q is assumed to be a positive constant (*We used summation convention.*). *We need some further assumption for the coefficients* $a^{ij}(z)$: $a^{ij}(z)$ belongs to class C^∞ with respect to z, and satisfies the following ellipticity and bounded conditions, $0 < \lambda|\xi|^2 \leq a^{ij}(z)\xi_i\xi_j \leq \Lambda|\xi|^2$ for all $\xi \in R^n - \{0\}$, moreover $[\dot{a}^{ij}(z)]$, the derivative of $[a^{ij}(z)]$ with respect to z, is positive definite. We call this the strong one sided condition.

Under these assumptions, we find a minimizer in the function set K, where $K = \{u \in L^2_{\text{loc}}(\Omega) | \nabla u \in L^2(\Omega), u = u^0 \text{ on } S\}$. Here u^0 is a given function with $u^0 \in L^2_{\text{loc}}(\Omega), \nabla u^0 \in L^2(\Omega)$, and $0 \leq u^0 \leq \sup_\Omega u^0 < +\infty$, and S is a subset of $\partial\Omega$ with a positive $n-1$ dimensional Hausdorff measure.

We show that if Ω is 2 dimensional, the free boundary of the minimizer J is a $C^{1,\beta}$ curve in any compact subset Ω.

2. Regularity of a minimizer

The existence theorem is a direct conclusion of the lower semicontinuity of the functional J under an assumption $J(u^0) < \infty$(see [1].). The boundedness of a minimizer is obtained in the same way as in [1], using the test function $u + \min(\sup_\Omega u^0 - u, 0)$ and $u - \min(u, 0)$.

We can treat u by the method of Ladyzhenskaya and Ural'tseva and obtained the Hölder continuity of a minimizer.

THEOREM 2.1. *If u is the minimizer, then there exists $\alpha > 0$ depending on $\tilde{\Omega}$, such that $u \in C^{\alpha}(\tilde{\Omega})$, where $\tilde{\Omega}$ is a subdomain whose closure is compactly contained in Ω.*

By the theorem above, $\Omega(u > 0)$ should be an open set, then u satisfies the following equation;

$$\int_{\Omega(u>0)} \left(-a^{ij}(u) D_i u D_j \varphi - \frac{1}{2} \dot{a}^{ij}(u) D_i u D_j u \varphi \right) dL^n = 0.$$

for all $\zeta \in C_0^{\infty}(\{u > 0\})$. (In the sequel, we denote left hand side Lu.) By using this equation, the higher regularity can be easily obtained (see [12] and [14]). In other words, $u \in C^{\infty}(\Omega(u > 0))$.

Since $0 \leq J(u - \varepsilon\zeta) - J(u)$ for $\forall \zeta \in C_0^{\infty}(\Omega)$, $\zeta \geq 0$ and $\varepsilon > 0$, we have $Lu \geq 0$ in *Omega*. From this equation, we cannot obtain further regularity results by using usual methods. To obtain the Lipschitz continuity we should use the method of Alt-Caffarelli-Friedman(see [2]).

THEOREM 2.2. *Let u be a minimizer, and choose $x_0 \in \Omega$ arbitrary with* $\mathrm{dist}(x_0, \Omega(u = 0)) < \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega)$, *then there is a constant $C = C(n, \lambda, \Lambda)$ such that*

$$u(x_0) \leq C \mathrm{dist}(x_0, \Omega(u = 0)).$$

By using the Lipschitz continuity of the miminum, we have a nondegeneracy theorem.

THEOREM 2.3. *For any $p > 1$ and for any $0 < \kappa < 1$, there is a constant $C_\kappa = C(n, \kappa)$, such that for any balls B_r with radius r contained in Ω,*

$$\frac{1}{r} \left(\frac{1}{|B_r|} \int_{B_r} u^p \right)^{\frac{1}{p}} dx \leq C_\kappa \text{ implies } u = 0 \text{ in } B_{\kappa r}, \text{ provided that } u \text{ is a minimizer.}$$

3. Identification of the differential q_u

Our aim of this paper is now to prove that the free boundary of a minimizer, $\partial\Omega(u > 0) = \Omega \cap \partial\{x \in \Omega; u(x) > 0\}$, becomes locally the graph of a $C^{1,\alpha}-$ function ($\alpha \in (0,1)$)). First, we will show that $\partial\Omega(u > 0)$ is an (n-1)-dimensional surface in some weak sense(See [16].). For this, we will introduce the following Radon measure:

$$\lambda(D) = \sup_{\varphi \in C_0^1(D), |\varphi| \leq 1} \int_D \left(-a^{ij}(u) D_i u D_j \varphi - \frac{1}{2} \dot{a}^n(u) D_i u D_j u \varphi \right) dL^n,$$

where D is an arbitrary open set, which is compactly contained in Ω. On this Radon measure λ, the following fact is proved in [2] and [15]: For any Borel measurable set $E \subset \partial\Omega(u > 0) \cap D$

$$cH^{n-1}(E) \leq \int_E d\lambda \leq CH^{n-1}(E), \tag{3.1}$$

where c and C depend only on D. In particular the left inequality of (3.1) indicates the local finiteness of the free boundary with respect to the $n-1$ dimensional Hausdorff measure. From this fact, we can conclude that the free boundary $\partial\Omega(u > 0)$ is the (n-1)-dimensional surface with locally finite perimeter in Ω(See [9].). Moreover (3.1) shows that the Radon measure λ is absolutely continuous with respect to $H^{n-1} \lfloor \partial\Omega(u > 0)$. Thus we obtain the following representation:

$$\int_\Omega \left(-a^{ij}(u) D_i u D_j \varphi - \frac{1}{2} \dot{a}^{ij}(u) D_i u D_j u \varphi \right) dL^n = \int_{\partial\Omega(u>0)} \varphi q_u dH^{n-1} \quad \text{for all } \varphi \in C_0^{\infty}(\Omega)$$

where

$$q_u(x) = \lim_{\rho \to 0} \frac{\lambda(B_\rho(x))}{H^{n-1}(B_\rho(x) \cap \partial\Omega(u > 0))} \qquad (x \in \partial\Omega(u > 0)).$$

Now we introduce the blow up of the minimum u:

$$u_{m,x_0}(x) = \frac{1}{\rho_m} u(x_0 + \rho_m x) \qquad (\rho_m \to 0).$$

Without loss of generality, by an adequate change of coordinates, we can assume $a^{ij}(0) = \delta^{ij} a(0)$. We can show that the blow up limit u_{x_0} achieves the minimum of the following functional which is related to the Laplace-equation:

$$I(w) = \int_{B_R(0)} \left(|\nabla w|^2 + \frac{Q^2}{a(0)} \chi_{w>0} \right) dL^n. \tag{3.2}$$

Moreover, for a.e. $x_0 \in \partial\Omega$, the blow up limit u_{x_0} is represented by a following linear function: $u_{x_0}(x) = q_u(x_0)/a(0) \max(< x, \nu_u(x_0) >, 0)$. Thus we get the next equality, or so-called Identification: $q_u \equiv \sqrt{a(0)} Q$ a.e. $\partial\Omega(u > 0)$.

4. Blow up limit of a minimizer (n=2)

In this section, we will mention the blow up limit in the special case $n = 2$. Since the blow up limit of a minimizer u_{x_0} is represented as (3.2), we can proceed in the same way as in [1].

As the first step we get the next equality using the notion of the blow up limit:

$$\overline{\lim_{x \to x_0, u(x)>0}} |\nabla u(x)| = \frac{Q}{\sqrt{a(0)}} \tag{4.1}$$

for $x_0 \in \partial\Omega(u > 0)$.

Secondly we obtain the following estimate which holds only in the case $n = 2$:

$$\frac{1}{L^n(B_r(u > 0))} \int_{B_r(u>0)} \left(\frac{Q^2}{a(0)} - |\nabla u|^2 \right)^+ \leq \frac{C}{\log \frac{1}{r}}, \tag{4.2}$$

where u is the minimizer of the functional J and B_r is a sufficiently small n-dimensional ball contained in Ω with the center on the free boundary.

From (4.1) and (4.2), only in the case $n = 2$ we conclude that for all $x_0 \in \partial\Omega(u > 0)$, the blow up limit of a minimizer u_{x_0} is the half plane solution.

5. Regularity of the free boundary

We can show that all free boundary points have their normal vector a.e.H^{n-1}. In this section, we will show the Hölder continuity of the normal vector of the free boundary. The notion of non-homogeneous blow up plays an essential role of this proof. (See [1-2].) Here, we need some definitions for non-homogeneous blow up.

DEFINITION 5.1. *Let $\sigma_0, \sigma_+ \in (0, 1]$ and $\tau > 0$. We say that the minimum u belongs to $F(\sigma_0, \sigma_+; \tau)$ in $B_\varrho(0)$ with respect to e_n, if u satisfies following conditions.*

$$u(x) = 0 \text{ in } B_\varrho(x_n \geq \sigma_0 \varrho),$$

$$u(x) \geq \frac{Q}{\sqrt{a(0)}} ((-x_n) - \sigma_- \varrho) \text{ in } B_\varrho(x_n < -\sigma_+ \varrho),$$

$$|\nabla u| \leq \frac{Q}{\sqrt{a(0)}} (1 + \tau) \text{ in } B_\varrho.$$

Using the method in [2], we obtain the following theorem, an improvement of the plus flatness condition.

THEOREM 5.2. *Let $\varrho < 1$ and $\sigma < \min(\frac{1}{10}, \sigma_0(n,\lambda,M))$, and satisfying $\varrho < \sigma$, then there exits $C = C(n,\lambda,M)$ such that $u \in F(\sigma,1;\sigma)$ in B_ϱ w.r.t. ν implies $u \in F(2\sigma, C\sigma;\sigma)$ in $B_{\frac{\varrho}{2}}$ w.r.t. ν.*

DEFINITION 5.3(NON-HOMOGENEOUS BLOW UP). *Let $u_k \in F(\sigma_k,\sigma_k;\tau_k)$ in $B_{\varrho_k}(y_k)$, where $\{\sigma_k\}$ is a sequence which is chosen $\sigma_k \to 0$ as $n \to \infty$ and $\varrho_k < \sigma_k$ for all k and $\tau_k = O(\sigma_k)$. Then we define*

$$f_k^+(\overline{x}) = sup\{x_n | (\varrho_k \overline{x}, \sigma_k \varrho_k x_n) \in \partial\{u_k > 0\}\},$$

$$f_k^-(\overline{x}) = inf\{x_n | (\varrho_k \overline{x}, \sigma_k \varrho_k x_n) \in \partial\{u_k > 0\}\}.$$

Using Theorem 5.2, it is easy to see that there is a subsequence such that

$$f := \limsup_{\substack{j\to 0\\ z\to\overline{x}}} f_j^+(z) = \liminf_{\substack{j\to 0\\ z\to\overline{x}}} f_j^-(z).$$

Using f defined above, we can show the following lemma which is essential for the improvement of zero flatness condition.

LEMMA 5.4. *Let u_k be the sequence of non-homogeneous blow up which satisfies the following conditions; $u_k \in F(\sigma_k,\sigma_k;\tau_k)$ in $B_{\varrho_k}(x_k)$ w.r.t. ν_k and $\varrho_k = o(\sigma_k)$, $\tau_k = o(\sigma_k^2)$, then we have*

$$\int_0^{\frac{1}{4}} \frac{1}{r^2}\Big[Av_r f(\overline{x}) - f(\overline{x})\Big] dr \le C \qquad (\overline{x} \in \mathcal{B}_{\frac{1}{4}}(0))$$

where $\mathcal{B}_r$ is a $n-1$ dimensional ball and $Av_r f(\overline{x})$ is the average of the integration of f on $\partial\mathcal{B}_r(\overline{x})$.

Combining theorem 5.2 and Lemma 5.4, we can easily obtain that $f \in C^{0,1}(\mathcal{B}_{\frac{1}{4}}(0))$ and for all $\theta > 0$, there exists a positive number c_θ such that $f(\overline{x}) \le l \cdot \overline{x} + \frac{1}{2}\theta r \qquad \overline{x} \in \mathcal{B}_r(0)$ for some $r \in [c_\theta, \theta]$ and l is the vector in R^{n-1} with $|l| \le c(n)$. Using these facts, we immediately follow the next lemma, the improvement of zero flatness condition.

LEMMA 5.5 (IMPROVEMENT OF ZERO FLATNESS CONDITIONS).

For all $\theta > 0$, there exists a positive number c_θ and σ_θ such that If $u \in F(\sigma,\sigma;\tau)$ in B_ρ w.r.t. ν, (for $\forall\sigma \le \sigma_\theta, \forall\tau \le \sigma_\theta\sigma^2, \forall\rho \le c(n)\tau^{\frac{1}{2}}$), then $u \in F(\theta\sigma, 1;\tau)$ in $B_{\overline{\rho}}$ w.r.t. ν (for some $\overline{\rho} \in [c_\theta\rho, \theta\rho], \overline{\nu}$ with $|\overline{\nu} - \nu| \le c(n)\sigma$).

Using the iteration method, we obtain the theorem.

THEOREM 5.6 (IMPROVEMENT OF ALL FLATNESS CONDITIONS).

For all $\theta > 0$, there exists a positive number c_θ and σ_θ such that If $u \in F(\sigma,1;\tau)$ in B_ρ w.r.t. ν, (for $\forall\sigma \le \sigma_\theta, \forall\tau \le \sigma_\theta\sigma^2, \forall\rho \le c(n)\tau^{\frac{1}{2}}$), then $u \in F(\theta\sigma, \theta\sigma, \theta^2\tau)$ in $B_{\overline{\rho}}$ w.r.t. ν (for some $\overline{\rho} \in [c_\theta\rho, \frac{1}{4}\rho], \overline{\nu}$ with $|\overline{\nu} - \nu| \le c(n)\sigma$).

Finally we can show the conclusion of this paper, by using theorem 5.6 and the well-known method by Federer([8]).

THEOREM 5.7 (REGULARITY OF THE FREE BOUNDARY).

Let D be the arbitrarily fixed subdomain compactly contained in Ω, then there exists a positive number $\sigma_0(n,\alpha) > 0$, such that $u \in F(\sigma,1;\infty)$ in $B_\rho(x_0) \subset D$ w.r.t. ν, ($\forall\sigma \le \sigma_0$ and $\forall\rho \le \sigma_0\sigma^{\frac{2}{\alpha}}$) implies that there exists positive number $\nu(x_0), \beta = \beta(n), C = C(n)$ such that

$$\left|\Big\langle (x - x_0), \nu(x_0)\Big\rangle\right| \le \frac{C\sigma}{\rho^\beta}|x - x_0|^{1+\beta} \quad \Big(x \in B_{\frac{\rho}{2}}(x_0) \cap \partial\{u > 0\}\Big).$$

This immediately follows that Free Boundary is a $C^{1,\beta}$ surface.

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On The Mean Curvature Flow of "Thin" Doughnuts

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There has recently been much interest in mean curvature flow or curve shortening. The form of the problem is as follows: Let M be a compact oriented manifold without boundary and assume that $F_0 : M \to \mathbf{R}^{n+1}$ smoothly immerses M as a hypersurface in $\mathbf{R}^{n+1}$. Then we want to find a family of smooth immersions $F(x,t)$ with corresponding hypersurfaces $M_t = F(\,\cdot\,,t)(M)$ such that

$$(1) \qquad \begin{cases} \dfrac{\partial}{\partial t}F(x,t) = -H(x,t)\cdot N(x,t) \\ F(x,0) = F_0(x), \end{cases}$$

where $H(x,t)$ is the mean curvature and $N(x,t)$ is the outer unit normal at $x \in M_t$. Equation (1) is parabolic and so the theory of partial differential equations ensures the existence of solutions for some short time interval. We therefore study the long-time behaviour.

When M is a convex hypersurface in $\mathbf{R}^{n+1}$ for $n \geq 2$, G.Huisken [**H**] showed that (1) has a solution on finite time interval and that the M_t's shrink to a point. While M.Gage and R.S.Hamilton [**GH**], in the corresponding one dimensional problem, proved that if M is a smooth convex plane curve then the equation (1) shrinks M to a point within finite time. Later M.Grayson [**G1**] generalized the result of Gage and Hamilton so that the convexity assumption can be removed. But he also showed [**G2**] that the same generalization does not necessarily hold in higher dimension; the convexity assumption in Huisken's theorem cannot be discarded.

The aim of our talk is to discuss the behaviour when M is an embedding of a 2-torus $F_0 : T^2 \to \mathbf{R}^3$ such that M is rotationally symmetric about the z-axis and represented by

$$F_0(u,\phi) = (f(u)\cos\phi, f(u)\sin\phi, g(u)),$$

where $u \in S^1$ is a parameter modulo 2π and $0 \leq \phi < 2\pi$. We then call M a doughnut. There are two sectional curvature on the doughnut. One is

a meridional sectional curvature k_m, a curvature of the generating curve. The other is a latitudinal sectional curvature k_l, a curvature around the axis of revolution. At first thinking, the effect of k_m is rather dominant than that of k_l (we then call M "thin"), then M shrinks to a circle, and the effect of k_l is rather dominant than that of k_m (we call M "fat"), then M shrinks to a connected surface with singularity on the z-axis. These conjectures are strengthened by a computer simulation but the analytical proof has not been obtained. We here consider the "thin" case and show analytically that it really shrinks to a circle. To be precise we prove

THEOREM. *Suppose M satisfies the following assumption (A), then the mean curvature flow shrinks M to a circle within finite time.*

(A) There exists a positive constant ϵ such that $f(u) > \epsilon$, and $k_m > \frac{1+\sqrt{5}}{2\epsilon}$.

DEFINITION. *When the assumption (A) is satisfied we call M a "thin" doughnut.*

We sketch the idea of the proof. We consider the generating curve C of M. k_m is nothing but the curvature of C. We regard the equation k_m satisfies as the perturbed equation of the plane curve shortening one. Then the method of Gage and Hamilton [GH] can be applied and we conclude that as long as the area enclosed by the curve C is positive the equation (1) has a smooth solution to determine a hypersurface. Although the calculation is complicated the discussion well proceeds under our assumption.

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ON CERTAIN INTEGRAL EQUATIONS RELATED TO NONLINEAR WAVE EQUATIONS

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§1. Introduction

In this talk I shall discuss the global in time existence of solutions for integral equations related to the Cauchy problem for nonlinear wave equations. The results stated here are joint works by prof. R.Agemi, K.Kubota and me. For details of the proof, see [1].

In order to describe integral equations we introduce some notations. For a function $\varphi(x,t)$ of $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, we define, dividing into two cases of odd or even space dimensions,

$$M(\varphi|x,r;t) = \begin{cases} \int_{|\omega|=1} \varphi(x+r\omega,t)dS_\omega & (n=2m+1), \\ \int_{|\xi|\leq 1} \frac{\varphi(x+r\xi,t)}{\sqrt{1-|\xi|^2}}d\xi & (n=2m), \end{cases}$$

where dS_ω stands for the surface element of the unit sphere in $\mathbb{R}^n$. When $\varphi(x)$ is independent of t, we denote $M(\varphi|x,r;t)$ by $M(\varphi|x,r)$.

We consider the integral equations for scalar unknowns $u(x,t)$ of the form

$$u(x,t) = v(x,t) + L(F(u))(x,t), \ (x,t) \in \mathbb{R}^n \times [0,\infty), \tag{1.1}$$

where

$$L(F(u))(x,t) = A_n \int_0^t (t-\tau)M(F(u)|x,t-\tau;\tau)d\tau. \tag{1.2}$$

Moreover, v and F are given functions and A_n is a given positive constant. Note that L is a positive linear operator.

We now specify the constant A_n as follows;

$$A_n = \frac{1}{(n-2)\omega_n} \ (n=2m+1), \quad A_n = \frac{2}{(n-1)\omega_{n+1}} \ (n=2m),$$

where ω_n stands for the measure of the unit sphere in $\mathbb{R}^n$. Let $f(x)$ and $g(x)$ be given functions with compact support. And let $v = v_0(x,t)$ be a unique solution to the Cauchy

problem for a linear wave equation

$$
(1.3) \qquad \begin{aligned} &\partial_t^2 v_0(x,t) - \triangle v_0(x,t) = G(x,t), && (x,t) \in \mathbb{R}^n \times [0,\infty), \\ &v_0(x,0) = f(x),\ \partial_t v_0(x,0) = g(x), && x \in \mathbb{R}^n, \end{aligned}
$$

where

$$
G(x,t) = 2(m-1)A_n M(F(f)|x,t).
$$

Then we find that a solution $u(x,t)$ to the integral equation (1.1) is a solution to the Cauchy problem for a nonlinear wave equation of the form

$$
(1.4) \qquad \begin{aligned} &\partial_t^2 u(x,t) - \triangle u(x,t) = F(u)(x,t) - H(x,t), && (x,t) \in \mathbb{R}^n \times [0,\infty), \\ &u(x,0) = f(x),\ \partial_t u(x,0) = g(x), && x \in \mathbb{R}^n, \end{aligned}
$$

where

$$
H(x,t) = 2(m-1)A_n \int_0^t M(\partial_t(F(u))|x,t-\tau;\tau)d\tau.
$$

The uniqueness of solutions to the Cauchy problem (1.4) follows from Appendix in [5]. Note that G and H vanish for $n = 2$ or 3.

When $F(u)$ is of the form $A|u|^p$ $(A > 0)$, F. John [4] has proved the global existence of solutions to (1.4) in three space dimensions provided $p > 1 + \sqrt{2}$ and initial data are small. R.T. Glassey [3] has also proved the same results in two space dimensions for $p > (3+\sqrt{17})/2$. Moreover, Y. Choquet-Bruhat [2] has studied the global existence in the Sobolev spaces for higher dimensions.

Let $p_0(n)$ be the positive root of

$$
(1.5) \qquad (n-1)p^2 - (n+1)p - 2 = 0.
$$

Then it follows that $1 < p_0(n) \leqq 2$ for $n \geqq 4$ and the equality holds only for $n = 4$. In order to show the global existence of a C^1-solution to the integral equation (1.1), we require the following hypothesis $(H)_1$ on F:

$(H)_1$ $F(s)$ is of class C^1 with Hölder exponent δ $(0 < \delta < 1)$ and $F(0) = F'(0) = 0$. Hence there exists a positive constant A such that $|F^{(j)}(s)| \leqq A|s|^{p-j}$ $(j = 0, 1)$ for $p = 1 + \delta > p_0(n)$, $|s| \leqq 1$.

Note that, for $n = 4$, a hypothesis $(H)_2$, stated below, holds and hence (1.1) has a global solution of class C^2.

$(H)_2$ $F(s)$ is of class C^2 with Hölder exponent δ $(0 < \delta < 1)$ and there exist positive constants p and A such that $p_0(n) < p < 2 + \delta$ and $|F^{(j)}(s)| \leqq A|s|^{p-j}$ for $|s| \leqq 1, 0 \leqq j < p$.

We note that a typical example $F(s) = s^2$ for $n \geqq 5$ satisfies $(H)_2$ and the condition $p > p_0(n)$ guarantees the integrability of a function $s^{-pq(n,p)}$ over $[1, \infty)$,where

$$q(n,p) = \frac{n-1}{2}p - \frac{n+1}{2}. \tag{1.6}$$

Moreover we find the global existence of solutions to the nonlinear wave equations (1.4) provided some derivatives of f and g are small.

§2. Statement of Main Results

Throughout this talk we assume $n \geqq 4$. In order to state main results we introduce the following norm for $u \in C^0(\mathbb{R}^n \times [0, \infty))$ with supp $u \subset \{(x,t) : |x| \leqq t + k\}$;

$$||u|| = \sup_{(x,t)\in\mathbb{R}^n\times[0,\infty)} [(\frac{t+r+2k}{k})^{(n-1)/2} N(\frac{t-r+2k}{k})|u(x,t)|] \tag{2.1}$$

where $r \equiv |x|$ and k is a fixed positive constant. The function $N(s)$ of $s \in [1, \infty)$ in (2.1) is defined by dividing into three cases. For the odd dimensional case, we set

$$N(s) = s^{q(n,p)} \quad \text{if} \quad p > p_0(n). \tag{2.2}$$

For the even dimensional case, we first set

$$N(s) = \begin{cases} s^{q(n,p)} & \text{if} \quad p_0(n) < p < \frac{2n}{n-1}, \\ \frac{s^{(n-1)/2}}{\log(1+s)} & \text{if} \quad p = \frac{2n}{n-1}, \\ s^{(n-1)/2} & \text{if} \quad p > \frac{2n}{n-1}. \end{cases} \tag{2.3}$$

When $n = 2, 3$, the above norms are essentially the same ones as in [3], [4]. However, in order to discuss the solution to the equation (1.4), we need another function $N(s)$ for the even dimensional case. For a fixed number $\overline{q}$ which satisfies

$$\frac{1}{p_0(n)} \leqq \overline{q} < \frac{n-1}{2}, \tag{2.4}$$

we next set

$$(2.5) \qquad N(s) = \begin{cases} s^{q(n,p)} & \text{if} \quad p_0(n) < p < \frac{2}{n-1}(\overline{q} + \frac{n+1}{2}), \\ s^{\overline{q}} & \text{if} \quad p \geqq \frac{2}{n-1}(\overline{q} + \frac{n+1}{2}). \end{cases}$$

We here give some remarks on the above norms and relations between p, $q(n,p)$ and $\overline{q}$. First of all, since

$$\frac{2}{n-1}(\overline{q} + \frac{n+1}{2}) < \frac{2n}{n-1} \quad \text{and}$$

$$\begin{aligned} &\overline{q} \leqq q(n,p) && \text{if and only if} \quad p \geqq \frac{2}{n-1}(\overline{q} + \frac{n+1}{2}), \\ &\frac{n-1}{2} < q(n,p) && \text{if and only if} \quad p > \frac{2n}{n-1}, \end{aligned}$$

we know that the norm (2.1) with (2.5) is weaker than that with (2.3). Next, the factor $(t+r+2k)^{(n-1)/2}$ in (2.1) indicates the decay rate of a solution v_0 to (1.3) in its support and $N((t-r+2k)/k)$ is closely related to the decay rate of v_0 inside of the solid characteristic cone $\{(x,t)|r < t-k\}$. Finally, since (1.5) and (1.6) imply $q(n,p) > q(n,p_0(n)) = 1/p_0(n)$ for $p > p_0(n)$, we know that $pq(n,p) > 1, pq > 1$ if $p > p_0(n), q \geqq 1/p_0(n)$.

For each $j = 1,2$ let X_j be a Banach space defined by

$$\begin{aligned} X_j = &\{u \in C^j(\mathbb{R}^n \times [0,\infty)) : \text{supp } u \subset \{(x,t) \in \mathbb{R}^n \times [0,\infty) : r \leqq t+k\}, \\ &||D_x^\alpha u|| < \infty \text{ for } |\alpha| \leqq j\} \end{aligned}$$

equipped with a norm $||u||_{X_j} = \sum_{|\alpha| \leqq j} ||D_x^\alpha u||$. Now we state our theorems.

THEOREM 1. *Assume the hypothesis $(H)_j$, where $j = 1$ or 2. Then the integral equation (1.1) is uniquely and globally solvable in X_j, provided $v \in X_j$ and $||v||$ does not exceed a certain positive number which depends on A, k, n, p and $\overline{q}$.*

This is proved by using the following a priori estimate and the classical iteration method by Picard.

LEMMA 2.1. *Let L be the linear integral operator defined by (1.2). Assume that $u \in C^0(\mathbb{R}^n \times [0,\infty))$ with* supp $u \subset \{(x,t) \in \mathbb{R}^n \times [0,\infty) : |x| \leqq t+k\}$ *and $||u|| < \infty$. Then*

there exists a positive constant C depending only on n, p and $\overline{q}$ such that

$$\text{(BE)} \qquad ||L(|u|^p)|| \leqq Ck^2||u||^p \quad \text{if} \quad p > p_0(n).$$

REMARK 2.1: When n is even, the basic estimate (BE) does not hold, if $N(s) = s^q$ and $q > (n-1)/2$. Besides, if $n = 3$ then (BE) coincides in essence with (50a) of John [4].

THEOREM 2. *Assume that $f \in C^{m+3}(\mathbb{R}^n)$, $g \in C^{m+2}(\mathbb{R}^n)$ and supports of f and g are contained in $\{x \in \mathbb{R}^n : |x| \leqq k\}$. Furthermore, assume that $F \in C^{m+1}(\mathbb{R})$ and F satisfies the inequality in $(H)_2$. Let the norm (2.1) be given by (2.5) with $\overline{q} = (n-3)/2$ in even space dimensions. Then there exists a unique solution $u \in X_2$ to the Cauchy problem (1.4) provided $|D_x^\alpha f|$ $(|\alpha| \leqq m+1)$, $|D_x^\beta g|$ $(|\beta| \leqq m)$ and $|D_x^\gamma F(f)|$ $(|\gamma| \leqq m-1)$ are sufficiently small.*

REMARK 2.2: In the theorem 2, the number $\overline{q} = (n-3)/2$ is the maximal decay rate for the solution v_0 to the linear wave equation (1.3) in the solid characteristic cone $\{(x,t) \in \mathbb{R}^n \times [0,\infty) : |x| < t-k\}$.

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Blow-up of solutions to nonlinear wave equations in two space dimensions

By

Rentaro Agemi

1. Introduction

The present paper is concerned with blow-up of solutions to the initial value problem for nonlinear wave equations of the form

$$
(1.1)\qquad \begin{aligned} &\partial_t^2 u(x,t) - \Delta u(x,t) = \partial_t |au(x,t) + b\partial_t u(x,t)|^p, && x \in \mathbb{R}^n, \quad t \geq 0 \\ &u(x,0) = f(x), \quad \partial_t u(x,0) = g(x), && x \in \mathbb{R}^n, \end{aligned}
$$

where a, b and p are certain constants such that $(a,b) \neq (0,0)$ and $p > 1$. We also consider the equations of the form

$$
(1.2)\qquad \partial_t^2 v(x,t) - \Delta v(x,t) = |a\partial_t v(x,t) + b\partial_t^2 v(x,t)|^p.
$$

F. John [2] has proved in the case where $n = 3$ and $p = 2$ that the classical solutions to (1.1) blow up at finite time provided f and g have compact support and satisfy

$$
(1.3)\qquad \int_{\mathbb{R}^3} h_2(x)dx \geq 0,
$$

where

$$
(1.4)\qquad h_p(x) = g(x) - |af(x) + bg(x)|^p.
$$

Applying the above results to the equation (1.2), he also proved that a C^3-solution v to (1.2) blows up at finite time if initial values $v(x,0), \partial_t v(x,0)$ and $\partial_t^2 v(x,0)$ have compact support.

On the other hand, S. Klainerman [4] [5] has established the Sobolev inequalities in the Minkowski space and proved the global in time existense of solutions to nonlinear wave equations of the form

$$\partial_t^2 v(x,t) - \Delta v(x,t) = F(\partial_t u, \nabla_x u). \tag{1.5}$$

More precisely one can prove the followings (also see F. John [3], Chap. 3). Let $F(w)$ be a smooth function of $w = (w_0, w_1, \cdots, w_n)$ such that

$$|D^\alpha F(w)| \leq A|w|^{p-|\alpha|} \quad \text{for} \quad p > |\alpha|, |w| \leq 1. \tag{1.6}$$

Then there exists a unique global solution to (1.5) provided initial data with compact support are sufficiently small and

$$\frac{n-1}{2} > \frac{1}{p-1} \quad \text{i.e.} \quad p > \frac{n+1}{n-1}. \tag{1.7}$$

The results stated above show that the number $p = 2$ is critical for the equations (1.2) with $b = 0$ in three space dimensions.

The aim of the present paper is to show that the number $p = 3$ is critical for the equations (1.2) with $b = 0$ in two space dimensions. More precisely we prove the following theorem and its corollary.

Theorem. *Let $2 \leq p \leq 3$ if $b \neq 0$ and $1 < p \leq 3$ if $b = 0$. Moreover, let $u(x,t)$ be a global C^2-solution to (1.1) with initial data $f \in C^3(\mathbb{R}^2)$ and $g \in C^2(\mathbb{R}^2)$. Then $u(x,t)$ vanishes identically provided f and g have compact support and satisfy*

$$f(x) \geq 0, \quad h_p(x) \geq 0 \quad \text{for} \quad x \in \mathbb{R}^2. \tag{1.8}$$

Corollary. *Let $2 \leq p \leq 3$ if $b \neq 0$ and $1 < p \leq 3$ if $b = 0$. Moreover, let $v(x,t)$ be a global C^3-solution to (1.2). Then $v(x,t)$ vanishes identically provided initial values $v(x,0), \partial_t v(x,0)$ and $\partial_t^2 v(x,0)$ have compact support and satisfy*

$$v(x,0) = 0, \quad \partial_t v(x,0) \geq 0 \quad \text{for} \quad x \in \mathbb{R}^2. \tag{1.9}$$

A key of the proof of Theorem is to derive an integral inequality for a nonnegative function $U(r)$ of the form

$$\sqrt{r}U(r) \geq C \int_c^r \sqrt{\lambda} U(\lambda)^p d\lambda, \tag{1.10}$$

where c and C are positive constants. The integral inequality used in [2] is as follows.

$$rU(r) \geq C \int_c^r \lambda U(\lambda)^2 d\lambda.$$

To derive the integral inequality (1.10) we use the positiveness of Riemann function in two space dimensions and the fundamental identity for iterated spherical means (F. John [1], p.81). We associate with a function $\varphi(x)$ of $x \in \mathbb{R}^n$ its spherical means at the origin with raduis r

$$\tilde{\varphi}(r) = \frac{1}{\omega_n} \int_{|\omega|=1} \varphi(r\omega) dS_\omega \tag{1.11}$$

where ω_n and dS_ω stands for the surface area and the surface element of the unit sphere in $\mathbb{R}^n$, respectively. The fundamental identity is

$$\begin{aligned} &\frac{1}{\omega_n^2} \int_{|\zeta|=1} \int_{|\omega|=1} \varphi(r\zeta + \rho\omega) dS_\omega dS_\zeta \\ &= \frac{2\omega_{n-1}}{\omega_n (2r\rho)^{n-2}} \int_{|\rho-r|}^{\rho+r} \lambda h(\rho, \lambda; r)^{(n-3)/2} \tilde{\varphi}(\lambda) d\lambda, \end{aligned} \tag{1.12}$$

where

$$h(\rho, \lambda; r) = (\rho^2 - (\lambda - r)^2)((\lambda + r)^2 - \rho^2). \tag{1.13}$$

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Asymptotic Behavior of Solutions to Semilinear Diffusion Equations of Volterra Type

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1. Problem

This lecture is concerned with the asymptotic analysis for some reaction-diffusion equations with time-delays which are represented by Volterra integrals. Especially, my interest lies in studying what kind of serious effects are brought about by the presence of time-delays. I will take a simple example from mathematical biology.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let $u = u(x,t)$ $(x \in \Omega,\ t > 0)$ satisfy

$$(P)\quad \begin{cases} u_t = d\Delta u + au\{1 - \alpha u - (1-\alpha)k*u\} & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 \geq 0 \quad (\not\equiv 0) & \text{in } \Omega, \end{cases}$$

where a and d are positive constants, α is a number satisfying $0 \leq \alpha \leq 1$ and the convolution $k*u$ is defined by

$$k*u(t) = \int_0^t k(t-s)u(s)ds.$$

It is assumed that k is a smooth nonnegative function on $(0,\infty)$ such that both k and tk are integrable over $(0,\infty)$. We normalize k so that $\int_0^\infty k(t)dt = 1$. We also assume that u_0 is a nonnegative $L^\infty(\Omega)$-function.

Problem (P) appears in population dynamics. In such a model, u represents the population density of some species and its growth rate, obeying the logistic law, is affected by a memory effect. Typical examples of kernels k are given by

$$\text{(K.1)} \qquad k(t) = \frac{1}{T} e^{-t/T},$$

$$\text{(K.2)} \qquad k(t) = \frac{1}{T^2} te^{-t/T}.$$

(When k is given by (K.1), a similar problem to (P) appears in a nuclear reactor model.)

It is very easy to establish the existence and uniqueness of nonnegative global solutions for (P). Moreover, it is possible to show that every solution u of (P) satisfies

$$\lim_{t\to\infty} u(\cdot,t) = 0 \qquad \text{uniformly in } \Omega,$$

if $a \le d\lambda_1$, where λ_1 is the principal eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition. Therefore, I will concentrate myself on the study of asymptotic behavior of solutions of (P) in the case $a > d\lambda_1$.

2. Preliminary results and related works

Suppose that there are no time-delays in (P); that is, $\alpha = 1$. It is well known that, if $a > d\lambda_1$, then every solution u of (P) satisfies

$$\lim_{t\to\infty} u(\cdot,t) = \varphi \qquad \text{uniformly in } \Omega,$$

where φ is a unique positive solution of

$$\text{(SP)} \qquad d\Delta\varphi + a\varphi(1-\varphi) = 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega.$$

(Such φ exists if and only if $a > d\lambda_1$.) This fact implies that the unique positive steady-state φ is a global attractor for (P) when there are no time-delays. Here it should be noted that the corresponding stationary problem (SP) is the same even if time-delays are concerned.

By the above consideration, our problem is closely related to the following question;

"Do the time-delays give any influence on the stability of φ ?"

If the Neuman condition is imposed on the boundary, there is a pretty lot of information on this question; say, stability of φ, destabilization of φ and Hopf bifurcation (e.g., [1], [2], [4], [5]). However, the Dirichlet boundary condition is concerned, the level of understanding seems very poor because the stability analysis of φ requires delicate calculations. As far as I know, there are few works except for Schiaffino and Tesei [3], in which they have proved that φ is a global attractor for (P) if $\alpha > 1/2$.

3. Global attractivity

Before stating results, define the Laplace transform $\hat{k}(p)$ of k by

$$\hat{k}(p) = \int_0^\infty e^{-pt} k(t)dt.$$

When k is given by (K.1) (resp. (K.2)),

$$k(p) = 1/(1+pT) \qquad (\text{resp. } \hat{k}(p) = 1/(1+pT)^2).$$

Theorem 1. Assume that there exists a positive constant c_0

such that

$$(*) \qquad \alpha + (1-\alpha)\mathrm{Re}\,\hat{k}(i\eta) \geq c_0 \qquad \text{for all} \quad \eta \in \mathbb{R}.$$

Then every solution u of (P) satisfies

$$\lim_{t\to\infty} u(\cdot,t) = \varphi \qquad \text{uniformly in} \quad \Omega.$$

Remark. (1) For general k, $\mathrm{Re}\,\hat{k}(i\eta) \geq -1$; so that $(*)$ is satisfied if $\alpha > 1/2$. This implies that Theorem 1 extends the result of Schiaffino and Tesei.

(2) If k is nonnegative, non-increasing and convex, then $\mathrm{Re}\,\hat{k}(i\eta) > 0$. Therefore, such a kernel (including (K.1)) satisfies $(*)$ if $\alpha > 0$.

(3) If k is given by (K.2), then $(*)$ is equivalent to

$$1/9 < \alpha < 1.$$

Theorem 2. Let $\alpha = 0$. Suppose that $\mathrm{Re}\,\hat{k}(i\eta) \geq 0$ for all $\eta \in \mathbb{R}$. Then

$$\sup_{t>1} \|u(t)\|_\infty < \infty.$$

Moreover, if $dk/dt \in L^1(0,\infty)$ and $\mathrm{Re}\,\{\hat{k}(i\eta)^{-1}\} \geq c_1$ for all $\eta \in \mathbb{R}$ with some $c_1 > 0$, then the same conclusion as Theorem 1 holds true.

Remark. When k is given by (K.1), Theorems 1 and 2 assure the global attractivity of φ for every $0 \leq \alpha \leq 1$.

4. Local stability

The abstract theory for the local stability of φ asserts the following: φ is asymptotically stable (in a suitable

totopolgy) if, for μ with $\mathrm{Re}\,\mu \geq 0$, the "characteristic problem"

$$\text{(CP)}\quad \begin{aligned} [\mu - d\Delta - a\{1-(1+\alpha)\varphi-(1-\alpha)\hat{k}(\mu)\varphi\}]w &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

has no non-trivial solutions. The abstract theory for Hopf bifurcations can be also developed. However, the analysis of (CP) is very delicate; so that it is not so easy to get any substantial results about the stability of φ. When k is defined by (K.2), we can show

Proposition 3. Define k by (K.2). For $0 \leq \alpha \leq 1/9$, if

$$\sqrt{aT\ \|\varphi\|_\infty} < \frac{2\sqrt{2}}{\sqrt{1-\alpha} + \sqrt{1-9\alpha}},$$

then φ is asymptotically stable.

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On one dimensional nonlinear thermoelasticity

By

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In this note, I would like to report recent works by the auther and R. Racke, Bonn University ([3], [5]), concerning a global existence of small and smooth solutions to one dimensional nonlinear thermoelastic equations in the case of a bounded reference configuration. Let us recall the equations of one dimensional nonlinear thermoelasticity. Let $\Omega = (0, 1)$ be a unit interval in one dimensional Euclidean space $\mathbb{R}$, which is identified with the reference configuration $\mathcal{R}$. The thermoelastic motion is described by the deformation map: $x \in \Omega \longmapsto X(t,x) \in \mathbb{R}$ and the absolute temperature $T(t,x) \in \mathbb{R}$ of the material point of coordinate $X(t,x)$, where t denotes time variable. Then, the equations of balance of linear momentum and balance of energy are given by (cf. Carlson [1]):

$$\text{(B.M)} \qquad \rho_{\mathcal{R}} X_{tt} = \tilde{S}_x + \rho_{\mathcal{R}} b,$$

$$\text{(B.E)} \qquad (\tilde{\varepsilon} + (\rho_{\mathcal{R}}/2)X_t^2)_t = (\tilde{S}X_t)_x + \tilde{q}_x + \rho_{\mathcal{R}} r,$$

where we use the following notation: The subscripts t and x denote differentiations with respect to t and x, respectively. $\rho_{\mathcal{R}}$ is the material density. The b and r are specific body force and heat supply, respectively. For simplicity, I assume that $\rho_{\mathcal{R}} = 1$ and that $b = r = 0$, below. $\tilde{\varepsilon}$ is the specific internal energy. $\tilde{q}$ is the heat flux. $\tilde{S}$ is the Piola-Kirchhoff stress tensor. According to 2nd Law of Thermodynamics and Coleman's theorem [2], I make the following assumptions.

Assumptions (1) There exists a so called Helmholtz energy function $\psi(F,T)$, which is real-valued and in $C^{\infty}(G(B))$, such that

(A.1) $\tilde{S} = S(X_x(t,x),T(t,x))$ and $\tilde{\varepsilon} = \varepsilon(X_x(t,x),T(t,x))$ where

(A.2) $S(F,T) = (\partial\psi/\partial F)(F,T)$, $\varepsilon(F,T) = \psi(F,T) - T(\partial\psi/\partial T)(F,T)$ $\quad (F = X_x)$,

$$G(B) = \{ (F,T) \in \mathbb{R}^2 \mid |F-1| + |T-T_0| < B,\ T > T_0/2 \}.$$

T_0 is a positive constant denoting the natural temperature of the reference body $\mathcal{R}$ and B is another positive constant. Moreover, I assume that

(A.3) $(\partial^2\psi/\partial F^2)(F,T) > 0$, $(\partial^2\psi/\partial T^2)(F,T) < 0$, $(\partial^2\psi/\partial F\partial T)(F,T) \neq 0$ for $(F,T) \in G(B)$.

(2) There exists a positive function $Q(F,T) \in C^{\infty}(G(B))$ such that

(A.4) $\tilde{q} = Q(X_x(t,x),T(t,x))T_x(t,x)$.

And then, (B.M) and (B.E) are rewritten as follows: for $t > 0$ and $x \in \Omega$,

(B.M)' $X_{tt} = S(X_x,T)_x$,

(B.E)' $(\varepsilon(X_x,T) + \frac{1}{2}X_t^2)_t = (S(X_x,T)X_t)_x + (Q(X_x,T)T_x)_x$.

If you use the entropy: $N(F,T) = -(\partial\psi/\partial T)(F,T)$, (B.E)' can be rewritten by:

(B.E)" $TN(X_x,T)_t = (Q(X_x,T)T_x)_x$.

In fact, multiplying (B.M)' by X_t implies that $\frac{1}{2}(X_t^2)_t = S_xX_t$. Using the constitutive relations (A.2), you have the identity: $\varepsilon(X_x,T)_t = TN(X_x,T)_t + S(X_x,T)X_{tx}$. Since $(S(X_x,T)X_t)_x = S(X_x,T)_xX_t + S(X_x,T)X_{tx}$, (B.E)" follows from (B.M)' and (B.E)'. Obviously, (B.E)' follows also from (B.M)' and (B.E)".

Put $u = X - x$ and $\theta = T - T_0$. As boundary conditions, I consider here the following

four type: for $t > 0$ and $x = 0$ and 1,

(D.D) $u = 0$ and $\theta = 0$,

(D.N) $u = 0$ and $\theta_x = 0$,

(N.D) $u_x = 0$ and $\theta = 0$,

(N.N) $S = 0$ and $\theta_x = 0$.

Since S can be represented by using the Taylor expansion as follows: $S = S_1 u_x + N_1\theta$, (N.D) is equivalent to what $S = 0$ and $\theta = 0$ at $x = 0$ and 1. In (N.N) case, in addition to (A.1)-(A.3), I assume that

(A.5) $S(1,T_0) = 0$.

In other cases, you may assume without loss of generality that (A.5) is valid. In fact, you can consider:

(B.M)" $X_{tt} = [S(X_x,T) - S(1,T_0)]_x$

instead of (B.M)' if (A.5) is not satisfied. But, in (N.N) case, if you consider (B.M)" instead of (B.M)', you must consider the boundary condition: $S(X_x,T) - S(1,T_0) = 0$ at $x = 0$ and 1 instead of (N.N). Since it is inhomogeneous, in general you can not expect to get the decay properties of solutions to linearized equations, and then the global existence theorem can not be expected in general.

As initial conditions, I put

(I.C) $X(0,x) = x + u_0(x)$, $X_t(0,x) = u_1(x)$, $T(0,x) = T_0 + \theta_0(x)$ for $x \in \Omega$,

where u_0, u_1 and θ_0 are given functions. In cases of (N.D) and (N.N), we assume that

(A.6) $\int_0^1 u_1(x)dx = 0$.

In fact, if you integrate (B.M)' under the boundary condition (N.D) or (N.N), you have

$$\int_0^1 X_t(t,x)\,dx = \int_0^1 u_1(x)\,dx.$$

Since I expect that $X_t(t,x) \to 0$ as $t \to \infty$, (A.6) is needed. Since X does not appear in (B.M)' and (B.E)", if we put $X' = X - (\int_0^1 u_1(x)dx)t$, then X' and T satisfy (B.M)', (B.E)", boundary conditions (N.D) or (N.N) and

$$\text{(I.C)}' \quad X'(0,x) = x + u_0(x),\ X_t'(0,x) = u_1(x) - \int_0^1 u_1(x)dx,\quad T(0,x) = T_0 + \theta_0(x).$$

Moreover, you have $\int_0^1 X_t'(t,x)dx = 0$. So, (A.6) is not an essential assumption.

Now, let us discuss the equilibrium state. In all the cases, $X = x$ and $T = T_0$ are solutions for initial data: $u_0 = u_1 = \theta_0 = 0$. In cases of (D.N) and (N.N), integrating (B.E)' on $(0,t) \times \Omega$, you have

$$\text{(1.1)} \quad \int_0^1 \{\varepsilon(X_x(t,x),T(t,x)) + \tfrac{1}{2}X_t^2(t,x)\}\,dx = c(u_0,u_1,\theta_0) \text{ where}$$

$$c_0(u_0,u_1,\theta_0) = \int_0^1 \{\varepsilon(1 + u_0'(x),T_0 + \theta_0(x)) + \tfrac{1}{2}u_1(x)^2\}dx,\ u_0' = du_0/dx,$$

as long as the solutions exist. If you expect that $X_t \to 0$, $X_x \to X_\infty$ and $T \to T_\infty$, X_∞ and T_∞ being constants, letting $t \to \infty$ in (1.1), you see that X_∞ and T_∞ should satisfy:

$$\text{(1.2.a)} \quad (X_\infty,T_\infty) = c(u_0,u_1,\theta_0),$$

$$\text{(1.2.b)} \quad (X_\infty,T_\infty) \in G(B).$$

In (N.N) case, in addition to (1.2.a) and (1.2.b), what $S = 0$ at $x = 0$ and 1 implies the condition:

$$\text{(1.2.c)} \quad S(X_\infty,T_\infty) = 0.$$

On the other hand, if you consider the map : $(1, T) \in G(B) \longmapsto \varepsilon(1,T) \in \mathbb{R}$ in (D.N) case and the map : $(F, T) \in G(B) \longmapsto (S(F,T), \varepsilon(F,T)) \in \mathbb{R}^2$ in (N.N) case, respectively,

the implicit function theorem tells you the unique existence of (X_∞, T_∞) satisfying (1.2) provided that $|u_0(x)|$, $|u_1(x)|$, $|\theta_0(x)|$ are sufficiently small, especially $X_\infty = 1$ in (D.N) case. Because, $(\partial\psi/\partial T)(1,T_0) = -T_0(\partial^2\psi/\partial T^2)(1,T_0) \neq 0$ in (D.N) case and the Jacobian $\partial(S,\varepsilon)/\partial(F,T)$ is equal to

$$-T_0(\partial^2\psi/\partial T^2)(1,T_0)(\partial^2\psi/\partial F^2)(1,T_0) + T_0(\partial^2\psi/\partial F\partial T)(1,T_0)^2 \neq 0$$

under the assumption (A.5) in (N.N) case.

I shall say that X and T will be global smooth solutions if X and T satisfy (B.M)', (B.E)' for $t \in (0,\infty)$ and $x \in \Omega$, one of the boundary conditions: (D.D), (D.N), (N.D) and (N.N) for $t \in (0,\infty)$ and $x = 0$ and 1, and the initial condition (I.C) for $x \in \Omega$, and if X and T belong to $C^2([0,\infty) \times \overline{\Omega})$ and $(X(t,x),T(t,x)) \in G(B)$ for all $(t,x) \in [0,\infty) \times \overline{\Omega}$.

Roughly speaking, I got the following theorem.

<u>Theorem</u>. <u>If initial data</u> u_0, u_1, θ_0 <u>are sufficiently small and smooth and satisfy the suitable compatibility conditions, then there exists a unique pair of global smooth solutions</u> $(X(t,x),T(t,x))$. <u>Moreover, they have the following asymptotic behaviours:</u>

(D.D) $X_t(t,x) \to 0$, $X_x(t,x) \to 1$, $T(t,x) \to T_0$ <u>as</u> $t \to \infty$ <u>for</u> $x \in \Omega$;

(D.N) $X_t(t,x) \to 0$, $X_x(t,x) \to 1$, $T(t,x) \to T_\infty$ <u>as</u> $t \to \infty$ <u>for</u> $x \in \Omega$;

(N.D) $X_t(t,x) \to 0$, $X_x(t,x) \to 1$, $T(t,x) \to T_0$ <u>as</u> $t \to \infty$ <u>for</u> $x \in \Omega$;

(N.N) $X_t(t,x) \to 0$, $X_x(t,x) \to X_\infty$, $T(t,x) \to T_\infty$ <u>as</u> $t \to \infty$ <u>for</u> $x \in \Omega$.

<u>Remark</u>. (1) The theorem was proved by M. Slemrod [4] in cases of (D.N) and (N.D), by R. Racke and the auther [3] in (D.D) case and by the auther [5] in (N.N) case.
(2) In case of Cauchy problem, according to the results due to Dafermos and Hsiao, Quart. Appl. Math. 44 (1986), 463-474 and Hrusa and Messaoudi, Arch. Rational Mech. Anal.

111 (1990), 135-151, for large initial data the smooth solutions blow up in finite time. From this point of view, the smallness assumption is necessary to get the global existence theorem.

Roughly spoken, if E(t) denotes a typical energy term, the L^2-energy decay method used by Slemrod [4] tries to prove an estimates of the type: $E(t) \leqq \text{const.}\ E(0)$ (uniformly on the interval of local existence) directly by differentiating the differential equations with respect to the time variable t and the space variable x, multiplying in L^2 with appropriate derivatives of the solutions and performing partial integrations. These partial integrations are possible in the case of the boundary conditions: (D.N) and (N.D) studied by Slemrod [4], but they lead to ill-behaved boundary terms in the cases of (D.D) and (N.N). So, after Slemrod's work in 1981, the problem for (D.D) and (N.N) was open about 10 years. Again very roughly spoken, the methods in [3] and [5] considers the inequality: $E(t) \leqq \text{const.}E(0)\exp(\int_0^t V(s)ds)$, which is obtained for a local solution using Gronwall's inequality, with V involving lower order derivatives of the solutions. The integral is shown to be bounded independent of t with the help of the decay properties of solutions to the linearized problem. And the decay properties can be shown by using the spectral analysis to the corresponding ordenary differential equations with spectral parameter to the linearized equations.

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