Infinite Dimensional Analysis on an Exterior Bundle and Supersymmetric Quantum Field Theory

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Series ♯ 34. August, 1994

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Infinite Dimensional Analysis on an Exterior Bundle and Supersymmetric Quantum Field Theory

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I. Introduction

In a recent development of quantum field theory (QFT), a new concept of symmetry, called *supersymmetry*, has been introduced [WZ1, WZ2] and various aspects of it have been discussed in both the physics and the mathematics literatures†. The supersymmetry is a symmetry which treats bosons and fermions on an equal footing. In some models of supersymmetric QFT (SSQFT), cancelations of ultraviolet divergences occur without ad hoc renormalizations. For some reasons including those just mentioned, there has been a growing belief that supersymmetry should play an important role in constructing a unified theory of elementary particles (e.g.,[We], [K]).

Mathematical studies on models of SSQFT have been made by some authors. Jaffe et al. gave detailed analyses of the Wess-Zumino models from the view-point of constructive QFT ([JL1], [JL2] and references therein). In [A8] the author has developed a general theory of infinite dimensional analysis on the abstract Boson-Fermion Fock space (cf. [A1] and [A5] as preliminary work), which, in application to SSQFT, gives a mathematically unified description of some models of SSQFT. In this theory we introduced an operator of Dirac-Kähler type acting in the abstract Boson-Fermion Fock space and analyzed operator-theoretical aspects of it. In particular, we derived an index formula for the Dirac-Kähler operator in terms of path (functional) integral representations. In [HK] a formulation of supersymetry within a functional approach of Euclidean QFT is given.

The abstract theory given in [A8] has been developed in some directions. In [AM1], we established decomposition theorems of de Rham-Hodge-Hodaira type with respect to (w.r.t.) exterior differential operators and Laplacians on an infinite dimensional exterior bundle, which include an extension of a decomposition theorem in [Sh], and discussed test functional spaces for distribution theories on infinite dimensional spaces (see also [M]). In [AM2], a comparison theorem was proven between a test functional space of Malliavin type and one of Hida type [HPS] in the framework of [AM1] ([A7]).

Another direction was given in [A4] and [A10], where supersymmetric extensions of non-supersymmetric qunatum field models were discussed.

In this note we present in a general framework a brief review of some basic results contained in the previous papers [A5–A8, A10, A11, AM1]. As for analysis on the Boson-

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† There is a vast literature concerning supersymmetry, see, e.g., [We], [F] and references therein for fundamental physical aspects of supersymmetry. A detailed survey for mathematical discussions of *supersymmetric quantum mechanics* is given in Chapter 5 of [Th] (cf. also references therein).

Fermion Fock space, which is a special case of the theory presented below, more detailed reviews are given in [A9] and [A13].

II. An Infinite Complex and Laplacians

2.1. A random process and a gradient operator

Let $\mathcal{H}$ be a separable real Hilbert space with inner product $(\cdot, \cdot)_{\mathcal{H}}$ and $\{\phi(f)|f \in \mathcal{H}\}$ be a family of random variables on a probability space (E, Σ, μ) with the following properties:

(ϕ.1) $\{\phi(f)|f \in \mathcal{H}\}$ is full, i.e., the Borel field Σ coincides with the one generated by $\{\phi(f)|f \in \mathcal{H}\}$.

(ϕ.2) If $f_n \to f$ in $\mathcal{H}$ as $n \to \infty$, then there exists a subsequence $\{f_{n_k}\}$ such that $\phi(f_{n_k}) \to \phi(f)$ a.e. as $k \to \infty$.

Let $C_b^\infty(\mathbf{R}^n)$ be the set of infinitely differentiable functions on $\mathbf{R}^n$ with all the partial derivatives being bounded on $\mathbf{R}^n$. For a linear operator T from $\mathcal{H}_c$ (the complexification of $\mathcal{H}$) to another Hilbert space with domain $D(T)$, we denote by $\mathfrak{C}_{b,T}^\infty$ the subspace spanned by functions of the form

$$F(f_1, \cdots, f_n) := F(\phi(f_1), \cdots, \phi(f_n)),\ F \in C_b^\infty(\mathbf{R}^n), f_j \in D(T) \cap \mathcal{H}, j = 1, \cdots, n,\ n \geq 1.$$

For the case $D(T) = \mathcal{H}_c$, we set $\mathfrak{C}_{b,T}^\infty = \mathfrak{C}_b^\infty$. If $D(T) \cap \mathcal{H}$ is dense in $\mathcal{H}$, then $\mathfrak{C}_{b,T}^\infty$ is dense in $L^2(E, d\mu)$ (see, e.g., [S, Lemma I.5]).

We say that $f \in \mathcal{H}$ is *well-μ-admissible* if there exists a function $\varrho_f \in L^2(E, d\mu)$ such that, for all $F \in C_b^\infty(\mathbf{R}^n), f_j \in \mathcal{H}, j = 1, \cdots, n,\ n \geq 1$,

$$\int_E \sum_{j=1}^n (\partial_j F)(f_1, \cdots, f_n)(f, f_j)_{\mathcal{H}} d\mu = \int_E F(f_1, \cdots, f_n)\varrho_f d\mu \tag{2.1}$$

(cf. [AR].) For each well-μ-admissible vector $f \in \mathcal{H}$, the function ϱ_f is uniquely determined and real. Condition (2.1) may be regarded as an integration by parts formula w.r.t. the measure μ.

We denote by $\mathcal{H}_\mu$ the set of all well-μ-admissible vectors in $\mathcal{H}$, which is a subspace of $\mathcal{H}$. We assume the following:

Assumption A. The subspace $\mathcal{H}_\mu$ is dense in $\mathcal{H}$.

Under this assumption, we can define a linear operator ("gradient operator") $\nabla : L^2(E, d\mu) \to L^2(E, d\mu; \mathcal{H}_c)$ with domain $\mathfrak{C}_b^\infty$ such that

$$\nabla F(f_1, \cdots, f_n) = \sum_{j=1}^n (\partial_j F)(\phi(f_1), \cdots, \phi(f_n)) f_j, \quad F(f_1, \cdots, f_n) \in \mathfrak{C}_b^\infty.$$

One can easily show that ∇ is closable.

Example 2.1. *Boson Fock space.* Let $\{\phi(f)|f \in \mathcal{H}\}$ be the Gaussian random process indexd by $\mathcal{H}$ and (E, Σ, μ_0) be the underlying probability space, so that

$$\int_E e^{i\phi(f)} d\mu_0 = e^{-\|f\|^2_{\mathcal{H}}/2}, \quad f \in \mathcal{H}.$$

Then every $f \in \mathcal{H}$ is well-μ_0-admissible with $\varrho_f = \phi(f)$. Hence $\mathcal{H}_{\mu_0} = \mathcal{H}$ and Assumption A is satisfied. There exist probability measures which are absolutely continuous w.r.t. μ_0 and satisfy Assumption A. The Hilbert space $L^2(E, d\mu_0)$ is isomorphic in a natural way to the Boson Fock space over $\mathcal{H}_c$ [S, §I.3].

Example 2.2. *QFT models of $P(\phi)$-type.* Let $\mathcal{S}_{\rm real}(\mathbf{R}^d)$ be the Schwartz space of rapidly decreasing real-valued C^∞-functions on $\mathbf{R}^d$ and $\mathcal{S}_{\rm real}(\mathbf{R}^d)'$ be its topological dual space. Then, for each constant $m > 0$, there exists a probability measure μ_m on $\mathcal{S}_{\rm real}(\mathbf{R}^d)'$ such that

$$\int_{\mathcal{S}_{\rm real}(\mathbf{R}^d)'} e^{i<\phi,f>} d\mu_m(\phi) = e^{-<f,(-\Delta+m^2)^{-d/2}f>/4}, \quad f \in \mathcal{S}_{\rm real}(\mathbf{R}^d),$$

where $< \cdot, \cdot >$ is the canonical duality pairing between $\mathcal{S}_{\rm real}(\mathbf{R}^d)'$ and $\mathcal{S}_{\rm real}(\mathbf{R}^d)$ and Δ is the d-dimensional Laplacian. The measure μ_m is a concrete realization of μ_0 in Example 1, which is related to QFT. In the case $d = 1$, the measure μ_m describes the *time-zero free Bose field* with mass m on one-dimensional space $\mathbf{R}$. In the case $d = 2$, μ_m gives an *Euclidean free field theory* on $\mathbf{R}^2$.

Let P be a polynomial on $\mathbf{R}$ and bounded from below. For each nonnegative function $g \in L^q(\mathbf{R}^d)(1 < q \le 2)$, one can define a functional $V_g = \int_{\mathbf{R}^d} : P(\phi(x)) :_{\mu_m} g(x)dx$ on $\mathcal{S}_{\rm real}(\mathbf{R}^d)'$ ([S],[GJ]), which describes a self-interaction of the Bose field $\phi(x)$. Then the probability measure $d\mu := \exp(-V_g)d\mu_m / \int_{\mathcal{S}_{\rm real}(\mathbf{R}^d)'} e^{-V_g} d\mu_m$ satisfies Assumption A. See [S], [GJ] for more details.

Example 2.3. *Measures defined by ground states of quantum scalar field theories.* See [A10].

2.2. An infinite complex

Let $\mathcal{K}$ be a separable real Hilbert space and $\bigwedge^p(\mathcal{K}_c)$ $(p = 0, 1, 2, \cdots)$ be the p-fold antisymmetric tensor product of $\mathcal{K}_c$ ($\bigwedge^0(\mathcal{K}_c) := \mathbf{C}$). For $u_j \in \mathcal{K}_c, j = 1, \cdots, p$, we define their exterior product $u_1 \wedge \cdots \wedge u_p \in \bigwedge^p(\mathcal{K}_c)$ by

$$u_1 \wedge \cdots \wedge u_p = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma) u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(p)}$$

where $\mathfrak{S}_p$ is the symmetric group of order p and $\varepsilon(\sigma)$ is the sign of the permutation σ. For each $p = 0, 1, 2, \cdots$, we set

$$\bigwedge^p(E, \mathcal{K}) := L^2\left(E, d\mu; \bigwedge^p(\mathcal{K}_c)\right) = L^2(E, d\mu) \otimes \bigwedge^p(\mathcal{K}_c).$$

Let $\mathsf{C}(\mathcal{H}_c, \mathcal{K}_c)$ denote the set of densely defined closed linear operators from $\mathcal{H}_c$ to $\mathcal{K}_c$. We introduce a subset of $\mathsf{C}(\mathcal{H}_c, \mathcal{K}_c)$: we say that an operator $S \in \mathsf{C}(\mathcal{H}_c, \mathcal{K}_c)$ is in the set $\mathsf{C}_\mu(\mathcal{H}_c, \mathcal{K}_c)$ if it satisfies the following conditions (S.1) and (S.2):

(S.1) For all $f \in D(S)$, $J_{\mathcal{H}} f \in D(S)$, where $J_{\mathcal{H}}$ is the natural conjugation on $\mathcal{H}_c$.
(S.2) The subspace $\mathcal{K}_{S,\mu} := \{u \in D(S^*) | S^* u \in \mathcal{H}_{\mu,c}\}$ is dense in $\mathcal{K}_c$.

In what follows, we assume that $\mathsf{C}_\mu(\mathcal{H}_c, \mathcal{K}_c)$ is not empty. Let

$$\bigwedge\nolimits_0^p(\mathcal{K}_c) = \mathcal{L}\{u_1 \wedge u_2 \wedge \cdots \wedge u_p | u_j \in \mathcal{K}_c, j = 1, \cdots, p\},$$

where $\mathcal{L}\{\cdots\}$ denotes the subspace spanned by the vectors in the set $\{\cdots\}$ ($\bigwedge_0^0(\mathcal{K}) := \mathbb{C}$). For $S \in \mathsf{C}_\mu(\mathcal{H}_c, \mathcal{K}_c)$, we define

$$\mathfrak{D}_{S,p} = \mathcal{L}\left\{ F\psi \middle| F \in C_{b,S}^\infty, \psi \in \bigwedge\nolimits_0^p(\mathcal{K}_c) \right\}, \quad p \geq 0,$$

which is dense in $\bigwedge^p(E, \mathcal{K})$. We define an operator $d_{S,p}$ with domain $\mathfrak{D}_{S,p}$ by

$$d_{S,p} F\psi == \sqrt{p+1}(S\nabla F) \wedge \psi, \quad F \in \mathfrak{C}_{b,S}^\infty, \psi \in \bigwedge\nolimits_0^p(\mathcal{K}_c).$$

Proposition 2.1. *For all $p \geq 0$, $d_{S,p}$ is well defined as an operator from $\bigwedge^p(E, \mathcal{K})$ to $\bigwedge^{p+1}(E, \mathcal{K}_c)$ and the equation*

$$d_{S,p+1} d_{S,p} = 0 \tag{2.2}$$

holds. Moreover, $d_{S,p}$ is closable.

Proof. Eq.(2.2) follows from a direct calculation. Using (2.1), we can compute $d_{S,p}^*$ and show that $D(d_{S,p}^*)$ is dense, which implies the closability of $d_{S,p}$. ∎

We denote the closure of $d_{S,p}$ by the same symbol. Thus we obtain an infinite complex $\{d_{S,p}, D(d_{S,p})\}_{p=0}^\infty$. The p-th cohomology space of this complex may be defined by

$$H_S^p = \ker d_{S,p} / \overline{R(d_{S,p-1})}, \quad p \geq 1,$$

where $R(T)$ denotes the range of the operator T and $\overline{\{\cdots\}}$ the closure of the set $\{\cdots\}$.

Remark. An important point in the definition of the operator $d_{S,p}$ lies in that it is "parameterized" by a densely defined closed linear operator $S \in \mathsf{C}_\mu(\mathcal{H}_c, \mathcal{K}_c)$. This freedom allows us to produce, in concrete realizations, various infinite dimensional Laplacians associated with $\{d_{S,p}\}_{p=0}^\infty$ by changing S, see [A8], [AM1]. In this sense, the operator S plays a role of a "deformation parameter".

2.3. Laplacians and decomposition theorems of de Rham-Hodge-Kodaira type

As in the case of analysis on finite dimensional manifolds, it is natural to introduce the operators $d_{S,p}^* d_{S,p} + d_{S,p-1} d_{S,p-1}^*, p = 0, 1, 2, \cdots$, as the Laplacians associated with the complex $\{d_{S,p}\}_{p=0}^{\infty}$ ($d_{S,-1} := 0$). In the present case, however, it may happen that $D(d_{S,p}^* d_{S,p}) \cap D(d_{S,p-1} d_{S,p-1}^*)$ is not dense in $\bigwedge^p(E, \mathcal{K})$. An idea to avoid this difficulty is to use quadratic form technique. Indeed, we can show that, for each $p \geq 0$, there exists a unique nonnegative self-adjoint operator $\Delta_{S,p}$ acting in $\bigwedge^p(E, \mathcal{K})$ such that $D(\Delta_{S,p}^{1/2}) = D(d_{S,p}) \cap D(d_{S,p-1}^*)$ and

$$(\Delta_{S,p}^{1/2} \Psi, \Delta_{S,p}^{1/2} \Phi) = (d_{S,p} \Psi, d_{S,p} \Phi) + (d_{S,p-1}^* \Psi, d_{S,p-1}^* \Phi), \quad \Psi, \Phi \in D(\Delta_{S,p}^{1/2}).$$

(see [AM1].) We call $\Delta_{S,p}$ the *p-th Laplacian associated with the complex* $\{d_{S,p}\}_{p=0}^{\infty}$.

Proposition 2.2 ([A8],[AM1]). *For each* $p \geq 0$, *the Hilbert space* $\bigwedge^p(E, \mathcal{K})$ *admits the orthogonal decomposition*

$$\bigwedge^p(E, \mathcal{K}) = \overline{R(d_{S,p-1})} \bigoplus \overline{R(d_{S,p}^*)} \bigoplus \ker \Delta_{S,p}.$$

It follows from this proposition that

$$\ker d_{S,p} = \overline{R(d_{S,p-1})} \bigoplus \ker \Delta_{S,p},$$

which implies the vector space isomorphism

$$H_S^p \cong \ker \Delta_{S,p}.$$

Remark. Consider the case $\mu = \mu_0$ (Example 2.1) and let $d\Gamma_b(\cdot)$ be the second quantization operator in the Boson Fock space $L^2(E, d\mu_0)$ [S, §I.4]. Then we can show that $\Delta_{S,0} = d\Gamma_b(S^*S)$ and

$$\Delta_{S,p} = d\Gamma_b(S^*S) \otimes I + I \otimes \left(\sum_{k=1}^{p} I \otimes \cdots \otimes I \otimes \overset{k}{\check{SS^*}} \otimes I \otimes \cdots \otimes I \right), p \geq 1,$$

Using this expression, we can explicitly identify $\ker \Delta_{S,p}$ [A8]. An explicit form of $\Delta_{S,p}$ for a more general μ is given in [A6] and [A7].

The subspace

$$\Omega_S^p(E, \mathcal{K}) := \cap_{n=1}^{\infty} D(\Delta_{S,p}^n)$$

is a countably Hilbert space with a suitable family of inner products and may be regarded as a fundamental space of $\bigwedge^p(\mathcal{K}_c)$-valued functions on (E, Σ). We denote by $\sigma(\Delta_{S,p})$ the spectrum of $\Delta_{S,p}$. For the space $\Omega_S^p(E, \mathcal{K})$, a decomposition theorem of de Rham-Hodge-Kodaira type holds:

Theorem 2.3 [AM1]. *Assume that* $\inf \sigma(\Delta_{S,p}) \setminus \{0\} > 0$. *Then*

$$\Omega_S^p(E,\mathcal{K}) = \Delta_{S,p}\Omega_S^p(E,\mathcal{K}) \bigoplus \ker \Delta_{S,p}.$$

Remark. In the present framework, we can also define a fundamental space of Hida type [HPS], which admits a decomposition as above (see [AM1]).

III. Operator of Dirac-Kähler Type

Let

$$\bigwedge(E,\mathcal{K}) = \bigoplus_{p=0}^{\infty} \bigwedge^p(E,\mathcal{K}).$$

Introducing the Fermion Fock space over $\mathcal{K}_c$

$$\bigwedge(\mathcal{K}_c) = \bigoplus_{p=0}^{\infty} \bigwedge^p(\mathcal{K}_c),$$

we can identify $\bigwedge(E,\mathcal{K})$ as

$$\bigwedge(E,\mathcal{K}) = L^2\left(E, d\mu; \bigwedge(\mathcal{K}_c)\right) = L^2(E,d\mu) \otimes \bigwedge(\mathcal{K}_c).$$

In the case $\mu = \mu_0$, $\bigwedge(E,\mathcal{K})$ is called the *Boson-Fermion Fock space* over $\{\mathcal{H},\mathcal{K}\}$[A8].

The Hilbert space $\bigwedge(E,\mathcal{K})$ admits the orthogonal decomposition

$$\bigwedge(E,\mathcal{K}) = \bigwedge_+(E,\mathcal{K}) \bigoplus \bigwedge_-(E,\mathcal{K}) \tag{3.1}$$

with $\bigwedge_+(E,\mathcal{K}) = \bigoplus_{p=0}^{\infty} \bigwedge^{2p}(E,\mathcal{K}), \bigwedge_-(E,\mathcal{K}) = \bigoplus_{p=0}^{\infty} \bigwedge^{2p+1}(E,\mathcal{K})$. Let $P_\pm$ be the orthogonal projections from $\bigwedge(E,\mathcal{K})$ onto $\bigwedge_\pm(E,\mathcal{K})$ and set

$$\Gamma = P_+ - P_-.$$

Then Γ is a bounded self-adjoint operator on $\bigwedge(E,\mathcal{K})$ with $\Gamma^2 = I$, which means that Γ is a grading operator on $\bigwedge(E,\mathcal{K})$ w.r.t the decomposition (3.1). Note that

$$\Gamma = (-1)^N,$$

where N is the degree operator on $\bigwedge(E,\mathcal{K})$, i.e., $N \restriction \bigwedge^p(E,\mathcal{K}) := p$.

The complex $\{d_{S,p}\}_{p=0}^{\infty}$ gives an operator d_S acting in $\bigwedge(E,\mathcal{K})$:

$$D(d_S) = \left\{ \Psi = \{\Psi^{(p)}\}_{p=0}^{\infty} \in \bigwedge(E,\mathcal{K}) \middle| \Psi^{(p)} \in D(d_{S,p}), \sum_{p=0}^{\infty} \|d_{S,p}\Psi^{(p)}\|^2 < \infty \right\},$$

$$(d_S\Psi)^{(p)} = d_{S,p-1}\Psi^{(p-1)}, \quad \Psi \in D(d_S),\ p \geq 0.$$

Proposition 3.1 ([A6], [A7]). *The operator d_S is densely defined, closed, and satisfies $d_S^2 = 0$.*

Remark. The operator $\widetilde{d_S} := d_S(N+1)^{-1/2}$ is an antiderivation on a suitable domain in $\bigwedge(E, \mathcal{K})$ and may be regarded as an exterior differential operator on the exterior bundle $E \times \bigwedge(\mathcal{K}_c)$.

In the same way as in the case of $\Delta_{S,p}$, we can show that there exists a unique nonnegative self-adjoint operator Δ_S acting in $\bigwedge(E, \mathcal{K})$ such that $D(\Delta_S^{1/2}) = D(d_S) \cap D(d_S^*)$ and

$$(\Delta_S^{1/2}\Psi, \Delta_S^{1/2}\Phi) = (d_S\Psi, d_S\Phi) + (d_S^*\Psi, d_S^*\Phi), \quad \Psi, \Phi \in D(\Delta_S^{1/2}).$$

We call Δ_S the *Laplacian associated with d_S*.

Proposition 3.2 ([A6], [A7]). *The operator equality $\Delta_S = \bigoplus_{p=0}^{\infty} \Delta_{S,p}$ holds.*

As is well known, Dirac-Kähler operators are important objects in analysis on finite dimensional manifolds. In the present framework of our infinite dimensional analysis, an operator of Dirac-Kähler type is defined by

$$Q_S = d_S + d_S^*$$

with $D(Q_S) = D(d_S) \cap D(d_S^*)$. Note that $Q_{iS} = i(d_S - d_S^*)$.

Proposition 3.3. *The operator Q_S is closed and symmetric. Moreover, $D(Q_S)$ is left invariant by Γ and*

$$Q_S\Gamma\Psi + \Gamma Q_S\Psi = 0, \quad \Psi \in D(Q_S). \tag{3.2}$$

For all $\Psi, \Phi \in D(Q_S)(= D(Q_{iS}))$, we have

$$(Q_S\Psi, Q_{iS}\Phi) + (Q_{iS}\Psi, Q_S\Phi) = 0.$$

A relation between Q_S and the Laplacian Δ_S is given by the following Theorem.

Theorem 3.4 [A6]. *The operator equalities $\Delta_S = Q_S^*Q_S = Q_{iS}^*Q_{iS}$ hold.*

Theorem 3.5 [A8]. *Consider the case $\mu = \mu_0$. Then*

$$\Delta_S = d\Gamma_b(S^*S) \otimes +I \otimes d\Gamma_f(SS^*),$$

where $d\Gamma_f(\cdot)$ is the second quantization operator on the Fermion Fock space $\bigwedge(\mathcal{K}_c)$. Moreover, Q_S is self-adjoint and essentially self-adjoint on any core of Δ_S.

Remark. (i) In the case where μ is not Gaussian, we can construct self-adjoint extensions of Q_S [A6](cf. also [A10]). But it is an open problem to prove the self-adjointness of Q_S in this case. See also [A11].

(ii) We can also consider perturbations of Q_S in the same way as in the case of $\mu = \mu_0$ discussed in [A8] (cf. also [A9] and [A13]).

IV. Supersymmetry and Index of the Dirac-Kähler Operator

In application to physics, the framework given in the last section has a connection with supersymmetry. We recall an abstract definition of *supersymmetric quantum theory* (SSQT)([Wi], [A2–A4], [GP]). A SSQT with N-supersymmetry is a quadruple $\{\mathcal{X}, \{Q_n\}_{n=1}^N, H, N_F\}$ consisting of a complex Hilbert space $\mathcal{X}$, a set of self-adjoint operators $\{Q_n\}_{n=1}^N$ (*supercharges*) and self-adjoint operators H (*supersymmetric Hamiltonian*), N_F (*Fermion number operator*) acting in $\mathcal{X}$ with the following properties:

(i) $N_F^2 = I$.

(ii) Each $D(Q_n)$ $(n = 1, \cdots, N)$ is left invariant by N_F and

$$Q_n N_F \psi + N_F Q_n \psi = 0, \quad \psi \in D(Q_n).$$

(iii) $H = Q_n^2, \quad n = 1, \cdots, N.$

(iv) For $n \neq m$ $(n, m = 1, \cdots, N)$, Q_n and Q_m anticommutes in the sense of quadratic form: $(Q_n\psi, Q_m\phi) + (Q_m\psi, Q_n\phi) = 0, \psi, \phi \in D(Q_n) \cap D(Q_m)$.

Remark. (i) In a relativistic supersymmetry, condition (iii) must be replaced by a more complicated one (see, e.g., [F], [We]). An abstract operator-theoretical analysis in such a case is made in [A15] (cf. also [A14]).

(ii) The matheamtical meaning of condition (iv) can be made clear in the light of the theory of anticommuting self-adjoint operators [A12].

Proposition 3.3 and Theorem 3.4 imply the following:

Proposition 4.1. *If Q_S is self-adjoint, then the quadruple $\{\bigwedge(E, \mathcal{K}), \{Q_S, Q_{iS}\}, \Delta_S$ $\Gamma\}$ is a SSQT.*

Remark. One can construct self-adjoint extensions of Q_S which anticommute with Γ and hence give a SSQT [A6]. We can apply this result to construct supersymmetric extensions of quantum scalar field theories [A10].

Assume that Q_S is self-adjoint. Then, by (3.2), there exists a unique densely defined closed linear operator $Q_{S,+} : \bigwedge_+(E, \mathcal{K}) \to \bigwedge_-(E, \mathcal{K})$ such that

$$Q_S = \begin{pmatrix} 0 & Q_{S,+}^* \\ Q_{S,+} & 0 \end{pmatrix},$$

where the matrix representation is relative to the decomposition (3.1). Then it is an interesting problem to consider under what conditions $Q_{S,+}$ is Fredholm and, in that case, to compute

$$\text{index } Q_{S,+} := \dim\ker Q_{S,+} - \dim\ker Q_{S,+}^*,$$

the index of $Q_{S,+}$, which, in the context of SSQFT, is related to the exsistence of supersymmetric states (zero-energy states).

In the case $\mu = \mu_0$, this problem is exactly soluble [A8]. Moreover, we can consider perturbations of $Q_{S,+}$ and establish index formulas for the perturbed Dirac-Kähler operators in terms of path (functional) integral representations [A8].

Acknowledgements

I would like to thank Rémi Léandre, Sylvie Paycha, and Tilmann Wurzbacher for inviting me to lecture at the conference *Journées Mathématiques de Strasbourg – "Espaces de Lacets"* and for their warm hospitality during my one month stay in Strasbourg.

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