Totally umbilical submanifolds in normal contact Riemannian Manifolds

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction.

It is well known that any complete hypersurface of an Euclidean space is isometric to a sphere if it is umbilical and has non-vanishing mean curvature. M. Okumura [3, 4] studied totally umbilical surfaces in a Kaehlerian manifold, and in a locally product manifold. Y. Watanabe [7] studied totally umbilical submanifolds in a normal contact Riemannian manifold and proved the following theorems:

THEOREM. *Let M be a complete connected totally umbilical hypersurface in a normal contact Riemannian manifold. If M is of constant mean curvature H, then M is isometric to a sphere of radius* $1/\sqrt{1+H^2}$ *in an Euclidean space.*

THEOREM. *Let M be a* $(2n-1)$*-dimensional complete connected totally umbilical surface in a* $(2n+1)$*-dimensional normal contact Riemannian manifold. Suppose that the covariant derivative of the mean curvature vector field of M is tangent to M, and that the mean curvature h does not vanish. Then either of the following two cases occurs*

(1) *M is isometric to a sphere of radius* $1/\sqrt{1+h^2}$ *in an Euclidean space*

(2) *M is homothetic to a* $(2n-1)$*-dimensional normal contact Riemannian manifold.*

The purpose of this paper is to give a generalization of the above theorems for submanifolds of codimension greater than 2 in a normal contact Riemannian manifold.

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§ 2. Normal contact Riemannian manifolds.

A $(2n+1)$-dimensional differentiable manifold $\widetilde{M}$ is called a contact Riemannian manifold if there exists a structure (ϕ, ξ, η, G), $\phi=(\phi_\lambda{}^\mu)$ being a (1, 1)-type tensor field, $\xi=(\xi^\lambda)$ a contravariant vector field, $\eta=(\eta_\lambda)$ a 1-form and $G=(G_{\lambda\mu})$ a Riemannian metric tensor field, on $\widetilde{M}$ such that

$$\phi_\lambda^\kappa \xi^\lambda = 0\,, \qquad \phi_\lambda^\kappa \eta_\kappa = 0\,, \qquad \xi^\lambda \eta_\lambda = 1\,, \tag{2.1}$$

$$\phi_\lambda^\kappa \phi_\nu^\lambda = -\delta_\nu^\kappa + \eta_\nu \xi^\kappa\,, \tag{2.2}$$

$$G_{\lambda\kappa}\phi_\mu^\lambda \phi_\nu^\kappa = G_{\mu\nu} - \eta_\mu \eta_\nu\,, \tag{2.3}$$

$$G_{\lambda\kappa}\xi^\kappa = \eta_\lambda\,, \tag{2.4}$$

$$\phi_{\lambda\mu} = G_{\mu\kappa}\phi_\lambda^\kappa = \frac{1}{2}(\partial_\lambda \eta_\mu - \partial_\mu \eta_\lambda)\,, \tag{2.5}$$

where (ϕ_λ^κ), (ξ^λ), (η_λ) and $(G_{\lambda\mu})$ denote respectively the component of ϕ, ξ, η and G with respect to local coordinates $\{X^\lambda\}$ and δ_ν^κ means the Kronecker delta.

If in a contact Riemannian manifold the tensor, defined by

$$N_{\mu\lambda}{}^\kappa = \phi_\mu{}^\nu(\partial_\nu \phi_\lambda^\kappa - \partial_\lambda \phi_\nu^\kappa) - \phi_\lambda{}^\nu(\partial_\nu \phi_\mu{}^\kappa - \partial_\mu \phi_\nu^\kappa) + \partial_\lambda \xi^\kappa \eta_\mu - \partial_\mu \xi^\kappa \eta_\lambda\,, \tag{2.6}$$

vanishes everywhere on $\widetilde{M}$, then the manifold $\widetilde{M}$ is called a normal contact Riemannian manifold.

We have the following theorem by Y. Hatakeyama, Y. Ogawa and S. Tanno [1]:

THEOREM. *A contact Riemannian manifold is normal if and only if the conditions*

$$\nabla_\lambda \eta_\kappa = \phi_{\lambda\kappa}\,, \qquad \nabla_\nu \phi_{\lambda\kappa} = \eta_\lambda G_{\kappa\nu} - \eta_\kappa G_{\lambda\nu}\,, \tag{2.7}$$

are satisfied.

§ 3. Submanifolds in a Riemannian manifold.

Let $\widetilde{M}$ be a $(m+k)$-dimensional orientable Riemannian manifold and M be an m-dimensional orientable submanifold in $\widetilde{M}$. In terms of local coordinates $(X^1, \cdots, X^{m+k})$ of $\widetilde{M}$ and $(x^1, \cdots, x^m)$ of M, M is locally expressed by equations

$$X^\lambda = X^\lambda(x^i) \qquad \begin{pmatrix} \lambda = 1, \cdots, m+k \\ i = 1, \cdots, m \end{pmatrix}. \tag{3.1}$$

If we put

$$B_i^\kappa = \partial X^\kappa / \partial x^i\,, \tag{3.2}$$

then B_i^κ are linearly independent local vector fields tangent to M. A Riemannian metric g_{ji} on M induced from the Riemannian metric $G_{\lambda\kappa}$ on $\widetilde{M}$ is given by

(3. 3) $$g_{ji}=G_{\lambda\kappa}B_j^{\lambda}B_i^{\kappa}.$$

We choose k mutually orthogonal unit normal vectors $N_A^{\lambda}(A=m+1,\cdots,m+k)$. Then we find

(3. 4) $$\begin{aligned} &G_{\lambda\kappa}B_j^{\lambda}N_A^{\kappa}=0, && G_{\lambda\kappa}N_A^{\lambda}N_B^{\kappa}=\delta_{AB},\\ &B_i^{\lambda}B^h{}_{\lambda}=\delta_i^h, && N_A^{\lambda}N_{B\lambda}=\delta_{AB},\\ &B_i^{\lambda}N_{A\lambda}=0, && N_A^{\lambda}B^i{}_{\lambda}=0,\\ &B_i^{\lambda}B^i{}_{\mu}+\sum_A N_A^{\lambda}N_{A\mu}=\delta_{\mu}^{\lambda}, \end{aligned}$$

where we have put $B^i{}_{\lambda}=G_{\lambda\kappa}B_j^{\kappa}g^{ji}$, $N_{A\lambda}=G_{\lambda\kappa}N_A^{\kappa}$.

Let H_{Aji} $(A=m+1,\cdots,m+k)$ be the second fundamental tensors and L_{ABi} the third fundamental tensors. Then the Gauss and Weingarten equation are given respectively by

(3. 5) $$\nabla_j B_i^{\kappa}=\sum_A H_{Aji}N_A^{\kappa},$$

(3. 6) $$\nabla_j N_A^{\kappa}=-H_{Aj}{}^{i}B_i^{\kappa}+\sum_B L_{ABj}N_B^{\kappa},$$

where $\nabla_j B_i^{\kappa}$ and $\nabla_j N_A^{\kappa}$ are defined respectively by

$$\nabla_j B_i^{\kappa}=\partial_j B_i^{\kappa}-\left\{{h \atop j\,i}\right\}B_h^{\kappa}+\left\{{\kappa \atop \lambda\,\mu}\right\}B_j^{\lambda}B_i^{\mu},$$

$$\nabla_j N_A^{\kappa}=\partial_j N_A^{\kappa}+\left\{{\kappa \atop \lambda\,\mu}\right\}B_j^{\mu}N_A^{\lambda},$$

$\left\{{h \atop j\,i}\right\}$ and $\left\{{\kappa \atop \lambda\,\mu}\right\}$ being the Christoffel's symbols of M and $\widetilde{M}$ respectively.

§4. Submanifolds in a normal contact Riemannian manifold.

Let $\widetilde{M}$ be a $(2n+1)$-dimensional normal contact Riemannian manifold with normal contact Riemannian structure $(\phi_{\lambda}^{\kappa},\xi^{\lambda},\eta_{\lambda},G_{\lambda\kappa})$ and M be a submanifold of codimension $k(>2)$ in $\widetilde{M}$. The transform $\phi_{\lambda}^{\kappa}B_i^{\lambda}$ of the tangent vector fields B_i^{λ} by ϕ_{λ}^{κ} can be represented as a sum of its tangential part and its normal part, that is,

(4. 1) $$\phi_{\lambda}^{\kappa}B_i^{\lambda}=f_i^h B_h^{\kappa}+\sum_A f_{Ai}N_A^{\kappa}.$$

In the same way, we can put

(4. 2) $$\phi_{\lambda}^{\kappa}N_A^{\lambda}=h_A^i B_i^{\kappa}+\sum_B h_{AB}N_B^{\kappa}.$$

From (4. 1) and (4. 2) we have

(4. 3) $$f_i^h=B^h{}_{\kappa}\phi_{\lambda}^{\kappa}B_i^{\lambda},$$

(4. 4) $$f_{Ai}=N_{A\kappa}\phi_{\lambda}^{\kappa}B_i^{\lambda},$$

(4. 5) $$h_A{}^i = B^i{}_\kappa \phi_\lambda{}^\kappa N_A{}^\lambda = -f_{Aj} g^{ji},$$

(4. 6) $$h_{AB} = N_{B\kappa} \phi_\lambda{}^\kappa N_A{}^\lambda = -h_{BA}.$$

On the other hand, we can put

(4. 7) $$\xi^\kappa = u^h B_h{}^\kappa + \sum_A u_A N_A{}^\kappa,$$

from the above relation we have

(4. 8) $$u_i = \xi^\kappa B^i{}_\kappa, \qquad u_i = u^j g_{ji} = \eta_\kappa B_i{}^\kappa,$$

(4. 9) $$u_A = \eta_\kappa N_A{}^\kappa.$$

Transforming both members of (4. 1) by $\phi_\kappa{}^\mu$, we have

$$-\delta_i{}^j B_j{}^\mu + u_i u^j B_j{}^\mu + \sum_B u_i u_B N_B{}^\mu$$
$$= (f_i{}^h f_h{}^j - \sum_A f_{Ai} f_A{}^j) B_j{}^\mu + \sum_B (f_i{}^h f_{Bh} + \sum_A f_{Ai} h_{AB}) N_B{}^\mu,$$

by means of (2. 2), (4. 1), (4. 2), (4. 5) and (4. 7). Because $B_j{}^\mu$, $N_A{}^\mu$ are linearly independent, we find

(4. 10) $$f_i{}^h f_h{}^j = -\delta_i{}^j + u_i u^j + \sum_A f_{Ai} f_A{}^j,$$

(4. 11) $$f_i{}^h f_{Ah} = u_A u_i - \sum_B f_{Bi} h_{BA}.$$

Similarly from (4. 2), we have

$$-\sum_B \delta_{AB} N_B{}^\mu + u_A u^j B_j{}^\mu + \sum_B u_A u_B N_B{}^\mu$$
$$= -(f_A{}^i f_i{}^j + \sum_B h_{AB} f_B{}^j) B_j{}^\mu + \sum_B (-f_A{}^i f_{Bi} + \sum_C h_{AC} h_{CB}) N_B{}^\mu,$$

which implies

(4. 12) $$f_A{}^i f_i{}^j = -\sum_B h_{AB} f_B{}^j - u_A u^j,$$

(4. 13) $$f_A{}^i f_{Bi} = \delta_{AB} - u_A u_B + \sum_C h_{AC} h_{CB}.$$

On the other hand, making use of (4. 1) and (4. 7), condition (2. 1) can be written respectively as

$$0 = \phi_\lambda{}^\kappa \xi^\lambda = \phi_\lambda{}^\kappa (u^i B_i{}^\lambda + \sum_A u_A N_A{}^\lambda)$$
$$= (u^i f_i{}^h - \sum_A u_A f_A{}^h) B_h{}^\kappa + \sum_B (u^i f_{Bi} + \sum_A u_A h_{AB}) N_B{}^\kappa,$$
$$1 = \eta_\lambda \xi^\lambda = (u^i B_{i\lambda} + \sum_A u_A N_{A\lambda})(u^j B_j{}^\lambda + \sum_B u_B N_B{}^\lambda) = u_i u^i + \sum_A u_A{}^2,$$

from which we find

(4. 14) $$u^i f_i{}^h = \sum_A u_A f_A{}^h,$$

(4. 15) $$u^i f_{Ai} = -\sum_B u_B h_{BA},$$

(4. 16) $$u_i u^i = 1 - \sum_A u_A^2.$$

Differentiating (4. 1) covariantly and making use of (2. 7), (4. 2) and (4. 7), we have

$$u_i \delta_j^h B_h^\kappa - g_{ji} u^h B_h^\kappa - \sum_A u_A g_{ji} N_A^\kappa - \sum_A H_{Aji} f_A^h B_h^\kappa + \sum_A \sum_B H_{Bji} h_{BA} N_A^\kappa$$
$$= \nabla_j f_i^h B_h^\kappa + \sum_A f_i^h H_{Ajh} N_A^\kappa + \sum_A \nabla_j f_{Ai} N_A^\kappa - \sum_A f_{Ai} H_{Aj}^h B_h^\kappa + \sum_A \sum_B f_{Bi} L_{BAj} N_A^\kappa,$$

which implies

(4. 17) $$\nabla_j f_{ih} = u_i g_{jh} - u_h g_{ji} - \sum_A (f_{Ah} H_{Aji} - f_{Ai} H_{Ajh})$$

(4. 18) $$\nabla_j f_{Ai} = -u_A g_{ji} + \sum_B (H_{Bji} h_{BA} - f_{Bi} L_{BAj}) - f_i^h H_{Ajh}.$$

Differentiating (4. 2) covariantly and making use of (2. 7), (4. 1), (4. 2) and (4. 7), we have

$$u_A B_j^\kappa - H_{Aj}^h (f_h^i B_i^\kappa + \sum_B f_{Bh} N_B^\kappa) + \sum_B L_{ABj} (-f_B^i B_i^\kappa + \sum_C h_{BC} N_C^\kappa)$$
$$= -\nabla_j f_A^h B_h^\kappa - \sum_B (f_A^i H_{Bji} - \nabla_j h_{AB}) N_B^\kappa + \sum_B h_{AB} (-H_{Bj}^i B_i^\kappa + \sum_C L_{BCj} N_C^\kappa),$$

from which we find

(4. 19) $$\nabla_j f_A^i = -u_A \delta_j^i + H_{Aj}^h f_h^i + \sum_B (h_{BA} H_{Bj}^i - L_{BAj} f_B^i),$$

(4. 20) $$\nabla_j h_{AC} = f_A^i H_{Cji} - f_C^i H_{Aji} + \sum_B (L_{ABj} h_{BC} - L_{BCj} h_{AB}).$$

Differentiating (4. 7) covariantly and making use of (2. 7) and (4. 1), we have

$$f_j^i B_i^\kappa + \sum_A f_{Aj} N_A^\kappa = \nabla_j u^i B_i^\kappa + \sum_A u^i H_{Aji} N_A^\kappa$$
$$+ \sum_A \left\{ \nabla_j u_A N_A^\kappa + u_A (-H_{Aj}^i B_i^\kappa + \sum_B L_{BAj} N_B^\kappa) \right\},$$

which implies

(4. 21) $$\nabla_j u^i = f_j^i + \sum_A u_A H_{Aj}^i,$$

(4. 22) $$\nabla_j u_A = f_{Aj} - u^i H_{Aji} - \sum_B u_B L_{BAj}.$$

§ 5. Totally umbilical submanifolds in a normal contact Riemannian manifold.

A submanifold M is called a totally umbilical submanifold if the second fundamental tensors H_{Aji} are proportional to the metric tensor g_{ji}, that is, satisfying the following condition

(5. 1) $$H_{Aji} = H_A g_{ji}.$$

Then the mean curvature vector field H^λ of M is given by

(5. 2) $$H^\lambda = \sum_A H_A N_A^\lambda.$$

LEMMA. *Let M be a totally umbilical submanifold in $\tilde{M}$. In order that the covariant derivative $\nabla_j H^\lambda$ of the mean curvature vector field H^λ of M is tangent to M, it is necessary and sufficient that*

(5. 3) $$\nabla_j H_B = -\sum_A H_A L_{ABj}.$$

PROOF. Differentiating (5. 2) covariantly, we have

$$\begin{aligned}\nabla_j H^\lambda &= \sum_A \nabla_j H_A N_A^\lambda + \sum_A H_A(-H_A B_j^\lambda + \sum_B L_{ABj} N_B^\lambda)\\ &= -\sum_A H_A^2 B_j^\lambda + \sum_B (\nabla_j H_B + \sum_A H_A L_{ABj}) N_B^\lambda,\end{aligned}$$

which proves the assertion of Lemma.

THEOREM. *Let M be a totally umbilical submanifold in $\tilde{M}$. If the covariant derivative $\nabla_j H^\lambda$ of the mean curvature vector field H^λ of M is tangent to M, then the mean curvature h of M is constant.*

PROOF. The mean curvature h of M is given by

(5. 4) $$h^2 = \sum_A H_A^2.$$

Differentiating (5. 4) covariantly and making use of (5. 3), we have

$$\begin{aligned}\nabla_j h^2 &= 2\sum_A H_A \nabla_j H_A = -2\sum_A H_A(\sum_B H_B L_{BAj})\\ &= -2\sum_A\sum_B H_A H_B L_{ABj} = 0,\end{aligned}$$

by means of $L_{ABj} = -L_{BAj}$. This proves that h is constant.

We know the following Obata's theorem.

THEOREM (M. Obata [2]). *Let M be a complete connected Riemannian manifold of dimension $n(>2)$. In order that M admits a non-trivial solution of the system of differential equations*

$$\nabla_\lambda \nabla_\kappa \phi + k\phi G_{\lambda\kappa} = 0, \qquad (k = const. > 0)$$

it is necessary and sufficient that M is isometric to a sphere S^n of radius $1/\sqrt{k}$ in the Euclidean $(n+1)$-space.

Now we shall prove the following theorem:

THEOREM. *Let $\tilde{M}$ be a normal contact Riemannian manifold and M be a complete connected totally umbilical submanifold of codimension $k(>2)$ in $\tilde{M}$. If the covariant derivative $\nabla_j H^\lambda$ of the mean curvature vector field*

H^{λ} of M is tangent to M and $H^{\lambda}\eta_{\lambda}$ does not vanish identically, then M is isometric to a sphere of radius $1/\sqrt{1+\sum_{A} H_A^2}$ in the Euclidean space.

PROOF. Making use of (5. 1), from (4. 18), (4. 21) and (4. 22), we have

$$\nabla_j f_{Ai} = -u_A g_{ji} + \sum_B (H_B h_{BA} g_{ji} - f_{Bi} L_{BAj}) + H_A f_{ji}, \tag{5. 5}$$

$$\nabla_j u_i = f_{ji} + \sum_A u_A H_A g_{ji}, \tag{5. 6}$$

$$\nabla_j u_A = f_{Aj} - H_A u_j - \sum_B u_B L_{BAj}. \tag{5. 7}$$

We put

$$\phi = H^{\lambda}\eta_{\lambda} = \sum_A u_A H_A. \tag{5. 8}$$

Differentiating (5. 8) covariantly and making use of (5. 3) and (5. 7), we have

$$\begin{aligned}\nabla_i \phi &= \sum_A \nabla_i u_A H_A + \sum_A u_A \nabla_i H_A \\ &= \sum_A (f_{Ai} - H_A u_i - \sum_B u_B L_{BAi}) H_A - \sum_A \sum_B u_A H_B L_{BAi} \\ &= \sum_A H_A f_{Ai} - \sum_A H_A^2 u_i.\end{aligned}$$

On the other hand, we can see from the preceding theorem that the mean curvature of M is constant. Then, differentiating the above equation covariantly and making use of (5. 5), (5. 6) and (5. 7), we have

$$\begin{aligned}\nabla_j \nabla_i \phi &= \sum_A \nabla_j H_A f_{Ai} + \sum_A H_A \nabla_j f_{Ai} - \sum_A H_A^2 \nabla_j u_i \\ &= -\sum_A \sum_B H_B L_{BAj} f_{Ai} + \sum_A H_A \Big\{ -u_A g_{ji} + \sum_B (H_B h_{BA} g_{ji} - f_{Bi} L_{BAj}) + H_A f_{ji} \Big\} \\ &\quad - \sum_A H_A^2 (f_{ji} + \sum_B u_B H_B g_{ji}) \\ &= -\sum_A u_A H_A g_{ji} - \sum_A H_A^2 \sum_B u_B H_B g_{ji}.\end{aligned}$$

Therefore, by means of (5. 8), we get

$$\nabla_j \nabla_i \phi + (1 + \sum_A H_A^2) \phi g_{ji} = 0.$$

This shows that M is isometric to a sphere of radius $1/\sqrt{1+\sum_A H_A^2}$ in the Euclidean space by means of Obata's theorem.

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